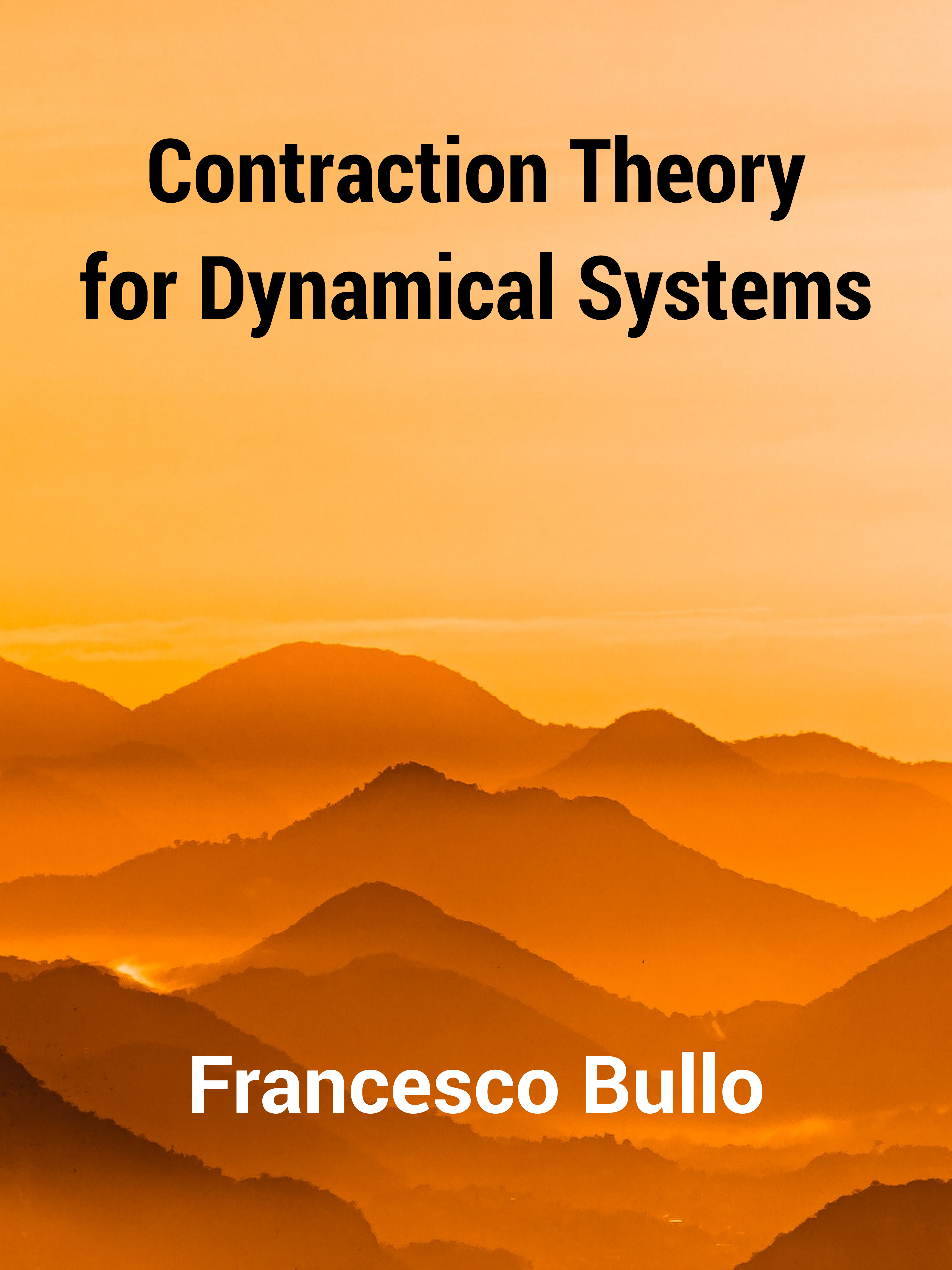
Contraction Theory for Dynamical Systems



Francesco Bullo

Contraction Theory for Dynamical Systems
Francesco Bullo
Edition 1.1, Mar 2023
Kindle Direct Publishing (print-on-demand publishing service)
ISBN 979-8836646806

(This document generated on March 3, 2023 with minor corrections and additions.)

Citation Information:

F. Bullo, *Contraction Theory for Dynamical Systems*, ed. 1.1, Kindle Direct Publishing, 2023, ISBN 979-8836646806

Book Website: Electronic versions of this document and updates are available at:

`http://motion.me.ucsb.edu/book-ctds`

To Gabriella, Marcello, and Lily

Preface

> *Books which try to digest, coordinate, get rid of the duplication, get rid of the less fruitful methods and present the underlying ideas clearly of what we know now, will be the things the future generations will value.* (Richard Hamming 1986)

These lecture notes provide a mathematical introduction to contraction theory for dynamical systems. Special emphasis is given to continuous-time differential equations arising in the study of network multi-agent systems, monotone dynamics, and semi-contracting systems. This document is in its initial version 1.0 on June 1, 2022. It is envisioned that future updates will include additional content.

These notes present in a unified framework: (i) a review of the Banach contraction theorem and fixed point theory, (ii) a comprehensive treatment of induced norms and induced logarithmic norms of matrices, (iii) definition and properties of strongly contracting dynamics over finite-dimensional vector spaces endowed with Euclidean and non-Euclidean norms, (iii) extensions to weakly-contracting dynamics and monotone dynamics, and (iv) extensions to semicontracting, perpendicularly and partially contracting systems. Numerous examples are presented in detail, including Hopfield neural networks, systems in Lure' form, interconnected contracting systems, gradient and primal dual flows of convex functions, Lotka-Volterra population dynamics, Daganzo traffic models, averaging flows, and diffusively-coupled synchronizing systems.

Contraction analysis for dynamical systems

These lecture notes aim to popularize a powerful set of tools applicable to a broad range of examples. The resulting framework has the potential to substantially push the boundaries of what can be rigorously analyzed in dynamical and control systems. This key tool is modern, geometric *contraction theory for dynamical systems*.

A dynamical system is contracting if any two solutions approach one another at an exponential rate. This single property implies highly ordered transient and asymptotic behavior, including (1) existence and global exponential stability of an equilibrium for time-invariant vector fields, (2) existence and global exponential stability of a limit cycle for time-varying periodic vector fields, (3) input-to-state stability and finite system gain for systems subject to state-independent disturbances as well as robustness with respect to unmodeled dynamics, (4) modularity and interconnection properties, and more. An objective of the theory is to provide explicit easily-checkable conditions for the key contractivity properties.

There are numerous advantages to contraction-based robust stability analysis rather than generic Lyapunov stability analysis.

(i) First, contraction analysis provides a single condition (for the system to satisfy) that implies numerous beneficial properties. In other words, the true distinction in the analysis of dynamical systems is not between linear and nonlinear systems, but rather between contracting and non-contracting systems.

Consider for example a generic nonlinear system or maybe even a complex large-scale network. In such a general setting, one does not know: whether an equilibrium point exists, for what parameter values does it exist, and how does it depend upon parameters. Contractivity analysis is a useful tool to simultaneously ensure the existence, uniqueness and exponential stability of an operating condition.

(ii) Second, while much control theory is focused on identifying the weakest possible conditions guaranteeing a given stability property of interest, in engineering and scientific applications of control principles one is often interested in devising models or controllers such that the final system enjoys multiple robustness properties. In this sense, contraction theory for dynamical systems is the right set of tools for non-specialists. Independently of rigorous Lyapunov stability definitions in control theory, when non-control scientists say "stable system," they essentially mean "contracting systems."

(iii) Contraction theory for dynamical systems naturally extends to systems on Banach spaces or on Riemannian manifolds, to stochastic systems, and to differential-algebraic models. Recent notions of higher-order contractivity allow us to study multistability and other rich dynamical behaviors. Contraction theory for dynamical systems is also inherently tightly connected with fixed point theory and monotone operator theory. (In the interest of brevity, this book only briefly touches upon some of these extensions.)

Historical notes

The study of contraction theory initiated with the ground-breaking work by (Stefan Banach 1922). While fixed point theory has received much attention over the decades, e.g., see the excellent texts (Zeidler, 1986; Khamsi and Kirk, 2001; Granas and Dugundji, 2003; Berinde, 2007), contraction theory for the analysis of dynamical systems is a much less explored field.

This text presents as a body of work that builds on the classic works by (Demidovič, 1961; Krasovskiĭ, 1963; Desoer and Haneda, 1972; Chua and Green, 1976; Vidyasagar, 1978a; Wu and Chua, 1995; Fang and Kincaid, 1996), the pioneering efforts by (Lohmiller and Slotine, 1998, 2000), and the important advances by (Pavlov et al., 2004; Wang and Slotine, 2005; Pham and Slotine, 2007; Russo et al., 2010; Russo et al., 2013; Forni and Sepulchre, 2014; Aminzare and Sontag, 2014a; Manchester and Slotine, 2017; Jafarpour et al., 2022; Wu et al., 2022b; Davydov et al., 2022a). Specifically, recent notable advances have shown how contraction theory is well suited to study *monotone systems* (Angeli and Sontag, 2003; Hirsch and Smith, 2005; Sontag, 2007; Como, 2017; Coogan, 2019; Kawano et al., 2020; Jafarpour et al., 2023), including traffic and infrastructure models, and (state-space models of) coupled oscillators (Turing, 1952; Goodwin, 1965; FitzHugh, 1961; Aminzare and Sontag, 2014a; Di Bernardo et al., 2016), including biological and neural networks.

This text is also influenced by the recent surveys (Aminzare and Sontag, 2014b; Di Bernardo et al., 2016; Tsukamoto et al., 2021; Giesl et al., 2023) and the doctoral theses (Russo, 2010; Aminzare, 2015; Wu, 2022), among numerous others. For the interested readers, each chapter ends with a section on "Historical notes and further reading."

Finally, we provide an incomplete list a topics not covered in this text and corresponding references for the interested reader. First, most of the systems and control literature treats contraction with respect to state-dependent metrics, i.e., contraction on Riemannian manifolds; see (Lohmiller and Slotine, 1998; Simpson-Porco and Bullo, 2014; Tsukamoto et al., 2021). On a related note, we refer to (Aylward et al., 2008; Giesl et al., 2023) for computational methods for contraction metrics. Contraction theory for stochastic systems is discussed in (Pham et al., 2009; Aminzare, 2022). Control contractions and transverse contraction are introduced in (Manchester and Slotine, 2017; Manchester and Slotine, 2014, 2018); see also (Lopez and Slotine, 2021; Tsukamoto and Chung, 2021; Singh et al., 2020). Based on the seminal work (Muldowney, 1990; Li and Muldowney, 1996), we refer to (Wu et al., 2022a) for recent work on k-contraction, i.e., contraction of k-dimensional bodies, and to (Wu et al., 2022b) for the notion of α-contraction, including contraction to fractal sets. Contraction theory for reachability analysis is studied in (Maidens and Arcak, 2015; Fan et al., 2018). Contraction after transients is studied by Margaliot et al. (2016).

Teaching instructions This book is intended for first or second-year graduate students in science, technology, engineering, and mathematics programs. The required background includes (i) competency in linear algebra and calculus and (ii) basic notions of dynamical systems and Lyapunov stability theory. The treatment is essentially self-contained and does not require a complete nonlinear systems course.

These lecture notes are an outgrowth of a graduate course that I taught at UC Santa Barbara a few times over the last years. Typically, I cover the main four chapters in 20-25 hours of lecture time.

Book versions These lecture notes are provided in the following formats:

- a *printed paperback version*, published by Kindle Direct Publishing (7in×10in, gray-scale), ISBN 979-8836646806, for sale at https://www.amazon.com/dp/B0B4K1BTF4,
- a *tablet PDF version*, optimized for viewing on tablets (3×4 aspect ratio, color), freely downloadable from the book website, and
- a *slides PDF version*, that is an abbreviated version suited for displaying on a classroom projector, freely downloadable from the book website.

The book website is:

http://motion.me.ucsb.edu/book-ctds

Self-publishing print-on-demand books There are several reasons why I decided to self-publish this book via the print-on-demand book publishing service by Kindle Direct Publishing. I appreciate being able to

(i) retain the full copyrights of all the material,
(ii) make the document freely available online in multiple versions (see above),
(iii) retain direct anytime control over the paperback publication costs, retain a majority of the royalties after low fix costs, and set a low price,
(iv) update the book revision at any time (simply re-upload the PDF file, minimal turn-around time, no need for any permission, and never out-of-print), and
(v) make available a high-quality paperback book through a broad distribution network.

Similar arguments are presented in my previous self-published book (Bullo, 2022) and in the write-up *Why I Self-Publish My Mathematics Texts With Amazon* by Robert Ghrist, the self-publishing author of (Ghrist, 2014).

This document is in its initial version 1.0, June, 2022. I envision that future updates will include additional chapters.

Acknowledgments

I would like to thank all my collaborators on the topics in this manuscript, including: Saber Jafarpour (Cisneros-Velarde et al., 2022a; Jafarpour et al., 2022; Cisneros-Velarde et al., 2022b; Davydov et al., 2022a; Jafarpour et al., 2023, 2021b; Davydov et al., 2022b), Alexander Davydov (Davydov et al., 2022a; Jafarpour et al., 2023, 2021b; Davydov et al., 2022c,b), Anton Proskurnikov (Davydov et al., 2022c; Jafarpour et al., 2021b; Davydov et al., 2022b), Pedro Cisneros-Velarde (Cisneros-Velarde et al., 2022b; Jafarpour et al., 2022; Cisneros-Velarde et al., 2022a), Kevin D. Smith (Smith et al., 2022; De Pasquale et al., 2022; Smith and Bullo, 2023), Veronica Centorrino and Giovanni Russo (Centorrino et al., 2022), Giulia De Pasquale and M. Elena Valcher (De Pasquale et al., 2022), Michael Margaliot and Ron Ofir (Ofir et al., 2022). Lastly, special thanks go to John W. Simpson-Porco who first introduced me to this topic (Simpson-Porco and Bullo, 2014).

Additionally, I would like to thank Samuel Coogan, Xiaoming Duan, Mario Di Bernardo, Rami Kats, Ian Manchester, and Karl J. Obermeyer for fruitful discussions and contributions.

This document would not be possible without the extensive support of students, staff and faculty at the University of California at Santa Barbara. This effort is based upon work partially supported by AFOSR grant FA9550-22-1-0059 and DTRA grant HDTRA1-19-1-0017; special thanks to Dr. Frederic Leve and Dr. Paul Tandy for their support and advice. Finally, I wish to thank the contributing authors and maintainers of LaTeX, Unsplash.com, StockSnap.io, Openclipart.org, Ghostscript, ImageMagick, and Wikipedia.

Finally, I thank Gabriella, Marcello, Lily, and my whole family for their loving support.

Santa Barbara, California, USA
September 1, 2018 – June 1, 2022

Francesco Bullo

Comments on revision 1.1 Since the first edition on June 1, 2022, I have corrected a theorem statement and improved several proofs. Regarding exercises, four are improved, three more have solutions and ten new ones are added. I have reorganized the treatment of the variations of the Banach Contraction Theorem working towards a more parallel treatment of discrete and continuous time problems.

For their extensive help in helping me understand this material, I wish to thank Veronica Centorrino, Alexander Davydov, Giulia De Pasquale, Robin Delabays, Anand Ghokale, Giovanni Russo, Kevin D. Smith, and M. Elena Valcher.

Detailed list of changes: In Chapter 1, added Remark 1.2 "Continuity and related concepts", reorganized Section 1.4 "Variations on the theme by Banach", added Figure 1.1, expanded E1.1, added solution to E1.6, added E1.7 and E1.8. In Chapter 2, improved the proof of Lemma 2.1, improved Figure 2.3, added E2.4, improved E2.9, E2.14 and E2.26, added E2.28, E2.29, E2.30 and E2.31. In Chapter 3, added Remark 3.6 "Existence and uniqueness theorems for ordinary differential equations" and clarified smoothness assumptions on vector fields throughout the chapter, added E3.4, improved E3.5, added solutions to E3.12, added E3.15 and E3.16, In Chapter 4, removed an incorrect claim in Theorem 4.4, added solution to E4.4. In Chapter 5, corrected the orthonormal basis in the example in Section 5.2.

March 1, 2023, Santa Barbara, California, USA

Francesco Bullo

Contents

Part I

Preliminaries

Chapter 1

A Primer on Fixed Point Theory and the Banach Contraction Theorem

Nous allons démontrer que la suite $\{X_n\}$ converge suivant la norme vers un certain élément X. (Stefan Banach 1922)[1]

We make no attempt to deal in the broadest generalities, but rather, we will try to expose the essential core simply, without trivializing it. (Eberhard Zeidler 1986)

Banach's Contraction Mapping Principle is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. (Mohamed A. Khamsi and William A. Kirk 2001)

1.1 Introduction

Motivated by the analysis of static and dynamic nonlinear systems, this chapter presents models, concepts and theorems about fixed point problems and iterative algorithms for their solution. Specifically, we provide a self-contained introduction to the Banach contraction theorem and numerous variations.

We start with a simple problem statement. Let $\mathcal{X}$ be a non-empty set and consider a map $T\colon \mathcal{X} \to \mathcal{X}$. The point $x \in \mathcal{X}$ is a *fixed point* of T if

$$T(x) = x. \tag{1.1}$$

Given an initial value $x_0 \in \mathcal{X}$, the sequence generated by

$$x_{k+1} = T(x_k) \tag{1.2}$$

is known as the *Picard iteration* for the map T starting at x_0.

For example, consider the scalar map $T\colon \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x/3 + 2$ and the sequence $x_0 = 1$, $x_1 = T(x_0) = 7/3$, $x_2 = T(x_1) = 25/9$, and so forth. We will see how this sequence converges to $x^* = 3$, the unique solution to the fixed point problem: $x = T(x)$.

The scientific literature contains numerous *fixed point theorems*, that is, numerous approaches to characterize the structure of the set $\mathcal{X}$ and the properties of the map T in order to (i) ensure the existence of fixed points, (ii) provide methods to compute them such as the Picard iteration, and (iii) characterize their properties (uniqueness, dependency upon parameters, etc).

In this chapter we mostly focus on the *Banach Contraction Theorem* because of its following desirable properties:

[1] Translation: We are going to show that the sequence $\{X_n\}$ converges according to the norm towards a certain element X.

(i) the theorem assumptions are remarkably simple to state and generalize. Specifically, the theorem describes "contracting maps" over "complete metric spaces" — for now we only comment that (a) contractivity is often easy to test and (b) there are many practical examples of complete metric spaces;

(ii) the theorem provides a constructive method, namely the so-called Picard iteration, to compute the fixed point. Moreover, the theorem provides rate of convergence and error estimates for the Picard iteration. These estimates are practically useful;

(iii) the theorem is the most widely-applied fixed point theorem in functional analysis. Specifically, the theorem provides the standard and canonical approach in the study of differential and integral equations;

(iv) under mild additional conditions, the resulting fixed point enjoys strong robustness properties with respect to parameters and data describing the fixed point problem.

As we mention in (i) and show below, there are many examples of complete metric spaces. In this book we will focus on finite dimensional vector spaces endowed with a norm. Future extensions of this document may include treatments of fixed point problems and dynamical systems on (complete) Riemannian manifolds or (infinite-dimensional) Banach spaces.

As we mention in (iii), the Banach Contraction Theorem and its variations have extensive, diverse and fundamental applications. As an incomplete list, the theorem is used in standard proofs for:

(i) the Picard-Lindelöf Theorem about existence and uniqueness of solutions to ordinary differential equations, see a brief review in Exercise E1.2,

(ii) the Perron-Frobenius Theorem about positive matrices, see a brief review in Exercise E1.3,

(iii) the inverse and implicit function theorems in nonlinear analysis,

(iv) the stable manifold and center manifold theorems in the study of nonlinear dynamics,

(v) existence theorems in PDEs and differential geometry, and

(vi) the convergence analysis of basic algorithms for dynamic programming.

1.2 Preliminary notions

Definition 1.1 (Metric space). *Let $\mathcal{X}$ be a non-empty set. A map $d\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}$ is a metric (or a distance) on $\mathcal{X}$ if*

$$\text{(separation):}\qquad d(x,y)=0 \text{ if and only if } x=y, \tag{1.3a}$$

$$\text{(symmetry):}\qquad d(x,y)=d(y,x) \text{ for all } x,y\in\mathcal{X}, \text{ and} \tag{1.3b}$$

$$\text{(triangle inequality):}\qquad d(x,y)\leq d(x,z)+d(z,y) \text{ for all } x,y,z\in\mathcal{X}. \tag{1.3c}$$

Given a metric d on a set $\mathcal{X}$, the pair $(\mathcal{X},d)$ is called a metric space.

One can show that, if d is a metric satisfying the three axioms in (1.3), then automatically $d(x,y)\geq 0$ for all $x,y\in\mathcal{X}$.

Examples of metric spaces include:

(i) the set of real numbers $\mathbb{R}$ with metric $d(x,y)=|x-y|$, where $|\cdot|$ is the absolute value;

(ii) the set of positive real numbers $\mathbb{R}_{>0}$ with metric $d_{\log}(x,y)=|\log(x/y)|$;

(iii) the Euclidean space $\mathbb{R}^n$ with the Euclidean metric $d_{\mathrm{E}}(x,y)=\Big(\sum_{i=1}^n(x_i-y_i)^2\Big)^{1/2}$;

(iv) any finite-dimensional vector space V with a norm $\|\cdot\|$ is a metric space with distance $d_{\|\cdot\|}(x,y) = \|x-y\|$; we will properly define norms and their properties in later chapters.

Two metrics d_1 and d_2 on the same set $\mathcal{X}$ are *equivalent* (or metrically equivalent or uniformly equivalent) if there exists constants $M > m > 0$ such that

$$md_1(x,y) \leq d_2(x,y) \leq Md_1(x,y) \quad \text{for all } x,y \in \mathcal{X}. \tag{1.4}$$

Remark 1.2 (Continuity and related concepts). *The open ball centered at $x \in \mathcal{X}$ with radius $r > 0$ is $B_r(x) = \{z \in \mathcal{X} \,:\, d(z,x) < r\}$. A set $N \subset \mathcal{X}$ is a neighborhood of a point $x \in \mathcal{X}$ if it contains an open ball centered at x with strictly positive radius. A function $f\colon \mathcal{X} \to \mathbb{R}$ is continuous at an element $x \in \mathcal{X}$ if, for every $\varepsilon > 0$, there is a neighborhood U of x such that f maps U into $(f(x)-\varepsilon, f(x)+\varepsilon)$. Given a metric space $(\mathcal{X}, d)$, the following facts are simple to show: the set $\mathcal{X} \times \mathcal{X}$ is a metric space with distance function given by the component-wise sum of distances, and the metric function $d\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is continuous.* •

Definition 1.3 (Convergence properties of sequences). *Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in a metric space $(\mathcal{X}, d)$. The sequence $\{x_k\}_{k\in\mathbb{N}}$ is*

(i) convergent to a point $x^* \in \mathcal{X}$ if, for any $\varepsilon > 0$, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that $d(x_k, x^*) < \varepsilon$ for each $k > k_0(\varepsilon)$,

(ii) a Cauchy sequence if, for any $\varepsilon > 0$, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that $d(x_k, x_{k+h}) < \varepsilon$ for each $k > k_0(\varepsilon)$ and $h \in \mathbb{N}$.

Note that, in an arbitrary metric space, any convergent sequence is a Cauchy sequence, but the converse is not true in general. For example, consider the set of rational numbers $\mathbb{Q}$ with the absolute value distance $|\cdot|$. On this metric space, the sequence $\{a_k = (1+1/k)^k\}_{k\in\mathbb{N}} \subset \mathbb{Q}$ is a Cauchy sequence but converges to Euler's number $\mathrm{e} \in \mathbb{R} \setminus \mathbb{Q}$.

Definition 1.4 (Complete metric space). *A metric space $(\mathcal{X}, d)$ is complete if each Cauchy sequence in $\mathcal{X}$ is convergent, that is, converges to a point in $\mathcal{X}$.*

Each example metric space listed above, that is, $(\mathbb{R}, |\cdot|)$, $(\mathbb{R}_{>0}, d_{\log})$, $(\mathbb{R}^n, d_{\mathrm{E}})$ and $(V, d_{\|\cdot\|})$, is complete. On the other hand, $(\mathbb{Q}, |\cdot|)$ is not a complete metric space. It is known that every metric space has a unique completion, that is, a complete space that contains the given space as a dense subset. Indeed, the completion of $\mathbb{Q}$ is $\mathbb{R}$.

Definition 1.5 (Lipschitz maps). *Given a metric space $(\mathcal{X}, d)$, a map $T\colon \mathcal{X} \to \mathcal{X}$ is*

(i) Lipschitz if there exists $\ell \geq 0$, called a Lipschitz constant of T, such that

$$d(T(x), T(y)) \leq \ell\, d(x,y) \quad \textit{for all } x,y \in \mathcal{X}, \tag{1.5}$$

(ii) a contraction if it is Lipschitz with constant $\ell < 1$. In this case, ℓ is called the contraction factor of T.

The minimal Lipschitz constant of T, denoted by $\mathsf{Lip}(T)$ is defined by

$$\mathsf{Lip}(T) := \sup_{x\neq y} \frac{d(T(x), T(y))}{d(x,y)} \in \mathbb{R}_{\geq 0}. \tag{1.6}$$

1.3 The Banach Contraction Theorem and its proof

We are now ready to present a complete form of the main theorem in this chapter.

Theorem 1.6 (Banach Contraction Theorem). *Let $(\mathcal{X}, d)$ be a complete metric space. If $T\colon \mathcal{X} \to \mathcal{X}$ is a contraction with contraction factor ℓ, then*

(i) T has a unique fixed point x^* in $\mathcal{X}$;

(ii) the sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by the Picard iteration $x_{k+1} = T(x_k)$ converges to x^* for all initial conditions $x_0 \in \mathcal{X}$;

(iii) the following error estimates hold for all $k \in \mathbb{N}$:

$$\text{(geometric convergence):} \qquad d(x_k, x^*) \leq \ell^k d(x_0, x^*), \tag{1.7a}$$

$$\text{(a-priori upper bound):} \qquad d(x_k, x^*) \leq \frac{\ell^k}{1-\ell} d(x_0, x_1), \tag{1.7b}$$

$$\text{(a-posteriori upper bound):} \qquad d(x_k, x^*) \leq \frac{\ell}{1-\ell} d(x_{k-1}, x_k). \tag{1.7c}$$

Note: The a-priori upper bound (1.7b) shows that the approximation error $d(x_k, x^*)$ at iterate k is upper bounded as a function only of the contraction factor ℓ and the initial displacement $d(x_1, x_0)$. The a-posteriori upper bound (1.7c) is useful to design a stopping criterion for the Picard iteration: given a desired error tolerance $0 < \varepsilon \ll 1$, as soon as the sequence satisfies $d(x_{k-1}, x_k) \leq (1-\ell)\varepsilon/\ell$, we know that the approximation error satisfies $d(x_k, x^*) \leq \varepsilon$.

Note: In the language of dynamical systems, the function $x \mapsto d(x, x^*)$ is a Lyapunov function for the discrete-time dynamical system defined by T and its level sets are T-invariant.

Proof of Theorem 1.6. Regarding statement (i), we start by showing that T has at most one fixed point. Indeed, assume x^* and y^* are distinct fixed points and note the contradiction:

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq \ell d(x^*, y^*) < d(x^*, y^*).$$

Next, we prove existence of a fixed point. For any initial value x_0, the Lipschitz condition (1.5) implies

$$d(x_2, x_1) = d(T(x_1), T(x_0)) \leq \ell d(x_1, x_0)$$

and, by induction, for any $k \in \mathbb{N}$,

$$d(x_{k+1}, x_k) \leq \ell^k d(x_1, x_0). \tag{1.8}$$

Therefore, for every number k and $h \geq 1$, we compute

$$\begin{aligned} d(x_{k+h}, x_k) &\overset{(1.3c)}{\leq} \sum_{i=k}^{k+h-1} d(x_{i+1}, x_i) \overset{(1.8)}{\leq} \sum_{i=k}^{k+h-1} \ell^i d(x_1, x_0) \\ &= \ell^k \Big(\sum_{i=0}^{h-1} \ell^i \Big) d(x_1, x_0) \leq \frac{\ell^k}{1-\ell} d(x_1, x_0), \end{aligned} \tag{1.9}$$

where the last inequality follows from $\sum_{i=0}^{h-1} \ell^i \leq \sum_{i=0}^{\infty} \ell^i = 1/(1-\ell)$. Since $0 \leq \ell < 1$, we know $d(x_{k+h}, x_k) \to 0$ as $k \to \infty$. Therefore, the sequence $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence and, since $(\mathcal{X}, d)$ is complete, there exists $x^* \in \mathcal{X}$ such that $x_k \to x^*$ as $k \to \infty$. Next, for any k, compute

$$d(T(x^*), x^*) \overset{(1.3c)}{\leq} d(T(x^*), x_k) + d(x_k, x^*) \overset{(1.5)}{\leq} \ell d(x^*, x_{k-1}) + d(x_k, x^*)$$

and note that, since $x_k \to x^*$, then $T(x^*) = x^*$. This equality completes the proof of existence in statement (i) (uniqueness was already established) and convergence in statement (ii).

Finally, we prove the error estimates in statement (iii). First, for all $k \in \mathbb{N}$,

$$d(x_k, x^*) = d(T(x_{k-1}), x^*) \leq \ell d(x_{k-1}, x^*),$$

and equation (1.7a) follows from applying this inequality repeatedly. Next, we consider (1.9) and, using the continuity of the metric (see Remark 1.2), take the limit $h \to \infty$ to obtain

$$d(x^*, x_k) \leq \frac{\ell^k}{1-\ell} d(x_1, x_0),$$

that is, inequality (1.7b). Finally, we note that, by induction from (1.5),

$$d(x_{k+h}, x_{k+h-1}) \leq \ell^h d(x_k, x_{k-1}), \tag{1.10}$$

so that

$$d(x_{k+h}, x_k) \leq (\ell + \ell^2 + \cdots + \ell^h) d(x_k, x_{k-1}) \leq \frac{\ell}{1-\ell} d(x_k, x_{k-1}). \tag{1.11}$$

The inequality (1.7c) follows in the limit as $h \to \infty$. ■

Remark 1.7 (On the assumptions of the Banach Contraction Theorem). *(i) The assumption of completeness of the metric space cannot be removed from the theorem. As a counterexample, consider the linear map $T\colon]0,1] \to]0,1]$ defined by $T(x) = x/2$. It is a contraction with factor 2, but does not have a fixed point in $]0,1]$. Indeed $T(x) = x$ implies $x = 0$.*

(ii) The assumption of uniform contraction factor $\ell < 1$ cannot be removed from the theorem. As a counterexample, consider the map $T\colon [1,\infty[\to [1,\infty[$ defined by $T(x) = x + 1/x$. Simple calculations show

$$|T(x) - T(y)| = \Big(1 - \frac{1}{xy}\Big)|x-y| \leq |x-y|.$$

However, there does not exist $\ell < 1$ independent of x and y such that $|T(x) - T(y)| \leq \ell |x-y|$ and, indeed, the map does not have a fixed point (since $x = x + 1/x$ has no solution in $\mathbb{R}$). •

1.4 Variations on the theme by Banach

We now present some simple variations on the main contraction theorem.

1.4.1 Parametrized contractions

In this subsection, we focus on solvability of fixed-point problems that depend upon parameters. To this aim, we consider a family of contractions parametrized by an element of a second metric space, i.e., we consider the parametrized fixed point problem:

$$x = T(x, u), \tag{1.12}$$

where x and u take value in appropriate metric spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{U}, d_{\mathcal{U}})$, respectively, and where $T\colon \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is contracting with respect to its first argument. The following notions will be useful.

Definition 1.8. *Given the metric spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{U}, d_{\mathcal{U}})$, the map $T\colon \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is*

(i) *uniformly Lipschitz in its first argument if there exists* $\ell_{\mathcal{X}} > 0$ *such that*

$$d_{\mathcal{X}}\big(T(x,u), T(y,u)\big) \leq \ell_{\mathcal{X}} d_{\mathcal{X}}(x,y), \qquad \text{for all } x,y \in \mathcal{X} \text{ and } u \in \mathcal{U}, \tag{1.13}$$

(ii) *uniformly Lipschitz in its second argument if there exists* $\ell_{\mathcal{U}} > 0$ *such that*

$$d_{\mathcal{X}}\big(T(x,u), T(x,v)\big) \leq \ell_{\mathcal{U}} d_{\mathcal{U}}(u,v), \qquad \text{for all } x \in \mathcal{X} \text{ and } u,v \in \mathcal{U}. \tag{1.14}$$

Next, we study the dependency of the solutions to the parametrized fixed point problem (1.12) as a function of the parameter u.

Lemma 1.9 (Parametrized contractions). *Given a complete metric space* $(\mathcal{X}, d_{\mathcal{X}})$ *and a metric space* $(\mathcal{U}, d_{\mathcal{U}})$*, let* $T\colon \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ *be*

(i) *uniformly contracting in its first argument, that is, uniformly Lipschitz in its first argument with* $\ell_{\mathcal{X}} < 1$*, and*

(ii) *uniformly Lipschitz in its second argument with constant* $\ell_{\mathcal{U}}$.

Then there exists a unique solution $x^*\colon \mathcal{U} \to \mathcal{X}$ *to the parametrized fixed point problem (1.12) that is a Lipschitz map with constant* $\ell_{\mathcal{U}}(1-\ell_{\mathcal{X}})^{-1}$*, that is:*

$$d_{\mathcal{X}}(x^*(u), x^*(v)) \leq \frac{\ell_{\mathcal{U}}}{1-\ell_{\mathcal{X}}} d_{\mathcal{U}}(u,v) \quad \text{for all } u,v \in \mathcal{U}.$$

Proof. The existence of the solution $x^*\colon \mathcal{U} \to \mathcal{X}$ is an immediate consequence of the uniform contractivity assumption. Next, we compute

$$\begin{aligned} d_{\mathcal{X}}(x^*(u), x^*(v)) &= d_{\mathcal{X}}(T(x^*(u),u), T(x^*(v),v)) \\ &\leq d_{\mathcal{X}}(T(x^*(u),u), T(x^*(v),u)) + d_{\mathcal{X}}(T(x^*(v),u), T(x^*(v),v)) \\ &\overset{(1.13) \text{ and } (1.14)}{\leq} \ell_{\mathcal{X}} d_{\mathcal{X}}(x^*(u), x^*(v)) + \ell_{\mathcal{U}} d_{\mathcal{U}}(u,v) \end{aligned}$$

and the result follows. ■

Note: in certain applications the parameter u may well represent a disturbance and we are interested in upper bounding its effects on the fixed point.

1.4.2 Local contractions and invariance

We now consider contractivity over a subset of the metric space and combine this weaker property with the property that "the center of a ball is not displaced too far." In what follows, a set $\mathcal{S} \subset \mathcal{X}$ is *invariant* under a map $T\colon \mathcal{X} \to \mathcal{X}$ if $T(s) \in \mathcal{S}$ for each $s \in \mathcal{S}$.

Lemma 1.10 (Local contractions and invariance). *Let* $(\mathcal{X}, d)$ *be a complete metric space and consider a map* $T\colon \mathcal{X} \to \mathcal{X}$*. Assume there exists a point* $z_0 \in \mathcal{X}$ *and a radius* $r_0 > 0$ *such that*

(i) T *is a contraction with factor* $\ell < 1$ *when restricted to the open ball*

$$B_{r_0}(z_0) = \{x \in \mathcal{X} \,:\, d(x,z_0) < r_0\},$$

that is, $d(T(x),T(y)) \leq \ell d(x,y)$ *for all* $x,y \in B_{r_0}(z_0)$*, and*

(ii) $d(T(z_0), z_0) < r_0(1-\ell)$.

Then the closed ball $\overline{B}_r(z_0)$ with radius $r := d(T(z_0), z_0)/(1-\ell)$ is invariant under T and the Banach Contraction Theorem applies to the map T restricted to $\overline{B}_r(z_0)$.

Proof. By assumption (ii) we note $r < r_0$ so that $\overline{B}_r(z_0) \subset B_{r_0}(z_0)$. For any $x \in \overline{B}_r(z_0)$ we compute

$$d(T(x), z_0) \overset{(1.3c)}{\leq} d(T(x), T(z_0)) + d(T(z_0), z_0) \overset{\text{(i) and (ii)}}{\leq} \ell d(x, z_0) + (1-\ell)r \leq r. \tag{1.15}$$

Therefore, T maps $\overline{B}_r(z_0)$ onto itself. The conclusion follows from noting that $\overline{B}_r(z_0)$ is a complete metric space. ■

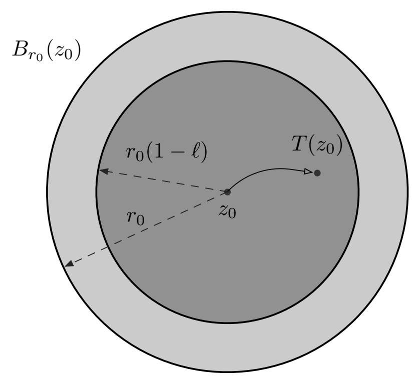


Figure 1.1: Illustration of the local contractivity Lemma 1.10 and, specifically, of the sufficient condition (ii) stating that "the center of the ball is not displaced too far" so that $T(z_0)$ is strictly inside the ball of radius $r_0(1-\ell)$.

1.4.3 Pseudocontractions

In this subsection we consider a different weaker requirement than contractivity.

Corollary 1.11 (Convergence of pseudocontractions). *Given a metric space $\mathcal{X}$, assume the map $T\colon \mathcal{X} \to \mathcal{X}$ is a pseudocontraction, that is,*

(i) T has a fixed point $x^* \in \mathcal{X}$, and

(ii) $d(T(x), x^*) \leq \ell d(x, x^*)$, for some contraction factor $\ell \in [0, 1[$ and for all $x \in \mathcal{X}$.

Then

(i) T has no other fixed point;

(ii) the Picard iteration $\{x_k\}_{k\in\mathbb{N}}$ defined by T converges to x^* for all initial conditions $x_0 \in \mathcal{X}$;

(iii) for all $k \in \mathbb{N}$, $d(x_k, x^*) \leq \ell^k d(x_0, x^*)$.

We leave the proof of this corollary to the reader.

1.4.4 Strict, but not uniformly strict, contractions

In the Banach Contraction Theorem, it is possible to remove the requirement that a uniform contraction factor $\ell < 1$ exists, but at the expense of additional assumptions. Specifically, assume that the map $T\colon \mathcal{X} \to \mathcal{X}$ on the complete metric space $(\mathcal{X}, d)$ satisfies

$$d\big(T(x), T(y)\big) < d(x, y) \quad \text{for all } x \neq y \in \mathcal{X}, \tag{1.16}$$

and, additionally, that

(i) $\mathcal{X}$ is a compact metric space,[2]

(ii) for some $k \geq 2$, the map T^k has Lipschitz constant $\ell < 1$, where T^k is the *kth iterate* of T, defined recursively by $T^k(x) = T(T^{k-1}(x))$, or

(iii) there exists $x_0 \in \mathcal{X}$ such that the sequence of Picard iterates $\{T^k(x_0)\}_{k\in\mathbb{N}}$ has a convergent subsequence.

Then the map T has a unique fixed point x^* and its corresponding Picard iteration converges to x^* from all initial conditions.

1.5 The Brouwer fixed point theorem

We now motivate and present a fixed point theorem that does not require the existence of a metric, but rather it is related to the topological properties of the space.

Example 1.12. We start with a simple application of the Banach Contraction Theorem. Consider a continuously-differentiable function $f\colon [0,1] \to [0,1]$ satisfying $|f'(x)| < 1$ for all $x \in [0,1]$. We claim that f is a contraction with factor $\max_{x\in[0,1]} |f'(x)| < 1$ and it therefore admits a fixed point. The reason why the claim holds is that, via the mean-value theorem, one can show the existence of a point $z \in]0,1[$ such that $f(x) - f(y) = f'(z)(x-y)$. In sum, every $f\colon [0,1] \to [0,1]$ satisfying $|f'(x)| < 1$ for all $x \in [0,1]$ has a fixed point. We will present a general version of the analysis of contraction factors via Jacobians in later chapters. •

Next, we present an important generalized version of this result that (i) requires continuity (instead of a bounded derivative) and (ii) allows for arbitrary convex compact domains (instead of a scalar interval). We present without proof a general version of this theorem (for fixed point problems defined over a convex compact set).

Theorem 1.13 (Brouwer Fixed Point Theorem). *Let $S \subset \mathbb{R}^n$ be convex and compact and let $T\colon S \to S$ be continuous. Then T has a fixed point in S.*

Note: This theorem does not provide a computational way to compute the equilibrium.

1.6 Historical notes and further reading

The original source for the Banach contraction theorem is (Banach, 1922), which contains Banach's doctoral thesis. The theorem is also sometimes referred to as the Picard-Banach-Caccioppoli, because of the earlier work by Picard on the "method of successive approximations" and the later independent work by Caccioppoli (1930). According to (Khamsi and Kirk, 2001), Banach was the first to understand the important role played by the completeness property and, recognizing his contribution, complete normed vector spaces are now referred to as Banach spaces.

This chapter follows the remarkable texts (Zeidler, 1986; Khamsi and Kirk, 2001; Granas and Dugundji, 2003; Berinde, 2007) and their treatment of the Banach Contraction Theorem. Notably, (Zeidler, 1986) is part of a landmark comprehensive effort in nonlinear functional analysis and (Khamsi and Kirk, 2001; Granas and Dugundji, 2003) contain valuable examples and insightful historical commentaries.

Regarding Subsection 1.4.4 about strictly, but not uniformly strictly, contracting maps, we refer to (Berinde, 2007, Section 2.2). Lemma 1.10 is taken from (Agarwal et al., 2001, Theorem 1.3) or (Berinde, 2007, Corollary 2.1). Corollary 1.11 is taken from (Bertsekas and Tsitsiklis, 1997, Proposition 1.2). Lemma 1.9 is important in the study of the implicit function theorem; for more details we refer to analysis of the continuation method in (Granas and Dugundji, 2003, Section 3 in Chapter 1) and to the so-called Lim's Lemma in (Lim, 1985).

[2] A metric space $(\mathcal{X}, d)$ is *compact* if every sequence in $\mathcal{X}$ contains a subsequence having a limit. Here, a subsequence of $\{x_k\}_{k\in\mathbb{N}}$ is any sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where $\{n_k\}_{k\in\mathbb{N}}$ is strictly increasing.

1.7 Exercises

E1.1 **Examples of complete metric spaces.** Show that the following spaces and distance functions are complete metric spaces:

(i) the set of positive real numbers $\mathbb{R}_{>0}$ with distance function $d(x,y) = |\log(x/y)|$;

(ii) any finite dimensional normed vector space with distance function $d(x,y) = \|x-y\|$;

(iii) the set of bounded functions $f\colon \mathcal{S} \to \mathcal{X}$ with distance function $d(f,g) = \sup_{s\in\mathcal{S}} d_{\mathcal{X}}(f(s),g(s))$. Here, $\mathcal{S}$ is a non-empty set, $(\mathcal{X}, d_{\mathcal{X}})$ is a metric space, and f is bounded if its image is a bounded subset of $\mathcal{X}$;

(iv) the unit sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ with arc-length distance function $d(x,y) = \arccos(x^\top y)$. In other words, the arc-length distance is the length of the shortest arc on the sphere joining x and y; and

(v) the punctured plane $\mathbb{R}^2 \setminus \{\mathbb{0}_2\}$ with distance defined by $d(x,y) = |r_x - r_y| + |\theta|$, where $r_x = \|x\|_2$, $r_y = \|y\|_2$ and θ is the smallest angle defined by the two segments connecting x to the origin and y to the origin.

E1.2 **The Picard-Lindelöf Theorem via the Banach Contraction Theorem.** In this exercise we use the Banach Contraction Theorem to prove the classic Picard-Lindelöf Theorem (also called the Cauchy–Lipschitz Theorem) about the existence and uniqueness of the solutions to ordinary differential equations.

Theorem 1.14 (Picard-Lindelöf Theorem). *Let the map $f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be continuous in its first argument and Lipschitz continuous in its second argument with constant M. Given $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^n$ and $\delta < 1/M$, there exists a unique continuously-differentiable solution $\phi_{t_0,t}(x_0)\colon]t_0-\delta, t_0+\delta[\to \mathbb{R}^n$ of the differential equation $\dot{x} = f(t,x)$ with initial condition $\phi_{t_0,t_0}(x_0) = x_0$.*

Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, consider the space $\mathcal{F}$ of continuous functions $\phi\colon]t_0-\delta, t_0+\delta[\to \mathbb{R}^n$ with max-norm $\|\phi\|_{\max} = \sup_{|t-t_0|<\delta} \|\phi\|$.
Next, given $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^n$, we define the *Picard operator* $\mathsf{P}_{t_0,x_0}\colon \mathcal{F} \to \mathcal{F}$ by

$$\mathsf{P}_{t_0,x_0}(\phi) = x_0 + \int_{t_0}^{t} f(\tau, \phi(\tau)) d\tau. \tag{E1.1}$$

Show that:

(i) the space of continuous functions $\mathcal{F}$ with max-norm $\|\cdot\|_{\max}$ is a complete metric space,

(ii) P_{t_0,x_0} is a contraction on $(\mathcal{F}, \|\cdot\|_{\max})$ and, specifically,

$$\|\mathsf{P}_{t_0,x_0}(\phi_1) - \mathsf{P}_{t_0,x_0}(\phi_2)\|_{\max} \leq M\delta \|\phi_1 - \phi_2\|_{\max}, \tag{E1.2}$$

(iii) the Banach Contraction Theorem 1.6 implies the Picard-Lindelöf Theorem 1.14, that is, show that the fixed point of the Picard iteration is the solution of the differential equation and vice versa, and

(iv) the unique solution in $\mathcal{F}$ is continuously differentiable.

Note: *We refer to (Agarwal et al., 2001, Chapter 1) for a detailed treatment.*

E1.3 **The Perron Theorem via the Banach Contraction Theorem.** In this exercise we use the Banach Contraction Theorem to prove the Perron Theorem for positive matrices; we leave to the reader the extension to the case of nonnegative irreducible matrices.

Theorem 1.15 (Perron Theorem). *Let $A \in \mathbb{R}^{n\times n}$, $n \geq 2$, have positive entries. Then*

(i) there exists a simple positive eigenvalue $\lambda > |\mu|$ for all other eigenvalues μ,

(ii) the right and left eigenvectors v and w of λ are unique and positive, up to rescaling.

Define the *n-simplex* $\Delta_n = \{x \in \mathbb{R}^n_{\geq 0} : \|x\|_1 = 1\}$ and its interior $\operatorname{int}(\Delta_n)$. Given a positive matrix $A \in \mathbb{R}^{n\times n}$, define the *power iteration* $\mathsf{P}_A\colon \Delta_n \to \Delta_n$ by

$$\mathsf{P}_A(x) = \frac{Ax}{\|Ax\|_1}.$$

Define the *Hilbert's projective metric* on $\operatorname{int}(\Delta_n)$, denoted $\mathrm{d_H}\colon \operatorname{int}(\Delta_n) \times \operatorname{int}(\Delta_n) \to \mathbb{R}_{\geq 0}$, by

$$\mathrm{d_H}(x, y) = \log \frac{\max_i x_i/y_i}{\min_j x_j/y_j} = -\log\left(\left(\min_j \frac{x_j}{y_j}\right)\left(\min_j \frac{y_j}{x_j}\right)\right).$$

Show that:

(i) $(\operatorname{int}(\Delta_n), \mathrm{d_H})$ is a complete metric space,

(ii) P_A is a contraction on $(\operatorname{int}(\Delta_n), \mathrm{d_H})$,

(iii) the Banach Contraction Theorem 1.6 implies the Perron Theorem 1.15, that is,

a) the conditional eigenvalue problem $Ax = \lambda x$ for $x \in \operatorname{int}(\Delta_n)$, has a unique solution $x^* \in \operatorname{int}(\Delta_n)$ and $\lambda^* > 0$, and

b) $\lim_{n\to\infty} \mathsf{P}_A^n(x_0) = x^*$ for all $x_0 \in \Delta_n$.

Note: *Further references include (Birkhoff, 1957; Bushell, 1973; Kohlberg and Pratt, 1982), this presentation is taken from (Krause, 1986, 1994, 2001). See also (Lemmens and Nussbaum, 2012, Chapter 2).*

E1.4 **Metrics and contractions over positive-definite matrices.** In this exercise we study the set of $n \times n$ positive definite matrices $\mathbb{S}^n_{\succ 0}$. First we comment that $\mathbb{S}^n_{\succ 0}$ is a *positive cone*, in the sense that: (i) $\mathbb{S}^n_{\succ 0}$ is a subset of the real vector space $\mathbb{R}^{n\times n}$, (ii) any positive linear combination of elements of $\mathbb{S}^n_{\succ 0}$ is in $\mathbb{S}^n_{\succ 0}$, and (iii) $\mathbb{S}^n_{\succ 0} \cap (-\mathbb{S}^n_{\succ 0}) = \{\mathbb{0}_{n\times n}\}$.

Next, we define the *Thompson metric* d_{T} on $\mathbb{S}^n_{\succ 0}$ by

$$d_{\mathrm{T}}(P, Q) = \log \max\{\lambda_1(Q^{-1}P), \lambda_1(P^{-1}Q)\}, \tag{E1.3}$$

were we let $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ denote the eigenvalues of any diagonalizable matrix A. The Thompson metric is related to Hilbert's projective metric in Exercise E1.3, can be generalized to arbitrary partially-ordered vector spaces (where indeed it was originally defined), and it enjoys remarkable properties, including invariance under matrix inversion and congruence transformations, nonpositive curvature, and contractivity under translations:

$$\begin{aligned} d_{\mathrm{T}}(P, Q) = d_{\mathrm{T}}(P^{-1}, Q^{-1}) &= d_{\mathrm{T}}(MPM^{\mathsf{T}}, MQM^{\mathsf{T}}) && \text{for all invertible } T,\\ d_{\mathrm{T}}(P^t, Q^t) &\leq t d_{\mathrm{T}}(P, Q) && \text{for all } t \in [0,1],\\ d_{\mathrm{T}}(P + S, Q + S) &\leq \frac{\alpha}{\alpha+\beta} d_{\mathrm{T}}(P, Q) && \text{for all } S \in \mathbb{S}^n_{\succ 0}, \end{aligned}$$

where $\alpha = \max\{\lambda_1(P), \lambda_1(Q)\}$ and $\beta = \lambda_n(S)$.

Next, we define the linear-fractional *Riccati map* $\mathsf{R}\colon \mathbb{S}^n_{\succ 0} \to \mathbb{S}^n_{\succ 0}$ by

$$\mathsf{R}(P) = M(P^{-1} + Q_0)^{-1}M^{\mathsf{T}} + Q_1, \tag{E1.4}$$

where $M \in \mathbb{R}^{n\times n}$ and $Q_0, Q_1 \in \mathbb{S}^n_{\succ 0}$.

Show

(i) $(\mathbb{S}^n_{\succ 0}, d_{\mathrm{T}})$ is a complete metric space,

(ii) R is a contraction on $(\mathbb{S}^n_{\succ 0}, d_{\mathrm{T}})$,

(iii) the fixed point for the Riccati map is the *discrete algebraic Riccati equation* $P = M(P^{-1} + Q_0)^{-1}M^{\mathsf{T}} + Q_1$.

Note: *The original work on the Thomson metric over partially-ordered vector spaces is (Thompson, 1963). Recent relevant developments include (Bougerol, 1993; Bhatia, 2003; Lawson and Lim, 2007; Lee and Lim, 2008).*

E1.5 **Dynamic programming.** Let $\mathcal{S}$ be an arbitrary set and $B(\mathcal{S})$ be the Banach space of the bounded functions $f\colon \mathcal{S} \to \mathbb{R}$. Consider the metric on $B(\mathcal{S})$ given by the supremum norm of the difference between elements and the natural entrywise partial order on $B(\mathcal{S})$, whereby $f \leq g$ if $f(s) \leq g(s)$ for all $s \in \mathcal{S}$.

Let E be the vector subspace of $B(\mathcal{S})$ containing all constant functions and $F\colon E \to E$ be a map satisfying

(i) $f \leq g$ implies $F(f) \leq F(g)$ for all $f, g \in E$,

(ii) there exists $q \in [0, 1[$ such that, for any constant function $c(s) = c$,

$$F(f + c) \leq F(f) + qc, \quad \text{for all } f \in E. \tag{E1.5}$$

Show that F has a unique fixed point.
Note: *This exercise is taken from (Granas and Dugundji, 2003, Exercise C.6). A more complete treatment is given in (Bertsekas and Tsitsiklis, 1997, Chapter 4).*

E1.6 **Expansive maps.** Let $(\mathcal{X}, d)$ be a complete metric space and let a map $T\colon \mathcal{X} \to \mathcal{X}$ be surjective and *expansive*, that is, $d(T(x), T(y)) \geq \beta d(x, y)$ for some $\beta > 1$ and for all $x, y \in \mathcal{X}$. Prove

(i) T is bijective, and
(ii) T has a unique fixed point x^* and $T^{-n}(x_0) \to x^*$ as $n \to \infty$ for all $x_0 \in \mathcal{X}$, where T^{-n} is the nth iterate of the inverse map T^{-1}.

Note: *This exercise is taken from (Granas and Dugundji, 2003, Exercise A.8).*

Answer: Regarding statement (i), we need to show that T is injective, i.e., we need to show that, for all $x_1, x_2 \in \mathcal{X}$, $x_1 \neq x_2$ implies $T(x_1) \neq T(x_2)$. But this property follows from the expansivity property of T and the separation property of the distance: first we note $d(T(x_1) \neq T(x_2)) \geq \beta d(x, y) > 0$ because $x_1 \neq x_2$ and then we note that $d(T(x_1) \neq T(x_2)) > 0$ implies $T(x_1) \neq T(x_2)$.

Regarding statement (ii), since T is a bijection we know T^{-1} is well defined and we claim that T^{-1} is a contraction with factor $1/\beta < 1$. For, let $\bar{x} = T(x)$ and note $T^{-1}(\bar{x}) = x$. We compute

$$d(\bar{x}, \bar{y}) = d(T(x), T(y)) \overset{\text{expansiveness}}{\geq} \beta d(x, y) = \beta d(T^{-1}(\bar{x}), T^{-1}(\bar{y})) \tag{E1.6}$$

which proves $d(T^{-1}(\bar{x}), T^{-1}(\bar{y})) \leq \frac{1}{\beta} d(\bar{x}, \bar{y})$ for each $\bar{x}$ and $\bar{y} \in \mathcal{X}$. Finally, since T^{-1} is a bijection, it has a unique fixed point x^*, which is a fixed point also for T.

E1.7 **The incremental small gain theorem over the space of p-th power integrable functions.** For $p \in [1, \infty]$, the L_p space, called the *set of p-th power integrable functions*, is the set of measurable functions $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

$$\|f\|_p := \begin{cases} \int_{\mathbb{R}_{\geq 0}} |f(x)|^p dx & \text{if } p < \infty \\ \inf\{\ell \;:\; |f(x)| < \ell \text{ for almost all } x \in \mathbb{R}_{\geq 0}\} & \text{if } p = \infty. \end{cases} \tag{E1.7}$$

The space L_p is a vector space; the function $\|\cdot\|_p$ is a seminorm (i.e., it is nonnegative and satisfies the triangle inequality, also called the Minkovski's inequality in Exercise E2.4). Modulo the set of functions with zero measure, the space L_p is a Banach space. By the Riesz-Fischer Theorem, the space L_p is complete.

Consider two maps $\mathscr{F}_1\colon L_p \to L_p$ and $\mathscr{F}_2\colon L_p \to L_p$ interconnected in a feedback loop, as illustrated in Figure E1.1. Assume

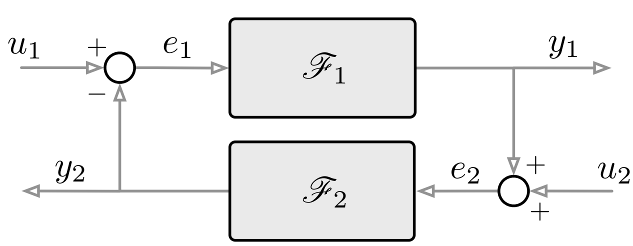


Figure E1.1: Illustration of a feedback loop interconnection and, specifically, of a negative feedback interconnection (since $e_1 = u_1 - y_2$).

(SG1) (Lipschitzness) $\mathsf{Lip}(\mathscr{F}_1) = \gamma_1$ and $\mathsf{Lip}(\mathscr{F}_2) = \gamma_2$, and
(SG2) (small gain condition) $\gamma_1\gamma_2 < 1$,

Show that, for each input pair u_1, u_2, the system has a unique internal signal solution e_1, e_2 and unique output solution y_1, y_2.
Note: *The same small gain condition is sufficient for the case of positive feedback.*

Answer: The equations defining the feedback loop are:

$$e_1 = u_1 - \mathscr{F}_2(e_2), \tag{E1.8}$$
$$e_2 = u_2 - \mathscr{F}_1(e_1), \tag{E1.9}$$

together with the output equations $y_1 = \mathscr{F}_1(e_1)$ and $y_1 = \mathscr{F}_1(e_1)$. Given the inputs u_1, u_2, we first look for solutions e_1, e_2. Eliminating e_2 from the previous equations we obtain the equality:

$$e_1 = u_1 - \mathscr{F}_2\big(u_2 - \mathscr{F}_1(e_1)\big) := \mathscr{G}(e_1). \tag{E1.10}$$

Note that, given u_1 and u_2, the equation (E1.10) is a fixed point problem in e_1 defined by the map $\mathscr{G}\colon L_p \to L_p$. We compute the Lipschitz constant of $\mathscr{G}$ as follows:

$$\begin{aligned}
\|\mathscr{G}(x_1) - \mathscr{G}(x_2)\| &= \big\|-\mathscr{F}_2\big(u_2 - \mathscr{F}_1(x_1)\big) + \mathscr{F}_2\big(u_2 - \mathscr{F}_1(x_2)\big)\big\| && \text{(E1.11)}\\
&\leq \gamma_2 \|-\mathscr{F}_1(x_1) + \mathscr{F}_1(x_2)\| && \text{(E1.12)}\\
&\leq \gamma_2\gamma_1 \|x_1 - x_2\|. && \text{(E1.13)}
\end{aligned}$$

The small gain assumption now implies that the map $\mathscr{G}$ is a contraction and, since L_p is complete, we know a unique solution exists e_1^* to equation (E1.10). In turn, unique solutions exist for e_2 and the outputs y_1, y_2.

E1.8 **A continuous-time Lipschitz property.** Consider the normed vector spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, and the Lipschitz map $T\colon \mathcal{X} \to \mathcal{Y}$ with constant $\ell > 0$. Given a locally Lipschitz curve $x\colon \mathbb{R}_{\geq 0} \to \mathcal{X}$,

(i) $y(t) = T(x(t))$, for $t \in \mathbb{R}_{\geq 0}$ is locally Lipschitz;
(ii) $\|\dot{y}(t)\|_{\mathcal{Y}} \leq \ell \|\dot{x}(t)\|_{\mathcal{X}}$, for almost all $t \in \mathbb{R}_{\geq 0}$.

Answer: Statement (i) is an immediate consequence of the definition of Lipschitz function. To prove statement (ii) we first note that statement (i) implies that $\dot{y}(t)$ exists almost everywhere. Next, for almost all $t \in \mathbb{R}_{\geq 0}$ we compute

$$\begin{aligned}
\|\dot{y}(t)\|_{\mathcal{Y}} &= \Big\| \lim_{h\to 0^+} \frac{y(t+h) - y(t)}{h} \Big\|_{\mathcal{Y}} = \lim_{h\to 0^+} \Big\| \frac{y(t+h) - y(t)}{h} \Big\|_{\mathcal{Y}}\\
&\leq \lim_{h\to 0^+} \frac{\ell}{h} \big\|x(t+h) - x(t)\big\|_{\mathcal{X}} = \ell \Big\| \lim_{h\to 0^+} \frac{x(t+h) - x(t)}{h} \Big\|_{\mathcal{X}} = \ell \|\dot{x}(t)\|_{\mathcal{X}},
\end{aligned}$$

where we have used the Lipschitzness of T and the continuity of the norm on $\mathcal{X}$ and $\mathcal{Y}$.

Part II

Contracting Dynamical Systems on Normed Vector Spaces

Chapter **2**

Norms and Induced Matrix Norms

Если в множестве матриц введена норма, удовлетворяющая аксиомам 1)—3) [...], то число $\gamma(A)$, определяемое для всякой матрицы формулой (1), назовем ***логарифмической нормой*** *матрицы A* (Sergei Mikhailovich Lozinskii 1958)[1]

The function $\mu[A]$ plays a certain role in the following. (Germund Dahlquist 1958)

The simplicity and flexibility of the measure $\mu(\cdot)$ led us to more general results. (Charles A. Desoer and Hiromasa Haneda 1972)

2.1 Introduction

In this chapter we define and study the properties of norms on vector spaces and induced norms of matrices. As it is widely known, the concept of norm over a vector space provides a measure of magnitude, proximity and error and is therefore very useful in applications. Along similar lines, induced matrix norms provide a natural way to estimate the magnitude of solutions of linear ordinary differential equation. We will review induced matrix norms and the less commonly studied induced logarithmic norms. For continuous-time dynamical systems, logarithmic norms are a basic tool required to define the infinitesimal contraction property and to infer various dynamical systems properties. We will see how logarithmic norms play a central role in robust stability analysis. The treatment pays special attention to weighted ℓ_p norms and to a parallel treatment of the properties of induced matrix norms (for discrete-time dynamics) and log norms (for continuous-time dynamics).

We start with some simple loose reasoning; we will give complete definitions and be more precise about all following derivations later in the book. (For example, we will formally define induced norms in Section 2.4.) We aim to upper bound the evolution of a linear dynamical system — and we do so first for discrete-time and then for continuous-time systems.

For discrete-time systems, given a norm $\|\cdot\|$ on $\mathbb{R}^n$ we define its *induced matrix norm* as a norm on $\mathbb{R}^{n\times n}$ by

$$\|A\| = \max_{x\in\mathbb{R}^n, \|x\|=1} \|Ax\|. \tag{2.1}$$

Recall that a norm $\|\cdot\|$ on $\mathbb{R}^n$ and its induced matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ enjoy many useful properties, including the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$ and the submultiplicativity property $\|Ay\| \leq \|A\|\|y\|$ for any matrix A and vector y.

[1] Translation: If a norm is introduced in the set of matrices, satisfying the axioms 1)-3) [...], then the number $\gamma(A)$ defined for any matrix by formula (1) is called the *logarithmic norm* of the matrix A.

Lemma 2.1 (Upper bounding the evolution of a discrete-time linear system). *Given a norm $\|\cdot\|$ and a matrix A, consider the discrete-time system*

$$x(k+1) = Ax(k) + u(k), \tag{2.2}$$

where $x(k)$ is the state and $u(k)$ the input signal. Then, for all $k \in \mathbb{N}$,

(i) $\|x(k+1)\| \leq \|A\|\|x(k)\| + \|u(k)\|$,

(ii) $\|x(k)\| \leq \|A\|^k\|x(0)\| + \sum_{s=0}^{k-1}\|A\|^{k-s-1}\|u(s)\|$,

(iii) at zero input $u(k) = \mathbb{0}_n$ for all k, we have $x(k) = A^k x(0)$, $\|A^k\| \leq \|A\|^k$, and $\|x(k)\| \leq \|A\|^k\|x(0)\|$,

(iv) at zero initial conditions $x(0) = \mathbb{0}_n$, we have $\sup_{k\in\mathbb{N}}\|x(k)\| \leq \dfrac{1}{1-\|A\|}\sup_{k\in\mathbb{N}}\|u(k)\|$ when $\|A\| < 1$.

Proof. The proof of statement (i) relies precisely upon the triangle inequality and the submultiplicativity of the induced norm:

$$\begin{aligned}\|x(k+1)\| = \|Ax(k) + u(k)\| &\overset{\text{(triangle ineq)}}{\leq} \|Ax(k)\| + \|u(k)\| \\ &\overset{\text{(submultiplicativity)}}{\leq} \|A\|\|x(k)\| + \|u(k)\|.\end{aligned}$$

Statement (ii) is a direct consequence of (i). We leave statement (iii) to the reader. Regarding statement (iv), we compute, for any k,

$$\|x(k)\| \leq \Big(\sup_{h\leq k}\|u(h)\|\Big)\sum_{s=0}^{k-1}\|A\|^{k-s-1} \leq \Big(\sup_{h\in\mathbb{N}}\|u(h)\|\Big)\sum_{s=0}^{\infty}\|A\|^{s},$$

and the result follows from the geometric series $\sum_{s=0}^{\infty} a^s = \frac{1}{1-a}$ when $|a| < 1$. ■

Next, we provide the equivalent discussion and the equivalent upper bounds for the setting of continuous-time linear systems. To do so, we need two notions. First, given a norm $\|\cdot\|$ and a matrix A, we define the *induced logarithmic norm*[2] of A by:

$$\mu(A) = \lim_{h\to 0^+}\frac{\|I_n + hA\| - 1}{h}. \tag{2.3}$$

(Here I_n is the identity matrix of dimension n.) Much of this chapter is dedicated to studying the properties of the logarithmic norm. Second, given a continuous (and not differentiable everywhere) function $\psi\colon \mathbb{R} \to \mathbb{R}$, we define the *upper-right Dini derivative* of ψ by

$$\big(D^+\psi\big)(t) = \limsup_{h\to 0^+}\frac{\psi(t+h) - \psi(t)}{h}. \tag{2.4}$$

For our purposes here, it suffices to say that $\big(D^+\psi\big)(t) = \frac{d}{dt}\psi(t)$ at all times t where ψ is differentiable. Unfortunately, functions of the form $t \mapsto \|x(t)\|$ are in general only continuous (even when $x(t)$ is a differentiable curve) and therefore their derivative with respect to time is well defined only in the sense of (2.4). We review Dini derivatives in Exercise E2.1 and Appendix A.7.

[2] We show this definition is well posed in Exercise E2.6.

Lemma 2.2 (Upper bounding the evolution of a continuous-time linear system). *Given a norm $\|\cdot\|$ and a matrix A, consider the continuous-time system*

$$\dot{x}(t) = Ax(t) + u(t), \tag{2.5}$$

where, as before, $x(t)$ is the state and $u(t)$ is the input signal. Then, for all $t \in \mathbb{R}_{\geq 0}$,

(i) $D^+\|x(t)\| \leq \mu(A)\|x(t)\| + \|u(t)\|$,

(ii) $\|x(t)\| \leq \mathrm{e}^{\mu(A)t}\|x(0)\| + \int_0^t \mathrm{e}^{(t-\tau)\mu(A)}\|u(\tau)\|d\tau$,

(iii) at zero input $u(t) = \mathbb{0}_n$ for all t, we have $x(t) = \mathrm{e}^{At}\, x(0)$, $\|\mathrm{e}^{At}\| \leq \mathrm{e}^{t\mu(A)}$, and $\|x(t)\| \leq \mathrm{e}^{t\mu(A)}\|x(0)\|$,

(iv) at zero initial condition $x(0) = \mathbb{0}_n$, we have $\sup_{t\in\mathbb{R}_{\geq 0}} \|x(t)\| \leq -\frac{1}{\mu(A)} \sup_{t\in\mathbb{R}_{\geq 0}} \|u(t)\|$ when $\mu(A) < 0$.

Note: the bounds in Lemma 2.2 are useful because we will see that there exists matrices and norms for which $\mu(A) < 0$. Next, we sketch a short proof, based upon the idea of avoiding the triangle inequality $\|I_n + hA\| \leq 1 + h\|A\|$ and using instead a sharper bounding method.

Proof of Lemma 2.2. Regarding statement (i), for $h > 0$ we compute

$$\begin{aligned}
D^+\|x\| &\overset{\text{(def Dini (2.4))}}{=} \limsup_{h\to 0^+} \frac{\|x(t+h)\| - \|x(t)\|}{h} \\
&\overset{\text{(Taylor expansion in } t)}{=} \limsup_{h\to 0^+} \frac{\|x(t) + h\dot{x}(t)\| - \|x(t)\|}{h} \\
&\overset{\text{(triangle ineq)}}{\leq} \limsup_{h\to 0^+} \frac{\|(I_n + hA)x(t)\| - \|x(t)\|}{h} + \|u\| \\
&\overset{\text{(submultiplicativity)}}{\leq} \limsup_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}\|x(t)\| + \|u\| \overset{(2.3)}{\leq} \mu(A)\|x(t)\| + \|u\|.
\end{aligned}$$

This concludes the proof of statement (i). Statement (ii) is a direct application of the Grönwall Comparison Lemma in Exercise E2.1. We leave to the reader the proof of statement (iii).

Regarding statement (iv), we compute

$$\begin{aligned}
\sup_{t\in\mathbb{R}_{\geq 0}} \|x(t)\| &\overset{\text{(statement (ii))}}{\leq} \sup_{t\in\mathbb{R}_{\geq 0}} \int_0^t \mathrm{e}^{(t-\tau)\mu(A)}\|u(\tau)\|d\tau \\
&\leq \sup_{t\in\mathbb{R}_{\geq 0}} \|u(t)\| \cdot \sup_{t\in\mathbb{R}_{\geq 0}} \int_0^t \mathrm{e}^{(t-\tau)\mu(A)}\, d\tau,
\end{aligned}$$

and $\int_0^t \mathrm{e}^{(t-\tau)\mu(A)}\, d\tau = -\frac{1}{\mu(A)}\, \mathrm{e}^{(t-\tau)\mu(A)}\Big|_{\tau=0}^{\tau=t} = -\frac{1}{\mu(A)}(1-\mathrm{e}^{\mu(A)t})$. The result follows from noting that $\sup_{t\in\mathbb{R}_{\geq 0}}(1-\mathrm{e}^{\mu(A)t}) = 1$, for $\mu(A) < 0$. ■

We conclude this introductory section with a more general statement of Lemma 2.2. Given a continuous map $t \mapsto A(t) \in \mathbb{R}^{n\times n}$, recall that its associated *state transition matrix* $\Phi(t, t_0)$, $t \geq t_0$, satisfies $x(t) = \Phi(t, t_0)x(t_0) \in \mathbb{R}^n$, where $x(t)$ is the solution to

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) \in \mathbb{R}^n, \text{ and } t \geq t_0 \in \mathbb{R}. \tag{2.6}$$

Theorem 2.3 (Coppel's inequalities (Coppel, 1965)). *Given a vector norm $\|\cdot\|$ on $\mathbb{R}^n$, let $\|\cdot\|$ and μ denote its induced matrix norm and log norm, respectively. The state transition matrix $\Phi(t,t_0)$, $t \geq t_0$, of a continuous map $t \mapsto A(t) \in \mathbb{R}^{n\times n}$ satisfies*

$$\exp\Big(\int_{t_0}^{t} -\mu(-A(\tau))d\tau\Big) \leq \|\Phi(t,t_0)\| \leq \exp\Big(\int_{t_0}^{t} \mu(A(\tau))d\tau\Big), \tag{2.7}$$

and, if A is time-invariant,

$$\mathrm{e}^{-\mu(-A)t} \leq \|\mathrm{e}^{At}\| \leq \mathrm{e}^{\mu(A)t}. \tag{2.8}$$

Coppel's inequalities explain why the log norm is also referred to as logarithmic norm. Indeed, if $\dot{x} = Ax$, the Coppel's upper bound can be rewritten as

$$\frac{d}{dt}\log\|x\| \leq \mu(A), \tag{2.9}$$

that is, the maximal growth rate of $\log\|x\|$ is $\mu(A)$.

2.2 Elements of matrix theory

We let I_n denote the n-dimensional identity matrix and $A \in \mathbb{R}^{n\times n}$ denote a square $n \times n$ matrix with real entries $\{a_{ij}\}$, $i,j \in \{1,\dots,n\}$. For two matrices (or vectors) A, B, we let $A \circ B$ and $A \oslash B$ be entrywise multiplication and division, respectively. For a vector $\eta \in \mathbb{R}^n$, we define $[\eta] \in \mathbb{R}^{n\times n}$ to be the *diagonal matrix with diagonal entries equal* to η.

The square matrix $A \in \mathbb{R}^{n\times n}$ is

(i) *non-negative* (respectively *positive*), denoted by $A \geq 0$ (respectively $A > 0$), if $a_{ij} \geq 0$ (respectively $a_{ij} > 0$) for all i and j in $\{1,\dots,n\}$;

(ii) *Metzler* if $a_{ij} \geq 0$ for all $i \neq j$ in $\{1,\dots,n\}$.

For a matrix $A \in \mathbb{R}^{n\times n}$, the *absolute value majorant* $|A| \in \mathbb{R}^{n\times n}$ and the *Metzler majorant* $|A|_{\mathrm{M}} \in \mathbb{R}^{n\times n}$ are defined by

$$(|A|)_{ij} = |a_{ij}|, \qquad (|A|_{\mathrm{M}})_{ij} = \begin{cases} a_{ii}, & \text{if } i = j \\ |a_{ij}|, & \text{if } i \neq j \end{cases}$$

In other words, $|A|$ is the entrywise absolute value of A and $|A|_{\mathrm{M}}$ is the off-diagonal entrywise absolute value of A. Given a square matrix A, a *principal submatrix* of A is a square matrix obtained by removing some rows and the corresponding columns from A.

A symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is

(i) *positive definite*, denoted by $P \succ 0$, if $x^\top P x > 0$ for all $x \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}$,

(ii) *positive semidefinite*, denoted by $P \succeq 0$, if $x^\top P x \geq 0$ for all $x \in \mathbb{R}^n \setminus \{\mathbb{0}_n\}$, and

(iii) *negative definite* or *negative semidefinite* if the corresponding analogous properties are satisfied.

We let $\mathbb{S}^n_{>0}$ denote the set of $n \times n$ positive definite matrices. It is known that P is positive definite (respectively semidefinite) if and only if all eigenvalues of P are strictly positive (respectively nonnegative).

A *linear matrix inequality (LMI)* is an expression of the form

$$A_0 + x_1A_1 + \dots + x_mA_m \succeq 0, \tag{2.10}$$

where $x \in \mathbb{R}^n$ is a real vector of coefficients and $A_0, A_1, \dots, A_n \in \mathbb{R}^{n \times n}$ are real symmetric matrices.

We will discuss linear programs (LPs) and semidefinite programs. We refer to (Boyd and Vandenberghe, 2004) for information about convex optimization and to (Grant and Boyd, 2014) for an example software package to solve convex programs.

Stability notions

We here review some stability notions.

Given a matrix $A \in \mathbb{R}^{n \times n}$, its *spectrum* $\operatorname{spec}(A)$ is the set of its eigenvalues, and its *spectral radius* and *spectral abscissa* are $\rho(A) = \max\{|\lambda| : \lambda \in \operatorname{spec}(A)\}$ and $\alpha(A) = \max\{\Re(\lambda) : \lambda \in \operatorname{spec}(A)\}$.

Regarding continuous-time stability properties, the matrix $A \in \mathbb{R}^{n \times n}$ is

(i) *Hurwitz*, denoted by $A \in \mathcal{H}$, if $\alpha(A) < 0$. It is well-known that A is Hurwitz if and only if there exists $P \in \mathbb{S}^n_{\succ 0}$ satisfying the *(continuous-time) Lyapunov stability LMI*

$$A^\top P + PA \prec 0, \tag{2.11}$$

(ii) *totally Hurwitz*, denoted by $A \in \mathcal{TH}$, if all principal submatrices of A are Hurwitz,

(iii) *Lyapunov diagonally Hurwitz*, denoted by $A \in \mathcal{LDH}$, if there exists a diagonal $P \in \mathbb{S}^n_{\succ 0}$ satisfying the LMI (2.11), and

(iv) *M-Hurwitz*, denoted by $A \in \mathcal{MH}$, if $\alpha(|A|_{\mathrm{M}}) < 0$.

The following result is presented for example in (Boyd et al., 1994, Section 2.5.2).

Lemma 2.4 (Lyapunov stability LMI). *Given $A \in \mathbb{R}^{n \times n}$, the weak Lyapunov stability LMI in the variable $P \in \mathbb{S}^n_{\succ 0}$ is the linear matrix inequality*

$$A^\top P + PA \preceq 0. \tag{2.12}$$

The weak Lyapunov stability LMI is

(i) feasible if and only if all eigenvalues of A have nonpositive real part and those with zero real part are semisimple;[3] *and*

(ii) strictly feasible (i.e., feasible with a strict inequality sign) if and only if all eigenvalues of A have negative real part, i.e., A is Hurwitz.

Lemma 2.5 (Inclusions for classes of matrices). *The inclusion relationships in Figure 2.1 are correct, that is, $(A \in \mathcal{MH})$ implies $(A \in \mathcal{LDH})$, $(A \in \mathcal{LDH})$ implies $(A \in \mathcal{TH})$, and $(A \in \mathcal{TH})$ implies $(A \in \mathcal{H})$.*

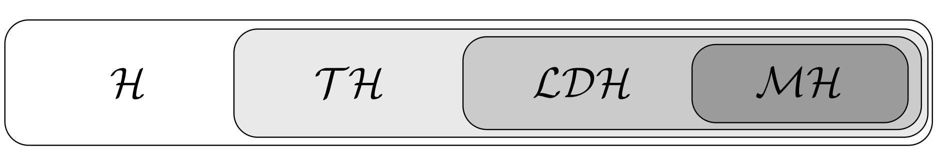


Figure 2.1: Inclusion relationships between the subsets of Hurwitz matrices.

Remark 2.6 (Related notions and alternative nomenclature). *From the literature,*

[3]An eigenvalue is *semisimple* (or non-defective) if its algebraic multiplicity equals its geometric multiplicity.

(i) *according to (Pastravanu and Voicu, 2006), a matrix $A \in \mathcal{MH}$ is said to be Hurwitz diagonally stable with respect to p-norms;*

(ii) *according to (Moylan, 1977) a matrix $A \in \mathbb{R}^{n\times n}$ is quasidominant if there exists a vector $\eta \in \mathbb{R}^n_{>0}$ such that*

$$\eta_i a_{ii} > \sum\nolimits_{j=1,j\neq i}^{n} \eta_j |a_{ij}|, \quad \text{for all } i \in \{1,\dots,n\}.$$

This is equivalent to $|-A|_{\mathrm{M}}\eta < \mathbb{0}_n$, which, by the Hurwitz Metzler Theorem 2.7, is equivalent to $\alpha(|-A|_{\mathrm{M}}) < 0$, i.e., $-A \in \mathcal{MH}$; and

(iii) *according to (Horn and Johnson, 1994), A is an H-matrix if its comparison matrix $M(A)$ is a nonsingular M-matrix where $(M(A))_{ii} = |a_{ii}|$ and $(M(A))_{ij} = -|a_{ij}|$ for all $i \neq j$. This implies $\alpha(-M(A)) < 0$. Therefore, if $A \in \mathcal{MH}$, both A and $-A$ are H-matrices.*

•

Finally, we note that there exists an equivalent discrete-time treatment. We only provide some basic definitions. The matrix $A \in \mathbb{R}^{n\times n}$ is

(i) *Schur* if $\rho(A) < 1$. It is well-known that A is Schur if and only if there exists $P \in \mathbb{S}^n_{>0}$ satisfying the *(discrete-time) Lyapunov stability LMI*

$$A^\top PA + P \prec 0. \tag{2.13}$$

(ii) *Lyapunov diagonally Schur* if there exists a diagonal $P \in \mathbb{S}^n_{>0}$ satisfying the LMI (2.13), and

(iii) *M-Schur* if $\rho(|A|) < 1$.

Properties of Hurwitz Metzler matrices

We present here the most famous equivalent conditions for a Metzler matrix to be Hurwitz. For more information we refer to (Bullo, 2022, Section 10.4).

Theorem 2.7 (Hurwitz Metzler Theorem). *For a Metzler matrix $M \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*

(i) *M is Hurwitz,*

(ii) *M is invertible and $-M^{-1} \geq 0$,*

(iii) *for all $b \geq \mathbb{0}_n$, there exists a unique $x^* \geq \mathbb{0}_n$ solving $Mx^* + b = \mathbb{0}_n$,*

(iv) *there exists $\xi \in \mathbb{R}^n$ such that $\xi > \mathbb{0}_n$ and $M\xi < \mathbb{0}_n$,*

(v) *there exists $\eta \in \mathbb{R}^n$ such that $\eta > \mathbb{0}_n$ and $\eta^\top M < \mathbb{0}_n^\top$,*

(vi) *there exists a diagonal matrix $P \succ 0$ such that $M^\top P + PM \prec 0$.*

Consider for example a 2×2 Metzler matrix parametrized as $M = \begin{bmatrix} -c_1 & m_{12} \\ m_{21} & -c_2 \end{bmatrix}$, where c_1, c_2, m_{12}, m_{21} are positive. One can easily see that Hurwitzness of M is equivalent to the small gain condition $c_1 c_2 > m_{12} m_{21}$.

2.3 Norms on vector spaces

In this section we review norms on vector spaces and induced matrix norms. Unless specified, proofs for all results are given in (Horn and Johnson, 2012, Chapter 5).

A function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a *norm* on $\mathbb{R}^n$ if, for all $v, w \in \mathbb{R}^n$ and $a \in \mathbb{R}$:

$$\text{(positive definiteness)} \qquad \|v\| \geq 0 \text{ and } \|v\| = 0 \text{ if and only if } v = \mathbb{0}_n, \tag{2.14a}$$

$$\text{(homogeneity)} \qquad \|av\| = |a|\,\|v\|, \text{ and} \tag{2.14b}$$

$$\text{(subadditivity)} \qquad \|v + w\| \leq \|v\| + \|w\|. \tag{2.14c}$$

The subadditivity inequality is also referred to as the *triangle inequality*.

2.3.1 Weighted ℓ_p norms

Classic examples of norms on $\mathbb{R}^n$ are the ℓ_p norms for $p \in [1, \infty]$. Specifically, the ℓ_1, ℓ_2 and ℓ_∞ norms are given by

$$\|v\|_1 = \sum\nolimits_{i=1}^n |v_i|, \quad \|v\|_2 = \sqrt{\sum\nolimits_{i=1}^n v_i^2}, \quad \|v\|_\infty = \max_{i \in \{1,...,n\}} |v_i|, \tag{2.15}$$

and $\|v\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p}$ for every other value of $1 < p < \infty$. We show the unit disks of these norms in Figure 2.2.

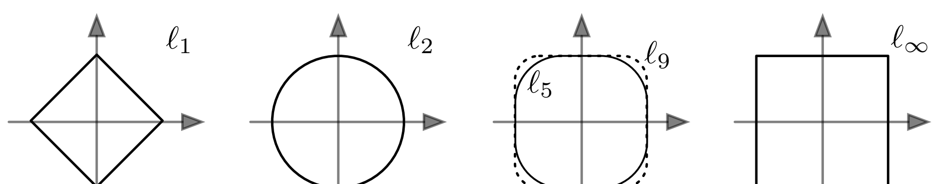


Figure 2.2: The unit disks $\{v \in \mathbb{R}^2 \;:\; \|v\| = 1\}$ in the ℓ_1, ℓ_2, ℓ_p for $p \in \{5, 9\}$, and ℓ_∞ norms.

We will be specifically interested in weighted norms.

Definition 2.8 (Weighted norms). *Given an invertible matrix $R \in \mathbb{R}^{n \times n}$ and a norm $\|\cdot\|$, the weighted norm $\|\cdot\|_R$ is defined by $\|x\|_R = \|Rx\|$. Specifically, given an invertible matrix $R \in \mathbb{R}^{n \times n}$ and number $p \in [1, \infty]$, the weighted ℓ_p norm $\|\cdot\|_{p,R}$ is defined by*

$$\|x\|_{p,R} = \|Rx\|_p. \tag{2.16}$$

Note that definition (2.16) is different from the commonly used notation $\|x\|_P := \sqrt{x^\top P x}$, for a positive definite $P \in \mathbb{S}^n_{\succ 0}$. Indeed, according to definition (2.16), we have $\|x\|_{2,R} = \|Rx\|_2 = \sqrt{x^\top R^\top R x}$. For later use, we note the two equalities: (1) given a positive definite P, $\|x\|_P = \|x\|_{2,P^{1/2}}$, and (2) given an invertible R, $\|x\|_{2,R} = \|x\|_{R^\top R}$. (Recall that the square root of a positive definite symmetric matrix is well posed.)

Finally, we are interested in diagonally-weighted norms. Given $\eta \in \mathbb{R}^n_{>0}$, the diagonally weighted ℓ_1, ℓ_2 and ℓ_∞ norms are

$$\|x\|_{1,[\eta]} = \sum_{i=1}^n \eta_i |x_i|, \quad \|x\|_{2,[\eta]} = \left(\sum_{i=1}^n \eta_i x_i^2\right)^{1/2}, \quad \text{and} \quad \|x\|_{\infty,[\eta]^{-1}} = \max_{i \in \{1,...,n\}} \frac{1}{\eta_i}|x_i|. \tag{2.17}$$

2.3.2 Properties of norms

(i) Each norm is a continuous convex function. Indeed, homogeneity and subadditivity imply convexity: $\|av + (1-a)w\| \leq a\|v\| + (1-a)\|w\|$ for all $v, w \in \mathbb{R}^n$ and $a \in [0,1]$. And any convex scalar function is continuous.

(ii) Each norm satisfies the *reverse triangle inequality*: $\|v - w\| \geq |\|v\| - \|w\||$, for each $v, w \in \mathbb{R}^n$. The proof follows from simple calculations and so does the equivalent inequality $\|v + w\| \geq |\|v\| - \|w\||$. The reverse triangle inequality implies that each norm is Lipschitz continuous with Lipschitz constant 1.

(iii) The unit disk of any norm is a convex centrally symmetric set, see Exercise E2.2.

(iv) The following properties are equivalent (see Exercise E2.3 for the proof):

a) $\|\cdot\|$ is *monotonic*, that is, for every $v, w \in \mathbb{R}^n$,

$$|v_i| \leq |w_i|, i \in \{1, \dots, n\} \quad \implies \quad \|v\| \leq \|w\|, \tag{2.18}$$

b) $\|\cdot\|$ is *absolute*, that is, for every $v, w \in \mathbb{R}^n$,

$$|v_i| = |w_i|, i \in \{1, \dots, n\} \quad \implies \quad \|v\| = \|w\|, \tag{2.19}$$

or, equivalently,

$$\|(v_1, \dots, v_n)\| = \|(|v_1|, \dots, |v_n|)\| = \||v|\|. \tag{2.20}$$

Lemma 2.9. *For each $p \in [1, \infty]$ and positive vector $\eta \in \mathbb{R}^n_{>0}$, the weighted norm $\|\cdot\|_{p,[\eta]}$ is monotonic.*

(v) Every two norms on $\mathbb{R}^n$ are *equivalent*, in the sense that for any norms $\|\cdot\|_a$ and $\|\cdot\|_b$ there exist positive constants $c_1 \leq c_2$ such that $c_1\|v\|_a \leq \|v\|_b \leq c_2\|v\|_a$ for all $v \in \mathbb{R}^n$. For example, for all $v \in \mathbb{R}^n$, it is known

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty. \tag{2.21}$$

(vi) If $\langle\!\langle \cdot\, ; \cdot \rangle\!\rangle$ is an inner product on $\mathbb{R}^n$, then the function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ defined by $\|v\| = \sqrt{\langle\!\langle v\, ; v \rangle\!\rangle}$ is a norm on $\mathbb{R}^n$. A norm derived from an inner product satisfies the *parallelogram identity*

$$\tfrac{1}{2}\big(\|v + w\|^2 + \|v - w\|^2\big) = \|v\|^2 + \|w\|^2, \tag{2.22}$$

for all $v, w \in \mathbb{R}^n$. Conversely, if a norm satisfies the parallelogram identity, then it is derived from an inner product.

(vii) A norm $\|\cdot\|$ is *polyhedral* if its unit disk $\{v \in \mathbb{R}^n : \|v\| \leq 1\}$ is a polyhedron. The non-Euclidean norms ℓ_1 and ℓ_∞ are polyhedral. Polynomial norms, defined by the dth root of a degree-d homogeneous polynomial, are studied for example by Ahmadi et al. (2019). So-called canonical homogeneous norms, induced by dilations, are studied for example by Polyakov (2018).

(viii) A norm on $\mathbb{R}^n$ is called a *symmetric gauge function* if it is invariant under permutations and sign changes of coordinates.

(ix) We will be interested also in hierarchical norms over Cartesian product spaces $\mathbb{R}^n \times \mathbb{R}^k$, and in tensor norms over tensor product spaces $\mathbb{R}^n \otimes \mathbb{R}^k$; e.g., see (Lancaster and Farahat, 1972).

2.4 Matrix and logarithmic norms

2.4.1 Basic definitions

We now return to the study of upper bounds on the evolution of dynamical systems using norms. Given a matrix $A \in \mathbb{R}^{n\times n}$, we consider the discrete-time and continuous-time linear dynamics

$$x(k+1) = Ax(k) \quad \text{and} \quad \dot{x}(t) = Ax(t),$$

with respective solutions

$$x(k) = A^k x(0) \quad \text{and} \quad x(t) = \mathrm{e}^{At}\, x(0).$$

As in equations (2.1) and (2.3) in Section 2.1, given a $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, the induced matrix norm and induced matrix logarithmic norm are defined by

$$\|A\| = \max_{x\in\mathbb{R}^n, \|x\|=1} \|Ax\| \quad \text{and} \quad \mu(A) = \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}. \tag{2.23}$$

These definitions immediately imply the bounds:

$$\|x(k)\| \leq \|A\|^k \|x(0)\| \quad \text{and} \quad \|x(t)\| \leq \mathrm{e}^{\mu(A)t}\|x(0)\|.$$

Moreover, one can show that these are *least upper bounds* in the following sense:

$$\|A\| = \min\{b \in \mathbb{R} \,:\, \|x(k)\| \leq b^k \|x(0)\|, \quad \text{for all } \; k \in \mathbb{Z}_{\geq 0}, x(0) \in \mathbb{R}^n\}, \tag{2.24}$$

$$\mu(A) = \min\{b \in \mathbb{R} \,:\, \|x(t)\| \leq \mathrm{e}^{bt}\|x(0)\|, \quad \text{for all } \; t \in \mathbb{R}_{\geq 0}, x(0) \in \mathbb{R}^n\}. \tag{2.25}$$

Remark 2.10 (Interpretation of the log norm). *The log norm can be interpreted as the directional derivative of the matrix norm in the direction of A and evaluated at I_n or as the derivative of the norm of the matrix exponential, more precisely:*

$$\mu(A) = \frac{d}{dh}\|I_n + hA\|\Big|_{h=0^+} = \frac{d}{dh}\|\exp(hA)\|\Big|_{h=0^+}. \tag{2.26}$$

Accordingly, here's an informative Taylor expansion, valid for $h > 0$ in a neighborhood of $h = 0$,

$$\|I_n + hA\| = 1 + h\mu(A) + o(h). \tag{2.27}$$

For the specific case of a 1×1-dimensional matrices $A = [a]$, the matrix norm and logarithmic norm become the magnitude and real part of the complex number a, respectively. •

2.4.2 Basic properties

There are many important similarities between matrix norm and matrix log norm, but also critical differences.

As first critical difference, it is useful to start with the values of norm and log norm on simple matrices. Given the identity matrix I_n and a scalar $c \in \mathbb{R}$, an immediate consequence of homogeneity of the norm is

$$\|cI_n\| = |c| \quad \text{and} \quad \mu(cI_n) = c. \tag{2.28}$$

Therefore,

$$\|\mathbb{0}_{n\times n}\| = \mu(\mathbb{0}_{n\times n}) = 0,$$
$$\|I_n\| = \mu(I_n) = 1, \quad \text{whereas} \quad \|-I_n\| = 1 \neq \mu(-I_n) = -1.$$

This is an illustration of the fact that the matrix norm is always non-negative, whereas the log norm can take negative values.

The following lemma explains how the induced matrix norm is a matrix norm and the log norm is not.

Lemma 2.11 (Basic norm and log norm properties). *Given a norm, the induced matrix norm $\|\cdot\|: \mathbb{R}^{n\times n} \to \mathbb{R}$ is a matrix norm in the sense that is satisfies, for all $A, B \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}$:*

(positive definiteness)	$\|A\| \geq 0$ *and* $\|A\| = 0$ *if and only if* $A = \mathbb{0}_{n\times n}$,	(2.29a)
(homogeneity)	$\|aA\| = \|a\| \, \|A\|$,	(2.29b)
(subadditivity)	$\|A+B\| \leq \|A\| + \|B\|$,	(2.29c)
(sub-multiplicativity)	$\|AB\| \leq \|A\|\|B\|$.	(2.29d)

The log norm $\mu(\cdot): \mathbb{R}^{n\times n} \to \mathbb{R}$ is not a matrix norm, but it satisfies, for all $A, B \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}$:

(positive homogeneity)	$\mu(aA) = \|a\| \, \mu(\mathrm{sign}(a)A)$,	(2.30a)
(subadditivity)	$\mu(A+B) \leq \mu(A) + \mu(B)$,	(2.30b)
(translation property)	$\mu(A + aI_n) = \mu(A) + a$,	(2.30c)
(product property)	$\|Ax\| \geq \max\{-\mu(A), -\mu(-A)\}\|x\|, \;\; \forall x \in \mathbb{R}^n$,	(2.30d)
(norm of difference property)	$\|\mu(A) - \mu(B)\| \leq \|A - B\|$,	(2.30e)
(uniform monotonicity property)	$\mu(A) < 0 \implies \|A^{-1}\| \leq -1/\mu(A)$,	(2.30f)

where, as usual, $\mathrm{sign}: \mathbb{R} \to \{-1, 0, +1\}$ *is defined so that* $\mathrm{sign}(0) = 0$.

Proof. Properties (2.29a)-(2.29d) are standard. Proofs of the properties (2.30a)–(2.30f) is postponed to Exercise E2.7 (or (Desoer and Vidyasagar, 1975, Chapter 2)). ■

Note: both norm and log norm are positive homogeneous and sub-linear. Therefore they are both *convex*, that is, for all $A, B \in \mathbb{R}^{n\times n}$ and $\theta \in [0,1]$;

(norm convexity)	$\|\theta A + (1-\theta)B\| \leq \theta\|A\| + (1-\theta)\|B\|$,	(2.31)
(log norm convexity)	$\mu(\theta A + (1-\theta)B) \leq \theta\mu(A) + (1-\theta)\mu(B)$.	(2.32)

It is known that any convex scalar function is continuous, so that the induced norm and the induced log norm are continuous. Moreover, the reverse triangle inequality (i.e., $|\|A\| - \|B\|| \leq \|A - B\|$ for all matrices A and B) and the norm of difference property (2.30e) imply that the induced norm and the induced log norm are Lipschitz continuous with Lipschitz constant equal to 1. (Basic counterexamples show that, in general, induced norm and induced log norm are not differentiable.)

Note: The norm sub-multiplicativity property and the log norm product property together imply that, for all $x \in \mathbb{R}^n$,

$$\max\{-\mu(A), -\mu(-A)\}\|x\| \leq \|Ax\| \leq \|A\|\|x\|. \tag{2.33}$$

2.4.3 Weighted ℓ_p norms

Formulas for ℓ_p norms We start with some straightforward formulas that are well known and relatively easy to derive. We let $\|\cdot\|_p$ and $\mu_p(\cdot)$ denote the matrix norm and log norm induced by the ℓ_p norm for $p \in [1, \infty]$. The proof of the expressions in Table 2.1 is left as Exercise E2.8.

Vector norm	Induced matrix norm	Log norm
$\|x\|_1 = \sum_{i=1}^n \lvert x_i \rvert$	$\|A\|_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n \lvert a_{ij}\rvert$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \Big(a_{jj} + \sum_{i=1,i\neq j}^n \lvert a_{ij}\rvert\Big)$ $=$ max column "absolute sum" of A
$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$	$\|A\|_2 = \sqrt{\lambda_{\max}(A^\top A)}$	$\mu_2(A) = \lambda_{\max}\Big(\frac{A+A^\top}{2}\Big)$
$\|x\|_\infty = \max_{i\in\{1,\dots,n\}} \lvert x_i\rvert$	$\|A\|_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^n \lvert a_{ij}\rvert$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \Big(a_{ii} + \sum_{j=1,j\neq i}^n \lvert a_{ij}\rvert\Big)$ $=$ max row "absolute sum" of A

Table 2.1: Vector norms, induced matrix norms, and log matrix norms, for $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.

Several observations follow from Table 2.1:

(i) if A is skew symmetric or A is symmetric, negative semi-definite and singular, then $\mu_2(A) = 0$;

(ii) for a matrix A with negative diagonal entries, $\mu_\infty(A) < 0$ if and only if A is strictly row diagonally dominant (and $\mu_1(A) < 0$ if and only if A is strictly column diagonally dominant);

(iii) The formulas in Table 2.1 apply to a complex-valued matrix A if $A^\top$ is replaced by the conjugate transpose A^*, and the diagonal entry a_{ii} is replaced by $\Re(a_{ii})$, where $\Re(\lambda)$ is the real part of λ.

Formulas for weighted ℓ_p norms Given an invertible matrix R and a norm $\|\cdot\|$, we let $\|\cdot\|_R$ and $\mu_R(\cdot)$ denote the weighted induced matrix norm and log matrix norm. Specifically, for $p \in [1,\infty]$, we let $\|\cdot\|_{p,R}$ and $\mu_{p,R}(\cdot)$ denote the weighted ℓ_p induced matrix norm and log matrix norm. We present a useful lemma, whose proof is postponed to Exercise E2.9.

Lemma 2.12 (Weighted matrix and log norms). *Given an invertible matrix R and a norm $\|\cdot\|$,*

$$\|A\|_R = \|RAR^{-1}\| \quad \textit{and} \quad \mu_R(A) = \mu(RAR^{-1}). \tag{2.34}$$

Given these formulas, for a invertible R, define the positive definite $P = R^\top R \in \mathbb{S}^n_{\succ 0}$, and compute in the formula for $\mu_{2,R}(A)$. the weighted ℓ_2 norm and log norm as

$$\begin{aligned}\|A\|_{2,R} &= \|RAR^{-1}\|_2 = \sqrt{\lambda_{\max}((R^{-\top}A^\top R^\top)(RAR^{-1}))}\\ &= \sqrt{\lambda_{\max}(R^{-1}(R^{-\top}A^\top PAR^{-1})R)} = \sqrt{\lambda_{\max}(P^{-1}A^\top PA)},\end{aligned} \tag{2.35}$$

$$\begin{aligned}\mu_{2,R}(A) &= \mu_2(RAR^{-1}) = \lambda_{\max}\Big(\frac{RAR^{-1} + R^{-\top}A^\top R^\top}{2}\Big)\\ &= \lambda_{\max}\Big(R^\top \frac{RAR^{-1} + R^{-\top}A^\top R^\top}{2} R^{-\top}\Big) = \lambda_{\max}\Big(\frac{PAP^{-1} + A^\top}{2}\Big),\end{aligned} \tag{2.36}$$

where we used the property that the spectrum of a matrix is invariant under similarity transformations (so that $\lambda_{\max}(B) = \lambda_{\max}(R^\top BR^{-\top})$, for any matrix $B = B^\top$). Second, given a positive vector $\eta \in \mathbb{R}^n_{>0}$, for the diagonally weighted cases, we compute the weighted norms as

$$\|A\|_{1,[\eta]} = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n \frac{\eta_i}{\eta_j}\lvert a_{ij}\rvert, \qquad \|A\|_{\infty,[\eta]^{-1}} = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^n \frac{\eta_j}{\eta_i}\lvert a_{ij}\rvert, \tag{2.37}$$

and the log norm as

$$\mu_{1,[\eta]}(A) = \mu_1([\eta]A[\eta]^{-1}) = \max_{j\in\{1,...,n\}} \Big(a_{jj} + \frac{1}{\eta_j}\sum_{i=1,i\neq j}^{n} |a_{ij}|\eta_i\Big), \tag{2.38}$$

$$\mu_{\infty,[\eta]^{-1}}(A) = \mu_\infty([\eta]^{-1}A[\eta]) = \max_{i\in\{1,...,n\}} \Big(a_{ii} + \frac{1}{\eta_i}\sum\nolimits_{j=1,j\neq i}^{n} |a_{ij}|\eta_j\Big), \tag{2.39}$$

where we used the equality $\big([\eta]A[\eta]^{-1}\big)_{ij} = a_{ij}\eta_i/\eta_j$ and we adopted the shorthand

$$[\eta] = \operatorname{diag}(\eta) = \begin{bmatrix} \eta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \eta_n \end{bmatrix}.$$

Formulas based upon absolute value and Metzler majorants Finally, we provide equivalent expressions for the diagonally-weighted ℓ_p matrix norm and log norm of general matrices and of nonnegative and Metzler matrices. From Section 2.2 recall the notions of *absolute value majorant* $|A|$ and of *Metzler majorant* $|A|_{\mathrm{M}}$ of a matrix $A \in \mathbb{R}^{n\times n}$.

It is immediate to write the following formulas, equivalent to those in Table 2.1,

$$\|A\|_{1,[\eta]} = \||A|\|_{1,[\eta]} = \max(\eta^\top |A| [\eta]^{-1}), \tag{2.40a}$$

$$\|A\|_{\infty,[\eta]^{-1}} = \||A|\|_{\infty,[\eta]^{-1}} = \max([\eta]^{-1}|A|\eta), \tag{2.40b}$$

and

$$\mu_{1,[\eta]}(A) = \mu_{1,[\eta]}(|A|_{\mathrm{M}}) = \max(\eta^\top |A|_{\mathrm{M}}[\eta]^{-1}), \tag{2.41a}$$

$$\mu_{\infty,[\eta]^{-1}}(A) = \mu_{\infty,[\eta]^{-1}}(|A|_{\mathrm{M}}) = \max([\eta]^{-1}|A|_{\mathrm{M}}\eta), \tag{2.41b}$$

as well as their unweighted versions:

$$\|A\|_1 = \max(\mathbb{1}^\top |A|), \qquad \|A\|_\infty = \max(|A|\mathbb{1}), \tag{2.42}$$

$$\mu_1(A) = \max(\mathbb{1}^\top |A|_{\mathrm{M}}), \qquad \mu_\infty(A) = \max(|A|_{\mathrm{M}}\mathbb{1}). \tag{2.43}$$

2.4.4 Composite norms

In this section we discuss composite norms for vector spaces that are Cartesian products of vector spaces. This setting is useful to analyze systems composed of subsystems as we illustrate in Figure 2.3.

Given r positive integers $n_1, \dots, n_r$ such that $n_1 + \cdots + n_r = n$, consider the decomposition

$$\mathbb{R}^n = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}, \tag{2.44}$$

so that any $x \in \mathbb{R}^n$ can be written as $x = \begin{bmatrix} x_1^\top & \ldots & x_r^\top \end{bmatrix}^\top$ with $x_i \in \mathbb{R}^{n_i}$. Suppose we are given

(i) r *local norms* $\|\cdot\|_i$ on $\mathbb{R}^{n_i}$ with induced matrix norm $\|\cdot\|_i$ and log norm $\mu_i(\cdot)$ for $i \in \{1, \dots, r\}$, and

(ii) an *aggregating norm* $\|\cdot\|_{\mathrm{agg}}$ on $\mathbb{R}^r$ with associated log norm $\mu_{\mathrm{agg}}(\cdot)$.

We define the *composite norm* $\|\cdot\|_{\mathrm{cmpst}}$ on $\mathbb{R}^n$ with associated log norm $\mu_{\mathrm{cmpst}}(\cdot)$ by

$$\|x\|_{\mathrm{cmpst}} = \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} \right\|_{\mathrm{cmpst}} = \left\| \begin{bmatrix} \|x_1\|_1 \\ \vdots \\ \|x_r\|_r \end{bmatrix} \right\|_{\mathrm{agg}}. \tag{2.45}$$

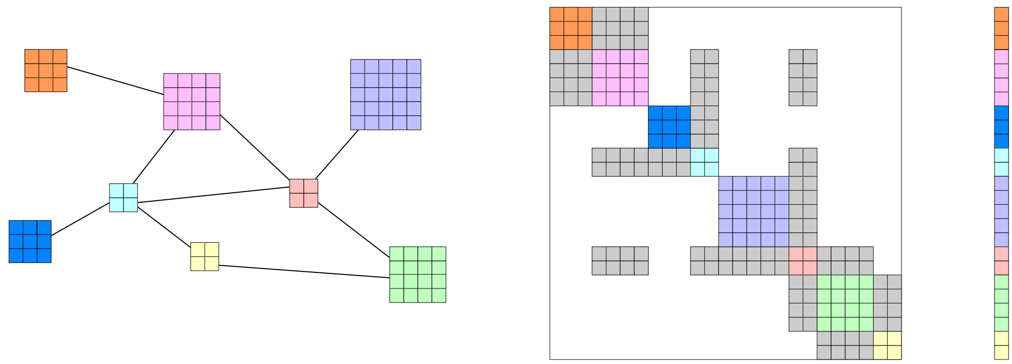


Figure 2.3: Composite norms are useful to analyze interconnected systems and tackle computational complexity. Left panel: An interconnected system is described by a graph: each node i is described by variables taking value in $\mathbb{R}^{n_i}$ and edges between the r nodes describe interactions between nodes. Center panel: a matrix description of the system has a sparsity pattern that reflects the interconnections. Right panel: The state of the interconnected system takes value in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}$.

Next, we consider an $n \times n$ block matrix A with blocks of the same dimension as the decomposition of $\mathbb{R}^n$:

$$A = \begin{bmatrix} A_{11} & \dots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \dots & A_{rr} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad A_{ij} \in \mathbb{R}^{n_i \times n_j}.$$

We define the *aggregate majorant* $\lceil A \rceil$ and *aggregate Metzler majorant* $\lceil A \rceil_{\mathrm{M}}$ in $\mathbb{R}^{r \times r}$ by

$$(\lceil A \rceil)_{ij} = \|A_{ij}\|_{j \to i}, \tag{2.46}$$

$$(\lceil A \rceil_{\mathrm{M}})_{ij} = \begin{cases} \mu_i(A_{ii}), & \text{if } j = i, \\ \|A_{ij}\|_{j \to i}, & \text{if } j \neq i, \end{cases} \tag{2.47}$$

where, for any i and j, $\|A_{ij}\|_{j \to i}$ is the *norm of $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ induced by the j-th local norm on $\mathbb{R}^{n_j}$ and the i-th local norm on $\mathbb{R}^{n_i}$*, defined by

$$\|A_{ij}\|_{j \to i} = \max\{\|A_{ij} y_j\|_i \, : \, y_j \in \mathbb{R}^{n_j} \text{ s.t. } \|y_j\|_j = 1\}. \tag{2.48}$$

We report explicit formulas for some common $j \to i$ induced matrix norms in Exercise E2.9.

Clearly, the aggregate majorant $\lceil A \rceil$ is non-negative and the aggregate Metzler majorant $\lceil A \rceil_{\mathrm{M}}$ is Metzler. Moreover, when $r = n$, $n_i = 1$ and $\|\cdot\|_i = |\cdot|$ for $i \in \{1, \dots, n\}$, the aggregate majorant and the aggregate Metzler majorant are generalizations of the absolute value and the Metzler majorant $|A|_{\mathrm{M}}$ defined in Section 2.2.

Theorem 2.13 (Composite induced norms and log norms). *For any set of local norms $\|\cdot\|_i$ and an aggregating norm $\|\cdot\|_{\mathrm{agg}}$ over a decomposition of $\mathbb{R}^n$, consider a matrix $A \in \mathbb{R}^{n \times n}$,*

(i) the composite norm $\|\cdot\|_{\mathrm{cmpst}}$ is a well-defined, i.e., it satisfies the norm properties;

(ii) if the aggregating norm $\|\cdot\|_{\mathrm{agg}}$ is monotonic, then

$$\max_{i \in \{1,\dots,r\}} \|A_{ii}\|_i \leq \|A\|_{\mathrm{cmpst}} \leq \|\lceil A \rceil\|_{\mathrm{agg}}, \tag{2.49}$$

$$\max_{i \in \{1,\dots,r\}} \mu_i(A_{ii}) \leq \mu_{\mathrm{cmpst}}(A) \leq \mu_{\mathrm{agg}}(\lceil A \rceil_{\mathrm{M}}). \tag{2.50}$$

We provide additional results on optimal composite norms later in Corollary 2.34.

Proof of Theorem 2.13. Regarding statement (i), it is easy to verify the properties in equation (2.14). Regarding statement (ii), we prove the log norm result and leave the induced matrix norm result to the reader. For $\mathbf{x} \in \mathbb{R}^n$ and $h > 0$, we compute

$$\begin{aligned}
&\|\mathbf{x} + hA\mathbf{x}\|_{\text{cmpst}} \\
&\quad = \left\| \begin{bmatrix} \vdots \\ \|x_i + hA_{ii}x_i + \sum_{j\neq i} hA_{ij}x_j\|_i \\ \vdots \end{bmatrix} \right\|_{\text{agg}} \\
&\quad \overset{(\|\cdot\|_{\text{agg}}\text{ is monotonic})}{\leq} \left\| \begin{bmatrix} \vdots \\ \|x_i + hA_{ii}x_i\|_i + \big\| \sum_{j\neq i} hA_{ij}x_j \big\|_i \\ \vdots \end{bmatrix} \right\|_{\text{agg}} \\
&\quad \overset{(\text{Taylor expansion } (2.27))}{\leq} \left\| \begin{bmatrix} \vdots \\ (1 + h\mu_i(A_{ii}))\|x_i\|_i + h\sum_{j\neq i} \|A_{ij}\|_{ij}\|x_j\|_j \\ \vdots \end{bmatrix} + o(h) \right\|_{\text{agg}} \\
&\quad = \left\| (I_n + h\lceil A \rceil_{\text{M}}) \begin{bmatrix} \|x_1\|_1 \\ \vdots \\ \|x_r\|_r \end{bmatrix} \right\|_{\text{agg}} + o(h).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mu_{\text{cmpst}}(A) &= \lim_{h\to 0^+} \frac{\|I_n + hA\|_{\text{cmpst}} - 1}{h} = \lim_{h\to 0^+} \max_{\|\mathbf{x}\|_{\text{cmpst}}=1} \frac{\|\mathbf{x} + hA\mathbf{x}\|_{\text{cmpst}} - 1}{h} \\
&\leq \lim_{h\to 0^+} \max_{y \geq \mathbb{0}_n, \|y\|_{\text{agg}}=1} \frac{\|y + h\lceil A \rceil_{\text{M}} y\|_{\text{agg}} - 1}{h} = \mu_{\text{agg}}(\lceil A \rceil_{\text{M}}).
\end{aligned}$$

On the other hand, pick $\mathbf{x} \in \mathbb{R}^n$ such that $x_j = \mathbb{0}_j$ for all $j \neq i$, and observe

$$\|\mathbf{x} + hA\mathbf{x}\|_{\text{cmpst}} = \left\| \begin{bmatrix} \|A_{1i}x_i\|_1 \\ \vdots \\ \|x_i + hA_{ii}x_i\|_i \\ \vdots \\ \|A_{ri}x_i\|_r \end{bmatrix} \right\|_{\text{agg}} \geq \|x_i + hA_{ii}x_i\|_i.$$

Hence

$$\begin{aligned}
\mu_{\text{cmpst}}(A) &= \lim_{h\to 0^+} \max_{\|\mathbf{x}\|_{\text{cmpst}}=1} \frac{\|\mathbf{x} + hA\mathbf{x}\|_{\text{cmpst}} - 1}{h} \\
&\geq \lim_{h\to 0^+} \max_{\|x_i\|_i=1} \frac{\|x_i + hA_{ii}x_i\|_i - 1}{h} = \lim_{h\to 0^+} \frac{\|I_i + hA_{ii}\|_i - 1}{h} = \mu_i(A_{ii}).
\end{aligned}$$

This completes the proof of statement (ii). ■

2.4.5 Norms and log norms of matrix sets

In this section we review some results about the worst-case norm and log norm of matrices taking value in a set. Recall that a *polytope* is a geometric object with flat sides (faces) and that polytopes are the generalization of three-dimensional polyhedra to any number of dimensions. In what follows, we focus on convex polytopes.

Lemma 2.14 (Norm and log norm inequalities for polytopes of matrices). *Consider the convex polytope $\mathcal{P}$ defined by the convex hull of a collection of matrices $A_1, \dots, A_m \in \mathbb{R}^{n\times n}$ (e.g., see Figure 2.4):*

$$\mathcal{P} = \Big\{ \sum\nolimits_{j=1}^m \beta_j A_j \;:\; \beta_j \geq 0, \sum\nolimits_{j=1}^m \beta_j = 1 \Big\}.$$

Then

(i) for any norm and log norm, each $A \in \mathcal{P}$ satisfies

$$\begin{aligned} \|A\| &\leq \max_{j\in\{1,\dots,m\}} \|A_j\|, \\ \mu(A) &\leq \max_{j\in\{1,\dots,m\}} \mu(A_j); \end{aligned}$$

(ii) if each matrix $A \in \mathcal{P}$ is Hurwitz and $A^\top \in \mathcal{P}$, then $\mu_2(A_j) < 0$ for all $j \in \{1, \dots, m\}$.

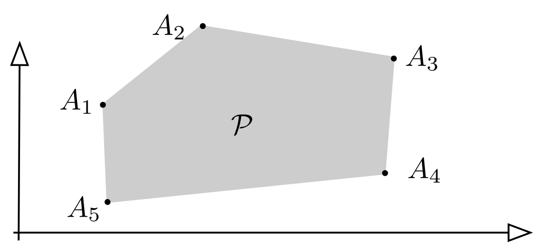


Figure 2.4: Convex polytope of matrices

Proof. Regarding statement (i), pick a matrix $A \in \mathcal{P}$ so that $A = \sum_{j=1}^m \beta_j A_j$ for some $\beta_j \geq 0$ satisfying $\sum_{j=1}^m \beta_j = 1$. The subadditivity and scaling properties of the norm (2.29c) and (2.29b) and of the log norm (2.30b) and (2.30a) imply

$$\begin{aligned} \|A\| &= \|\sum\nolimits_{j=1}^m \beta_j A_j\| \leq \sum\nolimits_{j=1}^m \beta_j \|A_j\| \leq \max_j \|A_j\|, \\ \mu(A) &= \mu(\sum\nolimits_{j=1}^m \beta_j A_j) \leq \sum\nolimits_{j=1}^m \beta_j \mu(A_j) \leq \max_j \mu(A_j). \end{aligned}$$

Regarding statement (ii), pick any matrix A in the convex hull of $A_1, \dots, A_m$. But then also $A^\top$ is in the convex hull $\mathcal{P}$ and so is $(A + A^\top)/2$. We now note that $(A + A^\top)/2$ is Hurwitz and symmetric, therefore its maximum eigenvalue is negative. In sum, $0 > \max_i \lambda_i((A + A^\top)/2) = \mu_2(A)$. ■

A set of square matrices $\mathcal{S}$ is *Hurwitz* if each matrix in $\mathcal{S}$ is Hurwitz. Lemma 2.14(i) implies the following sufficient condition: $\mathcal{P}$ is Hurwitz if there exists a log norm μ such that $\mu(A_j) < 0$ for all $j \in \{1, \dots, m\}$.

Next, as a specific example, we consider the matrix polytope

$$\mathcal{P} = \text{convex hull of } \{[d]A \in \mathbb{R}^{n\times n} \;:\; d \in [0,1]^n\}, \tag{2.51}$$

defined by a matrix $A \in \mathbb{R}^{n\times n}$. Note that there exist 2^n matrices of the form $[d]A \in \mathbb{R}^{n\times n}$, for $d \in [0,1]^n$. We now show how only 1 or 2 of the 2^n vertices need to be checked to compute the maximum value of the ℓ_1 or ℓ_∞ induced normsover $\mathcal{P}$; see Figure 2.5.

Lemma 2.15 (Log norm of multiplicatively-scaled matrix). *For* $A \in \mathbb{R}^{n\times n}$,

$$\max_{d\in[0,1]^n} \mu_1([d]A) = \max\{\mu_1(A), \mu_1(A - (I_n \circ A))\}, \tag{2.52}$$

$$\max_{d\in[0,1]^n} \mu_\infty([d]A) = \max\{0, \mu_\infty(A)\}. \tag{2.53}$$

Additionally, $\max_{d\in[0,1]^n} \|[d]A\|_p = \|A\|_p$ *for* $p \in \{1,\infty\}$.

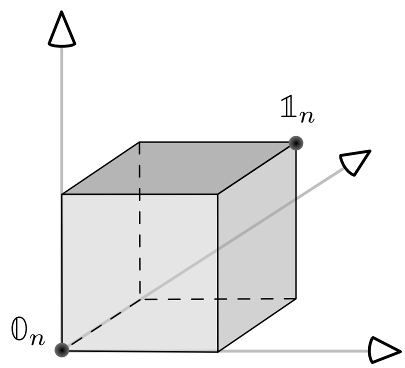


Figure 2.5: The hypercube $[0,1]^n$ has 2^n vertices, including the minimal vertex $\mathbb{0}_n = (0,\dots,0)$ and the maximal vertex $\mathbb{1}_n = (1,\dots,1)$.
The polytope $\mathcal{P}$ defined in equation (2.51) is a linear transformation of the hypercube $[0,1]^n$.
Note $\mu([\mathbb{0}_n]A) = \mu(\mathbb{0}_{n\times n}) = 0$, $\mu([\mathbb{1}_n]A) = \mu(A)$ and $\mu_1(A - (I_n \circ A)) = \mu_1([\mathbb{1}_n - \mathbb{e}_{i^*}]A)$, for an appropriate $i^* \in \{1,\dots,n\}$ (as defined in Exercise E2.22).

Note that corresponding formulas for $\mu_p(A[d])$, $p \in \{1,\infty\}$, can be obtained via the equality $\mu_1(A) = \mu_\infty(A^\top)$ and that we state a more general version of this lemma in Exercise E2.22.

Proof of Lemma 2.15. Recall the entrywise matrix product $(A\circ B)_{ij} = a_{ij}b_{ij}$ and recall from Table 2.1 that the ℓ_1 lognorm is a max column "absolute sum." We compute:

$$\begin{aligned}
\max_{d\in[0,1]^n} \mu_1([d]A) &\overset{\text{(by def)}}{=} \max_{d\in[0,1]^n} \max_j \Big(d_j a_{jj} + \sum\nolimits_{i=1,i\neq j}^n d_i |a_{ij}|\Big) \\
&\overset{(d_j\in[0,1] \text{ and } d_i\leq 1)}{\leq} \max_j \begin{cases} a_{jj} + \sum_{i=1,i\neq j}^n |a_{ij}|, & \text{if } a_{jj}\geq 0 \\ \sum_{i=1,i\neq j}^n |a_{ij}|, & \text{if } a_{jj} < 0 \end{cases} \\
&\overset{\text{(dropping the if clause)}}{\leq} \max\{\max_j \big(a_{jj} + \sum\nolimits_{i\neq j} |a_{ij}|\big), \max_j \big(\sum\nolimits_{i\neq j} |a_{ij}|\big)\} \\
&\leq \max\{\mu_1(A), \mu_1(A - (I_n \circ A))\}.
\end{aligned}$$

Next, for each row i, we define the absolute row-sum of A by $\mathbb{r}_i = a_{ii} + \sum_{j=1,j\neq i}^n |a_{ij}| \in \mathbb{R}$. Since $d_i \geq 0$ and $([d]A)_{ij} = d_i a_{ij}$, the absolute row-sum of $[d]A$ is $d_i \mathbb{r}_i$. From Table 2.1 we recall $\mu_\infty(A) = \max_i \mathbb{r}_i$ and compute

$$\begin{aligned}
\max_{d\in[0,1]^n} \mu_\infty([d]A) &\overset{\text{(by def)}}{=} \max_{d\in[0,1]^n} \max_i d_i \mathbb{r}_i \\
&\overset{\text{(the } n \text{ functions are decoupled)}}{=} \max_i \max_{d_i\in[0,1]} d_i \mathbb{r}_i \\
&\overset{(d_i\in[0,1])}{=} \max_i \begin{cases} \mathbb{r}_i, & \text{if } \mathbb{r}_i \geq 0 \\ 0, & \text{if } \mathbb{r}_i < 0 \end{cases} \\
&\overset{\text{(dropping the if clause)}}{\leq} \max\{\max_i \mathbb{r}_i, 0\} = \max\{\mu_\infty(A), 0\}.
\end{aligned}$$

We have now established upper bounds for $\max_{d\in[0,1]^n} \mu_p([d]A)$, $p \in \{1,\infty\}$. In each of the two cases, one can show that the upper bound is tight because it corresponds to two specific vertices of the polytope $[d]A$; we leave the details to the reader in Exercise E2.22.

Finally, the two bounds on the induced norms are immediate consequences of the formulas in Table 2.1. ■

2.5 Weak pairings and quadratic forms

2.5.1 Quadratic forms based on inner products

In this section we show how induced norms and log norms are related to certain quadratic forms.

We start with some useful calculations for the weighted ℓ_2 norm. Given a matrix A and a positive definite matrix P, we compute, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} &\|Ax\|_{2,P^{1/2}} \leq \|A\|_{2,P^{1/2}} \|x\|_{2,P^{1/2}} \\ \iff\quad & \|Ax\|_{2,P^{1/2}}^2 \leq \|A\|_{2,P^{1/2}}^2 \|x\|_{2,P^{1/2}}^2 \\ \iff\quad & x^\top A^\top P A x \leq \|A\|_{2,P^{1/2}}^2 x^\top P x \\ \iff\quad & A^\top P A \preceq \|A\|_{2,P^{1/2}}^2 P. \end{aligned} \tag{2.54}$$

Similarly, recalling the *Courant minimax principle*[4] we compute from (2.36) (with $R = P^{1/2}$)

$$\begin{aligned} \mu_{2,P^{1/2}}(A) = \mu_2(P^{1/2}AP^{-1/2}) &= \lambda_{\max}\Big(\frac{P^{1/2}AP^{-1/2} + (P^{1/2}AP^{-1/2})^\top}{2}\Big) \\ &= \max_{y^\top y=1} y^\top \frac{P^{1/2}AP^{-1/2} + (P^{1/2}AP^{-1/2})^\top}{2} y \\ &= \max_{x^\top P x=1} x^\top \frac{PA + A^\top P}{2} x \end{aligned}$$

where we changed variable from y to $x = R^{-1}y$. The following lemma summarizes and interprets these calculations.

Lemma 2.16 (Norms and log norm as a quadratic form in ℓ_2). *For any $A \in \mathbb{R}^{n\times n}$, $P \in \mathbb{S}^n_{\succ 0}$,*

$$\begin{aligned} &(\|\cdot\|_{2,P^{1/2}} \textit{ quadratic form}) & \|A\|_{2,P^{1/2}} &= \max\{(x^\top A^\top P A x)^{1/2} \;:\; x \in \mathbb{R}^n \text{ s.t. } x^\top P x = 1\} & \text{(2.55a)} \\ &(\|\cdot\|_{2,P^{1/2}} \textit{ LMI form}) & &= \min\{b \in \mathbb{R}_{\geq 0} \;:\; A^\top P A \preceq b^2 P\}, & \text{(2.55b)} \end{aligned}$$

and

$$\begin{aligned} &(\mu_{2,P^{1/2}}(\cdot) \textit{ quadratic form}) & \mu_{2,P^{1/2}}(A) &= \max\{x^\top P A x \;:\; x \in \mathbb{R}^n \text{ s.t. } x^\top P x = 1\} & \text{(2.56a)} \\ &(\mu_{2,P^{1/2}}(\cdot) \textit{ LMI form}) & &= \min\{b \in \mathbb{R} \;:\; A^\top P + PA \preceq 2bP\}. & \text{(2.56b)} \end{aligned}$$

Formula (2.56b) implies that, for all $A \in \mathbb{R}^{n\times n}$, $P \in \mathbb{S}^n_{\succ 0}$, and $b \in \mathbb{R}$,

$$\mu_{2,P^{1/2}}(A) \leq b \quad\iff\quad \tfrac{1}{2}\big(A^\top P + PA\big) \preceq bP. \tag{2.57}$$

We refer to the right condition as to the *Lyapunov contractivity LMI* in the Lyapunov matrix P. Finally, we note that it is possible to generalize (2.56a) to arbitrary inner products:

$$\mu(A) = \sup_{x\in\mathbb{R}^n, x\neq \mathbb{0}_n} \frac{\langle\!\langle Ax\,;\,x\rangle\!\rangle}{\langle\!\langle x\,;\,x\rangle\!\rangle} = \max\big\{\langle\!\langle Ax\,;\,x\rangle\!\rangle \;:\; x \in \mathbb{R}^n \text{ s.t. } \langle\!\langle x\,;\,x\rangle\!\rangle = 1\big\}.$$

[4] For any symmetric matrix S, $\lambda_{\max}(S) = \max\{y^\top S y \;:\; y \in \mathbb{R}^n \text{ s. t. } y^\top y = 1\}$.

2.5.2 Weak pairings on normed spaces

We now generalize this treatment to normed spaces endowed with a so-called "weak pairing" structure.

Before introducing the notion of weak pairing, it is useful to quickly review the stronger properties of an inner product. An *inner product* on $\mathbb{R}^n$ is a map $\langle\langle \cdot\,;\,\cdot \rangle\rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying:

(i) (*additivity*) $\langle\langle x_1 + x_2\,;\,y \rangle\rangle = \langle\langle x_1\,;\,y \rangle\rangle + \langle\langle x_2\,;\,y \rangle\rangle$, for all $x_1, x_2, y \in \mathbb{R}^n$,

(ii) (*homogeneity*) $\langle\langle \alpha x\,;\,y \rangle\rangle = \alpha \langle\langle x\,;\,y \rangle\rangle$ for all $x, y \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$,

(iii) (*positive definiteness*) $\langle\langle x\,;\,x \rangle\rangle > 0$, for all $x \neq \mathbb{0}_n$,

(iv) (*symmetry*) $\langle\langle x\,;\,y \rangle\rangle = \langle\langle y\,;\,x \rangle\rangle$ for all $x, y \in \mathbb{R}^n$.

We now introduce an operation that has weaker properties.

Definition 2.17. *A weak pairing on $\mathbb{R}^n$ is a map $[\![\cdot\,;\,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying:*

(WP1) *(sub-additivity and continuity of first argument) $[\![x_1 + x_2\,;\,y]\!] \leq [\![x_1\,;\,y]\!] + [\![x_2\,;\,y]\!]$, for all $x_1, x_2, y \in \mathbb{R}^n$ and $[\![\cdot\,;\,\cdot]\!]$ is continuous in its first argument,*

(WP2) *(weak homogeneity) $[\![\alpha x\,;\,y]\!] = [\![x\,;\,\alpha y]\!] = \alpha\,[\![x\,;\,y]\!]$ and $[\![-x\,;\,-y]\!] = [\![x\,;\,y]\!]$, for all $x, y \in \mathbb{R}^n$, $\alpha \geq 0$,*

(WP3) *(positive definiteness) $[\![x\,;\,x]\!] > 0$, for all $x \neq \mathbb{0}_n$,*

(WP4) *(Cauchy-Schwarz inequality) $|\,[\![x\,;\,y]\!]\,| \leq [\![x\,;\,x]\!]^{1/2}\,[\![y\,;\,y]\!]^{1/2}$, for all $x, y \in \mathbb{R}^n$.*

For every norm $\|\cdot\|$ on $\mathbb{R}^n$, there exists a (possibly not unique) *compatible* weak pairing $[\![\cdot\,;\,\cdot]\!]$ such that $\|x\|^2 = [\![x\,;\,x]\!]$, for every $x \in \mathbb{R}^n$. If the norm is induced by an inner product, the weak pairing coincides with the inner product, that is,

$$[\![x\,;\,y]\!]_{2,P^{1/2}} = x^\top P y.$$

In what follows we mostly focus on the nonEuclidean norms ℓ_p, $p \in [1, \infty]$ on $\mathbb{R}^n$. While weak pairings for these norms are not unique, we now propose specific choices that have numerous advantageous properties.

Definition 2.18 (Standing assumptions on weak pairing). *A weak pairing $[\![\cdot\,;\,\cdot]\!]$, compatible with the norm $\|\cdot\|$, satisfies:*

(WP5) *the curve norm derivative formula if, for every differentiable curve $x : \,]a, b[\, \to \mathbb{R}^n$ and for almost every $t \in \,]a, b[$,*

$$\|x(t)\| D^+ \|x(t)\| = [\![\dot{x}(t)\,;\,x(t)]\!]\,; \tag{2.58}$$

(WP6) *Lumer's equality if, for every matrix $A \in \mathbb{R}^{n \times n}$,*

$$\mu(A) = \sup_{x \in \mathbb{R}^n, x \neq \mathbb{0}_n} \frac{[\![Ax\,;\,x]\!]}{[\![x\,;\,x]\!]} = \sup_{\|x\|=1} [\![Ax\,;\,x]\!]\,; \tag{2.59}$$

(WP7) *Deimling's inequality if, for all $x, y \in \mathbb{R}^n$,*

$$[\![x\,;\,y]\!] \leq \|y\| \lim_{h \to 0^+} \frac{\|y + hx\| - \|y\|}{h}. \tag{2.60}$$

Every inner product, regarded as a weak pairing, satisfies the additional properties (WP5)–(WP7). Next, we focus on the nonEuclidean norms ℓ_1 and ℓ_∞ on $\mathbb{R}^n$.

Definition 2.19 (Sign and max pairings). *For $\eta \in \mathbb{R}^n_{>0}$, we define the sign and max pairings $[\![\cdot\,;\,\cdot]\!]_{1,[\eta]}$ and $[\![\cdot\,;\,\cdot]\!]_{\infty,[\eta]^{-1}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by*

$$[\![x\,;\,y]\!]_{1,[\eta]} = \|y\|_{1,[\eta]} \operatorname{sign}(y)^\top [\eta] x, \tag{2.61}$$

$$[\![x\,;\,y]\!]_{\infty,[\eta]^{-1}} = \max_{i \in I_\infty([\eta]^{-1} y)} \eta_i^{-2} y_i x_i, \tag{2.62}$$

where $I_\infty(x) = \{i \in \{1,\dots,n\} \,:\, |x_i| = \|x\|_\infty\}$.

Lemma 2.20. *For $\eta \in \mathbb{R}^n_{>0}$, the sign and max pairings satisfy:*

- *compatibility with the weighted ℓ_1 and ℓ_∞ norms respectively, that is, for all $x \in \mathbb{R}^n$,*

$$[\![x\,;\,x]\!]_{1,[\eta]} = \|x\|^2_{1,[\eta]} \quad \text{and} \quad [\![x\,;\,x]\!]_{\infty,[\eta]^{-1}} = \|x\|^2_{\infty,[\eta]^{-1}};$$

- *the characteristic properties (WP1)–(WP4) of a weak pairing, and*
- *the additional properties (WP5)–(WP7).*

For the sign and max pairings in ℓ_1 and ℓ_∞ respectively, we write Lumer's equality in detail and establish the following quadratic form equalities.

Lemma 2.21 (Log norm as quadratic form in ℓ_1/ℓ_∞). *For any $A \in \mathbb{R}^{n\times n}$, $\eta \in \mathbb{R}^n_{>0}$, the quadratic and LP forms of the ℓ_1 and ℓ_∞ norms are, respectively,*

$$(\mu_{1,\eta}(\cdot) \text{ quadratic form}) \qquad \mu_{1,[\eta]}(A) = \max\big\{ \operatorname{sign}(x)^\top [\eta] A \,:\, x \in \mathbb{R}^n \text{ s.t. } \|x\|_{1,[\eta]} = 1 \big\} \tag{2.63a}$$

$$(\mu_{1,[\eta]}(\cdot) \text{ LP form}) \qquad = \min\{ b \in \mathbb{R} \,:\, \eta^\top |A|_{\mathrm{M}} \leq b \eta^\top \}, \tag{2.63b}$$

and

$$(\mu_{\infty,[\eta]^{-1}}(\cdot) \text{ quadratic form}) \qquad \mu_{\infty,[\eta]^{-1}}(A) = \max\Big\{ \max_{i \in I_\infty([\eta]^{-1}x)} \frac{x_i}{\eta_i} ([\eta]^{-1} A x)_i \,:\, x \in \mathbb{R}^n \text{ s.t. } \|x\|_{\infty,[\eta]^{-1}} = 1 \Big\} \tag{2.64a}$$

$$(\mu_{\infty,[\eta]^{-1}}(\cdot) \text{ LP form}) \qquad = \min\{ b \in \mathbb{R} \,:\, |A|_{\mathrm{M}} \eta \leq b \eta \}. \tag{2.64b}$$

Note: Formulas (2.63b) and (2.64b) imply that, for all $A \in \mathbb{R}^{n\times n}$, $\eta \in \mathbb{R}^n_{>0}$, and $b \in \mathbb{R}$,

$$\mu_{1,[\eta]}(A) \leq b \iff \eta^\top |A|_{\mathrm{M}} \leq b\eta^\top, \quad \text{and} \quad \mu_{\infty,[\eta]^{-1}}(A) \leq b \iff |A|_{\mathrm{M}} \eta \leq b\eta. \tag{2.65}$$

Proof. To prove equation (2.63b), for any b such that $\mu_{1,[\eta]}(A) \leq b$, we compute

$$\begin{aligned} \mu_{1,[\eta]}(A) \leq b \quad &\iff \quad \max(\eta^\top |A|_{\mathrm{M}} [\eta]^{-1}) \leq b \\ &\iff \quad \eta^\top |A|_{\mathrm{M}} [\eta]^{-1} \leq b \mathbb{1}_n^\top \\ &\iff \quad \eta^\top |A|_{\mathrm{M}} \leq b \mathbb{1}_n^\top [\eta] = b \eta^\top. \end{aligned}$$

The minimum over all such scalars b is equal to $\mu_{1,[\eta]}(A)$. We leave the proof of equivalence (2.64b) to the reader. ■

We conclude this section with Table 2.2 summarizing the quadratic form results.

Log norms	Quadratic forms (for all $x \in \mathbb{R}^n$)	Ref
$\mu_{2,P^{1/2}}(A) = \lambda_{\max}\Big(\dfrac{PAP^{-1}+A^\top}{2}\Big)$	$\mu_{2,P^{1/2}}(A) = \max\{x^\top PAx \,:\, x^\top Px = 1\}$ $= \min\{b \in \mathbb{R} \,:\, A^\top P + PA \preceq 2bP\}$	Lemma 2.16
$\mu_{1,[\eta]}(A) = \max\big(\eta^\top \lvert A\rvert_{\mathrm{M}}[\eta]^{-1}\big)$	$\mu_{1,[\eta]}(A) = \sup\{\mathrm{sign}(x)^\top[\eta]Ax \,:\, \lVert x\rVert_{1,[\eta]} = 1\}$ $= \min\{b \in \mathbb{R} \,:\, \eta^\top \lvert A\rvert_{\mathrm{M}} \leq b\eta^\top\}$	Lemma 2.21
$\mu_{\infty,[\eta]^{-1}}(A) = \max\big([\eta]^{-1}\lvert A\rvert_{\mathrm{M}}\eta\big)$	$\mu_{\infty,[\eta]^{-1}}(A) = \max\big\{\max\limits_{i\in I_\infty(x)} x_i(Ax)_i \,:\, \lVert x\rVert_{\infty,[\eta]^{-1}} = 1\big\}$ $= \min\{b \in \mathbb{R} \,:\, \lvert A\rvert_{\mathrm{M}}\eta \leq b\eta\}$	Lemma 2.21

Table 2.2: Table of quadratic forms for weighted ℓ_1, ℓ_2, and ℓ_∞ log norms; $P \in \mathbb{S}^n_{\succ 0}$ and $\eta \in \mathbb{R}^n_{>0}$. We adopt the shorthand $I_\infty(x) = \{i \in \{1,\dots,n\} \,:\, |x_i| = \|x\|_\infty\}$ as the set of indices where x takes its maximal absolute value.

Remark 2.22 (More on the curve norm derivative formula). *As mentioned, for $p \in \{1,2,\infty\}$, the curve norm derivative formula holds*

$$\tfrac{1}{2}D^+\|x(t)\|_p^2 = [\![\dot{x}(t)\,;\,x(t)]\!]_p\,. \tag{2.66}$$

Therefore, given a trajectory $x(t)$ solution to $\dot{x} = Ax$, the curve norm derivative formula and the quadratic form equality (2.59) imply

$$\tfrac{1}{2}D^+\|x(t)\|_p^2 = [\![Ax(t)\,;\,x(t)]\!]_p \leq \mu_p(A)\|x(t)\|_p^2. \tag{2.67}$$

For example, we have

$$\mu_1(A) \leq b \quad\Longrightarrow\quad \tfrac{1}{2}D^+\|x\|_1^2 = \|x\|_1\,\mathrm{sign}(x)^\top\dot{x} \leq b\|x\|_1^2, \tag{2.68}$$
$$\mu_2(A) \leq b \quad\Longrightarrow\quad \tfrac{1}{2}D^+\|x\|_2^2 = x^\top\dot{x} \leq b\|x\|_2^2, \tag{2.69}$$
$$\mu_\infty(A) \leq b \quad\Longrightarrow\quad \tfrac{1}{2}D^+\|x\|_\infty^2 = \max_{i\in I_\infty(x)}\{x_i\dot{x}_i\} \leq b\|x\|_\infty^2. \tag{2.70}$$

Using the curve norm derivative formula we will study generalizations of these bounds for nonlinear systems. •

2.6 Spectral and monotonicity properties

2.6.1 Spectral properties

Theorem 2.23 (Spectrum-norm properties). *Given a matrix $A \in \mathbb{R}^{n\times n}$ and a norm $\|\cdot\|$,*

(i) for any eigenvalue λ of A, the spectral-radius norm property is

$$\text{(spectral-radius norm property)} \qquad 0 \leq |\lambda| \leq \rho(A) \leq \|A\|, \tag{2.71}$$

and, if A is invertible,

$$0 \leq 1/\|A^{-1}\| \leq |\lambda| \leq \rho(A) \leq \|A\|; \tag{2.72}$$

(ii) for any eigenvalue λ of A, the spectral-abscissa log-norm property is

$$\textit{(spectral-abscissa log-norm property)} \qquad -\|A\| \leq -\mu(-A) \leq \Re(\lambda) \leq \alpha(A) \leq \mu(A) \leq \|A\|; \tag{2.73}$$

(iii) if the norm $\|\cdot\|$ is monotonic and A is diagonal, then

$$\|A\| = \max_{i\in\{1,...,n\}} |A_{ii}| = \rho(A), \tag{2.74}$$

$$\mu(A) = \max_{i\in\{1,...,n\}} A_{ii} = \alpha(A). \tag{2.75}$$

Proof. For the proof of statement (i) we refer to (Horn and Johnson, 2012, Theorem 5.6.9). For the proof of statement (ii) we refer to (Desoer and Vidyasagar, 1975, Chapter 2).

Regarding statement (iii), equation (2.74) is in (Horn and Johnson, 2012, Theorem 5.6.36), where it is actually shown that the condition is not only sufficient, but also necessary. Regarding equation (2.75), we know $\|I_n + h\Lambda\| = \max_i |1 + h\lambda_i|$. Next, we note the Taylor expansion $|1 + h\lambda_i| - 1 = h\Re(\lambda_i) + o(h^2)$ for small h, and compute

$$\mu(\Lambda) = \lim_{h\to 0^+} \frac{\|I_n + h\Lambda\| - 1}{h} = \lim_{h\to 0^+} \frac{\max_i |1 + h\lambda_i| - 1}{h} = \max_i \Re(\lambda_i). \tag{2.76}$$

■

Properties (2.71) and (2.73) are especially important and are illustrated in Figures 2.6 and 2.7. Two consequences of property (2.73) are that: (i) the log norm of a matrix is an upper bound on the spectral abscissa and it is a tighter upper bound than the norm, and (ii) a matrix with negative log norm is Hurwitz.

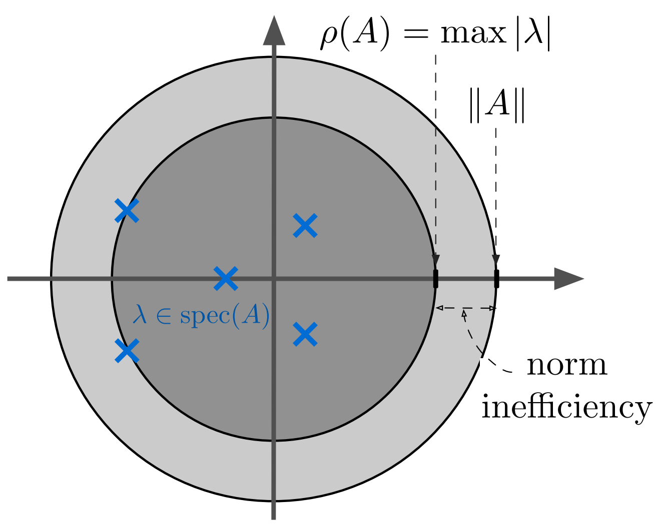


Figure 2.6: Illustration of the norm/spectrum inequalities (2.71). We refer to the gap between induced norm and spectral radius of A as the inefficiency of the norm.

While the set of Hurwitz matrices $\{A : \alpha(A) < 0\}$ is not convex, the set $\{A : \mu(A) < 0\}$ is convex. Moreover, for any $\gamma > 0$, we have the set inequality $\{A : \|A\| < \gamma\} \subset \{A : \mu(A) < \gamma\}$.

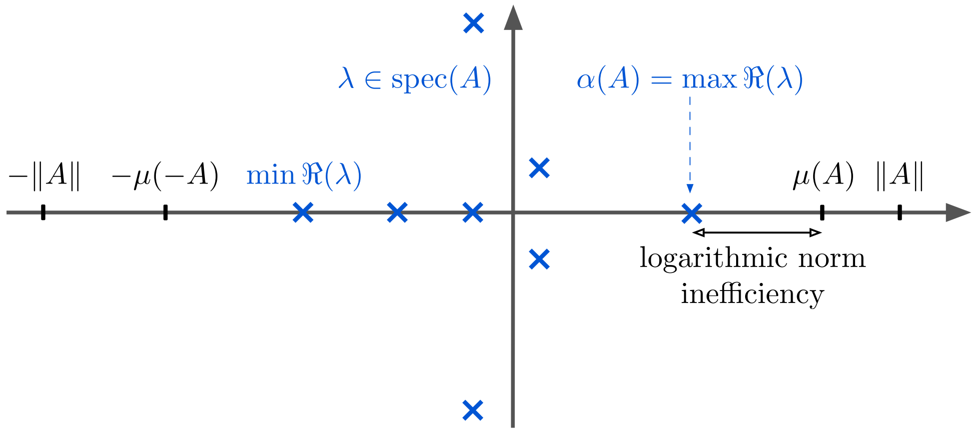


Figure 2.7: Illustration of the norm/spectrum inequalities (2.73): spectrum of a matrix A and upper and lower bounds given by log norms and induced matrix norms.

2.6.2 Monotonicity properties

We now present some useful monotonicity properties.

Theorem 2.24 (Monotonicity properties). *Consider a monotonic norm $\|\cdot\|$, a matrix $A \in \mathbb{R}^{n\times n}$, and a non-negative matrix $B \in \mathbb{R}^{n\times n}_{\geq 0}$. Then*

(monotonicity property of spectral radius) $$\rho(A) \leq \rho(|A|) \leq \rho(|A| + B), \tag{2.77a}$$

(monotonicity property of induced norm) $$\|A\| \leq \||A|\| \leq \||A| + B\| \tag{2.77b}$$

and

(monotonicity property of spectral abscissa) $$\alpha(A) \leq \alpha(|A|_{\mathrm{M}}) \leq \alpha(|A|_{\mathrm{M}} + B), \tag{2.78a}$$

(monotonicity property of log norms) $$\mu(A) \leq \mu(|A|_{\mathrm{M}}) \leq \mu(|A|_{\mathrm{M}} + B). \tag{2.78b}$$

Note: regarding the inequalities (2.77a) and (2.78b), there exist matrices such that $\rho(A) < \rho(|A|)$ and $\mu(A) < \mu(|A|_{\mathrm{M}})$. For example, the matrix $A_* = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ has eigenvalues $\{1+\mathrm{i}, 1-\mathrm{i}\}$ whereas $|A_*|$ has eigenvalues $\{2, 0\}$. Therefore, $\rho(A_*) = \sqrt{2} < 2 = \rho(|A_*|)$ and $\alpha(A_*) = 1 < 2 = \alpha(|A_*|)$. Regarding the inequality (2.77b), one can also check $\|A_*\|_2 = \sqrt{2} < 2 = \||A_*|\|_2$.

Proof of Theorem 2.24. Equation (2.77a) is in (Horn and Johnson, 2012, Theorem 8.1.18), essentially based upon Gelfand's formula $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$. Regarding equation (2.77b), we compute

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\||x|\|=1} \||Ax|\| \leq \max_{\||x|\|=1} \||A|\,|x|\| \leq \max_{\||x|\|=1} \||A|\|\||x|\| = \||A|\|, \tag{2.79}$$

$$\begin{aligned} \||A|\| &= \max_{\|x\|=1} \||A|x\| \leq \max_{\||x|\|=1} \||A||x|\| \\ &\leq \max_{\||x|\|=1} \|(|A|+B)|x|\| \leq \max_{\||x|\|=1} \||A|+B\|\||x|\| = \|(|A|+B)\|. \end{aligned} \tag{2.80}$$

Regarding equation (2.78a), for the inequality $\alpha(|A|_{\mathrm{M}}) \leq \alpha(|A|_{\mathrm{M}} + B)$ we refer to (Bullo, 2022, Exercise 10.5). We prove $\alpha(A) \leq \alpha(|A|_{\mathrm{M}})$ as follows. Pick $\gamma > \max_i |A_{ii}|$ and define $\bar{A} = A + \gamma I_n$ so that $|\bar{A}| = |A|_{\mathrm{M}} + \gamma I_n$.

We note that $\alpha(\bar{A}) \leq \rho(\bar{A})$ (which is always true) and, from inequality (2.77a) in Theorem 2.23, we know

$$\alpha(A) + \gamma = \alpha(\bar{A}) \leq \rho(\bar{A}) \leq \rho(|\bar{A}|) = \alpha(|\bar{A}|) = \alpha(|A|_{\mathrm{M}} + \gamma I_n) = \alpha(|A|_{\mathrm{M}}) + \gamma. \tag{2.81}$$

Here $\rho(|\bar{A}|) = \alpha(|\bar{A}|)$ follows from the Perron-Frobenius Theorem for non-negative matrices.

Regarding equation (2.78b), for small $h > 0$, we note $|I_n + hA| = I_n + h|A|_{\mathrm{M}}$. It is also easy to see that $|I_n + hA| = I_n + h|A|_{\mathrm{M}} \leq I_n + h(|A|_{\mathrm{M}} + B)$, for any non-negative B. Since $\|\cdot\|$ is monotonic, Equation (2.77b) implies

$$\|I_n + hA\| \leq \||I_n + hA|\| = \|I_n + h|A|_{\mathrm{M}}\| \leq \|I_n + h(|A|_{\mathrm{M}} + B)\|. \tag{2.82}$$

This immediately implies equation (2.78b). ■

We conclude this section with a table summarizing the monotonicity results.

spectral radius	spectral abscissa
$\rho(A) \leq \rho(\lvert A\rvert) \leq \rho(\lvert A\rvert + B)$	$\alpha(A) \leq \alpha(\lvert A\rvert_{\mathrm{M}}) \leq \rho(\lvert A\rvert_{\mathrm{M}} + B)$
$\rho(A) \leq \\|A\\|$	$\alpha(A) \leq \mu(A)$
induced norm	log norm
$\\|A\\| \leq \\|\lvert A\rvert\\| \leq \\|\lvert A\rvert + B\\|$	$\mu(A) \leq \mu(\lvert A\rvert_{\mathrm{M}}) \leq \mu(\lvert A\rvert_{\mathrm{M}} + B)$

Table 2.3: Relationships between spectral radius vs. induced norm and spectral abscissa vs. log norm, for arbitrary matrix $A \in \mathbb{R}^{n\times n}$ and nonnegative matrix $B \in \mathbb{R}^{n\times n}$

2.7 Optimal norms: analysis and design

We are now interested in analyzing and designing norms with the property that, given a matrix A, the induced norm $\|A\|$ approximates closely the spectral radius $\rho(A)$ and the log norm $\mu(A)$ approximates closely the spectral abscissa $\alpha(A)$.

Given a matrix $A \in \mathbb{R}^{n\times n}$ with spectral radius $\rho(A)$ and spectral abscissa $\alpha(A)$, and given a scalar $\varepsilon > 0$, the norm $\|\cdot\|$ is

(i) *ε-optimal* if $\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon$, and *optimal* if $\rho(A) = \|A\|$; and

(ii) *logarithmically ε-optimal* $\alpha(A) \leq \mu(A) \leq \alpha(A) + \varepsilon$ and *logarithmically optimal* if $\mu(A) = \alpha(A)$

Example 2.25. The ℓ_2 norm is optimal and logarithmically optimal for any symmetric matrix A since

$$A = A^{\mathsf{T}} \quad\implies\quad \rho(A) = \|A\|_2 \text{ and } \alpha(A) = \mu_2(A). \tag{2.83}$$

To show these simple facts, recall a symmetric A has real eigenvalues and satisfies $A = Q\Lambda Q^{\mathsf{T}}$, where Q is orthogonal and Λ is a diagonal matrix with the eigenvalues of A. Then

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^{\mathsf{T}}A)} = \sqrt{\lambda_{\max}(Q\Lambda^2 Q^{\mathsf{T}})} = \sqrt{\lambda_{\max}(\Lambda^2)} = \max_i |\lambda_i| = \rho(A), \tag{2.84}$$

$$\mu_2(A) = \tfrac{1}{2}\lambda_{\max}(A + A^{\mathsf{T}}) = \lambda_{\max}(A) = \alpha(A). \tag{2.85}$$

Example 2.26 (Logarithmic optimality of weighted ℓ_2 log norms). For $a \in \mathbb{R}$, consider the matrix

$$A = \begin{bmatrix} -1 & a \\ 0 & -2 \end{bmatrix}.$$

Clearly, $\operatorname{spec}(A) = \{-1, -2\}$ and $\alpha(A) = -1$. One can compute $\operatorname{spec}((A + A^{\mathsf{T}})/2) = \{(-3 \pm \sqrt{a^2+1})/2\}$ so that

$$\mu_2(A) = \frac{1}{2}(\sqrt{a^2+1} - 3).$$

Therefore $\mu_2(A)$ is an accurate estimate on the spectral abscissa only for small values of a; for example, $\mu_2(A) > 0$ for $|a| > \sqrt{2}$ and $\mu_2(A)$ grows linearly in $|a|$ for large $|a|$.

Next, we diagonalize the matrix A as:

$$RAR^{-1} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{where } R = \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}, \tag{2.86}$$

which implies that $\mu_{2,R}(\cdot)$ is logarithmically optimal for each matrix $A = A(a)$, $a \in \mathbb{R}$. •

2.7.1 Design of norms via Jordan and Schur normal forms

In this section we propose optimal norms by adopting Jordan and Schur normal forms of matrices. The approach is based upon three ideas. First, as established in Theorem 2.23(iii), any monotonic norm is optimal for a diagonal matrix. Therefore, if we can find a similarity transformation to put the given matrix into diagonal form, then that same transformation can serve as an optimal weight for any monotonic norm. Second, both the Jordan and Schur normal forms put the matrix into a form that, while not diagonal, is close enough to be useful. (Recall that the Schur normal form is complex upper triangular.) The third idea is to adopt an appropriately-designed diagonal scaling matrix to turn an upper triangular matrix into a diagonal matrix.

Specifically, given a scalar number $\delta > 0$ and a dimension $n \in \mathbb{N}$, define the *scaled diagonal matrix*

$$\mathsf{sdg}(\delta) = \begin{bmatrix} \delta & 0 & \dots & 0 \\ 0 & \delta^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \delta^n \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{2.87}$$

The following algebraic properties will be useful. First we note $\mathsf{sdg}(\delta)^{-1} = \mathsf{sdg}(\delta^{-1})$. Second, for any $A \in \mathbb{R}^{n \times n}$, we compute

$$\begin{aligned} \mathsf{sdg}(\delta^{-1}) A \, \mathsf{sdg}(\delta) &= \begin{bmatrix} 1/\delta & 0 & \dots & 0 \\ 0 & 1/\delta^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1/\delta^n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{(n-1)n} \\ a_{n1} & \dots & a_{n(n-1)} & a_{nn} \end{bmatrix} \begin{bmatrix} \delta & 0 & \dots & 0 \\ 0 & \delta^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \delta^n \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \delta a_{12} & \dots & \delta^{n-1} a_{1n} \\ \delta^{-1} a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta a_{(n-1)n} \\ \delta^{-n+1} a_{n1} & \dots & \delta^{-1} a_{n(n-1)} & a_{nn} \end{bmatrix}. \end{aligned} \tag{2.88}$$

Lemma 2.27 (Optimal weighted norms via the Jordan normal form). *Given a matrix $A \in \mathbb{R}^{n \times n}$, a monotonic norm $\|\cdot\|$, and $\varepsilon > 0$, define*

$$T \in \mathbb{C}^{n \times n} \text{ as an invertible matrix such that } TAT^{-1} \text{ is in Jordan normal form,} \tag{2.89}$$

$$Q \in \mathbb{C}^{n \times n} \text{ as an unitary matrix such that } QAQ^{-1} \text{ is in Schur normal form, and} \tag{2.90}$$

$$\delta = \varepsilon / \|J_{0,n}\| > 0, \text{ where } J_{0,n} \text{ is a Jordan block with eigenvalue } 0 \text{ and dimension } n. \tag{2.91}$$

Then

(i) the norm $\|\cdot\|_{\mathsf{sdg}(\delta^{-1})T}$ is ε-optimal and ε-logarithmically optimal;

(ii) if A is diagonalizable, the norm $\|\cdot\|_T$ is optimal and logarithmically optimal; and

(iii) the norm $\|\cdot\|_{\mathsf{sdg}(\delta^{-1})Q}$ is ε-optimal and ε-logarithmically optimal for sufficiently small δ.

Proof. We decompose $TAT^{-1} = \Lambda + U$, where the diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ contains the eigenvalues of A and $U \in \{0,1\}^{n\times n}$ has zero diagonal and off-diagonal entries equal to those of J. Then property (2.88) implies the *scaled Jordan form* formula:

$$\mathsf{sdg}(\delta^{-1})TAT^{-1}\,\mathsf{sdg}(\delta) = \Lambda + \delta U. \tag{2.92}$$

We now consider the weight matrix $\mathsf{sdg}(\delta^{-1})T$ and compute

$$\begin{aligned}
\|A\|_{\mathsf{sdg}(\delta^{-1})T} &= \|\mathsf{sdg}(\delta^{-1})TA(\mathsf{sdg}(\delta^{-1})T)^{-1}\| \\
&= \|\Lambda + \delta U\| && \text{(scaled Jordan formula (2.92))}\\
&\leq \|\Lambda\| + \delta\|U\| && \text{(subadditivity (2.29c) and scaling (2.29b))}\\
&= \rho(A) + \delta\|U\| && \text{(Theorem 2.23(iii) because } \mu \text{ is monotonic)}
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\mathsf{sdg}(\delta^{-1})T}(A) &= \mu(\Lambda + \delta U) && \text{(scaled Jordan formula (2.92))}\\
&\leq \mu(\Lambda) + \delta\mu(U) && \text{(subadditivity (2.30b) and scaling (2.30a))}\\
&= \alpha(A) + \delta\mu(U) && \text{(Theorem 2.23(iii) because } \mu \text{ is monotonic)}\\
&\leq \alpha(A) + \delta\|U\|. && \text{(spectral-abscissa log-norm property (2.73))}
\end{aligned}$$

If A is diagonalizable, then $U = \mathbb{0}_{n\times n}$, we set $\delta = 1$ and statement (ii) is established. If instead A is not diagonalizable, then, because $\|\cdot\|$ is monotonic, one can show $\|U\| \leq \|J_{0,n}\|$. This concludes the proof of statement (i).

Regarding statement (iii), the calculations are similar to those for the Jordan normal form. An example derivation for the ℓ_∞ norm is left to the reader as Exercise E2.15. ■

We conclude with some remarks.

Remark 2.28. *(i) The matrices T and Q are in general complex valued matrices and so the resulting norms are to be understood as norms on $\mathbb{C}^n$. When $A \in \mathbb{R}^{n\times n}$ is diagonalizable, then T can be selected real in statement (ii).*

(ii) The scaled Jordan decomposition (2.92) essentially provides a constructive proof that the set of diagonalizable matrices is dense in $\mathbb{C}^{n\times n}$.

(iii) Recall that, if a finite set of matrices commute, then they can be simultaneously upper triangularized. Then, it is possible to find a single norm that is optimal and log optimal for any matrix in the convex hull of the finite set of matrices.

Specifically, if $A_1,\dots,A_n$ commute, then there exists a unitary Q and a sufficiently small δ such that, for each A in the convex hull of $A_1,\dots,A_n$,

a) Q upper triangularizes A,

b) $\|\cdot\|_{\mathsf{dg}(\delta^{-1})Q}$ is optimal and log optimal for A, and

c) $\mu_{\mathsf{dg}(\delta^{-1})Q}(A) \leq \max_i \alpha(A_i)$. •

2.7.2 Design of norms via quasiconvex optimization

Given a convex set $X \subset \mathbb{R}^n$, a function $f\colon X \to \mathbb{R}$ is

(i) *convex* if $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$ for all $x, y \in X$ and $\theta \in [0,1]$,

(ii) *quasiconvex* if $f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$ for all $x, y \in X$ and $\theta \in [0,1]$. Equivalently, f is quasiconvex if the sublevel set $f_{\leq}^{-1}(\ell) = \{x \in X \,:\, f(x) \leq \ell\}$ is convex for all $\ell \in \mathbb{R}$.

We already studied the convexity of $\mu(\cdot)\colon \mathbb{R}^{n\times n} \to \mathbb{R}$. For example, for $P \in \mathbb{S}^n_{\succ 0}$ and $\eta \in \mathbb{R}^n_{>0}$, the following functions are convex in A:

$$A \mapsto \mu_{2,P^{1/2}}(A) = \lambda_{\max}\Big(\frac{PAP^{-1} + A^{\mathsf{T}}}{2}\Big), \quad \text{and} \quad A \mapsto \mu_{1,[\eta]}(A) = \max(\eta^{\mathsf{T}} |A|_{\mathrm{M}} [\eta]^{-1}).$$

The next result discusses the a distinct convexity property.

Lemma 2.29 (Quasiconvex dependence upon matrix weights). *Given any $A \in \mathbb{R}^{n\times n}$,*

(i) the function $P \in \mathbb{S}^n_{\succ 0} \;\mapsto\; \mu_{2,P^{1/2}}(A)$ is quasiconvex with sublevel sets

$$\{P \in \mathbb{S}^n_{\succ 0} \,:\, \mu_{2,P^{1/2}}(A) \leq b\} = \{P \in \mathbb{S}^n_{\succ 0} \,:\, A^{\mathsf{T}}P + PA \preceq 2bP\}, \tag{2.93}$$

(ii) the functions $\eta \in \mathbb{R}^n_{>0} \;\mapsto\; \mu_{1,[\eta]}(A)$ and $\eta \in \mathbb{R}^n_{>0} \;\mapsto\; \mu_{\infty,[\eta]^{-1}}(A)$ are quasiconvex with sublevel sets

$$\{\eta \in \mathbb{R}^n_{>0} \,:\, \mu_{1,[\eta]}(A) \leq b\} = \{\eta \in \mathbb{R}^n_{>0} \,:\, \eta^{\mathsf{T}}|A|_{\mathrm{M}} \leq b\eta^{\mathsf{T}}\}, \tag{2.94}$$

$$\{\eta \in \mathbb{R}^n_{>0} \,:\, \mu_{\infty,[\eta]^{-1}}(A) \leq b\} = \{\eta \in \mathbb{R}^n_{>0} \,:\, |A|_{\mathrm{M}}\eta \leq b\eta\}. \tag{2.95}$$

Proof. For any $b \in \mathbb{R}$, the formulas for the sublevel sets follow immediately from the equivalences (2.57) and (2.65). The first set is the set of feasible solutions to a linear matrix inequality LMI in P, and the second set is the set feasible solutions to a linear program in η. In both cases, the sublevel sets are convex and therefore the functions are quasiconvex. ■

The quasiconvex dependency can be exploited to design iterative algorithms to approximately solve the following optimization problem. Given a matrix A and a log norm $\mu(\cdot)$ with quasiconvex dependence on matrix weights in $\mathcal{R}$ (as in Lemma 2.29), we consider the optimization problem:

$$\inf_{R \in \mathcal{R}} \mu_R(A) \tag{2.96}$$

Note that the set of weights $\mathcal{R}$ is open and therefore the operator is in general an infimum.

We need to make the following assumption: an algorithm is available to check if a given sublevel set of the quasiconvex function $R \mapsto \mu_R(A)$ is empty. When $\mu(\cdot) = \mu_2(\cdot)$ and $\mathcal{R} = \mathbb{S}^n_{\succ 0}$ in Lemma 2.29(i), this requirement amounts to checking if an LMI is feasible. For the ℓ_1/ℓ_∞ log norms with diagonal positive weights in Lemma 2.29(ii), this requirement amounts to checking if an LP is feasible. Checking whether an LMI or an LP is feasible are tasks that can be executed efficiently and that are implemented for example in CVX (Grant and Boyd, 2014).

We are now ready to provide a bisection algorithm, well known in the context of convex optimization (Boyd and Vandenberghe, 2004).

Bisection method for computation of log norm weights

Input: matrix A, tolerance ε, log norm $\mu(\cdot)$ with quasiconvex dependence on matrix weights in $\mathcal{R}$
Output: a weight matrix $R^* \in \mathcal{R}$ satisfying $\mu_{R^*}(A) \leq \inf\{\mu_R(A) \;:\; R \in \mathcal{R}\} + \varepsilon$

1: $b_{\min} :=$ any lower bound on $\alpha(A)$, for example, $-\|A\|$
2: $b_{\max} := \mu_{R_{\text{tmp}}}(A)$ for any initial weight $R_{\text{tmp}} \in \mathcal{R}$, for example, $R_{\text{tmp}} := I_n$
3: **repeat**
4: $\quad b := (b_{\min} + b_{\max})/2$
5: $\quad$ solve convex feasibility problem at b, that is, does $R \in \mathcal{R}$ exists such that $\mu_R(A) \leq b$?
6: $\quad$ **if** the problem is feasible, **then** $b_{\max} := b$ and $R_{\text{tmp}} :=$ any feasible R from step 5:
7: $\quad$ **if** the problem is not feasible, **then** $b_{\min} := b$
8: **until** $b_{\max} - b_{\min} < \varepsilon$
9: **return** R_{tmp}

We make the following observations. This bisection method:

(i) computes a ε-suboptimal solution to the quasi-convex optimization problem (2.96),

(ii) performs a sequence of convex feasibility problems,

(iii) requires exactly $\lceil \log_2((b_{\max} - b_{\min})/\varepsilon) \rceil$ iterations, where $b_{\min}$ and $b_{\max}$ are at their starting values defined at steps 1: and 2:,

(iv) does not require knowledge of the spectral abscissa $\alpha(A)$ and, indeed, the output $\mu_R(A)$ is an ε overapproximation of it.

Optimal Euclidean log norms via quasiconvex optimization

In this section we assume that the spectral abscissa of the matrix A is known (or can be computed by other means) and show how to compute a logarithmically optimal ℓ_2 weight via a single convex feasibility problem. (The case of optimal ℓ_2 norms is similar and omitted.)

Lemma 2.30 (Weighted ℓ_2 log norms and Lyapunov inequalities). *Given a matrix $A \in \mathbb{R}^{n\times n}$ with spectral abscissa $\alpha(A)$, define for any nonnegative tolerance $\varepsilon \geq 0$*

$$P_\varepsilon := \textit{any element of } \{P \in \mathbb{S}^n_{>0} \;:\; AP^\top + PA \preceq 2(\alpha(A)+\varepsilon)P\}. \tag{2.97}$$

Then

(i) for any $\varepsilon > 0$, P_ε is well defined and $\|\cdot\|_{2,P_\varepsilon^{1/2}}$ is logarithmically ε-optimal for A,

(ii) if each eigenvalue $\lambda_i(A)$ with $\Re(\lambda_i(A)) = \alpha(A)$ is semisimple, then P_0 is well defined and $\|\cdot\|_{2,P_0^{1/2}}$ is logarithmically optimal for A.

Proof. Regarding statement (i), for any $b \in \mathbb{R}$

$$A^\top P + PA \preceq 2bP \quad\iff\quad A^\top P + PA - b(I_nP + PI_n) \preceq 0 \quad\iff\quad (A - bI_n)^\top P + P(A-bI_n) \preceq 0,$$

that is, the inequality $A^\top P + PA \preceq 2bP$ is the Lyapunov inequality for the matrix $A - bI_n$. For any $b = \alpha(A)+\varepsilon$, we know that all eigenvalues of $A - bI_n$ have negative real part. Therefore, there exists a positive definite P to the the Lyapunov inequality for the matrix $A - bI_n$ and, in turn, equation (2.57) implies $\mu_{2,P^{1/2}}(A) \leq b$. This proves that $\|\cdot\|_{2,P^{1/2}}$ is logarithmically ε-optimal with respect to A.

Regarding statement (ii), for $b = \alpha(A)$, the matrix $A - \alpha(A)I_n$ has at least one eigenvalue with zero real part. In this case, the Lyapunov inequality for the matrix $A - \alpha(A)I_n$ has a solution if and only if each eigenvalue with zero real part is semisimple. ■

2.7.3 Design of norms via Perron eigenvectors

In this section we show how to use the Perron-Frobenius theorem to design optimal norms. We consider both induced norms of nonnegative matrices and log norms of Metzler matrices.

Lemma 2.31 (Optimal diagonally-weighted norms for non-negative and Metzler matrices). *Consider a nonnegative matrix $A \in \mathbb{R}^{n\times n}_{\geq 0}$ and a Metzler matrix $M \in \mathbb{R}^{n\times n}$. For any $p \in [1,\infty]$ and $\delta > 0$, define*

$$\begin{aligned}
&v \text{ and } w \in \mathbb{R}^n_{>0} \ \text{ to be the right and left dominant eigenvectors of } A + \delta \mathbb{1}_n \mathbb{1}_n^\top \\
&\qquad\qquad (\text{respectively, } M + \delta \mathbb{1}_n \mathbb{1}_n^\top), \\
&q \in [1,\infty] \ \text{ to satisfy } 1/p + 1/q = 1 \text{ (with the convention } 1/\infty = 0\text{), and} \\
&\eta_\delta = \left(\frac{w_1^{1/p}}{v_1^{1/q}}, \dots, \frac{w_n^{1/p}}{v_n^{1/q}} \right) \in \mathbb{R}^n_{>0}.
\end{aligned}$$

Then

(i) for sufficiently small δ, the norm $\|\cdot\|_{p,[\eta_\delta]}$ is ε-optimal for A (respectively, ε-logarithmically optimal for M), and

(ii) if A (respectively, M) is irreducible, then the norm $\|\cdot\|_{p,[\eta_0]}$ is optimal for A (respectively, logarithmically optimal for M).

Specifically, for $p \in \{1,2,\infty\}$ and for an irreducible A with spectral radius $\rho(A)$ and an irreducible M with spectral abscissa $\alpha(A)$,

$$\rho(A) = \|A\|_{1,[w]} = \|A\|_{\infty,[v]^{-1}} = \|A\|_{2,[w\oslash v]^{1/2}}, \tag{2.98}$$

$$\alpha(M) = \mu_{1,[w]}(M) = \mu_{\infty,[v]^{-1}}(M) = \mu_{2,[w\oslash v]^{1/2}}(M). \tag{2.99}$$

(Recall that, for w and $v \in \mathbb{R}^n_{>0}$, the entrywise division $w \oslash v$ is defined by $(w \oslash v)_i = w_i/v_i$.)

Proof. We omit the full proof, based on the Hölder's inequality for convex functions, and refer to (Stoer and Witzgall, 1962, Theorem 2), (Albrecht, 1996), and (Pastravanu and Voicu, 2006, Lemma 3). The final statement can be verified by direct calculation, using the equalities $Av = \rho(A)v$ and $A^\top w = \rho(A)w$. We report here the calculations only for equation (2.98):

$$\begin{aligned}
\|A\|_{1,[w]} &= \max\left(([w]A[w]^{-1})^\top \mathbb{1}_n \right) = \max\left([w]^{-1} A^\top [w] \mathbb{1}_n \right) \\
&= \max\left([w]^{-1} A^\top w \right) = \rho(A) \max\left([w]^{-1} w \right) = \rho(A), \\
\|A\|_{\infty,[v]^{-1}} &= \max\left([v]^{-1} A [v] \mathbb{1}_n \right) = \max\left([v]^{-1} A v \right) = \rho(A) \max\left([v]^{-1} v \right) = \rho(A).
\end{aligned}$$

Regarding the ℓ_2 result, we compute

$$\|A\|_{2,[w\oslash v]^{1/2}} = \sqrt{\rho(A^\top [w][v]^{-1} A)} =: \sqrt{\rho(B)}.$$

Note that $Bv = A^\top [w][v]^{-1} A v = \rho(A)^2 v$ implies $\rho(B) \geq \rho(A)^2$ and, in turn, that $\|A\|_{2,[w\oslash v]^{1/2}} \geq \rho(A)$. The conclusion follows since, in general, $\|A\|_{2,[w\oslash v]^{1/2}} \leq \rho(A)$. ■

Note: regarding algorithmic and computational aspects, Bunse (1981) and Van At (1991) analyze and compare numerical algorithms to compute the Perron eigenvalue and eigenvector of a nonnegative irreducible matrix; see also (Ipsen and Selee, 2011; Nesterov and Nemirovski, 2015).

Remark 2.32 (Reinterpreting the Hurwitz Metzler Theorem). *Lemma 2.31 is consistent with the Hurwitz Metzler Theorem 2.7. We reinterpret here some of the equivalent statements in that theorem. Given a Metzler M,*

(i) *M is Hurwitz,*

(ii) *there exists $\eta \in \mathbb{R}^n_{>0}$ such that $\eta^\top M < \mathbb{0}_n^\top$ or, equivalently, $\mu_{1,[\eta]}(M) < 0$,*

(iii) *there exists $\eta \in \mathbb{R}^n_{>0}$ such that $M\eta < \mathbb{0}_n$ or, equivalently, $\mu_{\infty,[\eta]^{-1}}(M) < 0$, and*

(iv) *there exists $\eta \in \mathbb{R}^n_{>0}$ such that $M^\top[\eta] + [\eta]M \prec 0$ or, equivalently, $\mu_{2,[\eta]^{1/2}}(M) < 0$.* •

Note: for a non-negative matrix, at optimal diagonal weights, each ℓ_p norms becomes equally optimal.

2.7.4 Implications

We provide a simple corollary summarizing some implications of the design results in the previous sections. We present the continuous-time version and leave the discrete-time equivalent to the reader as an exercise.

Corollary 2.33 (Hurwitz matrices and log norms). *Given $A \in \mathbb{R}^{n\times n}$ and $p \in [1,\infty]$,*

(i) *the following inequalities hold*

$$\begin{aligned}\alpha(A) &= \inf_{T\in\mathbb{C}^n} \mu_{p,T}(A) = \inf_{T\in\mathbb{R}^n} \mu_{2,T}(A)\\ &\leq \inf_{\eta\in\mathbb{R}^n_{>0}} \mu_{2,[\eta]^{1/2}}(A)\\ &\leq \alpha(|A|_{\mathrm{M}}) = \inf_{\eta\in\mathbb{R}^n_{>0}} \mu_{p,[\eta]}(A),\end{aligned}$$

(ii) *A is Hurwitz if and only if there exists an invertible $T \in \mathbb{C}^n$ such that $\mu_{p,T}(A) < 0$,*

(iii) *A is Hurwitz if and only if there exists an invertible $T \in \mathbb{R}^n$ such that $\mu_{2,T}(A) < 0$;*

(iv) *A is Lyapunov diagonally Hurwitz if and only if there exists $\eta \in \mathbb{R}^n_{>0}$ such that $\mu_{2,[\eta]}(A) < 0$;*

(v) *A is M-Hurwitz if and only if there exists $\eta \in \mathbb{R}^n_{>0}$ such that $\mu_{p,[\eta]}(|A|_{\mathrm{M}}) < 0$.*

Corollary 2.34 (Composite induced norms and log norms: continued). *Under the same assumptions as in Theorem 2.13, consider the non-negative aggregate majorant $\lceil A\rceil$ and the Metzler aggregate majorant $\lceil A\rceil_{\mathrm{M}}$. Fix $p \in [1,\infty]$ and $\varepsilon > 0$ and define $\eta > 0$ as in Lemma 2.31 for $\lceil A\rceil$ (respectively $\lceil A\rceil_{\mathrm{M}}$). Then*

(i) *the aggregating norm $\|\cdot\|_{\mathrm{agg}} = \|\cdot\|_{p,[\eta]}$ is ε-optimal for $\lceil A\rceil$ (respectively ε-logarithmically optimal for $\lceil A\rceil_{\mathrm{M}}$) and*

$$\rho(A) \leq \|A\|_{\mathrm{cmpst}} \leq \|\lceil A\rceil\|_{p,[\eta]} \leq \rho(\lceil A\rceil) + \varepsilon, \tag{2.100}$$

$$\alpha(A) \leq \mu_{\mathrm{cmpst}}(A) \leq \mu_{p,[\eta]}(\lceil A\rceil_{\mathrm{M}}) \leq \alpha(\lceil A\rceil_{\mathrm{M}}) + \varepsilon; \tag{2.101}$$

(ii) *if $\lceil A\rceil$ is Schur, then A is Schur; and*
if $\lceil A\rceil_{\mathrm{M}}$ is Hurwitz, then A is Hurwitz.

Proof. Statement (i) is an immediate consequence of Theorem 2.13 and Lemma 2.31. Statement (ii) is an immediate consequence of statement (i). ■

2.8 Historical notes and further reading

The origins The *logarithmic norm* of a matrix was independently and simultaneously introduced in 1958 by Dahlquist in Chapter 1 of his PhD dissertation and by Lozinskii in a landmark paper. Indeed, the insightful calculations in Section 2.1 are essentially present in these two works (see also (Söderlind, 2006)).

Coppel's book with his famed inequality appeared in 1965 (Theorem 2.3). Desoer and Haneda wrote their landmark paper in 1972 (with ISS results, fixed point theorems, and convergence proofs for Euler method and Newton-basic). The basic properties of log norms in Lemma 2.11 were essentially known by then. Influenced by (Desoer and Haneda, 1972), Dahlquist wrote an impactful paper in 1976 introducing the *one-sided Lipschitz conditions* (which will be discussed in the next chapter).

An early outstanding survey on lognorms is by Ström (1975). The books by Desoer and Vidyasagar (1975) and Vidyasagar (1978b) included discussions of log norms, but essentially much of these ideas went forgotten by the control community for two decades. Instead, the influential 1984 and 1993 textbook by Dekker and Verwer and Hairer, Nørsett, and Wanner on numerical methods for differential equations are fully informed by logarithmic norms and one-sided Lipschitz constants.

Composite or hierarchical norms were originally studied by Ström (1975) where an early version of Theorem 2.13 is given; see also (Russo et al., 2013). An early reference to a closely related concept is the method of vector Lyapunov functions in (Šiljak, 1978, Theorem 2.11). A catalog of explicit formulae for the nine possible induced matrix norms corresponding to the ℓ_1, ℓ_2, and ℓ_∞ norms is given by Lewis (2010). Log norms for matrix pencils are studied in (Higueras and Garcia-Celayeta, 1999).

Lognorms in systems and control The role of log norms in systems and control theory was revitalized in Fang et al. (1993) and Fang and Kincaid (1996). Fang et al. (1993) studies the stability of interval dynamical systems. Fang and Kincaid (1996) develops some basic results in contraction theory, including showing the usefulness of the log norm of the Jacobian being negative and giving log norm conditions for Hopfield neural networks (essentially our Lemma 2.15 and Theorem 2.23). These two papers did not receive much attention in the literature. Later on, log norms for matrix sets with multiplicative uncertainty (of relevance in neural networks) were studied in (Qiao et al., 2001; He and Cao, 2009; Jafarpour et al., 2021b; Davydov et al., 2022c).

Finally, Lohmiller and Slotine in 1998 brought into focus the importance of log norms in the context of contraction theory; more historical details on this topic are in the next chapter.

The standing assumptions in Definition 2.18 can be proved with limited effort from an additional basic assumptions on the weak pairing. The relationship between log norms, Lumer pairings and one-sided Lipschitz constants is discussed in (Davydov et al., 2022a; Jafarpour et al., 2023) and the short tutorial (Bullo et al., 2021), building on Söderlind (2006); Aminzare and Sontag (2014b). Our Section 2.5.2 adopts the framework proposed in (Davydov et al., 2022a), to which we refer for the relationship between weak pairings and the Gateaux differential of norms.

The importance of non-Euclidean log norms in systems and control is highlighted for example in (Russo et al., 2010; Aminzare and Sontag, 2014a; Jafarpour et al., 2021b).

Log norms are very useful to estimate the evolution of time-varying linear systems; e.g., see (Pastravanu and Matcovschi, 2010; Vrabel, 2020).

Efficiency and monotonicity Logarithmically efficient log norms are widely studied and various results rediscovered; an incomplete list of works includes (Ström, 1975; Deutsch, 1975; Kågström, 1977; Vidyasagar, 1978a; Li and Wang, 1998; Zahreddine, 2003; Hu and Liu, 2008). For example, the proof of Lemma 2.27 is taken from (Desoer and Vidyasagar, 1975, Section II.7) and (Horn and Johnson, 2012, Theorem 5.6.10). Results on log norms for Metzler matrices are given in (Vidyasagar, 1978a; Pastravanu and Voicu, 2006; Coogan, 2019). Regarding Lemma 2.31 on efficient diagonally-weighted norms for non-negative and Metzler matrices, the first version of this result for positive matrices is given in (Stoer and Witzgall, 1962, Theorem 2). The extension to irreducible nonnegative matrices is stated in (Albrecht, 1996). The final version, as we present, is essentially due to (Pastravanu and Voicu, 2006, Lemma 3). The quasi-convexity results in Lemma 2.29 are classic; the nonEuclidean version is presented in (Davydov et al., 2022c). The design of optimal positive definite matrices in Lemma 2.30 is a classic result, e.g., see (Boyd et al., 1994).

The monotonicity properties of induced norms in Theorem 2.24 are classic results; the corresponding version for log norms have only recently been presented in (Davydov et al., 2022c).

We will rely upon results on Hurwitz Metzler matrices. We refer to (Berman and Plemmons, 1994, Chapter 6) for an extended treatment (with 50 equivalent statements) of Hurwitz Metzler matrices (referred to as M-matrices, modulo a minus sign), to (Bullo, 2022, Chapter 10) for some results with a systems theoretic interpretation, and to (Duan et al., 2021) for a recent graph-theoretical characterization.

Nomenclature: Log norm vs. matrix measure Some remarks are in order regarding the name of this operation. Dahlquist (1958) did not give it a name, but adopted the symbol $\mu(\cdot)$ for it; it is worth recalling that μ is the commonly adopted symbol for measures in measure theory. Coppel (1965) explicitly states that this operator is not a "measure" of a matrix. For unclear reasons, the name "the measure of a matrix" is adopted by (Desoer and Haneda, 1972) and (Desoer and Vidyasagar, 1975). However, the logarithmic norm is not a measure and, for example, can take negative values. We adopt Lozinskii's nomenclature of *logarithmic norm* (abbreviated log norm) with Dahlquist's symbol $\mu(\cdot)$. It is worth noting that Dahlquist himself refers to this operation as the logarithmic norm in (Dahlquist, 1983, 1985).

2.9 Exercises

E2.1 **Grönwall Comparison Lemma for absolutely continuous functions.** Recall the notion of upper right Dini derivative given in (2.4) and its properties reviewed in Appendix A.7. Given continuous functions of time $t \mapsto a(t) \in \mathbb{R}$ and $t \mapsto \gamma(t) \in \mathbb{R}$, assume the absolutely continuous function $t \mapsto z(t)$ satisfies the differential inequality

$$D^+ z(t) \leq a(t)z(t) + \gamma(t). \tag{E2.1}$$

Define $A(s,t) = \int_s^t a(\tau)d\tau$ for all $0 \leq s \leq t \in \mathbb{R}_{\geq 0}$, and show that, for $t \in [t_0, \infty)$,

$$z(t) \leq \mathrm{e}^{A(t_0,t)}\, z(t_0) + \int_{t_0}^t \mathrm{e}^{A(\tau,t)}\, \gamma(\tau)d\tau. \tag{E2.2}$$

Note: *In other words, $z(t)$ is upper bounded by the solution to the corresponding differential equality. When a and γ are time-invariant and $t_0 = 0$, the upper bound reads $z(t) \leq \mathrm{e}^{at}\, z(0) + (1 - \mathrm{e}^{at})\gamma/a$.*

Answer: We first show that, if the absolutely continuous signal $t \mapsto x(t)$ satisfies the differential inequality

$$D^+ x(t) \leq a(t)x(t),$$

then

$$x(t) \leq x(t_0) \exp\Big(\int_{t_0}^t a(\tau)d\tau\Big), \qquad \text{for all } t \in \mathbb{R}_{\geq 0}. \tag{E2.3}$$

To do so, we define the signal $y(t) = \exp\Big(\int_{t_0}^t a(\tau)d\tau\Big)$, $t \geq t_0 \in \mathbb{R}_{\geq 0}$, and note that $y(t_0) = 1$, $y(t) > 0$ for all $t \geq t_0$, and

$$\frac{d}{dt}y(t) = \exp\Big(\int_{t_0}^t a(\tau)d\tau\Big)a(t) = a(t)y(t),$$

where we used $\frac{d}{dt}\exp(f(t)) = \exp(f(t))\dot{f}(t)$. Using the product rule for the Dini derivative (Giorgi and Komlósi, 1992, equation (2a)) with $y(t) > 0$, we now compute

$$D^+ \frac{x(t)}{y(t)} = \frac{y(t)D^+x(t) - x(t)\dot{y}(t)}{y(t)^2} = \frac{y(t)D^+x(t) - x(t)a(t)y(t)}{y(t)^2} = \frac{\dot{x}(t) - x(t)a(t)}{y(t)} \leq 0.$$

Since the upper right Dini derivative of $\frac{x(t)}{y(t)}$ is non-positive, we know from (Giorgi and Komlósi, 1992, Theorem 1.14) that

$$\frac{x(t)}{y(t)} \leq \frac{x(0)}{y(0)} = x(0).$$

This concludes the proof of the inequality (E2.3).

Next, consider the signal $z(t) - \int_{t_0}^t \mathrm{e}^{A(\tau,t)}\, \gamma(\tau)d\tau$ and compute

$$\begin{aligned} D^+\Big(z(t) - \int_{t_0}^t \mathrm{e}^{A(\tau,t)}\, \gamma(\tau)d\tau\Big) &\leq a(t)z(t) + \gamma(t) - \Big(\mathrm{e}^{A(\tau,t)}\, \gamma(\tau)\Big|_{\tau=t} + \int_{t_0}^t \mathrm{e}^{A(\tau,t)}\, \big(\frac{d}{dt}A(\tau,t)\big)\gamma(\tau)d\tau\Big) \\ &= a(t)z(t) + \gamma(t) - \Big(\gamma(t) + \int_{t_0}^t \mathrm{e}^{A(\tau,t)}\, a(t)\gamma(\tau)d\tau\Big) \\ &= a(t)\Big(z(t) - \int_{t_0}^t \mathrm{e}^{A(\tau,t)}\, \gamma(\tau)d\tau\Big). \end{aligned}$$

One can then apply the inequality (E2.3) and establish (E2.2).

E2.2 **Unit disks are convex centrally symmetric sets.** A set $\mathcal{S} \subset \mathbb{R}^n$ is *centrally symmetric* if $v \in \mathcal{S}$ implies $-v \in \mathcal{S}$. Show that

(i) for any norm $\|\cdot\|$ on $\mathbb{R}^n$, the unit open disk $\{v \in \mathbb{R}^n : \|v\| < 1\}$ is bounded, open, convex, and centrally symmetric.

Conversely, let $\mathcal{S} \subset \mathbb{R}^n$ be bounded, open, convex and centrally symmetric. Show that

(ii) the function $\|v\|_{\mathcal{S}} = \inf\{\lambda > 0 : v/\lambda \in \mathcal{S}\}$ is a norm on $\mathbb{R}^n$ and $\mathcal{S} \subset \{v \in \mathbb{R}^n : \|v\|_{\mathcal{S}} < 1\}$.

Note: *For a related discussion see (Desoer and Vidyasagar, 1975, Chapter 2).*

Answer:

(i) Set $D = \{v \in \mathbb{R}^n : \|v\| < 1\}$. Recall that all norms on $\mathbb{R}^n$ are equivalent. Hence, there exist $0 < c < C$ such that

$$\forall x \in \mathbb{R}^n \qquad c\|x\|_2 \leq \|x\| \leq C\|x\|_2 \tag{E2.4}$$

a) *D is bounded*: Given $v \in D$, (E2.4) implies $\|v\|_2 < c^{-1}$.

b) *D is open*: Given $v \in D$, consider the open ball $B_{v,2} = \left\{z : \|z - v\|_2 < \frac{1-\|v\|}{C}\right\}$. For any $z \in B_{v,2}$ we have

$$\|z - v\| \overset{(E2.4)}{\leq} C\,\|z - v\|_2 < 1 - \|v\| \qquad \Longrightarrow \qquad \|z\| \leq \|z - v\| + \|v\| < 1$$

Hence $B_{v,2} \subseteq D$. In sum, we have show that, for any $v \in D$, there exists a ball in the 2 norm (which is open in the metric topology), centered at v with strictly positive radius, and contained within D. Hence, D is a union of open 2-balls and is open in the metric topology. This implies that D is open.

c) *D is convex*: Take $v, w \in D$ and $\theta \in (0, 1)$. By convexity of the norm

$$\|\theta v + (1 - \theta)w\| \leq \theta\|v\| + (1 - \theta)\,\|w\| < \theta + (1 - \theta) = 1.$$

Hence, $\theta v + (1 - \theta)w \in D$.

d) *D is centrally symmetric*: Follows from homogeneity of the norm $\|\cdot\|$.

(ii) We first show that $\mathcal{S} \subset \{v \in \mathbb{R}^n : \|v\|_{\mathcal{S}} < 1\}$. Given $v \in \mathcal{S}$, consider the function $\psi\colon (0, \infty) \to \mathbb{R}^n$ by $\psi(\lambda) = \lambda^{-1}v$. Since $\mathcal{S}$ is open, ψ is continuous and $1 \in \psi^{-1}(\mathcal{S})$, we have that $1 - \varepsilon \in \mathcal{S}$ for some small $\varepsilon > 0$. Hence, $\frac{1}{1-\varepsilon}v \in \mathcal{S}$ implies $\|v\|_{\mathcal{S}} \leq 1 - \varepsilon$. Next, we show that $\|\cdot\|_{\mathcal{S}}$ is a norm.

a) *$\|v\|_{\mathcal{S}} \geq 0$:* Follows from definition of an infimum. Moreover, if $\|v\|_{\mathcal{S}} = 0$, there exists a sequence $\lambda_n \to 0^+$ such that $\lambda_n^{-1}v \in \mathcal{S}$. Since $\mathcal{S}$ is bounded, it must be the case that $v = 0$.

b) *Homogeneity:* Take $v \in \mathbb{R}^n$ and $c \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned}\|cv\|_{\mathcal{S}} &= \inf\left\{\lambda > 0 : \frac{c}{\lambda}v \in \mathcal{S}\right\}\\ &= \inf\left\{|c|\frac{\lambda}{|c|} > 0 : \frac{|c|}{\lambda}v \in \mathcal{S}\right\} \overset{\mu=\frac{\lambda}{|c|}}{=} |c|\inf\{\mu > 0 : \mu^{-1}v \in \mathcal{S}\} = |c|\,\|v\|_{\mathcal{S}}\,,\end{aligned}$$

where the second equality follows from central symmetry of $\mathcal{S}$.

c) *Triangle inequality:* Let $v, w \in \mathbb{R}^n$. Fix $\varepsilon > 0$ and $\lambda_v, \lambda_w > 0$ such that

$$\begin{aligned}\|v\|_{\mathcal{S}} \leq \lambda_v \leq \|v\|_{\mathcal{S}} + \varepsilon,\ \tfrac{1}{\lambda_v}v \in \mathcal{S},\\ \|w\|_{\mathcal{S}} \leq \lambda_w \leq \|w\|_{\mathcal{S}} + \varepsilon,\ \tfrac{1}{\lambda_w}w \in \mathcal{S}.\end{aligned}$$

Now,

$$\tfrac{1}{\lambda_v+\lambda_w}(v + w) = \tfrac{\lambda_v}{\lambda_v+\lambda_w}\left(\tfrac{1}{\lambda_v}v\right) + \tfrac{\lambda_w}{\lambda_v+\lambda_w}\left(\tfrac{1}{\lambda_w}w\right) \in \mathcal{S}$$

where we used convexity of $\mathcal{S}$. Hence

$$\|v + w\|_{\mathcal{S}} \leq \lambda_v + \lambda_w \leq \|v\|_{\mathcal{S}} + \|w\|_{\mathcal{S}} + 2\varepsilon.$$

Taking $\varepsilon \to 0^+$ finishes the proof.

E2.3 **Monotonic and absolute norms (Bauer et al., 1961).** For a norm $\|\cdot\|$ on $\mathbb{C}^n$, show that the following statements are equivalent:

(i) $\|\cdot\|$ is *monotonic*, that is, for every $x, y \in \mathbb{C}^n$,

$$|x_i| \leq |y_i|, i \in \{1, \dots, n\}, \quad \implies \quad \|x\| \leq \|y\|,$$

(ii) $\|\cdot\|$ is *absolute*, that is, for every $x, y \in \mathbb{C}^n$,

$$|x_i| = |y_i|, i \in \{1, \dots, n\} \quad \implies \quad \|x\| = \|y\|,$$

or, equivalently, for each $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ $\|(x_1, \dots, x_n)\| = \|(|x_1|, \dots, |x_n|)\|$.

Answer: Regarding (i) $\implies$ (ii), assume that $\|\cdot\|$ is monotonic and that $|y| = |x|$ (that is, $|x_i| = |y_i|, i \in \{1, \dots, n\}$). Then $|y| \leq |x|$ and $|y| \geq |x|$, so that $\|y\| \leq \|x\|$ and $\|x\| \leq \|y\|$. This shows that $\|\cdot\|$ is absolute.

Regarding (ii) $\implies$ (i), pick $j \in \{1, \dots, n\}$ and $0 < \alpha < 1$. For any $x \in \mathbb{R}^n$, since $\|\cdot\|$ is absolute, we compute

$$\begin{aligned}
\left\| \begin{bmatrix} x_1 \\ \vdots \\ \alpha x_k \\ \vdots \\ x_n \end{bmatrix} \right\| &= \left\| \frac{1-\alpha}{2} \begin{bmatrix} x_1 \\ \vdots \\ -x_k \\ \vdots \\ x_n \end{bmatrix} + \frac{1-\alpha}{2} x + \alpha x \right\| \\
&\leq \frac{1-\alpha}{2} \left\| \begin{bmatrix} x_1 \\ \vdots \\ -x_k \\ \vdots \\ x_n \end{bmatrix} \right\| + \frac{1-\alpha}{2}\|x\| + \alpha\|x\| = \frac{1-\alpha}{2}\|x\| + \frac{1-\alpha}{2}\|x\| + \alpha\|x\| = \|x\|.
\end{aligned}$$

E2.4 **Hölder and Minkovski inequalities for ℓ_p norms.** For $p \in [1, \infty]$, define the functions $\|\cdot\|_p \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$\|x\|_\infty = \max_{i \in \{1, \dots, n\}} |x_i|, \qquad \text{and} \qquad \|x\|_p = \big(\textstyle\sum_{i=1}^n |x_i|^p \big)^{1/p}, \quad \text{for } 1 \leq p < \infty.$$

Show that:

(i) for each $x, y \in \mathbb{R}^n$ and $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$ (with the convention $1/\infty = 0$), the *Hölder inequality* holds

$$x^\top y \leq \|x\|_p \|y\|_q, \tag{E2.5}$$

with equality (for $1 < p < \infty$) if and only if there exists a constant $\mu > 0$ such that $\mu |x_i|^p = |y_i|^q$ and x_i and y_i have the same sign for all $i \in \{1, \dots, n\}$.

Hint: *The weighted arithmetic and geometric means inequality states that, for every $a, b > 0$,*

$$a^{1-\lambda} b^\lambda \leq (1-\lambda)a + \lambda b, \qquad \text{for every } 0 < \lambda < 1, \tag{E2.6}$$

with equality if and only if $a = b$.

(ii) for each $x, y \in \mathbb{R}^n$ and $p \in [1, \infty]$, the *Minkovski's inequality* holds

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p,$$

with equality (when $x \neq \mathbb{0}_n$ and $p > 1$) if and only if there exists a constant $\mu > 0$ such that $\mu |x_i| = |y_i|$ and x_i and y_i have the same sign for all $i \in \{1, \dots, n\}$.

(iii) for each $p \in [1, \infty]$, the function $\|\cdot\|_p \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a norm.

***Note:** The weighted arithmetic and geometric means inequality (E2.6) follows from the strict concavity of the logarithm function.*

Answer: The Hölder inequality (E2.5) is trivial when $p \in \{1, \infty\}$. Therefore, we set $1 < p < \infty$. It now suffices to show the inequality (E2.5) for nonnegative $x, y \in \mathbb{R}^n_{\geq 0}$ satisfying $\|x\|_p = \|y\|_q = 1$. We set $a_i = x_i^p$, $b_i = y_i^q$, $\lambda = 1 - 1/p = 1/q$ (so that $1 - \lambda = 1/p$), and compute

$$x_i y_i = a_i^{1/p} b_i^{1-1/p} = a_i^{1-\lambda} b_i^{\lambda} \overset{\text{(E2.6)}}{\leq} \frac{1}{p} a_i + \Big(1 - \frac{1}{p}\Big) b_i = \frac{1}{p} x_i^p + \Big(1 - \frac{1}{p}\Big) y_i^q. \tag{E2.7}$$

Next we compute

$$x^\mathsf{T} y = \sum_{i=1}^n x_i y_i \leq \frac{1}{p} \sum_{i=1}^n x_i^p + \Big(1 - \frac{1}{p}\Big) \sum_{i=1}^n y_i^q = \frac{1}{p} \|x\|_p^p + \Big(1 - \frac{1}{p}\Big) \|y\|_q^q$$

The Hölder inequality (E2.5) now follows from substituting $\|x\|_p = \|y\|_q = 1$. By the weighted arithmetic and geometric means inequality, the inequality (E2.7) is an equality if and only if $a_i = b_i$ for each i, that is $x_i^p = y_i^q$ for each i. The proof of statement (i) follows from generalizing the result to the case of vectors x and y with arbitrary signs and norms.

Regarding statement (ii), we compute

$$\begin{aligned}
\|x + y\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p = \sum_{i=1}^n |x_i + y_i|^{p-1} |x_i + y_i| \leq \sum_{i=1}^n |x_i + y_i|^{p-1} \big(|x_i| + |y_i|\big) \\
&= \sum_{i=1}^n |x_i + y_i|^{p-1} |x_i| + \sum_{i=1}^n |x_i + y_i|^{p-1} |y_i|.
\end{aligned} \tag{E2.8}$$

We now use Hölder inequality (with q such that $1/p + 1/q = 1$) to obtain

$$\begin{aligned}
&\sum_{i=1}^n |x_i + y_i|^{p-1} |x_i| \leq \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} \Big(\sum_{i=1}^n |x_i + y_i|^{q(p-1)}\Big)^{1/q} = \|x\|_p \|x + y\|_p^{p-1}, \\
&\sum_{i=1}^n |x_i + y_i|^{p-1} |y_i| \leq \|y\|_p \|x + y\|_p^{p-1}.
\end{aligned}$$

Here we used the arithmetic identities $(p-1)q = p$ and $1/q = (p-1)/p$. We now substitute these two upper bounds into the inequality (E2.8) to obtain

$$\|x + y\|_p^p \leq \big(\|x\|_p + \|y\|_p\big) \|x + y\|_p^{p-1},$$

which implies the Minkovski's inequality.

Finally, statement (iii) is an immediate consequence of statement (ii), since the Minkovski's inequality is the triangle inequality (i.e., the subadditivity property) for the ℓ_p norm. Positive definiteness and homogeneity are immediate to verify.

E2.5 **Dual norms.** Given a vector space V, let $\|\cdot\|$ be a norm on V and let V^* denote the dual space of V. Define the *dual norm* $\|\cdot\|_* \colon V^* \to \mathbb{R}$ by

$$\|\omega\|_* = \max_{x \in V, \|x\| = 1} \langle \omega, x \rangle. \tag{E2.9}$$

Show that

(i) the dual norm is a norm on V^*.

Next, consider the usual identification $V = \mathbb{R}^{n \times 1}$ and $V^* = \mathbb{R}^{1 \times n}$, so that $x \in V$ is a column vector, $\omega \in V^*$ is a row vector, and $\langle \omega, x \rangle = \omega x$. For any $p \in [1, \infty]$, define $q \in [1, \infty]$ to satisfy $1/p + 1/q = 1$ (with the convention $1/\infty = 0$) and show that

(ii) the dual norm $\|\cdot\|_{p*}$ to $\|\cdot\|_p$ satisfies
$$\|\omega\|_{p*} = \|\omega^\top\|_q,$$
(iii) each linear operator $A\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies
$$\|A\|_p = \|A^\top\|_q \quad \text{and} \quad \mu_p(A) = \mu_q(A^\top).$$

Answer:

(i) We show that $\|\cdot\|_*$ is a norm.

- Let $\omega \in V^*$. If $\omega = 0$, then $\langle \omega, x\rangle = 0$ for all $x \in V$. If $\omega \neq 0$, there exists some $x \in V$ such that $\|x\| = 1$ and $\langle \omega, x\rangle \neq 0$. Without loss of generality, we assume $\langle \omega, x\rangle > 0$ (otherwise we can replace x with $-x$ and use linearity of ω). Hence, $\|\omega\|_* > 0$.
- *Homogeneity:* Given any $\omega \in V^*$ and scalar $c \neq 0$, we compute
$$\begin{aligned}\|c\omega\|_* &\overset{(E2.9)}{=} \max_{x\in V, \|x\|=1} \langle c\omega, x\rangle = \max_{x\in V, \|x\|=1} \langle |c|\omega, \operatorname{sign}(c)x\rangle \\ &\overset{y=\operatorname{sign}(c)x}{=} |c| \max_{y\in V, \|y\|=1} \langle \omega, y\rangle = |c|\,\|\omega\|_*\,.\end{aligned}$$
- *Triangle inequality:* Let $v, \omega \in V^*$ and $x \in V$ satisfying $\|x\| = 1$. Then
$$\langle \omega + v, x\rangle = \langle \omega, x\rangle + \langle v, x\rangle \leq \|\omega\|_* + \|v\|_*.$$
The property follows by taking the maximum over $x \in V$ satisfying $\|x\| = 1$ on the left-hand side.

(ii) We show the claim for $p, q \in (1, \infty)$. The remaining cases are similar. Let $\omega \in \mathbb{R}^{1\times n}$ and $x \in \mathbb{R}^n$ satisfy $\|x\|_p = 1$. First, the Hölder inequality (E2.5) implies
$$\omega x \leq |\omega x| \overset{(E2.5)}{\leq} \left(\sum_{i=1}^n |\omega_i|^q\right)^{1/q} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \|\omega^\top\|_q \cdot 1,$$
which implies $\|\omega\|_{p^*} \leq \|\omega^\top\|_q$. For the other direction, define
$$x = \begin{cases} 0, & \omega_i = 0, \\ \operatorname{sign}(\omega_i)\,|\omega_i|^{q-1}, & \text{otherwise.}\end{cases}$$
Note that $1/p + 1/q = 1$ implies $q = (q-1)p$. Therefore,
$$\|x\|_p^p = \sum_{i=1}^n |\omega_i|^{p(q-1)} = \sum_{i=1}^n |\omega_i|^q = \|\omega^\top\|_q^q.$$
Setting $\bar{x} = \|\omega^\top\|_q^{-\frac{q}{p}} x$, we have $\|\bar{x}\|_p = 1$. The latter gives
$$\|\omega\|_{p^*} \geq \omega\bar{x} = \|\omega^\top\|_q^{-\frac{q}{p}} \sum_{i=1}^n |\omega_i|\,|\omega_i|^{q-1} = \|\omega^\top\|_q^{-\frac{q}{p}} \|\omega^\top\|_q^q = \|\omega^\top\|_q^{q\left(1-\frac{1}{p}\right)} = \|\omega^\top\|_q.$$

(iii) It suffices to show that $\|A\|_p \leq \|A^\top\|_q$ and then, replacing A with $A^\top$ and p with q, we obtain the reversed inequality. Fix $x \in \mathbb{R}^n$ satisfying $\|x\|_p = 1$ and $\|Ax\|_p = \|A\|_p$. Then statement (ii) implies
$$\begin{aligned}\|A\|_p = \|Ax\|_p = \|x^\top A^\top\|_q &\overset{(E2.9)}{=} \max_{y\in\mathbb{R}^n, \|y\|_q=1} x^\top(A^\top y) \\ &\overset{(E2.5)}{\leq} \max_{y\in\mathbb{R}^n, \|y\|_q=1} \|A^\top y\|_q = \|A^\top\|_q.\end{aligned}$$
Regarding the logarithmic norm, we compute
$$\lim_{h\to 0^+} \frac{\|I_n + hA\|_p - 1}{h} = \lim_{h\to 0^+} \frac{\|I_n + hA^\top\|_q - 1}{h} = \mu_q(A^\top).$$

E2.6 **The logarithmic norm is well-defined.** Show that, for any matrix A and any norm $\|\cdot\|$, the limit in the definition (2.3) of the induced matrix logarithmic norm exists and is finite (and therefore the definition is well posed).

Answer: We define the function $h \mapsto f(h) = h^{-1}(\|I_n + hA\| - 1)$ and prove that the limit $\lim_{h\to 0^+} f(h)$ is well posed. For $0 < \theta < 1$ and $h > 0$, the triangle inequality implies

$$\|I_n + \theta hA\| = \|(1-\theta)I_n + \theta(I_n + hA)\| \leq (1-\theta) + \theta\|I_n + hA\|,$$

and, after some simple rescaling,

$$\begin{aligned} f(\theta h) = (\theta h)^{-1}(\|I_n + \theta hA\| - 1) &\leq (\theta h)^{-1}\Big((1-\theta) + \theta\|I_n + hA\| - 1\Big) \\ &= (\theta h)^{-1}\Big(\theta\|I_n + hA\| - \theta\Big) = h^{-1}(\|I_n + hA\| - 1) = f(h). \end{aligned}$$

Therefore, f is weakly increasing in h. Next, using again the triangle inequality, we note $|f(h)| \leq |h^{-1}(1 + h\|A\| - 1)| \leq \|A\|$. In sum, f is weakly increasing and takes value in the bounded interval $[-\|A\|, \|A\|]$. Hence, it has a finite limit as $h \to 0^+$.

E2.7 **Properties of the logarithmic norm.** Given the definition of logarithmic norm, prove properties (2.30a)–(2.30f) in Lemma 2.11.

Answer:

Positive homogeneity (2.30a)***:*** Let $A \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}$. If $a = 0$, the claim is trivial. Otherwise, applying the definition (2.3), we have

$$\begin{aligned} \mu(aA) &= \lim_{h\to 0^+} \frac{\|I_n + haA\| - 1}{h} = \lim_{h\to 0^+} \frac{\|I_n + h|a|\operatorname{sign}(a)A\| - 1}{h} \\ &\overset{\tau = h|a|}{=} |a| \lim_{\tau\to 0^+} \frac{\|I_n + \tau\operatorname{sign}(a)A\| - 1}{\tau} \overset{\text{def}}{=} |a|\mu(\operatorname{sign}(a)A). \end{aligned}$$

Subadditivity (2.30b)**:** We begin by showing the convexity of the lognorm (2.32). Let $A, B \in \mathbb{R}^{n\times n}$ and $\theta \in (0,1)$. Applying the definition (2.3), we have

$$\begin{aligned} \mu(\theta A + (1-\theta)B) &= \lim_{h\to 0^+} \frac{\|I_n + h[\theta A + (1-\theta)B]\| - 1}{h} \\ &= \lim_{h\to 0^+} \frac{\|\theta[I_n + hA] + (1-\theta)[I_n + hB]\| - \theta - (1-\theta)}{h} \\ &\overset{\text{triangle ineq.}}{\leq} \theta \lim_{\tau\to 0^+} \frac{\|I_n + hA\| - 1}{h} + (1-\theta) \lim_{\tau\to 0^+} \frac{\|I_n + hB\| - 1}{h} \\ &\overset{\text{def}}{=} \theta\mu(A) + (1-\theta)\mu(B). \end{aligned}$$

We now show (2.30b). By using (2.30a) and (2.32), we have

$$\mu(A+B) \overset{(2.30a)}{=} 2\mu\left(\frac{1}{2}A + \frac{1}{2}B\right) \overset{(2.32)}{\leq} 2\mu\left(\frac{1}{2}A\right) + 2\mu\left(\frac{1}{2}B\right) \overset{(2.30a)}{=} \mu(A) + \mu(B)$$

Translation property (2.30c)**:** Let $A \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}$. We apply (2.30b) twice. First,

$$\mu(A + aI_n) \leq \mu(A) + \mu(aI_n) \overset{(2.28)}{=} \mu(A) + a.$$

Second,

$$\mu(A) = \mu([A + aI_n] - aI_n) \leq \mu(A + aI_n) + \mu(-aI_n) \overset{(2.28)}{=} \mu(A + aI_n) - a.$$

Product property (2.30d)**:** Without loss of generality, assume $\|x\| = 1$. Let $h > 0$ satisfy $h\,\|Ax\| < 1$. Using the reverse triangle inequality, we have

$$\begin{aligned}\frac{\|I_n + hA\| - 1}{h} &\overset{\|x\|=1}{\geq} \frac{\|(I_n + hA)\,x\| - 1}{h}\\ &\geq \frac{\|x\| - \|hAx\| - 1}{h} = \frac{-h\,\|Ax\|}{h} = -\,\|Ax\|\,.\end{aligned}$$

Taking $h \to 0^+$ yields $\mu(A) \geq -\,\|Ax\|$. Repeating the argument for the matrix $-A$ completes the proof.

Norm of difference property (2.30e)**:** Without loss of generality, assume $\mu(A) \geq \mu(B)$ and let $h > 0$. Using the reverse triangle inequality, we have

$$\begin{aligned}\frac{\|I_n + hA\| - 1}{h} - \frac{\|I_n + hB\| - 1}{h} &= \frac{\|I_n + hA\| - \|I_n + hB\|}{h}\\ &\leq \frac{\|[I_n + hA] - [I_n + hB]\|}{h} = \frac{h\,\|A - B\|}{h} = \|A - B\|\,.\end{aligned}$$

Taking $h \to 0^+$ yields $\mu(A) - \mu(B) \leq \|A - B\|$.

Uniform monotonicity property (2.30f)**:** We first claim that A is invertible. Otherwise, there exists some $x \in \mathbb{R}^n$ satisfying $\|x\| = 1$ and $Ax = 0$. Using the product property, we obtain $\mu(A) \geq -\|Ax\| = 0$, which is a contradiction.

To prove the bound on $\left\|A^{-1}\right\|$, let $h > 0$. Then,

$$\begin{aligned}\left\|A^{-1}\right\|\frac{1 - \|I_n + hA\|}{h} &\leq \frac{\left\|A^{-1}\right\| - \left\|A^{-1}\,[I_n + hA]\right\|}{h}\\ &\leq \frac{\left\|A^{-1} - \left[A^{-1} + hI_n\right]\right\|}{h} = \frac{h\,\|I_n\|}{h} = 1.\end{aligned}$$

Here, the first inequality follows from $\left\|A^{-1}\,[I_n + hA]\right\| \leq \left\|A^{-1}\right\|\,\|I_n + hA\|$, whereas the second one follows from the triangle inequality. Taking $h \to 0^+$, we obtain $-\mu(A)\,\left\|A^{-1}\right\| \leq 1$.

E2.8 **Formulas for the basic logarithmic norms.** Based on the formulas for the induced norms, prove the formulas for the $\mu_1(\cdot)$, $\mu_2(\cdot)$ and $\mu_\infty(\cdot)$ logarithmic norms in Table 2.1.

Answer: Regarding the formula for $\mu_2(\cdot)$, note that the eigenvalues of

$$(I_n + hA)^\top(I_n + hA) = I_n + h(A + A^\top) + h^2A^\top A,$$

converge to $1 + h\lambda_i$ for $h \to 0^+$, where λ_i are the eigenvalues of $A + A^\top$. Therefore, since $\|A\|_2 = \sqrt{\rho(A^\top A)}$, we compute

$$\begin{aligned}\mu_2(A) &= \lim_{h\to 0^+}\frac{\|I_n + hA\|_2 - 1}{h}\\ &= \lim_{h\to 0^+}\frac{\rho(I_n + h(A + A^\top) + h^2A^\top A)^{1/2} - 1}{h}\\ &= \lim_{h\to 0^+}\frac{1 + \frac{1}{2}h\lambda_{\max}(A + A^\top) + \mathcal{O}(h^2) - 1}{h}\\ &= \tfrac{1}{2}\lambda_{\max}(A + A^\top),\end{aligned}$$

where we used $\sqrt{1 + ah + bh^2} = 1 + ah/2 + \mathcal{O}(h^2)$.

Regarding the formulas for $\mu_1(\cdot)$ and $\mu_\infty(\cdot)$, they follow from the formulas for the induced norms and from the observation that the presence of the identity matrix ensures, for h sufficiently small, that the absolute value is taken only on the off-diagonal entries. Specifically, for $\mu_\infty(\cdot)$

$$\begin{aligned}\mu_\infty(A) &= \lim_{h\to0^+} \frac{\|I_n + hA\|_\infty - 1}{h} \\ &= \lim_{h\to0^+} \frac{\max_{i\in\{1,...,n\}} \big(|1+ha_{ii}| + \sum_{j=1,j\neq i}^n |ha_{ij}|\big) - 1}{h} \\ &= \lim_{h\to0^+} \frac{\max_{i\in\{1,...,n\}} \big(ha_{ii} + \sum_{j=1,j\neq i}^n |ha_{ij}|\big)}{h} \\ &= \max_i \Big(a_{ii} + \sum_{j=1,j\neq i}^n |a_{ij}|\Big).\end{aligned}$$

The proof of the formula for $\mu_1(\cdot)$ is essentially identical.

E2.9 **Additional formulas for induced matrix norms.** Consider the following induced matrix norm formulas.

(i) For each matrix $A \in \mathbb{R}^{n\times n}$, invertible matrix $R \in \mathbb{R}^{n\times n}$ and norm $\|\cdot\|$, show the equalities (2.34) in Lemma 2.12, i.e.,

$$\|A\|_R = \|RAR^{-1}\|, \qquad \mu_R(A) = \mu(RAR^{-1}). \tag{E2.10}$$

(ii) For each matrix $A \in \mathbb{R}^{n\times m}$, invertible matrices $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$, and norms $\|\cdot\|_m$ on $\mathbb{R}^m$ and $\|\cdot\|_n$ on $\mathbb{R}^n$, show that the norm of A induced by the weighted norm $\|\cdot\|_{m,M}$ on $\mathbb{R}^m$ and the weighted norm $\|\cdot\|_{n,N}$ on $\mathbb{R}^n$ satisfies

$$\|A\|_{(m,M)\to(n,N)} = \max_{x\neq \mathbb{0}_m} \frac{\|Ax\|_{n,N}}{\|x\|_{m,M}} = \|NAM^{-1}\|_{m\to n}. \tag{E2.11}$$

(iii) Prove the nine formulas in Table 2.4 for the induced matrix norm $\|A\|_{p\to q}$, for a matrix $A \in \mathbb{R}^{n\times m}$ and for $p, q \in \{1, 2, \infty\}$.

$\|A\|_{p\to q}$	$q=1$	$q=2$	$q=\infty$
$p=1$	$\max_{j\in\{1,...,m\}} \|A_{:,j}\|_1$	$\max_{j\in\{1,...,m\}} \|A_{:,j}\|_2$	$\max_{i\in\{1,...,n\},j\in\{1,...,m\}} \|A_{ij}\|$
$p=2$	$\max_{u\in\{-1,1\}^n} \|A^\top u\|_2$	$\max\{\sqrt{\lambda} : \lambda \in \operatorname{spec}(A^\top A)\}$	$\max_{i\in\{1,...,n\}} \|A_{i,:}\|_2$
$p=\infty$	$\max_{u\in\{-1,1\}^m} \|Au\|_1$	$\max_{u\in\{-1,1\}^m} \|Au\|_2$	$\max_{i\in\{1,...,n\}} \|A_{i,:}\|_1$

Table 2.4: Formulas for the induced matrix norm $\|A\|_{p\to q}$, for a matrix $A \in \mathbb{R}^{n\times m}$ and for $p, q \in \{1, 2, \infty\}$. For $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, we let $A_{i,:} \in \mathbb{R}^m$ and $A_{:,j} \in \mathbb{R}^n$ denote the ith row and jth column of A, respectively.

Answer: Regarding statement (i) and the claimed equality $\|A\|_{p,R} = \|RAR^{-1}\|$ in equation (E2.10), we compute

$$\begin{aligned}\|A\|_R := \sup_{\|x\|_R=1} \|Ax\|_R = \sup_{\|Rx\|=1} \|RAx\| = \sup_{\|Rx\|=1} \|RAR^{-1}Rx\| \\ = \sup_{\|y\|=1} \|RAR^{-1}y\| = \|RAR^{-1}\|.\end{aligned}$$

Similarly, regarding the claimed equality $\mu_R(A) = \mu(RAR^{-1})$, we compute

$$\begin{aligned}\|I_n + hA\|_R &:= \sup_{\|x\|_R=1} \|x + hAx\|_R = \sup_{\|Rx\|=1} \|R(x + hAx)\| \\ &= \sup_{\|Rx\|=1} \|Rx + hRAR^{-1}Rx\| = \sup_{\|y\|=1} \|y + hRAR^{-1}y\| \\ &= \sup_{\|y\|=1} \|(I_n + hRAR^{-1})y\| = \|I_n + hRAR^{-1}\|.\end{aligned}$$

The claimed identity now follows from the definition of logarithmic norm.

Regarding statement (ii), we compute

$$\|A\|_{(m,M)\to(n,N)} = \max_{x\neq \mathbb{0}_m} \frac{\|Ax\|_{n,N}}{\|x\|_{m,M}} \overset{(2.16)}{=} \max_{x\neq \mathbb{0}_m} \frac{\|NAx\|_n}{\|Mx\|_m} \tag{E2.12}$$

$$\overset{y=Mx}{=} \max_{y\neq \mathbb{0}_m} \frac{\|NAM^{-1}y\|_n}{\|y\|_m} = \|NAM^{-1}\|_{m\to n} \tag{E2.13}$$

Finally, regarding statement (iii), we refer to (Lewis, 2010) for the proof (and complexity analysis) for each of the nine formulas in Table 2.4.

E2.10 **Quadratic form for non-Euclidean logarithmic norms.** Prove equation (2.63a) and (2.64a) in Lemma 2.21.

Answer: We report equation (2.63a) here for convenience here:

$$\mu_{1,[\eta]}(A) = \max\left\{ \operatorname{sign}(x)^\mathsf{T}[\eta]A \;:\; x \in \mathbb{R}^n \text{ s.t. } \|x\|_{1,[\eta]} = 1\right\}$$

In what follows we prove the following statement: for all $b \in \mathbb{R}$, we have

$$\mu_{1,Q}(A) \leq b \quad\iff\quad \operatorname{sign}(Qv)^\mathsf{T}(QA)v \leq b\|v\|_{1,Q}, \quad \text{for all } v \in \mathbb{R}^n, \tag{E2.14}$$

which is a stronger statement than equation (2.63a) in that it allows for an arbitrary matrix Q (not necessarily diagonal positive-definite).

Regarding equation (E2.14), we first prove the implication $\implies$ for the case $Q = I_n$. Let $\tilde{\jmath} \in \{1, \dots, n\}$ be an index satisfying $\mu_1(A) = \left(a_{\tilde{\jmath}\tilde{\jmath}} + \sum_{i=1,i\neq\tilde{\jmath}}^n |a_{i\tilde{\jmath}}|\right)$. For any $v \in \mathbb{R}^n$ with $\|v\|_1 = 1$, let $s_i = \operatorname{sign}(v_i)$ for brevity of notation, note $v_i = s_i|v_i|$ (when $v_i = 0$, then both $s_i = |v_i| = 0$), and compute

$$\begin{aligned}\operatorname{sign}(v)^\mathsf{T}Av &= \sum_{j=1}^n \Big(\sum_{i=1}^n a_{ij}s_is_j\Big)|v_j| \\ &= \sum_{j=1}^n \Big(a_{jj}s_j^2 + \sum_{i=1,i\neq j}^n a_{ij}s_is_j\Big)|v_j| \\ &= \sum_{j=1}^n \Big(a_{jj} + \sum_{i=1,i\neq j}^n a_{ij}s_is_j\Big)|v_j| && \text{(because } s_j^2 = +1 \text{ whenever } |v_j| \neq 0) \\ &\leq \sum_{j=1}^n \Big(a_{jj} + \sum_{i=1,i\neq j}^n |a_{ij}|\Big)|v_j|. && \text{(because } a_{ij}s_is_j \leq |a_{ij}| \text{ and } |v_j| \geq 0)\end{aligned}$$

Finally, we compute

$$\sup_{\|v\|_1=1} \operatorname{sign}(v)^\mathsf{T}Av \leq \max_{\eta\in\Delta_n} \sum_{j=1}^n \Big(a_{jj} + \sum_{i=1,i\neq j}^n |a_{ij}|\Big)\eta_j = \max_{j\in\{1,\dots,n\}} \Big(a_{jj} + \sum_{i=1,i\neq j}^n |a_{ij}|\Big) = \mu_1(A),$$

where the maximum is achieved at $\eta = \mathbb{e}_{\tilde{j}}$. This concludes the proof of the implication $\implies$ for the case $Q = I_n$. Next, we prove the implication $\impliedby$ for the case $Q = I_n$. Let $A_{\tilde{j}} = A\mathbb{e}_{\tilde{j}}$ denote is the $\tilde{j}$ column of A (corresponding to a maximum column absolute sum). For $0 < \varepsilon \ll 1$, define $w_\varepsilon = \mathbb{e}_{\tilde{j}} + \varepsilon \operatorname{sign}(A_{\tilde{j}})$, $v_\varepsilon = w_\varepsilon/\|w_\varepsilon\|_1$, and compute

$$\begin{aligned}
Av_\varepsilon &= \frac{1}{\|w_\varepsilon\|_1} A\mathbb{e}_{\tilde{j}} + \frac{\varepsilon}{\|w_\varepsilon\|_1} A \operatorname{sign}(A_{\tilde{j}}) = A_{\tilde{j}} + O(\varepsilon),\\
\operatorname{sign}(v_\varepsilon)_i &= \begin{cases} +1, & \text{if } i = \tilde{j},\\ \operatorname{sign}(a_{i\tilde{j}}), & \text{otherwise,} \end{cases}\\
\operatorname{sign}(v_\varepsilon)^\top Av_\varepsilon &= \operatorname{sign}(v_\varepsilon)^\top A_{\tilde{j}} + O(\varepsilon) = \Big(a_{\tilde{j}\tilde{j}} + \sum_{i=1, i\neq\tilde{j}}^{n} |a_{i\tilde{j}}|\Big) + O(\varepsilon) = \mu_1(A) + O(\varepsilon).
\end{aligned}$$

By assumption we know $\operatorname{sign}(v_\varepsilon)^\top Av_\varepsilon \leq b$ for all $\varepsilon > 0$, and so we obtain

$$\mu_1(A) = \lim_{\varepsilon\to 0^+} \operatorname{sign}(v_\varepsilon)^\top Av_\varepsilon \leq b.$$

This concludes the proof of equation (E2.14) and therefore (2.63a) for the case $Q = I_n$. Next, for the case of a general invertible Q, we recall that $\mu_{1,Q}(A) = \mu_1(QAQ^{-1})$ by Lemma 2.12. But we have just proved that $\mu_1(QAQ^{-1}) \leq b$ if and only if $\operatorname{sign}(w)^\top(QAQ^{-1})w \leq b$ for all $w \in \mathbb{R}^n$ with $\|w\|_1 = 1$. Choosing $w = Qv$, we obtain the stated result. This concludes the proof of equation (2.63a).

Next, we report equation (2.64a) here for convenience here:

$$\mu_{\infty,[\eta]^{-1}}(A) = \max\Big\{ \max_{i\in I_\infty([\eta]^{-1}x)} \frac{x_i}{\eta_i}([\eta]^{-1}Ax)_i \;:\; x \in \mathbb{R}^n \text{ s.t. } \|x\|_{\infty,[\eta]^{-1}} = 1\Big\}$$

In what follows we prove the following statement: for all $b \in \mathbb{R}$, we have

$$\mu_\infty(A) \leq b \quad\iff\quad \max_{i\in I_\infty(x)} x_i(Ax)_i \leq b\|v\|_\infty, \quad \text{for all } v \in \mathbb{R}^n. \tag{E2.15}$$

We first prove the implication $\implies$. Assuming $\mu_\infty(A) = \max_{i\in\{1,...,n\}} \big(a_{ii} + \sum_{j=1,j\neq i}^{n} |a_{ij}|\big) \leq b$, we consider an arbitrary $v \in \mathbb{R}^n$. For $i \in I_\infty(v)$, that is, for $i \in \{1,\dots,n\}$ such that $|v_i| = \|v\|_\infty$, we compute

$$\begin{aligned}
\operatorname{sign}(v_i)\,(Av)_i &= \operatorname{sign}(v_i)\sum_{j=1}^{n} a_{ij}v_j = a_{ii}\|v\|_\infty + \sum_{j=1,j\neq i}^{n} a_{ij}\operatorname{sign}(v_i)v_j\\
&\leq a_{ii}\|v\|_\infty + \sum_{j=1,j\neq i}^{n} |a_{ij}|\|v\|_\infty = \mu_\infty(A)\|v\|_\infty \leq b\|v\|_\infty.
\end{aligned}$$

This proves that $\max_{i\in I_\infty(v)}\{\operatorname{sign}(v_i)\,(Av)_i\} \leq b\|v\|_\infty$ for all $v \in \mathbb{R}^n$. Next, we prove the implication $\impliedby$. Let $\tilde{\imath} \in \{1,\dots,n\}$ be an index satisfying $\mu_\infty(A) = \big(a_{\tilde{\imath}\tilde{\imath}} + \sum_{j=1,j\neq\tilde{\imath}}^{n} |a_{\tilde{\imath}j}|\big)$. Define $\tilde{v} \in \{-1,0,+1\}^n \subset \mathbb{R}^n$ by

$$\tilde{v}_j = \begin{cases} 1, & \text{if } j = \tilde{\imath},\\ \operatorname{sign}(a_{\tilde{\imath}j}), & \text{otherwise.} \end{cases}$$

Note $\operatorname{sign}(\tilde{v}_{\tilde{\imath}}) = +1$ and compute

$$\operatorname{sign}(\tilde{v}_{\tilde{\imath}})\,(A\tilde{v})_{\tilde{\imath}} = (+1)\sum_{j=1}^{n} a_{\tilde{\imath}j}\tilde{v}_j = a_{\tilde{\imath}\tilde{\imath}} + \sum_{j=1,j\neq\tilde{\imath}}^{n} |a_{\tilde{\imath}j}| = \mu_\infty(A).$$

Since $\tilde{v}_{\tilde{\imath}} = +1$, we know $\tilde{\imath} \in I_\infty(\tilde{v})$. In summary, we compute

$$\mu_\infty(A) = \operatorname{sign}(\tilde{v}_{\tilde{\imath}})\,(A\tilde{v})_{\tilde{\imath}}\Big|_{\tilde{\imath}\in I_\infty(\tilde{v}),\|\tilde{v}\|_\infty=1} \leq \max_{i\in I_\infty(v),\|v\|_\infty=1} \{\operatorname{sign}(v_i)\,(Av)_i\} \leq b.$$

This concludes our proof of equation (E2.15). We leave to the reader the generalization to equation (2.64a).

E2.11 **On the logarithmic norm of minus a matrix.** Given $A \in \mathbb{R}^{n\times n}$ and a norm $\|\cdot\|$, show that

$$\mu(-A) \geq -\mu(A),$$
$$\mu(-A) = -\lim_{h\to 0^-} \frac{\|I_n + hA\| - 1}{h}.$$

Answer: Clearly, $0 = \mu(\mathbb{0}_{n\times n}) = \mu(A - A) \leq \mu(A) + \mu(-A)$ by subadditivity. Next, we compute

$$\mu(-A) = \lim_{\tau\to 0^+} \frac{\|I_n + \tau(-A)\| - 1}{\tau} = \lim_{\tau\to 0^+} \frac{\|I_n - \tau A\| - 1}{\tau} = \lim_{h\to 0^-} \frac{\|I_n + hA\| - 1}{-h},$$

where, in the last equality, we define $h = -\tau$.

E2.12 **Relationship between Euclidean and nonEuclidean logarithmic norms.** For any $A \in \mathbb{R}^{n\times n}$,

(i) $\mu_1(A) = \mu_\infty(A^\top)$,
(ii) if $A = A^\top$, then $\mu_2(A) = \alpha(A) \leq \mu_{\|\cdot\|}(A)$, for any norm $\|\cdot\|$,
(iii) $\mu_2(A) = \mu_2(\frac{1}{2}(A + A^\top))$, and
(iv) $\mu_2(A) \leq \frac{1}{2}\big(\mu_1(A) + \mu_\infty(A)\big)$.

Note: *The inequality (iii) and the Geršgorin Disks Theorem together imply* $\mu_2(A) \leq \max_i \big(a_{ii} + \frac{1}{2}\sum_{j\neq i} |a_{ij} + a_{ji}|\big)$.

Answer: Statement (i) follows from the formulas in Table 2.1. Regarding statement (ii), if A is symmetric, then

$$\mu_2(A) = \lambda_{\max}\Big(\frac{A + A^\top}{2}\Big) = \lambda_{\max}(A) = \alpha(A).$$

Therefore, $\mu_2(\cdot)$ is logarithmically optimal and, for any other norm, $\mu(A)$ cannot be smaller than $\alpha(A)$ by property (2.73).

Regarding statement (iii), since $S + S^\top = 2S$ if $S = S^\top$, we compute

$$\begin{aligned}\mu_2(\tfrac{1}{2}(A + A^\top)) &= \lambda_{\max}\big(\tfrac{1}{2}(S + S^\top)\big)\Big|_{S=\frac{1}{2}(A+A^\top)} \\ &= \lambda_{\max}(S)\Big|_{S=\frac{1}{2}(A+A^\top)} = \lambda_{\max}\Big(\frac{A + A^\top}{2}\Big) = \mu_2(A).\end{aligned}$$

Regarding statement (iii), we compute

$$\begin{aligned}\mu_2(A) &= \frac{1}{2}\mu_2(A + A^\top) && \text{(from statement (iii))} \\ &\leq \frac{1}{2}\mu_\infty(A + A^\top) && \text{(from statement (ii))} \\ &\leq \frac{1}{2}\big(\mu_\infty(A) + \mu_\infty(A^\top)\big) && \text{(because of the subadditivity property (2.30b))} \\ &= \frac{1}{2}\big(\mu_\infty(A) + \mu_1(A)\big). && \text{(from statement (i))}\end{aligned}$$

E2.13 **Logarithmic norm results for monotonic norms.** Given $A \in \mathbb{R}^{n\times n}$ and a monotonic norm $\|\cdot\|$, show

$$\mu_R(A) = \mu(A), \qquad \text{if } R \text{ is the product of a permutation and reflection matrix,} \tag{E2.16}$$
$$\mu(A) \leq \mu(A + D), \qquad \text{if } D \text{ is diagonal with non-negative diagonal entries,} \tag{E2.17}$$
$$\mu_2(A) \leq \mu_2(A + S), \qquad \text{if } \mu_2(-S) \leq 0 \iff S + S^\top \succeq 0, \tag{E2.18}$$
$$\mu_1(A) \leq \mu_1(A + \Xi), \qquad \text{if } \mu_1(-\Xi) \leq 0 \iff \Xi_{jj} \geq \sum\nolimits_{i=1, i\neq j}^{n} |\Xi_{ij}| \text{ for all } j; \tag{E2.19}$$
$$\mu_\infty(A) \leq \mu_\infty(A + \Xi), \qquad \text{if } \mu_\infty(-\Xi) \leq 0 \iff \Xi_{ii} \geq \sum\nolimits_{j=1, j\neq i}^{n} |\Xi_{ij}| \text{ for all } i. \tag{E2.20}$$

(Here a *permutation and reflection matrix* is a square matrix with exactly one entry equal to $+1$ or -1 in every row and every column and all other entries equal to 0.)

Answer: Since the norm is monotonic, Theorem 2.23(iii) implies $\mu(-D) = \max_i(-D_{ii}) \leq 0$. Therefore,

$$\mu(A) = \mu(A + D - D) \leq \mu(A + D) + \mu(-D) \leq \mu(A + D). \tag{E2.21}$$

Equation (E2.18) follows from the same inequality (E2.21) where $D = S$ and $\mu_2(-S) \leq 0$. We also note that $-\mu_2(-S) = -\lambda_{\max}(-S-S^\top)/2 = \lambda_{\min}(S+S^\top) \geq 0$ if and only if $S+S^\top \succeq 0$, as claimed. Similarly, equations (E2.19) and (E2.20) follow from inequality (E2.21) where $D = \Xi$ and simple diagonal dominance inequalities.

E2.14 **Induced norms of rank 1 matrices (Fiore et al., 2016).** For any $x, y \in \mathbb{R}^n$, $x, y \neq \mathbb{0}_n$, show

(i) the spectrum of $xy^\top$ is $\{0, \dots, 0, x^\top y\}$,
(ii) $\|xy^\top\|_p = \|x\|_p \|y\|_q$, for each $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$ (with the convention $1/\infty = 0$),
(iii) $\mu(xy^\top) \geq 0$ for all logarithmic norms, and
(iv) given $P = P^\top \succ 0$, $\mu_{2,P^{1/2}}(xy^\top) = 0$ if and only if $x = -Py$.

Note: *It is an open problem extend statement (iv) to the setting of arbitrary ℓ_p norms.*

Answer: Statement (i) is well known and a consequence of the simple equalities: $(xy^\top)x = (x^\top y)x$ and $(xy^\top)v = 0$ for all $v \perp y$.

Regarding (ii), we compute

$$\|xy^\top\|_p = \sup_{\|v\|_p=1} \|(xy^\top)v\|_p = \|x\|_p \sup_{\|v\|_p=1} |y^\top v| \tag{E2.22}$$

The result follows from Hölder inequality (E2.5), which holds as equality for appropriate v.

Regarding (iii), the spectral property (2.73) states that $\mu(xy^\top)$ is an upper bound on all eigenvalues of $xy^\top$ and, therefore, statement (iii) is a consequence of the fact that $xy^\top$ has $n-1$ eigenvalues equal to 0.

Regarding (iv), (Fiore et al., 2016) first shows that $\mu_2(xy^\top) = 0$ if and only if $x = -y$ (and then treats the weighted case). The key idea is writing the characteristic polynomial of $xy^\top$ using (Bernstein, 2009, Fact 4.9.16). The proof for the weighted case follows directly from the definition of weighted logarithmic norm; we refer to (Fiore et al., 2016).

E2.15 **Optimal logarithmic norms for upper triangular matrices.** Let $A \in \mathbb{R}^{n\times n}$ be upper triangular. Recall the definition of sdg from equation (2.87). For $\varepsilon > 0$, define $\delta = \min\{1, \varepsilon/((n-1)\bar{a})\}$ and $\bar{a} = \max_{i<j} |a_{ij}|$. Show that $\|\cdot\|_{\infty,\mathsf{sdg}(\delta^{-1})}$ is logarithmically ε-optimal for A.

Answer: We compute

$$\mathsf{sdg}(\delta^{-1}) A \,\mathsf{sdg}(\delta) = \begin{bmatrix} a_{11} & \delta a_{12} & \dots & \delta^{n-1} a_{1n} \\ 0 & a_{22} & \ddots & \delta^{n-2} a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

and, in turn,

$$\begin{aligned} \mu_{\infty,\mathsf{sdg}(\delta^{-1})}(A) &= \max_i \Big(a_{ii} + \sum_{j>i} \delta^{j-1} |a_{ij}| \Big) \\ &\leq \max_i \Big(a_{ii} + \bar{a} \sum_{j>i} \delta^{j-1} \Big) \\ &\leq \max_i \Big(a_{ii} + (n-1)\bar{a}\delta \Big) \leq \alpha(A) + \varepsilon. \end{aligned}$$

E2.16 **Polyhedral norms as Lyapunov functions for linear systems.** Let $\mu(\cdot)$ be the logarithmic norm of a norm $\|\cdot\|$. Show that the function $\|Wx\|$, $W \in \mathbb{R}^{m\times n}$, $m \geq n$, W full rank, is a global Lyapunov function for the system $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$ if there exists a matrix $\bar{A} \in \mathbb{R}^{m\times m}$ such that

$$WA = \bar{A}W, \tag{E2.23}$$

$$\mu(\bar{A}) < 0. \tag{E2.24}$$

Note: *References include (Molchanov and Pyatnitsky, 1986; Kiendl et al., 1992; Polanski, 1997). Converse statements are studied in (Polanski, 1997; Loskot et al., 1998). Note that sublevel sets of a Lyapunov function of the form $\|Wx\|_\infty$ are positively-invariant polyhedral sets. The setting of $m = n$ and time-dependent W is studied in (Pastravanu and Matcovschi, 2010).*

Answer: One direct proof is by defining $y = Wx$ and noting that $\dot{y} = \bar{A}y$ because of equation (E2.23). Then the Coppel inequality (2.7) completes the proof that the norm of y decreases to zero exponentially fast.

Here is an alternative proof, based on the notion of weak pairing satisfying the curve norm derivative formula. From $\dot{x} = Ax$, and from (E2.23), we obtain:

$$\|Wx(t)\| D^+\|Wx(t)\| = [\![W\dot{x}(t)\,;\, Wx(t)]\!] = [\![WAx(t)\,;\, Wx(t)]\!] = [\![\bar{A}Wx(t)\,;\, Wx(t)]\!]\,.$$

By the homogeneity of the weak pairing and by Lumer's equality (2.59), we obtain

$$\|Wx(t)\| D^+\|Wx(t)\| = [\![\bar{A}Wx(t)\,;\, Wx(t)]\!] \leq \sup_{\|y\|=1} [\![\bar{A}y\,;\, y]\!]\ \|Wx(t)\|^2 = \mu(\bar{A})\|Wx(t)\|^2.$$

Therefore, applying a generalization of the Grönwall comparison lemma in Exercise E2.1, we obtain

$$D^+\|Wx(t)\| \leq \mu(\bar{A})\|Wx(t)\| \quad \Longrightarrow \quad \|Wx(t)\| \leq \|Wx(0)\|\, e^{\mu(\bar{A})t}\,.$$

E2.17 **On the matrix exponential and commutator (Moler and Loan, 2003, Lemma 7).** Let $\mu(\cdot)$ be the logarithmic norm of a norm $\|\cdot\|$. Show that, for any matrix $X, Y \in \mathbb{R}^{n\times n}$,

$$\|[e^X, Y]\| \leq e^{\mu(X)}\|[X,Y]\| \quad \text{and} \quad \|[e^X, e^Y]\| \leq e^{\mu(X)}\, e^{\mu(Y)}\|[X,Y]\|.$$

Answer: One can see by direct integration that

$$\begin{aligned}[e^X, Y] = e^{tX}\, Y\, e^{(1-t)X}\Big|_0^1 &= \int_0^1 \Big(\frac{d}{dt}\, e^{tX}\, Y\, e^{(1-t)X}\Big) dt \\ &= \int_0^1 \Big(e^{tX}[X,Y]\, e^{(1-t)X}\Big) dt.\end{aligned}$$

We now upper bounds the right hand side as follows:

$$\|[e^X, Y]\| \leq \|[X,Y]\| \int_0^1 \Big(\|e^{tX}\|\|e^{(1-t)X}\|\Big) dt.$$

The claim now follows from Coppel inequality $\|e^{tX}\| \leq e^{\mu(X)t}$.

E2.18 **Logarithmic norms of principal submatrices and totally Hurwitz matrices (Davydov et al., 2022c).** Given a matrix $A \in \mathbb{R}^n$ and a non-empty index set $\mathcal{I} \subset \{1,\dots,n\}$, let $A_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{I}|}$ denote the *principal submatrix* obtained by removing the rows and columns of A which are not in the index $\mathcal{I}$. Next, given a non-empty $\mathcal{I} \subset \{1,\dots,n\}$, define the *zero-padding map* $\mathrm{pad}_{\mathcal{I}}\colon \mathbb{R}^{|\mathcal{I}|} \to \mathbb{R}^n$ as follows: $\mathrm{pad}_{\mathcal{I}}(y)$ is obtained by inserting zeros among the entries of y corresponding to the indices in $\{1,\dots,n\} \setminus \mathcal{I}$. For example, with $n = 3$ and $\mathcal{I} = \{1,3\}$, we define $\mathrm{pad}_{\{1,3\}}(y_1,y_2) = (y_1, 0, y_2)$.

Given a norm $\|\cdot\|$ on $\mathbb{R}^n$ and a non-empty $\mathcal{I} \subset \{1,\dots,n\}$, show that

(i) the map $\|\cdot\|_{\mathcal{I}} : \mathbb{R}^{|\mathcal{I}|} \to \mathbb{R}_{\geq 0}$ defined by $\|y\|_{\mathcal{I}} = \|\mathrm{pad}_{\mathcal{I}}(y)\|$ is a norm on $\mathbb{R}^{|\mathcal{I}|}$.

Next, assume $\|\cdot\|$ is monotonic, let $\mu(\cdot)$ and $\mu_{\mathcal{I}}(\cdot)$ denote the logarithmic norms associated to $\|\cdot\|$ and $\|\cdot\|_{\mathcal{I}}$ respectively, and show that any matrix $A \in \mathbb{R}^{n\times n}$ satisfies

(ii) $\|A_{\mathcal{I}}\|_{\mathcal{I}} \leq \|A\|$,

(iii) $\mu_{\mathcal{I}}(A_{\mathcal{I}}) \leq \mu(A)$,

(iv) if $\mu(A) < 0$, then A is *totally Hurwitz*, i.e., all its principal submatrices are Hurwitz.

Note: *For a related discussion see (Johnson, 1975). Note that if a matrix A is totally Hurwitz, then $-A$ is known to be a $\mathcal{P}$-matrix (i.e., a matrix all of whose principal minors are positive), see (Fiedler and Ptak, 1962; Tsatsomeros, 2004).*

Answer: Regarding statement (i), it is immediate to check that the restricted map $\|\cdot\|_{\mathcal{I}}$ indeed satisfies four properties of a norm.

Regarding statement (i), let $D_{\mathcal{I}}$ denote the diagonal matrix with entries $(D_{\mathcal{I}})_{ii} = 1$ if $i \in \mathcal{I}$ and $(D_{\mathcal{I}})_{ii} = 0$ if $i \notin \mathcal{I}$. It is useful to see this definitions in action; given an arbitrary matrix $A = \left[\begin{smallmatrix}1&2&3\\4&5&6\\7&8&9\end{smallmatrix}\right] \in \mathbb{R}^{3\times 3}$ and the arbitrary index set $\{1,3\}$, write

$$A_{\{1,3\}} = \begin{bmatrix}1 & 3\\ 7 & 9\end{bmatrix}, \quad D_{\{1,3\}} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}, \quad \text{and } D_{\{1,3\}}AD_{\{1,3\}} = \begin{bmatrix}1&0&3\\0&0&0\\7&0&9\end{bmatrix}.$$

Note also

$$\mathrm{pad}_{\{1,3\}}\left(A_{\{1,3\}}\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) = \begin{bmatrix}1y_1+3y_2\\0\\7y_1+9y_2\end{bmatrix} = (D_{\{1,3\}}AD_{\{1,3\}})\begin{bmatrix}y_1\\0\\y_2\end{bmatrix}.$$

With this notation, we are ready to compute

$$\begin{aligned}
\|A_{\mathcal{I}}\|_{\mathcal{I}} &= \max_{y\in\mathbb{R}^{|\mathcal{I}|}, \|y\|_{\mathcal{I}}=1} \|A_{\mathcal{I}}\, y\|_{\mathcal{I}} \\
&= \max_{y\in\mathbb{R}^{|\mathcal{I}|}, \|y\|_{\mathcal{I}}=1} \|\mathrm{pad}_{\mathcal{I}}(A_{\mathcal{I}}y)\| && \text{(by the definition of } \|\cdot\|_{\mathcal{I}}\text{)}, \\
&= \max_{y\in\mathbb{R}^{|\mathcal{I}|}, \|\mathrm{pad}_{\mathcal{I}}(y)\|=1} \|(D_{\mathcal{I}}AD_{\mathcal{I}})\ \mathrm{pad}_{\mathcal{I}}(y)\| && \text{(by direct inspection)} \\
&\leq \max_{x\in\mathbb{R}^n, \|x\|=1} \|(D_{\mathcal{I}}AD_{\mathcal{I}})\, x\| && \text{(relaxing the constraint)} \\
&= \|D_{\mathcal{I}}AD_{\mathcal{I}}\| \leq \|D_{\mathcal{I}}\|\|A\|\|D_{\mathcal{I}}\| = \|A\|.
\end{aligned}$$

The last equality holds because, from Theorem 2.23(iii), the monotonicity of $\|\cdot\|$ implies $\|D_{\mathcal{I}}\| = 1$. This concludes the proof of (ii).

Statement (iii) follows from the definition of logarithmic norm and applying statement (ii) to the matrix $I_{|\mathcal{I}|} + hA_{\mathcal{I}}$ as a principal submatrix of $I_n + hA$:

$$\mu_{\mathcal{I}}(A_{\mathcal{I}}) := \lim_{h\to 0^+} \frac{\|I_{|\mathcal{I}|} + hA_{\mathcal{I}}\|_{\mathcal{I}} - 1}{h} \leq \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h} = \mu(A).$$

Finally, statement (iv) is an immediate consequence of (iii).

E2.19 **A sign contractivity property of Metzler matrices (Jafarpour, 2019).** Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix and let $\xi > \mathbb{0}_n$ satisfy $\xi^{\top} M \leq \mathbb{0}_n^{\top}$. Recall the sign pairing $[\![x\,;\,y]\!]_{1,[\eta]} = \|y\|_{1,[\eta]}\,\mathrm{sign}(y)^{\top}[\eta]x$. Show that the following statements hold:

(i) every $x \in \mathbb{R}^n$ satisfies

$$[\![Mx\,;\,x]\!]_{1,[\xi]} \leq 0, \quad \text{or equivalently} \quad \mathrm{sign}(x)^{\top}\,\mathrm{diag}(\xi)Mx \leq 0, \tag{E2.25}$$

(ii) the equality (E2.25) holds if and only if $x \in \ker(M)$.

Note the following three implications:

(i) if M is Hurwitz, then there exists $\xi > 0$ such that $\xi^\top M < \mathbb{0}_n^\top$ and $\ker(M) = \{\mathbb{0}_n\}$,

(ii) if M is an outflow-connected compartmental matrix, then $\mathbb{1}_n^\top M \leq \mathbb{0}_n^\top$ and $\ker(M) = \{\mathbb{0}_n\}$, and

(iii) if M is a transpose Laplacian matrix (not Hurwitz), then $\mathbb{1}_n^\top M = \mathbb{0}_n^\top$ and $\ker(M) = \operatorname{span}\{\mathbb{1}_n\}$.

Note: *A weak version of this inequality is used in (Como, 2017, Proposition 2); earlier references include (Maeda et al., 1978, Theorem 2) and (Jacquez and Simon, 1993, Appendix 4).*

Answer: Statement (i) follows from an application of Lumer's equality (WP6) in equation (2.59). Indeed, Lumer's equality implies that, for each x,

$$[\![Ax\,;\,x]\!]_{1,[\xi]} \leq \mu_{1,[\xi]}(M)\|x\|_{1,[\xi]}^2 \leq 0.$$

where the second inequality follows from recalling that $\xi^\top M \leq \mathbb{0}_n^\top$ if and only if $\mu_{1,[\xi]}(M) < 0$.

We leave the remaining steps to the reader.

E2.20 **A max contractivity property of Metzler matrices (Smith et al., 2022).** Let M be an irreducible Metzler matrix with zero row sums $M\mathbb{1}_n = \mathbb{0}_n$. For a given vector $x \in \mathbb{R}^n$, define $I_\infty(x) = \{i \in \{1,\dots,n\} \,:\, |x_i| = \|x\|_\infty\}$ as the set of indices where x takes its maximal absolute value. Show that

(i) every $x \in \mathbb{R}^n$ satisfies

$$\max_{i\in I_\infty(x)} \{\operatorname{sign}(x_i)\,(Mx)_i\} \leq 0,$$

(ii) the equality holds if and only if $x \in \operatorname{span}\{\mathbb{1}_n\}$.

Answer: The following proof is taken from (Smith et al., 2022, Lemma 3). For every $i \in I_\infty(x)$, we compute

$$\operatorname{sign}(x_i)(Mx)_i = -\sum_{j\neq i} m_{ij}|x_i| + \sum_{j\neq i} m_{ij}x_j(t)\operatorname{sign}(x_i(t)),$$

and, in turn,

$$\begin{aligned}\operatorname{sign}(x_i)(Mx)_i &\leq -\sum_{j\neq i} m_{ij}|x_i| + \sum_{j\neq i} m_{ij}|x_j| \\ &= \sum_{j=1}^n m_{ij}(|x_j| - |x_i|).\end{aligned}$$

Since $i \in I_\infty(x)$, we know $|x_i| = \|x\|_\infty \geq |x_j|$ and therefore $\sum_{j=1}^n m_{ij}(|x_j| - |x_i|) \leq 0$. This proves that $\operatorname{sign}(x_i)(Mx)_i \leq 0$, for every $i \in I_\infty(x)$. To prove the strict inequality, note that $\max_{i\in I_\infty(x)} \operatorname{sign}(x_i)(Mx)_i = 0$ if and only if, for every $i \in I_\infty(x)$,

$$\operatorname{sign}(x_i)\Big(\sum_{j=1}^n m_{ij}(x_i - x_j)\Big) = 0. \tag{E2.26}$$

If $\operatorname{sign}(x_i) = 0$ for some $i \in I_\infty(x)$, then $x_i = 0$ and as a result $x(t) = \mathbb{0}_n$. Now suppose that, for every $i \in I_\infty(x)$, $\operatorname{sign}(x_i) \neq 0$. In this case, using the equality (E2.26), we obtain

$$\sum_{j=1}^n m_{ij}(x_i - x_j) = 0, \qquad \text{for all } i \in I_\infty(x).$$

Note that, for every $(i,j) \in \mathcal{E}$, we have $m_{ij} > 0$. Therefore, for every $i \in I_\infty(x)$ and every j such that $(i,j) \in \mathcal{E}$, we have $x_i = x_j$. This means that, for every $i \in I_\infty(x)$, the out-neighbors of i in G are also in the set $I_\infty(x)$. Since G is a strongly connected digraph, we conclude that $I_\infty(x) = \{1,\dots,n\}$. Therefore, $x = \mathbb{0}_n$.

E2.21 **Direct parametrization of matrices satisfying μ_∞ constraint (Jafarpour et al., 2021b).** For a matrix $A \in \mathbb{R}^{n\times n}$ and $\eta \in \mathbb{R}^n_{>0}$, prove that the following statements are equivalent:

(i) $\mu_{\infty,[\eta]^{-1}}(A) \leq \gamma$,

(ii) there exists $T \in \mathbb{R}^{n\times n}$ s.t. $A = [\eta]T[\eta]^{-1} + |T|\mathbb{1}_n + \gamma I_n$,

(iii) there exists $T \in \mathbb{R}^{n\times n}$ s.t. $A = T + [[\eta]^{-1}|T|[\eta]\mathbb{1}_n] + \gamma I_n$.

Answer: This lemma is a generalization of Lemma 9 in (Jafarpour et al., 2021b).

E2.22 **Log norms of matrices multiplied by a diagonal uncertainty (Davydov et al., 2022c).** For $A \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, $0 \leq d_{\min} \leq d_{\max} \in \mathbb{R}$, and $\eta \in \mathbb{R}^n_{>0}$, show the following generalizations of Lemma 2.15:

$$\max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]}([c]+[d]A) = \max\big\{\mu_{\infty,[\eta]}([c]+d_{\min}A), \mu_{\infty,[\eta]}([c]+d_{\max}A)\big\}, \tag{E2.27}$$

$$\max_{d\in[d_{\min},d_{\max}]^n} \mu_{1,[\eta]}([c]+A[d]) = \max\big\{\mu_{1,[\eta]}([c]+d_{\min}A), \mu_{1,[\eta]}([c]+d_{\max}A)\big\}, \tag{E2.28}$$

$$\max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]}([c]+A[d]) = \max\{\mu_{\infty,[\eta]}([c]+d_{\max}A), \mu_{\infty,[\eta]}([c]+d_{\max}A-(d_{\max}-d_{\min})(I_n\circ A))\}, \tag{E2.29}$$

$$\max_{d\in[d_{\min},d_{\max}]^n} \mu_{1,[\eta]}([c]+[d]A) = \max\{\mu_{1,[\eta]}([c]+d_{\max}A), \mu_{1,[\eta]}([c]+d_{\max}A-(d_{\max}-d_{\min})(I_n\circ A))\}. \tag{E2.30}$$

Note: *These formulas generalize previous results (Fang and Kincaid, 1996, Theorem 3.8), (He and Cao, 2009, Lemma 3) and (Jafarpour et al., 2021b, Lemma 8).*

Answer: We start by proving formula (E2.27). Since $0 \leq d_{\min} \leq d_{\max}$, we know the off-diagonal terms satisfy $|([c]+[d]A)|_{ij} = |0 + d_i a_{ij}| = d_i|a_{ij}|$ for $j \neq i$. For each row i, we define the *weighted absolute row-sum* $\mathbb{r}_i = a_{ii} + \sum_{j=1,j\neq i}^n |a_{ij}|\eta_i/\eta_j \in \mathbb{R}$ and rewrite

$$\begin{aligned}\mu_{\infty,[\eta]}([c]+[d]A) &= \max_{i\in\{1,...,n\}} \Big\{([c]+[d]A)_{ii} + \sum\nolimits_{j=1,j\neq i}^n |([c]+[d]A)|_{ij}\eta_i/\eta_j\Big\} \\ &= \max_{i\in\{1,...,n\}} \{c_i + d_i\mathbb{r}_i\}.\end{aligned}$$

We consider separately the cases of $\mathbb{r}_i > 0$ and $\mathbb{r}_i \leq 0$:

$$\begin{aligned}\max_{d\in[d_{\min},d_{\max}]^n} \max_{i:\mathbb{r}_i>0} \{c_i + d_i\mathbb{r}_i\} &= \max_{i:\mathbb{r}_i>0} \max_{d_i\in[d_{\min},d_{\max}]} \{c_i + d_i\mathbb{r}_i\} \\ &= \max_{i:\mathbb{r}_i>0} \{c_i + d_{\max}\mathbb{r}_i\} \leq \mu_{\infty,[\eta]}([c]+d_{\max}A), \\ \max_{d\in[d_{\min},d_{\max}]^n} \max_{i:\mathbb{r}_i\leq 0} \{c_i + d_i\mathbb{r}_i\} &= \max_{i:\mathbb{r}_i\leq 0} \max_{d_i\in[d_{\min},d_{\max}]} \{c_i + d_i\mathbb{r}_i\} \\ &= \max_{i:\mathbb{r}_i>0} \{c_i + d_{\min}\mathbb{r}_i\} \leq \mu_{\infty,[\eta]}([c]+d_{\min}A).\end{aligned}$$

In summary, we obtain

$$\begin{aligned}\max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]}([c]+[d]A) &= \max_{d\in[d_{\min},d_{\max}]^n} \max_{i\in\{1,...,n\}} \{c_i + d_i\mathbb{r}_i\} \\ &= \max_{d\in[d_{\min},d_{\max}]^n} \max\Big\{\max_{i:\mathbb{r}_i\leq 0}\{c_i + d_i\mathbb{r}_i\}, \max_{i:\mathbb{r}_i>0}\{c_i + d_i\mathbb{r}_i\}\Big\} \\ &\leq \max\big\{\mu_{\infty,[\eta]}([c]+d_{\min}A), \mu_{\infty,[\eta]}([c]+d_{\max}A)\big\}.\end{aligned}$$

On the other hand, the maximum of the function $d \mapsto f(d) = \mu_{\infty,[\eta]}([c]+[d]A)$ is lower-bounded by the value of the function at two specific points $d_{\min}\mathbb{1}_n$ and $d_{\max}\mathbb{1}_n$ so that

$$\begin{aligned}\max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]}([c]+[d]A) &\geq \max\big\{\mu_{\infty,[\eta]}([c]+d_{\min}[\mathbb{1}_n]A), \mu_{\infty,[\eta]}([c]+d_{\max}[\mathbb{1}_n]A)\big\} \\ &= \max\big\{\mu_{\infty,[\eta]}([c]+d_{\min}A), \mu_{\infty,[\eta]}([c]+d_{\max}A)\big\}.\end{aligned}$$

This completes the proof of the first equality (E2.27). Next, regarding formula (E2.28), recall that $\mu_{1,[\eta]}(B) = \mu_{\infty,[\eta]^{-1}}(B^\top)$ for any matrix $B \in \mathbb{R}^{n\times n}$ and compute

$$\begin{aligned}\max_{d\in[d_{\min},d_{\max}]^n} \mu_{1,[\eta]}([c]+A[d]) &= \max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]^{-1}}([c]+[d]A^\top)\\ &= \max\left\{\mu_{\infty,[\eta]^{-1}}([c]+d_{\min}A^\top), \mu_{\infty,[\eta]^{-1}}([c]+d_{\max}A^\top)\right\}\\ &= \max\left\{\mu_{1,[\eta]}([c]+d_{\min}A), \mu_{1,[\eta]}([c]+d_{\max}A)\right\}.\end{aligned}$$

This concludes the proof of equality (E2.28).

Regarding formula (E2.29), we compute

$$\begin{aligned}\max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]}([c]+A[d]) &= \max_{d\in[d_{\min},d_{\max}]^n}\ \max_{i\in\{1,...,n\}} \left\{c_i + a_{ii}d_i + \sum\nolimits_{j=1,j\neq i}^n \frac{\eta_i}{\eta_j}|a_{ij}d_j|\right\}\\ &\leq \max_{i\in\{1,...,n\}}\ \max_{d\in[d_{\min},d_{\max}]^n} \left\{c_i + a_{ii}d_i + \sum\nolimits_{j=1,j\neq i}^n \frac{\eta_i}{\eta_j}|a_{ij}d_j|\right\}\\ &\leq \max_{i\in\{1,...,n\}}\ \max_{d_i\in[d_{\min},d_{\max}]} \left\{c_i + a_{ii}d_i + \sum\nolimits_{j=1,j\neq i}^n \frac{\eta_i}{\eta_j}|d_{\max}a_{ij}|\right\}\\ &\leq \max_{i\in\{1,...,n\}} \begin{cases} c_i + d_{\max}a_{ii} + \sum\nolimits_{j=1,j\neq i}^n \frac{\eta_i}{\eta_j}|d_{\max}a_{ij}|, & \text{if } a_{ii}\geq 0\\ c_i + d_{\min}a_{ii} + \sum\nolimits_{j=1,j\neq i}^n \frac{\eta_i}{\eta_j}|d_{\max}a_{ij}|, & \text{if } a_{ii}< 0\end{cases}\\ &\leq \max\{\mu_{\infty,[\eta]}([c]+d_{\max}A), \mu_{\infty,[\eta]}([c]+d_{\max}A-(d_{\max}-d_{\min})(I_n\circ A))\}.\end{aligned}$$

On the other hand, compute $i^\star \in \operatorname{argmax}_{i\in\{1,...,n\}} d_{\min}a_{ii} + \sum_{j\neq i}\frac{\eta_i}{\eta_j}|d_{\max}a_{ij}|$ and define $d^\star = d_{\max}\mathbb{1}_n - (d_{\max} - d_{\min})\mathbb{e}_{i^\star}$, where $\mathbb{e}_{i^\star}$ is the vector with $i^\star$-th entry equal to 1 and all other entries equal to 0. With this definition of $d^\star$, we compute

$$\begin{aligned}\max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]}([c]+A[d]) &\geq \max\{\mu_{\infty,[\eta]}([c]+d_{\max}A[\mathbb{1}_n]), \mu_{\infty,[\eta]}([c]+d_{\max}A[d^\star])\}\\ &= \max\{\mu_{\infty,[\eta]}([c]+d_{\max}A), \mu_{\infty,[\eta]}([c]+d_{\max}A-(d_{\max}-d_{\min})(I_n\circ A))\},\end{aligned}$$

thus concluding the proof of formula (E2.29). Regarding formula (E2.30), we again leverage the fact that $\mu_{1,[\eta]}(B) = \mu_{\infty,[\eta]^{-1}}(B^\top)$ to see that

$$\begin{aligned}\max_{d\in[d_{\min},d_{\max}]^n} \mu_{1,[\eta]}([c]+[d]A) &= \max_{d\in[d_{\min},d_{\max}]^n} \mu_{\infty,[\eta]^{-1}}([c]+A^\top[d])\\ &= \max\{\mu_{\infty,[\eta]^{-1}}([c]+d_{\max}A^\top), \mu_{\infty,[\eta]^{-1}}([c]+d_{\max}A^\top-(d_{\max}-d_{\min})(I_n\circ A))\}\\ &= \max\{\mu_{1,[\eta]}([c]+d_{\max}A), \mu_{1,[\eta]}([c]+d_{\max}A-(d_{\max}-d_{\min})(I_n\circ A))\}.\end{aligned}$$

Thus, formulas (E2.27)-(E2.30) have been proved.

E2.23 **ℓ_2 log norms of matrices multiplied by a diagonal uncertainty.** Given $A\in\mathbb{R}^{n\times n}$, define $|A|_{\mathrm{M+}} \in \mathbb{R}^n_{\geq 0}$ by

$$(|A|_{\mathrm{M+}})_{ij} = \begin{cases}(a_{ii})_+, & \text{if } i=j,\\ |a_{ij}|, & \text{if } i\neq j.\end{cases} \tag{E2.31}$$

Show that, for any monotonic norm (and in particular for ℓ_2) with weight $\eta\in\mathbb{R}^n_{>0}$,

(i) $\max\{\mu_{[\eta]}([d]A), \mu_{[\eta]}(A[d])\} \leq \mu_{[\eta]}(|A|_{\mathrm{M+}})$ for all $d\in[0,1]^n$;

(ii) $\inf_{\eta\in\mathbb{R}^n_{>0}}\ \max_{d\in[0,1]^n} \mu_{[\eta]}([d]A) \leq \alpha(|A|_{\mathrm{M+}})$;

(iii) $\max\{\mu_{[\eta]}([d]A), \mu_{[\eta]}(A[d])\} \leq \|A\|_{[\eta]}$ for all $d\in[0,1]^n$.

Answer: We apply repeatedly the monotonicity inequalities (2.78b) in Theorem 2.24 to compute

$$\begin{aligned}\mu_{[\eta]}([d]A) &\leq \mu_{[\eta]}(|[d]A|_{\mathrm{M}}) = \mu_{[\eta]}([d]|A|_{\mathrm{M}}) \\ &\leq \mu_{[\eta]}([d]|A|_{\mathrm{M}^+}) \leq \mu_{[\eta]}(|A|_{\mathrm{M}^+}).\end{aligned}$$

The proof for the $A[d]$ matrix is identical. This proves statement (i).

Statement (ii) is an immediate consequence because of Lemma 2.31:

$$\inf_{\eta\in\mathbb{R}^n_{>0}} \max_{d\in[0,1]^n} \mu_{[\eta]}([d]A) \leq \inf_{\eta\in\mathbb{R}^n_{>0}} \mu_{[\eta]}(|A|_{\mathrm{M}^+}) = \alpha(|A|_{\mathrm{M}^+}).$$

Regarding statement (iii), we apply the spectral-abscissa log-norm property (2.73)

$$\begin{aligned}\mu_{[\eta]}([d]A) &\leq \|[d]A\|_{[\eta]} && \text{(spectral-abscissa log-norm property (2.73))} \\ &\leq \|[d]\|_{[\eta]}\|A\|_{[\eta]} && \text{(sub-multiplicativity property (2.29d))} \\ &= \|A\|_{[\eta]} && \text{(because the norm is monotonic, by (2.74))}\end{aligned}$$

The proof for the $A[d]$ matrix is identical. This proves statement (iii).

E2.24 **Norms and log norms conditions (Jafarpour et al., 2021b).** Consider $A(\delta,\gamma) = \begin{bmatrix}\delta & \gamma \\ \gamma & \delta\end{bmatrix}$, for $\delta,\gamma\in\mathbb{R}$, and recall $\|A(\delta,\gamma)\|_\infty = |\delta|+|\gamma|$ and $\mu_\infty(A(\delta,\gamma)) = \delta+|\gamma|$. Define the average matrix $A_\alpha(\delta,\gamma)$, for $\alpha\in(0,1]$, by $A_\alpha(\delta,\gamma) = (1-\alpha)I_2 + \alpha A(\delta,\gamma)$. Figure E2.1 compares the set of parameters (δ,γ) such that $\|A(\delta,\gamma)\|_\infty < 1$, $\|A_\alpha(\delta,\gamma)\|_\infty < 1$, and $\mu_\infty(A(\delta,\gamma)) < 1$.

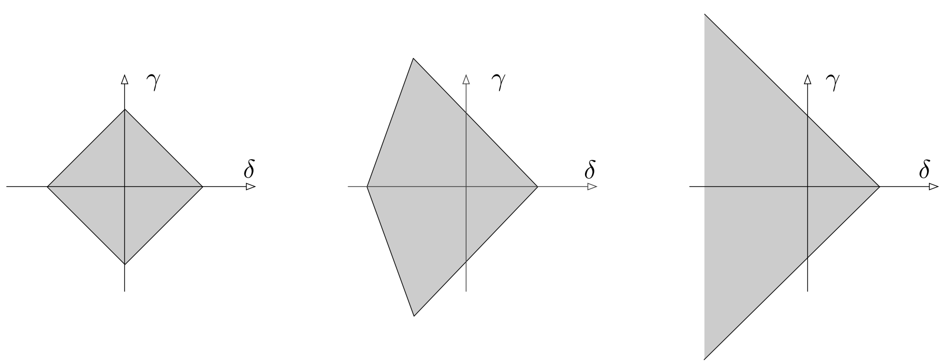


Figure E2.1: The sets $\{(\delta,\gamma) : \|A(\delta,\gamma)\|_\infty < 1\}$, $\{(\delta,\gamma) : \|A_\alpha(\delta,\gamma)\|_\infty < 1\}$ and $\{(\delta,\gamma) : \mu_\infty(A(\delta,\gamma)) < 1\}$.

Show that, as $\alpha\to 0^+$, the set $\{(\delta,\gamma) : \|A_\alpha(\delta,\gamma)\|_\infty < 1\}$ converges to the set $\{(\delta,\gamma) : \mu_\infty(A(\delta,\gamma)) < 1\}$.

Answer: We leave the answer to the reader.

E2.25 **Monotonicity of norms and weak pairings (Jafarpour et al., 2023).** For any monotonic norm and compatible weak pairing satisfying the Deimling's inequality, show

(i) for every $x \leq \mathbb{0}_n$ and $y > \mathbb{0}_n$, we have $[\![x\,;\,y]\!] \leq 0$; and
(ii) for every $x_1 \leq x_2$ and $y > \mathbb{0}_n$, we have $[\![x_1\,;\,y]\!] \leq [\![x_2\,;\,y]\!]$.

Answer: Regarding statement (i), for any sufficiently small positive h, we have $|(y+hx)_i| \leq |y_i|$ for all i. Since the norm is monotonic, this implies $\|y+hx\| \leq \|y\|$. The result then follows from Deimling's inequality:

$$[\![x\,;\,y]\!] \leq \|y\| \lim_{h\to 0^+} \frac{\|y+hx\|-\|y\|}{h} \leq 0.$$

Regarding part (ii), define $w \geq \mathbb{0}_n$ such that $x_2 = x_1 + w$. Using the subadditivity of the weak pairing, we obtain

$$[\![x_1\,;\,y]\!] = [\![x_2 - w\,;\,y]\!] \leq [\![x_2\,;\,y]\!] + [\![-w\,;\,y]\!] \leq [\![x_2\,;\,y]\!]\,,$$

where the last inequality holds because of statement (i).

E2.26 **Optimal norms for products of symmetric matrices.** Let $A_1 = SP \in \mathbb{R}^{n\times n}$ and $A_2 = PS \in \mathbb{R}^{n\times n}$ where $S = S^\top$ and $P = P^\top \succ 0$. Show that, for each $i \in \{1,2\}$,

(i) $\operatorname{spec}(A_i)$ is real and has the same number of negative, zero, positive eigenvalues as S.

Hint: *Use the Wikipedia: Sylvester's law of inertia: Two symmetric $n \times n$ matrices B and C are congruent if there exists an invertible matrix T such that $B = T^\top CT$. Then, two symmetric $n \times n$ matrices are congruent if and only if they have the same number of positive, negative and zero eigenvalues.*

(ii) the norm $\|\cdot\|_{2,P^{1/2}}$ is optimal and log optimal for A_i, that is

$$\|A_i\|_{2,P^{1/2}} = \rho(A_i), \qquad \text{and} \qquad \mu_{2,P^{1/2}}(A_i) = \alpha(A_i). \tag{E2.32}$$

Answer: We show both statements for $i = 1$; the proof for $i = 2$ is a straightforward adaptation. Regarding statement (i), note that A_1 is similar to the symmetric matrix $P^{1/2}SP^{1/2}$, hence $\operatorname{spec}(A_1)$ is real. Additionally, the matrix $P^{1/2}SP^{1/2}$ is congruent to S and the claim follows from Sylvester's law of inertia.

Regarding statement (ii), we compute

$$\|A_1\|^2_{2,P^{1/2}} = \lambda_{\max}(P^{-1}A_1^\top PA_1) = \lambda_{\max}\Big(P^{-1}(PS)P(SP)\Big) = \lambda_{\max}((SP)^2) = \rho(SP)^2,$$

where the last equality follows from the fact that $(SP)^2$ has the same eigenvectors as SP and real eigenvalues equal to the square of the real eigenvalues of SP. Finally, we compute

$$\begin{aligned}\mu_{2,P^{1/2}}(A_1) &= \lambda_{\max}\Big(\frac{PA_1P^{-1} + A_1^\top}{2}\Big) = \lambda_{\max}\Big(\frac{PSPP^{-1} + PS}{2}\Big)\\ &= \lambda_{\max}(PS) = \lambda_{\max}(PSPP^{-1}) = \lambda_{\max}(PA_1P^{-1}) = \lambda_{\max}(A_1) = \alpha(A_1).\end{aligned}$$

This concludes the proof of statement (ii).

E2.27 **Norms and log norms of circulant matrices.** For $c \in \mathbb{R}^n$, let $C = \operatorname{circ}(c) \in \mathbb{R}^{n\times n}$ denote the *circulant matrix* whose first row is equal to c. Recall that circulant matrices commute and that the set of circulant matrices is closed under scalar product and sum. Show

(i) the ℓ_2 norm is optimal and logarithmically optimal;

(ii) $\|C\|_1 = \|C\|_\infty = \sum_{j=1}^n |c_j|$ and $\mu_1(C) = \mu_\infty(C) = c_1 + \sum_{j=2}^n |c_j|$;

(iii) $\mu_2(\operatorname{circ}(0,1,-1)) = 0$, whereas $\mu_1(\operatorname{circ}(0,1,-1)) = 2$;

(iv) if $c_2,\dots,c_n \geq 0$, then C is Metzler and $\sum_{i=1}^n c_i = \alpha(C) = \mu_p(C)$, for $p \in \{1,2,\infty\}$.

Answer: Statement (i) follows from the fact that C is normal and it is known that $\|C\|_2 = \rho(C)$ for any normal matrix C. We leave the proof of the log norm optimality to the reader.

Statements (ii)-(iii) follow from applying the formulas for the log norms. Statement (iv) follows from noting that $\sum_{i=1}^n c_i$ is the eigenvalue of $\operatorname{circ}(c)$ with eigenvector $\mathbb{1}_n$.

E2.28 **The ℓ_2 logarithmic norm of block matrices with off-diagonal skew-symmetric structure.** Consider the block matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)},$$

Assume $P_1 = P_1^\top \succ 0$ in $\mathbb{R}^{n\times n}$ and $P_2 = P_2^\top \succ 0$ in $\mathbb{R}^{m\times m}$ verify the off-diagonal skew-symmetric equality:

$$P_1 A_{12} = -A_{21}^\top P_2. \tag{E2.33}$$

Show that $\mu_{2,P^{1/2}}(A) = \max\left\{\mu_{2,P_1^{1/2}}(A_{11}), \mu_{2,P_2^{1/2}}(A_{22})\right\}$.

Note: *(Kozachkov et al., 2020) consider the case where P_2 and A_{22} are diagonal and $P_1 = I_n$. Saddle point matrices, arising for example in primal dual optimization problems, are example of matrices satisfying (E2.33), see Exercise E4.8.*

Answer: First, we note that $P_1 A_{12} = -A_{21}^\top P_2$ implies $P_1^{1/2} A_{12} P_2^{-1/2} = -P_1^{-1/2} A_{21}^\top P_2^{1/2} = -(P_2^{1/2} A_{21} P_1^{-1/2})^\top$. Next, we compute

$$\begin{aligned}
\mu_{2,P^{1/2}}(A) &= \mu_2\left(\begin{bmatrix} P_1^{1/2} & 0 \\ 0 & P_2^{1/2}\end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix} P_1^{-1/2} & 0 \\ 0 & P_2^{-1/2}\end{bmatrix}\right) \\
&= \mu_2\left(\begin{bmatrix} P_1^{1/2} A_{11} P_1^{-1/2} & P_1^{1/2} A_{12} P_2^{-1/2} \\ P_2^{1/2} A_{21} P_1^{-1/2} & P_2^{1/2} A_{22} P_2^{-1/2}\end{bmatrix}\right) \\
&= \mu_2\left(\begin{bmatrix} P_1^{1/2} A_{11} P_1^{-1/2} & 0 \\ 0 & P_2^{1/2} A_{22} P_2^{-1/2}\end{bmatrix} + \begin{bmatrix} 0 & P_1^{1/2} A_{12} P_2^{-1/2} \\ P_2^{1/2} A_{21} P_1^{-1/2} & 0\end{bmatrix}\right) \\
&= \mu_2\left(\begin{bmatrix} P_1^{1/2} A_{11} P_1^{-1/2} & 0 \\ 0 & P_2^{1/2} A_{22} P_2^{-1/2}\end{bmatrix}\right),
\end{aligned}$$

where the last equality follows from Exercise E2.12(iii) because the matrix

$$\begin{bmatrix} 0 & P_1^{1/2} A_{12} P_2^{-1/2} \\ P_2^{1/2} A_{21} P_1^{-1/2} & 0\end{bmatrix}$$

is skew symmetric. Finally, from the LMI characterization of the ℓ_2 logarithmic norm, we obtain

$$\mu_{2,P^{1/2}}(A) = \mu_2\left(\begin{bmatrix} P_1^{1/2} A_{11} P_1^{-1/2} & 0 \\ 0 & P_2^{1/2} A_{22} P_2^{-1/2}\end{bmatrix}\right) = \max\left\{\mu_{2,P_1^{1/2}}(A_{11}), \mu_{2,P_2^{1/2}}(A_{22})\right\}.$$

E2.29 **Small gain condition with respect to ℓ_2.** Consider a symmetric matrix $A \in \mathbb{R}^{n\times n}$ partitioned in blocks $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22}\end{bmatrix}$. Show that $\mu_2(A) < 0$ if

$$\lambda_{\max}(A_{11}) < 0, \quad \lambda_{\max}(A_{22}) < 0, \quad \text{and} \quad \lambda_{\max}(A_{12}^\top A_{12}) < \lambda_{\max}(A_{11})\lambda_{\max}(A_{22}). \tag{E2.34}$$

Answer: As in Section 2.4.4 on composite norms, let n_i denote the dimension of A_{ii}, $i \in \{1,2\}$, with $n_1 + n_2 = n$.

First, we note that, if each local norm (on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$) is the ℓ_2 norm and the aggregating norm on $\mathbb{R}^2$, is the ℓ_2 norm, then the composite norm is the ℓ_2 norm.

Second, we compute the aggregate Metzler majorant of A to be

$$\lceil A \rceil_{\mathrm{M}} = \begin{bmatrix} \mu_2(A_{11}) & \|A_{12}\|_{2,2} \\ \|A_{21}\|_{2,2} & \mu_2(A_{22}). \end{bmatrix} = \begin{bmatrix} \lambda_{\max}(A_{11}) & \sqrt{\lambda_{\max}(A_{12}^\top A_{12})} \\ \sqrt{\lambda_{\max}(A_{12}^\top A_{12})} & \lambda_{\max}(A_{22}) \end{bmatrix}. \tag{E2.35}$$

Here, from Table 2.1, we used the facts that $A_{ii} = A_{ii}^\top$ implies $\mu_2(A_{ii}) = \lambda_{\max}(A_{ii})$ and $\|A_{12}\|_{2,2} = \|A_{12}\|_2 = \sqrt{\lambda_{\max}(A_{12}^\top A_{12})}$. Additionally, we used the fact that $\|A_{12}^\top\|_2 = \|A_{12}\|_2$.

Clearly, condition (E2.34) implies that $\lceil A \rceil_{\mathrm{M}}$ is Hurwitz (since it has negative trace and positive determinant). The claim now follows from Theorem 2.13 on composite induced log norms: $\mu_2(A) \leq \mu_2(\lceil A \rceil_{\mathrm{M}}) = \lambda_{\max}(\lceil A \rceil_{\mathrm{M}}) < 0$. (Note that, here again, we used the fact that the ℓ_2 norm is log optimal for symmetric matrices.)

Beyond the statement above, one can show that

$$\lambda_{\max}(\lceil A \rceil_{\mathrm{M}}) = \frac{\lambda_{\max}(A_{11}) + \lambda_{\max}(A_{22})}{2} + \sqrt{\lambda_{\max}(A_{12}^\top A_{12}) + (\lambda_{\max}(A_{11}) - \lambda_{\max}(A_{22}))/4} \tag{E2.36}$$

and provide an explicit negative upper bound to $\mu_2(A)$.

E2.30 **The signed Neumann Lemma.** Given an $n \times n$ matrix A and a norm $\|\cdot\|$, if $\mu(A) < 1$, then

(i) $I_n - A$ is invertible, and

(ii) $\|(I_n - A)^{-1}\| \leq \dfrac{1}{1 - \mu(A)}$.

Note: *The classic Neumann Lemma states that $\|A\| < 1$ implies $\|(I_n - A)^{-1}\| \leq \frac{1}{1-\|A\|}$. The bound $\frac{1}{1-\mu(A)}$ in this exercise is at least as tight or tighter because $\mu(A) \leq \|A\|$. Specifically, $0 \leq \|A\| < 1$ always implies $\frac{1}{1-\|A\|} \geq 1$, whereas $\frac{1}{1-\mu(A)} < 1$ when $\mu(A) < 0$.*

Answer: The proof follows from the uniform monotonicity property (2.30f) of the log norm. The details are as follows. Since $\mu(A) < 1$, the translation property (2.30c) implies $\mu(A - I_n) < 0$ so that $A - I_n$ is Hurwitz by the spectral-abscissa log-norm property (2.73). This concludes the proof of statement (i).

Regarding statement (ii),

$$\|(I_n - A)^{-1}\| = \|(A - I_n)^{-1}\| \overset{(2.30f)}{\leq} -\frac{1}{\mu(A - I_n)} = -\frac{1}{\mu(A) - 1} = \frac{1}{1 - \mu(A)}.$$

This concludes the proof of statement (ii).

E2.31 **More properties of weak pairings (Davydov et al., 2022a).** Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with weak pairing $[\![\cdot\,;\cdot]\!]$ satisfying the standing assumptions in Definition 2.18. Show that

(i) for each $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$[\![cx\,; x]\!] = c\|x\|^2; \tag{E2.37}$$

(ii) for each $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$[\![x + cy\,; y]\!] = [\![x\,; y]\!] + c\|y\|^2. \tag{E2.38}$$

Answer: Regarding statement (i), if $c \geq 0$, the result follows from weak homogeneity (WP2). Without loss of generality, assume $c = -1$. Lumer's equality (WP6) in Definition 2.18 with $A = -I_n$ implies that, for all $x \in \mathbb{R}^n$

$$\sup_{x \neq \mathbb{0}_n} \frac{[\![-x\,; x]\!]}{\|x\|^2} = \mu(-I_n) = -1 \quad \implies \quad [\![-x\,; x]\!] \leq -\|x\|^2. \tag{E2.39}$$

Regarding the inequality with the opposite direction, we compute

$$\begin{aligned} [\![x\,; x]\!] = [\![x - x + x\,; x]\!] &\overset{(\mathrm{WP1})}{\leq} [\![-x\,; x]\!] + 2\,[\![x\,; x]\!] \\ \implies [\![-x\,; x]\!] &\geq -\,[\![x\,; x]\!] = -\|x\|^2. \end{aligned}$$

Regarding statement (ii), the inequality $[\![x + cy\,; y]\!] \leq [\![x\,; y]\!] + c\|y\|^2$ follows from subadditivity (WP1) and statement (i). Additionally,

$$\begin{aligned} [\![x\,; y]\!] = [\![x + cy - cy\,; y]\!] &\overset{(\mathrm{WP1})}{\leq} [\![x + cy\,; y]\!] + [\![-cy\,; y]\!] \\ &= [\![x + cy\,; y]\!] - c\|y\|^2, \end{aligned}$$

where the final equality holds again by statement (i). Rearranging the inequality implies the result.

Chapter 3

Strongly Contracting Systems

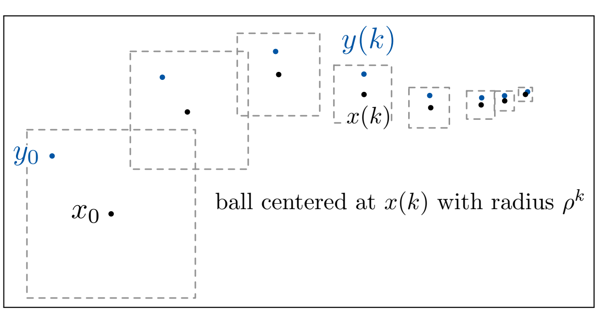
One object of the present paper is to give a result analogous to Krasovskii's. (Howard H. Rosenbrock 1973)

Based on a differential analysis of convergence, these results may be viewed as generalizing the classical Krasovskii theorem, and, more loosely, linear eigenvalue analysis. (Winfried Lohmiller and Jean-Jacques E. Slotine 1998)

Leaving the classical theory, we first turn to nonlinear maps. For a long time linearization was used, with the logarithmic norm applied to the Jacobian. The modern approach, however, is functional analytic. (Gustaf Söderlind 2006)

3.1 Introduction

In this chapter we study the properties of dynamical systems with various contraction properties. The treatment is based upon the modern approach proposed in (Davydov et al., 2022a; Jafarpour et al., 2023). We illustrate discrete and continuous-time contracting systems in Figure 3.1.



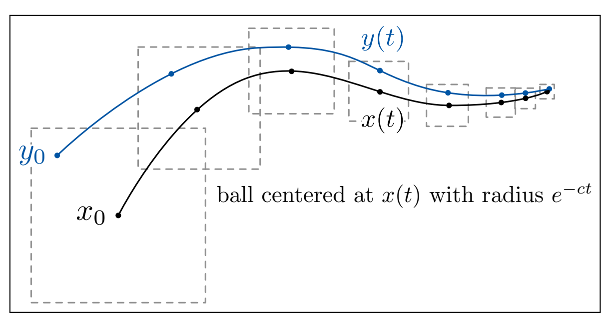


Figure 3.1: We characterize contraction in discrete and continuous time.

Given a differentiable map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, we denote its *Jacobian matrix* by

$$Df(x) = \frac{\partial f}{\partial x}(x).$$

We will relate the log norm of the Jacobian of a map to certain properties of the map itself. We will study the properties of maps whose Jacobian has uniformly upper-bounded log norm. We will then show how these notions are useful in the study of dynamical systems.

Throughout the chapter, we consider a generic norm $\|\cdot\|$ on $\mathbb{R}^n$ with induced matrix norm $\|\cdot\|$ and induced log norm $\mu(\cdot)$. Associated to the norm is a compatible weak pairing $[\![\cdot\,;\,\cdot]\!]$ satisfying the additional standing assumptions.

3.2 Lipschitz and one-sided Lipschitz maps

In this section we are interested in continuous scalar functions and maps with a special type of monotone behavior and bounded derivatives.

3.2.1 Introduction for scalar functions

As a starting point, we illustrate the basic idea with a scalar continuous function $f\colon \mathbb{R} \to \mathbb{R}$. As reviewed in Chapter 1, the classic Lipschitz bound, given by a non-negative constant $b \geq 0$ (or possibly $b = +\infty$ if the function has unbounded variations), is:

$$|f(x) - f(y)| \leq b|x - y| \quad \text{for all } x \text{ and } y \in \mathbb{R}. \tag{3.1}$$

When f is differentiable, the Lipschitz bound is equivalently written in terms of the derivative of f, denoted by f':

$$|f'(x)| \leq b, \quad \text{for all } x \in \mathbb{R}. \tag{3.2}$$

Also well know is that b is *a Lipschitz constant* of f and that the *minimum Lipschitz bound* is defined by any one of the following equivalent formulas:

$$\begin{aligned} &\inf\{b \in \mathbb{R} \,:\, |f(x) - f(y)| \leq b|x - y|, \text{ for all } x \text{ and } y \in \mathbb{R}\} \\ &= \inf\{b \in \mathbb{R} \,:\, |f'(x)| \leq b\} \\ &= \sup\nolimits_{x \in \mathbb{R}^n} |f'(x)|. \end{aligned} \tag{3.3}$$

For reasons that will soon become clear, we are interested in the *one-sided Lipschitz bound*:

$$f'(x) \leq b, \qquad \text{for all } x \in \mathbb{R}. \tag{3.4}$$

Simple calculations show the following equivalences

$$f'(x) \leq b, \qquad \text{for all } x \tag{3.5}$$

$$\iff \begin{cases} f(x) - f(y) \leq b(x - y), & \text{for all } x > y \\ f(x) - f(y) \geq b(x - y), & \text{for all } x < y \end{cases} \tag{3.6}$$

$$\iff (f(x) - f(y))(x - y) \leq b(x - y)^2, \qquad \text{for all } x, y \tag{3.7}$$

The definitions of Lipschitz and one-sided Lipschitz functions share many similarities. For example, as before, the equivalent conditions (3.6) and (3.7) hold for any function f, whereas condition (3.5) is well-posed only when f is differentiable. Also as before, it is possible and useful to define the minimum one-sided Lipschitz constant.

However, there are also notable differences between the two notions. For example

- a one-sided Lipschitz function does not need to be Lipschitz, For example, the continuous scalar function $f(x) = -x^3$ is one-sided Lipschitz with constant 0 but it is not Lipschitz continuous;
- if f is Lipschitz with constant b, then it is also one-sided Lipschitz with constant b, (since obviously $|f'(x)| < b$ implies $f'(x) < b$); and
- but b can be negative. For example, for $b < 0$, the function f satisfies the one-sided bound with $b < 0$ if and only if it is strictly decreasing. We illustrate such a function in Figure 3.2.

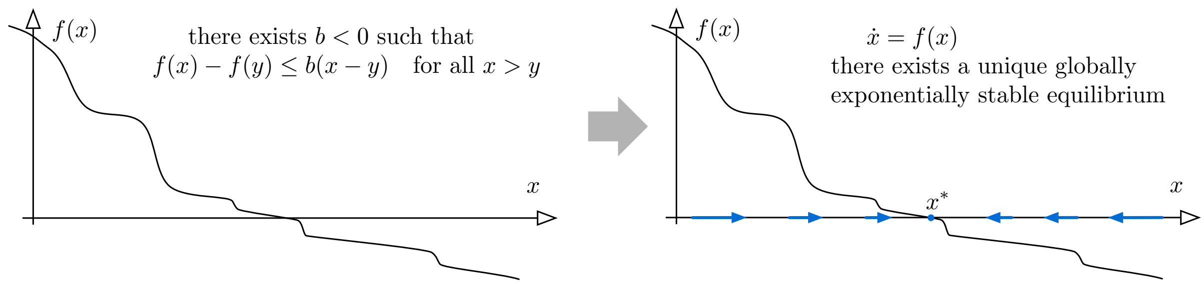


Figure 3.2: Left: A strictly decreasing function with negative one-sided Lipschitz constant. Right: The corresponding differential equation has a unique globally exponentially stable equilibrium.

3.2.2 Demidovich and one-sided Lipschitz conditions

In this section we generalize the ideas presented for scalar functions to maps of the form $f\colon C \subset \mathbb{R}^n \to \mathbb{R}^n$, where $C \subset \mathbb{R}^n$ is convex. We provide the next result without proof, since it is a special case of a more general result (Lemma 3.4) that we prove later.

Lemma 3.1 (Demidovic Lemma). *Let $C \subset \mathbb{R}^n$ be convex. Let $f\colon C \to \mathbb{R}^n$ be continuously-differentiable. Pick a matrix $P = P^\top \succ 0$ and a scalar $b \in \mathbb{R}$. The following statements are equivalent*

(i) the Jacobian of f has weighted ℓ_2 log norm uniformly bounded by b, that is,

$$\mu_{2,P^{1/2}}(Df(x)) \leq b, \quad \textit{for all } x \in C, \tag{3.8}$$

(ii) f satisfies the Demidovich condition

$$PDf(x) + Df(x)^\top P \preceq 2bP, \quad \textit{for all } x \in C, \tag{3.9}$$

(iii) f satisfies the one-sided Lipschitz condition (also referred to as the integral Demidovich condition)

$$\big(f(x) - f(y)\big)^\top P(x-y) \leq b\|x-y\|^2_{2,P^{1/2}}, \quad \textit{for all } x, y \in C. \tag{3.10}$$

By minimizing over the constants b satisfying conditions (3.8) and (3.10), one can restate these equivalence as:

$$\sup_{x \in C} \mu_{2,P^{1/2}}(Df(x)) = \sup_{x,y \in C, x \neq y} \frac{\big(f(x) - f(y)\big)^\top P(x-y)}{\|x-y\|^2_{2,P^{1/2}}}.$$

3.2.3 Lipschitz and one-sided Lipschitz maps

In this section we review the Lipschitz continuity property of a map (defined in Definition 1.5 over metric spaces) as well as a weaker property called one-sided Lipschitzness. This weaker property plays a key role in our analysis of continuous-time dynamical systems.

Definition 3.2 (Lipschitz and one-sided Lipschitz maps). *Given a convex set $C \subset \mathbb{R}^n$ and a norm with compatible weak pairing, a continuous map $f\colon C \to \mathbb{R}^n$ is*

(i) Lipschitz (or Lipschitz continuous) if there exists $b \in \mathbb{R}_{\geq 0}$, called a Lipschitz constant of f, such that

$$\|f(x) - f(y)\| \leq b\|x-y\| \quad \textit{for all } x, y \in C, \tag{3.11}$$

(ii) one-sided Lipschitz if there exists $b \in \mathbb{R}$, called a one-sided Lipschitz constant of f, such that

$$[\![f(x) - f(y)\,;\, x - y]\!] \leq b\|x - y\|^2 \quad \text{for all } x, y \in C. \tag{3.12}$$

The minimal Lipschitz constant of f, denoted by $\mathsf{Lip}(f)$, and the minimal one-sided Lipschitz constant of f, denoted by $\mathsf{osLip}(f)$, are defined by

$$\mathsf{Lip}(f) := \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \in \mathbb{R}_{\geq 0}, \tag{3.13}$$

$$\mathsf{osLip}(f) := \sup_{x \neq y} \frac{[\![f(x) - f(y)\,;\, x - y]\!]}{\|x - y\|^2} \in \mathbb{R}. \tag{3.14}$$

Note: The Cauchy-Schwartz inequality for weak pairings implies

$$\mathsf{osLip}(f) \leq \mathsf{Lip}(f), \tag{3.15}$$

where we allow the possibility that $\mathsf{Lip}(f) = +\infty$ or $\mathsf{osLip}(f) = +\infty$ for maps that fail to be Lipschitz or one-sided Lipschitz. Therefore, if a map f is Lipschitz with constant b, then f is also one-sided Lipschitz with constant upper bounded by b. The converse statement is not true, as illustrated by the counterexamples in Exercise E3.12.

Example 3.3 (Examples of Lipschitz and one-sided Lipschitz maps). (i) Let $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the identity map, and pick a scalar $c \in \mathbb{R}$. Then

$$\mathsf{Lip}(c\,\mathrm{id}) = |c| \quad \text{and} \quad \mathsf{osLip}(c\,\mathrm{id}) = c. \tag{3.16}$$

Therefore,

$$\mathsf{Lip}(\mathrm{id}) = \mathsf{osLip}(\mathrm{id}) = 1, \quad \text{whereas} \quad \mathsf{Lip}(-\mathrm{id}) = 1 \neq \mathsf{osLip}(-\mathrm{id}) = -1.$$

These examples illustrate that the Lipschitz constant is always non-negative, whereas the one-sided Lipschitz constant can take negative values.

(ii) Any affine map $f_A(x) = Ax + d$, with $A \in \mathbb{R}^{n \times n}$ and $d \in \mathbb{R}^n$, is Lipschitz and one-sided Lipschitz with respect to any norm on $\mathbb{R}^n$ and with minimal constants

$$\mathsf{Lip}(f_A) = \sup_{x \neq y} \frac{\|A(x-y)\|}{\|x-y\|} = \sup_{z \neq \mathbb{0}_n} \frac{\|Az\|}{\|z\|} = \|A\|,$$

$$\mathsf{osLip}(f_A) = \sup_{x \neq y} \frac{[\![A(x-y)\,;\, x-y]\!]}{\|x-y\|^2} = \sup_{z \neq \mathbb{0}_n} \frac{[\![Az\,;\, z]\!]}{\|z\|^2} = \mu(A).$$

The latter statement holds by Lumer's equality (2.59). •

After the definition and some examples, we provide two key lemma establishing the important properties of one-sided Lipschitz maps. As already illustrated for affine maps, the Lipschitz constant and the one-sided Lipschitz constant have a direct relationship with induced matrix norm and log norm, respectively. The first lemma generalizes to normed spaces the Demidovich Lemma 3.1.

Lemma 3.4 (Lipschitz and one-sided Lipschitz constants of differentiable maps). *Let $C \subset \mathbb{R}^n$ be convex. Let $f\colon C \to \mathbb{R}^n$ be continuously-differentiable. The norm and log norm of the Jacobian of f satisfy:*

$$\mathsf{Lip}(f) = \sup\nolimits_{x \in C} \|Df(x)\|, \tag{3.17}$$

$$\mathsf{osLip}(f) = \sup\nolimits_{x \in C} \mu(Df(x)). \tag{3.18}$$

Note: The equality (3.18) implies that, when the map f is continuously-differentiable and C is convex, the two definitions (3.14) and (3.18) are equivalent and, therefore, $\mathsf{osLip}(f)$ does not depend on the choice of weak pairing – it instead depends only on the norm.

Before proceeding, for completeness we present the result of this lemma also in the equivalent "upper bound formulation." Given $b \in \mathbb{R}$,

$$\begin{aligned}
\mu_1(Df(x)) \leq b \text{ for all } x \quad &\iff \quad \operatorname{sign}(x-y)^\top (f(x) - f(y)) \leq b\|x-y\|_1 \text{ for all } x \neq y, \\
\mu_\infty(Df(x)) \leq b \text{ for all } x \quad &\iff \quad \max_{i \in I_\infty(x-y)} (x_i - y_i)\,(f_i(x) - f_i(y)) \leq b\|x-y\|_\infty^2 \text{ for all } x \neq y.
\end{aligned}$$

Additionally, we summarize the three cases ℓ_1, ℓ_2, and ℓ_∞ in Table 3.1.

Log norm condition	Demidovich condition	osLip condition	Reference
$\mu_{2,P^{1/2}}(Df(x)) \leq b$	$PDf(x) + Df(x)^\top P \preceq 2bP$	$(x-y)^\top P\big(f(x) - f(y)\big) \leq b\|x-y\|_{2,P^{1/2}}^2$	Lemma 3.1
$\mu_1(Df(x)) \leq b$	$\operatorname{sign}(v)^\top Df(x) v \leq b\|v\|_1$	$\operatorname{sign}(x-y)^\top (f(x) - f(y)) \leq b\|x-y\|_1$	Lemma 3.4, E3.11
$\mu_\infty(Df(x)) \leq b$	$\max_{i \in I_\infty(v)} \operatorname{sign}(v_i)\,(Df(x)v)_i \leq b\|v\|_\infty$	$\max_{i \in I_\infty(x-y)} (x_i - y_i)\,(f(x) - f(y))_i \leq b\|x-y\|_\infty^2$	Lemma 3.4

Table 3.1: Table of one-sided Lipschitz conditions and equivalent log norm and Demidovich conditions. Each row contains three equivalent statements, to be understood for all $x, y \in \mathbb{R}^n$ and all $v \in \mathbb{R}^n$. For the third row on the ℓ_∞ norm, we adopt the shorthand $I_\infty(v) = \{i \in \{1, \dots, n\} \,:\, |v_i| = \|v\|_\infty\}$. The matrix P is positive definite and the vector η is positive.

Proof of Lemma 3.4. Statement (3.17) is widely known. Regarding statement (3.18), the Mean Value Theorem for vector-valued continuously-differentiable functions in Exercise E3.8 states $f(x) - f(y) = \big(\int_0^1 Df\big(y + s(x-y)\big)ds\big)(x-y)$ for any x, y in the convex set C. Accordingly, we compute

$$\begin{aligned}
\mathsf{osLip}(f) &= \sup_{x \neq y} \frac{\big[\!\!\big[\big(\int_0^1 Df\big(y + s(x-y)\big)ds\big)(x-y)\,;\, x-y\big]\!\!\big]}{\|x-y\|^2} \\
&\leq \sup_{x \neq y} \int_0^1 \frac{\big[\!\!\big[Df\big(y + s(x-y)\big)(x-y)\,;\, x-y\big]\!\!\big]}{\|x-y\|^2} ds && \text{(subadditivity of } [\![\cdot\,;\,\cdot]\!] \text{ (WP1))} \\
&\leq \int_0^1 \sup_{x \neq y} \frac{\big[\!\!\big[Df\big(y + s(x-y)\big)(x-y)\,;\, x-y\big]\!\!\big]}{\|x-y\|^2} ds \\
&\leq \int_0^1 \sup_{y \in \mathbb{R}^n} \sup_{z \neq \mathbb{0}_n} \frac{\big[\!\!\big[Df\big(y + sz\big)z\,;\, z\big]\!\!\big]}{\|z\|^2} ds \\
&= \int_0^1 \sup_{y \in \mathbb{R}^n, z \neq \mathbb{0}_n} \mu\big(Df\big(y + sz\big)\big) ds && \text{(Lumer's equality (2.59) (WP6))} \\
&= \sup\nolimits_{x \in \mathbb{R}^n} \mu(Df(x)).
\end{aligned}$$

Vice versa, recall $Df(y)v = \lim_{h \to 0^+}(f(y + hv) - f(y))/h$. Pick $x = y + hv$ for arbitrary $v \in \mathbb{R}^n$, $\|v\| = 1$, and

$h > 0$, and compute

$$\begin{aligned}
\mathsf{osLip}(f) &= \sup_{y\in\mathbb{R}^n, v\in\mathbb{R}^n, \|v\|=1, h>0} \frac{[\![f(x)-f(y)\,;\, x-y]\!]}{\|x-y\|^2}\Big|_{x=y+hv} \\
&\geq \sup_{y\in\mathbb{R}^n, v\in\mathbb{R}^n, \|v\|=1} \lim_{h\to 0^+} \frac{[\![f(y+hv)-f(y)\,;\, v]\!]}{h} && \text{(weak homogeneity (WP2))} \\
&= \sup_{y\in\mathbb{R}^n, v\in\mathbb{R}^n, \|v\|=1} [\![Df(y)v\,;\, v]\!] && \text{(continuity of } [\![\cdot\,;\,\cdot]\!] \text{ (WP1))} \\
&= \sup_{y\in\mathbb{R}^n} \mu(Df(y)). && \text{(Lumer's equality (2.59) (WP6))}
\end{aligned}$$

This concludes the proof of statement (3.18). ■

Next, we present a second set of results establishing the basic properties of Lipschitz and one-sided Lipschitz maps. These properties are analogous to those of matrix induced norm and matrix log norm, as stated in Lemma 2.11. We leave the proof of the following lemma as Exercise E3.1.

Lemma 3.5 (Basic properties of Lipschitz and one-side Lipschitz maps). *Let $C \subset \mathbb{R}^n$ be convex. Let $f, g\colon C \to \mathbb{R}^n$ be Lipschitz. Then*

(positive definiteness)	$\mathsf{Lip}(f) \geq 0$ *and* $\mathsf{Lip}(f) = 0 \iff f$ *is a constant map,*	(3.19a)
(homogeneity)	$\mathsf{Lip}(af) = \lvert a\rvert\ \mathsf{Lip}(f)$, *for all* $a \in \mathbb{R}$,	(3.19b)
(subadditivity)	$\mathsf{Lip}(f+g) \leq \mathsf{Lip}(f) + \mathsf{Lip}(g)$,	(3.19c)
(sub-multiplicativity)	$\mathsf{Lip}(f\circ g) \leq \mathsf{Lip}(f)\,\mathsf{Lip}(g)$.	(3.19d)

Next, let $f, g\colon C \to \mathbb{R}^n$ be one-sided Lipschitz. Then

(positive homogeneity)	$\mathsf{osLip}(af) = \lvert a\rvert\ \mathsf{osLip}(\mathrm{sign}(a)f)$, *for all* $a \in \mathbb{R}$,	(3.20a)
(subadditivity)	$\mathsf{osLip}(f+g) \leq \mathsf{osLip}(f) + \mathsf{osLip}(g)$,	(3.20b)
(translation property)	$\mathsf{osLip}(f + a\,\mathrm{id}) = \mathsf{osLip}(f) + a$, *for all* $a \in \mathbb{R}$,	(3.20c)
(uniform monotonicity property)	$\mathsf{osLip}(f) < 0 \implies f$ *injective and* $\mathsf{Lip}(f^{-1}) \leq -1/\mathsf{osLip}(f)$,	(3.20d)

where $\mathrm{sign}\colon \mathbb{R} \to \{-1, 0, +1\}$ *is defined so that* $\mathrm{sign}(0) = 0$ *and where,* $\mathsf{osLip}(f) < 0$, *the inverse map* f^{-1} *is well defined as a map from* $f(\mathbb{R}^n)$ *to* $\mathbb{R}^n$.

3.3 On Lipschitzness and dynamical systems

In this section we consider the time-varying dynamical systems

$$x(k+1) = f(k, x(k)), \tag{3.21a}$$
$$\dot{x}(t) = f(t, x(t)). \tag{3.21b}$$

Given a time instant $k \in \mathbb{Z}_{\geq 0}$ or $t \in \mathbb{R}_{\geq 0}$, we define the time-invariant map or vector field $x \mapsto f_k(x) = f(k, x)$ or $x \mapsto f_t(x) = f(t, x)$. When the map $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in its second argument, we denote its Jacobian (with respect to the second argument) by

$$Df(t, x) = Df_t(x) = \frac{\partial f}{\partial x}(t, x).$$

In what follows, we aim to study the properties of the solutions of the dynamical systems (3.21) — under the assumption that they exist and unique for at least small times. We offer a remark on this matter.

Remark 3.6 (Existence and uniqueness theorems for ordinary differential equations). *While the discrete-time model (3.21a) with an initial condition always has a unique solution, the continuous time model (3.21b) may fail to do so. So it is useful to briefly review existence and uniqueness results for time-varying ordinary differential equation defined by $(t,x) \mapsto f(t,x)$ with a given initial condition. The following results deal with local properties and local solutions in the (t,x) space:*

Peano's Existence Theorem: *If f is continuous in (t,x), then a solution exists, but it is not necessarily unique.*

Picard–Lindelöf Theorem: *If f is continuous in t at each x and Lipschitz in x at each t, then a solution exists, is unique and continuously differentiable; see Exercise E1.2;*

Uniqueness from one-sided Lipschitzness: *If f is continuous in (t,x) and f is one-sided Lipschitz in x, then a solution exists and is unique; see Exercise E3.4;*

Carathéodory's Existence Theorem: *This theorem provides even weaker conditions than the Peano's Theorem for existence of solutions. Typically, f is continuous in x for all t, measurable in t for all x, and its norm satisfies a summability condition with respect to t.*

Weaker versions of these theorems hold when the time-dependency in f is piecewise continuous, instead of continuous — but, in this text, we will assume continuity for simplicity of exposition. •

In summary from Remark 3.6, in what follows we will (i) typically assume that the vector field is continuous and satisfies a one-sided Lipschitzness condition (so that a solution exists locally and is unique) and (ii) will implicitly assume that solutions exist for all time. We let $t \mapsto \phi_{t_0,t}(x_0)$ denote the flow of (3.21b) starting from initial condition $x(t_0) = x_0$.

We are now ready state the main equivalent result in this section.

Theorem 3.7 (Main equivalence between solutions and Lipschitz properties). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with compatible weak pairing. Let $C \subset \mathbb{R}^n$ be convex. Let $f\colon \mathbb{R}_{\geq 0} \times C \to \mathbb{R}^n$ be continuous. For $b \in \mathbb{R}$, the following discrete-time statements are equivalent:*

(i) $\mathsf{Lip}(f_k) \leq b$, *for all* $k \geq 0$,

(ii) any two solutions $x(k)$ and $y(k)$ of (3.21a) taking value in C satisfy, for all $k \geq 0$,

$$\|x(k+1) - y(k+1)\| \leq b\|x(k) - y(k)\|; \tag{3.22}$$

and the following continuous-time statements are equivalent:

(iii) $\mathsf{osLip}(f_t) \leq b$, *for all* $t \geq 0$,

(iv) any two solutions $x(t)$ and $y(t)$ of (3.21b) taking value in C satisfy, for all $t \geq 0$,

$$D^+\|x(t) - y(t)\| \leq b\|x(t) - y(t)\|. \tag{3.23}$$

Proof. We omit the proof of the discrete-time statements for brevity and simplicity. Regarding (iii) $\implies$ (iv), given two solutions $x(t)$ and $y(t)$, we apply the curve norm derivative (WP5) to the curve $t \mapsto x(t) - y(t)$ to obtain

$$\begin{aligned}\|x(t) - y(t)\| D^+\|x(t) - y(t)\| &= [\![f(t,x) - f(t,y)\,;\, x(t) - y(t)]\!] \\ &\leq b\|x(t) - y(t)\|^2.\end{aligned}$$

Regarding (iv) $\implies$ (iii), let $x_0, y_0 \in C$, $t_0 \geq 0$ be arbitrary. The Deimling's inequality (WP7) implies

$$[\![f(t_0,x_0) - f(t_0,y_0)\,;\, x_0 - y_0]\!] \leq \|x_0 - y_0\| \lim_{h \to 0^+} \frac{\|x_0 - y_0 + h(f(t_0,x_0) - f(t_0,y_0))\| - \|x_0 - y_0\|}{h}. \tag{3.24}$$

We now let $t \mapsto \phi_{t_0,t}(x_0)$ denote the flow of (3.21) starting from initial condition $x(t_0) = x_0$. For $h \geq 0$, the Taylor expansion of the flow map gives

$$\begin{aligned}\|\phi_{t_0,t_0+h}(x_0) - \phi_{t_0,t_0+h}(y_0)\| &= \|x_0 - y_0 + h(f(t_0,x_0) - f(t_0,y_0)) + \mathcal{O}(h^2)\| \\ &= \|x_0 - y_0 + h(f(t_0,x_0) - f(t_0,y_0))\| + \mathcal{O}(h^2).\end{aligned}$$

On the other hand, the assumption (iv) and the Grönwall comparison Lemma E2.1 together imply, for $h \geq 0$,

$$\|\phi_{t_0,t_0+h}(x_0) - \phi_{t_0,t_0+h}(y_0)\| \leq e^{bh}\|x_0 - y_0\|.$$

Plugging these two results into (3.24), we obtain

$$\begin{aligned}[\![f(t_0,x_0) - f(t_0,y_0)\,;\,x_0 - y_0]\!] &\leq \|x_0 - y_0\| \lim_{h\to 0^+} \frac{\|\phi_{t_0,t_0+h}(x_0) - \phi_{t_0,t_0+h}(y_0)\| - \|x_0 - y_0\| + \mathcal{O}(h^2)}{h} \\ &\leq \|x_0 - y_0\| \lim_{h\to 0^+} \frac{e^{bh}\|x_0 - y_0\| - \|x_0 - y_0\|}{h} \\ &= \lim_{h\to 0^+} \frac{e^{bh} - 1}{h}\|x_0 - y_0\|^2 = b\|x_0 - y_0\|^2.\end{aligned}$$

The claims follows since t_0, x_0 and y_0 are arbitrary. ■

3.4 Strongly contracting systems

In this section, we study continuous-time dynamical systems whose Jacobians have log norm uniformly upper bounded by a negative constant. All following results have discrete-time analogues, which we omit for brevity and simplicity. In the language of Theorem 3.7, we assume $b = -c$, for a constant $c > 0$ called a contraction rate, and we discuss stability and Lyapunov functions.

Definition 3.8 (Strong infinitesimal contractivity). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with compatible weak pairing. Let $C \subset \mathbb{R}^n$ be convex. Let $f\colon \mathbb{R}_{\geq 0} \times C \to \mathbb{R}^n$ be continuous. The vector field f is strongly infinitesimally contracting on C if*

$$\mathsf{osLip}(f_t) \leq -c, \qquad \text{for all } t \in \mathbb{R}_{\geq 0}, \tag{3.25}$$

or, equivalently for differentiable f,

$$\mu(Df_t(x)) \leq -c, \qquad \text{for all } x \in C \text{ and } t \in \mathbb{R}_{\geq 0}. \tag{3.26}$$

We refer to $c > 0$ as the contraction rate.

Note: the strong infinitesimal contractivity condition can be regarded as a uniform Hurwitz condition.

3.4.1 Property #1: Globally exponentially stable equilibrium

Recall from Section 1.3 that, given a closed set $C \subset \mathbb{R}^n$, a map $T\colon C \to C$ is a *contraction* if there exists $q \in [0,1[$, called the *contraction factor*, such that, for all $x, y \in C$, $\|T(x) - T(y)\| \leq q\|x - y\|$.

Theorem 3.9 (From strong contraction to global stability and flow map contractivity). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with compatible weak pairing. Let $f\colon \mathbb{R}_{\geq 0} \times C \to \mathbb{R}^n$ be continuous. Let $C \subset \mathbb{R}^n$ be convex and f-invariant.*

If f is strongly infinitesimally contracting with rate $c > 0$ on C, then

(i) *for any $t \geq t_0 \geq 0$, the flow map $x \mapsto \phi_{t_0,t}(x)$ is a contraction mapping with contraction factor $\mathrm{e}^{-c(t-t_0)}$, that is, for any $x_0, y_0 \in C$,*

$$\|\phi_{t_0,t}(x_0) - \phi_{t_0,t}(y_0)\| \leq \mathrm{e}^{-c(t-t_0)}\|x_0 - y_0\|; \tag{3.27}$$

(ii) *if f is time invariant and C is closed, then f has a unique globally exponentially stable equilibrium $x^* \in C$ with global Lyapunov functions*

$$x \mapsto \|x - x^*\| \quad \text{and} \quad x \mapsto \|f(x)\|. \tag{3.28}$$

Specifically, for all $t \in \mathbb{R}_{\geq 0}$, letting $x \mapsto \phi_t(x)$ denote the flow of $\dot{x} = f(x)$ starting from initial condition x at time t, we have

$$\|\phi_t(x) - x^*\| \leq \mathrm{e}^{-ct}\|x - x^*\| \quad \text{and} \quad \|f(\phi_t(x))\| \leq \mathrm{e}^{-ct}\|f(x)\|. \tag{3.29}$$

We illustrate Theorem 3.9 in Figure 3.3.

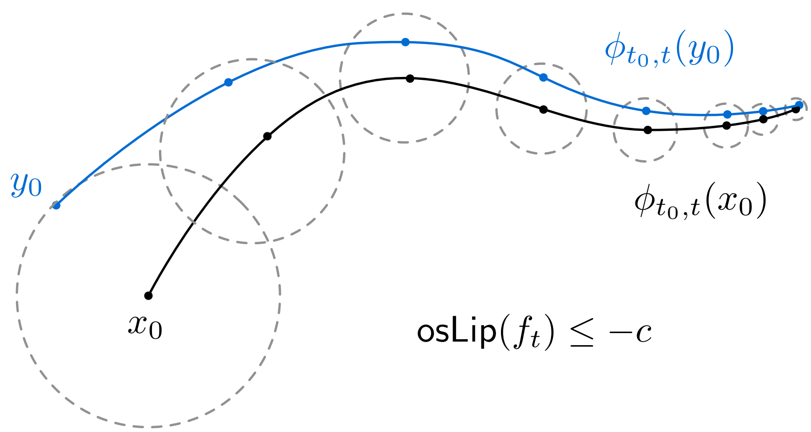


Figure 3.3: For a strongly infinitesimally contracting systems, any two trajectories converge.

Note: sometimes we will use the following equivalence: if V is a Lyapunov function, then so is V^2. This transformation is usually adopted to define the Lyapunov function $x \mapsto \|x - x^*\|^2$. The two Lyapunov functions in equation (3.28) are sometime referred to as the *Krasovskiĭ Lyapunov functions*.

Proof of Theorem 3.9. Statement (i) is an immediate consequence of the Main Equivalence Theorem 3.7. Regarding statement (ii), we start by proving the existence and uniqueness of the equilibrium point x^*. We reason as follows:

- note that $(C, \|\cdot\|)$ is complete metric space,
- let ϕ denote the unit-flow map of the vector field f,
- the Banach Fixed Point Theorem 1.6 implies there exists a unique x^* fixed point of ϕ,
- $\phi(x^*) = x^*$ implies that either x^* is an equilibrium or it is a point in a periodic orbit with period 1,
- by contradiction, assume a periodic orbit of period 1 exists. Then each point in the orbit is a fixed point of ϕ, which violates the uniqueness of x^* as a fixed point,
- hence, x^* is the unique equilibrium of f.

Next, the global exponential stability of x^* is an immediate consequences of the contractivity of the flow map in statement (i). Regarding the claim that both functions are global Lyapunov functions, clearly they both are globally positive definite about x^* (e.g., since x^* is the unique equilibrium, $\|f(x)\|$ cannot vanish anywhere else in C). The radial unboundedness of $x \mapsto \|f(x)\|$ is established by Theorem 3.26 in Exercise E3.2. Finally, it remains to establish the exponential decrease conditions in equation (3.29). The first equation in (3.29) is an immediate consequence of the contraction inequality (3.27). Regarding the second equation in (3.29), we first note

$$\frac{d}{dt}f(x(t)) = Df(x(t))\dot{x}(t) = Df(x(t))f(x(t)). \tag{3.30}$$

Finally, we compute

$$\begin{aligned}
\|f(x(t))\| D^+\|f(x(t))\| &= [\![\tfrac{d}{dt}f(x(t))\,;\,f(x(t))]\!] && \text{(by the curve norm derivative (WP5))}\\
&= [\![Df(x(t))f(x(t))\,;\,f(x(t))]\!]\\
&= \mu(Df(x(t)))\,[\![f(x(t))\,;\,f(x(t))]\!] && \text{(by Lumer equality (WP6))}\\
&\leq \sup_{z\in\mathbb{R}^n}\mu(Df(z))\|f(x(t))\|^2 = -c\|f(x(t))\|^2.
\end{aligned}$$

This concludes the proof of statement (ii). ■

Remark 3.10 (Alternative proofs). *There are other ways to prove the existence and uniqueness of the equilibrium point x^*. For example, here is an instructive approach to proving existence. We start by recalling that every two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^n$ are equivalent in the sense that there exist constants m and M such that $m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$, for every $x \in \mathbb{R}^n$. Also recall that the length of the curve $t \mapsto \phi_t(x_0)$ is*

$$\text{Length}(\phi_\cdot(x_0)) = \int_0^\infty \Big\|\frac{d}{dt}\phi_t(x_0)\Big\|_2 dt = \int_0^\infty \big\|f(\phi_t(x_0))\big\|_2 dt.$$

From equation (3.29) in Theorem 3.9 we know that, along each solution in C, the quantity $t \mapsto \|f(\phi_t(x_0))\|$ is decreasing to zero exponentially fast with respect to the norm $\|\cdot\|$ and, therefore also with respect to the ℓ_2 norm. Because the integrand is exponentially vanishing, the curve $t \mapsto \phi_t(x_0)$ has bounded length. Now, note that each continuously-differentiable curve in $\mathbb{R}^n$ with a finite length has a limit (and because C is closed the limit is in C); a proof of this intuitive fact is given in (Bhat and Bernstein, 1999, Lemma 2.1) and is based on the completeness of the Euclidean space. But, if $x^* = \lim_{t\to\infty}\phi_t(x_0)$, then, by continuity of f, $f(x^*) = f(\lim_{t\to\infty}\phi_t(x_0)) = \lim_{t\to\infty} f(\phi_t(x_0)) = \mathbb{0}_n$. This establishes the existence of an equilibrium point $x^* \in C$.

Alternatively, if the set C is compact, then the existence of an equilibrium point can be equivalently established from Yorke's fixed point Theorem 3.30. Analogously, if the set C is $\mathbb{R}^n$, then the existence of an equilibrium point can be equivalently established from Desoer-Haneda global inverse function Theorem 3.26. In the latter case, the map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a global diffeomorphism (so that a unique equilibrium $x^* = f^{-1}(\mathbb{0}_n)$ exists). •

3.4.2 Simple examples

We now review some examples of strongly infinitesimally contracting systems.

Example 3.11 (Time-invariant affine systems). Given a matrix $A \in \mathbb{R}^{n\times n}$, consider the time-invariant affine system

$$\dot{x} = Ax + d =: f_A(x). \tag{3.31}$$

Assume A is Hurwitz, that is, $\alpha(A) < 0$. Pick an ε sufficiently small so that $\alpha(A) + \varepsilon < 0$ and, as in Lemma 2.30, compute a positive definite matrix P that is ε-optimal for A with respect to the ℓ_2 norm. In sum, compute P so that

$$\mathsf{osLip}_{2,P^{1/2}}(f_A) \leq \alpha(A) + \varepsilon. \tag{3.32}$$

Then f_A is strongly infinitesimally contracting with respect to the norm $\|\cdot\|_{2,P^{1/2}}$ with rate $\alpha(A)+\varepsilon$. As a consequence, the system has a unique exponentially stable equilibrium $x^* = -A^{-1}d$ with global Lyapunov functions

$$x \mapsto \|x - x^*\|_{2,P^{1/2}}^2 = \|x + A^{-1}d\|_{2,P^{1/2}}^2 \quad \text{and} \quad x \mapsto \|f_A(x)\|_{2,P^{1/2}}^2 = \|Ax + d\|_{2,P^{1/2}}^2.$$

•

As second example, we present a classic generalization of the first example to nonlinear dynamical systems. Again the contractivity is with respect to the Euclidean ℓ_2 norm.

Example 3.12 (Weighted ℓ_2 contraction of nonlinear systems, aka Krasovskiĭ's method). Consider the continuously-differentiable time-invariant dynamical system $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let $C \subset \mathbb{R}^n$ be a convex and f-invariant set. Given a matrix $P = P^\top \succ 0$ and a positive scalar $c > 0$, assume f satisfies the Demidovich condition

$$PDf(x) + Df(x)^\top P \preceq -2cP, \qquad \text{for all } x \in C. \tag{3.33}$$

Then

(i) f is strongly infinitesimally contracting with contraction rate c with respect to $\|\cdot\|_{2,P^{1/2}}$ and the distance in the $\|\cdot\|_{2,P^{1/2}}$ norm between any two solutions in C is exponentially decreasing with rate c,

(ii) f has a unique globally exponentially stable equilibrium $x^* \in C$ with global Lyapunov functions

$$x \mapsto \|x - x^*\|_{2,P^{1/2}}^2 \quad \text{and} \quad x \mapsto \|f(x)\|_{2,P^{1/2}}^2. \tag{3.34}$$

The proof of these statements is immediate. Given the equivalences in Lemma 3.1, statements (i) and (ii) follow from Theorem 3.9. •

The fact that Demidovich's condition (3.33) implies that the two functions in equation (3.34) are global Lyapunov functions is usually attributed to (Krasovskiĭ, 1963) and referred to as the *Krasovskiĭ's method for the design of Lyapunov functions*; see also Exercise E3.9.

Example 3.13 (The Markus-Yamabe conjecture). In this chapter we have studied the stability properties of a differentiable vector field as a function of the log norm of its Jacobian. Here is a related classic conjecture.

> **Conjecture 3.14 (Markus–Yamabe conjecture).** *Consider a dynamical system $(\mathbb{R}^n, f)$ with continuously differentiable f. Assume that the origin is an equilibrium $f(\mathbb{0}_n) = \mathbb{0}_n$ and that the Jacobian $Df(x)$ is Hurwitz for every $x \in \mathbb{R}^n$. Then the origin $\mathbb{0}_n$ is a globally asymptotically stable equilibrium.*

This conjecture, originated by Markus and Yamabe (1960), is known to be true for $n = 2$. However, the conjecture is false for $n \geq 3$ as counter-examples were discovered starting in 1995. For more detail, please read Wikipedia: Markus-Yamabe conjecture and references therein.

A modified conjecture is easily true by Theorem 3.9. The origin $\mathbb{0}_n$ is a globally asymptotically stable equilibrium if there exists a norm $\|\cdot\|$ such that

$$\mu\big(Df(t,x)\big) \leq -c < 0.$$

for all $x \in \mathbb{R}^n$ and $t \geq 0$. In closing we comment that even weaker requirements are sufficient. •

3.4.3 Property #2: Entrainment to periodic forcing

We now study the response of a strongly infinitesimally contracting system to a periodic input and show a basic theoretical result about periodic orbits. Given a duration $T > 0$, a time-varying vector field $(t, x) \mapsto f(t, x)$ is *T-periodic* if $f(t + T, x) = f(t, x)$ for all $t \geq 0$ and $x \in C$. We illustrate the following theorem in Figure 3.4.

Theorem 3.15 (Contractivity implies entrainment to periodic forcing). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with compatible weak pairing. Let $C \subset \mathbb{R}^n$ be convex. Let $f : \mathbb{R}_{\geq 0} \times C \to \mathbb{R}^n$ be continuous. Given a duration $T > 0$, assume that*

(i) C is convex, closed and f-invariant;

(ii) f is strongly infinitesimally contracting with contraction rate $c > 0$; and

(iii) f is T-periodic in its first argument.

Then there exists a unique periodic solution $x^* : \mathbb{R}_{\geq 0} \to C$ with period T to $\dot{x} = f(t, x)$ and, for every initial condition $x_0 \in C$ at time t_0,

$$\|\phi_{t_0,t}(x_0) - x^*(t)\| \leq \mathrm{e}^{-c(t-t_0)} \|x_0 - x^*(t_0)\|. \tag{3.35}$$

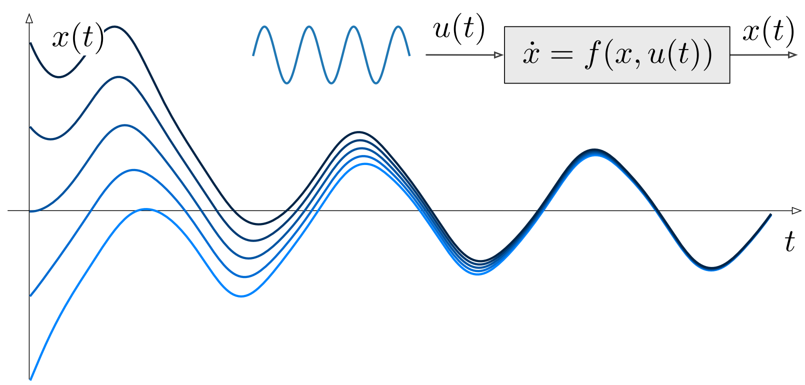


Figure 3.4: Given any T-periodic input, a strongly infinitesimally contracting system will feature a unique globally exponentially stable T-periodic solution. Example here: $\dot{x} = -(1.1 + \cos(2t))x/4 + \sin(t)$ with initial conditions $-3, -1.5, 0, 1.5, 3$ at time 0.

Proof. We sketch a proof outline here and refer to (Russo et al., 2010) for details. Define the T-flow map $\Phi : C \to C$ by $\Phi(x) := \phi_{0,T}(x)$. It is immediate to see that, for all $k \in \mathbb{N}$,

$$\Phi^k(x) = \phi_{0,kT}(x). \tag{3.36}$$

Indeed, by induction,

$$\Phi^{k+1}(x) = \Phi(\Phi^k(x)) = \phi_{0,kT}(\phi_{0,kT}(x)) = \phi_{0,(k+1)T}(x). \tag{3.37}$$

As established in Theorem 3.9(i), the map Φ is a contraction with contraction factor $\mathrm{e}^{-Tc} < 1$. By the contraction mapping theorem, there exists a unique fixed point $\bar{x} \in C$. Define $x^*(t) = \phi_{0,t}(\bar{x})$ and note that this solution is T-periodic since $x^*(T) = \Phi(\bar{x}) = \bar{x} = x^*(0)$. By the main contraction Theorem 3.9(i), the distance between any trajectory and $x^*(t)$ is exponentially decreasing; this statement also implies uniqueness. ■

3.4.4 Property #3: Stability with respect to external inputs

We now present a generalization of the Main Equivalence Theorem to the setting of systems with an input. This generalization is crucial to obtain various input-to-state stability results.

Theorem 3.16 (Response to external inputs). *Let $C \subset \mathbb{R}^n$ be convex and let $\mathcal{U} \subset \mathbb{R}^k$. Let $f\colon \mathbb{R}_{\geq 0} \times C \times \mathcal{U} \to \mathbb{R}^n$ be continuous. Consider the time- and input-dependent dynamics*

$$\dot{x} = f(t, x, u(t)), \tag{3.38}$$

where u takes values in $\mathcal{U}$. Assume there exist a norm $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with compatible weak pairing $[\![\cdot\,;\cdot]\!]$, a norm $\|\cdot\|_{\mathcal{U}}\colon \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ such that

(A1) *at fixed t, u, the map $x \mapsto f(t,x,u)$ is strongly infinitesimally contracting with rate c, that is,*

$$[\![f(t,x,u) - f(t,y,u)\,;\, x-y]\!] \leq -c\|x-y\|^2, \quad \textit{for all } t \in \mathbb{R}_{\geq 0},\, x, y \in \mathbb{R}^n,\, u \in \mathbb{R}^k,$$

(A2) *at fixed t, x, the map is $u \mapsto f(t,x,u)$ Lipschitz with constant ℓ, that is,*

$$\|f(t,x,u) - f(t,x,v)\| \leq \ell\|u-v\|_{\mathcal{U}}, \quad \textit{for all } t \in \mathbb{R}_{\geq 0},\, x \in \mathbb{R}^n,\, u, v \in \mathbb{R}^k.$$

Then any two solutions $x(t)$ and $y(t)$ to (3.38) with input signals $u_x, u_y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^k$ satisfy

$$D^+\|x(t) - y(t)\| \leq -c\|x(t) - y(t)\| + \ell\|u_x(t) - u_y(t)\|_{\mathcal{U}}, \quad \textit{for all } t \geq 0. \tag{3.39}$$

Proof. Using the properties of the weak pairing, we compute

$$\begin{aligned}
&\|x(t) - y(t)\| D^+\|x(t) - y(t)\| \\
&\quad = [\![f(t,x,u_x) - f(t,y,u_y)\,;\, x-y]\!] && \text{(curve norm derivative (WP5))} \\
&\quad \leq [\![f(t,x,u_x) - f(t,y,u_x)\,;\, x-y]\!] \\
&\qquad\quad + [\![f(t,y,u_x) - f(t,y,u_y)\,;\, x-y]\!] && \text{(subadditivity (WP1))} \\
&\quad \leq -c\|x-y\|^2 + \|f(t,y,u_x) - f(t,y,u_y)\|\,\|x-y\| && \text{((A1), Cauchy-Schwartz (WP4))} \\
&\quad \leq -c\|x-y\|^2 + \ell\|u_x - u_y\|_{\mathcal{U}}\|x-y\|. && \text{((A2))}
\end{aligned}$$

■

Corollary 3.17 (Input-state stability properties). *Under the same assumptions as in Theorem 3.16, with trajectories $x(t)$ and $y(t)$ with initial conditions $x(0) = x_0$ and $y(0) = y_0$,*

(i) the Grönwall Comparison Lemma E2.1 implies

$$\|x(t) - y(t)\| \leq \mathrm{e}^{-ct}\|x_0 - y_0\| + \ell \int_0^t \mathrm{e}^{-c(t-\tau)}\|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}} d\tau; \tag{3.40}$$

(ii) f is incrementally input-to-state stable, in the sense that, from any initial conditions $x_0, y_0 \in \mathbb{R}^n$,

$$\|x(t) - y(t)\| \leq e^{-ct}\|x_0 - y_0\| + \frac{\ell}{c}(1 - e^{-ct}) \sup_{\tau \in [0,t]} \|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}}; \tag{3.41}$$

In other words, bounded disturbances generate bounded responses.

(iii) *if* $\|u_x(t) - u_y(t)\|_{\mathcal{U}} \leq H\,\mathrm{e}^{-ht}$ *for some positive* h *and* H*, then*

$$\|x(t) - y(t)\| \leq e^{-ct}\|x_0 - y_0\| + \ell H \frac{\mathrm{e}^{-ht} - \mathrm{e}^{-ct}}{c - h}, \tag{3.42}$$

that is, the distance between trajectories is upper bounded by exponentially decaying signals with rates c *and* h*. In other words, exponentially decaying disturbances generate exponentially decaying responses.*

Proof. The statements are immediate consequences of the Grönwall Comparison Lemma E2.1 applied to the differential inequality (3.39). ∎

We conclude with an additional result, whose proof is omitted; see (Davydov et al., 2022a).

In order to state this result we need to review notions of signal norms and system gains; for a comprehensive treatment, we refer the reader to (Doyle et al., 1990, Chapter 2). Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, let $\mathcal{L}^q_{\|\cdot\|}$, $q \in [1, \infty]$, denote the vector space of piecewise-continuous signals $u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfying $\int_0^\infty \|u(t)\|^q dt < \infty$. This space has a norm given by $\|u(\cdot)\|_{\cdot,q} = \big(\int_0^\infty \|u(t)\|^q dt\big)^{1/q}$. An input-output system has a $\mathcal{L}^q_{\|\cdot\|}$-induced gain upper bounded by $\gamma > 0$ if, for all inputs $u \in \mathcal{L}^q_{\|\cdot\|}$, the output y (from zero initial state) also belongs to $\mathcal{L}^q_{\|\cdot\|}$ and $\|y(\cdot)\|_{\cdot,q} \leq \gamma \|u(\cdot)\|_{\cdot,q}$.

Corollary 3.18 (Input-state stability properties: Continued). *Under the same assumptions as in Theorem 3.16 and Corollary 3.17,*

(iv) f *has incremental* $\mathcal{L}^q_{\|\cdot\|}$ *gain equal to* ℓ/c*, for* $q \in [1, \infty]$*, in the sense that solutions with* $x(0) = y(0)$ *satisfy*

$$\|x(\cdot) - y(\cdot)\|_{\cdot,q} \leq \frac{\ell}{c}\,\|u_x(\cdot) - u_y(\cdot)\|_{\cdot,q}. \tag{3.43}$$

3.4.5 Property #4: Robustness to unmodeled dynamics

The next result provides sufficient conditions for contractivity under perturbations.

Theorem 3.19 (Robustness properties of contracting systems). *Consider the dynamics* $\dot{x} = f(t, x) + g(t, x)$*, where both* f *and* g *are continuous on* $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$*. If* f *has one-sided Lipschitz constant* $-c$ *and* g *has one-sided Lipschitz constant* $d \in \mathbb{R}$ *with respect to the same norm, then*

(i) *(contractivity under perturbations) if* $d < c$*, then* $f + g$ *is strongly infinitesimally contracting with rate* $c - d$*,*

(ii) *(equilibrium point under perturbations) if additionally* f *and* g *are time-invariant, then the unique equilibrium point* x^* *of* f *and* x^{**} *of* $f + g$ *satisfy*

$$\|x^* - x^{**}\| \leq \min\Big\{\frac{\|g(x^*)\|}{c - d}, \frac{\|g(x^{**})\|}{c}\Big\}. \tag{3.44}$$

Proof. Statement (i) is an immediate consequence of the sub-additivity property of osLip (3.20b). Regarding statement (ii), recall Corollary 3.17(ii) and take the limit as $t \to \infty$ in equation (3.41) to obtain

$$\lim_{t\to\infty} \|x(t) - y(t)\| \leq \frac{\ell}{c} \sup_{\tau \in [0,t]} \|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}}, \tag{3.45}$$

for any two trajectories of an input-dependent vector field that is strongly infinitesimally contracting with rate c and with input Lipschitz constant ℓ. First, consider the vector field $F_1(x,u) = f(x) + g(x) - g(x^*) + u$ with $u = \mathbb{0}_n$ and $u = g(x^*)$ (hence with input Lipschitz constant $\ell = 1$):

$$\begin{aligned}\dot{x} &= f(x) + g(x) - g(x^*),\\ \dot{y} &= f(y) + g(y) - g(x^*) + g(x^*) = f(y) + g(y).\end{aligned}$$

Note that F_1 has contraction rate $c - d$ and unique equilibrium x^*, because $f(x^*) + g(x^*) - g(x^*) = f(x^*) = \mathbb{0}_n$. Equation (3.45) now implies

$$\|x^* - x^{**}\| \leq \frac{1}{c-d}\|g(x^*)\|. \tag{3.46}$$

Repeating this argument with vector field $F_2(x,u) = f(x) + u$ with $u = \mathbb{0}_n$ and $u = g(x^{**})$ implies statement (ii). ■

Example 3.20 (Local contractivity for systems with disturbances). This example is inspired by (Ström, 1975). Consider the dynamical system

$$\dot{x} = Ax + g(x). \tag{3.47}$$

Assume g continuous and there exist $k > 0$ and a norm such that $\mu(A) < -k < 0$ and $\|g(x)\| \leq k\|x\|$ in a neighborhood $\mathbb{0}_n$.

Then there exists a neighborhood of $\mathbb{0}_n$ which is forward invariant and inside which the system is strongly infinitesimally contracting with rate $k - |\mu(A)|$ and with equilibrium $\mathbb{0}_n$. •

3.5 Example: Hopfield neural networks

We consider models of neural circuits and give sufficient conditions on their synaptic matrix that ensure the dynamics are infinitesimally strongly contracting.

We start by introducing the continuous-time *Hopfield* and *firing rate neural network* models:

$$\dot{x} = -x + A\Psi(x) + u =: f_{\mathrm{H}}(x,u), \tag{3.48}$$
$$\dot{x} = -x + \Psi(Ax + u) =: f_{\mathrm{FR}}(x,u), \tag{3.49}$$

where

- $A \in \mathbb{R}^{n\times n}$ is an arbitrary *synaptic matrix*,
- $u \in \mathbb{R}^n$ is a (possibly time-varying) input, and
- $\Psi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal map, that is, $\Psi(x) = [\psi_1(x_1), \dots, \psi_n(x_n)]$ and each component function $\psi_i\colon \mathbb{R} \to \mathbb{R}$ is referred to as an *activation function*.

Figure 3.5 illustrates a possible physical realization of the Hopfield model.

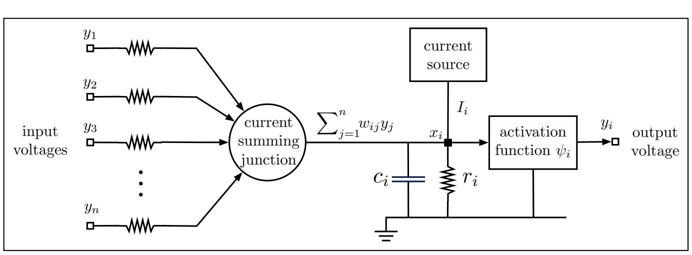


Figure 3.5: Electric circuit model of a neuron based upon a current summing junction. Based upon first principles, the circuit equations in the voltage variable x_i are written as $c_i\dot{x}_i + \frac{1}{r_i}x_i = I_i + \sum_{j=1}^n w_{ij}\psi_j(x_j)$ and are equivalent to the continuous-time Hopfield model in equation (3.48) (after simple parameter rescaling).

Typical activation functions (hyperbolic tangent and ReLU) are illustrated in Figure 3.6. We assume that[1] the typical activation function $\psi\colon \mathbb{R} \to \mathbb{R}$ is:

(i) differentiable,

(ii) weakly increasing, so that $\frac{d\psi_i}{dy}(y) \geq 0$ for all $y \in \mathbb{R}$, and

(iii) slope-restricted in the sense that $\frac{d\psi_i}{dy}(y) \leq 1$ for all $y \in \mathbb{R}$.

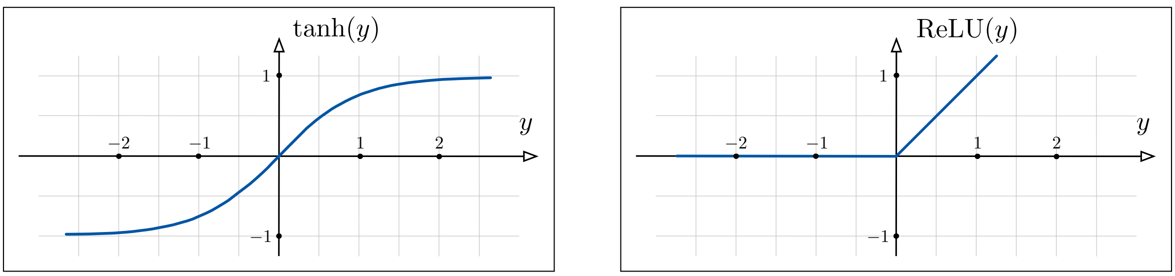


Figure 3.6: Left: the hyperbolic tangent $\psi(y) = \tanh(y)$. Right: the *rectified linear unit* (ReLU) activation function $\psi(y) = \max\{y, 0\} = y_+$. Note: The hyperbolic tangent is an odd function satisfying: $-1 \leq \tanh(y) \leq 1$, $0 \leq \frac{d}{dy}\tanh(y) \leq 1$, $\frac{d}{dy}\tanh(y) = 1 - \tanh(y)^2$.

The following lemma relies upon lognorms results from Lemma 2.15 to establish contractivity of neural network models. We analyze the firing rate model and leave to the reader the contractivity analysis of the Hopfield model f_{H} with respect to ℓ_1 and ℓ_∞. We refer the reader to E3.10 for a more general treatment. We illustrate an example phase portrait of a firing rate model in Figure 3.7.

Lemma 3.21 (Contracting firing rate models). *The firing rate model f_{FR} with synaptic matrix A in equation (3.49) is strongly infinitesimally contracting*

(i) with respect to $\|\cdot\|_\infty$ and rate $1 - \max\{\mu_\infty(A), 0\}$, if $\max\{\mu_\infty(A), 0\} < 1$; and

(ii) with respect to $\|\cdot\|_1$ and rate $1 - \max\{\mu_1(A), \mu_1(A - (I_n \circ A))\}$, if $\max\{\mu_1(A), \mu_1(A - (I_n \circ A))\} < 1$.

Additionally, for $|A|_{\mathrm{M}}$ irreducible with right dominant eigenvector η,

(iii) if $\alpha(|A|_{\mathrm{M}}) < 1$, then f_{FR} is strongly infinitesimally contracting with rate $1 - \alpha(|A|_{\mathrm{M}})_+$ with respect to $\|\cdot\|_{\infty,[\eta]^{-1}}$.

Proof of Lemma 3.21. First, for arbitrary $u \in \mathbb{R}^n$, we compute

$$\begin{aligned}\mathsf{osLip}(f_{\mathrm{FR}}) &= \sup_{x\in\mathbb{R}^n} \mu\big(Df_{\mathrm{FR}}(x,u)\big)\\ &= \sup_{x\in\mathbb{R}^n} \mu\big(-I_n + (D\Psi(Ax+u))A\big)\\ &\leq -1 + \max_{d\in[0,1]^n} \mu([d]A).\end{aligned}$$

[1] Note: the following treatment can be generalized to more general activation functions, including locally Lipschitz functions slope-restricted in the sense that $d_{\min} := \operatorname{ess\,inf}_{y\in\mathbb{R}} \frac{d\psi_i}{dy}(y)$ and $d_{\max} := \operatorname{ess\,sup}_{y\in\mathbb{R}} \frac{d\psi_i}{dy}(y) < \infty$. We refer to (Davydov et al., 2022c) for more details.

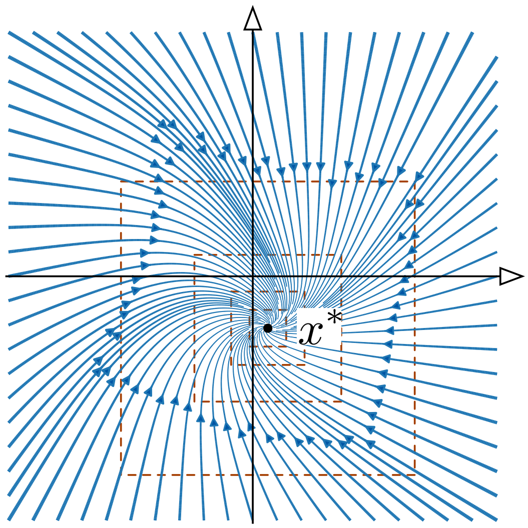


Figure 3.7: Phase portrait of the firing rate model $\dot{x} = f_{\mathrm{FR}}(x,u) = -x + \Psi(Ax+u)$ with synaptic matrix $A = \begin{bmatrix} 0 & .9 \\ -.9 & 0 \end{bmatrix}$ and bias $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since $\mu_\infty(A) = .9 < 1$, this vector fields is strongly infinitesimally contracting with respect to ℓ_∞. Accordingly there exists a unique globally exponentially stable equilibrium x^*. The dashed boxes around x^* are the level sets of the Lyapunov function $x \mapsto \|x - x^*\|_\infty$.

Then we use equation (2.53) from Lemma 2.15 stating $\max_{d \in [0,1]^n} \mu_\infty([d]A) = \max\{0, \mu_\infty(A)\} = \mu_\infty(A)_+$. This completes the proof of statements (i) and (ii).

Statement (iii) is proved in two steps. First, we generalize Lemma 2.15 to establish the weighted lognorm equality $\max_{d \in [0,1]^n} \mu_{\infty,[\eta]^{-1}}([d]A) = \mu_{\infty,[\eta]^{-1}}(A)_+$; it is easy to generalize the proof of Lemma 2.15 in this direction. Then we optimize the weight η to minimize the one-side Lipschitz constant, in a manner similar to Lemma 2.31. ■

3.6 Modularity and interconnections of strongly contracting systems

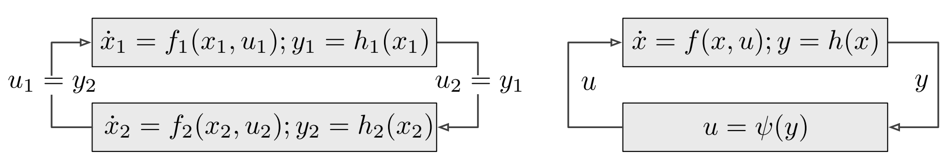


Figure 3.8: Interconnections of contracting dynamical systems (left image) and nonlinear systems in Lur'e form (right image).

In this section we review various modularity properties that contracting systems naturally inherit from the properties of the one-sided Lipschitz constant.

We will show how

(i) positive "parallel" combinations, positive scaling, and, therefore, convex combinations of contracting systems are contracting; see Section 3.6.1;

(ii) general "Lipschitz" interconnections under Hurwitz assumption on resulting gain matrix and, as special case, series interconnections of contracting systems are contracting; see the left image in Figure 3.8 and Section 3.6.2;

(iii) certain static nonlinear interconnections of contracting systems in so-called in Lur'e form are contracting, see the right image in Figure 3.8 and Section 3.7.

Additionally, any combination of the above interconnection retains contractivity.

3.6.1 Parallel and convex combinations of contracting systems

The first formal result is an immediate consequence of the basic properties of osLip.

Corollary 3.22 (Parallel combinations, positive scalings and convex combinations of contracting systems). *Given a norm $\|\cdot\|$, consider a family of vector fields $f_1, \dots, f_m$ that are strongly infinitesimally contracting with respect to $\|\cdot\|$ with rates $c_1, \dots, c_m$. Then, for any positive scalar numbers $a_1, \dots, a_m$, the vector field $a_1 f_1 + \dots + a_m f_m$ is strongly infinitesimally contracting with respect to $\|\cdot\|$ with rate $a_1 c_1 + \dots + a_m c_m$.*

3.6.2 Networks of contracting systems

Recall the discussion about composite norms from Section 2.4.4: Given positive integers $n_1, \dots, n_r$ such that $n_1 + \dots + n_r = n$, consider the decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_r}$, so that any $x \in \mathbb{R}^n$ can be written as $x = \begin{bmatrix} x_1^\top & \dots & x_r^\top \end{bmatrix}^\top$ with $x_i \in \mathbb{R}^{n_i}$. Additionally assume that we are given a local norm $\|\cdot\|_i$ on $\mathbb{R}^{n_i}$, for each $i \in \{1, \dots, r\}$, with log norm $\mu_i(\cdot)$ and weak pairing $[\![\cdot\,;\cdot]\!]_i$.

We now consider the *interconnection of r continuous dynamical systems*

$$\dot{x}_i = f_i(t, x_i, x_{-i}), \quad \text{for } i \in \{1, \dots, r\}, \tag{3.50}$$

where $x_i \in \mathbb{R}^{n_i}$ and $x_{-i} \in \mathbb{R}^{n-n_i}$. We let x_{-i} denote the vector with all components x_j of x, except x_i. We assume

(A1) (*contractivity-at-each-node*) at fixed x_{-i} and t, each map $x_i \mapsto f_i(t, x_i, x_{-i})$ is strongly infinitesimally contracting with rate c_i with respect to $\|\cdot\|_i$. In other words,

$$[\![f_i(t, x_i, x_{-i}) - f_i(t, y_i, x_{-i})\,; x_i - y_i]\!]_i \leq -c_i \|x_i - y_i\|_i^2, \quad \text{for all } x_i, y_i \in \mathbb{R}^{n_i},$$

(A2) (*Lipschitz interconnections*) at fixed x_i and t, each map $x_{-i} \mapsto f_i(t, x_i, x_{-i})$ is Lipschitz in the sense that there exist $\ell_{ij} \in \mathbb{R}_{\geq 0}$, $j \neq i$, satisfying

$$\|f_i(t, x_i, x_{-i}) - f_i(t, x_i, y_{-i})\|_i \leq \sum_{j=1, j\neq i}^{n} \ell_{ij} \|x_j - y_j\|_j, \quad \text{for all } x_j, y_j \in \mathbb{R}^{n_j}.$$

Note: for example, a vector field of the form $f_i(t, x_i, x_{-i}) = g_i(t, x_i) + \sum_{j=1, j\neq i}^{n} H_{ij} x_j$ satisfies Assumption (A2), with $\ell_{ij} = \|H_{ij}\|_{ij}$, that is, the induced gain of H_{ij} as a map from $\mathbb{R}^{n_j}$ to $\mathbb{R}^{n_i}$ (see equation (2.48)).

Next, for any interconnected system satisfying the two assumptions, we define three useful quantities.

Step I: First, we define the *gain matrix*

$$\Gamma = \begin{bmatrix} -c_1 & \dots & \ell_{1r} \\ \vdots & & \vdots \\ \ell_{r1} & \dots & -c_r \end{bmatrix} \in \mathbb{R}^{r \times r}.$$

We regard such a matrix as an aggregate Metzler majorant for the nonlinear system.

Step II: Second, we compute a logarithmically optimal norm for Γ. We pick ε arbitrarily small, consider the 2-norm, and compute an optimal weight as in Lemma 2.31 for the matrix Γ. Specifically, we assume that the norm $\|\cdot\|_{2,[\eta]^{1/2}}$ is ε-log optimal for Γ:

$$\begin{aligned} &\mu_{2,[\eta]^{1/2}}(\Gamma) \leq \alpha(\Gamma) + \varepsilon \\ \iff \qquad &[\eta]\Gamma + \Gamma^\top[\eta] \preceq 2(\alpha(\Gamma) + \varepsilon)[\eta]. \end{aligned}$$

Step III: Third and final, given $\eta \in \mathbb{R}^r_{>0}$, we define the weighted ℓ_2 composite norm $\|\cdot\|_\eta$ on $\mathbb{R}^n$ by

$$\|(x_1, \dots, x_r)\|_\eta^2 = \sum\nolimits_{i=1}^r \eta_i \|x_i\|_i^2. \tag{3.51}$$

One can show that a natural choice of weak pairing associated to this norm is

$$[\![(x_1, \dots, x_r)\,;\,(y_1, \dots, y_r)]\!]_\eta = \sum\nolimits_{i=1}^r \eta_i \,[\![x_i\,;\,y_i]\!]_i\,. \tag{3.52}$$

With these definitions at hand, we are ready for the main result of this section.

Theorem 3.23 (Contractivity of interconnected system). *Consider the interconnected system in (3.50) satisfying the contractivity-at-each-node Assumption (A1) and the Lipschitz interconnections Assumption (A2). If the gain matrix Γ is Hurwitz, then the interconnected system is strongly infinitesimally contracting with respect to $\|\cdot\|_\eta$ and with rate $|\alpha(\Gamma) + \varepsilon|$.*

Note: Any interconnection of contracting systems that has the topology of a directed acyclic graph (aka DAG topology) is automatically strongly infinitesimally contracting (with rate equal to the minimum contraction rate of any subsystem). This property is a consequence of the fact that, after a renumber of nodes, the gain matrix is triangular.

Note: For the interconnection of any two contracting systems, the Hurwitzness requirement is easily transcribed into the small gain condition: $c_1 c_2 > \ell_{12}\ell_{21}$. In other words, the interconnected system is contracting if the contraction rates are larger than the interconnection gains.

Proof. Pick $i \in \{1, \dots, r\}$. Using Assumptions (A1) and (A2) as well as the subadditivity and Cauchy-Schwartz properties of the weak pairing, we compute

$$\begin{aligned}
&[\![f_i(t, x_i, x_{-i}) - f_i(t, y_i, y_{-i})\,;\,x_i - y_i]\!]_i \\
&\qquad \le [\![f_i(t, x_i, x_{-i}) - f_i(t, y_i, x_{-i})\,;\,x_i - y_i]\!]_i + [\![f_i(t, y_i, x_{-i}) - f_i(t, y_i, y_{-i})\,;\,x_i - y_i]\!]_i \\
&\qquad \le -c_i\|x_i - y_i\|_i^2 + \sum\nolimits_{j=1, j\neq i}^r \ell_{ij}\|x_j - y_j\|_j \|x_i - y_i\|_i.
\end{aligned}$$

Next, we check the one-sided Lipschitz condition for the vector field on $\mathbb{R}^n$:

$$\begin{aligned}
&\sum\nolimits_{i=1}^r \eta_i \,[\![f_i(t, x_i, x_{-i}) - f_i(t, y_i, y_{-i})\,;\,x_i - y_i]\!]_i \\
&\qquad \le -\sum\nolimits_{i=1}^r \eta_i c_i \|x_i - y_i\|_i^2 + \sum\nolimits_{i,j=1, j\neq i}^r \eta_i \ell_{ij}\|x_j - y_j\|_j\|x_i - y_i\|_i \\
&\qquad = \begin{bmatrix} \|x_1 - y_1\|_1 \\ \vdots \\ \|x_r - y_r\|_r \end{bmatrix}^\top \operatorname{diag}(\eta)\Gamma \begin{bmatrix} \|x_1 - y_1\|_1 \\ \vdots \\ \|x_r - y_r\|_r \end{bmatrix} \\
&\qquad = \begin{bmatrix} \|x_1 - y_1\|_1 \\ \vdots \\ \|x_r - y_r\|_r \end{bmatrix}^\top \frac{\operatorname{diag}(\eta)\Gamma + \Gamma^\top \operatorname{diag}(\eta)}{2} \begin{bmatrix} \|x_1 - y_1\|_1 \\ \vdots \\ \|x_r - y_r\|_r \end{bmatrix},
\end{aligned}$$

so that the interconnected system is contracting if the gain matrix Γ is diagonally stable. Additionally,

$$\begin{aligned}
&\sum\nolimits_{i=1}^{r} \eta_i \,[\![f_i(t,x_i,x_{-i}) - f_i(t,y_i,y_{-i})\,;\, x_i - y_i]\!]_i \\
&\qquad\leq (\alpha(\Gamma)+\varepsilon) \begin{bmatrix} \|x_1-y_1\|_1 \\ \vdots \\ \|x_r-y_r\|_r \end{bmatrix}^{\top} \operatorname{diag}(\eta) \begin{bmatrix} \|x_1-y_1\|_1 \\ \vdots \\ \|x_r-y_r\|_r \end{bmatrix} \\
&\qquad= (\alpha(\Gamma)+\varepsilon) \sum\nolimits_{i=1}^{r} \eta_i \|x_i-y_i\|_i^2 = (\alpha(\Gamma)+\varepsilon)\|(x_1-y_1,\dots,x_r-y_r)\|_\eta^2.
\end{aligned}$$

This concludes the proof of strong infinitesimal contractivity. ■

3.7 Example: Nonlinear systems in Lur'e form

In this section we study *nonlinear system in Lur'e form*:

$$\begin{aligned}
\dot{x} &= Ax + B\Psi(y) =: f_{\text{Lur'e}}(x), \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m, \\
y &= Cx,
\end{aligned} \tag{3.53}$$

where the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ are arbitrary, and where the nonlinearity $\Psi\colon \mathbb{R}^m \to \mathbb{R}^m$ is diagonal and at least continuous.

Note that the Lur'e nonlinearities describe the interconnection from the output y to the input $\Psi(y)$. In this sense, Lur'e' form can be easily regarded as a nonlinear networked system.

We start by studying the case of a single nonlinearity $m = 1$ described by the scalar function $\psi\colon \mathbb{R} \to \mathbb{R}$. For any $\sigma_1, \sigma_2 \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that $\sigma_1 \leq \sigma_2$, we say ψ is *(σ_1, σ_2) slope-restricted* if

$$\sigma_1 (y_1 - y_2)^2 \leq \big(\psi(y_1) - \psi(y_2)\big)(y_1 - y_2) \leq \sigma_2 (y_1 - y_2)^2, \qquad \text{for all } y_1, y_2 \in \mathbb{R}. \tag{3.54}$$

Note that, when ψ is differentiable, condition (3.54) is equivalent to

$$\sigma_1 \leq \psi'(y) \leq \sigma_2. \tag{3.55}$$

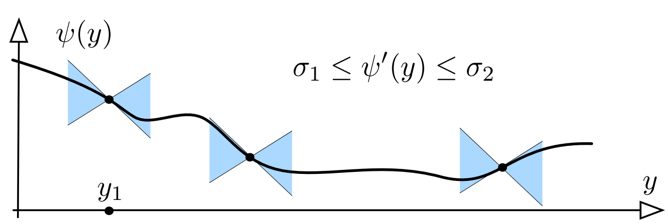


Figure 3.9: Illustration of the pointwise sector bound (3.55): at each y the graph of the function remains inside the sector defined by the lines at angular coefficients σ_1 and σ_2.

It is useful to transcribe the slope constraint into an equivalent linear matrix inequality. Specifically,

$$\begin{aligned}
\text{equation (3.54)} \quad &\iff \quad \begin{cases} \big(\psi(y_1) - \psi(y_2)\big) - \sigma_1(y_1 - y_2) \geq 0, \\ \sigma_2(y_1 - y_2) - \big(\psi(y_1) - \psi(y_2)\big) \geq 0, \end{cases} \\
&\iff \quad \Big(\big(\psi(y_1) - \psi(y_2)\big) - \sigma_1(y_1 - y_2)\Big)\Big(\sigma_2(y_1 - y_2) - \big(\psi(y_1) - \psi(y_2)\big)\Big) \geq 0, \\
&\iff \quad \begin{bmatrix} y_1 - y_2 \\ \psi(y_1) - \psi(y_2) \end{bmatrix}^{\top} M \begin{bmatrix} y_1 - y_2 \\ \psi(y_1) - \psi(y_2) \end{bmatrix} \geq 0,
\end{aligned}$$

where, given a positive multiplier m, the matrix M is defined by

$$M = m \begin{bmatrix} -\sigma_1\sigma_2 & (\sigma_1+\sigma_2)/2 \\ (\sigma_1+\sigma_2)/2 & -1 \end{bmatrix}. \tag{3.56}$$

Definition 3.24 (Incremental multiplier matrices). *The symmetric matrix $M \in \mathbb{R}^{2\times 2}$ is an incremental multiplier matrix for $\psi\colon \mathbb{R} \to \mathbb{R}$ if and only if, for all $y_1, y_2 \in \mathbb{R}$,*

$$\begin{bmatrix} y_1 - y_2 \\ \psi(y_1) - \psi(y_2) \end{bmatrix}^\top M \begin{bmatrix} y_1 - y_2 \\ \psi(y_1) - \psi(y_2) \end{bmatrix} \geq 0. \tag{3.57}$$

We summarize the calculations above and this definition as follows:

(i) for any (σ_1, σ_2) slope-restricted ψ, the matrix M (for example with $m = 1$) in equation (3.56) is an incremental multiplier;

(ii) if the nonlinearity ψ is *Lipschitz* with constant ℓ, then a suitable incremental multiplier matrix is $\begin{bmatrix} \ell^2 & 0 \\ 0 & -1 \end{bmatrix}$. This result follows from assuming $\sigma_1 = -\ell$, $\sigma_2 = \ell$, and $m = 1$ in equation (3.56); and

(iii) if the nonlinearity ψ is *weakly increasing*, then a suitable incremental multiplier matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This result follows from assuming $m = 2/\sigma_2$, $\sigma_1 = 0$, and taking the limit as $\sigma_2 \to +\infty$ in equation (3.56).

Theorem 3.25 (Contracting Lur'e models). *Consider a system in Lur'e form (3.53) with a nonlinearity ψ described by an incremental multiplier matrix M. If there exist $P = P^\top \succ 0$ and a scalar $c > 0$ satisfying*

$$\begin{bmatrix} PA + A^\top P + 2cP & PB \\ B^\top P & 0 \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top M \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{3.58}$$

then the system is strongly infinitesimally contracting with rate c with respect to $\|\cdot\|_{2,P^{1/2}}$.

Proof. For arbitrary $x_1, x_2 \in \mathbb{R}^n$, compute $\psi_i = \psi((Cx)_i)$, $i \in \{1,2\}$. Since M is an incremental multiplier matrix, for all x_1, x_2 and corresponding ψ_1, ψ_2,

$$\begin{bmatrix} x_1 - x_2 \\ \psi_1 - \psi_2 \end{bmatrix}^\top Q_M \begin{bmatrix} x_1 - x_2 \\ \psi_1 - \psi_2 \end{bmatrix} \geq 0, \quad \text{for } Q_M := \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top M \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}.$$

By assumption (3.58), for all x_1, x_2 and corresponding ψ_1, ψ_2,

$$\begin{bmatrix} x_1 - x_2 \\ \psi_1 - \psi_2 \end{bmatrix}^\top (Q_A + Q_M) \begin{bmatrix} x_1 - x_2 \\ \psi_1 - \psi_2 \end{bmatrix} \leq 0, \quad \text{for } Q_A := \begin{bmatrix} PA + A^\top P + 2cP & PB \\ B^\top P & 0 \end{bmatrix}.$$

But these two inequalities imply that, for all x_1, x_2 and corresponding ψ_1, ψ_2,

$$\begin{bmatrix} x_1 - x_2 \\ \psi_1 - \psi_2 \end{bmatrix}^\top Q_A \begin{bmatrix} x_1 - x_2 \\ \psi_1 - \psi_2 \end{bmatrix} \leq 0.$$

We now expand this inequality to obtain:

$$\begin{aligned} &(x_1 - x_2)^\top (PA + A^\top P + 2cP)(x_1 - x_2) + 2(x_1 - x_2)^\top PB(\psi_1 - \psi_2) \leq 0 \\ \iff \quad & \Big((Ax_1 + B\psi_1) - (Ax_2 + B\psi_2)\Big)^\top P(x_1 - x_2) \leq -c(x_1 - x_2)^\top P(x_1 - x_2) \\ \iff \quad & \mathsf{osLip}_{2,P^{1/2}}(f_{\text{Lur'e}}) \leq -c. \end{aligned}$$

This concludes the proof. ■

3.8 Historical notes and further reading

One-sided Lipschitz maps The concept of one-sided Lipschitz condition was apparently introduced by (Dahlquist, 1976). At the same time, the key idea appears in (Chua and Green, 1976), whereby minus the vector field is called *uniformly increasing* and in the classic work on discontinuous differential equations, see (Filippov, 1988, (Chapter 1, page 5)) and references therein. Smith (1986a) adopts the same condition to study periodic solutions of differential equations. The nomenclature, one-sided Lipschitz condition, is then adopted in the classic textbook (Hairer et al., 1993) on numerical analysis. Since these early references, related conditions have been studied in the literature under such various names as the *QUAD condition* in (Lu and Chen, 2006), the *nonlinear measure* (Qiao et al., 2001), the *dissipative Lipschitz condition* (Caraballo and Kloeden, 2005), and *incremental quadratic stability* in (D'Alto and Corless, 2013). A related unifying concept is the *logarithmic Lipschitz constant*, advocated in the excellent surveys (Söderlind, 2006; Aminzare and Sontag, 2014b) and, for example, in (Aminzare and Sontag, 2013). Comparisons between the Lipschitz conditions, the QUAD condition, and contraction theory are detailed in (DeLellis et al., 2011), see also (Hairer et al., 1993, Section 1.10, Exercise 6). The treatment in this chapter is based upon (Davydov et al., 2022a; Jafarpour et al., 2023).

Contraction theory in the systems and control literature Early references on infinitesimally contracting systems include the classic works by Lewis (1951), Demidovič (1961) and Yoshizawa (1966). A historical review is given by Pavlov et al. (2004).

Contraction theory applied to control problems was studied in the visionary work (Lohmiller and Slotine, 1998). Early works on incremental stability include (Fromion et al., 1996; Angeli, 2002). Notable surveys on contraction theory include (Sontag, 2010; Jouffroy and Fossen, 2010; Aminzare and Sontag, 2014b; Di Bernardo et al., 2016; Tsukamoto et al., 2021; Giesl et al., 2023). Differential geometric approaches include (Simpson-Porco and Bullo, 2014; Forni and Sepulchre, 2014).

Regarding the Main Equivalence Theorem 3.7, the implication *(iii)* implies *(iv)* is due to (Lohmiller and Slotine, 1998), and the reverse direction to (Aminzare and Sontag, 2014b). An alternative (and more common) proof based upon the log norm of the Jacobian and the Coppel's inequality is given in Exercise E3.5. The uniform monotonicity property originates in the work by Minty (1962, 1964). The Demidovich Lemma 3.1 is essentially given in the classic textbook (Hairer et al., 1993, Exercise 6, Chapter I.10). Lemma 3.4 with the generalization of the equality ($\mathsf{osLip}(f) = \sup_x \mu(Df(x))$ to normed spaces with weak pairings is due to (Davydov et al., 2022a), based upon the earlier works (Söderlind, 2006; Aminzare and Sontag, 2014b). Theorem 3.9 is due to (Lohmiller and Slotine, 1998), see also (Russo et al., 2010); the statement about the norm of the vector field being a Lyapunov function is given by (Coogan and Arcak, 2013). Theorem 3.15 is due to (Lohmiller and Slotine, 1998; Russo et al., 2010), closely related statements go back to (Smith, 1986b). Theorems 3.16 and 3.19 are stated and proved using tools from (Davydov et al., 2022a). The results on stability with respect to external inputs in Corollary 3.17 originates in (Desoer and Haneda, 1972, Theorem A in Appendix) and the more recent (Sontag, 2010; Hamadeh et al., 2015).

The modularity properties of contracting systems have been long investigated, starting for example with the early work (Slotine and Lohmiller, 2001). The main interconnection theorem is an outgrowth of the work on composite norms by (Ström, 1975, Section 5), the method of vector Lyapunov functions (Šiljak, 1978), as well as the results in (Russo et al., 2013) and (Simpson-Porco and Bullo, 2014, Lemma 3.2).

Non-Euclidean concepts The classic analysis setting for contraction theory is with respect to the Euclidean norm ℓ_2; LMI tools are adopted to design optimal weight matrices. The interest for non-Euclidean norms (e.g., ℓ_1, ℓ_∞ and polyhedral norms) is more recent and motivated by classes of network systems, such as biological transcriptional systems (Russo et al., 2010), Hopfield neural networks (Fang and Kincaid, 1996; Qiao et al., 2001), chemical reaction networks (Al-Radhawi and Angeli, 2016), traffic networks (Coogan and Arcak, 2015a; Como et al., 2015; Coogan, 2019), multi-vehicle systems (Monteil et al., 2019), and coupled oscillators (Russo et al., 2013; Aminzare and Sontag, 2014a).

Hopfield neural networks The treatment of Hopfield neural networks, including Lemma 3.21 and Exercise E3.10, follows closely the results in (Davydov et al., 2022c), which extends the earlier works (Fang and Kincaid, 1996; Qiao et al., 2001; Jafarpour et al., 2021b) and, ultimately, the original treatment by Hopfield (1982).

The Hopfield and firing-rate models are membrane potential models, widely studied in dynamical neuroscience. (Miller and Fumarola, 2012) provides a careful comparison between the two. Shortly after Hopfield's work (Hopfield, 1984), control-theoretic ideas were proposed by Michel et al. (1989). Later, Kaszkurewicz and Bhaya (1994); Forti et al. (1994); Forti and Tesi (1995) obtained various stability results based upon the Lyapunov diagonal stability of the synaptic matrix. Notably, Fang and Kincaid (1996) were the first to obtain stability results on ℓ_p logarithmic norms of the Jacobian for neural networks with smooth activation functions. Arik (2002) proposes a quasi-dominance condition on the synaptic matrix. Qiao et al. (2001) propose the notion of the nonlinear measure of a map to study global asymptotic stability. A comprehensive survey on continuous-time RNNs is (Zhang et al., 2014). Finally, contractivity of RNNs with respect to the ℓ_2 norm has been studied, e.g., see the early reference (Fang and Kincaid, 1996), the related discussion in (Revay et al., 2020), and the recent work (Kozachkov et al., 2021).

3.9 Exercises

E3.1 **Properties of the one-sided Lipschitz constant.** Given the definition of one-sided Lipschitz constant, prove properties (3.20a)–(3.20d) in Lemma 3.5.
Note: *Here is a consequence of the translation property (3.20c): if f satisfies $\mathsf{osLip}(f) \leq -c$ (so that it is strongly infinitesimally contracting with rate c), then $f(x) = g(x) - cx$ for all $x \in \mathbb{R}^n$, where g satisfies $\mathsf{osLip}(g) \leq 0$ (we will call such maps weakly infinitesimally contracting in the next Chapter).*

Answer: The proof of positive homogeneity (3.20a) and subadditivity (3.20b) is elementary from the properties of weak pairings and omitted. The translation property (3.20c) follows from the properties of weak pairings in Exercise E2.31. Regarding the uniform monotonicity property (3.20d), from the definition (3.14), we know that, for all $x, y \in C$,

$$|\,\mathsf{osLip}(f)|\|x-y\|^2 \leq |\,[\![f(x) - f(y)\,;\, x-y]\!]\,| \leq \|f(x)-f(y)\|\|x-y\| \quad \Longrightarrow \quad |\,\mathsf{osLip}(f)|\|x-y\| \leq \|f(x)-f(y)\|.$$

E3.2 **A global inverse function theorem based on contractivity (Desoer and Haneda, 1972).** The following facts are well known. A smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a *global diffeomorphism* (or simply a diffeomorphism) if it is a bijection of $\mathbb{R}^n$ onto itself with a smooth inverse. Given a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with Jacobian Df and a point $x \in \mathbb{R}^n$,

(i) f is a diffeomorphism in a neighborhood of x if and only if $Df(x)$ is full rank, and
(ii) f is a global diffeomorphism if and only if $Df(x)$ is full rank for all $x \in \mathbb{R}^n$ and $x \mapsto \|f(x)\|$ is radially unbounded.

Theorem 3.26 (Desoer-Haneda global inverse function theorem). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $\|\cdot\|$ be a norm on $\mathbb{R}^n$. If there exists a scalar function $c\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$c(\alpha) > 0 \quad \text{for all } \alpha \in \mathbb{R}_{\geq 0} \quad \text{and} \quad \int_0^{+\infty} c(\alpha) d\alpha = \infty,$$
$$\mu(Df(x)) \leq -c(\|x\|) < 0 \quad \text{for all } x \in \mathbb{R}^n,$$

then $Df(x)$ is Hurwitz at all $x \in \mathbb{R}^n$, $x \mapsto \|f(x)\|$ is radially unbounded, and f is a global diffeomorphism.

Prove the theorem, i.e., prove that the Jacobian is full rank everywhere and that the f is radially unbounded.

Answer: One can show that Df is full rank everywhere using the product property (2.30d) of the log norm. Radial unboundedness is proved using the Taylor expansion as well as the convexity property (2.32) and the product property (2.30d).
For the complete proof we refer to the original article (Desoer and Haneda, 1972) or (Desoer and Vidyasagar, 1975, pages 34-35).

E3.3 **Lipschitz vector fields and the one-sided Lipschitz condition.** Recall that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on $\mathbb{R}^n$ if there exists a constant $L > 0$ such that

$$\|f(x) - f(y)\|_2 \leq L\|x-y\|_2.$$

Given a Lipschitz continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with constant L and a positive definite matrix $P = P^\top \succ 0$, show that:

(i) there exists a constant $L_{2,P^{1/2}} > 0$ such that

$$\|f(x) - f(y)\|_{2,P^{1/2}} \leq L_{2,P^{1/2}}\|x-y\|_{2,P^{1/2}},$$

(ii) f satisfies the one-sided Lipschitz condition (3.10) with coefficient $L_{2,P^{1/2}} > 0$, that is,

$$\begin{aligned} &\mu_{2,P^{1/2}}(Df(t,x)) \leq L_{2,P^{1/2}} \qquad \text{for all } t \in \mathbb{R}_{\geq 0} \text{ and } x \in \mathbb{R}^n \\ \iff \quad & (x-y)^\top P\big(f(t,x) - f(t,y)\big) \leq L_{2,P^{1/2}}\|x-y\|_P^2, \quad \text{for all } t \in \mathbb{R}_{\geq 0} \text{ and } x, y \in \mathbb{R}^n. \end{aligned}$$

(iii) there exist vector fields that have negative one-sided Lipschitz constant and that are not Lipschitz.

Answer: Regarding statement (i), the statement follows by appropriate bounding and involves the largest and smallest eigenvalues of P.

Regarding statement (ii), the inequality is a consequence of the inequality $\langle\!\langle v\,;\,w\rangle\!\rangle \leq \|v\|\|w\|$ which holds on any inner product space.

Regarding statement (iii), an example is $\dot{x} = -x^3$.

E3.4 **One-sided Lipschitz uniqueness for differential equations.** Let $C \subset \mathbb{R}^n$ be convex and open. Given $t_0 \in \mathbb{R}_{\geq 0}$ and $a > 0$, let $f\colon [t_0, t_0 + a] \times C \to \mathbb{R}^n$ be continuous. Consider the initial value

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0 \in C. \tag{E3.1}$$

Assume there exists a norm $\|\cdot\|$ and a constant $b \in \mathbb{R}$ such that, for all (t, x) and $(t, \bar{x})$ in $[t_0, t_0 + a] \times C$,

$$\mathsf{osLip}(f_t) \leq b. \tag{E3.2}$$

Show that the initial value (E3.1) has at most one solution for time $[t_0, t_0 + a]$. The solution, if it exists, is continuously differentiable.

Note: *An early reference where one-sided Lipschitzness is shown to imply uniqueness is (Krasnoselskiĭ and Krein, 1955), see also the discussion in (Filippov, 1988, (Chapter 1, page 5)) and in (Agarwal and Lakshmikantham, 1993, Theorem 3.2.2).*

Answer: Consider two trajectories starting at the same initial condition. From Theorem 3.7, uniqueness follows from noting that the distance between these two trajectories starts at zero and obeys the linear differential inequality (3.23).

E3.5 **Main solution estimate, without weak pairings.** Consider the dynamical system $\dot{x}(t) = f(t, x(t))$ defined by a continuously-differentiable $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$. Assume $C \subset \mathbb{R}^n$ is open, convex and f-invariant. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with associated log norm μ. Then every two trajectories $x(t) = \phi_t(x_0)$ and $y(t) = \phi_t(y_0)$, with $x_0, y_0 \in C$, satisfy

(i) the main solution estimate: for all $t \in \mathbb{R}_{\geq 0}$,

$$D^+\|x(t) - y(t)\| \leq \int_0^1 \mu\Big(Df\big(t, \mathrm{seg}(t, \alpha)\big)\Big) d\alpha \;\cdot\; \|x(t) - y(t)\|, \tag{E3.3}$$

where $\mathrm{seg}(t, \alpha) = x(t) + \alpha(y(t) - x(t))$, $\alpha \in [0, 1]$, parametrizes the segment from $x(t)$ to $y(t)$ and the quantity $\int_0^1 \mu\Big(Df\big(t, \mathrm{seg}(t, \alpha)\big)\Big) d\alpha$ is the average of the lognorm of the Jacobian along this segment; and

(ii) the following special cases: if there exists $b \in \mathbb{R}$ such that

$$\mu(Df(t, x)) \leq b, \qquad \text{for all } (t, x) \in \mathbb{R}_{\geq 0} \times C, \tag{E3.4}$$

then, for all $t \in \mathbb{R}_{\geq 0}$,

$$D^+\|x(t) - y(t)\| \leq b\|x(t) - y(t)\| \quad \text{and} \quad \|x(t) - y(t)\| \leq \mathrm{e}^{bt}\|x_0 - y_0\|. \tag{E3.5}$$

Answer: As in the statement, for given fixed $t \geq 0$ and $\varepsilon > 0$, define the map $\mathrm{seg}\colon [t, t + \varepsilon] \times [0, 1] \to C$ by $\mathrm{seg}(\tau, \alpha) = \phi_{\tau - t}\big(x(t) + \alpha(y(t) - x(t))\big)$ parametrizes the segment from $x(\tau)$ to $y(\tau)$. This map is well posed because C is forward invariant and convex. We note the following properties:

$$\mathrm{seg}(\tau, 0) = x(\tau), \quad \mathrm{seg}(\tau, 1) = y(\tau), \tag{E3.6}$$

$$\frac{\partial}{\partial \alpha}\mathrm{seg}(t, \alpha) = \frac{\partial}{\partial \alpha}\big(x(t) + \alpha(y(t) - x(t))\big) = y(t) - x(t). \tag{E3.7}$$

We also study the evolution of an associated "variational vector:"

$$\frac{\partial}{\partial \tau}\frac{\partial}{\partial \alpha}\operatorname{seg}(\tau,\alpha) = \frac{\partial}{\partial \alpha}\frac{\partial}{\partial \tau}\operatorname{seg}(\tau,\alpha) = \frac{\partial}{\partial \alpha} f(\tau,\operatorname{seg}(\tau,\alpha)) = Df(\tau,\operatorname{seg}(\tau,\alpha))\frac{\partial}{\partial \alpha}\operatorname{seg}(\tau,\alpha).$$

Since this differential equation is linear time-varying, Lemma 2.2(i) implies

$$D^+\Big\|\frac{\partial}{\partial \alpha}\operatorname{seg}(\tau,\alpha)\Big\| \leq \mu\big(Df(\tau,\operatorname{seg}(\tau,\alpha))\big)\Big\|\frac{\partial}{\partial \alpha}\operatorname{seg}(\tau,\alpha)\Big\|. \tag{E3.8}$$

We are now ready to study how the distance from $x(t)$ to $y(t)$ varies. To do so, we study the length of the segment changes along the flow by invoking the fundamental theorem of calculus:

$$\|x(\tau)-y(\tau)\| = \|\operatorname{seg}(\tau,1)-\operatorname{seg}(\tau,0)\| = \Big\|\int_0^1 \frac{\partial \operatorname{seg}}{\partial \alpha}(\tau,\alpha)d\alpha\Big\|. \tag{E3.9}$$

We now note that

$$\Big\|\int_0^1 \frac{\partial \operatorname{seg}}{\partial \alpha}(\tau,\alpha)d\alpha\Big\| \leq \int_0^1 \Big\|\frac{\partial \operatorname{seg}}{\partial \alpha}(\tau,\alpha)\Big\|d\alpha, \tag{E3.10}$$

and that, at $\tau = t$, left and right hand side are equal to each other because they both are equal to $\|x(t)-y(t)\|$. These two facts[2] imply that the two upper right Dini derivatives (with respect to τ) satisfy:

$$D^+\|x(\tau)-y(\tau)\|\Big|_{\tau=t} \leq D^+\int_0^1 \Big\|\frac{\partial \operatorname{seg}}{\partial \alpha}(\tau,\alpha)\Big\|d\alpha\Big|_{\tau=t} = \int_0^1 D^+\Big\|\frac{\partial \operatorname{seg}}{\partial \alpha}(t,\alpha)\Big\|d\alpha. \tag{E3.11}$$

Plugging in the inequality (E3.8), we obtain

$$D^+\|x(t)-y(t)\| \leq \int_0^1 \mu\Big(Df\big(t,\operatorname{seg}(t,\alpha)\big)\Big)\underbrace{\Big\|\frac{\partial}{\partial \alpha}\operatorname{seg}(t,\alpha)\Big\|}_{\|x(t)-y(t)\|}d\alpha. \tag{E3.12}$$

This concludes the proof of statement (i). Statement (ii) is an immediate consequence.

E3.6 **The curve norm derivative formula for ℓ_1 and ℓ_∞.** This exercise asks the reader to provide explicit direct proof that the curve norm derivative formula holds almost everywhere for the sign and max pairings.

We adopt the usual definitions of the sign function $\operatorname{sign}\colon \mathbb{R} \to \{-1,0,+1\}$, defined by $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x < 0$, and $\operatorname{sign}(0) = 0$, and of the indicator function of a set A, defined by $\chi_A(x) = 1$, if $x \in A$, and $\chi_A(x) = 0$, otherwise. We start with a basic result.

Lemma 3.27 (Upper right Dini derivative of the absolute value function). *Given a differentiable* $y\colon]a,b[\to \mathbb{R}$,

$$\begin{aligned} D^+|y(t)| &= \begin{cases} \dot{y}(t), & \text{if } y(t) > 0 \\ |\dot{y}(t)|, & \text{if } y(t) = 0 \\ -\dot{y}(t), & \text{if } y(t) < 0 \end{cases} \\ &= \dot{y}(t)\operatorname{sign}(y(t)) + |\dot{y}(t)|\chi_{\{0\}}(y(t)). \end{aligned}$$

Proof. The result is trivial for all times t such that $y(t) \neq 0$. Next, note $|y(t)| = \max\{y(t),-y(t)\}$ and apply Danskin's Lemma A.16(iii) to this max function to prove that, at each time t when $y(t) = 0$, the upper right Dini derivative equals $\max\{\dot{y}(t),-\dot{y}(t)\} = |\dot{y}(t)|$. ■

Recall the sign pairing $[\![x\,;\,y]\!]_1 = \|y\|_1\, x^\top \operatorname{sign}(y)$. As in (Deimling, 1985, Example 13.1(b)), but with different notation, define the *$(\ell_1,+)$ pairing* on $\mathbb{R}^n$ by

$$(x,y)_{1,+} = \|y\|_1\Big(x^\top \operatorname{sign}(y) + \sum_{i=1}^n |x_i|\chi_{\{0\}}(y_i)\Big).$$

[2] Given differentiable functions $f_1, f_2\colon [t^*-\varepsilon, t^*+\varepsilon] \to \mathbb{R}$, if $f_1(t) \leq f_2(t)$ for $t \geq t^*$ and $f_1(t^*) = f_2(t^*)$, then $f_1'(t^*) \leq f_2'(t^*)$.

Lemma 3.28 (Curve norm derivative for ℓ_1). *Given a differentiable* $y\colon]a,b[\to \mathbb{R}^n$,

(i) $D^+\|y(t)\|_1 = \dot{y}(t)^\top \operatorname{sign}(y(t)) + \sum_{i=1}^n |\dot{y}_i(t)|\, \chi_{\{0\}}(y_i(t)) = \big(\dot{y}(t), y(t)\big)_{1,+}$;

(ii) $\|y(t)\|_1 D^+\|y(t)\|_1 = [\![\dot{y}(t)\,;\,y(t)]\!]_1$ *for almost all times.*

Note: statement (i) is more informative than (ii) since the statement (ii) loses information at $y(t) = \mathbb{0}_m$.

Next, recall the max pairing $[\![x\,;\,y]\!]_\infty = \|y\|_\infty \max_{i\in I_\infty(y)} x_i \operatorname{sign}(y_i)$, with the shorthand $I_\infty(y) = \{i \in \{1,\dots,n\} \,:\, |y_i| = \|y\|_\infty\}$.

Lemma 3.29 (Curve norm derivative for ℓ_∞). *Given a differentiable* $y\colon]a,b[\to \mathbb{R}^n$,

(i) $D^+\|y(t)\|_\infty = \max\limits_{i\in I_\infty(y(t))} \dot{y}_i(t) \operatorname{sign}(y_i(t)) + \chi_{\{\mathbb{0}_m\}}(y(t))\|\dot{y}(t)\|_\infty$;

(ii) $\|y(t)\|_\infty D^+\|y(t)\|_\infty = [\![\dot{y}(t)\,;\,y(t)]\!]_\infty$ *for almost all times.*

Prove Lemmas 3.28 and 3.29.

Answer:

Proof of Lemma 3.28. Statement (i) is an immediate consequence of Lemma 3.27, since $\|y(t)\|_1 = \sum_{i=1}^n |y_i(t)|$. Statement (ii) is an immediate implication of statement (i). ■

Proof of Lemma 3.29. From Danskin's Lemma A.16(iii), $f_{\max}(t) = \max\{f_1(t),\dots,f_m(t)\}$, with differentiable $f_1,\dots,f_m$, satisfies $D^+f_{\max}(t) = \max\left\{\frac{d}{dt}f_i(t) \,:\, i \in \operatorname{argmax}(f_{\max}(t))\right\}$. If the functions f_i are max functions themselves (e.g., absolute values), a simple argument shows

$$D^+f_{\max}(t) = \max\left\{D^+f_i(t) \,:\, i \in \operatorname{argmax}(f_{\max}(t))\right\}.$$

Now $\|y(t)\|_\infty = \max_{i\in\{1,\dots,m\}} |y_i(t)|$ implies

$$D^+\|y(t)\|_\infty = \max\left\{D^+|y_i(t)| \,:\, i \in \operatorname{argmax}(\|y(t)\|_\infty)\right\} = \max_{i\in I_\infty(y(t))} D^+|y_i(t)|,$$

where we use the simple fact $i \in \operatorname{argmax}(\|y(t)\|_\infty)$ if and only if $i \in I_\infty(y(t))$. Lemma 3.27 implies

$$\begin{aligned} D^+\|y(t)\|_\infty &= \max_{i\in I_\infty(y(t))} \Big(\dot{y}_i(t)\operatorname{sign}(y_i(t)) + |\dot{y}_i(t)|\chi_{\{0\}}(y_i(t))\Big) \\ &= \begin{cases} \max_{i\in I_\infty(y(t))} \dot{y}_i(t)\operatorname{sign}(y_i(t)), & \text{if } y(t) \neq \mathbb{0}_m, \\ \max_{i\in\{1,\dots,m\}} |\dot{y}_i(t)|, & \text{if } y(t) = \mathbb{0}_m. \end{cases} \\ &= \max_{i\in I_\infty(y(t))} \dot{y}_i(t)\operatorname{sign}(y_i(t)) + \chi_{\{\mathbb{0}_m\}}(y(t))\|\dot{y}(t)\|_\infty. \end{aligned}$$

This equality proves statement (i) and statement (ii) is an immediate consequence. ■

E3.7 **Grönwall–Bellman–Halanay inequality.** The following delay differential inequality from (Liu et al., 2011a) generalizes the Grönwall Comparison Lemma in Exercise E2.1 and the classic Halanay inequality (Halanay, 1966).

Let t_0 and r be nonnegative constants. Let $z\colon [t_0 - r, \infty) \to \mathbb{R}_{\geq 0}$ be continuous and satisfy

$$D^+z(t) \leq -\mu a(t)z(t) + \lambda a(t) \sup_{-r\leq s\leq 0} z(t+s) + \gamma(t), \tag{E3.13}$$

on $[t_0 - r, \infty)$, where $t \mapsto \gamma(t)$ and $t \mapsto a(t)$ are piecewise continuous functions with $\gamma(t) \geq 0$ and $a(t) \in (0,1]$ for all $t \geq 0$, and μ and λ are constants satisfying $\mu > \lambda > 0$. Define the shorthand $A(s,t) = \int_s^t a(\tau)d\tau$ for all $0 \leq s \leq t \in \mathbb{R}_{\geq 0}$. Show that, for $t \in [t_0, \infty)$,

$$z(t) \leq \mathrm{e}^{-\rho A(t_0,t)}\, z_0 + \int_{t_0}^t \mathrm{e}^{-\rho A(\tau,t)}\, \gamma(\tau) d\tau, \tag{E3.14}$$

where $\rho > 0$ is the root of $\rho = \mu - \lambda\, e^{\rho r}$ and $z_0 = \sup_{-r \leq s \leq 0} z(t_0 + s)$.

Answer: We refer to (Liu et al., 2011a) for a brief proof.

E3.8 **Mean value theorem for vector-valued functions.** For a continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with Jacobian Df, show that

$$f(x) - f(y) = \Big(\int_0^1 Df\big(y + s(x-y)\big) ds \Big)(x - y),$$

so that, if $y = \mathbb{0}_n$,

$$f(x) - f(\mathbb{0}_n) = \Big(\int_0^1 Df\big(sx\big) ds \Big) x. \tag{E3.15}$$

Note: *An example reference is (Abraham et al., 1988, Proposition 2.4.7)*

Answer: We leave the answer to the reader.

E3.9 **Revisiting the Krasovskiĭ's method for the design of Lyapunov functions.** Consider the continuously differentiable dynamical system $\dot{x} = f(x)$ with Jacobian Df. Given a matrix $P = P^\top \succ 0$ and a positive scalar $c > 0$, assume f satisfies the Demidovich condition

$$P Df(x) + Df(x)^\top P \preceq -2cP, \qquad \text{for all } x \in C \subset \mathbb{R}^n.$$

Prove directly (without invoking the contraction theorems) that

(i) there exists $x^* \in \mathbb{R}^n$ such that $f(x^*) = \mathbb{0}_n$,

(ii) if $f(\mathbb{0}_n) = \mathbb{0}_n$, then the functions $V_1(x) = \|x\|_P^2$ and $V_2(x) = \|f(x)\|_P^2$ are global Lyapunov functions so that the origin $\mathbb{0}_n$ is globally asymptotically stable.

Hint: *You are allowed to use Theorem 3.26 in Exercise E3.2 and/or Demidovich Lemma 3.1.*

Answer: Statement (i) is an immediate consequence of Exercise E3.2: under the state assumptions f is a global diffeomorphism and so $f^{-1}(\mathbb{0}_n)$ is an equilibrium for f.

Regarding statement (ii), we leave the study of the function V_1 to the reader and focus on proving that:

(F1) V_2 is globally positive definite about $\mathbb{0}_n$,

(F2) $\mathscr{L}_f V_2$ is globally negative definite about $\mathbb{0}_n$, and

(F3) V_2 is radially unbounded.

Regarding statement (F1), we know $f(\mathbb{0}_n) = \mathbb{0}_n$ and $\|f(x)\|_P^2 \geq 0$ for all $x \in \mathbb{R}^n$. We need to show that $V_2(x) = 0$ implies $x = \mathbb{0}_n$. By contradiction, assume there exists a $q \in \mathbb{R}^n$ satisfying $f(q) = \mathbb{0}_n$ and $q \neq \mathbb{0}_n$. But then we have $q^\top P q > 0$ and, from the integral Demidovich condition (3.10) in Lemma 3.1(iii) with $y = \mathbb{0}_n$, we obtain

$$0 = q^\top P f(q) \leq -c q^\top P q < 0,$$

which is a contradiction. Regarding statement (F2), it is easy to verify the statement directly. Given $V_2(x) = f(x)^\top P f(x)$, we compute:

$$\begin{aligned}
\mathscr{L}_f V_2(x) &= f(x)^\top P Df(x) f(x) + \Big(Df(x) f(x) \Big)^\top P f(x) \\
&= f(x)^\top P Df(x) f(x) + f(x)^\top Df(x)^\top P f(x) \\
&= f(x)^\top \big(P Df(x) + Df(x)^\top P \big) f(x) \leq -c f(x)^\top P f(x) = -c V_2(x),
\end{aligned}$$

where we used $\mathscr{L}_f f(x) = Df(x) f(x)$. Finally, regarding statement (F3), we proceed as follows. Suppose there exists an upper bound α such that $\|f(x)\|_2 \leq \alpha$ as $\|x\|_2 \to \infty$. Then

$$\frac{|x^\top P f(x)|}{\|x\|_2^2} \leq \frac{\|x\|_2 \|P\|_2 \alpha}{\|x\|_2^2} = \frac{\|P\|_2 \alpha}{\|x\|_2} \to 0 \quad \text{as } \|x\|_2 \to \infty.$$

Now, note $x^\top P x \geq \lambda_{\min}(P)\|x\|_2^2$ and, from the integral Demidovich condition (3.10) with $y = \mathbb{0}_n$, we know $x^\top P f(x) < -c\lambda_{\min}(P)\|x\|_2^2$ and therefore

$$\frac{x^\top P f(x)}{\|x\|_2^2} < -c\lambda_{\min}(P).$$

But this is a contradiction because $\frac{x^\top P f(x)}{\|x\|_2^2}$ cannot be upper bounded by a strictly negative number and, at the same time, have vanishing magnitude as $\|x\|_2 \to \infty$. Therefore $\|f(x)\|_2$ is unbounded as $\|x\|_2 \to \infty$ and so is $V_2(x) = \|f(x)\|_P$.

E3.10 **Contractivity of the firing rate model.** This exercises generalizes Lemma 3.21, is based upon the lognorm results in Exercise E2.22, and contains the analysis in (Davydov et al., 2022c). For the firing rate neural network

$$\dot{x} = -Cx + \Psi(Ax + u) =: f_{\mathrm{FR}}(x, u),$$

where A is arbitrary, C is a diagonal matrix of strictly positive *dissipation rates*, and Ψ is a differentiable activation function satisfying $d_{\min} := \inf_{y\in\mathbb{R}} \frac{d\psi_i}{dy}(y) \geq 0$ and $d_{\max} := \sup_{y\in\mathbb{R}} \frac{d\psi_i}{dy}(y) < \infty$. Show that:

(i) with respect to any norm and with respect to the weighted ℓ_∞ norm:

$$\mathsf{osLip}(f_{\mathrm{FR}}) = \max_{d\in[d_{\min},d_{\max}]^n} \mu(-C + [d]A), \tag{E3.16}$$

$$\mathsf{osLip}_{\infty,[\eta]^{-1}}(f_{\mathrm{FR}}) = \max\{\mu_{\infty,[\eta]^{-1}}(-C + d_{\min}A), \mu_{\infty,[\eta]^{-1}}(-C + d_{\max}A)\}; \tag{E3.17}$$

(ii) the minimum value and optimal weight $\eta \in \mathbb{R}^n_{>0}$ of $\mathsf{osLip}_{\infty,[\eta]^{-1}}(f_{\mathrm{FR}})$ is given by the quasiconvex optimization problem:

$$\begin{aligned} \inf_{b\in\mathbb{R},\eta\in\mathbb{R}^n_{>0}} \quad & b \\ \text{s.t.} \quad & (-C + d_{\min}|A|_{\mathrm{M}})\eta \leq b\eta \\ & (-C + d_{\max}|A|_{\mathrm{M}})\eta \leq b\eta \end{aligned}$$

(iii) for simplifying parameter choices,

$$\inf_{\eta\in\mathbb{R}^n_{>0}} \mathsf{osLip}_{\infty,[\eta]^{-1}}(f_{\mathrm{FR}}) = \begin{cases} \max\left\{\alpha(-C), \alpha(-C + d_{\max}|A|_{\mathrm{M}})\right\}, & \text{if } d_{\min} = 0, \\ -c + d_{\max}\max\left\{0, \alpha(|A|_{\mathrm{M}})\right\}, & \text{if } d_{\min} = 0 \text{ and } C = cI_n, \end{cases}$$

where the optimal weight η is computed using the results in Lemma 2.31. For example, if $|A|_{\mathrm{M}}$ is irreducible, then the optimal η is the right dominant eigenvector of $-C + d_{\max}|A|_{\mathrm{M}}$.

Answer: Equation (E3.16) is obtained as in the proof of Lemma 3.21. Equation (E3.17) is an immediate consequence of Exercise E2.22. Statements (ii) and (iii) follow from reasoning as in Sections 2.7.2 and 2.7.3 respectively. We refer to (Davydov et al., 2022c) for additional details.

E3.11 **ℓ_1 norm Demidovich condition.** Given a convex set $C \subset \mathbb{R}^n$, consider a continuously-differentiable map $f : C \to \mathbb{R}^n$. Pick an invertible matrix Q and a scalar b. Show by direct computation that the following statements are equivalent:

(i) the Jacobian of f has weighted ℓ_1 log norm uniformly bounded by b, that is,

$$\mu_{1,Q}(Df(x)) \leq b, \qquad \text{for all } x \in C, \tag{E3.18}$$

(ii) f satisfies the ℓ_1 norm Demidovich condition

$$\operatorname{sign}(Qv)^\top (QDf(x))\, v \leq b\|v\|_{1,Q}, \qquad \text{for all } w \in \mathbb{R}^n, \text{ for all } x \in C, \tag{E3.19}$$

(iii) f satisfies the integral ℓ_1 norm Demidovich condition

$$\operatorname{sign}(Q(x-y))^\top Q(f(x)-f(y)) \le b\|x-y\|_{1,Q}, \qquad \text{for all } (x,y)\in C\times C. \tag{E3.20}$$

Answer: The equivalence (i) $\iff$ (ii) is a consequence of the generalization in Exercise E2.10 to arbitrary weight matrices of equation (2.63a) Lemma 2.21.

Next, we show that (ii) $\iff$ (iii). First suppose that (ii) holds. The Mean-Value Theorem implies $f(x)-f(y) = \int_0^1 (Df(y+s(x-y))ds)\,(x-y)$ and, in turn,

$$\begin{aligned}\operatorname{sign}(Q(x-y))^\top Q(f(x)-f(y)) &= \int_0^1 \operatorname{sign}(Q(x-y))^\top \big(QDf(y+s(x-y))\big)(x-y)ds\\ &\le b\|x-y\|_{1,Q}.\end{aligned}$$

This means that (iii) hold. Now assume that (iii) holds, then we pick $v\in\mathbb{R}^n$, $h\in\mathbb{R}_{\ge 0}$, and $x=y+hv$. In this case, equation (E3.20) holds implies

$$\operatorname{sign}(Qv)^\top Q(f(y+hv)-f(y)) \le bh\|v\|_{1,Q}.$$

In the limit as h goes to zero, we obtain

$$\lim_{h\to 0^+} \operatorname{sign}(Qv)^\top Q\left(\frac{f(y+hv)-f(y)}{h}\right) \le b\|v\|_{1,Q}.$$

Using the fact that $\lim_{h\to 0^+}\frac{f(y+hv)-f(y)}{h} = Df(y)v$, we obtain

$$\operatorname{sign}(Qv)^\top\left(QDf(y)\right)v \le b\|v\|_{1,Q}.$$

This means that (ii) holds.

E3.12 **One-sided Lipschitz maps that fail to be Lipschitz (Chu et al., 2016).** As illustrated by the bound $\mathsf{osLip}(f) \le \mathsf{Lip}(f)$ in equation (3.15), we know that a Lipschitz map is always one-sided Lipschitz. The converse is not necessarily true as illustrated by the following examples.

(i) Consider the scalar function $g(x) = -x - x^3$. Show that g is not globally Lipschitz because $\mathsf{Lip}(g) = +\infty$, but it is one-sided Lipschitz with $\mathsf{osLip}(g) = -1$.

(ii) Pick $a>0$ and define the function $f\colon (0,a)\to\mathbb{R}_{>0}$ by $f(x) = 1/\sqrt{x}$. Show that f is not Lipschitz and but it is one-side Lipschitz with negative one-sided Lipschitz constant $-\frac{1}{2a\sqrt{a}}$.

Answer: Statement (i) follows from noting that $g'(x) = -1-3x^2 \le 1$ for all $x\in\mathbb{R}$, whereas $|g'(x)|$ is unbounded over $x\in\mathbb{R}$.

Regarding statement (ii), for any $x_1,x_2\in(0,a)$, we compute

$$|f(x_1)-f(x_2)| = \left|\frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{x_2}}\right| = \frac{|\sqrt{x_2}-\sqrt{x_1}|(\sqrt{x_2}+\sqrt{x_1})}{\sqrt{x_1x_2}(\sqrt{x_2}+\sqrt{x_1})} = \frac{|x_2-x_1|}{\sqrt{x_1x_2}(\sqrt{x_2}+\sqrt{x_1})}.$$

Now we note that, for every $M>0$, there exists sufficiently small $x_1,x_2\in(0,a)$ such that $|f(x_1)-f(x_2)| \ge M|x_2-x_1|$. Therefore, f is not Lipschitz. On the other hand, we compute

$$\begin{aligned}(f(x_1)-f(x_2))(x_1-x_2) &= \left(\frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{x_2}}\right)(x_1-x_2)\\ &= \frac{x_2-x_1}{\sqrt{x_1x_2}(\sqrt{x_2}+\sqrt{x_1})}(x_1-x_2) \le -\tfrac{1}{2a\sqrt{a}}(x_1-x_2)^2.\end{aligned}$$

E3.13 **Nagumo's Theorem.** Loosely speaking, Nagumo's Theorem (Nagumo, 1942) states that a set is invariant for a vector field if, at each point on the boundary of the set, the vector field is pointed towards the inside of the set or is tangent to the set boundary. This exercise provides precise definitions for these concepts.

Let S be a closed and convex set with boundary ∂S. At each point $x \in \partial S$, the *tangent cone* to S at x is defined by

$$\mathcal{C}_S(x) = \Big\{ z \in \mathbb{R}^n \; : \; \liminf_{h \to 0^+} \frac{\operatorname{dist}(x + hz, S)}{h} = 0 \Big\}.$$

where the distance from a point to a set is defined by $\operatorname{dist}(x, S) = \inf_{y \in S} \|x - y\|_2$. Show that

(i) a set S is positively invariant for f if and only if

$$f(x) \in \mathcal{C}_S \quad \text{for all } x \in \partial S, \tag{E3.21}$$

(ii) the positive orthant $\mathbb{R}^n_{\geq 0}$ is positively invariant for f if and only if

$$f_i(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n_{\geq 0} \text{ such that } x_i = 0. \tag{E3.22}$$

Answer: For the proof of this exercise and a modern treatment to set-theoretic methods in control we refer to (Blanchini, 1999; Blanchini and Miani, 2015).

E3.14 **Yorke's fixed point theorem.** The next theorem about fixed points of vector fields combines (1) an application of the Brouwer Fixed Point Theorem 1.13 on a sequence of flows of the vector field with (2) an estimate on the period of periodic orbits based on the Lipschitz constant of the vector field.

Theorem 3.30 (Yorke's fixed point theorem (Yorke, 1969; Lajmanovich and Yorke, 1976)). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. If*

(A1) *f is continuously differentiable , and*

(A2) *there exists a compact convex f-invariant set C,*

then f has an equilibrium $x^* \in C$.

E3.15 **Parametrized infinitesimal contractions.** As in Lemma 1.9 for contractions (discrete-time dynamical systems), this exercises considers continuous-time dynamical systems that are strongly infinitesimally contracting and depend upon a parameter.

Given two normed vector spaces $(\mathcal{X} = \mathbb{R}^n, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{U} = \mathbb{R}^k, \|\cdot\|_{\mathcal{U}})$ and a continuous parameter-dependent map $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, consider the parametrized equilibrium problem

$$\mathbb{0}_n = f(x, u), \quad x \in \mathcal{X}, u \in \mathcal{U}. \tag{E3.23}$$

Assume f is

(i) uniformly strongly infinitesimally contracting in its first argument with constant $\mathsf{osLip}_x(f) < 0$, and

(ii) uniformly Lipschitz in its second argument with constant $\mathsf{Lip}_u(f)$.

Show that there exists a unique solution $x^* : \mathcal{U} \to \mathcal{X}$ to the parametrized equilibrium problem (E3.23) that is a Lipschitz map with constant $\mathsf{Lip}_u(f)/|\,\mathsf{osLip}_x(f)|$, that is:

$$\|x^*(u) - x^*(v)\|_{\mathcal{X}} \leq \frac{\mathsf{Lip}_u(f)}{|\,\mathsf{osLip}_x(f)|} \|u - v\|_{\mathcal{U}} \quad \text{for all } u, v \in \mathcal{U}.$$

Answer: Consider the dynamical systems $\dot{x} = f(x, u)$ where u is constant. Theorem 3.9(ii) implies that there exists a unique point $x^*(u) \in \mathcal{X}$ satisfying the equilibrium equation (E3.23). Given two constant inputs u and v, the two equilibrium solutions are $x^*(u)$ and $x^*(v)$. The assumptions of Theorem 3.16 are satisfied with $c = -\,\mathsf{osLip}_x(f)$ and $\ell = \mathsf{Lip}_u(f)$, and the differential inequality (3.39) implies

$$0 \leq -c\|x^*(u) - x^*(v)\|_{\mathcal{X}} + \ell\|u - v\|_{\mathcal{U}}$$

This concludes the proof.

E3.16 **Local infinitesimal contractivity and invariance.** As in Lemma 1.10 for contractions (discrete-time dynamical systems), this exercises considers continuous-time dynamical systems that are strongly infinitesimally contracting over a set.

Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, assume the (time-invariant) continuously-differentiable vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is strongly infinitesimally contracting with rate c over a closed set $S \subset \mathbb{R}^n$ in the sense that

$$\sup_{x\in S} \mu(Df(x)) \leq -c < 0. \tag{E3.24}$$

Note: *In the interest of generality, we do not assume that S is convex. From Lemma 3.4, recall that $\mathsf{osLip}(f|_S) = \sup_{x\in S}\mu(Df(x))$ holds only when S is convex. When S is not convex, the proof method of Lemma 3.4 implies only that $\mathsf{osLip}(f|_S) \geq \sup_{x\in S}\mu(Df(x))$. Therefore, in this exercise, we clarify that we mean contractivity in the sense of equation (E3.24).*

Show that

(i) if there exists an equilibrium $x^* \in \operatorname{int}(S)$ of f, then any closed ball centered at x^* inside S is f-invariant;

(ii) if there exists a closed ball $\overline{B}_r(x^*)$ with center $x^* \in \operatorname{int}(S)$ and radius $r > 0$ such

$$\overline{B}_r(x^*) \subseteq S \quad \text{and} \quad \|f(x^*)\| \leq cr, \tag{E3.25}$$

then $\overline{B}_r(x^*)$ is f-invariant and, therefore, it contains a unique exponentially stable equilibrium. Moreover, each closed ball of radius $\bar{r}$ with $r \leq \bar{r} \leq \|f(x^*)\|/c$ is forward invariant and there exists a unique exponentially stable equilibrium inside $\overline{B}_{\|f(x^*)\|/c}(x^*)$;

(iii) in each convex subset of S there exists at most a unique equilibrium of f; and

(iv) if there exists an initial condition $x_0 \in S$ such that $\phi_t(x_0) \in S$ for all $t \geq 0$, then $x^* := \lim_{t\to+\infty}\phi_t(x_0) \in S$ is an equilibrium of f.

Note: *Statement (i) has the following consequence. Consider a continuously-differentiable vector field f with an equilibrium x^* such that $Df(x^*)$ is Hurwitz. Pick a sufficiently small $\varepsilon > 0$ and, as discussed in Lemma 2.30, compute $P = P^\top \succ 0$ such that $\mu_{2,P^{1/2}}(Df(x^*)) \leq \alpha(Df(x^*)) + \varepsilon$. By the continuity of Df, there exists a radius $r > 0$ such that $\mu_{2,P^{1/2}}(Df(x)) < 0$ in a ball of radius r centered at x^* with respect to the norm $\|\cdot\|_{2,P^{1/2}}$. Statement (i) implies that each trajectory starting inside this ball converges to x^*.*

Note: *Statement (ii) has the following consequence. Pick a point x^* in the interior of S and compute the distance r from x^* to the boundary of S. If $\|f(x^*)\| \leq cr$, then $\overline{B}_r(x^*)$ is f-invariant.*

Note: *Additional results on locally contracting systems are based on the notion of K-reachable subsets proposed by (Sontag, 2010).*

Answer: Statement (i) is a special case of statement (ii), which we now prove using the Nagumo Theorem in Exercise E3.13 on the closed ball $\overline{B}_r(x^*)$. For any point x on the boundary of $\overline{B}_r(x^*)$, the triangle inequality implies

$$\|\phi_t(x) - x^*\| \leq \|\phi_t(x) - \phi_t(x^*)\| + \|\phi_t(x^*) - x^*\|, \tag{E3.26}$$

with equality when $t = 0$. Therefore, it suffices to show that the time-derivative of the right hand side is nonpositive at time $t = 0$. We compute

$$D^+\big|_{t=0}\|\phi_t(x^*) - x^*\| = \|f(x^*)\|, \qquad \text{(derivative of distance)} \tag{E3.27}$$

$$D^+\big|_{t=0}\|\phi_t(x) - \phi_t(x^*)\| \leq -c\|\phi_t(x) - \phi_t(x^*)\|\big|_{t=0} = -cr, \qquad \text{(contractivity)} \tag{E3.28}$$

where we used the fact $r = \|x - x^*\|$. We also recall from Lemma A.16 that each continuous function h of time satisfies $D^+\big|_{t=0}h(t) \leq 0$ implies $h(t) \leq h(0)$ for sufficiently small time t. The result now follows from Nagumo's Theorem in Exercise E3.13.

Regarding statement (iii), by contradiction, suppose there exist two equilibria x^* and y^* inside a convex set C contained in S. Recall that an equilibrium is a constant trajectory for the system. But then the iISS properties in Theorem 3.16 hold and the distance between the two trajectories originating at x^* and y^* should be decreasing. This is a contradiction.

Statement (iv) follows from the inequality $D^+\|f(\phi_t(x_0))\| \leq -c\|f(\phi_t(x_0))\|$ that holds for any trajectory $\phi_t(x_0)$ inside the set S (and from the continuity of the norm).

Chapter 4

Weakly Contracting and Monotone Systems

The theory of compartmental systems in the areas of biology, medicine, and ecology is concerned with the exchanges of material (or biomass, or energy) among various compartments and an environment. (Irwin W. Sandberg 1978)

One naturally suspects that nonoscillation and asymptotic stability are intrinsic properties not only of the linear class but of more general compartmental systems. (Hajime Maeda, Shinzo Kodama, and Yuzo Ohta 1978)

Monotone subsystems have appealing properties as components of larger networks, since they exhibit robust dynamical stability and predictability of responses to perturbations. (Eduardo Sontag 2007)

4.1 Introduction

In this chapter we study dynamical systems that are weakly infinitesimally contracting in the sense that their contraction rate is zero. We first present the general case and then focus on monotone systems.

4.1.1 Motivating examples

In this section we consider systems that enjoy conservation or invariance properties and that, therefore, cannot be strongly infinitesimally contracting. Three examples are illustrated in Figure 4.1.

Lemma 4.1 (Loss of contraction). *For a continuously-differentiable vector field f and vectors $\eta, \xi \in \mathbb{R}^n$, $\eta, \xi \neq \mathbb{0}_n$, the following equivalences hold:*

$$(\textit{conservation property}) \qquad \eta^\top f(x) = \eta^\top f(y) \;\; \forall x, y \in \mathbb{R}^n \qquad \iff \qquad \eta^\top Df(x) = \mathbb{0}_n^\top \;\; \forall x \in \mathbb{R}^n; \tag{4.1}$$

$$(\textit{invariance property}) \qquad f(x+\alpha\xi) = f(x) \;\; \forall x \in \mathbb{R}^n, \alpha \in \mathbb{R} \qquad \iff \qquad Df(x)\xi = \mathbb{0}_n \;\; \forall x \in \mathbb{R}^n. \tag{4.2}$$

Moreover, if f satisfies a conservation or invariance property, then

(i) $Df(x)$ *has eigenvalue* 0 *at each* x,

(ii) $\mathsf{osLip}(f) \geq 0$ *with respect to any norm,*

(iii) additionally, if $Df(x)$ is a Metzler matrix[1] *for all $x \in \mathbb{R}^n$ and f satisfies a the conservation property with $\eta > \mathbb{0}_n$ (respectively, the invariance property with $\xi > \mathbb{0}_n$), then $\mathsf{osLip}_{1,[\eta]}(f) = 0$ (respectively, $\mathsf{osLip}_{\infty,[\xi]^{-1}}(f) = 0$).*

[1] Recall a matrix is Metzler if all its off-diagonal entries are nonnegative.

(a) Transmission lines in a power grid, where an energy conservation law is at play (modulo small losses). Image in the public domain by Pok Rie from stocksnap.io.

(b) Traffic networks, as an example of a flow network with a conservation law. Image licensed under the Unsplash license, by Denys Nevozhai Hire from unsplash.com.

(c) Fish School, as an example of a network of agents moving in Euclidean space in a manner that is typically invariant under translations. Image licensed under the CC BY-SA 3.0 license, by Epipelagic from commons.wikimedia.

Figure 4.1: Example systems that enjoy conservation or invariance properties

Note: We refer to (4.1) as a conservation property for the following reason. If there exists $y \in \mathbb{R}^n$ such that $f(y) = \mathbb{0}_n$, then $\eta^\top f(x) = \eta^\top f(y) = 0$ and therefore $\eta^\top \dot{x} = \eta^\top f(x) = 0$. In turn, $\eta^\top x(t)$ is a conserved quantity along the solutions to $\dot{x} = f(x)$.

Note: Consider a Laplacian matrix L with associated Laplacian flow (continuous-time averaging flow)

$$\dot{x} = -Lx =: f_{-L}(x). \tag{4.3}$$

Then we know that $L\mathbb{1}_n = \mathbb{0}_n$ so that f_{-L} satisfies an invariance property (4.2) with $\xi = \mathbb{1}_n$. Additionally one can see that $-L$ is Metzler and so the dynamics (4.3) satisfy $\mathsf{osLip}_\infty(f_{-L}) = \mu_\infty(-L) = 0$. Its phase portrait is shown in Figure 4.2.

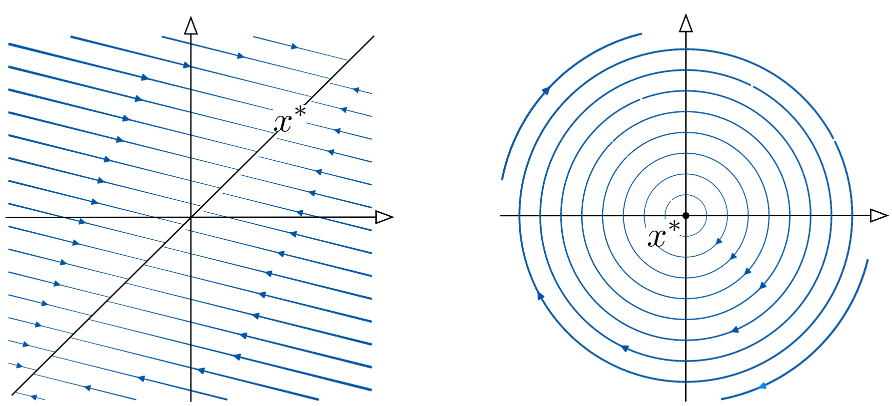


Figure 4.2: Left: Phase portrait of the *Laplacian flow* $\dot{x} = -Lx$ for Laplacian $\begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}$, with $\mu_\infty(-L) = 0$. Right: Phase portrait of the *harmonic oscillator* $\dot{x} = Ax$ for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with $\mu_2(A) = 0$. In both cases, the distance between trajectories remains constant.

Proof of Lemma 4.1. The equivalences in equations (4.1) and (4.2) are proved using the Mean Value Theorem and an argument based upon differentiation. The details are similar to those in the proof of Lemma 3.4 and left to the reader. We prove a more general version of these correspondences in Exercise E5.8.

Statements (i), (ii), and (iii) follow from the formulas for the ℓ_1 and ℓ_∞ log norms. ■

4.2 Weakly contracting systems

As in the previous section, we consider a time-varying dynamical system described by

$$\dot{x} = f(t, x). \tag{4.4}$$

We now extend some of the previous results to weakly infinitesimally contracting systems.

Definition 4.2 (Infinitesimal weak contractivity). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with compatible weak pairing. Let $C \subset \mathbb{R}^n$ be convex. Let $f : \mathbb{R}_{\geq 0} \times C \to \mathbb{R}^n$ be continuous. The vector field f is weakly infinitesimally contracting on C (also referred to as infinitesimally non-expansive) if*

$$\mathsf{osLip}(f_t) \leq 0 \quad \textit{for all } t \in \mathbb{R}_{\geq 0}. \tag{4.5}$$

It is immediate to see that, if f is weakly infinitesimally contracting, then the bound (3.23) in the Main Equivalent Theorem 3.7 holds with $c = 0$, that is, for any trajectories $x(t)$ and $y(t)$

$$\|x(t) - y(t)\| \leq \|x(0) - y(0)\|.$$

Despite this limitation, weakly infinitesimally contracting systems have numerous other properties, as we now illustrate.

Theorem 4.3 (Dichotomy theorem for weakly infinitesimally contracting systems). *Consider a continuously differentiable time-invariant vector field f and a norm $\|\cdot\|$ with associated log norm μ. Assume*

(A1) *there exists a convex and f-invariant set C,*

(A2) *f is weakly infinitesimally contracting on the set C with respect to $\|\cdot\|$, so that (i) the distance between any two solutions is non-increasing in time, and (ii) the function $x \mapsto \|f(x)\|$ is non-increasing along the solutions.*

Then precisely one of the following mutually-exclusive conditions hold: either

(C1) *f has no equilibrium point in C and every trajectory starting in C is unbounded, or*

(C2) *f has at least one equilibrium point $x^* \in C$ and every trajectory starting in C is bounded.*

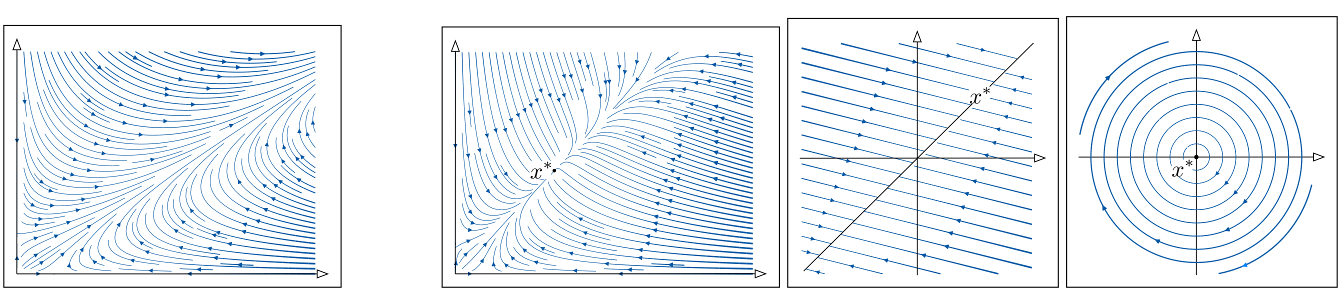


Figure 4.3: Illustration of the dichotomy in Theorem 4.3. Left image: Theorem 4.3(C1): f has no equilibrium point and every trajectory starting is unbounded. Three images in center and right: Theorem 4.3(C2): one equilibrium exists and it is either asymptotically stable, or not unique, or Lyapunov stable only.

Sketch of the proof of Theorem 4.3. First, suppose that f has one equilibrium in C, say x^*. Pick any initial condition x_0 at time 0. Then the solution $\phi_t(x_0)$ satisfies $\|\phi_t(x_0) - x^*\| \leq \|x_0 - x^*\|$ for all $t \geq 0$. But then

$$\|\phi_t(x_0)\| \leq \|\phi_t(x_0) - x^*\| + \|x^*\| \leq \|x_0 - x^*\| + \|x^*\|.$$

This inequality proves that if an equilibrium exists, all solutions are bounded.

Next, suppose instead that no equilibrium exists in C. It suffices to show that every trajectory is unbounded; by contradiction assume there exists a bounded trajectory. Under these assumptions (namely a time-invariant weakly infinitesimally contracting system with a bounded trajectory), one can show the existence of compact convex f-invariance set $W \subset C$; for the proof of this statement we refer to (Jafarpour et al., 2022, Lemma 17). Finally, based upon the existence of such a set W, Yorke's Fixed Point Theorem 3.30 in Exercise E3.14 implies the existence of an equilibrium for f. This completes the proof by contradiction. ■

We recall the notions of weak, local and global Lyapunov function, as introduced in the Lyapunov Stability Theorem A.4 in Appendix A.

Theorem 4.4 (Convergence theorem for weakly contracting systems). *Under the same assumptions as in Theorem 4.3, the weakly infinitesimally contracting f has at least one equilibrium point $x^* \in C$. Then*

(i) each equilibrium x^{**} is stable with weak Lyapunov function $x \mapsto \|x - x^{**}\|$;

(ii) if the norm $\|\cdot\|$ is a (p, R)-norm with $p \in \{1, \infty\}$ and f is piecewise real analytic, then every trajectory converges to the set of equilibrium points;

(iii) if x^* is locally asymptotically stable, then x^* is globally asymptotically stable in C;

(iv) if $\mu\big(Df(x^*)\big) < 0$, then $x \mapsto \|x - x^*\|$ is a global Lyapunov function (including being monotonically decreasing in time) and $x \mapsto \|f(x)\|$ is a local Lyapunov function.

Note: in statement (ii) the assumption $p \in \{1, \infty\}$ cannot be relaxed and, indeed, the harmonic oscillator in Figure 5.3 is an example of a weakly contracting system whose solutions do not converge to the unique equilibrium.

Proof of Theorem 4.4. Statement (i) follows from noting that each solution $\phi_t(x_0)$ satisfies $\|\phi_t(x_0) - x^*\| \leq \|x_0 - x^*\|$. Therefore the function $x \mapsto \|x - x^{**}\|$ is positive definite and its derivative along the flow is locally negative semidefinite.

Regarding statement (ii), we refer to (Jafarpour et al., 2022) and provide only a sketch of the argument. First, using the boundedness property from the Dichotomy Theorem, the LaSalle Invariance Principle applied with the function $x \mapsto \|f(x)\|$ ensures convergence of each trajectory to a largest invariant set. Second, using the piecewise analyticity assumption and the properties of the norms ℓ_1/ℓ_∞, one proves that each point in this largest invariant set is indeed an equilibrium. Third and final, any trajectory that approaches a set of Lyapunov stable equilibria must converge to a specific point.

Regarding statement (iii), since x^* is locally asymptotically stable, there exists $\varepsilon > 0$ such that each trajectory starting in the closed ball $\overline{B}_\varepsilon(x^*) = \{z \in C \,:\, \|z - x^*\| \leq \varepsilon\}$ centered at x^* of radius ε converges asymptotically to x^*. Hence, there exists a finite time T_ε such that each trajectory starting in $\overline{B}_\varepsilon(x^*)$ is inside $\overline{B}_{\varepsilon/2}(x^*)$ at and after time T_ε. Now, pick an initial condition $x_0 \in C$ outside $\overline{B}_\varepsilon(x^*)$ and define the point $x_{\text{tmp}} = x^* + \varepsilon \frac{(x_0 - x^*)}{\|x_0 - x^*\|}$, as in Figure 4.4.

Note $\|x_{\text{tmp}} - x^*\| = \varepsilon$, $\|\phi_{T_\varepsilon}(x_{\text{tmp}}) - x^*\| \leq \varepsilon$, and $\|x_0 - x_{\text{tmp}}\| = \|x_0 - x^*\| - \varepsilon$. We claim that $\|\phi_t(x_0) - x^*\|$ must decrease by at least $\varepsilon/2$ at time T_ε because of the triangle inequality and because the distance between any two solutions is non-increasing:

$$\begin{aligned}
\|\phi_{T_\varepsilon}(x_0) - x^*\| &= \|\phi_{T_\varepsilon}(x_0) - \phi_{T_\varepsilon}(x_{\text{tmp}}) + \phi_{T_\varepsilon}(x_{\text{tmp}}) - x^*\| \\
&\leq \|\phi_{T_\varepsilon}(x_0) - \phi_{T_\varepsilon}(x_{\text{tmp}})\| + \|\phi_{T_\varepsilon}(x_{\text{tmp}}) - x^*\| \\
&\leq (\|x_0 - x^*\| - \varepsilon) + \varepsilon/2 = \|x_0 - x^*\| - \varepsilon/2.
\end{aligned}$$

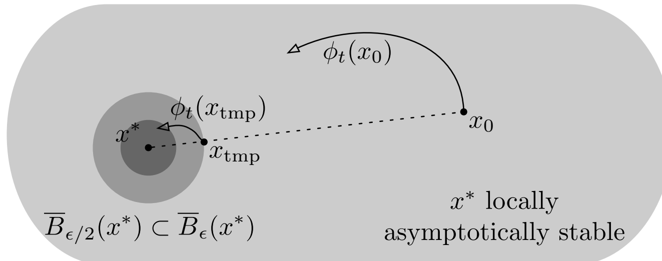


Figure 4.4: Illustration of the proof of statement (iii).

Because $\|\phi_t(x_0) - x^*\|$ has decreased by the finite amount $\varepsilon/2$ in time T_ε, either the trajectory originating from x_0 is inside $\overline{B}_\varepsilon(x^*)$ at time T_ε or it will be so after a finite number of such steps. We have proved that all trajectories originating in C approach x^*; this concludes the proof of statement (iii).

Regarding statement (iv), $\mu\big(Df(x^*)\big) < 0$ implies $\mu\big(Df(x)\big) < 0$ for each x in a neighborhood $\mathcal{U}$ of x^*. Therefore f is strongly infinitesimally contracting in $\mathcal{U}$ and Theorem 3.9 implies that both functions $x \mapsto \|x - x^*\|$ and $x \mapsto \|f(x)\|$ are Lyapunov functions over $\mathcal{U}$, that is, local Lyapunov functions. Additionally, one can see that $x \mapsto \|x - x^*\|$ is globally positive definite about x^* and proper. It remains to prove that its Lie derivative is globally negative definite. To do so, we strengthen the argument in the proof of statement (iii): we pick as local neighborhood $\mathcal{U} \cap \overline{B}_\varepsilon(x^*)$ and note that the global distance $\|\phi_t(x_0) - x^*\|$ is strictly continuously decreasing in time since the local distance $\|\phi_t(x_{\text{tmp}}) - x^*\|$ is strictly continuously decreasing in time. ■

Example 4.5 (Affine averaging systems). Let L be the Laplacian matrix of a weighted digraph with a globally reachable node. According to the Perron-Frobenius Theorem, let v be the dominant left eigenvector of L satisfying $\mathbb{1}_n^\top v = 1$. Let $d \in \mathbb{R}^n$. Then we claim that the *affine averaging system*

$$\dot{x} = -Lx + d, \tag{4.6}$$

(i) is weakly infinitesimally contracting with respect to $\|\cdot\|_\infty$,

(ii) if $v^\top d \neq 0$, then every trajectory is unbounded,

(iii) if $v^\top d = 0$, then the trajectory starting from $x(0) = x_0$ converges to the equilibrium point $L^\dagger d + (v^\top x_0)\mathbb{1}_n$.

We verify these three claims as follows. With the shorthand $f_{-L}(x) := -Lx + d$, the first claim (i) is an immediate consequence of $\mu_\infty(Df_{-L}(x)) = \mu_\infty(-L) = 0$.

Second, regarding the second claim (ii), if $v^\top d \neq 0$, then there does not exist $x \in \mathbb{R}^n$ such that $-Lx + d = \mathbb{0}_n$ because $v^\top(-Lx + d) = v^\top d \neq 0$. Therefore, Theorem 4.3(C1) implies that every trajectory of (4.6) is unbounded.

Regarding the third claim (iii), since G has a reachable node, L has a simple eigenvalue 0 with right eigenvector $\mathbb{1}_n$ and its other eigenvalues have strictly positive real parts. Therefore, if $v^\top d = 0$, then $-Lx + d = \mathbb{0}_n$ has solutions $x = L^\dagger d + \beta\mathbb{1}_n$, for $\beta \in \mathbb{R}$. Thus, by Theorem 4.3(C2), every trajectory of (4.6) converges to an equilibrium point in $\operatorname{span}(\mathbb{1}_n)$. Let $t \mapsto x(t)$ be a trajectory of (4.6). Since $v^\top L = 0$, we have $v^\top x(t) = v^\top x_0$. Thus, $t \mapsto x(t)$ converges to the equilibrium point $L^\dagger b + (v^\top x_0)\mathbb{1}_n$. •

4.3 Example: Convex functions and their gradient flows

In this section we study convex functions and contractivity of their negative gradient flow.

4.3.1 Convex functions

Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a scalar function defined over a convex set $C \subset \mathbb{R}^n$. If V is differentiable (resp. twice differentiable), let $\operatorname{grad} V$ (resp. $\operatorname{Hess} V$) denote the *gradient* (resp. *Hessian*) of V. The function V is

(i) *convex* if
$$V(\theta x + (1-\theta)y) \leq \theta V(x) + (1-\theta)V(y),$$

(ii) *strictly convex* if
$$V(\theta x + (1-\theta)y) < \theta V(x) + (1-\theta)V(y),$$

(iii) *strongly convex with parameter $m > 0$* if
$$V(\theta x + (1-\theta)y) \leq \theta V(x) + (1-\theta)V(y) - \tfrac{1}{2} m\theta(1-\theta)\|x-y\|_2^2, \tag{4.7}$$

for all $x \neq y$ in $C \subset \mathbb{R}^n$ and for all $\theta \in [0,1]$.

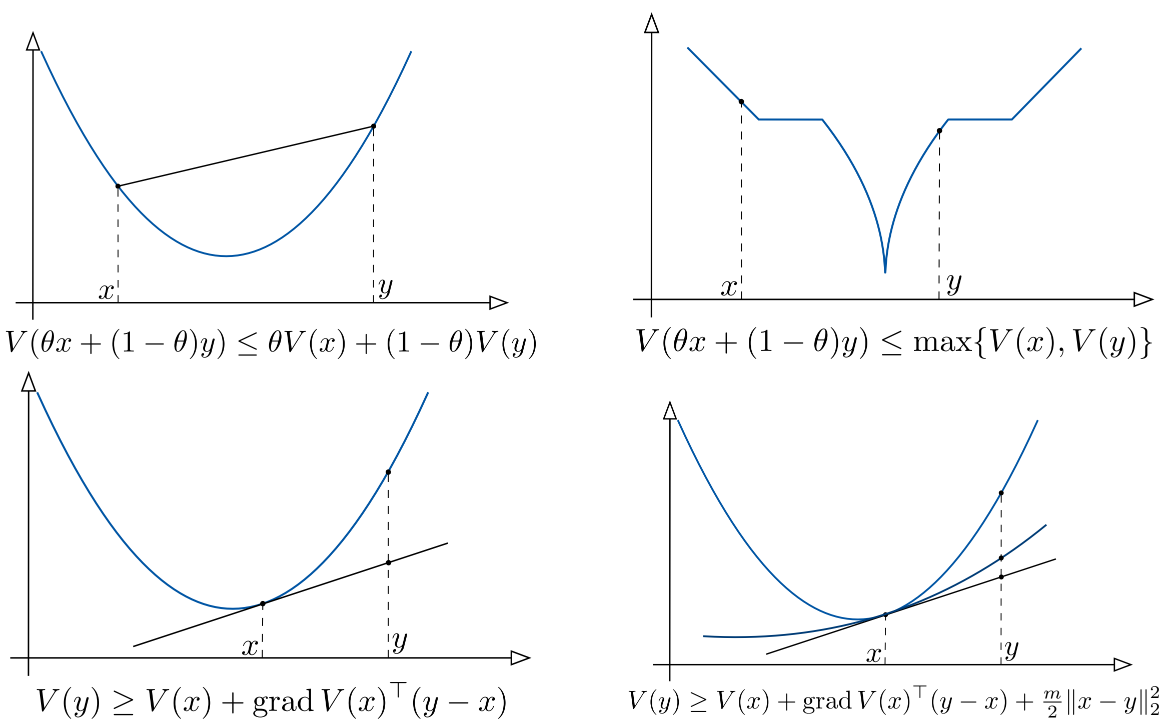


Figure 4.5: Convex, quasi-convex, convex differentiable, and strongly-convex differentiable functions

Lemma 4.6 (Convexity and contractivity of the negative gradient). *For a differentiable V over a convex set C, the following statements are equivalent:*

(C1) *V is strongly convex with parameter m;*

(C2) *$x \mapsto V(x) - \frac{m}{2}\|x\|_2^2$ is convex;*

(C3) *$V(x) \geq V(y) + \operatorname{grad} V(y)^\mathsf{T}(x-y) + \frac{m}{2}\|x-y\|_2^2$ for all $x, y \in C$; and*

(C4) $\big(\operatorname{grad} V(x) - \operatorname{grad} V(y)\big)^\mathsf{T}(x-y) \geq m\|x-y\|_2^2$*, that is, $-\operatorname{grad} V$ is one-sided Lipschitz with* $\mathsf{osLip}_2(-\operatorname{grad} V) = -m$.

If additionally V is twice differentiable, then another equivalent statement is

(C5) $\operatorname{Hess} V(x) \succeq m I_n$, *for all* $x \in C$, *that is,* $-\operatorname{grad} V$ *satisfies the* ℓ_2 *Demidovich condition with rate* m.

Note: the equivalence in Lemma 4.6 continue to hold also at $m = 0$. Specifically, a differentiable V is convex over a convex set C if and only if any of the following equivalent statements holds for all $x \in C$: (i) $V(x) \geq V(y) + \operatorname{grad} V(y)^\top (x-y)$, (ii) $\big(\operatorname{grad} V(x) - \operatorname{grad} V(y)\big)^\top (x-y) \geq 0$, or (iii) $\operatorname{Hess} V(x) \succeq 0$.

Note: The main equivalence (C3) $\iff$ (C4) is sometimes referred to as *Kachurovskii's Theorem* (Kachurovskii, 1960).

Proof of Theorem 4.6. The characterizations of strong convexity in Lemma 4.6 are well known, e.g., see (Bertsekas et al., 2003, Exercises 1.7 and 1.9) and (Ryu and Boyd, 2016, Appendix). We here focus only on the equivalence between the strong convexity condition (C4) and the contractivity condition (C4).

Regarding the implication (C3) $\implies$ (C4), if a differentiable V is convex, then, for all $x, y \in C$,

$$\begin{aligned}
V(x) &\geq V(y) + \operatorname{grad} V(y)^\top (x-y) + \frac{m}{2}\|x-y\|_2^2 \\
V(y) &\geq V(x) + \operatorname{grad} V(x)^\top (y-x) + \frac{m}{2}\|x-y\|_2^2 \\
&\implies 0 \geq \big(\operatorname{grad} V(y) - \operatorname{grad} V(x)\big)^\top (x-y) + m\|x-y\|_2^2 \\
&\iff \big(-\operatorname{grad} V(x) + \operatorname{grad} V(y)\big)^\top (x-y) \leq -m\|x-y\|_2^2,
\end{aligned}$$

that is, $-\operatorname{grad} V$ satisfies the ℓ_2 integral Demidovich condition.

Regarding the implication (C4) $\implies$ (C3), let x and y be two distinct points in C and define $z(t) = x + t(y-x) \in C$, for $t \in [0,1]$. Note $z(t_2) - z(t_1) = (t_2 - t_1)(y-x)$. Define $h\colon [0,1] \to \mathbb{R}$ by $h(t) = V(z(t))$. For $t_2 > t_1$ in $[0,1]$, we compute

$$\begin{aligned}
\Big(\frac{dh(t_2)}{dt} - \frac{dh(t_1)}{dt}\Big)(t_2 - t_1) &= \Big(\operatorname{grad} V(z(t_2))^\top (y-x) - \operatorname{grad} V(z(t_1))^\top (y-x)\Big)(t_2 - t_1) \\
&\geq \big(\operatorname{grad} V(z(t_2)) - \operatorname{grad} V(z(t_1))\big)^\top (z(t_2) - z(t_1)) \geq m\|z(t_2) - z(t_1)\|_2^2,
\end{aligned}$$

where we used the short-hand $z' = z(t_2)$ and assumption (C4). We rewrite the last equality as

$$\frac{dh(t_2)}{dt} - \frac{dh(t_1)}{dt} \geq m(t_2 - t_1)\|x-y\|_2^2. \tag{4.8}$$

We now integrate inequality (4.8) with respect to t_2 on the interval $[t,1]$, set $t_1 = t$, and obtain

$$h(1) \geq h(t) + \frac{dh(t)}{dt}(1-t) + (1-t)^2 \frac{m}{2}\|x-y\|_2^2. \tag{4.9}$$

We then integrate inequality (4.8) with respect to t_1 on the interval $[0,t]$, set $t_2 = t$, and obtain

$$h(0) \geq h(t) - \frac{dh(t)}{dt} t + t(1-t)\frac{m}{2}\|x-y\|_2^2. \tag{4.10}$$

We multiply inequality (4.9) by t and inequality (4.10) by $1-t$, sum the results and obtain

$$t h(1) + (1-t) h(0) \geq h(t) + t(1-t)\frac{m}{2}\|x-y\|_2^2.$$

This is condition (4.7) with $t = \alpha$, $\beta = 1-t$ and $h(t) = V(x + t(y-x))$. ■

4.3.2 Gradient flows of convex functions

Next, building on the previous lemma, we present contraction results for the negative gradient flow of a convex function.

Corollary 4.7. *Let $V:\mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. If V is strongly convex with parameter $m>0$ over $\mathbb{R}^n$, then*

(i) $-\operatorname{grad} V$ *is ℓ_2 strongly infinitesimally contracting with rate m,*

(ii) any two trajectories approach each other exponentially fast, and

(iii) there exists a unique globally exponentially stable equilibrium (i.e., the global minimum of V).

Corollary 4.8 (Dichotomy in the negative gradient flow of a convex function). *Let $V:\mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. If V is convex over $\mathbb{R}^n$, then $-\operatorname{grad} V$ is ℓ_2 weakly infinitesimally contracting, the distance between any two trajectories is non-increasing, and precisely one of the following mutually-exclusive conditions hold: either*

(C1) $-\operatorname{grad} V$ *has no equilibrium point in $\mathbb{R}^n$, i.e., V has no minimum, and every trajectory starting in C is unbounded, or*

(C2) $-\operatorname{grad} V$ *has at least one equilibrium point $x^* \in \mathbb{R}^n$ and the following properties hold:*

a) every trajectory is bounded and each equilibrium x^{**} is stable with weak Lyapunov function $x \mapsto \|x - x^{**}\|_2$,

b) if x^* is locally asymptotically stable, then x^* is globally asymptotically stable in C,

c) if $\mu_2\big(-D\operatorname{grad} V(x^*)\big) = \mu_2(-\operatorname{Hess}(V)(x^*)) < 0$, then $x \mapsto \|x - x^*\|_2$ is a global Lyapunov function and $x \mapsto \|\operatorname{grad} V(x)\|_2$ is a local Lyapunov function.

Proof. The proof of these corollaries is a direct application of the strong and weak contraction theorems to the minus gradient vector field. ■

4.3.3 Distributed primal-dual dynamics

Finally, we consider a distributed version of the previous problem. We are given the objective function $V:\mathbb{R}^k \to \mathbb{R}$:

$$\min_{y\in\mathbb{R}^k} V(y) = \min_{y\in\mathbb{R}^k} \sum_{i=1}^{n} V_i(y). \tag{4.11}$$

We assume that n agents communicate over a connected undirected weighted graph G. Each agent i has access to only the function V_i. Then the centralized optimization problem (4.11) is equivalent to the distributed optimization problem:

$$\begin{aligned} \min_{x_i\in\mathbb{R}^k, i\in\{1,\dots,n\}} \quad & \sum_{i=1}^{n} V_i(x_i), \\ & x_i = x_j, \quad \text{for every edge } i,j. \end{aligned} \tag{4.12}$$

Under mild smoothness assumptions, the *primal-dual continuous-time algorithm* associated with the distributed optimization problem (4.12) is

$$\begin{aligned}\dot{x}_i &= -\operatorname{grad} V_i(x_i) - \sum_{i=1}^n a_{ij}(\nu_i - \nu_j),\\ \dot{\nu}_i &= \sum_{j=1}^n a_{ij}(x_i - x_j).\end{aligned} \tag{4.13}$$

Theorem 4.9 (Weak contractivity of the distributed primal-dual dynamics). *Consider the optimization problem (4.11) and let $f_{\mathrm{PD}}\colon \mathbb{R}^{2nk} \to \mathbb{R}^{2nk}$ denote the vector field for the distributed primal-dual dynamics (4.13). Assume*

(A1) *V has a minimum $x^* \in \mathbb{R}^k$,*

(A2) *for each $i \in \{1,\dots,n\}$, V_i is twice differentiable, $\operatorname{Hess} V_i(x) \succeq 0$ for all x, and $\operatorname{Hess} V_i(x^*) \succ 0$, and*

(A3) *the undirected weighted graph G is connected with Laplacian L.*

Then

(i) f_{PD} is weakly infinitesimally contracting with respect to ℓ_2,

(ii) each trajectory $(x(t),\nu(t))$ of (4.13) converges exponentially to $(\mathbb{1}_n \otimes x^*, \mathbb{1}_n \otimes \nu^*)$, where $\nu^* = \sum_{i=1}^n \nu_i(0)$.

Recall that the *Kronecker product* of $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{q\times r}$ is the $nq \times mr$ matrix $A \otimes B$ given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ddots & a_{nm}B \end{bmatrix}. \tag{4.14}$$

As a simple example, we write

$$I_n \otimes B = \begin{bmatrix} B & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & B \end{bmatrix} \in \mathbb{R}^{nq\times nr} \quad \text{and } A \otimes I_q = \begin{bmatrix} a_{11}I_q & \dots & a_{1m}I_q \\ \vdots & \ddots & \vdots \\ a_{n1}I_q & \ddots & a_{nm}I_q \end{bmatrix} \in \mathbb{R}^{nq\times mq}. \tag{4.15}$$

Additionally, for $v, w \in \mathbb{R}^n$, we have $v \otimes w = \begin{bmatrix} v_1 w \\ \vdots \\ v_n w \end{bmatrix} \in \mathbb{R}^{n^2}$. We review several useful properties of the Kronecker product in Exercises E4.5-E4.7.

Proof. We set $x = (x_1,\dots,x_n)^\top \in \mathbb{R}^{n\times k}$ and $\nu = (\nu_1,\dots,\nu_n)^\top \in \mathbb{R}^{n\times k}$, and define $\mathcal{V}\colon \mathbb{R}^{n\times k} \to \mathbb{R}$ by $\mathcal{V}(x) = \sum_{i=1}^n V_i(x_i)$. Then algorithm (4.13) can be written as

$$\begin{aligned}\dot{x} &= -\operatorname{grad} \mathcal{V}(x) - (L \otimes I_k)\nu,\\ \dot{\nu} &= (L \otimes I_k)x.\end{aligned} \tag{4.16}$$

Regarding (i), we compute

$$Df_{\mathrm{PD}}(x,\nu) = \begin{bmatrix} -\operatorname{Hess} \mathcal{V}(x) & -L \otimes I_k \\ L \otimes I_k & \mathbb{0}_{nk\times nk} \end{bmatrix} \in \mathbb{R}^{2nk\times 2nk}. \tag{4.17}$$

Because the Hessian matrix $\operatorname{Hess}\mathcal{V}(x)$ is a block diagonal matrix with individual blocks $\operatorname{Hess}V_i(x) \succeq 0$, we compute

$$\mu_2(Df_{\text{PD}}(x,\nu)) = \lambda_{\max}\begin{bmatrix} -\operatorname{Hess}\mathcal{V}(x) & \mathbb{0}_{nk\times nk} \\ \mathbb{0}_{nk\times nk} & \mathbb{0}_{nk\times nk}\end{bmatrix} \leq 0, \quad \text{for all } x,\nu. \tag{4.18}$$

Therefore, f_{PD} is weakly infinitesimally contracting with respect to the ℓ_2-norm.

Regarding part (ii), let $t \mapsto (x(t),\nu(t))$ be a trajectory of the system (4.13). Note that, for every $u \in \mathbb{R}^k$, we have $(\mathbb{1}_n \otimes u)^\top \dot{\nu}(t) = (\mathbb{1}_n \otimes u)^\top (L\otimes I_k)x(t) = \mathbb{0}_{nk}$. This implies that $\sum_{i=1}^n \nu_i(t) = \sum_{i=1}^n \nu_i(0)$ and the subspace

$$\mathcal{U} = \left\{(x,\nu) \in \mathbb{R}^{nk}\times\mathbb{R}^{nk} \;:\; \sum\nolimits_{i=1}^n \nu_i = 0\right\}$$

is invariant for system (4.13). For the Laplacian $L = L^\top$, define

$$R_{\mathcal{V}} = \begin{bmatrix} v_2 & \dots & v_n\end{bmatrix}^\top \in \mathbb{R}^{(n-1)\times n}. \tag{4.19}$$

where $\mathcal{V} = \{v_2,\dots,v_n\}$ are orthonormal eigenvectors of L. Define new coordinates $(\tilde{x},\tilde{v})$ on $\mathcal{U}$ by $\tilde{x} = x$ and $\tilde{\nu} = (R_{\mathcal{V}}\otimes I_k)\nu$. Thus, the dynamical system (4.13) restricted to $\mathcal{U}$ can be written in the new coordinate $(\tilde{x},\tilde{v})$ as

$$\begin{aligned}\dot{\tilde{x}} &= -\operatorname{grad}\mathcal{V}(\tilde{x}) - (LR_{\mathcal{V}}^\top \otimes I_k)\tilde{\nu}, \\ \dot{\tilde{\nu}} &= (R_{\mathcal{V}}L\otimes I_k)\tilde{x}.\end{aligned} \tag{4.20}$$

Let $(\dot{\tilde{x}},\dot{\tilde{\nu}}) := \tilde{f}_{\text{PD}}(\tilde{x},\tilde{\nu})$ Note that $(\tilde{x},\tilde{v}) = (\mathbb{1}_n\otimes x^*, \mathbb{0}_{(n-1)k})$ is an equilibrium point of the dynamical system (4.20) and

$$D\tilde{f}_{\text{PD}}(\tilde{x},\tilde{\nu}) = \begin{bmatrix} -\operatorname{Hess}\mathcal{V}(\tilde{x}) & -LR_{\mathcal{V}}^\top\otimes I_k \\ R_{\mathcal{V}}L\otimes I_k & \mathbb{0}\end{bmatrix}.$$

Again we note that $\mu_2(D\tilde{f}_{\text{PD}}(\tilde{x},\tilde{\nu})) = 0$, for every $(\tilde{x},\tilde{\nu}) \in \mathcal{U}$, and therefore, the reduced system (4.20) is weakly infinitesimally contracting with respect to the ℓ_2-norm. Moreover, $-\operatorname{Hess}\mathcal{V}(\mathbb{1}_n\otimes x^*) \prec 0$ and $\ker(LR_{\mathcal{V}}^\top\otimes I_k) = \emptyset$. Thus, as described in Exercise E4.8, the matrix $D\tilde{f}_{\text{PD}}(\mathbb{1}_n\otimes x^*, 0)$ is a Hurwitz saddle point matrix and the equilibrium point $(\mathbb{1}_n\otimes x^*, \mathbb{0}_{(n-1)k})$ is locally asymptotically stable for the dynamical system (4.20).

The Dichotomy Theorem 4.3(iii) applied to the weakly infinitesimally contracting system (4.20) implies that $(\mathbb{1}_n\otimes x^*, \mathbb{0}_{(n-1)k})$ is a globally exponentially stable equilibrium point of the system (4.20), i.e., $\lim_{t\to\infty}(\tilde{x}(t),\tilde{\nu}(t)) = (\mathbb{1}_n\otimes x^*, \mathbb{0}_{(n-1)k})$. Therefore, we obtain $\lim_{t\to\infty} x(t) = \mathbb{1}_n\otimes x^*$ and

$$\begin{aligned}\mathbb{0}_{nk} = (R_{\mathcal{V}}^\top\otimes I_k)\lim_{t\to\infty}\tilde{\nu} &= \lim_{t\to\infty}(R_{\mathcal{V}}^\top R_{\mathcal{V}}\otimes I_k)\nu(t) \\ &= \lim_{t\to\infty}\Big((I_n - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top)\otimes I_k\Big)\nu(t) \\ &= \lim_{t\to\infty}(\nu(t) - \mathbb{1}_n\otimes\nu^*),\end{aligned}$$

where the last equality holds because $\frac{1}{n}\sum_{i=1}^n \nu_i(t) = \nu^*$. As a result, we obtain $\lim_{t\to\infty}\nu(t) = \mathbb{1}_n\otimes\nu^*$. ■

4.4 Example: Lotka-Volterra population dynamics

The Lotka-Volterra population models are one the simplest and most widely adopted frameworks for modeling the dynamics of interacting populations in mathematical ecology. The starting point is the *logistic equation* describing the evolution of a single isolated species

$$\dot{x}(t) = rx(t)\big(1 - x(t)/\kappa\big).$$

We refer the reader to Exercise E4.2 for the basic properties of this model. Here, we consider a generalization of the logistic equation to include signed interaction terms among multiple species. The Lotka-Volterra population model is described by

$$\dot{x} = [x]\big(Ax + r\big) =: f_{\mathrm{LV}}(x), \tag{4.21}$$

where $x \in \mathcal{X} = \mathbb{R}^n_{\geq 0}$, the matrix $A = [a_{ij}]$ is called the interaction matrix, and the vector $r > \mathbb{0}_n$ is called the intrinsic growth rate. In components, $\dot{x}_i = x_i \sum_{j=1}^n (a_{ij}x_j + r_i)$. For later uses, We compute

$$\begin{aligned} Df_{\mathrm{LV}_{ij}}(x) &= x_i a_{ij}, \quad i \neq j \\ Df_{\mathrm{LV}_{ii}}(x) &= x_i a_{ii} + (Ax + r)_i \\ &\implies \qquad Df_{\mathrm{LV}}(x) = [x]A + [Ax + r]. \end{aligned}$$

(a) Common clownfish (*Amphiprion ocellaris*) near magnificent sea anemones (*Heteractis magnifica*) on the Great Barrier Reef, Australia. Clownfish and anemones provide an example of ecological mutualism in that each species benefits from the activity of the other. Public domain image from Wikipedia.

(b) The Canadian lynx (*Lynx canadensis*) is a major predator of the snowshoe hare (*Lepus americanus*). Historical records of animals captures indicate that the lynx and hare numbers rise and fall periodically; see (Odum, 1959). Public domain image from Rudolfo's Usenet Animal Pictures Gallery (no longer in existence).

(c) Subadult male lion (*Panthera Leo*) and spotted hyena (*Crocuta Crocuta*) compete for the same resources in the Maasai Mara National Reserve in Narok County, Kenya. Picture "Hyänen und Löwe im Morgenlicht" by lubye134, licensed under Creative Commons Attribution 2.0 Generic (BY 2.0).

Figure 4.6: Mutualism, predation and competition in population dynamics

As illustrated in Figure 4.6, for any two species i and j, the sign of a_{ij} and a_{ji} in the interaction matrix A is determined by which of the following three possible types of interaction is being modeled:

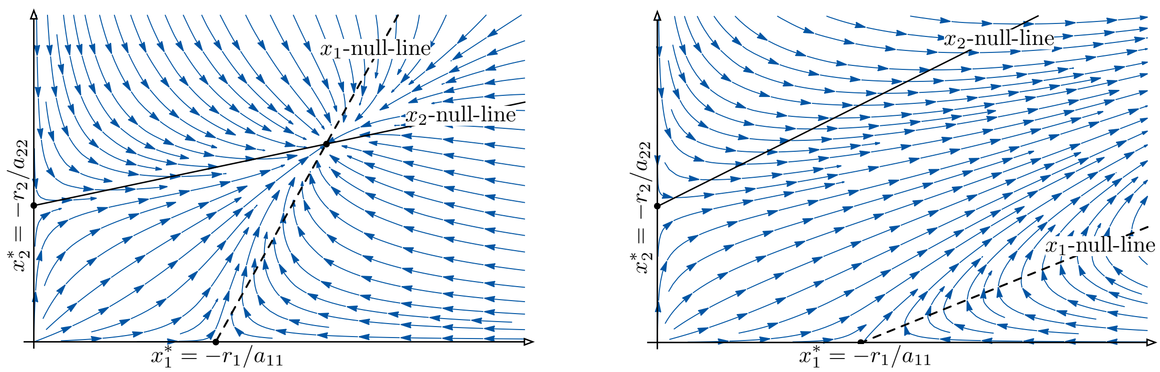
(+, +) = mutualism: for $a_{ij} > 0$ and $a_{ji} > 0$, the two species are in symbiosis and cooperation. The presence of species i has a positive effect on the growth of species j and vice versa.

(+,-) = predation: for $a_{ij} > 0$ and $a_{ji} < 0$, the species are in a predator-prey or host-parasite relationship. In other words, the presence of a prey (or host) species j favors the growth of the predator (or parasite) species i, wheres the presence of the predator species has a negative effect on the growth of the prey.

(-,-) = competition: for $a_{ij} < 0$ and $a_{ji} < 0$, the two species compete for a common resources of sorts and have therefore a negative effect on each other.

Note: the typical availability of bounded resources suggests it is ecologically meaningful to assume that the interaction matrix A is Hurwitz and that, to model the setting in which species live in isolation, the diagonal entries a_{ii} are negative.

For example, in the 2-dimensional case when $a_{12}, a_{21} \geq 0$, we identify two distinct parameter ranges corresponding to distinct dynamic behavior and illustrate them in Figure 4.7.



Case I: $a_{12} > 0$, $a_{21} > 0$, $a_{12}a_{21} < a_{11}a_{22}$. There exists a unique positive equilibrium point. All trajectories starting in $\mathbb{R}^2_{>0}$ converge to the equilibrium point.

Case II: $a_{12} > 0$, $a_{21} > 0$, $a_{12}a_{21} > a_{11}a_{22}$. There exists no positive equilibrium point. All trajectories starting in $\mathbb{R}^2_{>0}$ diverge.

Figure 4.7: Two possible cases of mutualism in the two-species Lotka-Volterra system

Theorem 4.10 (Contractivity of the Lotka-Volterra model). *Consider the Lotka–Volterra model (4.21) with interaction matrix A and intrinsic growth rate $r > \mathbb{0}_n$. Assume*

(A1) *the interaction matrix A is M-Hurwitz, i.e., $\alpha(|A|_{\mathrm{M}}) < 0$,*

(A2) *the equilibrium point is nonnegative, that is, $-A^{-1}r > \mathbb{0}_n$.*

By Assumption (A1), let $v \in \mathbb{R}^n_{>0}$ satisfy $v^\mathsf{T}|A|_{\mathrm{M}} < \mathbb{0}_n$. Then

(i) the open positive orthant $\mathbb{R}^n_{>0}$ is invariant,

(ii) $x^* = -A^{-1}r > \mathbb{0}_n$ is the unique globally asymptotically stable equilibrium point of (4.21) restricted to $\mathbb{R}^n_{>0}$,

(iii) the following distance between any two trajectories $x(t)$ and $\bar{x}(t)$ is decreasing:

$$d_{\mathrm{LV}}(x(t), \bar{x}(t)) = \sum_{i=1}^{n} v_i \left| \ln\left(x_i(t)/\bar{x}_i(t)\right) \right|, \tag{4.22}$$

(iv) the following functions are sum-separable global Lyapunov functions in the domain $\mathbb{R}^n_{>0}$:

$$x \mapsto \sum_{i=1}^{n} v_i |\ln(x_i/x_i^*)|, \qquad x \mapsto \sum_{i=1}^{n} v_i |(Ax + r)_i|.$$

Note: The Lyapunov function $x \mapsto \sum_{i=1}^n v_i |\ln(x_i/x_i^*)|$ is related to, but not exactly equal to, the classic Lyapunov functions adopted to study the Lotka-Volterra models (e.g., see (Bullo, 2022, Theorem 15.5 and Lemma 15.6)).

Proof of Theorem 4.10. Statement (i) follows from Nagumo's Theorem E3.13; we leave the details to the reader.

Regarding statement (ii), for $x \in \mathbb{R}^n_{>0}$, we consider the change of variable $y_i = \ln(x_i) \in \mathbb{R}$, for every $i \in \{1, \dots, n\}$ or in the matrix form $y = \ln(x)$. Then, under this change of variable, the Lotka–Volterra model (4.21) can be written as

$$\dot{y} = A\exp(y) + r := f_{\mathrm{LVexp}}(y) \tag{4.23}$$

where y and its entrywise exponential $\exp(y)$ are vectors in $\mathbb{R}^n$. The Jacobian of f_{LVexp} is

$$Df_{\text{LVexp}}(y) = A[\exp(y)].$$

Since $v \in \mathbb{R}^n_{>0}$ satisfies $v^\mathsf{T}|A|_\text{M} < \mathbb{0}_n$ by Assumption (A1), one can see[2] that $c = -\frac{\max(v^\mathsf{T}|A|_\text{M})}{\max(v)} > 0$ satisfies $v^\mathsf{T}|A|_\text{M} \leq -cv^\mathsf{T}$. From (2.41) we recall $\mu_{1,[v]}(B) = \max(v^\mathsf{T}|B|_\text{M}[v]^{-1})$ for any matrix B and compute

$$\begin{aligned}\mu_{1,[v]}(Df_{\text{LVexp}}(y)) &= \max(v^\mathsf{T}|A|_\text{M}[\exp(y)][v]^{-1}) = \max(v^\mathsf{T}|A|_\text{M}[v]^{-1}[\exp(y)]) \\ &\leq \max(-cv^\mathsf{T}[v]^{-1}[\exp(y)]) = \max(-c\mathbb{1}_n^\mathsf{T}[\exp(y)]) \\ &= \max(-c\exp(y)^\mathsf{T}) = -c\min(\exp(y)),\end{aligned}$$

for every $y \in \mathbb{R}^n$. Therefore, we have established that

$$\mathsf{osLip}_{1,[v]}(f_{\text{LVexp}}) \leq 0$$

and that the system (4.23) is weakly infinitesimally contracting on its entire domain with respect to $\|\cdot\|_{1,[v]}$.

Next, we consider the equilibrium point $x^* = -A^{-1}r > \mathbb{0}_n$, that is, at $\exp(y^*) = -A^{-1}r > \mathbb{0}_n$ (by Assumption (A2)). We note

$$\mu_{1,[v]}(Df_{\text{LVexp}}(y^*)) = -c\min(\exp(y)) < 0$$

so that $\mu_{1,[v]}\big(Df_{\text{LVexp}}(y^*)\big) < 0$. We now invoke Theorem 4.3(iv): f_{LVexp} is weakly infinitesimally contracting over its entire domain $\mathbb{R}^n$ and, with respect to the same norm, has an equilibrium point y^* satisfying a strict lognorm inequality $\mu_{1,[v]}\big(Df_{\text{LVexp}}(y^*)\big) < 0$. Therefore, f_{LVexp} has a unique globally asymptotically stable equilibrium with sum-separable global Lyapunov functions

$$y \mapsto \|y - y^*\|_{1,[v]} = \sum_{i=1}^n v_i|y_i - y_i^*|, \quad \text{and} \quad y \mapsto \|f_{\text{LVexp}}(y)\|_{1,[v]} = \sum_{i=1}^n v_i|(f_{\text{LVexp}})_i(y)|. \tag{4.24}$$

After a change of coordinates, we recover the results in statements (ii), (iii) and (iv). ■

4.5 Contracting monotone and positive monotone systems

In this section we study dynamical systems with a certain monotonicity property that arises from so-called mutualistic interactions. Quoting wikipedia, mutualism describes the ecological interaction between two or more species where each species has a net benefit. We illustrate mutual activation interactions, as well as mutual inhibition and activation-inhibition, in Figure 4.8

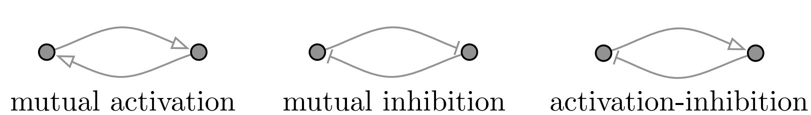


Figure 4.8: In biochemical and ecological networks, the interaction between two species may be one of mutual activation, mutual inhibition or activation-inhibition. Monotone dynamical systems, in their simplest formulation, feature interactions that are exclusively of mutual activation.

[2]Indeed, one can compute $-v^\mathsf{T}|A|_\text{M} > -\max(v^\mathsf{T}|A|_\text{M})\mathbb{1}_n^\mathsf{T} = -\frac{\max(v^\mathsf{T}|A|_\text{M})}{\max(v)}\max(v)\mathbb{1}_n^\mathsf{T} = c\max(v)\mathbb{1}_n^\mathsf{T} \geq cv^\mathsf{T}$.

4.5.1 Monotone systems

A set in $\mathbb{R}^n$ is *rectangular* if it is the Cartesian product of possibly-unbounded intervals with non-empty interiors, i.e., it is of the form $\mathcal{X} = \mathcal{X}_1 \times \dots \mathcal{X}_n$ where each $\mathcal{X}_i$ is an interval of positive length.

We consider a time-invariant continuously-differentiable dynamical system $(\mathcal{X}, f)$ where $\mathcal{X}$ is rectangular:

$$\dot{x} = f(x). \tag{4.25}$$

As usual, $\phi_t(x_0)$ is the solution at time t from initial condition $x_0 \in \mathcal{X}$ at time 0.

Definition 4.11 (Monotone systems). *Consider the continuously-differentiable dynamical system $\dot{x} = f(x)$ on an f-invariant rectangular domain $\mathcal{X}$. The system (or the vector field) is*

(i) monotone if, for all initial conditions $x_0, y_0 \in \mathcal{X}$,

$$x_0 \leq y_0 \quad \Longrightarrow \quad \phi_t(x_0) \leq \phi_t(y_0) \quad \text{for all } t \geq 0, \tag{4.26}$$

(ii) strongly monotone if $x_0 \leq y_0$ and $x_0 \neq y_0$ together imply $\phi_t(x_0) < \phi_t(y_0)$ for all $t > 0$.

In other words, trajectories of monotone systems maintain for all positive time the partial ordering between the initial conditions, see Figure 4.9.

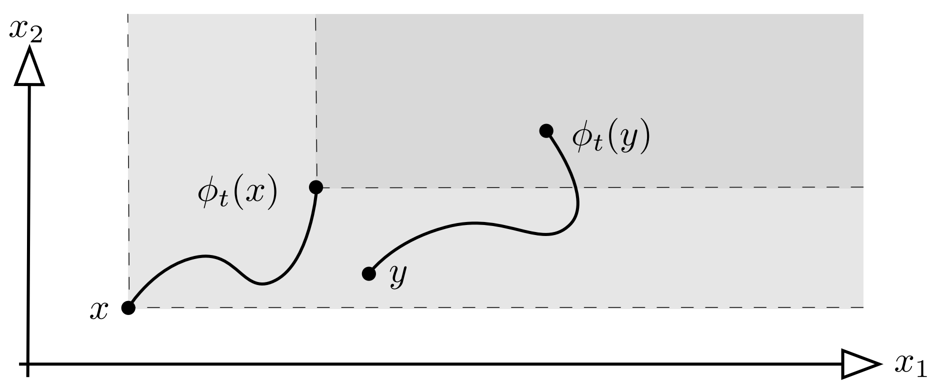


Figure 4.9: The flow of a monotone dynamical system is *order preserving* in the sense that, if $x \leq y$, then $\phi_t(x) \leq \phi_t(y)$.

Note: This nomenclature is standard for vector fields. For clarity, what we mean here by monotone vector field is not a monotone operator in functional analysis.

Theorem 4.12 (Kamke-Müller conditions). *Consider the continuously-differentiable dynamical system $\dot{x} = f(x)$ on an invariant rectangular domain $\mathcal{X}$. The following statements are equivalent:*

(i) f is monotone,

(ii) $f_i(y) \leq f_i(x)$ for all $x, y \in \mathcal{X}$ and for each $i \in \{1, \dots, n\}$ such that $y \leq x$ and $y_i = x_i$, and

(iii) $Df(x)$ is Metzler for all $x \in \mathcal{X}$.

Moreover, the system is strongly monotone if and only if $Df(x)$ is Metzler and irreducible for all $x \in \mathcal{X}$.

We refer to (ii) and (iii) as to the integral and differential Kamke-Müller conditions.

The condition (ii) can be restated as: at fixed x_i, the map $x_{-i} \mapsto f_i(x_i, x_{-i})$ is weakly increasing; see Figure 4.10.

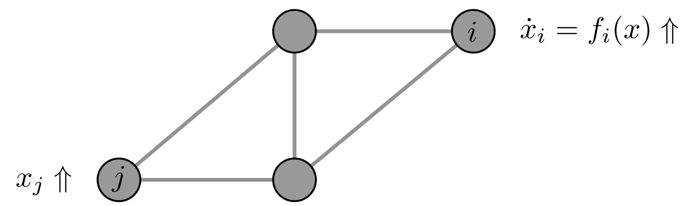


Figure 4.10: Illustration of the weak increasing property of $x_{-i} \mapsto f_i(x_i, x_{-i})$. Loosely speaking, when x_j increases, $\dot{x}_i$ increases or remains unchanged.

Proof of Theorem 4.12. We sketch the proof by contradiction of the implication (iii) $\implies$ (i). Aiming to identify a contradiction, we assume that $x_0 \le y_0$ does not imply $\phi_t(x_0) \le \phi_t(y_0)$ for some time $t > 0$. Therefore there exists a time $\tau > 0$ and an index i such that $\phi_\tau(x_0) \le \phi_\tau(y_0)$, $(\phi_\tau(x_0))_i = (\phi_\tau(y_0))_i$, and $\frac{d}{d\tau}\big((\phi_\tau(x_0))_i - (\phi_\tau(y_0))_i\big) > 0$. With the abbreviated notation $x = \phi_\tau(x_0)$, $y = \phi_\tau(y_0)$, and $Df(x,y) := \int_0^1 Df\big(y + s(x-y)\big)ds$, the Mean Value Theorem for vector-valued functions in Exercise E3.8 implies

$$\frac{d}{d\tau}\big((\phi_\tau(x_0))_i - (\phi_\tau(y_0))_i\big) = f_i(\phi_\tau(x_0)) - f_i(\phi_\tau(y_0)) = \mathbb{e}_i^\top Df(x,y)\,(x-y).$$

Note that $x - y = \phi_\tau(x_0) - \phi_\tau(y_0) \le 0$, $x_i = y_i$, and $Df(x,y)$ is Metzler. Therefore we compute

$$\frac{d}{d\tau}\big((\phi_\tau(x_0))_i - (\phi_\tau(y_0))_i\big) = \mathbb{e}_i^\top Df(x,y)(x-y) = \sum_{j=1}^n Df_{ij}(x_j - y_j) = \sum_{j=1, j\neq i}^n Df_{ij}(x,y)(x_j - y_j) \le 0.$$

This is a contradiction and so the implication (iii) $\implies$ (i) is established. We leave to the reader the other easier implications. ■

Remark 4.13 (Separable Lyapunov functions). *Consider a monotone system $(\mathcal{X}, f)$. A function $V\colon \mathcal{X} \to \mathbb{R}$ is agent sum-separable if it can be written as*

$$V(x) = \sum_{i=1}^n V_i(x_i, f_i(x)), \tag{4.27}$$

for a collection of appropriate functions V_i. If V_i are functions only of the state, then we call V sum-separable.

Similarly, a function $V\colon \mathcal{X} \to \mathbb{R}$ is agent max-separable if it can be written as

$$V(x) = \max_{i \in \{1,...,n\}} V_i(x_i, f_i(x)), \tag{4.28}$$

for a collection of appropriate functions V_i. If V_i are functions only of the state, then we call V max-separable. •

Example 4.14. Numerous interesting dynamical systems and networks are monotone. Here is an incomplete list:

(i) standard affine averaging and flow systems: $\dot{x} = -Lx + u$ and $\dot{x} = -L^\top x + u$, where L is a weighted directed Laplacian matrix or, even more general, a Metzler matrix (see Example 4.20 below);

(ii) numerous traffic networks (see Section 4.6 below);

(iii) Hopfield neural networks with only excitatory synaptic connections, that is, Metzler synaptic matrices;

(iv) Ecological models for the evolution of a single species migrating between distinct geographic patches: $\dot{x}_i = r_i x_i(1 - x_i/\kappa_i) + \sum_{j\neq i}(f_{ji}x_j - f_{ij}x_i)$, where $i, j \in \{1, \dots, n\}$ are indices describing the distinct geographic patches;

(v) epidemic models (we refer the reader to (Mei et al., 2017) and references therein). •

4.5.2 Positive monotone systems

Definition 4.15. *A dynamical system* $(\mathbb{R}^n, f)$ *is positive if the positive orthant* $\mathbb{R}^n_{\geq 0}$ *is* f*-invariant.*

In other words, if the initial condition is non-negative, the solution remains non-negative for all subsequent time. For example, networks where commodities (such as water, gas) flow, feature state variables that cannot ever be negative. For positive systems, it is customary to restrict the domain to the positive orthant and describe the system as the pair $(\mathbb{R}^n_{\geq 0}, f)$.

Lemma 4.16 (Invariant sets and basic properties of positive monotone systems). *Given a monotone vector field* $f: \mathbb{R}^n \to \mathbb{R}^n$ *and* $a, b \in \mathbb{R}^n$ *with* $a < b$,

(i) the rectangular set $\{x \in \mathbb{R}^n : a \leq x\}$, *for* $a \in \mathbb{R}^n$, *is invariant if and only if* $f(a) \geq \mathbb{0}_n$,

(ii) the rectangular set $\{x \in \mathbb{R}^n : a \leq x \leq b\}$ *is invariant if and only if* $f(a) \geq \mathbb{0}_n$ *and* $f(b) \leq \mathbb{0}_n$, *(see Figure 4.11)*

(iii) f *is positive if and only if* $f(\mathbb{0}_n) \geq \mathbb{0}_n$.

Moreover, for a positive monotone vector field f, *the zero-trajectory* $t \mapsto \phi_0(t) = \phi_t(\mathbb{0}_n)$

(iv) is entrywise monotonically non-decreasing,

(v) either converges to an equilibrium or is unbounded, and

(vi) lower bounds all solutions in the positive orthant, that is, $\mathbb{0}_m \leq \phi_0(t) \leq \phi_t(x_0)$ *for all* $x_0 \in \mathbb{R}^n_{\geq 0}$ *and* $t \geq 0$.

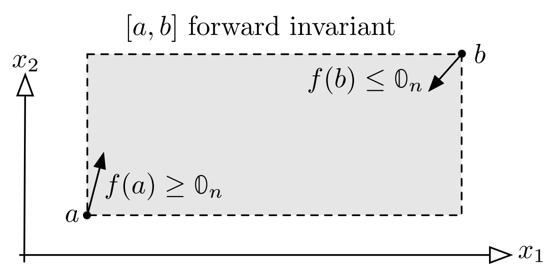


Figure 4.11: The rectangular set defined by $a < b$ is invariant for a monotone vector field f if and only if $f(a) \geq \mathbb{0}_n$ and $f(b) \geq \mathbb{0}_n$.

Proof. Statement (iii) is a consequence of the Kamke-Müller conditions in Theorem 4.12: pick any $y \geq \mathbb{0}_n$ satisfying $y_i = 0$ for some $i \in \{1, \dots, n\}$. Then by Theorem 4.12(ii) with $x = \mathbb{0}_n$, we know $f_i(y) \geq f_i(\mathbb{0}_n)$ which is nonnegative by assumption. Nagumo's Theorem in Exercise E3.13 completes the proof of statement (iii). We leave the other details to the reader.

Regarding statement (iv) and (v), pick nonnegative times $t \geq 0$ and $s \geq 0$. Since f is positive, we know $\phi_0(s) = \phi_s(\mathbb{0}_n) \geq \mathbb{0}_n$. Since f is monotone,

$$\phi_0(t+s) = \phi_{t+s}(\mathbb{0}_n) = \phi_t(\phi_s(\mathbb{0}_n)) \geq \phi_t(\mathbb{0}_n) = \phi_0(t).$$

This proves that $t \mapsto \phi_0(t)$ is entrywise monotonically non-decreasing. Therefore, $t \mapsto \phi_0(t)$ is either unbounded or it converges to a finite limit. In the latter case, since the convergence is monotone, it must be that also $\lim_{t \to +\infty} \dot{\phi}_0(t) = 0^+$. A standard continuity argument then proves that the limiting point must be an equilibrium. The proof of statement (vi) is left to the reader. ■

Note: for example, monotone systems with an equilibrium at the origin are automatically positive. In general, monotone systems are not positive and positive systems are not monotone. The case of affine systems is simple to characterize: The affine system $\dot{x} = Mx + b$ is

(i) monotone if and only if M is Metzler, and
(ii) positive if and only if M is Metzler and $b \geq \mathbb{0}_n$.

4.5.3 One-sided Lipschitzness for monotone maps

In the study of monotone vector fields, we will consider monotonic norms and compatible weak pairings $[\![\cdot\,;\,\cdot]\!]$ satisfying:

(WP8) *the invariance property*: for any permutation and reflection matrix[3] P,

$$[\![x\,;\,y]\!] = [\![Px\,;\,Py]\!], \quad \text{for all } x, y \in \mathbb{R}^n. \tag{4.29}$$

It is immediate to verify that the sign and max pairing satisfy this invariance property.

We now state the main result of this section. The proof is elegant, but elaborate and postponed to Appendix 4.8.

Lemma 4.17 (One-sided Lipschitz conditions for monotone maps). *Let $\|\cdot\|$ be a monotonic norm with associated weak pairing (satisfying the standing assumptions and the invariance property). Let the vector field f be continuous and monotone. Then*

$$\mathsf{osLip}(f) = \sup_{x>y} \frac{[\![f(x) - f(y)\,;\,x-y]\!]}{\|x-y\|^2}. \tag{4.30}$$

Specifically, for a positive vector $\eta \in \mathbb{R}^n_{>0}$,

$$\mathsf{osLip}_{1,[\eta]}(f) = \sup_{x>y, x,y\in\mathbb{R}^n} \frac{\eta^\mathsf{T}\big(f(x) - f(y)\big)}{\eta^\mathsf{T}(x-y)}, \tag{4.31}$$

$$\mathsf{osLip}_{\infty,[\eta]^{-1}}(f) = \sup_{y\in\mathbb{R}^n, \beta>0, x=y+\beta\eta} \; \max_{i\in\{1,...,n\}} \frac{f_i(x) - f_i(y)}{x_i - y_i}. \tag{4.32}$$

As before, we present the implications of this lemma the equivalent "upper bound formulation." For a monotone vector field f, a positive vector η, and a scalar $b \in \mathbb{R}$, the following conditions are equivalent with $\mathsf{osLip}_{1,[\eta]}(f) \leq b$:

$$\eta^\mathsf{T} Df(x) \leq b\eta^\mathsf{T} \qquad \text{for all } x \in C, \tag{4.33}$$

$$\eta^\mathsf{T}\big(f(x) - f(y)\big) \leq b\eta^\mathsf{T}(x-y) \qquad \text{for all } (x,y) \in C \times C \text{ s.t. } x \geq y. \tag{4.34}$$

And, the following conditions are equivalent with $\mathsf{osLip}_{\infty,[\eta]^1}(f) \leq b$:

$$Df(x)\eta \leq b\eta \qquad \text{for all } x \in C, \tag{4.35}$$

$$f(x) - f(y) \leq b(x-y) \qquad \text{for all } x, y \in C \text{ s.t. } x = y + \beta\eta, \text{ for some } \beta > 0. \tag{4.36}$$

The following table contains the results in Lemma 3.4 (which generalize those in Lemma 3.1) for the common ℓ_p norms, for $p \in \{1, 2, \infty\}$, and the special case of monotone vector fields, presented in Lemma 4.17.

[3] Recall that a *permutation and reflection matrix* is a square matrix with precisely one entry equal to $+1$ or -1 in every row and every column and all other entries equal to 0.

Log norm condition	Demidovich condition	osLip condition	Reference
$\mu_{2,P^{1/2}}(Df(x)) \leq b$	$PDf(x) + Df(x)^\top P \preceq 2bP$	$(x-y)^\top P\big(f(x)-f(y)\big) \leq b\|x-y\|^2_{2,P^{1/2}}$	Lemma 3.1
$\mu_1(Df(x)) \leq b$	$\operatorname{sign}(v)^\top Df(x)v \leq b\|v\|_1$	$\operatorname{sign}(x-y)^\top(f(x)-f(y)) \leq b\|x-y\|_1$	Lemma 3.4 and E3.11
$\mu_\infty(Df(x)) \leq b$	$\max\limits_{i\in I_\infty(v)} \operatorname{sign}(v_i)\,(Df(x)v)_i \leq b\|v\|_\infty$	$\max\limits_{i\in I_\infty(x-y)}(x_i-y_i)\,(f(x)-f(y))_i \leq b\|x-y\|^2_\infty$	Lemma 3.4
		Special Case: Monotone vector fields f	
$\mu_{1,[\eta]}(Df(x)) \leq b$	$\eta^\top Df(x) \leq b\eta^\top$	$\eta^\top\big(f(x)-f(y)\big) \leq b\eta^\top(x-y)$ for all $x \geq y$	Lemma 4.17
$\mu_{\infty,[\eta]}(Df(x)) \leq b$	$Df(x)\eta \leq b\eta$	$f(x)-f(y) \leq b(x-y)$ for all $x = y+\beta\eta, \beta > 0$	Lemma 4.17

Table 4.1: Cumulative table of one-sided Lipschitz conditions and equivalent log norm and Demidovich conditions. Each row contains three equivalent statements, to be understood for all $x, y \in \mathbb{R}^n$ and all $v \in \mathbb{R}^n$. For the third row on the ℓ_∞ norm, we adopt the shorthand $I_\infty(v) = \{i \in \{1,\dots,n\} \,:\, |v_i| = \|v\|_\infty\}$. The matrix P is positive definite and the vector η is positive.

4.5.4 Strongly contracting monotone systems

We now study the contractivity properties of monotone dynamical systems with respect to the 1 and ℓ_∞ norm. Combining the results on log norms for Metzler matrices (e.g., see Table 2.2) with the main contractivity Theorem 3.9, we are now ready to state the contractivity and stability results about nonlinear monotone systems.

Corollary 4.18 (ℓ_1 contraction of monotone systems). *Consider the monotone continuously-differentiable dynamical system $\dot{x} = f(x)$ on an invariant rectangular domain $\mathcal{X}$. Given a positive vector $\eta \in \mathbb{R}^n_{>0}$ and a positive scalar $c > 0$, assume f satisfies the Demidovich condition (see Lemma 4.17 for equivalent characterizations)*

$$\eta^\top Df(x) \leq -c\eta^\top \quad \textit{for all } x \in \mathcal{X}. \tag{4.37}$$

Then

(i) f is strongly infinitesimally contracting with contraction rate c on $\mathcal{X}$ with respect to $\|\cdot\|_{1,[\eta]}$ and the distance in the $\|\cdot\|_{1,[\eta]}$ norm between any two solutions is exponentially decreasing with rate c,

(ii) f has a unique globally exponentially stable equilibrium x^* with sum-separable global Lyapunov functions

$$x \mapsto \|x-x^*\|_{1,[\eta]} = \sum_{i=1}^n \eta_i|x_i - x_i^*|, \quad \textit{and} \quad x \mapsto \|f(x)\|_{1,[\eta]} = \sum_{i=1}^n \eta_i|f_i(x)|. \tag{4.38}$$

Corollary 4.19 (ℓ_∞ contraction of monotone systems). *Consider the monotone continuously-differentiable dynamical system $\dot{x} = f(x)$ on an invariant rectangular domain $\mathcal{X}$. Given a positive vector $\xi \in \mathbb{R}^n_{>0}$ and a positive scalar $c > 0$, assume f satisfies the Demidovich condition (see Lemma 4.17 for equivalent characterizations)*

$$Df(x)\xi \leq -c\xi \quad \textit{for all } x \in \mathcal{X}. \tag{4.39}$$

Then

(i) f *is strongly infinitesimally contracting with contraction rate* c *on* $\mathcal{X}$ *with respect to* $\|\cdot\|_{\infty,[\xi]^{-1}}$ *and the distance in the* $\|\cdot\|_{\infty,[\xi]^{-1}}$ *norm between any two solutions is exponentially decreasing with rate* c*,*

(ii) f *has a unique globally exponentially stable equilibrium* x^* *with max-separable global Lyapunov functions*

$$x \mapsto \|x - x^*\|_{\infty,[\xi]^{-1}} = \max_{i\in\{1,...,n\}} \tfrac{1}{\xi_i}|x_i - x_i^*|, \quad \textit{and} \quad x \mapsto \|f(x)\|_{\infty,[\xi]^{-1}} = \max_{i\in\{1,...,n\}} \tfrac{1}{\xi_i}|f_i(x)|. \tag{4.40}$$

We now present a Metzler version of Example 3.11.

Example 4.20 (Time-invariant monotone affine systems). Given a Metzler matrix $M \in \mathbb{R}^{n\times n}$, consider the time-invariant monotone affine system

$$\dot{x} = Mx + d =: f_M(x). \tag{4.41}$$

Assume M is Hurwitz, that is, $\alpha(M) < 0$. Pick an ε sufficiently small so that $\alpha(M)+\varepsilon < 0$. For any $p \in [1,\infty]$, as in Lemma 2.31, compute a positive vector $\eta \in \mathbb{R}^n_{>0}$ such that $\|\cdot\|_{p,[\eta]}$ is ε-optimal for M. In other words, compute η such that

$$\mathsf{osLip}_{p,[\eta]}(f_M) \leq \alpha(M) + \varepsilon. \tag{4.42}$$

Then f_M is strongly infinitesimally contracting with respect to the norm $\|\cdot\|_{p,[\eta]}$ with rate $\alpha(M)+\varepsilon$. As a consequence, the system has a unique exponentially stable equilibrium $x^* = -M^{-1}d$ with global Lyapunov function

$$x \mapsto \|x - x^*\|_{p,[\eta]} = \|x + M^{-1}d\|_{p,[\eta]} \quad \text{and} \quad x \mapsto \|f_M(x)\|_{p,[\eta]} = \|Mx + d\|_{p,[\eta]}.$$

As special cases, for $p \in \{1,\infty\}$, we obtain the sum-separable global Lyapunov functions

$$x \mapsto \|x - x^*\|_{1,[\eta]} = \sum_{i=1}^n \eta_i |x_i + M^{-1}b|_i \quad \text{and} \quad x \mapsto \|f(x)\|_{1,[\eta]} = \sum_{i=1}^n \eta_i |Mx + b|_i$$

and max-separable global Lyapunov functions

$$x \mapsto \|x - x^*\|_{\infty,[\xi]^{-1}} = \max_{i\in\{1,...,n\}} \frac{|x_i + M^{-1}b|_i}{\xi_i} \quad \text{and} \quad x \mapsto \|f(x)\|_{\infty,[\xi]^{-1}} = \max_{i\in\{1,...,n\}} \frac{|Mx+b|_i}{\xi_i},$$

for any positive vectors η and ξ such that $\eta^\top M < \mathbb{0}_n^\top$ and $M\xi < \mathbb{0}_n$. •

4.5.5 Weakly contracting monotone systems

Motivated by the discussion in the introduction to this chapter and in Lemma 4.1, we now focus on continuously-differentiable monotone dynamical systems $(\mathbb{R}^n, f)$ whose Jacobian satisfies a weak contractivity condition. Specifically, we consider systems whose Jacobian is a compartmental matrix everywhere in $\mathbb{R}^n$. In precise words, we assume, for all $x \in \mathbb{R}^n$,

$$Df_{ij}(x) = \frac{\partial f_i}{\partial x_j}(x) \geq 0 \text{ for all } i \neq j, \quad \text{and} \quad \mathbb{1}_n^\top Df(x) \leq \mathbb{0}_n^\top, \text{that is, } \sum_{i=1}^n \frac{\partial f_i}{\partial x_j}(x) \leq 0 \text{ for all } j. \tag{4.43}$$

Note: since the column sums are non-positive, the quantity $\mathbb{1}_n^\top x(t)$ is either conserved or diminishing with time along the solutions to $\dot{x} = f(x)$.

Recall that a matrix is compartmental if it is Metzler and if all its column sums are negative or zero.

We are now ready to state a monotone systems version of Theorem 4.3 on the Dichotomy and Theorem 4.4 on convergence for weakly contracting systems.

Theorem 4.21 (Dichotomy theorem for monotone weakly-contracting systems). *Consider the continuously-differentiable time-invariant dynamical system $\dot{x} = f(x)$. Assume*

(A1) *there exists an invariant rectangular domain $\mathcal{X}$,*

(A2) *the Jacobian $Df(x)$ is a compartmental matrix for all $x \in \mathcal{X}$ so that (i) the system is weakly infinitesimally contracting with respect to the ℓ_1 norm, (ii) the distance in the ℓ_1 norm between any two solutions is non-increasing in time, and (iii) the function $x \mapsto \|f(x)\|_1$ is non-increasing along the solutions.*

Then precisely one of the following mutually-exclusive conditions hold: either

(C1) *f has no equilibrium point in $\mathcal{X}$ and every trajectory starting in $\mathcal{X}$ is unbounded, or*

(C2) *f has at least one equilibrium point $x^* \in \mathcal{X}$ and the following properties hold:*

  a) each solution is bounded,
  b) each solution approaches the set of equilibria of $(\mathbb{R}^n_{\geq 0}, f)$,
  c) each equilibrium x^{**} is stable with weak Lyapunov function $x \mapsto \|x - x^{**}\|_1$, and
  d) if the compartmental matrix $Df(x^*)$ is Hurwitz, then the equilibrium x^* is globally exponentially stable,

Remark 4.22. *Regarding fact (C2)a, if the set of equilibria is finite (so that the equilibrium points are isolated), then each trajectory approaches one of the equilibrium points.*

Remark 4.23. *Linear compartmental systems satisfy the following asymptotic property from Theorem B.26: trajectories either (i) are unbounded (when the compartmental digraph has a trap and the inflow accumulates linearly in time inside each inflow-connected simple trap) or (ii) converge to a unique globally asymptotically stable equilibrium point. These properties are consistent with Theorem 4.21.*

Remark 4.24 (Comparison between Theorems 4.3/4.4 and 4.21). *It is useful to compare Theorems 4.3/4.4 on weakly infinitesimally contracting systems and Theorem 4.21 on monotone systems with compartmental Jacobians:*

(i) Theorems 4.3/4.4 do not require monotonicity nor the adoption of the ℓ_1 norm, so they are more general;

(ii) Theorem 4.21 is more specific in that it applies only to systems whose Jacobian is compartmental. However, numerous examples fall in this class and Theorem 4.21 has the advantage of not requiring piecewise analyticity for convergence to the set of equilibrium points. •

4.6 Example: Daganzo models for traffic networks

In this section we review the so-called Daganzo cell transmission model for traffic networks (Daganzo, 1994). The aim is to provide mathematical models and analysis of traffic phenomena on roads, including congestion phenomena over realistic street maps as illustrated in Figure 4.12.

Example 4.25 (Linear flow networks). Here we review key ideas about linear network flow systems from (Bullo, 2022, Chapter 10) and compartmental systems, as illustrated in Figure 4.13.

(i) The model is

$$\dot{q} = Cq + F_{\text{inflows}}, \tag{4.44}$$

where q_i is the amount/density of commodity at compartment i, C is a compartmental matrix (C is Metzler and $\mathbb{1}^\top C \leq \mathbb{0}_n$), and F_{inflows} are the (state-independent) inflows. Alternatively, we can decompose the compartmental matrix as $C := -(I_n - R^\top)D$, where R is a row-substochastic routing matrix and D is diagonal positive definite;

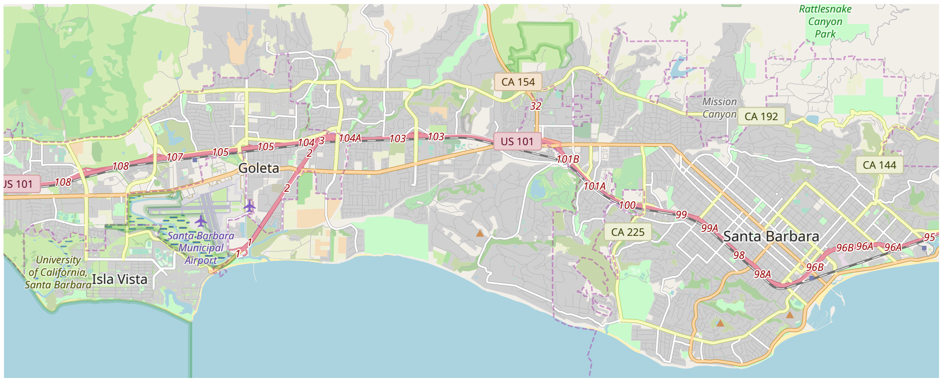


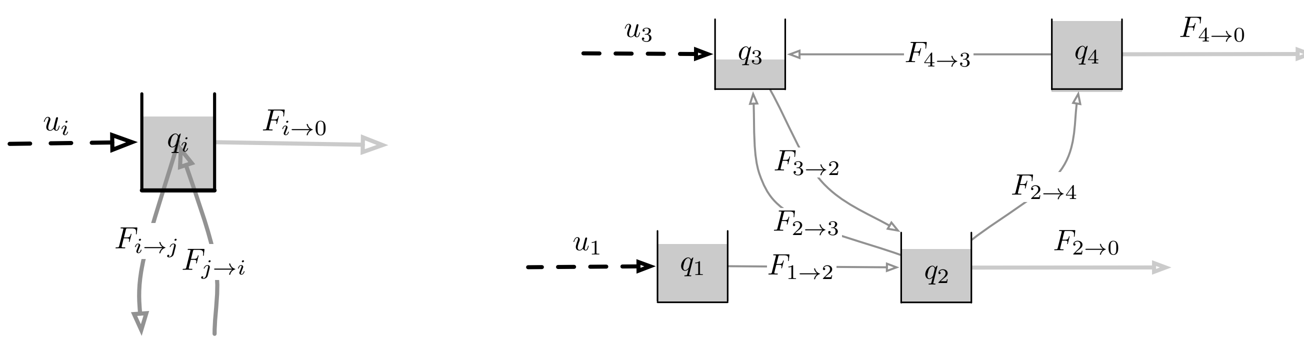
Figure 4.12: Example street map, courtesy of of the OpenStreetMap Foundation.

(ii) the system is monotone and positive;

(iii) C is Hurwitz if and only if the digraph is outflow connected if and only if the routing matrix R is Schur stable, and

(iv) when C is Hurwitz, the linear dynamics $\dot{q} = Cq + F_{\text{inflows}}$ have the unique globally exponentially stable equilibrium $-C^{-1}F_{\text{inflows}} \geq \mathbb{0}_n$.

Since the model is linear, it exhibits no congestion. •



(a) A compartment with inflow u_i, outflow $F_{i\to 0}$, and inter-compartmental flows $F_{i\to j}$

(b) A compartmental system with two inflows and two outflows

Figure 4.13: Example of a single compartment and of a dynamical flow system

4.6.1 Daganzo model with demand/supply on a path digraph

We start by considering a simple topology, as illustrated in Figure 4.14.

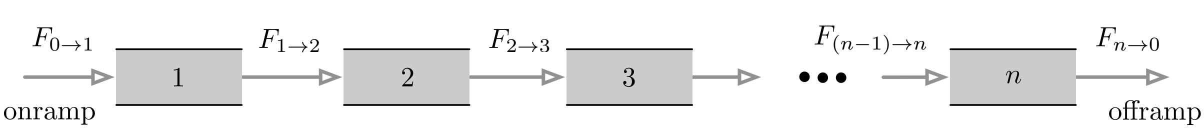


Figure 4.14: A traffic network on a path digraph. Each node is a segment of a road/freeway and is referred to as a *cell*.

Following the figure and the law of conservation of mass, we define the *Daganzo model with demand/supply on a path digraph*, denoted by $f_{\mathrm{D}}\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$, by:

$$\begin{aligned}\dot{q}_1 &= F_{0\to 1} - F_{1\to 2}(q_1, q_2),\\ \dot{q}_i &= F_{(i-1)\to i}(q_{i-1}, q_i) - F_{i\to(i+1)}(q_i, q_{i+1}), \qquad i \in \{2, \dots, n-1\},\\ \dot{q}_n &= F_{(n-1)\to n}(q_{n-1}, q_n) - F_{n\to 0}(q_n).\end{aligned} \tag{4.45}$$

where $q_i \geq 0$ is the density of commodity at compartment i and $F_{i\to j}$ is the flow from compartment i to j.

Next, we define the flow functions in equation (4.45) in terms of (continuous) *outflow demand functions* and *inflow supply functions*:

$$\begin{aligned}F_{i\to i+1}(q_i, q_{i+1}) &= \min\Big\{ \underbrace{\varphi_i(q_i)}_{\text{increasing outflow demand}}, \underbrace{\sigma_{i+1}(q_{i+1})}_{\text{decreasing inflow supply}} \Big\},\\ F_{n\to 0}(q_n) &= \underbrace{\varphi_n(q_n)}_{\text{increasing outflow demand}}.\end{aligned} \tag{4.46}$$

Outflow demand functions are weakly increasing, satisfy $\varphi_i(0) = 0$, and are typically upper bounded, thereby modeling *congested links*, see Figure 4.15. Accordingly, the *outflow capacity* of cell i is

$$\overline{\varphi}_i^{\mathrm{o}} = \sup_{q_i \in \mathbb{R}_{\geq 0}} \varphi_i(q_i) = \lim_{q_i \to \infty} \varphi_i(q_i). \tag{4.47}$$

Inflow supply functions are weakly decreasing and are equal to zero for large traffic densities. The *capacity* of cell i is

$$\overline{\varphi}_i = \max_{q_i \in \mathbb{R}_{\geq 0}} \min\{\varphi_i(q_i), \sigma_i(q_i)\}. \tag{4.48}$$

Indeed, at an equilibrium point for cell i with maximum flow through i, the inflow equals the outflow so that demand equals supply at maximum value.

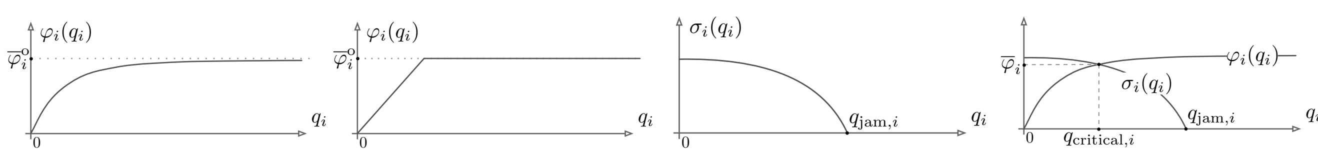


Figure 4.15: First two images: Prototypical outflow demand functions. Third image: Prototypical supply function. Fourth image: cell capacity.

We now study the Daganzo model defined by equation (4.45) with demand/supply flow functions defined by equation (4.46):

(i) f_{D} is *positive* under the natural assumptions: $F_{0\to 1} \geq 0$, $F_{i\to(i+1)}(0, q_{i+1}) = 0$ for all $q_{i+1} \geq 0$, and $F_{n\to 0}(0) = 0$. Moreover, f_{D} satisfies the law of *conservation of mass*. Recall $\mathbb{1}_n^\top q(t)$ is the total mass and compute:

$$\frac{d}{dt}\mathbb{1}_n^\top q(t) = \mathbb{1}_n^\top f_{\mathrm{D}}(q) = F_{0\to 1} - F_{n\to 0}(q_n); \tag{4.49}$$

(ii) f_{D} is *monotone* if it satisfies the Kamke-Müller conditions in Theorem 4.12. For continuously differentiable flows the differential Kamke-Müller condition is:

$$\begin{aligned}&\frac{\partial (f_{\mathrm{D}})_h}{\partial q_k}(q) \geq 0 \quad \text{for all } h \neq k \in \{1, \dots, n\}\\ \iff \quad &\frac{\partial F_{i\to(i+1)}}{\partial q_i}(q_i, q_{i+1}) \geq 0 \quad \text{and} \quad \frac{\partial F_{i\to(i+1)}}{\partial q_{i+1}}(q_i, q_{i+1}) \leq 0, \quad \text{for all } i \in \{1, \dots, n-1\};\end{aligned}$$

for non-differentiable flows, the integral Kamke-Müller condition is, for $i \in \{1, \dots, n-1\}$

$$F_{i\to i+1}(q_i, q_{i+1}) \text{ weakly increasing in } q_i \text{ and weakly decreasing in } q_{i+1};$$

(iii) finally, we say that f_{D} has *weakly increasing outflows* if $F_{n\to 0}(q_n)$ is a weakly increasing function.

Clearly, the demand/supply flow functions (4.46) satisfy the Kamke-Müller conditions and the weakly-increasing outflow property.

Lemma 4.26 (The Daganzo model with outflow/inflow over path digraph is ℓ_1 weakly contracting). *Consider the Daganzo model with demand/supply on a path digraph, defined by equations* (4.45) *and* (4.46). *Then*

(i) the model is positive, monotone and weakly infinitesimally contracting with respect to the ℓ_1 norm.

For constant F_{inflows}, precisely one of the following mutually-exclusive conditions hold:

(ii) if $F_{\text{inflows}} \geq \min_{i\in\{1,...,n\}} \overline{\varphi}_i$, then there exists no equilibrium point and every trajectory is unbounded; or

(iii) if $F_{\text{inflows}} < \min_{i\in\{1,...,n\}} \overline{\varphi}_i$, then there exists a unique globally asymptotically stable equilibrium point $q^* = \varphi^{-1}(F_{\text{inflows}})$.

Proof. We already showed how f_{D} is monotone and has monotonic outflows. For $q \geq s \geq \mathbb{0}_n$, we now compute the integral Demidovich condition with respect to ℓ_1:

$$\mathbb{1}_n^\top\big(f_{\mathrm{D}}(q) - f_{\mathrm{D}}(s)\big) = F_{n\to 0}(s_n) - F_{n\to 0}(q_n) \leq 0. \tag{4.50}$$

This concludes the proof of the weak contractivity property in statement (i).

Regarding statements (ii) and (iii), we reason as follows. First, if an equilibrium q^* exists, then the flow F_{inflows} must be upper bounded by both the demand and supply at each cell i, that is, $F_{\text{inflows}} \leq \min\{\varphi_i(q_i^*), \sigma_i(q_i^*)\}$. In turn, this implies $F_{\text{inflows}} \leq \max_{q_i\in\mathbb{R}_{\geq 0}} \min\{\varphi_i(q_i), \sigma_i(q_i)\} = \overline{\varphi}_i$. Second, if $F_{\text{inflows}} < \overline{\varphi}_i$ for all i, then define $q_i^* = \varphi_i^{-1}(F_{\text{inflows}})$. It is easy to verify that the inflow supply constraint are satisfied at each cell i and, therefore, q^* is an equilibrium. All other statements are immediate consequences of the Dichotomy Theorem 4.21 for monotone weakly infinitesimally contracting systems. ∎

Note: $F_{n\to 0}(s_n) - F_{n\to 0}(q_n)$ is not upper-bounded by $-c\mathbb{1}_n^\top(q-s)$, for $c > 0$, and so f_{D} is not strongly infinitesimally contracting.

4.6.2 The Daganzo model with demand over arbitrary digraphs (without inflow constraints)

We now consider a simplified model for the individual flows, but we allow an arbitrary topology. We assume the flow from cell i to cell j is

$$F_{i\to j}(q_i) = \underbrace{\varphi_i(q_i)}_{\text{total outflow from } i} \times \underbrace{R_{i\to j}}_{\text{fraction of } i\text{-outflow routed to } j}, \tag{4.51}$$

where

(i) as before, $\varphi_i\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the total outflow demand from i;

(ii) the (constant) *split ratio* $R_{i\to j}$ is the fraction of outflow from i that flows to j. The *routing matrix* $R \in \mathbb{R}^{n\times n}$ is row substochastic and strictly convergent if and only if the associated digraph is outflow connected;

With the decomposition (4.51), the network flow system (4.54) is rewritten as the *Daganzo model with outflow demand (and without inflow supply)*

$$\dot{q} = -(I_n - R^\top)\varphi(q) + F_{\text{inflows}}. \tag{4.52}$$

Lemma 4.27 (The Daganzo model with outflow demands is ℓ_1 weakly contracting). *Consider the Daganzo model (4.52) with strictly-increasing outflow demand functions with outflow capacity $\overline{\varphi}_i^{\mathrm{o}}$, $i \in \{1, \dots, n\}$, and constant routing matrix R (and without inflow supply). Then*

(i) the model is positive, monotone and weakly infinitesimally contracting with respect to the ℓ_1 norm.

For constant F_{inflows} and outflow-connected R, define $f^* = (I_n - R^\top)^{-1} F_{\text{inflows}}$. Then, precisely one of the following mutually-exclusive conditions hold:

(ii) if $f_i^* \geq \overline{\varphi}_i^{\mathrm{o}}$ for some cell i, then there exists no equilibrium point and every trajectory is unbounded; or

(iii) if $f_i^* < \overline{\varphi}_i^{\mathrm{o}}$ for each cell i, then there exists a unique globally asymptotically stable equilibrium point $q^* = \varphi^{-1}(f^*)$.

Proof. The proof follows from the Dichotomy Theorem 4.21 for monotone weakly infinitesimally contracting systems. The key step is the computation of the equilibrium operating condition of the network:

$$0 = F_{\text{inflows}} - (I_n - R^\top)\varphi(q^*) \quad \iff \quad \varphi(q) = (I_n - R^\top)^{-1} F_{\text{inflows}}. \tag{4.53}$$

A unique solution to the latter equation exists if and only if $f_i^* < \overline{\varphi}_i^{\mathrm{o}}$ since the functions φ are strictly monotonically increasing and upper bounded. All other statements are immediate consequences of the Dichotomy Theorem 4.21 for monotone weakly infinitesimally contracting systems. ∎

4.6.3 Monotone flow systems

We now present a unifying theory that generalizes the examples studied so far. A *network flow system* is a positive dynamical system described by

$$\dot{q}_i = \sum_{j=1, j\neq i}^{n} \Big(F_{j\to i}(q) - F_{i\to j}(q) \Big) + F_{0\to i} - F_{i\to 0}(q), \tag{4.54}$$

or, in vector format,

$$\dot{q} = \big(F_{\text{flows}}^\top(q) - F_{\text{flows}}(q)\big)\mathbb{1}_n + F_{\text{inflows}} - F_{\text{outflows}}(q),$$

where $F_{\text{flows}}(q)$ denotes the matrix function with components $F_{ij} = F_{i\to j}$. It is customary to require that $F_{i\to j}(q)$ vanishes whenever $q_i = 0$; this requirement ensures that the system is positive.

Note: $\mathbb{1}_n^\top(F_{\text{flows}}(q)^\top - F_{\text{flows}}(q))\mathbb{1}_n = 0$ so that the total commodity inside the network flow system is affected only by inflows and outflows: $\frac{d}{dt}\mathbb{1}_n^\top q = \mathbb{1}_n^\top\big(F_{\text{inflows}} - F_{\text{outflows}}(q)\big)$. We adopt the following notations:

(i) *donor/recipient-controlled flows* (as in equation (4.46)) are defined by $F_{j\to i}(q) = F_{j\to i}(q_j, q_i)$ for all i, j and $F_{i\to 0}(q) = F_{i\to 0}(q_i)$ for all i,

(ii) *donor-controlled flows* (as in equation (4.51)) are defined by $F_{j\to i}(q) = F_{j\to i}(q_j)$ for all i,

(iii) Specifically, *linear donor-controller flows* are defined by $F_{i\to j} = f_{ij}q_i$, that is, $F_{\text{flows}} = \operatorname{diag}(q)F$, where F is a matrix of flow rates, and $F_{i\to 0} = f_{0,i}q_i$. In this case, the *affine network flow system* reads

$$\dot{q} = (F^\top - \operatorname{diag}(F\mathbb{1}_n + f_0))q + F_{\text{inflows}} = Cq + F_{\text{inflows}},$$

as in Example 4.25.

Lemma 4.28 (Monotone network flow systems). *A network flow model (4.54) (satisfying the law of conservation of mass with state-independent inflows) is*

(i) monotone if and only if, for each i, the flow $\sum_{j=1,j\neq i}^{n}(F_{j\to i}(q) - F_{i\to j}(q)) - F_{i\to 0}(q)$ is weakly increasing in q_{-i}, and

(ii) ℓ_1 weakly contracting if it is monotone and each outflow $F_{i\to 0}(q)$ is weakly increasing in q.

Moreover, a network flow systems with donor/recipient-controlled flows and donor-controlled outflows is

(iii) monotone if and only if, for each i and $j \neq i$, the flow $F_{j\to i}(q_j, q_i) - F_{i\to j}(q_i, q_j)$ is weakly increasing in q_j, and

(iv) ℓ_1 weakly infinitesimally contracting if it is monotone and each outflow $F_{i\to 0}(q_i)$ is weakly increasing in q_i.

Remark 4.29 (Interpreting the dichotomy as classic traffic behavior). *Traffic models that are monotone and ℓ_1 weakly contracting exhibit the following dichotomy:*

(i) quick travel and no congestion in the approximately-linear free-flow regime; a globally attractive equilibrium exists if the nominal flows (computed as function of constant routing matrix) are below the flow capacity values (which depend upon the edge capacities). In this case, "inflows can be carried by the network," or

(ii) long wait times (in the congested regime); the wait time at one or more cells goes to infinity. •

4.6.4 Daganzo models with inflow supply constraints

Finally, we consider Daganzo models with *inflow supply constraints* into nodes. These constraints are especially important in so-called *diverging junctions*, that is, cells with multiple out-neighbors. We consider two models of *FIFO* (first-in first out) vs *non-FIFO priorities* and proportional scaling. Flows in these models are not donor/recipient controlled.

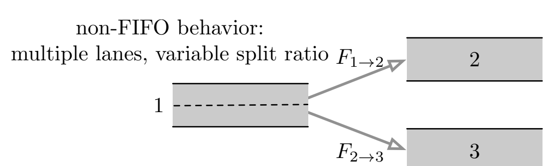


Figure 4.16: Non-FIFO junction: two lanes in cell 1 ensure that congestion in cell 2 does not affect negatively the flow from 1 to 3. When congestion is present in one of the two out-neighbors of 1, then the split ratio may not be preserved.

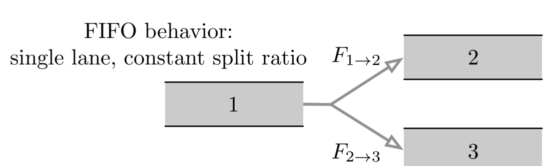


Figure 4.17: FIFO junction: single lane in cell 1, together with fixed split ratio, creates non-monotone behavior: congestion in cell 2 affects negatively the flow from 1 to 3.

Note: The Daganzo model with demand/supply over topology without diverging junctions is monotone.

Model with non-FIFO junctions For the model illustrated in Figure 4.16, given a nominal routing matrix R at free flow,

$$F_{i\to j}(q) = R_{ij}\overline{\sigma}_j(q)\varphi_i(q_i), \tag{4.55}$$
$$\overline{\sigma}_j(q) = \sup\Big\{\bar{\delta}\in[0,1] \,:\, \bar{\delta}\sum\nolimits_h R_{hj}\varphi_h(q_h) \le \sigma_j(q_j)\Big\}. \tag{4.56}$$

Here $\overline{\sigma}_j(q)$ scales down the outflow from i to j as follows: for all in-neighbors h of j, each inflow from h into j is scaled down proportionally. If node i has only one outflow to out-neighbor $i+1$ (as in the example), then this model simplifies to the one in equation (4.46), that is, $F_{i\to i+1}(q_i,q_{i+1}) = \min\big\{\varphi_i(q_i),\sigma_{i+1}(q_{i+1})\big\}$. The following result, based on the Dichotomy Theorem 4.21 for monotone weakly contracting systems, generalizes Lemmas 4.26 and 4.27 (but it is not an IFF condition).

Lemma 4.30 (The Daganzo model with inflow constraints and non-FIFO diverging junctions (Lovisari et al., 2014)). *Under usual assumptions, the model is monotone and ℓ_1 weakly infinitesimally contracting. For constant inflows, define $f^* = (I-R^\top)^{-1}F_{\text{inflows}}$. Then:*

(i) if $f_i^* < \overline{\varphi}_i$ for each cell i, then there exists a globally asymptotically stable equilibrium point $q^* = \varphi^{-1}(f^*)$, which corresponds to a free-flow equilibrium (in each cell $q_i^* \le q_{\text{critical},i}$);

(ii) if the inequality $f_i^* < \overline{\varphi}_i$ is violated for some i, there is no free-flow equilibrium point, but there might still be stable equilibrium points; and

(iii) if there exists no equilibrium point, then each solution is unbounded.

Model with FIFO junctions For the model illustrated in Figure 4.17:

$$\dot{q} = -(I_n - R^\top)\begin{bmatrix}\delta_1(q)\varphi_1(q_1)\\ \vdots \\ \delta_n(q)\varphi_n(q_n)\end{bmatrix} + F_{\text{inflows}}, \tag{4.57}$$

where the flow control functions are

$$\delta_i(q) = \sup\big\{\bar{\delta}\in[0,1] \,:\, \bar{\delta}\sum\nolimits_h R_{hk}\varphi_h(q_h) \le \sigma_k(q_k) \text{ for all } k \text{ s.t. } (i,k) \text{ edge}\big\}. \tag{4.58}$$

Here $\delta_i(q)$ scales down the outflow from i to all its out-neighbors as follows: (i) for each out-neighbor k, scale down the amount of traffic to ensure the inflow constraint of k is satisfied, (ii) scale down the traffic incoming into k from all in-neighbors h of k proportionally.

Results: in the free flow region $\{q\in\mathbb{R}^n_{\geq 0} \,:\, F_{\text{inflows}} + R^\top\varphi(q) \le \sigma(q)\}$ the model (4.57) is identical to the model (4.52) without inflow supply constraints and it is therefore monotone.

For general graph topologies, the model is not monotone on the entire positive orthant. Coogan and Arcak (2016) analyze this model over so-called polytree networks. Coogan and Arcak (2015b) propose the concept of mixed monotonicity.

4.7 Historical notes and further reading

Weak contractivity Lemma 4.1 is related to the more advanced treatments in (Mierczyński, 1987; Angeli and Sontag, 2008).

Early versions of the two main theorems about weakly contracting systems, namely Theorems 4.3 and 4.4, were presented in (Maeda et al., 1978; Sandberg, 1978; Jacquez and Simon, 1993); these works focused on monotone systems. Theorem 4.3 with the "(C1) or (C2)" dichotomy result is due to (Jafarpour et al., 2022). In Theorem 4.4 statement (ii) is due to (Jafarpour et al., 2022). Statement (iii) is a generalization of (Lovisari et al., 2014, Lemma 6) which focuses on the setting of monotone systems with compartmental Jacobians (the same proof method is adopted). Statement (iv) and a generalization to periodic trajectories is due to (Coogan, 2019, Theorem 7 and Corollary 8). Example 4.5 is taken from (Jafarpour et al., 2022). Related early results were given by (Borkar and Soumyanatha, 1997). Our presentation in Section 4.2 follows the treatment in (Jafarpour et al., 2022).

Distributed primal dual optimization A recent survey on distributed optimization algorithms including primal-dual methods is (Yang et al., 2019). Much recent interest has focused on primal-dual methods, including works on convergence analysis (Feijer and Paganini, 2010), centralized optimization (Qu and Li, 2019), and distributed optimization (Wang and Elia, 2011).

Our presentation in Section 4.3.3 follows the treatment in (Jafarpour et al., 2022); other analysis approaches based upon contraction-theory include (Cisneros-Velarde et al., 2022a; Nguyen et al., 2018).

Other examples of contraction theory applied to optimization problems includes the analysis of the gradient flow in (Simpson-Porco and Bullo, 2014), the Newton Raphson method (Desoer and Haneda, 1972), and of the dual ascent dynamics in (Como, 2017).

Lotka-Volterra models These models were originally developed in (Lotka, 1920; Volterra, 1928). Our presentation in Section 4.4 is an extension of the treatment in (Jafarpour et al., 2021a).

An early reference for the analysis of the 2-species model is (Goh, 1976). Early references for key stability results are (Takeuchi et al., 1978; Goh, 1979). Textbook treatment include (Goh, 1980; Takeuchi, 1996; Baigent, 2010). For a more complete treatment of the n-special model, we refer the interested reader to (Takeuchi, 1996; Baigent, 2010). For example, Baigent (2010) discusses conservative Lotka-Volterra models (Hamiltonian structure and existence of periodic orbits), competitive and monotone models.

We refer to the texts (Hofbauer and Sigmund, 1998; Sandholm, 2010) for comprehensive discussions about the connection between Lotka-Volterra models and evolutionary game dynamics.

Monotone systems Monotone systems are a generalization of cooperative/competitive systems, that naturally arise in ecological and biological networks. Note that linear averaging and network flow systems are monotone. Many nonlinear network systems are monotone either locally (e.g., over a compact set) or globally. The Kamke-Müller conditions and the concept of order-preserving flows were presented originally by (Müller, 1927) and then independently by (Kamke, 1932).

The rigorous study of monotone dynamical systems was initiated by Hirsch in a sequence of fundamental works (Hirsch, 1982, 1985, 1988); see also the survey (Smith, 1988) and texts (Smith, 1995; Hirsch and Smith, 2005).

Lemma 4.17 on the one-sided Lipschitz conditions for monotone maps is essentially due to (Jafarpour et al., 2023); see also the inspiring treatment in (Coogan, 2016). The treatment in Section 4.5.5 is related to the survey (Como, 2017), but also contains results and ideas from the survey (Jacquez and Simon, 1993) and from the original sources (Maeda et al., 1978; Sandberg, 1978). For example, parts of Theorem 4.21 are stated as Theorem 5 in (Jacquez and Simon, 1993) and attributed to (Maeda et al., 1978; Sandberg, 1978).

Examples of monotone systems arise in the study of biological and chemical reaction networks (Sontag, 2007).

Daganzo traffic models Inspired by the early work (Daganzo, 1994) on so-called cell transmission models, we study single-commodity first-order flow systems. An early reference on the use of monotonicity to study the dynamics of traffic networks is (Gomes et al., 2008).

Key references for our presentation include the classic works on compartmental systems (Sandberg, 1978; Maeda et al., 1978), the excellent survey (Jacquez and Simon, 1993), and the work on monotone systems with inputs and outputs (Angeli and Sontag, 2003). (A little-known early reference is also (Rosenbrock, 1963).) Also very relevant is the recent survey (Como, 2017). Other recent works on transportation networks include (Como et al., 2015; Coogan and Arcak, 2015a);

4.8 Appendix: Proofs

Proof of Lemma 4.17. First we note that, since f is continuous,

$$\mathsf{osLip}(f) = \sup_{x>y} \frac{[\![f(x)-f(y)\,;\,x-y]\!]}{\|x-y\|^2} = \sup_{x\geq y, x\neq y} \frac{[\![f(x)-f(y)\,;\,x-y]\!]}{\|x-y\|^2}. \tag{4.59}$$

To prove the latter equality, it suffices to show that, for each $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$ with at least one index i such that $x_i<y_i$, there exists $(\bar{x},\bar{y})\in\mathbb{R}^n\times\mathbb{R}^n$ with $\bar{x}\neq\bar{y}$ and $\bar{x}\geq\bar{y}$ such that

$$\frac{[\![f(x)-f(y)\,;\,x-y]\!]}{\|x-y\|^2} \leq \frac{[\![f(\bar{x})-f(\bar{y})\,;\,\bar{x}-\bar{y}]\!]}{\|\bar{x}-\bar{y}\|^2}. \tag{4.60}$$

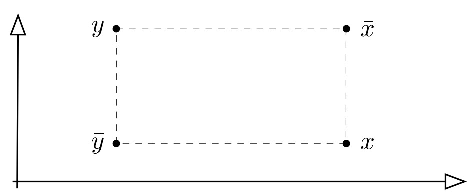


Figure 4.18: Entrywise maximum $\bar{x}$ and minimum $\bar{y}$, for arbitrary x and y.

Given such a pair (x,y), define $\bar{x}$ to be the entrywise maximum of $\{x,y\}$ and $\bar{y}$ to be the entrywise minimum of $\{x,y\}$.

Clearly, $\bar{x}\geq\bar{y}$, $\bar{y}\leq x$ and $\bar{x}\geq y$. Define a diagonal reflection matrix $S\in\{-1,0,+1\}^n$ by $S_{kk}=-1$ if $x_k<y_k$ and $S_{kk}=+1$ otherwise. We now have $S(x-y)=\bar{x}-\bar{y}$ and, by the invariance property (WP8), $\|x-y\|=\|\bar{x}-\bar{y}\|$ and

$$\frac{[\![f(x)-f(y)\,;\,x-y]\!]}{\|x-y\|^2} = \frac{[\![S(f(x)-f(y))\,;\,S(x-y)]\!]}{\|S(x-y)\|^2} = \frac{[\![S(f(x)-f(y))\,;\,\bar{x}-\bar{y}]\!]}{\|\bar{x}-\bar{y}\|^2}. \tag{4.61}$$

Because $\bar{x}\geq\bar{y}$ and because of the monotonicity property in Exercise E2.25(ii), the inequality (4.60) is implied by

$$S(f(x)-f(y)) \leq f(\bar{x})-f(\bar{y}). \tag{4.62}$$

We prove this inequality (4.62) holds in two steps. First, we pick an index i such that $x_i<y_i$. We need to show

$$\begin{aligned} &-\big(f_i(x)-f_i(y)\big) \leq f_i(\bar{x})-f_i(\bar{y}) \\ \iff\quad & f_i(\bar{y})-f_i(x) \leq f_i(\bar{x})-f_i(y). \end{aligned}$$

Since $\bar{y}\leq x$ and $\bar{y}_i=x_i$, the monotonicity of f implies that the left-hand side satisfies $f_i(\bar{y})-f_i(x)\leq 0$. Additionally, since $\bar{x}\geq y$ and $\bar{x}_i=y_i$, the monotonicity of f implies that right-hand side satisfies $f_i(\bar{x})-f_i(y)\geq 0$. Therefore, the inequality (4.62) holds for all indices i where $x_i<y_i$. Second, in general there might exist an index j such that $x_j\geq y_j$. We need to show

$$\begin{aligned} & f_j(x)-f_j(y) \leq f_j(\bar{x})-f_j(\bar{y}) \\ \iff\quad & f_j(x)-f_j(\bar{x}) \leq f_j(y)-f_j(\bar{y}). \end{aligned}$$

But again, since $\bar{x} \geq x$ and $x_j = \bar{x}_j$, the monotonicity of f implies that the left-hand side is negative or zero. Similarly, since $\bar{y} \leq x$ and $y_j = \bar{y}_j$, the monotonicity of f implies that the right-hand since is positive or zero. This completes the proof of inequality (4.62), which implies equation (4.60). In sum, this completes the proof of equation (4.30).

Next, regarding the weighted ℓ_1 case, we recalling the sign pairing

$$[\![v\,;\,w]\!]_{1,[\eta]} = \|w\|_{1,[\eta]}\,\mathrm{sign}(w)^\mathsf{T}[\eta]v$$

compatible with $\|v\|_{1,[\eta]} = \eta^\mathsf{T}|v|$ and we compute

$$\mathsf{osLip}_{1,[\eta]}(f) = \sup_{x>y} \frac{\|x-y\|_{1,[\eta]}\,\mathrm{sign}(x-y)^\mathsf{T}[\eta](f(x)-f(y))}{\|x-y\|_{1,[\eta]}^2} = \sup_{x>y} \frac{\eta^\mathsf{T}\big(f(x)-f(y)\big)}{\eta^\mathsf{T}(x-y)}.$$

This completes the proof of equation (4.31).

Finally, regarding the weighted ℓ_∞ case, we recall the max pairing

$$[\![v\,;\,w]\!]_{\infty,[\eta]^{-1}} = \max_{i\in I_\infty([\eta]^{-1}w)} \eta_i^{-2} v_i w_i,$$

where $I_\infty(w) = \{i \in \{1,\dots,n\} \,:\, |w_i| = \|w\|_\infty\}$. Note that this pairing is compatible with the norm $\|v\|_{\infty,[\eta]^{-1}} = \max_{i\in\{1,\dots,n\}} \eta_i^{-1}|v_i|$. We compute

$$\begin{aligned}\mathsf{osLip}_{\infty,[\eta]^{-1}}(f) &= \sup_{x>y} \frac{[\![f(x)-f(y)\,;\,x-y]\!]_{\infty,[\eta]^{-1}}}{\|x-y\|_{\infty,[\eta]^{-1}}^2} &(4.63)\\ &= \sup_{x>y} \frac{\max_{i\in I_\infty([\eta]^{-1}(x-y))} \eta_i^{-2}(f_i(x)-f_i(y))(x_i-y_i)}{\max_{j\in\{1,\dots,n\}} \eta_j^{-2}(x_j-y_j)^2}\end{aligned}$$

but $i^* = j^*$ at numerator and denominator, so that

$$\begin{aligned}&= \sup_{x>y} \max_{i\in I_\infty([\eta]^{-1}(x-y))} \frac{\eta_i^{-2}(f_i(x)-f_i(y))(x_i-y_i)}{\eta_i^{-2}(x_i-y_i)^2} &(4.64)\\ &= \sup_{x>y} \max_{i\in I_\infty([\eta]^{-1}(x-y))} \frac{f_i(x)-f_i(y)}{x_i-y_i} &(4.65)\end{aligned}$$

It now remains to prove that the supremum is achieved at $x = y + \beta\eta$ for some positive β, that is, $[\eta]^{-1}(x-y) = \beta\mathbb{1}_n$ and all indices belong to $I_\infty([\eta]^{-1}(x-y))$. By contradiction, assume there exists an index j such that $\eta_j^{-1}(x_j-y_j) < \eta_i^{-1}(x_i-y_i)$ for $i \in I_\infty([\eta]^{-1}(x-y))$. Define $\bar{x} \in \mathbb{R}^n$ to be equal to x with the exception of the jth entry, defined to satisfy $\eta_j^{-1}(\bar{x}_j - y_j) = \eta_i^{-1}(x_i-y_i)$. Note that $\bar{x}_j > x_j$. Since $\bar{x} \geq x$ and $\bar{x}_i = x_i$, the monotonicity of f ensures that $f_i(\bar{x}) \geq f_i(x)$. Therefore

$$\max_{i\in I_\infty([\eta]^{-1}(x-y))} \frac{f_i(x)-f_i(y)}{x_i-y_i} \leq \max_{i\in I_\infty([\eta]^{-1}(\bar{x}-y))} \frac{f_i(\bar{x})-f_i(y)}{\bar{x}_i-y_i}$$

This completes the proof of (4.32). ■

4.9 Exercises

E4.1 **On vector fields whose Jacobian is a compartmental matrix (Como et al., 2015).** Let W be a nonempty closed hyper-rectangle in $\mathbb{R}^n$. Let $f\colon W \to \mathbb{R}^n$ be Lipschitz (hence differentiable almost everywhere) and have a compartmental Jacobian $Df(x)$ for all x where f is differentiable.

(i) Prove that, for almost all $x, y \in W$,

$$\operatorname{sign}(x-y)^\top \big(f(x)-f(y)\big) \leq 0. \tag{E4.1}$$

(ii) Provide a direct proof that the function $x \mapsto V(x) = \|f(x)\|_1$ is non-increasing along the flow of a vector field f.

Hint: Use the sign contractivity property in Exercise E2.19.

Answer: The first result is an immediate consequence of Lemma E3.11.

Next, with the notation in Exercise E2.19, by setting $C = Df(x)$ and $y = f(x)$, we know that, for all $x \in \mathbb{R}^n_{\geq 0}$,

$$\operatorname{sign}(f(x))^\top Df(x) f(x) \leq 0. \tag{E4.2}$$

Recall that, for $z \in \mathbb{R} \setminus \{0\}$ and scalar g, we have $\frac{d}{dz}|g(z)| = \operatorname{sign}(g(z))\frac{d}{dz}g(z)$. At each point x where V is differentiable, we now compute

$$\begin{aligned}
\mathscr{L}_f V(x) &= \sum_{j=1}^n \Big(\frac{\partial}{\partial x_j} V(x)\Big) f_j(x) = \sum_{j=1}^n \frac{\partial}{\partial x_j} \sum_{i=1}^n |f_i(x)| f_j(x) \\
&= \sum_{i,j=1}^n \frac{\partial}{\partial x_j} |f_i(x)| f_j(x) = \sum_{i,j=1}^n \operatorname{sign}(f_i(x)) \frac{\partial f_i}{\partial x_j}(x) f_j(x) = \operatorname{sign}(f(x))^\top Df(x) f(x) \leq 0.
\end{aligned}$$

E4.2 **Logistic ordinary differential equation.** Given a growth rate $r > 0$ and a carrying capacity $\kappa > 0$, consider the logistic equation defined by

$$\dot{x} = rx(1 - x/\kappa),$$

with initial condition $x(0) \in \mathbb{R}_{\geq 0}$. Show that

(i) there are two equilibrium points 0 and κ,

(ii) the solution is

$$x(t) = \frac{\kappa x(0)\,\mathrm{e}^{rt}}{\kappa + x(0)(\mathrm{e}^{rt} - 1)}, \tag{E4.3}$$

and it takes value in $\mathbb{R}_{\geq 0}$,

(iii) all solutions with $0 < x(0) < \kappa$ are monotonically increasing and converge asymptotically to κ,

(iv) all solutions with $\kappa < x(0)$ are monotonically decreasing and converge asymptotically to κ, and

(v) if $x(0) < \kappa/2$, then the solution $x(t)$ has an inflection point when $x(t) = \kappa/2$.

Answer: Regarding statement (i), we set $rx(1 - x/\kappa) = 0$ and easily determine the two unique equilibrium points.

Regarding statement (ii), it suffices the expression for $x(t)$ to plug into the differential equation:

$$\begin{aligned}
\dot{x} &= \frac{d}{dt} \frac{\kappa x(0)\,\mathrm{e}^{rt}}{\kappa + x(0)(\mathrm{e}^{rt} - 1)} \\
&= rx(t) + x(t)(-1)\frac{rx(0)\,\mathrm{e}^{rt}}{\kappa + x(0)(\mathrm{e}^{rt} - 1)} \\
&= rx(t) - rx(t)\frac{x(0)\,\mathrm{e}^{rt}}{\kappa + x(0)(\mathrm{e}^{rt} - 1)} \\
&= rx(t) - \frac{r}{\kappa}x(t)\frac{\kappa x(0)\,\mathrm{e}^{rt}}{\kappa + x(0)(\mathrm{e}^{rt} - 1)} = rx(t) - \frac{r}{\kappa}x^2(t).
\end{aligned}$$

We leave the remaining steps to the reader.

E4.3 **Monotone systems with inputs (Angeli and Sontag, 2003).** We generalize Definition 4.11 and Theorem 4.12 as follows. A time-invariant system with input $\dot{x} = f(x, u)$ is *monotone* if

$$u_1(t) \leq u_2(t) \text{ and } x_1 \leq x_2 \quad \Longrightarrow \quad \phi(t, x_1, u_1) \leq \phi(t, x_2, u_2) \quad \text{for all } t \geq 0. \tag{E4.4}$$

where we let $\phi(t, x, u)$ denote the solution at time t from initial condition x at time 0, with input $u(\tau)$, $0 \leq \tau \leq t$. Show that, for a continuously differentiable system with inputs, the following statements are equivalent:

(i) the system with inputs is monotone, and
(ii) $D_x f(x, u) = \frac{\partial f}{\partial x}(x, u)$ is Metzler and $D_u f(x, u) = \frac{\partial f}{\partial u}(x, u)$ is nonnegative for all x and u.

Note: *Angeli and Sontag (2003) study interconnections of monotone input/output systems, including a small gain stability theorem.*

Answer: We leave the answer to the reader.

E4.4 **Multi-agent decision making.** This exercise is inspired by the treatment on multi-agent decision making by Gray et al. (2018). Given a strongly connected weighted digraph with (non-negative) adjacency matrix A and out-degree matrix $D = [A\mathbb{1}_n]$. Given an odd smooth activation function $\Psi \colon \mathbb{R}^n \to \mathbb{R}^n$ with $\mathbb{0}_n \leq \Psi'(x) \leq \mathbb{1}_n$ for all $x \in \mathbb{R}^n$, consider the dynamics

$$\dot{x} = -Dx + \nu\Psi(Ax) =: f_{\mathrm{FR}\nu}(x), \tag{E4.5}$$

where $\nu \in [0, 1]$ is an activation parameter. Show that

(i) for $0 \leq \nu < 1$, $\mathsf{osLip}_\infty(f_{\mathrm{FR}\nu}) < 0$, $f_{\mathrm{FR}\nu}$ is strongly infinitesimally contracting and $\mathbb{0}_n$ is globally exponentially stable,
(ii) for $\nu = 1$, $\mathsf{osLip}_\infty(f_{\mathrm{FR}\nu}) = 0$ and $f_{\mathrm{FR}\nu}$ is weakly infinitesimally contracting and $\mathbb{0}_n$ is Lyapunov stable, and
(iii) for $\nu = 1$ and Ψ piecewise-analytic, all trajectories converge to $\mathbb{0}_n$.

Answer: Note that $f_{\mathrm{FR}\nu}$ is very similar to the firing rate neural network vector field f_{FR} defined in equation (3.49) in Section 3.5. As for that system, we compute

$$\begin{aligned}
\mathsf{osLip}_\infty(f_{\mathrm{FR}\nu}) &= \sup_{x\in\mathbb{R}^n} \mu_\infty(Df_{\mathrm{FR}\nu}(x)) = \sup_{x\in\mathbb{R}^n} \mu_\infty(-D + \nu(D\Psi(Ax))A) \\
&\leq \max_{d\in[0,1]^n} \mu_\infty(-D + \nu[d]A) \\
&\overset{(\text{E2.27})}{=} \max\left\{\mu_\infty(-D), \mu_\infty(-D+\nu A)\right\} \\
&\overset{A\geq 0,\ \text{monotonicity (2.78b)}}{=} \mu_\infty(-D+\nu A),
\end{aligned}$$

where we used equation (E2.27) from Exercise E2.22. Next, we introduce the Laplacian matrix $L = D - A$ and recall that $\mu_\infty(-L) = 0$. One can see $-D + \nu A = -\nu L - (1-\nu)D$ so that

$$\begin{aligned}
\mathsf{osLip}_\infty(f_{\mathrm{FR}\nu}) &\leq \mu_\infty(-D+\nu A) = \mu_\infty(-\nu L - (1-\nu)D) \\
&\leq \mu_\infty(-\nu L) + \mu_\infty(-(1-\nu)D) = -(1-\nu) \min_{i\in\{1,...,n\}} d_i.
\end{aligned}$$

This inequality proves the contractivity properties in statements (i) and (ii). Finally, since Ψ is assumed odd, $\mathbb{0}_n$ is an equilibrium and its stability properties follow from the contractivity properties.

Regarding statement (iii), at $\nu = 1$, the piece-wise analyticity property in Theorem 4.4(ii) implies global asymptotic convergence.

E4.5 **The Kronecker product.** The *Kronecker product* of $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{q\times r}$ is the $nq \times mr$ matrix $A\otimes B$ given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ddots & a_{nm}B \end{bmatrix}. \tag{E4.6}$$

As simple example, we write

$$I_n \otimes B = \begin{bmatrix} B & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & B \end{bmatrix} \in \mathbb{R}^{nq\times nr} \quad \text{and } A\otimes I_q = \begin{bmatrix} a_{11}I_q & \dots & a_{1m}I_q \\ \vdots & \ddots & \vdots \\ a_{n1}I_q & \ddots & a_{nm}I_q \end{bmatrix} \in \mathbb{R}^{nq\times mq}. \tag{E4.7}$$

Additionally, for $v, w \in \mathbb{R}^n$, we have $v \otimes w = \begin{bmatrix} v_1 w \\ \vdots \\ v_n w \end{bmatrix} \in \mathbb{R}^{n^2}$.

The Kronecker product enjoys numerous properties, including for example

$$\begin{aligned}
&\text{the bilinearity property:} && (\alpha A + \beta B)\otimes(\gamma C + \delta D) = \alpha\gamma A\otimes C + \alpha\delta A \otimes D \\
& && \qquad + \beta\gamma B\otimes C + \beta\delta B\otimes D, && \text{(E4.8a)}\\
&\text{the associativity property:} && (A\otimes B)\otimes C = A\otimes(B\otimes C), && \text{(E4.8b)}\\
&\text{the transpose property:} && (A\otimes B)^\mathsf{T} = A^\mathsf{T}\otimes B^\mathsf{T}, && \text{(E4.8c)}\\
&\text{the mixed product property:} && (A\otimes B)(C\otimes D) = (AC)\otimes(BD), && \text{(E4.8d)}
\end{aligned}$$

where the A, B, C, D matrices have appropriate compatible dimensions.

The remarkable mixed product property (E4.8d) leads to many useful consequences. As first example, if $Av = \lambda v$ and $Bw = \mu w$, then property (E4.8d) implies

$$(A\otimes B)(v\otimes w) = (Av)\otimes(Bw) = (\lambda v)\otimes(\mu w) = \lambda\mu(v\otimes w).$$

Therefore, we know

$$\begin{aligned}
&\text{the eigenpair property:} && Av = \lambda v, Bw = \mu w \implies (A\otimes B)(v\otimes w) = \lambda\mu(v\otimes w), && \text{(E4.9)}\\
&\text{the spectrum property:} && \operatorname{spec}(A\otimes B) = \{\lambda\mu \,:\, \lambda\in\operatorname{spec}(A), \mu\in\operatorname{spec}(B)\}. && \text{(E4.10)}
\end{aligned}$$

A second consequence of property (E4.8d) is that, for square matrices A and B, $A\otimes B$ is invertible if and only if both A and B are invertible, in which case

$$\text{the inverse property:} \qquad (A\otimes B)^{-1} = A^{-1}\otimes B^{-1} \qquad . \tag{E4.11}$$

Prove properties (E4.8a)–(E4.8d), (E4.9), (E4.10) and (E4.11) of the Kronecker product.

Answer: a) Regarding the bilinear property (E4.8a):

$$\begin{aligned}
&(\alpha A + \beta B)\otimes(\gamma C + \delta D) \\
&\quad = \begin{bmatrix} \alpha a_{11} + \beta b_{11} & \alpha a_{12} + \beta b_{12} & \dots & \alpha a_{1m} + \beta b_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} + \beta b_{n1} & \alpha a_{n2} + \beta b_{n2} & \dots & \alpha a_{nm} + \beta b_{nm} \end{bmatrix} \otimes (\gamma C + \delta D) \\
&\quad = \begin{bmatrix} (\alpha a_{11} + \beta b_{11})(\gamma C + \delta D) & (\alpha a_{12} + \beta b_{12})(\gamma C + \delta D) & \dots & (\alpha a_{1m} + \beta b_{1m})(\gamma C + \delta D) \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha a_{n1} + \beta b_{n1})(\gamma C + \delta D) & (\alpha a_{n2} + \beta b_{n2})(\gamma C + \delta D) & \dots & (\alpha a_{nm} + \beta b_{nm})(\gamma C + \delta D) \end{bmatrix} \\
&\quad = \alpha\gamma A\otimes C + \alpha\delta A\otimes D + \beta\gamma B\otimes C + \beta\delta B\otimes D.
\end{aligned}$$

b) Regarding the associativity property (E4.8b):

$$(A\otimes B)\otimes C = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix} \otimes C = \begin{bmatrix} a_{11}b_{11}C & a_{11}b_{12}C & \dots & a_{1m}b_{1r}C \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}b_{q1}C & a_{n2}b_{q2}C & \dots & a_{nm}b_{qr}C \end{bmatrix}$$
$$= A\otimes \begin{bmatrix} b_{11}C & \dots & b_{1r}C \\ \vdots & \ddots & \vdots \\ b_{q1}C & \dots & b_{qr}C \end{bmatrix} = A\otimes(B\otimes C).$$

c) Regarding the transpose property (E4.8c):

$$(A\otimes B)^\top = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix}^\top = \begin{bmatrix} a_{11}B^\top & \dots & a_{n1}B^\top \\ \vdots & \ddots & \vdots \\ a_{1m}B^\top & \dots & a_{nm}B^\top \end{bmatrix} = A^\top \otimes B^\top.$$

d) Regarding mixed product property (E4.8d), for matrices with the following dimensions: $A\in\mathbb{R}^{n\times m}, B\in\mathbb{R}^{q\times r}, C\in\mathbb{R}^{m\times s}, D\in\mathbb{R}^{r\times t}$:

$$(A\otimes B)(C\otimes D) = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix} \begin{bmatrix} c_{11}D & \dots & c_{1s}D \\ \vdots & \ddots & \vdots \\ c_{m1}D & \dots & c_{ms}D \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{j=1}^m a_{1j}c_{j1}BD & \sum_{j=1}^m a_{1j}c_{j2}BD & \dots & \sum_{j=1}^m a_{1j}c_{js}BD \\ \sum_{j=1}^m a_{2j}c_{j1}BD & \sum_{j=1}^m a_{2j}c_{j2}BD & \dots & \sum_{j=1}^m a_{2j}c_{js}BD \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^m a_{nj}c_{j1}BD & \sum_{j=1}^m a_{nj}c_{j2}BD & \dots & \sum_{j=1}^m a_{nj}c_{js}BD \end{bmatrix}$$
$$= (AC)\otimes(BD).$$

e) Regarding the eigenpair property (E4.9) and the spectrum property (E4.10): The eigenvalues α and eigenvectors x of $(A\otimes B)$ satisfy $(A\otimes B)x = \alpha x$. If we set $x = v\otimes w$ where v and w denote the eigenvectors of A and B it follows from the mixed product property that

$$(A\otimes B)(v\otimes w) = (\lambda v)\otimes(\mu w) = \lambda\mu(v\otimes w). \tag{E4.12}$$

f) Regarding the inverse property (E4.11): The mixed product property reads $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$. Choosing $C = A^{-1}$ and $D = B^{-1}$ yields $(A\otimes B)(A^{-1}\otimes B^{-1}) = (AA^{-1})\otimes(BB^{-1}) = I$ where I denotes the identity matrix of appropriate dimension. Hence $(A\otimes B)^{-1} = (A^{-1}\otimes B^{-1})$.

E4.6 **Kronecker products and induced norms.** For any $A\in\mathbb{R}^{n\times m}$, $B\in\mathbb{R}^{q\times r}$, and $p\in[1,\infty]$, show

(i) $\|A\otimes B\|_p = \|A\|_p\|B\|_p$.

Additionally, given $n\times n$ matrix A and an $m\times m$ matrix B, their *Kronecker sum* is

$$A\oplus B = A\otimes I_m + I_n\otimes B \in \mathbb{R}^{nm\times nm}. \tag{E4.13}$$

Show the following identities:

(ii) $\exp(A)\otimes\exp(B) = \exp(A\oplus B)$;

(iii) $\operatorname{spec}(A\oplus B) = \{\lambda+\gamma \,:\, \lambda\in\operatorname{spec}(A) \text{ and } \gamma\in\operatorname{spec}(B)\}$; and

(iv) $\mu_p(A\oplus B) = \mu_p(A)+\mu_p(B)$, for any $p\in[1,\infty]$.

Answer: The proof of statement (i) is given in (Lancaster and Farahat, 1972); see also (Wu et al., 2022b, Proposition 5). For statements (ii)-(iv), we refer to (Wu et al., 2022b, Lemma 2, Proposition 5, and Theorem 6).

E4.7 **The vectorization operator, the Kronecker product, and the Sylvester equation.** Given a matrix $X \in \mathbb{R}^{n\times m}$, the *vectorization of X* is the vector of dimension mn obtained by stacking all columns of X, that is,

$$\operatorname{vec}(X) = [x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}, \dots, x_{1m}, \dots, x_{nm}]^\top \in \mathbb{R}^{mn}. \tag{E4.14}$$

Show that

(i) any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ satisfy $\operatorname{vec}(xy^\top) = y \otimes x$;

(ii) for any $X, Y \in \mathbb{R}^{n\times m}$, recall their Frobenius inner product defined by $\langle\langle X\,;\,Y\rangle\rangle = \operatorname{trace}(X^\top Y)$ and show $\langle\langle X\,;\,Y\rangle\rangle = \operatorname{vec}(X)^\top \operatorname{vec}(Y)$;

(iii) any matrices A, B and C, for which the product ABC is well defined, satisfy

$$\operatorname{vec}(ABC) = (C^\top \otimes A)\operatorname{vec}(B);$$

(iv) for any matrix function $A\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$, the Jacobian of the vector field $A(x)x$ satisfies

$$\frac{\partial A(x)x}{\partial x} = (x^\top \otimes I_n)\frac{\partial \operatorname{vec}(A(x))}{\partial x} + A(x).$$

Next, for $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, and $C \in \mathbb{R}^{n\times m}$, consider the *Sylvester equation*

$$AX + XB = C$$

in the matrix variable $X \in \mathbb{R}^{n\times m}$. Show that the Sylvester equation

(v) can be rewritten as

$$\big\{(I_m \otimes A) + (B^\top \otimes I_n)\big\}\operatorname{vec}(X) = \operatorname{vec}(C);$$

(vi) has a unique solution for all C if and only if A and $-B$ have no common eigenvalues.

Answer: Loosely speaking, statements (i) and (ii) are straightforward to prove in coordinates. Statement (iii) is a consequence of statement (i).

Statement (iv) follows from: $A(x)x = \operatorname{vec}(A(x)x) = \operatorname{vec}(I_n A(x)x) = (x^\top \otimes I_n)\operatorname{vec}(A(x))$.

Statement (v) is a immediate consequence of statement (iii).

Statement (vi) follows from statement (v): a unique solution exists if and only if the matrix $(I_m \otimes A) + (B^\top \otimes I_n)$ is invertible. In other words, all eigenvalues of $(I_m \otimes A) + (B^\top \otimes I_n)$ need to be different from 0. The result then follows from the spectrum property (E4.10) of the Kronecker product.

E4.8 **Spectrum of saddle point matrices.** Given a positive semidefinite matrix $\mathcal{S} = \mathcal{S}^\top \succeq 0$ in $\mathbb{R}^{n\times n}$ and a matrix $\mathcal{C}$ in $\mathbb{R}^{m\times n}$, define the *saddle point matrix* $\mathcal{A}_{\text{sp}} \in \mathbb{R}^{(n+m)\times(n+m)}$ by

$$\mathcal{A}_{\text{sp}} = \begin{bmatrix} \mathcal{S} & \mathcal{C}^\top \\ -\mathcal{C} & \mathbb{0}_{m\times m} \end{bmatrix}.$$

Then each eigenvalue λ of $\mathcal{A}_{\text{sp}}$ satisfies

(i) $\Re(\lambda) \geq 0$,

(ii) if $\ker(\mathcal{S}) \cap \operatorname{img}(\mathcal{C}^\top) = \{\mathbb{0}_n\}$, then either $\Re(\lambda) > 0$ or $\lambda = 0$; moreover, if $\lambda = 0$, then λ is semisimple, and

(iii) if $\mathcal{S}$ is positive definite and $\ker(\mathcal{C}^\top) = \{\mathbb{0}_m\}$, then $\Re(\lambda) > 0$.

Note: *Statements (i) and (iii) are (Benzi et al., 2005, Theorem 3.6). Statement (ii) is (Cherukuri et al., 2017, Lemma 5.3). Additional results on saddle point matrices are given in (Dörfler et al., 2018, Proposition 5.13).*

Answer: We start by collecting a few useful facts. First,

$$\frac{1}{2}(\mathcal{A}_{\text{sp}} + \mathcal{A}_{\text{sp}}^\top) = \begin{bmatrix} \mathcal{S} & \mathbb{0}_{n\times m} \\ \mathbb{0}_{m\times n} & \mathbb{0}_{m\times m} \end{bmatrix} \succeq 0. \tag{E4.15}$$

Next, let (λ, v) be a complex eigenpair of $\mathcal{A}_{\text{sp}}$ with $\|v\|_2 = 1$ and $v = [v_1^\top \ v_2^\top]^\top \in \mathbb{C}^{n+m}$. Here $\|v\|_2^2 = v^* v$, where v^* is the conjugate transpose of v. The equality $\lambda v = \mathcal{A}_{\text{sp}} v$ reads in components

$$\lambda v_1 = \mathcal{S} v_1 + \mathcal{C}^\top v_2, \tag{E4.16a}$$
$$\lambda v_2 = -\mathcal{C} v_1. \tag{E4.16b}$$

Finally, we note that $\mathcal{A}_{\text{sp}} v = \lambda v$ implies $v^* \mathcal{A}_{\text{sp}} v = \lambda$ and $(v^* \mathcal{A}_{\text{sp}} v)^* = v^* \mathcal{A}_{\text{sp}}^\top v = \lambda^*$. Therefore

$$\begin{aligned}\Re(\lambda) &= \frac{1}{2}(\lambda + \lambda^*) = \frac{1}{2} v^* (\mathcal{A}_{\text{sp}} + \mathcal{A}_{\text{sp}}^\top) v \\ &= \frac{1}{2}\Re(v)^\top (\mathcal{A}_{\text{sp}} + \mathcal{A}_{\text{sp}}^\top)\Re(v) + \frac{1}{2}\Im(v)^\top (\mathcal{A}_{\text{sp}} + \mathcal{A}_{\text{sp}}^\top)\Im(v) \\ &= \Re(v_1)^\top \mathcal{S}\, \Re(v_1) + \Im(v_1)^\top \mathcal{S}\, \Im(v_1) \geq 0. \end{aligned} \tag{E4.17}$$

Statement (i) is now precisely equation (E4.17). Regarding statement (iii), by contradiction assume there is an eigenvalue λ with zero real part. The assumption $\mathcal{S} \succ 0$ and equation (E4.17) together imply $\Re(\lambda) = 0$ if and only if $v_1 = \mathbb{0}_n$. But if $v_1 = \mathbb{0}_n$, then equation (E4.16a) implies $\mathcal{C}^\top v_2 = \mathbb{0}_n$ and, since $\ker(\mathcal{C}^\top) = \{\mathbb{0}_m\}$, $v_2 = \mathbb{0}_m$. But this is a contradiction since v has unit norm. Therefore, there exists no eigenvalue with zero real part. This concludes the proof of statement (iii).

Regarding statement (ii), we reason by contradiction. Let $\mathrm{i}\lambda$, with $\lambda \neq 0$, be an imaginary eigenvalue of $\mathcal{A}_{\text{sp}}$ with eigenvector $x + \mathrm{i}y$, where $x = [x_1^\top \ x_2^\top]^\top, y = [y_1^\top \ y_2^\top]^\top \in \mathbb{R}^{n+m}$. Then the real and imaginary parts of (E4.16) yield:

$$\mathcal{S} x_1 + \mathcal{C}^\top x_2 = -\lambda y_1, \tag{E4.18a}$$
$$\mathcal{S} y_1 + \mathcal{C}^\top y_2 = \lambda x_1, \tag{E4.18b}$$
$$-\mathcal{C} x_1 = -\lambda y_2, \tag{E4.18c}$$
$$-\mathcal{C} y_1 = \lambda x_2. \tag{E4.18d}$$

Pre-multiplying equations (E4.18a), respectively (E4.18c), by $x_1^\top$, respectively by $x_2^\top$, gives $x_1^\top \mathcal{S} x_1 = -\lambda x_1^\top y_1 - x_1^\top \mathcal{C}^\top x_2$, respectively $-x_2^\top \mathcal{C} x_1 = -\lambda x_2^\top y_2$. By combining these equations we get $x_1^\top \mathcal{S} x_1 = -\lambda(x_1^\top y_1 + x_2^\top y_2)$. By applying analogous modifications to equations (E4.18b) and (E4.18d), we obtain $y_1^\top \mathcal{S} y_1 = \lambda(x_1^\top y_1 + x_2^\top y_2)$. These two conditions imply that $x_1^\top \mathcal{S} x_1 = -y_1^\top \mathcal{S} y_1$. Since $\mathcal{S}$ is negative semidefinite by assumption, we obtain $x_1, y_1 \in \ker(\mathcal{S})$. Note that $x_1, y_1 \neq \mathbb{0}_n$ because otherwise we conclude that $x = y = \mathbb{0}_{n+m}$. By further using this fact in equations (E4.18a) and (E4.18b) we obtain $\mathcal{C}^\top x_2 = -\lambda y_1$ and $\mathcal{C}^\top y_2 = \lambda x_1$, that is, $x_1, y_1 \in \operatorname{img}(\mathcal{C}^\top)$ which is a contradiction.

Finally, we establish the semisimplicity of 0. Assume for the sake of contradiction that the zero eigenvalue is not semisimple. Then the evolution of $\dot{x} = -\mathcal{A}_{\text{sp}} x$ from a generic initial condition is unbounded (unstable and linearly growing). On the other hand, since $-(\mathcal{A}_{\text{sp}} + \mathcal{A}_{\text{sp}}^\top)$ is negative semidefinite, the function $V(x) = \frac{1}{2}\|x\|_2^2$ has a non-increasing derivative $\dot{V}(x) = -x^\top (\mathcal{A}_{\text{sp}} + \mathcal{A}_{\text{sp}}^\top) x \leq 0$ along the evolution of $\dot{x} = -\mathcal{A}_{\text{sp}} x$. Hence, all trajectories of $\dot{x} = -\mathcal{A}_{\text{sp}} x$ are bounded, and the zero eigenvalue must be semisimple.

Chapter 5

Semicontracting Systems

Our mathematical focus is an exploration of the contractivity ideas of [Andrei A. Markov, "Extensions of the law of large numbers to dependent quantities," Izvestiya Fiziko-matematicheskogo, 1906] in the context of finite stochastic matrices, and specifically of the structure and usage of the Markov-Dobrushin coefficient of ergodicity. (Eugene Seneta 2006)

In particular, we prove that systems with uniformly bounded Jacobians, with symmetric or mutual diffusive coupling that connects the entire array together, will synchronize for large enough coupling. (Chai Wah Wu and Leon O. Chua 1995)

We introduce the new concept of partial contraction, which extends contraction analysis to include convergence to behaviors or to specific properties (such as equality of state components or convergence to a manifold). (Wei Wang and Jean-Jacques E. Slotine 2005)

5.1 Introduction

In this chapter we define and study the properties of systems that are contracting only in a limited sense and, loosely speaking, only when restricted to a certain subspace. We are especially interested in understanding when interconnected systems achieve *asymptotic consensus* (whereby the states of each component system become equal to the same constant value, as time diverges) or *asymptotic synchrony* (whereby the states of each component system become equal to the same time-varying value, as time diverges).

To perform this study we introduce the notion of a seminorm, that is, a map that is homogeneous and subadditive, like a norm, but not necessarily positive definite. Seminorms whose kernel is equal to the consensus space allow us to measure naturally the "distance from consensus."

Seminorm allow us to study dynamical systems that are semicontracting in the sense that, for the discrete-time case, the induced matrix seminorm of the system Jacobian is less than unity and, for the continuous-time case, the induced matrix log seminorm of the system Jacobian is less than zero.

5.2 Vector subspaces

We will be discussing linear and affine subspaces in this chapter. Given a subspace $\mathcal{K} \subset \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$,

- $\mathcal{K}^\perp = \{w \in \mathbb{R}^n : w \perp v \text{ for all } v \in \mathcal{K}\}$ denotes the *perpendicular complement* to $\mathcal{K}$,

- $\Pi_\parallel$ and $\Pi_\perp$ denote the *orthogonal projection matrices*[1] onto $\mathcal{K}$ and $\mathcal{K}^\perp$,
- any vector $v \in \mathbb{R}^n$ satisfies the *orthogonal decomposition* $v = v_\perp + v_\parallel$, for unique $v_\perp = \Pi_\perp v \in \mathcal{K}^\perp$ and $v_\parallel = \Pi_\parallel v \in \mathcal{K}$,
- $x + \mathcal{K} = \{x + v \,:\, v \in \mathcal{K}\}$ is the *affine subspace* defined by x and $\mathcal{K}$,
- $\mathcal{K}$ is A-*invariant* if $A\mathcal{K} = \{Av \,:\, v \in \mathcal{K}\} \subseteq \mathcal{K}$. When $\mathcal{K}$ is A-invariant,
  (i) A, regarded as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, can be restricted to a map from $\mathcal{K}$ to $\mathcal{K}$; we let $A|_{\mathcal{K}} \colon \mathcal{K} \to \mathcal{K}$ denote this restricted map;
  (ii) the perpendicular complement $\mathcal{K}^\perp$ is $A^\top$-invariant and we let $A^\top|_{\mathcal{K}^\perp} \colon \mathcal{K}^\perp \to \mathcal{K}^\perp$ denote the corresponding restricted map.

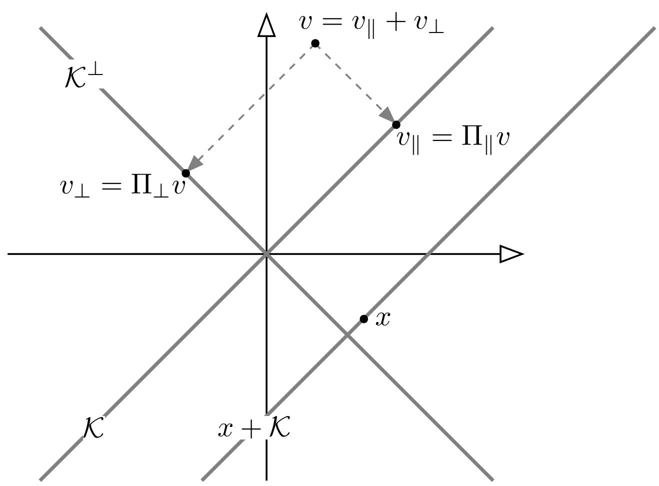


Figure 5.1: A vector subspace $\mathcal{K}$, its perpendicular complement $\mathcal{K}^\perp$, the corresponding orthogonal projections, and an affine subspace.

We then study some of these concepts in coordinates. For later purposes, we adopt the following assumption:

$$\mathcal{K} = \operatorname{span}\{v_1, \dots, v_k\} \subset \mathbb{R}^n \quad \text{and} \quad \mathcal{K}^\perp = \operatorname{span}\{v_{k+1}, \dots, v_n\} \subset \mathbb{R}^n, \tag{5.1}$$

where $v_1, \dots, v_n$ is a family of orthonormal vectors in $\mathbb{R}^n$. We then define the matrices $V_\parallel = \begin{bmatrix} v_1 & \dots & v_k \end{bmatrix} \in \mathbb{R}^{n\times k}$ and $V_\perp = \begin{bmatrix} v_{k+1} & \dots & v_n \end{bmatrix} \in \mathbb{R}^{n\times(n-k)}$. Note that $\begin{bmatrix} V_\parallel & V_\perp \end{bmatrix} \in \mathbb{R}^{n\times n}$ is an orthonormal matrix[2] and, specifically, $V_\parallel^\top V_\parallel = I_k$ and $V_\perp^\top V_\perp = I_{n-k}$.

(i) We start by giving coordinate representations of elements of the parallel and perpendicular spaces: $v_\parallel \in \mathcal{K}$ implies that $v_\parallel = V_\parallel \tilde{v}_\parallel$ where $\tilde{v}_\parallel \in \mathbb{R}^k$ and $v_\perp \in \mathcal{K}^\perp$ implies that $v_\perp = V_\perp \tilde{v}_\perp$ where $\tilde{v}_\perp \in \mathbb{R}^{n-k}$. More generally, given $z \in \mathbb{R}^n$, the components of z along $\mathcal{K}$ and $\mathcal{K}^\perp$ are $V_\parallel^\top z \in \mathbb{R}^k$ and $V_\perp^\top z \in \mathbb{R}^{n-k}$, respectively;

(ii) the orthogonal projection matrices can be written as $\Pi_\parallel = V_\parallel V_\parallel^\top$ and $\Pi_\perp = V_\perp V_\perp^\top$. Indeed one can directly verify that $\Pi_\parallel^2 = \Pi_\parallel$, $\Pi_\parallel = \Pi_\parallel^\top$, and $\operatorname{img}(\Pi_\parallel) = \mathcal{K}$ and the corresponding properties for $\Pi_\perp$;

(iii) the matrix A can be written in block form, after a similarity transformation, as

$$\begin{bmatrix} V_\parallel & V_\perp \end{bmatrix}^\top A \begin{bmatrix} V_\parallel & V_\perp \end{bmatrix} = \begin{bmatrix} V_\parallel^\top A V_\parallel & V_\parallel^\top A V_\perp \\ V_\perp^\top A V_\parallel & V_\perp^\top A V_\perp \end{bmatrix} =: \begin{bmatrix} A_{\parallel\parallel} & A_{\parallel\perp} \\ A_{\perp\parallel} & A_{\perp\perp} \end{bmatrix}; \tag{5.2}$$

[1] Recall that $\Pi \in \mathbb{R}^{n\times n}$ is an orthogonal projection matrix if it is symmetric $\Pi = \Pi^\top$ and idempotent $\Pi^2 = \Pi$.
[2] Recall that $V \in \mathbb{R}^{n\times n}$ is an *orthonormal* matrix if $V^{-1} = V^\top$.

(iv) $A\mathcal{K} \subseteq \mathcal{K}$ if and only if $A_{\perp\|} = V_\perp^\top A V_\| = \mathbb{0}_{(n-k)\times k}$; indeed this can be verified in components;

(v) when $\mathcal{K}$ is A-invariant, the coordinate representations of $A|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ and $A^\top|_{\mathcal{K}^\perp} : \mathcal{K}^\perp \to \mathcal{K}^\perp$ are the matrices $A_{\|\|} = V_\|^\top A V_\| \in \mathbb{R}^{k\times k}$ and $A^\top_{\perp\perp} = V_\perp^\top A^\top V_\perp \in \mathbb{R}^{(n-k)\times(n-k)}$, respectively.

We conclude with an example.

- Define the *consensus space* $\mathcal{K} = \operatorname{span}\{\mathbb{1}_n\} \subset \mathbb{R}^n$ to be the set of all vectors with equal entries. We also define $\mathcal{K}^\perp = \operatorname{span}\{\mathbb{1}_n\}^\perp$.
- The two orthogonal projection matrices are as follows. First, $\Pi_\| = \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$. Second, we adopt the convention in the literature that, for this particular choice of kernel, namely the consensus space, the perpendicular projection is denoted by the symbol Π_n. In summary:

$$\Pi_\perp = \Pi_n = I_n - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top = \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \dots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \dots & \frac{n-1}{n} \end{bmatrix} \succeq 0$$

- For $x \in \mathbb{R}^n$, define $x_{\text{avg}} = \frac{1}{n}\mathbb{1}_n^\top x \in \mathbb{R}$. Then $x = x_\| + x_\perp$ where $x_\| = \Pi_\| x = x_{\text{avg}}\mathbb{1}_n$ and $x_\perp = \Pi_n x = x - x_{\text{avg}}\mathbb{1}_n$.
- We also define the orthonormal basis $\{v_1, v_2, \dots, v_n\}$ for $\mathbb{R}^n$ by $v_1 = \frac{1}{\sqrt{n}}\mathbb{1}_n$ and, for $j \in \{2,\dots,n\}$,

$$(v_j)_i = \frac{1}{\sqrt{j(j-1)}}\begin{cases} 1 & \text{if } i < j, \\ 1-j & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \implies \quad v_2 = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; v_3 = \tfrac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \\ \vdots \\ 0 \end{bmatrix}, \; v_4 = \tfrac{1}{\sqrt{12}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \\ \vdots \end{bmatrix}, \quad \text{etc.}$$

  Correspondingly, we set

$$V_\| = \tfrac{1}{\sqrt{n}}\begin{bmatrix} \mathbb{1}_n \end{bmatrix} \quad \text{and} \quad V_\perp = \begin{bmatrix} v_2 & \dots & v_n \end{bmatrix} \in \mathbb{R}^{n\times(n-1)}.$$

- Finally, we compute $A_{\|\|} = \frac{1}{n}\mathbb{1}_n^\top A \mathbb{1}_n$ and we omit the coordinate expressions of $A_{\|\perp}$, $A_{\perp\|}$ and $A_{\perp\perp}$.

5.3 Seminorms, matrix and logarithmic seminorms

Definition 5.1 (Seminorms). *A function $|||\cdot||| : \mathbb{R}^n \to \mathbb{R}$ is a seminorm on $\mathbb{R}^n$ if, for all $v, w \in \mathbb{R}^n$ and $a \in \mathbb{R}$:*

$$\text{(homogeneity):} \quad |||av||| = |a|\,|||v|||, \text{ and} \tag{5.3a}$$
$$\text{(subadditivity):} \quad |||v+w||| \leq |||v||| + |||w|||. \tag{5.3b}$$

The kernel of a seminorm $|||\cdot|||$ is

$$\ker(|||\cdot|||) = \{v \in \mathbb{R}^n \; : \; |||v||| = 0\}.$$

As we did in Section 2.3.2 with the properties of norms, we now review the properties of seminorms:

(i) Homogeneity and subadditivity imply non-negativity, that is, $|||v||| \geq 0$ for all $v \in \mathbb{R}^n$.

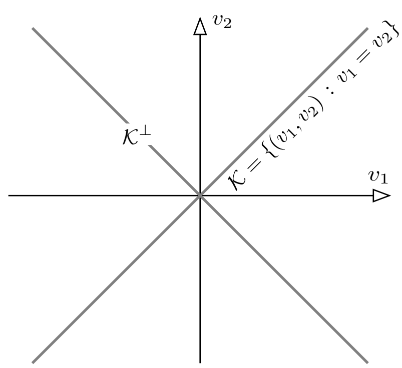


Figure 5.2: On $\mathbb{R}^2$, consider the scalar function $(v_1, v_2) \mapsto \sqrt{(v_1 - v_2)^2} = |v_1 - v_2|$. This function satisfies homogeneity and subadditivity and so it is a seminorm. To show homogeneity, note $|(v_1 + w_1) - (v_2 + w_2)| = |(v_1 + v_2) - (w_1 + w_2)| \leq |v_1 - v_2| + |w_1 - w_2|$. The kernel of the seminorm is $\mathcal{K} = \{(v_1, v_2) : v_1 = v_2\} = \operatorname{span}\{(1,1)^\top\}$ and the perpendicular subspace is $\mathcal{K}^\perp = \operatorname{span}\{(1,-1)^\top\}$. The orthogonal projection matrices onto $\mathcal{K}$ and $\mathcal{K}^\perp$ are

$$\Pi_\parallel = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \Pi_\perp = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Finally, we can introduce $V_\parallel = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $V_\perp = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(ii) Each seminorm is a continuous convex function: $|||av + (1-a)w||| \leq a|||v||| + (1-a)|||w|||$ for all $v, w \in \mathbb{R}^n$ and $a \in [0,1]$.

(iii) The kernel $\mathcal{K} := \ker(|||\cdot|||)$ is a linear subspace of $\mathbb{R}^n$. We let $\Pi_\parallel$ and $\Pi_\perp$ denote the orthogonal projection matrices onto $\mathcal{K}$ and $\mathcal{K}^\perp$, respectively.

(iv) Each seminorm satisfies the *reverse triangle inequality* $|||v \pm w||| \geq \big|\,|||v||| - |||w|||\,\big|$, for each $v, w \in \mathbb{R}^n$. Indeed,

$$|||v||| + |||w - v||| \overset{\text{sub-additivity}}{\geq} |||v + w - v||| = |||w||| \quad \Longrightarrow \quad |||w - v||| \geq |||w||| - |||v|||.$$

The claim follows from exchanging the roles of v and w and from noting that the sign of w is irrelevant. The reverse triangle inequality implies that, for all $v \in \mathbb{R}^n$ and $w \in \mathcal{K}$,

$$|||v + w||| = |||v||| \quad \text{or, equivalently,} \quad |||v||| = |||\Pi_\perp v|||. \tag{5.4}$$

Example 5.2 (Example seminorms). Seminorms can naturally arise from norms and rank-deficient matrices. Of specific interest in consensus and synchronization problems are seminorms on $\mathbb{R}^n$ whose kernel is $\operatorname{span}\{\mathbb{1}_n\}$.

(i) Given a norm $\|\cdot\|: \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and a matrix $R \in \mathbb{R}^{m \times n}$, define the *$R$-weighted seminorm* $\|\cdot\|_R: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$\|v\|_R = \|Rv\|, \qquad \text{for all } v \in \mathbb{R}^n. \tag{5.5}$$

If $\operatorname{rank}(R) = n$, then $\|\cdot\|_R$ is a norm on $\mathbb{R}^n$. If instead $\operatorname{rank}(R) < n$, then $\|\cdot\|_R$ is a seminorm. For convenience, we let $\|\cdot\|_R$ denote both the norm or seminorm, independently of the rank of R.

For example, for any $p \in [1, \infty]$, we let $\|\cdot\|_{p,R}$ denote the *R-weighted ℓ_p seminorm*. As usual, for a positive semidefinite matrix $P = P^\top \succeq 0$, we write

$$\|v\|^2_{2,P^{1/2}} = v^\top P v. \tag{5.6}$$

(Again, we emphasize that $\|v\|_{2,P^{1/2}}$ is a seminorm when P is singular.)

(ii) Recall that $\Pi_n = I_n - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$ denotes the orthogonal projection matrix onto $\operatorname{span}\{\mathbb{1}_n\}^\perp$ and define the *disagreement seminorm* by

$$\|x\|_{2,\Pi_n} := \|\Pi_n x\|_2 = \Big(\frac{1}{n}\sum_{i,j=1}^n (x_i - x_j)^2\Big)^{1/2}. \tag{5.7}$$

(iii) Let $B_{\mathrm{c}} \in \{-1, 0, +1\}^{n(n-1)}$ denote the oriented incidence matrix of the complete undirected graph on n nodes. Define the *max-min seminorm* by

$$\|x\|_{\infty, B_{\mathrm{c}}^\top} := \|B_{\mathrm{c}}^\top x\|_\infty = \max_{i\in\{1,...,n\}} x_i - \min_{i\in\{1,...,n\}} x_i. \tag{5.8}$$

Note: both disagreement and max-min seminorms satisfy $\ker(\|\!|\cdot|\!\|_{2,\Pi_n}) = \ker(\|\!|\cdot|\!\|_{\infty,B_{\mathrm{c}}^\top}) = \mathrm{span}\{\mathbb{1}_n\}$ so that they induce norms on $\mathrm{span}\{\mathbb{1}_n\}^\perp$. As defined above, the set $\mathrm{span}\{\mathbb{1}_n\}$ is called the consensus space and any element of it is called a *consensus state*.

Note: the orthogonal projection Π_n is also the Laplacian of the complete undirected graph (modulo a $1/n$ factor). Therefore both disagreement and max-min seminorms are related to the same graph. •

Lemma 5.3 (Seminorms and norms on perpendicular subspaces). *(i) A seminorm $\|\!|\cdot|\!\|$ on $\mathbb{R}^n$ with kernel $\mathcal{K}$ induces a restricted perpendicular norm $\|\cdot\|_\perp$ on $\mathcal{K}^\perp$ by restriction: $\|w\|_\perp = \|\!|w|\!\|$ for all $w \in \mathcal{K}^\perp$.*

(ii) Vice-versa, given a subspace $\mathcal{K} \subset \mathbb{R}^n$, a norm $\|\cdot\|_\perp$ on $\mathcal{K}^\perp$ induces a seminorm $\|\!|\cdot|\!\|$ on $\mathbb{R}^n$ with kernel $\mathcal{K}$ by projection: $\|\!|v|\!\| = \|\Pi_\perp v\|_\perp$ for all $v \in \mathbb{R}^n$.

Proof. Regarding the first statement, homogeneity and subadditivity follow from the fact that $\|\!|\cdot|\!\|$ is a seminorm. Positive definiteness follows from noting that, if $w \in \mathcal{K}^\perp$ and $w \neq \mathbb{0}_n$, then necessarily $\|\!|w|\!\| > 0$ (otherwise w would be an element of the kernel $\mathcal{K}$). For the second statement it suffices to verify homogeneity and subadditivity; we leave these steps to the reader. ■

It is useful to present some identities in coordinates for a seminorm $\|\!|\cdot|\!\|$ on $\mathbb{R}^n$ with kernel $\mathcal{K}$. As in equation (5.1), we assume $\mathcal{K} = \mathrm{span}\{v_1, \dots, v_k\}$, $\mathcal{K}^\perp = \mathrm{span}\{v_{k+1}, \dots, v_n\}$, and define the corresponding matrices $V_\parallel$ and $V_\perp$ (such that $\begin{bmatrix} V_\parallel & V_\perp \end{bmatrix}$ is orthonormal). We identify the restricted perpendicular norm $\|\cdot\|_\perp : \mathcal{K}^\perp \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with its representation in coordinates $\|\cdot\|_\perp : \mathbb{R}^{n-k} \to \mathbb{R}_{\geq 0}$. For a generic vector $\tilde{v}_\perp \in \mathbb{R}^{n-k}$, we recall $V_\perp \tilde{v}_\perp \in \mathcal{K}^\perp$ and write

$$\|\tilde{v}_\perp\|_\perp = \|\!|V_\perp \tilde{v}_\perp|\!\|. \tag{5.9}$$

For example, consider $P = P^\top \succeq 0$ with kernel $\mathcal{K}$. Since $\Pi_\perp v = V_\perp \tilde{v}_\perp$ for any $v \in \mathbb{R}^n$, we compute

$$\|\tilde{v}_\perp\|_\perp^2 = \|v\|_{2,P^{1/2}}^2 = v^\top P v = (V_\perp \tilde{v}_\perp)^\top P V_\perp \tilde{v}_\perp = \tilde{v}_\perp^\top (V_\perp^\top P V_\perp) \tilde{v}_\perp =: \|\tilde{v}_\perp\|_{2,P_{\perp\perp}^{1/2}}^2, \tag{5.10}$$

so that the restricted perpendicular norm is the norm $\|\cdot\|_{2,P_{\perp\perp}^{1/2}}$ in $\mathbb{R}^{n-k}$, where $P_{\perp\perp} = V_\perp^\top P V_\perp \succ 0$.

Definition 5.4 (Induced matrix seminorm and log seminorm). *Given a seminorm $\|\!|\cdot|\!\| : \mathbb{R}^n \to \mathbb{R}$ with kernel $\mathcal{K}$,*

(i) the (induced) matrix seminorm is

$$\|\!|A|\!\| = \max_{\substack{\|\!|v|\!\|=1 \\ v\perp\mathcal{K}}} \|\!|Av|\!\|; \tag{5.11}$$

(ii) the (induced) matrix log seminorm is

$$\mu_{\|\!|\cdot|\!\|}(A) = \lim_{h\to 0^+} \frac{\|\!|I_n + hA|\!\| - 1}{h}. \tag{5.12}$$

Given the identity matrix I_n and a scalar $c \in \mathbb{R}$,

$$\|\!|cI_n|\!\| = |c| \quad \text{and} \quad \mu_{\|\!|\cdot|\!\|}(cI_n) = c.$$

We now review basic properties of these induced operations. We postpone the proof to Exercise E5.1.

Lemma 5.5 (Basic properties of matrix seminorms and log seminorms). *Given a seminorm $|||\cdot|||$ with kernel $\mathcal{K}$, for each $A, B \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}$, $v \in \mathbb{R}^n$, and $t \geq 0$,*

(i) the induced matrix seminorm (5.11) satisfies

(homogeneity) $$|||aA||| = |a|\,|||A|||, \tag{5.13a}$$
(subadditivity) $$|||A+B||| \leq |||A||| + |||B|||, \tag{5.13b}$$
(simplified definition for $A\mathcal{K}\subseteq\mathcal{K}$) $$|||A||| = \max_{|||v|||\leq 1} |||Av|||, \tag{5.13c}$$
(conditional sub-multiplicativity for $A\mathcal{K}\subseteq\mathcal{K}$) $$|||AB||| \leq |||A||||||B|||, \tag{5.13d}$$
(conditional sub-multiplicativity for $A\mathcal{K}\subseteq\mathcal{K}$) $$|||Av||| \leq |||A||||||v|||, \tag{5.13e}$$
(conditional sub-multiplicativity for $A^{\mathsf{T}}\mathcal{K}\subseteq\mathcal{K}$) $$|||BA||| \leq |||B||||||A|||; \tag{5.13f}$$

(ii) the induced matrix log seminorm (5.12) is well defined and satisfies

(positive homogeneity) $$\mu_{|||\cdot|||}(aA) = |a|\,\mu_{|||\cdot|||}(\operatorname{sign}(a)A), \tag{5.14a}$$
(subadditivity) $$\mu_{|||\cdot|||}(A+B) \leq \mu_{|||\cdot|||}(A) + \mu_{|||\cdot|||}(B), \tag{5.14b}$$
(norm of difference property) $$|\mu_{|||\cdot|||}(A) - \mu_{|||\cdot|||}(B)| \leq \|A-B\|, \tag{5.14c}$$
(conditional Coppel inequality for $A\mathcal{K}\subseteq\mathcal{K}$) $$|||\,\mathrm{e}^{At}\,||| \leq \mathrm{e}^{t\mu_{|||\cdot|||}(A)}. \tag{5.14d}$$

The induced matrix seminorm is a seminorm since it satisfies homogeneity and subadditivity.

The conditional sub-multiplicativity property (5.13e) allows us to provide upper bounds on the evolution of discrete- and continuous-time linear systems, similarly to what we did in Lemmas 2.1 and 2.2 and in the Coppel inequality Theorem 2.3. We reason as follows. Given a seminorm $|||\cdot|||$ with kernel $\mathcal{K}$ and a matrix A satisfying the invariance property[3] $A\mathcal{K}\subseteq\mathcal{K}$, the solutions to the discrete-time equation

$$x(k+1) = Ax(k) + u(k)$$

satisfy

$$|||x(k+1)||| \leq |||A||||||x(k)||| + |||u(k)|||, \tag{5.15}$$
$$|||x(k)||| \leq |||A|||^k|||x(0)||| + \sum_{s=0}^{k-1} |||A|||^{k-s-1}|||u(s)|||. \tag{5.16}$$

Similarly, the solutions to the continuous-time differential equation

$$\dot{x}(t) = Ax(t) + u(t)$$

satisfy

$$D^+|||x(t)||| \leq \mu_{|||\cdot|||}(A)|||x(t)||| + |||u(t)|||, \tag{5.17}$$
$$|||x(t)||| \leq \mathrm{e}^{\mu_{|||\cdot|||}(A)t}\,|||x(0)||| + \int_0^t \mathrm{e}^{(t-\tau)\mu(A)}\,|||u(\tau)|||d\tau. \tag{5.18}$$

We provide here some equivalent approaches to computing induced matrix seminorms. We will require the notion of *Moore-Penrose pseudoinverse* (also called pseudoinverse) of a matrix. Given $R \in \mathbb{R}^{k\times n}$, we let $R^\dagger$ denote its pseudoinverse. Recall that $\Pi^\dagger = \Pi$ for all orthogonal projections Π. We refer the reader to Exercise E5.3 for the definition and basic properties.

[3] The consensus set is invariant under any row-stochastic or Laplacian matrix.

Lemma 5.6 (Computing matrix induced seminorms). *For all $A \in \mathbb{R}^{n\times n}$,*

(i) given a seminorm $|||\cdot|||$ on $\mathbb{R}^n$ with kernel $\mathcal{K}$, let $\|\cdot\|_\perp$ denote the restricted perpendicular norm on $\mathcal{K}^\perp$ and $\mu_\perp(\cdot)$ denote the corresponding log norm on $\mathcal{K}^\perp$. Given the block decomposition of the matrix A in equation (5.2),

$$|||A||| = \|A_{\perp\perp}\|_\perp \quad \text{and} \quad \mu_{|||\cdot|||}(A) = \mu_\perp(A_{\perp\perp}); \tag{5.19}$$

(ii) given a norm $\|\cdot\|$ on $\mathbb{R}^k$ and a full row-rank $R \in \mathbb{R}^{k\times n}$,

$$\|A\|_R = \|RAR^\dagger\| \quad \text{and} \quad \mu_R(A) = \mu(RAR^\dagger); \tag{5.20}$$

(iii) given a norm $\|\cdot\|$ on $\mathbb{R}^n$ and a subspace $\mathcal{K}$ satisfying

$$\Pi_\perp\{x \in \mathbb{R}^n : \|x\| \le 1\} = \{x \in \mathbb{R}^n : \|x\| \le 1, x \perp \mathcal{K}\}, \tag{5.21}$$

the seminorm $\|x\|_{\Pi_\perp} = \|\Pi_\perp x\|$ defined by the orthogonal projection $\Pi_\perp$ onto $\mathcal{K}^\perp$ satisfies

$$\|A\|_{\Pi_\perp} = \|\Pi_\perp A \Pi_\perp\|. \tag{5.22}$$

Proof. To show equation (5.19), we compute

$$|||A||| = \max_{\substack{|||v|||\le 1\\ v\perp\mathcal{K}}} |||Av||| \overset{v=V_\perp\tilde v_\perp}{=} \max_{\|\tilde v_\perp\|_\perp\le 1} |||AV_\perp \tilde v_\perp||| \overset{(5.9)}{=} \max_{\|\tilde v_\perp\|_\perp\le 1} \|V_\perp^\top A V_\perp \tilde v_\perp\|_\perp = \|V_\perp^\top A V_\perp\|_\perp.$$

We leave the proof for the lognorm case to the reader.

Regarding equation (5.20), Exercise E5.3(v) states that $R\colon \ker(R)^\perp \to \operatorname{img}(R)$ is a linear bijection with inverse $R^\dagger\colon \operatorname{img}(R) \to \ker(R)^\perp$ so that, for each $v \in \ker(R)^\perp$ there exists a unique $y \in \operatorname{img}(R)$ with $y = Rv$ and $v = R^\dagger y$. Hence

$$\begin{aligned}\|A\|_R &= \max\{\|RAv\| : \|Rv\| = 1, v \perp \ker(R)\}\\ &= \max\{\|RAR^\dagger y\| : \|y\| = 1, y \in \operatorname{img}(R)\} = \max\{\|RAR^\dagger y\| : \|y\| = 1\} = \|RAR^\dagger\|,\end{aligned}$$

where the constraint $y \in \operatorname{img}(R)$ can be dropped since, as stated in Exercise E5.3(vi), $\operatorname{img}(R)^\perp$ is the kernel of $R^\dagger$ and R is full rank. Next, we compute

$$\begin{aligned}\mu_R(A) &= \lim_{h\to 0^+} \frac{\|I_n + hA\|_R - 1}{h} = \lim_{h\to 0^+} \frac{\|RR^\dagger + hRAR^\dagger\| - 1}{h}\\ &= \lim_{h\to 0^+} \frac{\|I_k + hRAR^\dagger\| - 1}{h} = \mu(RAR^\dagger),\end{aligned}$$

since $RR^\dagger$ is the orthogonal projection onto $\operatorname{img}(R)$.

Regarding equation (5.22), we refer to (De Pasquale et al., 2022). ■

Next, as we did in Section 2.6 for the standard induced norm and lognorm, we study the spectral properties of induced seminorms and induced log seminorms.

Lemma 5.7 (Spectral properties of induced seminorms and log seminorms). *Consider a seminorm $|||\cdot|||$ with kernel $\mathcal{K}$ and a matrix $A \in \mathbb{R}^{n\times n}$ such that $\mathcal{K}$ is A-invariant. Then*

(i) $\operatorname{spec}(A) = \operatorname{spec}(A|_{\mathcal{K}}) \cup \operatorname{spec}(A^\top|_{\mathcal{K}^\perp})$,

(ii) *identifying $A^\top|_{\mathcal{K}^\perp}$ with its matrix representation, the following spectral bounds hold:*

$$\rho(A^\top|_{\mathcal{K}^\perp}) \leq |||A|||, \tag{5.23}$$
$$\alpha(A^\top|_{\mathcal{K}^\perp}) \leq \mu_{|||\cdot|||}(A). \tag{5.24}$$

Proof. The decomposition of the spectrum of A in statement (i) is a standard fact from linear algebra. As in equation (5.1), we assume $\mathcal{K} = \operatorname{span}\{v_1, \dots, v_k\}$, $\mathcal{K}^\perp = \operatorname{span}\{v_{k+1}, \dots, v_n\}$, and recall the block decomposition of the matrix A in equation (5.2). We recall that $A^\top|_{\mathcal{K}^\perp} : \mathcal{K}^\perp \to \mathcal{K}^\perp$ has coordinate representation $A_{\perp\perp}^\top = V_\perp^\top A^\top V_\perp$. We compute

$$\rho(A^\top|_{\mathcal{K}^\perp}) \overset{\text{in components}}{=} \rho((A_{\perp\perp})^\top) = \rho(A_{\perp\perp}) \overset{\text{spectral bound in (2.71)}}{\leq} \|A_{\perp\perp}\|_\perp \overset{(5.19)}{=} |||A|||.$$
$$\alpha(A^\top|_{\mathcal{K}^\perp}) \overset{\text{in components}}{=} \alpha((A_{\perp\perp})^\top) = \alpha(A_{\perp\perp}) \overset{\text{spectral bound in (2.73)}}{\leq} \mu_\perp(A_{\perp\perp}) \overset{(5.19)}{=} \mu_{|||\cdot|||}(A).$$

This completes the proof of statement (ii). ∎

Finally, we study the ℓ_2 norm case in some detail and provide both analysis and design results.

Lemma 5.8 (Computing and optimizing ℓ_2 matrix induced seminorms). *For all $A \in \mathbb{R}^{n\times n}$,*

(i) *for any $P = P^\top \succeq 0$ with kernel $\mathcal{K}$,*

$$\|A\|_{2,P^{1/2}} = \min\{b \geq 0 \,:\, \Pi_\perp A^\top P A \Pi_\perp \preceq b^2 P\}, \quad \textit{and} \tag{5.25a}$$
$$\mu_{2,P^{1/2}}(A) = \min\{b \in \mathbb{R} \,:\, P A \Pi_\perp + \Pi_\perp A^\top P \preceq 2bP\}, \tag{5.25b}$$

(ii) *for any $P = P^\top \succeq 0$ with kernel $\mathcal{K}$ such that $A\mathcal{K} \subseteq \mathcal{K}$,*

$$\|A\|_{2,P^{1/2}} = \min\{b \geq 0 \,:\, A^\top P A \preceq b^2 P\} \quad \textit{and} \tag{5.26a}$$
$$\mu_{2,P^{1/2}}(A) = \min\{b \in \mathbb{R} \,:\, PA + A^\top P \preceq 2bP\}, \tag{5.26b}$$

(iii) *for any subspace $\mathcal{K}$ such that $A\mathcal{K} \subseteq \mathcal{K}$ and for all $\varepsilon > 0$, there exists $P_\varepsilon = P_\varepsilon^\top \succeq 0$ with kernel $\mathcal{K}$ such that*

$$\rho(A^\top|_{\mathcal{K}^\perp}) \leq \|A\|_{2,P_\varepsilon^{1/2}} \leq \rho(A^\top|_{\mathcal{K}^\perp}) + \varepsilon, \tag{5.27}$$

and similarly, there exists $P_\varepsilon = P_\varepsilon^\top \succeq 0$ with kernel $\mathcal{K}$ such that $\alpha(A^\top|_{\mathcal{K}^\perp}) \leq \mu_{2,P_\varepsilon^{1/2}}(A) \leq \alpha(A^\top|_{\mathcal{K}^\perp}) + \varepsilon$.

Proof. Let $\mathcal{K} = \ker P$ and note $P = P\Pi_\perp = \Pi_\perp P$. For any $P_e \succ 0$ such that $P = P_e\Pi_\perp = \Pi_\perp P_e$ and therefore $\Pi_\perp P_e^{-1} = P_e^{-1}\Pi_\perp = P^\dagger$,

$$\begin{aligned}
\|A\|_{2,P^{1/2}} &\overset{(5.11)}{=} \max_{\substack{v^\top P v \leq 1 \\ v \perp \mathcal{K}}} \|Av\|_{2,P^{1/2}} \overset{(5.4)}{=} \max_{\substack{v^\top P v \leq 1 \\ v \perp \mathcal{K}}} \|\Pi_\perp A \Pi_\perp v\|_{2,P^{1/2}} \overset{P=\Pi_\perp P_e \Pi_\perp}{=} \max_{\substack{v^\top P_e v \leq 1 \\ v \perp \mathcal{K}}} \|\Pi_\perp A \Pi_\perp v\|_{2,P_e^{1/2}} \\
&\overset{w=P_e^{1/2}v}{=} \max_{\substack{w^\top w \leq 1 \\ w \perp \mathcal{K}}} \|P_e^{1/2}\Pi_\perp A \Pi_\perp P_e^{-1/2} w\|_2 \overset{(5.11)}{=} \|P_e^{1/2}\Pi_\perp A \Pi_\perp P_e^{-1/2}\|_{2,\Pi_\perp} \\
&\overset{(5.22)}{=} \|\underbrace{\Pi_\perp P_e^{1/2} A P_e^{-1/2} \Pi_\perp}_{=:\mathcal{A}}\|_2 \overset{(2.55b)}{=} \min\{b \in \mathbb{R}_{\geq 0} \,:\, \mathcal{A}^\top \mathcal{A} \preceq b^2 I_n\} \\
&= \min\{b \in \mathbb{R}_{\geq 0} \,:\, \Pi_\perp A^\top P A \Pi_\perp \preceq b^2 P_e\},
\end{aligned}$$

where we computed

$$\begin{aligned}\mathcal{A}^\top\mathcal{A} &= (\Pi_\perp P_{\mathrm{e}}^{1/2} A P_{\mathrm{e}}^{-1/2}\Pi_\perp)^\top (\Pi_\perp P_{\mathrm{e}}^{1/2} A P_{\mathrm{e}}^{-1/2}\Pi_\perp)\\ &= (\Pi_\perp P_{\mathrm{e}}^{-1/2} A^\top P^{1/2})(P^{1/2} A P_{\mathrm{e}}^{-1/2}\Pi_\perp) = \Pi_\perp P_{\mathrm{e}}^{-1/2} A^\top P A P_{\mathrm{e}}^{-1/2}\Pi_\perp\\ &= P_{\mathrm{e}}^{-1/2}\Pi_\perp A^\top P A \Pi_\perp P_{\mathrm{e}}^{-1/2}.\end{aligned}$$

The claim follows from noting that the inequality $\Pi_\perp A^\top P A \Pi_\perp \preceq b^2 P_{\mathrm{e}}$ is automatically satisfied for all b^2 on $\mathcal{K}$. This concludes the proof of the induced seminorm result in (5.25). The induced seminorm result in (5.26) follows from noting that, under the stated assumptions on P and $\Pi_\perp$, we have $P = \Pi_\perp P \Pi_\perp$ and $\Pi_\perp A \Pi_\perp = \Pi_\perp A$ for all A.

Finally, regarding statement (iii), we claim that

$$A^\top P A \leq b^2 P \quad\iff\quad A_{\perp\perp}^\top P_{\perp\perp} A_{\perp\perp} \leq b^2 P_{\perp\perp}. \tag{5.28}$$

where $P_{\perp\perp}$ is defined in (5.10). Indeed, for any $x \in \mathcal{K}$, the inequality $x^\top A^\top P A x \leq b^2 x^\top P x$ is satisfied for any b since it simply states $0 \leq 0$. We restrict the inequality $A^\top P A \leq b^2 P$ to $\mathcal{K}^\perp$ by left and right multiplying by $V_\perp^\top$ and $V_\perp$ respectively to obtain

$$V_\perp^\top A^\top P A V_\perp \leq b^2 V_\perp^\top P V_\perp.$$

This inequality is equivalent to $A_{\perp\perp}^\top P_{\perp\perp} A_{\perp\perp} \leq b^2 P_{\perp\perp}$ since $\Pi_\perp = V_\perp V_\perp^\top$, we decompose $P = \Pi_\perp P \Pi_\perp = V_\perp V_\perp^\top P V_\perp V_\perp^\top = V_\perp P_{\perp\perp} V_\perp^\top$. Finally, as already discussed in Lemma 2.30, the LMI $A_{\perp\perp}^\top P_{\perp\perp} A_{\perp\perp} \leq b^2 P_{\perp\perp}$ admits a solution $P_{\perp\perp} \succ 0$ for any $b > \rho(A_{\perp\perp}) = \rho(A^\top|_{\mathcal{K}^\perp})$. This concludes the proof. ■

5.4 Semicontracting systems

In this section we present two main theorems about time-varying nonlinear dynamics

$$\dot{x} = f(t,x), \quad t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n \tag{5.29}$$

that are contracting with respect to a seminorm, in some sense to be made precise. Treatment of the discrete-time version of these results is analogous.

We start with some useful definitions and equivalences. First, we introduce the notion of a semicontracting system and show an example in Figure 5.3.

Definition 5.9 (Semicontracting systems). *Let $|||\cdot|||$ be a semi-norm on $\mathbb{R}^n$ with kernel $\mathcal{K}$, and $\mu_{|||\cdot|||}$ be its associated log seminorm. The continuously-differentiable time-varying vector field $f\colon \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^n$ is strongly infinitesimally semicontracting with rate $c > 0$ if*

$$\mu_{|||\cdot|||}(Df(t,x)) \leq -c, \quad \textit{for all } t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n.$$

Note: the semicontraction property along is not sufficient to ensure convergence properties of the dynamics. Consider, for example, a linear system described by $A_{2\times 2} = \begin{bmatrix} a_{\|\|} & a_{\|\perp} \\ a_{\perp\|} & a_{\perp\perp}\end{bmatrix} \in \mathbb{R}^{2\times 2}$. Even knowing that $a_{\perp\perp} < 0$ is not sufficient to infer anything about the spectrum of $A_{2\times 2}$. In what follows we discuss some additional properties for the dynamics that, in the linear case, are tantamount to assuming $a_{\perp\|} = 0$. Note that, when $a_{\perp\|} = 0$, then $a_{\perp\perp} < 0$ does imply that $a_{\perp\perp} < 0$ is an eigenvalue of $A_{2\times 2}$.

Next we establish a useful equivalence.

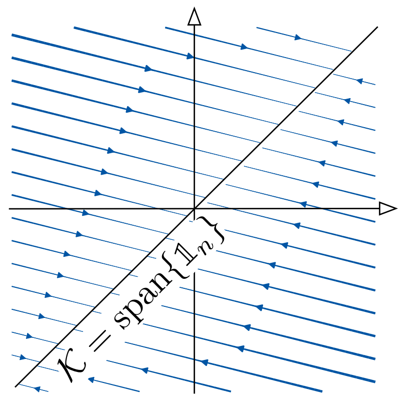


Figure 5.3: Phase portrait of the *Laplacian flow* $\dot{x} = -Lx$ with Laplacian matrix $\begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}$. As we will see later, this vector field is strongly infinitesimally semi-contracting with respect to a seminorm whose kernel is the consensus space $\mathrm{span}\{\mathbb{1}_n\}$.

Lemma 5.10 (Infinitesimal invariance and cascade dynamics). *Given a continuously-differentiable time-varying map $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ and a vector subspace $\mathcal{K} \subset \mathbb{R}^n$ (with perpendicular complement $\mathcal{K}^\perp$ and the orthogonal projection matrices $\Pi_\parallel$ and $\Pi_\perp$), the following properties are equivalent:*

(i) the subspace $\mathcal{K}$ is infinitesimally f-invariant in the sense that, for all $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}^n$,

$$Df(t,x)\mathcal{K} \subseteq \mathcal{K}, \qquad \text{or, equivalently,} \tag{5.30a}$$
$$f(t, x+\mathcal{K}) - f(t,x) \subseteq \mathcal{K}; \tag{5.30b}$$

(ii) the differential equation $\dot{x} = f(t,x)$ can be rewritten in cascade form as

$$\dot{x}_\perp = f_\perp(t, x_\perp), \tag{5.31}$$
$$\dot{x}_\parallel = f_\parallel(t, x_\perp + x_\parallel), \tag{5.32}$$

where $x = x_\perp + x_\parallel$ with $x_\parallel \in \mathcal{K}$ and $x_\perp \in \mathcal{K}^\perp$ and where the perpendicular and parallel vector fields $f_\perp\colon \mathbb{R}_{\geq 0} \times \mathcal{K}^\perp \to \mathcal{K}^\perp$ and $f_\parallel\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathcal{K}$ are defined by

$$f_\perp(t,y) = \Pi_\perp f(t,y), \qquad \text{and} \qquad f_\parallel(t,x) = \Pi_\parallel f(t,x). \tag{5.33}$$

Note: The infinitesimal f-invariance property can also be stated as

$$\Pi_\perp Df(t,x)\Pi_\parallel = \mathbb{0}_{n\times n}, \qquad \text{for all } t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n,$$

or, alternatively, if $\mathcal{K} = \mathrm{img}(V_\parallel)$ and $\mathcal{K}^\perp = \mathrm{img}(V_\perp)$ as in Section 5.2,

$$V_\perp^\top Df(t,x)V_\parallel = \mathbb{0}_{(n-k)\times k} \qquad \text{for all } t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n. \tag{5.34}$$

Proof of Lemma 5.10. The equivalence stated in equation (5.30) between statements (5.30a) and (5.30b) is established in Exercise E5.8. To show (i) $\implies$ (ii), we show that the vertical dynamics are independent of $x_\parallel$:

$$\dot{x}_\perp = \frac{d}{dt}\Pi_\perp x(t) = \Pi_\perp f(t, x(t)) \overset{(5.30b)}{=} \Pi_\perp f(t, \Pi_\perp x(t)) = \Pi_\perp f(t, x_\perp(t)). \tag{5.35}$$

The converse implication (i) $\impliedby$ (ii) follows from taking the Jacobian of the right-hand side of (5.31). ■

Theorem 5.11 (Perpendicular contraction). *Let $|||\cdot|||$ be a semi-norm on $\mathbb{R}^n$ with kernel $\mathcal{K}$. Let $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuously-differentiable time-varying vector field. Assume that*

(A1) *f is strongly infinitesimally semicontracting with rate c, and*

(A2) *$\mathcal{K}$ is infinitesimally f-invariant.*

Then

(i) the perpendicular vector field $f_\perp$ is strongly infinitesimally contracting with rate c with respect to $|||\cdot|||$ restricted to $\mathcal{K}^\perp$, and

(ii) for any initial conditions $x_0, y_0 \in \mathbb{R}^n$ at time $t_0 \geq 0$,

$$|||\phi_{t_0,t}(x_0) - \phi_{t_0,t}(y_0)||| \leq \mathrm{e}^{-c(t-t_0)}\, |||x_0 - y_0|||, \tag{5.36}$$

that is, the seminorm of the difference between any two trajectories is exponentially decaying with rate c.

Example 5.12 (Perpendicular and parallel contracting systems). With $g(x_i) = 3x_i - x_i^3$, we consider the two example dynamics

$$f_1 : \begin{cases} \dot{x}_1 = g(x_1) - (x_1 - x_2) \\ \dot{x}_2 = g(x_1) - (x_2 - x_1) \end{cases} \qquad f_2 : \begin{cases} \dot{x}_1 = g(x_1) - (x_1 - x_2) \\ \dot{x}_2 = g(x_2) - (x_2 - x_1) \end{cases} \tag{5.37}$$

with Jacobians $Df_1(x_1, x_2) = \begin{bmatrix} g(x_1)' - 1 & 1 \\ g(x_1)' + 1 & -1 \end{bmatrix}$ and $Df_2(x_1, x_2) = \begin{bmatrix} g(x_1)' - 1 & 1 \\ 1 & g(x_2)' - 1 \end{bmatrix}$. We consider the seminorm $|||x_1, x_2||| = |x_1 - x_2|$ with $V_\parallel = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $V_\perp = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, as in Figure 5.2. Note that the restricted perpendicular norm is simply the absolute value. We first verify that both dynamics are strongly infinitesimally semicontracting. We compute

$$\begin{aligned} \mu_{|||\cdot|||}(Df_1(x_1, x_2)) &= Df_1(x_1, x_2)_{\perp\perp} \overset{(5.19)}{=} V_\perp^\top Df_1(x_1, x_2) V_\perp \\ &= \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \begin{bmatrix} g(x_1)' - 1 & 1 \\ g(x_1)' + 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2, \\ \mu_{|||\cdot|||}(Df_2(x_1, x_2)) &= Df_2(x_1, x_2)_{\perp\perp} \overset{(5.19)}{=} V_\perp^\top Df_2(x_1, x_2) V_\perp \\ &= \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \begin{bmatrix} g(x_1)' - 1 & 1 \\ 1 & g(x_2)' - 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{g(x_1)' + g(x_2)'}{2} - 2. \end{aligned}$$

Therefore, f_1 is strongly infinitesimally semicontracting with rate 2 for all $g(x_1)$. Additionally, f_2 is strongly infinitesimally contracting since one can verify $|g(x_1)' + g(x_2)'| \leq 3 < 4$ for all x_1, x_2. Next, we check if the consensus space $\operatorname{span}\{\mathbb{1}_2\}$ is infinitesimally invariant under f_1 and/or f_2. We compute

$$\begin{aligned} Df_1(x_1, x_2)\mathbb{1}_2 &= \begin{bmatrix} g(x_1)' - 1 & 1 \\ g(x_1)' + 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = g(x_1)' \mathbb{1}_2, \\ Df_2(x_1, x_2)\mathbb{1}_2 &= \begin{bmatrix} g(x_1)' - 1 & 1 \\ 1 & g(x_2)' - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} g(x_1)' \\ g(x_2)' \end{bmatrix}, \end{aligned}$$

so that $\operatorname{span}\{\mathbb{1}_2\}$ is infinitesimally f_1-invariant but not f_2-invariant. For the vector field f_1, the perpendicular dynamics are well defined. We define $x_\perp = x_1 - x_2 \in \mathbb{R}$ and compute $\dot{x}_\perp = -2x_\perp$. Theorem 5.11 predicts that $|x_1(t) - x_2(t)| \leq \mathrm{e}^{-2t}\, |x_1(0) - x_2(0)|$. To analyze the vector field f_2 we introduce the notion of partially contracting vector fields in the next theorem. We illustrate the two phase portraits in Figure 5.4; note that, once on the consensus set $x_1 = x_2$, the remaining dynamics are $\dot{x}_1 = g(x_1) = 3x_1 - x_1^3$, which has equilibria at $x_0 = 0$ (unstable) and $x_\pm = \pm\sqrt{3}$ (exponentially stable). •

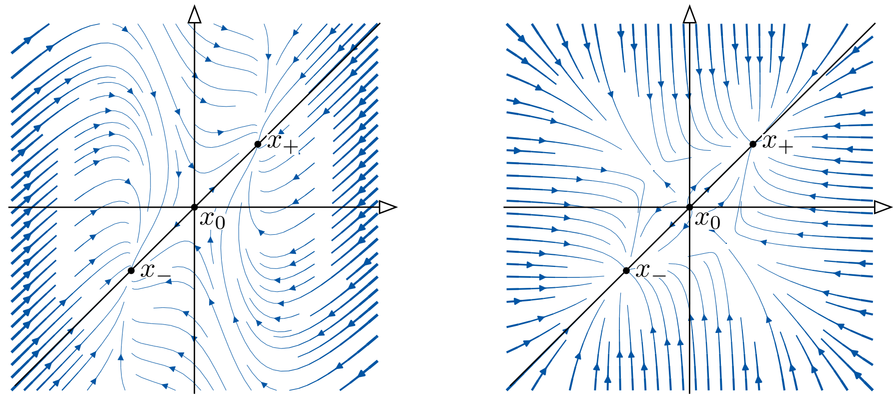


Figure 5.4: The phase portrait of the vector fields f_1 and f_2 in Example 5.12 as illustrations of the perpendicular and partial contraction Theorems 5.11 and 5.13.

Note: A consequence of the theorem is that all properties of strongly contracting vector fields apply directly to the perpendicular dynamics (e.g., existence of a globally exponentially stable equilibrium for time-invariant systems, etc).

Proof of Theorem 5.11. Regarding statement (i), compute

$$\begin{aligned}\mathsf{osLip}(f_\perp) &= \sup_{y\in\mathcal{K}^\perp,\, t\geq 0} \mu_{|||\cdot|||}(Df_\perp(t,y)) = \sup_{y\in\mathcal{K}^\perp,\, t\geq 0} \mu_{|||\cdot|||}(\Pi_\perp Df(t,y)) \\ &\overset{(5.4)}{=} \sup_{y\in\mathcal{K}^\perp,\, t\geq 0} \mu_{|||\cdot|||}(Df(t,y)) \leq \sup_{x\in\mathbb{R}^n,\, t\geq 0} \mu_{|||\cdot|||}(Df(t,x)) \leq -c.\end{aligned}$$

where in the last inequality we relaxed the constraints. Statement (ii) is an immediate consequence of the strong contractivity of the perpendicular dynamics. ■

Theorem 5.13 (Partially contracting systems). *Let $|||\cdot|||$ be a semi-norm on $\mathbb{R}^n$ with kernel $\mathcal{K}$. Let $f\colon \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^n$ be a continuously-differentiable time-varying vector field. Assume*

(A1) *f is strongly infinitesimally semicontracting with rate c, and*

(A2) *there exists an f-invariant affine subspace of the form $x^* + \mathcal{K}$, that is, there exists $x^* \in \mathcal{K}^\perp$, such that*

$$f(t, x^* + \mathcal{K}) \subseteq \mathcal{K}, \quad \textit{for all } t \in \mathbb{R}_{\geq 0}, \qquad \textit{or, equivalently,} \tag{5.38a}$$

$$\phi_t(x^* + \mathcal{K}) \subset x^* + \mathcal{K} \;\; \textit{for all } t \in \mathbb{R}_{\geq 0}, \quad \textit{(i.e., any trajectory in } x^* + \mathcal{K} \textit{ remains in } x^* + \mathcal{K}) \tag{5.38b}$$

Then for every $x_0 \in \mathbb{R}^n$ and $t \geq t_0 \in \mathbb{R}_{\geq 0}$,

$$|||\phi_{t_0,t}(x_0) - x^*||| \leq e^{-c(t-t_0)} |||x_0 - x^*|||, \tag{5.39}$$

that is, each trajectory converges to the affine subspace $x^* + \mathcal{K}$ with exponential rate c.

Remark 5.14 (Comparison of invariance conditions). *We report the two invariance conditions for convenience here:*

- *the subspace $\mathcal{K}$ is infinitesimally f-invariant if, $\forall x \in \mathcal{K}^\perp$, we have $f(x+\mathcal{K}) - f(x) \subset \mathcal{K}$,*
- *there exists x^* such that the affine subspace $x^* + \mathcal{K}$ is f-invariant, that is, $f(x^* + \mathcal{K}) \subset \mathcal{K}$.*

Next, we compare them:

(i) *when f is linear, the two notions are equivalent, with $x^* = 0$;*

(ii) *assume $\mathcal{K}$ is infinitesimally f-invariant, $f_\perp$ is time-invariant, and $f_\perp$ is strongly infinitesimally contracting. Then $f_\perp$ admits a unique equilibrium $x^* \in \mathcal{K}^\perp$ and, in turn, the affine subspace $x^* + \mathcal{K}$ is f-invariant.*

(iii) *we conclude with two examples:*

a) *consider the system in $\mathbb{R}^2$ given by*

$$\dot{x}_\perp = -(1 + x_\parallel^2)x_\perp, \tag{5.40a}$$
$$\dot{x}_\parallel = f_\parallel(x_\perp, x_\parallel). \tag{5.40b}$$

Note that the kernel $\{(x_\perp, x_\parallel) : x_\perp = 0, x_\parallel \in \mathbb{R}\}$ is forward invariant. However, the kernel is not infinitesimally f-invariant at any $(x_\perp, x_\parallel)$ such that $x_\parallel x_\perp \neq 0$, since $\partial(-(1+x_\parallel^2)x_\perp)/\partial x_\parallel = -2x_\parallel x_\perp$.

b) *consider the system in $\mathbb{R}^2$ given by*

$$\dot{x}_\perp = -x_\perp, \tag{5.41a}$$
$$\dot{x}_\parallel = x_\perp x_\parallel^2. \tag{5.41b}$$

This system is semicontracting and satisfies both invariance conditions. Therefore all trajectories approach the kernel, that is, $x_\perp(t) = x_\perp(0)\,\mathrm{e}^{-t}$. One can also integrate the parallel dynamics to obtain $x_\parallel(t) = x_\parallel(0)\,\mathrm{e}^{x_\perp(0)(1-\mathrm{e}^{-t})}$. Therefore, all trajectories with $x_\perp(0) > 0$ diverge to infinity. •

Proof of Theorem 5.13. We adopt the virtual system analysis approach, as described in Appendix 5.7. For simplicity of exposition we focus on the time-invariant case. We proceed in three steps.

Step I: Given an initial condition $x_0 \in \mathbb{R}^n$, define the time-varying *virtual vector field* $f_{\perp,x_0} : \mathbb{R}_{\geq 0} \times \mathcal{K}^\perp \to \mathcal{K}^\perp$ by

$$f_{\perp,x_0}(t, y) = \Pi_\perp f(y + \Pi_\parallel \phi_t(x_0)), \tag{5.42}$$

where $\phi_t(x_0)$ is the flow of f from initial condition x_0 at $t = 0$, that is, $\frac{d}{dt}\phi_t(x_0) = f(\phi_t(x_0))$. Next, we show that $f_{\perp,x_0}$ is strongly infinitesimally contracting with rate c with respect to $|||\cdot|||$ on $\mathcal{K}^\perp$. To show this claim, following the same steps as in the previous proof and using Assumption (A1), we claim that

$$\mathsf{osLip}(f_{\perp,x_0}) \leq -c; \tag{5.43}$$

we leave the details to the reader.

Step II: We now select two appropriate trajectories of the virtual system. First, we note that $x^* \in \mathcal{K}^\perp$ is an equilibrium of $f_{\perp,x_0}$, since Assumption (A2) implies $f(x^* - \Pi_\parallel \phi_t(x_0)) =: \kappa \in \mathcal{K}$ so that $f_{\perp,x_0}(t, x^*) = \Pi_\perp \kappa = \mathbb{0}_n$.

Second, we note that $t \mapsto \Pi_\perp \phi_t(x_0) \in \mathcal{K}^\perp$ is a solution of $f_{\perp,x_0}$, since

$$\begin{aligned}\tfrac{d}{dt}\Pi_\perp \phi_t(x_0) = \Pi_\perp \tfrac{d}{dt}\phi_t(x_0) = \Pi_\perp f(\phi_t(x_0)) &= \Pi_\perp f(\Pi_\perp \phi_t(x_0) + \Pi_\parallel \phi_t(x_0)) \\ &= f_{\perp,x_0}(t, \Pi_\perp \phi_t(x_0)).\end{aligned}$$

Step III: Finally we infer properties of the nominal system from the incremental stability of the two virtual trajectories. In other words, we now know that $f_{\perp,x_0}$ is strongly infinitesimally contracting on $\mathcal{K}^\perp$ and has two solutions: the equilibrium x^* and the projected trajectory $\Pi_\perp\phi_t(x_0)$. The exponential bound (5.39) follows from noting

$$|||\Pi_\perp\phi_t(x_0) - x^*||| = |||\phi_t(x_0) - x^*|||.$$

■

5.5 Example: Time-varying averaging dynamics

An important application of semi-norms arises in distributed systems and consensus problems.

5.5.1 Seminorms of adjacency and Laplacian matrices

Recall that a weighted directed graph G defines an adjacency matrix A and a Laplacian matrix L. Specifically, we will typically assume that the adjacency matrix A is row-stochastic, that is, $A \geq 0$ and $A\mathbb{1}_n = \mathbb{1}_n$, so that the Laplacian matrix $L = I_n - A$ has a nonpositive off-diagonal pattern and satisfies $L\mathbb{1}_n = \mathbb{0}_n$.

Lemma 5.15 (Induced seminorms of row-stochastic and Laplacian matrices). *Let $A \in \mathbb{R}^{n\times n}$ be row-stochastic and define $L = I_n - A$.*

(i) For the disagreement seminorm $|||\cdot|||_{2,\Pi_n}$, if A is symmetric and primitive,

$$\|A\|_{2,\Pi_n} = \big\|A - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top\big\|_2 = \min\{b \in \mathbb{R}_{\geq 0} \,:\, A^\top\Pi_n A \preceq b^2\Pi_n\} = \rho_{\text{ess}}(A) < 1, \tag{5.44a}$$
$$\mu_{2,\Pi_n}(-L) = -\lambda_2(L) = \min\{b \in \mathbb{R} \,:\, -L \preceq b\Pi_n\} = \alpha_{\text{ess}}(-L) < 0, \tag{5.44b}$$

where the essential spectral radius of A is $\rho_{\text{ess}}(A) = \max\{|\lambda| \,:\, \lambda \in \text{spec}(A) \setminus \{1\}\}$, the essential spectral abscissa of $-L$ is $\alpha_{\text{ess}}(-L) := \max\{\Re(\lambda) \,:\, \lambda \in \text{spec}(-L) \setminus \{0\}\}$, and $\lambda_2(L)$ is the algebraic connectivity of L.

(ii) For the disagreement seminorm $|||\cdot|||_{2,\Pi_n}$, if A is doubly-stochastic (that is, L is weight-balanced), irreducible, and with positive diagonal,

$$\|A\|_{2,\Pi_n} = \min\{b \in \mathbb{R}_{\geq 0} \,:\, A^\top\Pi_n A \preceq b^2\Pi_n\} = \rho_{\text{ess}}(A^\top A)^{1/2} < 1, \tag{5.45a}$$
$$\mu_{2,\Pi_n}(-L) = \min\{b \in \mathbb{R} \,:\, -\Pi_n L - L^\top\Pi_n \preceq 2b\Pi_n\} = -\tfrac{1}{2}\lambda_2(L + L^\top) < 0, \tag{5.45b}$$

where we note that $L + L^\top$ is a symmetric Laplacian matrix.

Proof. For a symmetric and row-stochastic A we compute

$$\|A\|_{2,\Pi_n} \overset{(5.22)}{=} \|\Pi_n A\Pi_n\|_2 \overset{\text{(modal decomposition)}}{=} \|A - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top\|_2 = \rho(A - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top) = \rho_{\text{ess}}(A). \tag{5.46}$$

Since A is assumed primitive, then it is known that $\rho_{\text{ess}}(A) < 1$. This concludes the proof of (5.44a).

Regarding the proof of (5.45a), for A doubly-stochastic, we compute

$$\|A\|_{2,\Pi_n} \overset{(5.22)}{=} \|\Pi_n A\Pi_n\|_2 = \sqrt{\lambda_{\max}(\Pi_n A^\top\Pi_n A\Pi_n)}. \tag{5.47}$$

From $\Pi_n = I_n - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$ we compute $\Pi_n A^\top\Pi_n A\Pi_n = \Pi_n A^\top(I_n - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top)A\Pi_n = \Pi_n A^\top A\Pi_n$. Additionally, since $A^\top A$ is doubly-stochastic and symmetric, $\Pi_n A^\top A\Pi_n = A^\top A - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$. Therefore, $\|A\|_{2,\Pi_n} = \lambda_{\max}(A^\top A -$

$\frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top)^{1/2}$. Exercise E5.5 implies that $A^\top A$ is primitive so that $\|A\|_{2,\Pi_n} = \rho_{\text{ess}}(A^\top A)^{1/2} < 1$. This concludes the proof of (5.45a).

Regarding the formulas for log seminorm of the (minus) Laplacian matrix in statements (i) and (ii), equation (5.44b) is a consequence of the Lyapunov LMI $\Pi_n L + L^\top \Pi_n \succeq 2\lambda_2 \Pi_n$ for symmetric L in E5.6 and equation (5.45b) is a consequence of the Lyapunov inequality $\Pi_n L + L^\top \Pi_n \succeq \lambda_2(L + L^\top)\Pi_n$ for weight-balanced digraphs in E5.7. ■

5.5.2 Linear time-varying Laplacian flows

We start by considering the Laplacian flow

$$\dot{x}(t) = -L(t)x(t) + u(t) =: f_{\text{L}}(t, x). \tag{5.48}$$

where $L(t)$ is a Laplacian matrix for all $t \in \mathbb{R}_{\geq 0}$. We will consider seminorms whose kernel is the consensus space $\operatorname{span}\{\mathbb{1}_n\}$. This system is the time-varying version of the affine averaging system (4.6) studied in Example 4.5.

(i) First, we note that $L(t)\mathbb{1}_n \subseteq \mathbb{1}_n$ for all t, that is, the consensus space is infinitesimally L-invariant. Therefore, the cascade decomposition in Lemma 5.10 is applicable and, as we calculate in Exercise E5.9, we write

$$\dot{x}_\perp = f_{\text{L},\perp}(t, x_\perp) := -\Pi_n L(t) x_\perp + u(t) - u_{\text{avg}}(t)\mathbb{1}_n \in \mathbb{1}_m^\perp, \tag{5.49a}$$
$$\dot{x}_{\text{avg}} = -\tfrac{1}{n}\mathbb{1}_n^\top L(t) x_\perp + u_{\text{avg}}(t) \qquad \in \mathbb{R}. \tag{5.49b}$$

where $x_{\text{avg}} = \frac{1}{n}\mathbb{1}_n^\top x$ is the average state.

(ii) Second, we assume that $\{L(t)\}_t$ is *uniformly irreducible* and weight-balanced in the sense that

$$L(t) \in \mathcal{L} \quad \text{for all } t \in \mathbb{R}_{\geq 0}, \tag{5.50}$$

where $\mathcal{L}$ denotes a closed set[4] of Laplacian matrices that are irreducible and weight-balanced. Under this assumption, the continuity of the Perron eigenvalue over the compact set $\mathcal{L}$ implies the existence of $c > 0$ such that

$$\sup_{x \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0}} \mu_{2,\Pi_n}(Df_{\text{L}}(t, x)) = \max_{L \in \mathcal{L}} \mu_{2,\Pi_n}(-L) = -c < 0. \tag{5.51}$$

Therefore, if $L(t)$ is uniformly irreducible and weight-balanced, then f_{L} is strongly infinitesimally semicontracting with respect to the disagreement seminorm $\|\cdot\|_{2,\Pi_n}$ with rate $c > 0$.

Corollary 5.16 (Time-varying affine Laplacian flow). *The affine time-varying Laplacian flow (5.48) defined by a uniformly irreducible and weight-balanced Laplacian $t \mapsto L(t)$ satisfies both assumptions of the perpendicular contraction Theorem 5.11. Therefore:*

(i) $f_{\text{L},\perp}$ is strongly infinitesimally contracting with rate c on $\operatorname{span}\{\mathbb{1}_n\}^\perp$, where c is defined in (5.51). Therefore, for example, any two trajectories of the Laplacian flow (5.48) satisfy

$$\|x(t) - y(t)\|_{2,\Pi_n} \leq e^{-ct}\|x(0) - y(0)\|_{2,\Pi_n},$$

(ii) if $u(t) = \mathbb{0}_n$, then $\mathbb{0}_n$ is an equilibrium of $f_{\text{L},\perp}$ and so each trajectory of the Laplacian flow (5.48) satisfies

$$\|x(t)\|_{2,\Pi_n} \leq e^{-ct}\|x(0)\|_{2,\Pi_n}.$$

Additionally, since $u_{\text{avg}} = 0$ and $L(t)x(t)$ is exponentially vanishing, $\lim_{t\to+\infty} x_{\text{avg}}(t) = x^*_{\text{avg}}$ and, in summary, $x(t) \to x^*_{\text{avg}}\mathbb{1}_n$ with exponential rate c.

[4] Typically one considers the set of matrices whose non-zero entries are lower-bounded by a strictly positive value.

Next, we revise this analysis for the time-invariant setting (that is, for the same model studied in Example 4.5):

$$\dot{x} = -Lx + u =: f_{\mathrm{L}}(x). \tag{5.52}$$

(i) It remains true that the consensus space is L-invariant and f_{L} can still be written in the cascade form (5.49).

(ii) Instead of uniform irreducibility and weight-balance, we assume L is the Laplacian of a weighted digraph with a globally reachable node (see Theorem B.16). This assumption implies $\alpha_{\text{ess}}(-L) < 0$. Then, for arbitrarily small ε, we invoke Lemma 5.8(iii) to compute a positive semidefinite P_ε with kernel $\operatorname{span}\{\mathbb{1}_n\}$ such that

$$\mu_{2,P_\varepsilon^{1/2}}(-L) \leq \alpha_{\text{ess}}(-L) + \varepsilon < 0 \tag{5.53}$$

Corollary 5.17 (Affine Laplacian flow). *The affine Laplacian flow (5.52) defined by the Laplacian matrix of a weighted digraph with a globally reachable node is strongly infinitesimally semicontracting with respect to the norm $\|\cdot\|_{2,P_\varepsilon}$. Moreover,*

(i) if $v^\top u \neq 0$, then every trajectory is unbounded, where v is the dominant left eigenvector of L satisfying $\mathbb{1}_n^\top v = 1$,

(ii) if $v^\top u = 0$, then the trajectory starting from $x(0) = x_0$ converges to the equilibrium point $L^\dagger u + (v^\top x_0)\mathbb{1}_n$ with exponential rate $-\alpha_{\text{ess}}(-L)$.

5.6 Example: Chua's diffusively-coupled dynamical systems

We here study synchronization phenomena in networks of identical diffusively-coupled systems. Diffusively-coupled dynamical systems are widespread in many disciplines including developmental biology, neuroscience, cellular systems, and cellular neural networks. Consider n agents connected through a weighted undirected graph with Laplacian matrix L. Given a vector field $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$:

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}, \tag{5.54}$$

Key points:

(i) the phenomenon of interest is asymptotic synchronization (as opposed to stability of the origin as in the setting of interconnected ISS systems). We say the system is *synchronized* when

$$x_i(t) = x_j(t), \quad \text{for all } i \neq j \text{ and } t \geq 0; \tag{5.55}$$

(ii) diffusive-coupling promotes synchronization (as opposed to having a destabilizing effect as in the setting of interconnected ISS systems);

(iii) loosely speaking, diffusively-coupled networks will synchronize depending on

a) the homogeneity of the internal dynamics,

b) the (level of) contractivity or expansivity of the internal dynamics, and

c) the strength of the diffusive coupling.

To model and perform computation on identical diffusively-coupled systems, it is convenient to use the operation of Kronecker product, which we reviewed in Exercises E4.5-E4.7. For example, the synchronous state belongs to the d-dimensional consensus space

$$\mathcal{K}_d = \operatorname{span}\{\mathbb{1}_n \otimes u \,:\, u \in \mathbb{R}^d\}. \tag{5.56}$$

Diffusively-coupled identical linear systems Given $\mathcal{A} \in \mathbb{R}^{d\times d}$ and a weighted undirected graph, consider the diffusively-coupled linear identical systems:

$$\dot{x}_i = \mathcal{A}x_i - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1,\dots,n\}. \tag{5.57}$$

From (Bullo, 2022, Theorem 8.4(ii)), the system (5.57) achieves asymptotic synchronization if and only if

$$\begin{aligned}
&\iff \quad \mathcal{A} - \lambda_i(L)I_d \text{ is Hurwitz, for all } i \geq 2,\\
&\iff \quad \mathcal{A} - \lambda_2(L)I_d \text{ is Hurwitz}\\
&\iff \quad \alpha(\mathcal{A}) < \lambda_2(L)\\
&\iff \quad \text{there exists } p \in [1,\infty] \text{ and } Q \text{ invertible such that } \mu_{p,Q}(\mathcal{A}) < \lambda_2(L)
\end{aligned}$$

If so, then the diffusively-coupled linear identical systems (5.57) achieve asymptotic synchronization with exponential rate $\lambda_2(L) - \alpha(\mathcal{A})$.

Note: $\mathcal{A}$ can be unstable and yet, above the threshold, diffusive coupling will achieve sync.

A useful norm on $\mathbb{R}^{nd} \simeq \mathbb{R}^n \otimes \mathbb{R}^d$ For $p \in [1,\infty]$, define the *$(2,p)$-tensor norm* $\|\cdot\|_{(2,p)}$ on $\mathbb{R}^{nd} \simeq \mathbb{R}^n \otimes \mathbb{R}^d$ by:

$$\|u\|_{(2,p)} = \inf\Big\{ \Big(\sum_{i=1}^{r} \|v^i\|_2^2 \|w^i\|_p^2 \Big)^{\frac{1}{2}} \;:\; u = \sum_{i=1}^{r} v^i \otimes w^i \Big\}. \tag{5.58}$$

The well-definiteness and properties of the $(2,p)$-tensor norm are studied in (Jafarpour et al., 2022).

The $(2,p)$-tensor norm is closely related to, but different from, the well-known projective tensor product norm (see (Ryan, 2002, Chapter 2) for definition and properties of this well-known norm). The $(2,p)$-tensor norm is also different from the aggregation norm discussed in Section 2.4.4.

Lemma 5.18 (Properties of $(2,p)$-tensor norms). *Consider the identification $\mathbb{R}^{nd} \simeq \mathbb{R}^n \otimes \mathbb{R}^d$ and the $(2,p)$-tensor norm defined in (5.58). Let $A \in \mathbb{R}^{nd\times nd}$, $\Lambda = \operatorname{blkdg}(\Lambda_1,\dots,\Lambda_n)$ such that $\Lambda_i \in \mathbb{R}^{d\times d}$, for every $i \in \{1,\dots,n\}$ and $U \in \mathbb{R}^{(n-1)\times n}$ is such that $UU^\top = I_{n-1}$, then*

(i) *$(2,p)$-tensor norm is well-defined;*
(ii) $\|U \otimes I_d\|_{(2,p)} = \|U^\top \otimes I_d\|_{(2,p)} = 1$;
(iii) $\|\Lambda\|_{(2,p)} \leq \max_i \|\Lambda_i\|_p$;
(iv) $\mu_{(2,p),U\otimes I_d}(A) \leq \mu_{(2,p)}(A)$;
(v) $\mu_{(2,p)}(\Lambda) \leq \max_i \mu_p(\Lambda_i)$.

Theorem 5.19 (Networks of diffusively-coupled identical dynamical systems (Jafarpour et al., 2022)). *Given a weighted undirected graph with Laplacian matrix L and identical internal dynamics $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$, consider*

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1,\dots,n\}. \tag{5.59}$$

If, for every $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d$,

$$\mu_{p,Q}(Df(t,x)) \leq \lambda_2(L) - c, \tag{5.60}$$

for some $p \in [1,\infty]$, invertible $Q \in \mathbb{R}^{d\times d}$, and $c > 0$, then

(i) *the system* (5.59) *is strongly infinitesimally semicontracting with rate* c *with respect to the seminorm* $\|\cdot\|_{(2,p),R_{\mathcal{V}}\otimes Q}$,

(ii) *the kernel* $\mathcal{K}_d = \operatorname{span}\{\mathbb{1}_n \otimes u \mid u \in \mathbb{R}^d\}$ *of the seminorm is invariant (so that the system is partially contracting as in Theorem 5.13),*

(iii) *system* (5.59) *achieves global exponential synchronization with rate* c*. Specifically, for each trajectory* $x(t) = (x_1(t), \dots, x_n(t))$,

$$\lim_{t\to\infty} \|x(t) - \mathbb{1}_n \otimes x_{\mathrm{avg}}(t)\|_2 = 0, \qquad \text{where } x_{\mathrm{avg}}(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t).$$

Remark 5.20 (Interpretation). (i) *Even if the graph is not connected ($\lambda_2(L) = 0$), identical infinitesimally-contracting systems will synchronize. If f is time-invariant, they will synchronize to the equilibrium point. If f is periodic, then they will synchronize to the unique periodic solution (see Section 3.4.3).*

(ii) *Even if dynamics are unstable (say with positive Jacobian matrix measure), there exists a threshold above which sufficiently-strong diffusive coupling will force asymptotic synchrony. Here sufficiently-strong means $\lambda_2(L)$ sufficiently large. This result is consistent with, but sharper than, the original findings in (Wu and Chua, 1995).*

(iii) *If $f(t, x_i) = Ax_i$, then the synchronization test* (5.60) *simplifies to $\alpha(A) < \lambda_2(L)$, i.e., the test for diffusively-coupled identical linear systems.* •

Proof. Set $x = (x_1^\top, \dots, x_n^\top)^\top \in \mathbb{R}^{nd}$ and define $x_{\mathrm{avg}} := \frac{1}{n}\sum_{i=1}^{n} x_i$. The dynamics (5.59) can be written as:

$$\dot{x} = F(t, x) - (L \otimes I_d)x, \tag{5.61}$$

where $F(t, x) = [f^\top(t, x_1), \dots, f^\top(t, x_n)]^\top$. Moreover,

$$DF(t, x) = \begin{bmatrix} Df(t, x_1) & \dots & \mathbb{0}_{d\times d} \\ \vdots & \ddots & \vdots \\ \mathbb{0}_{d\times d} & \dots & Df(t, x_n) \end{bmatrix} \in \mathbb{R}^{nd\times nd}.$$

Claim: the system (5.59) is semicontracting with respect to $|||\cdot|||_{(2,p),R_{\mathcal{V}}\otimes Q}$ with rate c, that is, for every $t \in \mathbb{R}_{\geq 0}$ and every $x \in \mathbb{R}^{nd}$,

$$\mu_{(2,p),R_{\mathcal{V}}\otimes Q}(DF(t, x) - L \otimes I_d) \leq -c.$$

As first step, note that

$$\mu_{(2,p),R_{\mathcal{V}}\otimes Q}(DF(t, x) - L \otimes I_d) \leq \mu_{(2,p),R_{\mathcal{V}}\otimes Q}(DF(t, x)) + \mu_{(2,p),R_{\mathcal{V}}\otimes Q}(-L \otimes I_d).$$

Let $R_{\mathcal{V}}$ be the matrix containing $n-1$ eigenvectors of L corresponding to the eigenvalues different from 0. One can show that $R_{\mathcal{V}}^\dagger = R_{\mathcal{V}}^\top (R_{\mathcal{V}} R_{\mathcal{V}}^\top)^{-1} = R_{\mathcal{V}}^\top$ and $R_{\mathcal{V}} L R_{\mathcal{V}}^\top = \Lambda = \operatorname{diag}(\lambda_2(L), \dots, \lambda_n(L))$. Using $\mu_R(A) = \mu(RAR^\dagger)$, this implies that

$$\begin{aligned} \mu_{(2,p),R_{\mathcal{V}}\otimes Q}(-L \otimes I_d) &= \mu_{(2,p)}(-R_{\mathcal{V}} L R_{\mathcal{V}}^\top \otimes QQ^{-1}) \\ &= \mu_{(2,p)}(-\Lambda \otimes I_d) \leq -\lambda_2(L). \end{aligned} \tag{5.62}$$

where the last inequality holds by Lemma 5.18(v).

As second step, we have $R_{\mathcal{V}} R_{\mathcal{V}}^{\top} = I_{n-1}$ and $R_{\mathcal{V}} \otimes Q = (R_{\mathcal{V}} \otimes I_d)(I_n \otimes Q)$. Thus, $\mu_R(A) = \mu(RAR^{\dagger})$ and Lemma 5.18(iv) imply that

$$\begin{aligned}\mu_{(2,p),R_{\mathcal{V}}\otimes Q}(DF(t,x)) &= \mu_{(2,p),R_{\mathcal{V}}\otimes I_d}((I_n \otimes Q)DF(t,x)(I_n \otimes Q^{-1})) \\ &\leq \mu_{(2,p)}((I_n \otimes Q)DF(t,x)(I_n \otimes Q^{-1}))\end{aligned}$$

Note that

$$\Gamma := (I_n \otimes Q)DF(t,x)(I_n \otimes Q^{-1}) = \begin{bmatrix} \Gamma_1 & \dots & \mathbb{0}_{d\times d} \\ \vdots & \ddots & \vdots \\ \mathbb{0}_{d\times d} & \dots & \Gamma_n \end{bmatrix},$$

where $\Gamma_i = QDf(t,x_i)Q^{-1} \in \mathbb{R}^{d\times d}$, for every $i \in \{1,\dots,n\}$. In turn, using Lemma 5.18(v),

$$\mu_{(2,p),R_{\mathcal{V}}\otimes Q}(DF(t,x)) \leq \mu_{(2,p)}(\Gamma) \leq \max_{i\in\{1,\dots,n\}} \{\mu_p(\Gamma_i)\} = \max_{i\in\{1,\dots,n\}} \mu_{p,Q}(Df(t,x_i)). \tag{5.63}$$

Thus, combining (5.62) and (5.63), we get

$$\mu_{(2,p),R_{\mathcal{V}}\otimes Q}(DF(t,x) - L\otimes I_d) \leq \max_{i\in\{1,\dots,n\}} \mu_{p,Q}(Df(t,x_i)) - \lambda_2(L).$$

This statement and the assumption in (5.60) concludes the proof of our claim.

As third and final step, we note that

$$\ker(|||\cdot|||_{(2,p),R_{\mathcal{V}}\otimes Q}) = \ker(R_{\mathcal{V}}\otimes Q) = \mathcal{K}_d$$

where $\mathcal{K}_d = \operatorname{span}\{\mathbb{1}_n \otimes u \;:\; u \in \mathbb{R}^d\}$ is invariant under the dynamical system (5.61). Thus, using Theorem 5.13, every trajectory of (5.61) converges exponentially with rate c to $\mathcal{K}_d$. Note that $\Pi_n \otimes I_d$ is the orthogonal projection onto $\mathcal{K}_d^{\perp}$. Thus, Theorem 5.13 implies that $\lim_{t\to\infty} \|(\Pi_n \otimes I_d)x(t)\|_2 = 0$ with rate c. Moreover, we know that

$$\begin{aligned}(\Pi_n \otimes I_d)x(t) &= x(t) - \tfrac{1}{n}(\mathbb{1}_n\mathbb{1}_n^{\top} \otimes I_d)x(t) \\ &= x(t) - (\mathbb{1}_n \otimes x_{\text{avg}}(t)).\end{aligned}$$

As a result, $\lim_{t\to\infty} \|x(t) - (\mathbb{1}_n \otimes x_{\text{avg}}(t))\|_2 = 0$ with rate c, which in turn means that $\lim_{t\to\infty} \|x_i(t) - x_j(t)\|_2 = 0$ with rate c, for every $i,j \in \{1,\dots,n\}$. ■

5.7 Appendix: Analysis via the virtual system approach

The *virtual system* analysis approach is a method to study the asymptotic behavior of a dynamical system that may not enjoy contracting properties.

The virtual system analysis approach is described as follows. We are given a time-varying dynamical system

$$\dot{x} = f(t,x), \quad x(0) = x_0 \in \mathbb{R}^n \tag{5.64}$$

and we let $\phi_{0,t}(x_0)$ denote the solution from initial condition $x(0) = x_0$ at time 0. We proceed in three steps:

(i) *design* a time-varying dynamical system, called the *virtual system*, of the form

$$\dot{y} = f_{\text{virtual}}(y, \phi_{0,t}(x_0)), \quad y \in \mathbb{R}^d \tag{5.65}$$

satisfying a strong infinitesimal contractivity property with respect to an appropriate norm:

$$\sup_{y\in\mathbb{R}^d,\, z\in\mathbb{R}^n} \mu(Df_{\text{virtual}}(y,z)) = -c, \quad c>0; \tag{5.66}$$

(The vector field is called virtual or sometimes auxiliary, since it is different from the nominal vector field f and does not necessarily correspond to any physically meaningful variation of f.)

(ii) *select* two specific solutions of the virtual system and state their incremental stability property:

$$\|y_1(t) - y_2(t)\| \leq \mathrm{e}^{-ct}\|y_1(0) - y_2(0)\|; \tag{5.67}$$

(iii) *infer* properties of the trajectory $\phi_{0,t}(x_0)$ of the nominal system.

For example, if $d = n$ and $f(x) = f_{\text{virtual}}(x,x)$, then one can see that $\phi_t(x_0)$ is a solution for both systems and is often selected as one of the two specific solutions in step (ii).

Case Study I: Equilibrium contraction We now consider a time-invariant vector field f with an equilibrium x^*. We assume that it is possible to factorize $f(x) = A(x)(x - x^*)$, where $A(x)$ is a matrix-valued function of x. Note that the Jacobian of f contains the Jacobian of A and we may be unable to obtain strong contraction properties of f. We illustrate and comment on the following lemma in Figure 5.5.

Lemma 5.21 (Equilibrium contraction). *Consider the dynamical system*

$$\dot{x} = A(x)(x - x^*) \tag{5.68}$$

and assume that, with respect to some norm,

$$\sup_{z\in\mathbb{R}^n} \mu(A(z)) \leq -c. \tag{5.69}$$

Then every solution converges to the equilibrium x^* with exponential rate c, that is, for all $x_0 \in \mathbb{R}^n$

$$\|\phi_t(x_0) - x^*\| \leq \mathrm{e}^{-ct}\|x_0 - x^*\|. \tag{5.70}$$

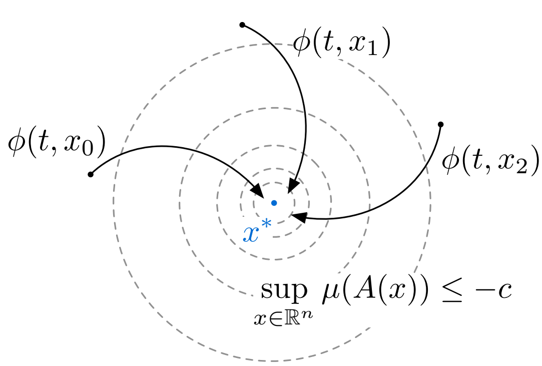


Figure 5.5: We refer to systems of the form (5.68) satisfying a log norm condition such as (5.69) as *equilibrium contracting*. For an equilibrium contracting system, all trajectories converge to the unique equilibrium.

Proof. We adopt the virtual system analysis approach:

(i) given an initial condition $x(0) = x_0$, we define the x_0-dependent virtual system

$$\dot{y} = A(\phi_t(x_0))(y - x^*) =: f_{\text{virtual}}(t,y), \tag{5.71}$$

and show that it is strongly infinitesimally contracting:

$$\mathsf{osLip}(f_{\text{virtual}}) = \sup_{t\geq 0,\, y\in\mathbb{R}^n} \mu(Df_{\text{virtual}}(t,y)) = \sup_{t\geq 0} \mu(A(\phi_t(x_0))) \leq \sup_{z\in\mathbb{R}^n} \mu(A(z)) = -c; \tag{5.72}$$

(ii) we select two specific solutions of the virtual system:

$$y_1(t) = \phi_t(x_0) \qquad \text{and} \qquad y_2(t) = x^*, \tag{5.73}$$

(iii) we translate the incremental stability of the two specific solutions of the virtual system into properties of the solution $\phi_t(x_0)$ of the nominal system:

$$\|y_1(t) - y_2(t)\| \leq \mathrm{e}^{-ct}\|y_1(0) - y_2(0)\| \qquad \Longrightarrow \qquad \|\phi_t(x_0) - x^*\| \leq \mathrm{e}^{-ct}\|x_0 - x^*\|. \tag{5.74}$$

This completes the proof. ∎

Case Study II: Nonlinear observer design We consider the differentiable control system $\dot{x} = f(x, u)$ with an unknown initial condition $x_0 \in \mathbb{R}^n$ and with inputs u taking value in a set $\mathcal{U}$.

Given knowledge of the dynamics f and assuming that we can measure a differentiable function of the state $h(x) \in \mathbb{R}^k$ and the input signal $u(t)$, we aim to design an observer (i.e., a dynamical system driven by the signals $h(x(t))$ and $u(t)$) that reconstructs the full state x of the system.

Given a gain matrix $K \in \mathbb{R}^{n\times k}$, we design the observer:

$$\dot{y} = f(y, u(t)) + K\big(h(y) - h(\phi_t(x_0))\big) := f_{\text{virtual}}(t, y, u), \tag{5.75}$$

and initialize it with an arbitrary initial condition $y(0) = y_0$.

Lemma 5.22 (Nonlinear observer design). *If* $\sup_{u\in\mathcal{U}, x\in\mathbb{R}^n} \mu(Df(x, u) + KDh(x)) = -c$, *for* $c > 0$, *then the state of the observer* (5.75) *converges exponentially fast to the unknown system state, that is, for any initial conditions* x_0 *and* y_0

$$\|y(t) - \phi_t(x_0)\| \leq \mathrm{e}^{-ct}\|y_0 - x_0\|. \tag{5.76}$$

Proof. We adopt the virtual system approach.

(i) We show that the virtual system is strongly infinitesimally contracting:

$$\sup_{t\geq 0, u\in\mathcal{U}, y\in\mathbb{R}^n} \mu(Df_{\text{virtual}}(t, x, u)) = \sup_{u\in\mathcal{U}, x\in\mathbb{R}^n} \mu(Df(x, u) + KDh(x)) = -c. \tag{5.77}$$

(ii) We select two trajectories of the virtual system: First we pick a generic trajectory $y_1(0) = y_0$ of the observer. Second, we pick $y_2(0) = x_0$. But then by uniqueness of the solutions to ordinary differential equations, we know $y_2(t) = \phi_t(x_0)$.

(iii) We infer properties of the trajectory $\phi_t(x_0)$ of the nominal system:

$$\|y_1(t) - y_2(t)\| \leq \mathrm{e}^{-ct}\|y_1(0) - y_2(0)\| \qquad \Longrightarrow \qquad \|y(t) - \phi_t(x_0)\| \leq \mathrm{e}^{-ct}\|y_0 - x_0\|. \tag{5.78}$$

This completes the proof. ∎

Case Study III: Synchronization in all-to-all identical systems Consider n identical smooth nonlinear systems with nonlinear diffusive coupling

$$\dot{x}_i = f(t, x_i) + \sum_{j=1}^{n} \big(h(x_j) - h(x_i)\big), \quad i \in \{1, \dots, n\}, \ x_i \in \mathbb{R}^d \tag{5.79}$$

In other words, assume all-to-all homogeneous coupling. Then one can rewrite equation (5.79) as:

$$\dot{x}_i = f(t, x_i) - nh(x_i) + \sum_{j=1}^{n} h(x_j). \tag{5.80}$$

In this rewriting the effect of all individual system is equal to a single averaged/cumulative effect. In this sense, the analysis of this equivalent formulation resembles ideas in the "mean field analysis" domain.

Theorem 5.23 (Synchronization of all-to-all coupled systems). *Consider all-to-all coupled identical systems in equation (5.79). Assume that, with respect to some norm,*

$$\sup_{x \in \mathbb{R}^k, t \geq 0} \mu\big(Df(t, x) - nDh(x)\big) \leq -c. \tag{5.81}$$

Then the all-to-all coupled systems achieve exponential state synchronization, in the sense that, for all $(x_1(0), \dots, x_n(0))$,

$$\|x_i(t) - x_j(t)\| \leq \mathrm{e}^{-ct} \|x_i(0) - x_j(0)\|. \tag{5.82}$$

Note: the log norm condition is on the d-dimensional dynamics of a single system, not on the full dn-dimensional dynamics of the coupled system.

Proof. We adopt the virtual system approach in three steps.

(i) Given an initial condition $(x_1(0), \dots, x_n(0))$ and corresponding solution $(x_1(t), \dots, x_n(t))$, we define a *virtual system* on $\mathbb{R}^d$ by

$$\dot{y} = f(t, y) - nh(y) + \sum_{j=1}^{n} h(x_j(t)). \tag{5.83}$$

The assumption (5.81) implies that the virtual system is strongly infinitesimally contracting with rate $c > 0$.

(ii) Next, we pick any $i, j \in \{1, \dots, n\}$ and select two virtual trajectories defined by $y_1(0) = x_i(0)$ and $y_2(0) = x_j(0)$.

(iii) Finally, we relate these two trajectories of the virtual system to the solution $(x_1(t), \dots, x_n(t))$ of the nominal original system:

$$\begin{aligned} y_1(0) = x_i(0) &\implies y_1(t) = x_i(t), \\ y_2(0) = x_j(0) &\implies y_2(t) = x_j(t). \end{aligned}$$

Because the virtual system is strongly infinitesimally contracting, we know that the distance between $y_1(t)$ and $y_2(t)$ is exponentially converging:

$$\|y_1(t) - y_2(t)\| \leq \mathrm{e}^{-ct} \|y_1(0) - y_2(0)\| \quad \implies \quad \|x_i(t) - x_j(t)\| \leq \mathrm{e}^{-ct} \|x_i(0) - x_j(0)\|. \tag{5.84}$$

This completes the proof. ∎

5.8 Historical notes and further reading

Seminorms, induced seminorms and ergodic coefficients An early reference on induced matrix seminorms is (Kolpakov, 1983), which obtains some of the results in Lemma 5.5. Logarithmic seminorms are defined in (Jafarpour et al., 2022); additional results on seminorms and their relationship to ergodic coefficients is given in the recent report (De Pasquale et al., 2022).

The study of semicontraction theory can be traced back all the way to the original work by Markov (1906) in the context of the Weak Law of Large Numbers. In this work and in the rich literature on Markov chains, e.g., see (Kolmogorov, 1931; Doeblin, 1937; Dobrushin, 1956), the induced matrix seminorm is essentially referred to as an ergodic coefficient of the matrix. The key results in this research area were extended and then reviewed by Seneta in the 80's, e.g., see (Seneta, 1981). A comprehensive survey of ergodicity coefficients is given by Ipsen and Selee (2011) and a recent treatment on their connection with spectral graph theory is given by Marsli and Hall (2020). Applications to so-called gossiping algorithms in consensus networks motivate the study of optimally deflated matrices and the characterization of "convergability" by Liu et al. (2011c,b). Applications to so-called nonlinear Markov chains and models of social power in influence networks motivate the study of ℓ_1-contractivity results by Askarzadeh et al. (2020). The relationship between induced matrix seminorms and ergodicity coefficients is established by (De Pasquale et al., 2022).

Partial contraction theory Partial contraction theory was introduced by Slotine (2003), Wang and Slotine (2005), and Slotine and Wang (2005). It was later elaborated by Pham and Slotine (2007). Chung and Slotine (2009) and Seo et al. (2010) apply partial contraction theory to the study of concurrent synchronization of Lagrangian systems and to central pattern generators in robotic locomotion. Russo and Slotine (2010) apply partial contraction theory to the study of quorum-sensing networks. The methodology is surveyed in (Di Bernardo et al., 2016).

The treatment of semicontracting systems and infinitesimal invariance by Jafarpour et al. (2022) are related to the notions of transverse contraction given by Manchester and Slotine (2014) and horizontal contraction on Finsler manifolds given by Forni and Sepulchre (2014) and elaborated by Wu (2022).

Synchronization of diffusively-coupled dynamics Diffusively-coupled dynamics appear in different areas of science and engineering. Examples include

(i) *chemical reaction-diffusion* in biological tissues and the process of morphogenesis in developmental biology (Turing, 1952),
(ii) variants of the well-known *Goodwin model* for oscillating autoregulated genes in cellular systems (Goodwin, 1965),
(iii) the well-known *FitzHugh–Nagumo model* describing neuronal interactions in the brain (FitzHugh, 1961), and
(iv) *cellular neural networks (CNNs)* for real-time large-scale signal processing in parallel computing (Chua and Yang, 1988).

Synchronization is arguably one of the most important collective behaviors in networks of diffusively-coupled dynamics. Finding sharp conditions that ensure asymptotic synchronization of diffusively-coupled dynamics is important for detecting stable pattern formations in morphogenesis, analyzing oscillatory behaviors in the Goodwin model of cellular systems, and preventing disorders such as Parkinson's disease in FitzHugh–Nagumo model of neurons.

Synchronization of diffusively-coupled dynamics has been extensively studied using contraction theory, e.g., see (Wu and Chua, 1995; Lu and Chen, 2006; Wang and Slotine, 2005; Scardovi and Sepulchre, 2009; DeLellis et al., 2011; Aminzare and Sontag, 2014a). Indeed the notion of partial contraction was developed by Wang and Slotine (2005) precisely to study this class of problems. Section 5.6 follows the treatment in (Jafarpour et al., 2022).

5.9 Exercises

E5.1 **Properties of the seminorm.** Given the definition of induced matrix seminorm and matrix log seminorm, prove the properties in equations (5.13) and (5.14) in Lemma 5.5.

Answer: The homogeneity property (5.13a) follows from the homogeneity of the seminorm. Regarding statement (5.13b), we compute

$$\begin{aligned}|||A+B||| &= \max\{|||(A+B)v|||\ :\ |||v|||=1,\ v\perp\mathcal{K}\} \leq \max\{|||Av|||+|||Bv|||\ :\ |||v|||=1,\ v\perp\mathcal{K}\}\\ &\leq \max\{|||Av|||\ :\ |||v|||=1,\ v\perp\mathcal{K}\}+\max\{|||Bv|||\ :\ |||v|||=1,\ v\perp\mathcal{K}\} = |||A|||+|||B|||,\end{aligned}$$

where we used the property $\max_{x\in\mathcal{X}}(f(x)+g(x)) \leq \max_{x\in\mathcal{X}} f(x) + \max_{x\in\mathcal{X}} g(x)$.

Regarding statement (5.13c), let $\Pi_{\|}$ and $\Pi_{\perp}$ denote the orthogonal projections onto $\mathcal{K}$ and $\mathcal{K}^{\perp}$, respectively, recall $|||v||| = |||\Pi_{\perp} v|||$ for all v, and compute

$$\begin{aligned}\max_{|||v|||\leq 1} |||Av||| &= \max_{|||v|||\leq 1} |||A(\Pi_{\|}v+\Pi_{\perp}v)||| \overset{A\mathcal{K}\subseteq\mathcal{K}}{=} \max_{|||v|||\leq 1} |||A\Pi_{\perp}v|||\\ &= \max_{|||\Pi_{\perp}v|||\leq 1} |||A\Pi_{\perp}v||| = \max_{|||w|||\leq 1, w\perp\mathcal{K}} |||Aw||| = |||A|||.\end{aligned}$$

Regarding (5.13d), if $|||B|||=0$, then B maps elements of $\mathcal{K}^{\perp}$ into elements of $\mathcal{K}$ and, since $\mathcal{K}$ is A-invariant, then $|||AB|||=0$ and the claim is satisfied. Next, assume $|||B|||>0$ and let v^* with $|||v^*|||=1$ and $v^*\perp\mathcal{K}$ satisfy

$$|||AB||| = \max\{|||(AB)v|||\ :\ |||v|||=1,\ v\perp\mathcal{K}\} = |||ABv^*|||.$$

Since $|||B|||>0$, we claim that it is possible to choose v^* satisfying also $\Pi_{\perp}Bv^*\neq 0$. For, if $Bv^*\in\mathcal{K}$, then we know $|||ABv^*|||=0$ which is in general a lower bound on $|||AB|||$. We now compute

$$\begin{aligned}|||AB||| = |||ABv^*||| &= |||A\Pi_{\perp}Bv^* + A\Pi_{\|}Bv^*||| \overset{A\mathcal{K}\subseteq\mathcal{K}}{=} |||A\Pi_{\perp}Bv^*|||\\ &\overset{\text{(homogeneity)}}{=} |||\Pi_{\perp}Bv^*|||\,|||A\frac{\Pi_{\perp}Bv^*}{|||\Pi_{\perp}Bv^*|||}|||.\end{aligned}$$

The claim now follows from noting that $|||\Pi_{\perp}Bv^*||| = |||Bv^*||| \leq |||B|||$ since $|||v^*|||=1$ and $v^*\perp\mathcal{K}$ and $|||A\frac{\Pi_{\perp}Bv^*}{|||\Pi_{\perp}Bv^*|||}||| \leq |||A|||$ since the vector $\frac{\Pi_{\perp}Bv^*}{|||\Pi_{\perp}Bv^*|||}$ has unit seminorm and is perpendicular to $\mathcal{K}$.

Regarding statement (5.13e), decompose $v = v_{\|}+v_{\perp}$ with $v_{\|}\in\mathcal{K}$ and $v_{\perp}\in\mathcal{K}^{\perp}$. If $v_{\perp}=0$, then the statement is immediate. Since $\mathcal{K}$ is invariant under A, we compute

$$|||Av||| \overset{(5.4)}{=} |||Av_{\perp}||| = |||A(v_{\perp}/|||v_{\perp}|||)|||\,|||v_{\perp}||| \overset{(5.11)}{\leq} |||A|||\,|||v_{\perp}||| = |||A|||\,|||v|||.$$

Regarding statement (5.13f), let v^* with $|||v^*|||=1$ and $v^*\perp\mathcal{K}$ satisfy

$$|||BA||| = \max\{|||BAv|||\ :\ |||v|||=1,\ v\perp\mathcal{K}\} = |||BAv^*|||.$$

We claim $Av^*\perp\mathcal{K}$. Indeed, for all $w\in\mathcal{K}$, we compute $w^{\top}Av^* = (A^{\top}w)^{\top}v^* = 0$ since $A^{\top}w\in\mathcal{K}$ and $v^*\perp\mathcal{K}$. Therefore, $|||Av^*|||\neq 0$. Finally we compute

$$|||BA||| = |||BAv^*||| = |||Av^*|||\,|||B\frac{Av^*}{|||Av^*|||}||| \leq |||A|||\,|||B|||.$$

This completes the proof of all properties in equation (5.13).

We leave the properties in equation (5.14) to the reader.

E5.2 **Total variation seminorm.** Consider the seminorm $\|\cdot\|_{1,\Pi_n}:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$. Show

(i) $\|x\|_{1,\Pi_n} = \sum_{i=1}^n |x_i - x_{\text{avg}}|$, for any $x\in\mathbb{R}^n$ and $x_{\text{avg}} = \frac{1}{n}\mathbb{1}_n^{\top}x$;

(ii) $\|x\|_{1,\Pi_n} = \sum_{i=1}^n |x_i - \frac{1}{n}|$, for any $x \in \Delta_n$;
(iii) $\|x-y\|_{1,\Pi_n} = \sum_{i=1}^n |x_i - y_i|$, for any $x, y \in \Delta_n$.

Answer: The proof is immediate from definition of $\Pi_n = I_n - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$.

E5.3 **The pseudoinverse and the singular value decomposition.** Let $A \in \mathbb{R}^{m\times n}$ have rank $r \leq \min\{m,n\}$. The *pseudoinverse* (also referred to as the *Moore-Penrose pseudoinverse*) of A is the unique matrix $A^\dagger \in \mathbb{R}^{n\times m}$ satisfying:

$$\begin{gathered} AA^\dagger A = A, \qquad A^\dagger AA^\dagger = A^\dagger, \\ AA^\dagger \text{ is symmetric, and } A^\dagger A \text{ is symmetric.} \end{gathered} \tag{E5.1}$$

Show

(i) if $m = n = r$, then $A^\dagger = A^{-1}$,
(ii) if $m > n = r$, then $A^\dagger = (A^\top A)^{-1}A^\top$,
(iii) if $r = m < n$, then $A^\dagger = A^\top(AA^\top)^{-1}$.

Next, adopting a geometric viewpoint, let $\oplus$ denote the operation of *direct sum* between subspaces, $\perp$ denote the *orthogonal complement*, ker and img for kernel and image of a linear map, respectively. Regarding the matrix A as a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$, show

(iv) $\mathbb{R}^n = \ker(A)^\perp \oplus \ker(A)$ and $\mathbb{R}^m = \operatorname{img}(A) \oplus \operatorname{img}(A)^\perp$,
(v) the restricted map $A\colon \ker(A)^\perp \to \operatorname{img}(A)$ is a linear bijection, that is, an invertible linear map,
(vi) the restricted map $A^\dagger\colon \operatorname{img}(A) \to \ker(A)^\perp$ is the inverse of $A\colon \ker(A)^\perp \to \operatorname{img}(A)$ and satisfies $\ker(A^\dagger) = \operatorname{img}(A)^\perp$, and
(vii) $AA^\dagger$ is the orthogonal projection onto $\operatorname{img}(A)$ and $A^\dagger A$ is the orthogonal projection onto $\ker(A)^\perp$.

Finally, a triple (U, Σ, V) is a *singular value decomposition (SVD)* of A if

$$A = U\Sigma V^\top,$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal and, for a positive diagonal matrix $\tilde{\Sigma} \in \mathbb{R}^{r\times r}$,

$$\Sigma = \begin{bmatrix} \tilde{\Sigma} & \mathbb{0}_{r\times(n-r)} \\ \mathbb{0}_{(m-r)\times r} & \mathbb{0}_{(m-r)\times(n-r)} \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Show

(viii) $A^\dagger = V\Sigma^\dagger U^\top$, where $\Sigma^\dagger = \begin{bmatrix} \tilde{\Sigma}^{-1} & \mathbb{0}_{r\times(m-r)} \\ \mathbb{0}_{(n-r)\times r} & \mathbb{0}_{(n-r)\times(m-r)} \end{bmatrix} \in \mathbb{R}^{n\times m}$.

Answer: These are all standard statements in matrix theory, reported for example at Wikipedia: Moore–Penrose inverse.

E5.4 **NonEuclidean induced matrix seminorms.** For all $A \in \mathbb{R}^{n\times n}$, for each $\eta \in \mathbb{R}^n_{\geq 0}$,

$$\|A\|_{1,[\eta]} = \max_{j:\eta_j\neq 0}\Big\{\eta_j \sum_{i:\eta_i\neq 0} \frac{|a_{ij}|}{\eta_i}\Big\} \quad\text{and}\quad \mu_{1,[\eta]}(A) = \max_{j:\eta_j\neq 0}\Big\{a_{jj} + \eta_j \sum_{i:\eta_i\neq 0, i\neq j} \frac{|a_{ij}|}{\eta_i}\Big\}, \tag{E5.2}$$

$$\|A\|_{\infty,[\eta]} = \max_{i:\eta_i\neq 0}\Big\{\eta_i \sum_{j:\eta_j\neq 0} \frac{|a_{ij}|}{\eta_j}\Big\} \quad\text{and}\quad \mu_{\infty,[\eta]}(A) = \max_{i:\eta_i\neq 0}\Big\{a_{ii} + \eta_i \sum_{j:\eta_j\neq 0, j\neq i} \frac{|a_{ij}|}{\eta_j}\Big\}. \tag{E5.3}$$

Answer: Equations (E5.2) and (E5.3) follow from Lemma 5.6(iii) as well as the formulas for the pseudoinverse of a diagonal matrix (e.g., see Exercise E5.3(viii)) and for the nonEuclidean norms ℓ_1/ℓ_∞.

E5.5 **Some properties of doubly-stochastic matrices.** For $A \in \mathbb{R}^{n\times n}$, show that:

(i) if A is doubly-stochastic, then the matrix $A^\top A$ is doubly-stochastic and its spectrum satisfies $\operatorname{spec}(A^\top A) \subset [0,1]$,

(ii) if A is doubly-stochastic and irreducible, then $A^\top A$ has positive diagonal and does not need to be irreducible (give a counterexample), and

(iii) if A is doubly-stochastic, irreducible and with positive diagonal, then $A^\top A$ is doubly-stochastic, irreducible and with positive diagonal.

Hint: *Show that, if* $\min_i a_{ii} = a_{\min} > 0$*, then* $A^\top A \geq a_{\min} A$.

Answer: Regarding (i), one can verify that the product of doubly-stochastic matrices is doubly-stochastic, hence $A^\top A$ is doubly-stochastic. Moreover, the matrix $A^\top A$ is symmetric and so all its eigenvalues are real and belong to the interval $[-1,1]$. Finally, all eigenvalues of $A^\top A$ are non-negative because, for all $x \in \mathbb{R}^n$, $x^\top A^\top A x = \|Ax\|_2^2 \geq 0$.

Regarding (ii), for any $i \in \{1,\dots,n\}$ we compute

$$(A^\top A)_{ii} = \sum_{k=1}^n (A^\top)_{ik} a_{ki} = \sum_{k=1}^n a_{ki}^2.$$

Since A is irreducible, each column of A has at least one element distinct from zero. Moreover, even if A is irreducible, $A^\top A$ does not have to be irreducible. A counterexample is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \implies \quad A^\top A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Regarding (iii), for any $i,j \in \{1,\dots,n\}$ we compute

$$(A^\top A)_{ij} = \sum_{k=1}^n (A^\top)_{ik} a_{kj} = \sum_{k=1}^n a_{ki} a_{kj} \geq a_{ki} a_{kj}\Big|_{k=i} = a_{ii} a_{ij} \geq a_{\min} a_{ij},$$

so that $A^\top A \geq a_{\min} A$. Since $a_{\min} > 0$ by assumption, $A^\top A$ is irreducible.

E5.6 **A symmetric Laplacian matrix is positive semidefinite.** Let G be a weighted undirected graph with symmetric Laplacian matrix $L \in \mathbb{R}^{n\times n}$. Assume G is connected and let (λ_2, v) denote the Fiedler eigenpair. Let $\Pi_n = I_n - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$. Show that

(i) $L \succeq 0$,

(ii) the Fiedler eigenvector satisfies $v \perp \mathbb{1}_n$ and $v^\top L v = \lambda_2 \|v\|_2^2$,

(iii) for any $x \in \mathbb{R}^n$ with $x_{\text{avg}} = \mathbb{1}_n^\top x/n$,
$$x^\top L x \geq \lambda_2 \big\| x - x_{\text{avg}} \mathbb{1}_n \big\|_2^2,$$
with equality if x is parallel to v,

(iv) $\Pi_n L = L \Pi_n = L$ and $L \succeq \lambda_2 \Pi_n$.

Note: *Statement (iv) implies that* $-L$ *satisfies the Lyapunov stability LMI:* $-\Pi_n L - L^\top \Pi_n \preceq -2\lambda_2 \Pi_n$.

Answer: Regarding statement (i), for an undirected graph we know $L^\top = L$ and simple calculations show $x^\top L x = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x_i - x_j)^2 \geq 0$, for all $x \in \mathbb{R}^n$. Hence, L is a symmetric positive semidefinite matrix.

Regarding statement (ii), the vector v is perpendicular to $\mathbb{1}_n$ because L is symmetric and $\mathbb{1}_n$ is the dominant eigenvector of the eigenvalue 0. The equality $v^\top L v = \lambda_2 \|v\|_2^2$ follows from $Lv = \lambda_2 v$.

Regarding statement (iii), we decompose $x = (x - x_{\text{avg}}\mathbb{1}_n) + x_{\text{avg}}\mathbb{1}_n$ and know that $(x - x_{\text{avg}}\mathbb{1}_n) \perp x_{\text{avg}}\mathbb{1}_n$. Recall that $L\mathbb{1}_n = \mathbb{0}_n$ and that λ_2 is the smallest eigenvalue of the operator L restricted to the subspace $\operatorname{span}\{\mathbb{1}_n\}$. Therefore,

$$\begin{aligned} x^\top L x &= \big((x - x_{\text{avg}}\mathbb{1}_n) + x_{\text{avg}}\mathbb{1}_n\big)^\top L \big((x - x_{\text{avg}}\mathbb{1}_n) + x_{\text{avg}}\mathbb{1}_n\big) \\ &= (x - x_{\text{avg}}\mathbb{1}_n)^\top L (x - x_{\text{avg}}\mathbb{1}_n) \\ &\geq \lambda_2 \big\| x - x_{\text{avg}}\mathbb{1}_n \big\|_2^2. \end{aligned}$$

Regarding statement (iv), statement (iii) and simple calculations show that, for all $x \in \mathbb{R}^n$,

$$x^\top L x \geq \lambda_2 \big\|x - x_{\text{avg}}\mathbb{1}_n\big\|_2^2 = \lambda_2 \big\|\Pi_n x\big\|_2^2 = \lambda_2 x^\top \Pi_n x.$$

E5.7 **The Laplacian matrix of a weight-balanced digraph.** Let G be a weighted digraph with Laplacian matrix L. Prove the following statements are equivalent:

(i) G is weight-balanced,

(ii) $L + L^\top$ is the Laplacian matrix of the undirected digraph associated to the adjacency matrix $A + A^\top$.

Moreover, with the notation in Exercise E5.6, show that

(iii) if G is weight-balanced, then L satisfies the Lyapunov LMI $\Pi_n L + L^\top \Pi_n \succeq \lambda_2(L + L^\top)\Pi_n$, and

(iv) if additionally G is weakly connected, then $\lambda_2(L + L^\top) > 0$.

Answer: Let D_{out} and D_{in} be the out-degree and in-degree matrices of G and recall that G is weight-balanced if $D_{\text{out}} = D_{\text{in}}$.

Regarding the implication (i) $\implies$ (ii), let $\bar{G}$ be the weighted undirected digraph with adjacency matrix $\bar{A} = A + A^\top$ and Laplacian matrix $\bar{L}$. Note that

$$\begin{aligned}
\bar{D}_{\text{out}} &= \operatorname{diag}(\bar{A}\mathbb{1}_n) = \operatorname{diag}(A\mathbb{1}_n) + \operatorname{diag}(A^\top\mathbb{1}_n) = D_{\text{out}} + D_{\text{in}} = 2D_{\text{out}},\\
\bar{L} &= \bar{D}_{\text{out}} - \bar{A} = 2D_{\text{out}} - (A + A^\top) = (D_{\text{out}} - A) + (D_{\text{out}} - A^\top) = L + L^\top.
\end{aligned}$$

This proves the implication (i) $\implies$ (ii).

Regarding implication (ii) $\implies$ (i), define $\bar{L} = L + L^\top$ and compute:

$$\begin{aligned}
\mathbb{1}_n^\top \bar{L}\mathbb{1}_n &= \mathbb{1}_n^\top (L + L^\top)\mathbb{1}_n = \mathbb{1}_n^\top L\mathbb{1}_n + \mathbb{1}_n^\top L^\top \mathbb{1}_n = \mathbb{0}_n,\\
\bar{L}\mathbb{1}_n &= L^\top \mathbb{1}_n = (D_{\text{out}} - A^\top)\mathbb{1}_n = A\mathbb{1}_n - A^\top\mathbb{1}_n.
\end{aligned}$$

Note that $\bar{L}\mathbb{1}_n = \mathbb{0}_n$ if and only if $A\mathbb{1}_n = A^\top\mathbb{1}_n$, that is, $D_{\text{out}} = D_{\text{in}}$.

Recall that, for any positive semidefinite matrix $\Delta = \Delta^\top \geq 0$, we know that $v^\top \Delta v = 0$ implies $\Delta v = \mathbb{0}_n$. Indeed, by contradiction, assume $\Delta v \neq \mathbb{0}_n$. Write $\Delta = U\Lambda U^\top$, define $\Gamma = U\Lambda^{1/2}U^\top \geq 0$ and note $\Delta = \Gamma^2$. Then $\Delta v \neq \mathbb{0}_n$ implies $\Gamma v \neq \mathbb{0}_n$, which implies $v^\top \Gamma\Gamma v \neq 0$, which is a contradiction.

In summary, we know that $\mathbb{1}_n^\top \bar{L}\mathbb{1}_n = 0$ implies that $\bar{L}\mathbb{1}_n = \mathbb{0}_n$ and, in turn, that G is weight-balanced. This concludes the proof of (ii) $\implies$ (i).

Finally, assuming G is weight-balanced, Exercise E5.6 implies

$$L + L^\top \succeq \lambda_2(L + L^\top)\Pi_n.$$

Additionally, one can see that $L\mathbb{1}_n = \mathbb{0}_n$ implies $L\Pi_n = L$, $\Pi_n L^\top = L^\top$, and, in turn, the Lyapunov inequality in statement (iii). Regarding statement (iv), recall from (Bullo, 2022, E3.6) that a weight-balanced and weakly connected digraph is strongly connected. Therefore, the undirected digraph associated to the adjacency matrix $A + A^\top$ is connected and $\lambda_2(L + L^\top) > 0$.

E5.8 **On the differential and integral notions of infinitesimal invariance.** Given a continuously-differentiable map $f \colon \mathbb{R}^n \to \mathbb{R}^n$, scalar $\gamma \in \mathbb{R}$, and vector $\eta \in \mathbb{R}^n$, show

$$\eta^\top f(x) = \eta^\top f(y) + \gamma\eta^\top(x - y) \quad \text{for all } x, y \in \mathbb{R}^n \qquad \iff \qquad \eta^\top Df(x) = \gamma\eta^\top \quad \text{for all } x \in \mathbb{R}^n; \tag{E5.4}$$

$$f(x + \beta\eta) = f(x) + \gamma\beta\eta \quad \text{for all } x \in \mathbb{R}^n, \beta \in [0,1] \qquad \iff \qquad Df(x)\eta = \gamma\eta \quad \text{for all } x \in \mathbb{R}^n. \tag{E5.5}$$

Additionally, for any subspace $\mathcal{K} \subset \mathbb{R}^n$,

$$f(x + \kappa) - f(x) \in \mathcal{K} \quad \text{for all } x \in \mathbb{R}^n, \kappa \in \mathcal{K} \qquad \iff \qquad Df(x)\mathcal{K} \subseteq \mathcal{K} \quad \text{for all } x \in \mathbb{R}^n. \tag{E5.6}$$

Note: *The equivalences (E5.4) and (E5.5) are similar to those obtained for monotone systems in Lemma 4.17. The differences are that (i) these are equality constraints, (ii) the vector field is not required to be monotone, and (iii) the vector η is not required to be positive.*

Answer: We show equation (E5.5); the proof of (E5.4) is very similar and left to the reader. To show the implication $\Longrightarrow$, we compute

$$Df(x)\eta = \lim_{h\to 0} \frac{f(x+h\eta)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)+h\gamma\eta - f(x)}{h} = \gamma\eta.$$

Vice versa regarding the implication $\Longleftarrow$, recall that the Mean Value Theorem for vector-valued functions in Exercise E3.8 states $f(y)-f(x) = \big(\int_0^1 Df\big(x+s(y-x)\big)ds\big)(y-x)$. For $y=x+\eta$, we compute

$$f(x+\eta)-f(x) = \Big(\int_0^1 Df\big(x+s\eta\big)ds\Big)\eta = \Big(\int_0^1 ds\Big)\gamma\eta.$$

The proof of equation (E5.6) generalizes these calculations as follows. Regarding the implication $\Longrightarrow$, we write

$$Df(x)\kappa = \lim_{h\to 0} \frac{f(x+h\kappa)-f(x)}{h} \in \mathcal{K}. \tag{E5.7}$$

and for the converse implication $\Longleftarrow$, at $y=x+\kappa$

$$f(x+\kappa)-f(x) = \Big(\int_0^1 Df\big(x+s\kappa\big)ds\Big)\kappa \in \mathcal{K}. \tag{E5.8}$$

E5.9 **Cascade decomposition of vector fields leaving the consensus space invariant.** Given a continuously-differentiable map $f\colon \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^n$ with the associated differential equation $\dot{x} = f(t,x)$,

(i) if $Df(t,x)\mathbb{1}_n \subseteq \operatorname{span}\{\mathbb{1}_n\}$ for all t, x, then

$$\begin{aligned} \dot{x}_\perp &= \Pi_\perp f(t,x_\perp) && \in \mathbb{1}_m^\perp, && \text{(E5.9)}\\ \dot{x}_{\text{avg}} &= f_{\text{avg}}(t, x_\perp + x_{\text{avg}}\mathbb{1}_n) && \in \mathbb{R}, && \text{(E5.10)} \end{aligned}$$

where $x_{\text{avg}} = \frac{1}{n}\mathbb{1}_n^\top x$ and $f_{\text{avg}}(t,x) = \frac{1}{n}\mathbb{1}_n^\top f(t,x)$;

(ii) if there exists[5] $\gamma\in\mathbb{R}$ such that $Df(t,x)\mathbb{1}_n = \gamma\mathbb{1}_n$ for all t,x, then

$$\begin{aligned} \dot{x}_\perp &= \Pi_\perp f(t,x_\perp) && \in \mathbb{1}_m^\perp, && \text{(E5.11)}\\ \dot{x}_{\text{avg}} &= \gamma x_{\text{avg}} + f_{\text{avg}}(t, x_\perp) && \in \mathbb{R}; && \text{(E5.12)} \end{aligned}$$

(iii) if $\dot{x} = -L(t)x + u(t)$, where $L(t)$ is Laplacian (so that $L(t)\mathbb{1}_n = 0$),

$$\begin{aligned} \dot{x}_\perp &= -\Pi_\perp L(t)x_\perp + u(t) - u_{\text{avg}}(t)\mathbb{1}_n && \in \mathbb{1}_m^\perp, && \text{(E5.13)}\\ \dot{x}_{\text{avg}} &= -\tfrac{1}{n}\mathbb{1}_n^\top L(t)x_\perp + u_{\text{avg}}(t) && \in \mathbb{R}. && \text{(E5.14)} \end{aligned}$$

Answer: Statement (i) is a consequence of Lemma 5.10.
Statement (ii) follows from Exercise E5.8.
Statement (iii) is a special case of statement (ii).

[5]As in Exercise E5.8, this is equivalent to $f(t,x+\beta\mathbb{1}_n) = f(x)+\gamma\beta\mathbb{1}_n$ for all t,x,β

E5.10 **Coppel's inequality for time-varying matrices.** Let $|||\cdot|||$ be a semi-norm on $\mathbb{R}^n$ with kernel $\mathcal{K}$. Let $\mu_{|||\cdot|||}$ denote the associated matrix log seminorm. Consider a time-dependent continuous matrix $t \mapsto A(t) \in \mathbb{R}^{n\times n}$ such that $A(t)\mathcal{K} \subseteq \mathcal{K}$ for all $t \geq t_0$.

Show that, for every $t \geq t_0$, the solution to $\dot{x}(t) = A(t)x(t)$ satisfies

$$|||x(t)||| \leq \exp\left(\int_{t_0}^{t} \mu_{|||\cdot|||}(A(\tau))d\tau\right) |||x(0)|||.$$

Moreover, if A is time-invariant, we have

$$|||x(t)||| \leq \mathrm{e}^{t\mu_{|||\cdot|||}(A)}\, |||x(0)|||.$$

Answer: Because $\mathcal{K}$ is $A(t)$-invariant for all t, we know the simplified definition (5.13c) of induced matrix seminorm holds. Based upon the simplified definition the proof of statement is entirely identical to the proof of the standard Coppel's inequality in Section 2.1.

E5.11 **Perturbation bounds on the left dominant eigenvector.** Let A and $A + \Delta$ be row-stochastic and irreducible; let $v(A)$ and $v(A + \Delta)$ denote their left dominant eigenvectors, normalized to have unit sum. If A is scrambling so that $\tau_1(A) < 1$, then

$$\|v(A) - v(A + \Delta)\|_1 \leq \frac{\|\Delta\|_\infty}{1 - \tau_1(A)}. \tag{E5.15}$$

Note: *This result is originally by Seneta (1988). A comparison of various perturbation bounds, also referred to sensitivity bounds or condition numbers, is given by Cho and Meyer (2001).*

Answer: Here is a sketch of the answer, following the proof of (Ipsen and Selee, 2011, Theorem 3.14): (1) write a formula for the difference and expand; (2) define the deflated matrix A_{def}, then compute the inverse of $I_n - A_{\text{def}}$ (this step is closely related to computing the fundamental matrix of the Markov chain); (3) write the Neumann series for the inverse; (4) upper bound each term of the series using the induced seminorm property of τ_1; (5) sum the series to prove that it equals $1/(1 - \tau_1)$.

E5.12 **A useful fact about positive semidefinite matrices.** Let P and $Q \succeq 0$ be symmetric positive semidefinite with $\ker(P) \subseteq \ker(Q)$. The following statements are equivalent:

(i) $P \preceq Q$,
(ii) $P \preceq Q + S$, for all $S \succeq 0$.

Answer: We leave the proof to the interested reader.

Part III

Appendices

Appendix **A**

Appendix: Stability Theory for Dynamical Systems

In this chapter we provide a brief self-contained review of stability theory for nonlinear dynamical systems. We review the key ideas and theorems in stability theory, including the Lyapunov Stability Criteria and the Krasovskiĭ-LaSalle Invariance Principle. We then apply these theoretical tools to a number of example systems, including linear and linearized systems, negative gradient systems, continuous-time averaging dynamics (i.e., the Laplacian flow) and positive linear systems described by Metzler matrices.

This chapter is not meant to provide a comprehensive treatment, e.g., we leave out matters of existence and uniqueness of solutions and we do not include proofs. Section A.8 below provides numerous references for further reading. We start the chapter by introducing a running example with three prototypical dynamical systems.

Example A.1 (Gradient and mechanical systems). We start by introducing a differentiable function $V : \mathbb{R} \to \mathbb{R}$; for example see Figure A.1. Based on V and on two positive coefficients m and d, we define three instructive and

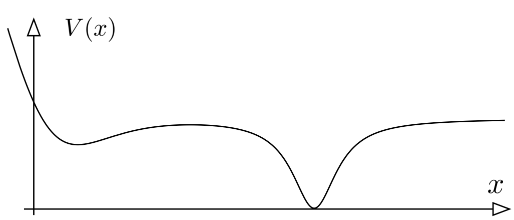


Figure A.1: A differentiable function V playing the role of a potential energy function (i.e., a function describing the potential energy stored) in a negative gradient system, a conservative mechanical systems or a dissipative mechanical systems. Specifically, $V(x) = -x\,\mathrm{e}^{-x}/(1+\mathrm{e}^{-x}) + (x-10)^2/(1+(x-10)^2)$.

prototypical dynamical systems:

$$\textit{negative gradient system:} \qquad \dot{x} = -\frac{\partial V}{\partial x}(x), \tag{A.1}$$

$$\textit{conservative mechanical system:} \qquad m\ddot{x} = -\frac{\partial V}{\partial x}(x), \tag{A.2}$$

$$\textit{dissipative mechanical system:} \qquad m\ddot{x} = -\frac{\partial V}{\partial x}(x) - d\dot{x}. \tag{A.3}$$

In the study of physical systems, the parameter m is an inertia, d is a damping coefficient, and the function V is the potential energy function, describing the potential energy stored in the system.

These example are also know as a (first order, second order, or second order dissipative) particle on an energy landscape, or the "rolling ball on a hill" examples. According to Newton's law, the correct physical systems are models (A.2) and (A.3), but we will also see interesting examples of first-order negative gradient systems (A.1). •

A.1 On sets and functions

Before proceeding we review some basic general properties of sets and functions. First, we recall that a set $W \subset \mathbb{R}^n$ is *bounded* if there exists a constant K such that each $w \in W$ satisfies $\|w\| \leq K$, *closed* if it contains its boundary (or, equivalently, if it contains all its limit points), and *compact* if it is bounded and closed.

Second, given a differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$, a *critical point* of V is a point $x^* \in \mathbb{R}^n$ satisfying

$$\frac{\partial V}{\partial x}(x^*) = \mathbb{0}_n.$$

A critical point x^* is a *local minimum point* (resp. *local strict minimum point*) of V if there exists a distance $\varepsilon > 0$ such that $V(x^*) \leq V(x)$ (resp. $V(x^*) < V(x)$) for all $x \neq x^*$ within distance ε of x^*. The point x^* is a global minimum if $V(x^*) < V(x)$ for all $x \neq x^*$. Local and global maximum points are defined similarly.

Given a constant $\ell \in \mathbb{R}$, we define the *ℓ-level set* of V and the *ℓ-sublevel set* of V by

$$V^{-1}(\ell) = \{y \in \mathbb{R}^n \,:\, V(y) = \ell\}, \quad \text{and} \quad V_{\leq}^{-1}(\ell) = \{y \in \mathbb{R}^n \,:\, V(y) \leq \ell\}.$$

These notions are illustrated in Figure A.2.

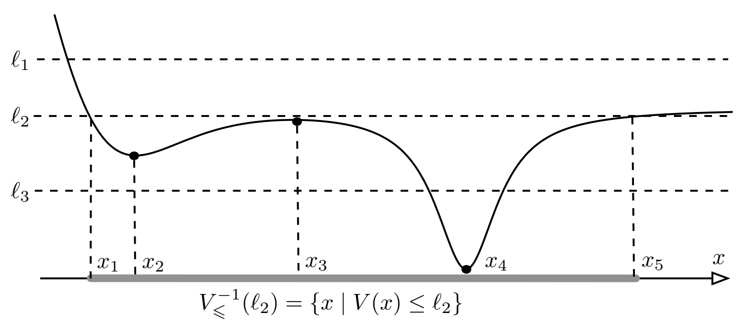


Figure A.2: A differentiable function, its sublevel set and its critical points. The sublevel set $V_{\leq}^{-1}(\ell_1) = \{x : V(x) \leq \ell_1\}$ is unbounded. The sublevel set $V_{\leq}^{-1}(\ell_2) = [x_1, x_5]$ is compact and contains three critical points (x_2 and x_4 are local minima and x_3 is a local maximum). Finally, the sublevel set $V_{\leq}^{-1}(\ell_3)$ is compact and contains a single critical point, the global minimum x_4.

Third, given a point $x_0 \in \mathbb{R}^n$, a function $V\colon \mathbb{R}^n \to \mathbb{R}$ is

(i) *locally positive-definite* (resp. *positive-semidefinite*) *about* x_0 if $V(x_0) = 0$ and if there exists a neighborhood U of x_0 such that $V(x) > 0$ (resp. $V(x) \geq 0$) for all $x \in U \setminus \{x_0\}$,

(ii) *globally positive-definite about* x_0 if $V(x_0) = 0$ and $V(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{x_0\}$, and

(iii) *locally* (resp. *globally*) *negative-definite* if $-V$ is *locally* (resp. *globally*) *positive-definite*; and *negative-semidefinite* if $-V$ is *positive-semidefinite*.

Note: Assume a differentiable V is locally positive-definite about x_0. Pick $\alpha > V(x_0)$. One can show that the sublevel set $V_{\leq}^{-1}(\alpha)$ contains a neighborhood of x_0. Indeed, in Figure A.2, V is locally positive-definite about x_4 and $V_{\leq}^{-1}(\ell_2)$ and $V_{\leq}^{-1}(\ell_3)$ are both compact intervals containing x_4.

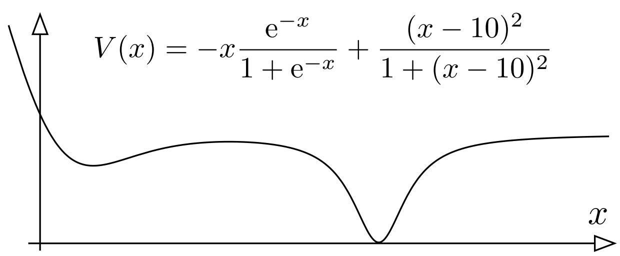
Fourth and finally, a non-negative continuous function $V\colon X \to \mathbb{R}_{\geq 0}$ is

(i) *radially unbounded* if $X = \mathbb{R}^n$ and $V(x) \to \infty$ along any trajectory such that $\|x\| \to \infty$, that is, any sequence $\{x_n\}_{n \in \mathbb{N}}$ with the property that $\lim_{n\to\infty} \|x_n\| = \infty$ satisfies $\lim_{n\to\infty} V(x_n) = \infty$, and

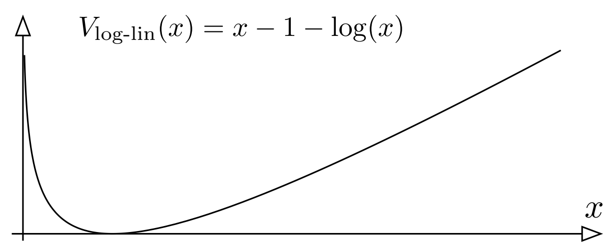
(ii) *proper* if, for all $\ell \in \mathbb{R}$, the ℓ-sublevel set of V is compact.

We illustrate these concepts in Figure A.3 and state a useful equivalence without proof.

Lemma A.2. *A continuous function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is proper if and only if it is radially unbounded.*



(a) This function $V: \mathbb{R} \to \mathbb{R}$ is not radially unbounded because $\lim_{x\to+\infty} V(x) = 1$.



(b) The function $V_{\text{log-lin}}\mathbb{R}_{>0} \to \mathbb{R}$ is proper on $X = \mathbb{R}_{>0}$ since each sublevel set is a compact interval.

Figure A.3: Example of proper and not proper functions

A.2 Dynamical systems and stability notions

Dynamical systems

A *(continuous-time) dynamical system* is a pair (X, f) where X, called the *state space*, is a subset of $\mathbb{R}^n$ and f, called the *vector field*, is a map from X to $\mathbb{R}^n$. Given an initial state $x_0 \in X$, the *solution* (also called *trajectory* or *evolution*) of the dynamical system is a curve $t \mapsto x(t) \in X$ satisfying the differential equation

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0.$$

A dynamical system (X, f) is *linear* if $x \mapsto f(x) = Ax$ for some square matrix A.

Typically, the map f is assumed to have some continuity properties so that the solution exists and is unique for at least small times. Moreover, some of our examples are defined on closed submanifolds of $\mathbb{R}^n$ (e.g., the Lotka-Volterra model (4.21) is defined over the positive orthant $\mathbb{R}^n_{\geq 0}$), and Kuramoto coupled oscillator are defined over the set of n angles) and additional assumptions are required to ensure that the solution exists for all times in X. We do not discuss these topics in great detail here, we simply assume the systems admit solutions inside X for all time, and refer to the references in Section A.8 below.

Equilibrium points and their stability

An *equilibrium point* for the dynamical systems (X, f) is a point $x^* \in X$ such that $f(x^*) = \mathbb{0}_n$. If the initial state is $x(0) = x^*$, then the solution exists unique for all time and is constant: $x(t) = x^*$ for all $t \in \mathbb{R}_{\geq 0}$.

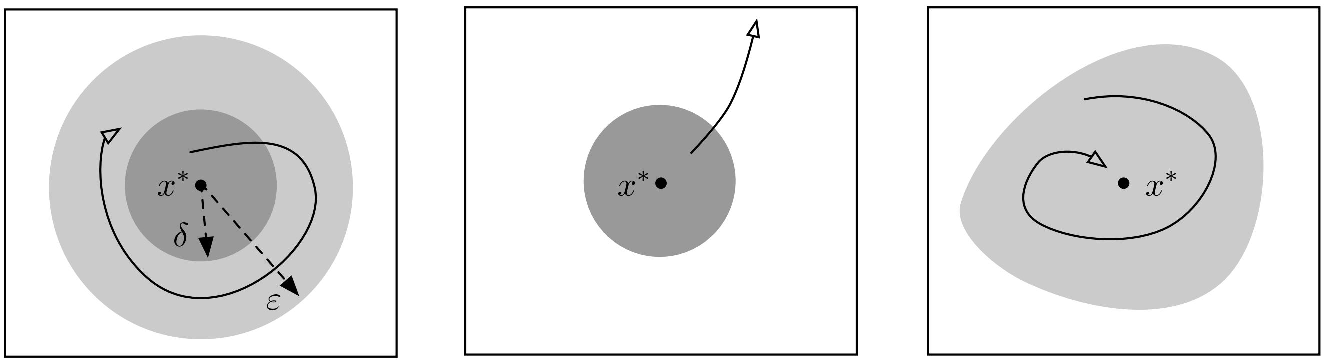
An equilibrium point x^* for the dynamical system (X, f) is

(i) *stable* (or *Lyapunov stable*) if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ so that if $\|x(0) - x^*\| < \delta$, then $\|x(t) - x^*\| < \varepsilon$ for all $t \geq 0$,

(ii) *unstable* if it is not stable, and

(iii) *locally asymptotically stable* if it is stable and if there exists $\delta > 0$ so that $\lim_{t\to\infty} x(t) = x^*$ for all trajectories satisfying $\|x(0) - x^*\| < \delta$.

These three concepts are illustrated in Figure A.4.



(a) Stable equilibrium: for all ε, each solution starting inside a sufficiently small δ-disk remains inside the ε-disk.

(b) Unstable equilibrium: no matter how small δ is, at least one solution starting inside the δ-disk diverges.

(c) Asymptotically stable equilibrium: solutions starting in a sufficiently small δ-disk converge asymptotically to the equilibrium.

Figure A.4: Illustrations of a stable, an unstable and an asymptotically stable equilibrium.

These first three notions are local in nature. To characterize global properties of a dynamical system (X, f), we introduce the following notions. Given a locally asymptotically stable equilibrium point x^*,

(i) the set of initial conditions $x_0 \in X$ whose corresponding solution $x(t)$ converges to x^* is called the *region of attraction* of x^*,

(ii) x^* is said to be *globally asymptotically stable* if its region of attraction is the whole space X, and

(iii) x^* is said to be *globally* (respectively, *locally*) *exponentially stable* if it is globally (respectively, locally) asymptotically stable and there exist positive constants c_1 and c_2 such that all trajectories starting in the region of attraction satisfy

$$\|x(t) - x^*\| \leq c_1 \|x(0) - x^*\| \, \mathrm{e}^{-c_2 t}.$$

Example A.3 (Gradient and mechanical systems: Example A.1 continued). It is instructive to report some numerical simulations of the three dynamical systems and state some conjectures about their equilibria and stability properties. These conjectures will be established in the next section.

•

A.3 The Lyapunov Stability Criteria

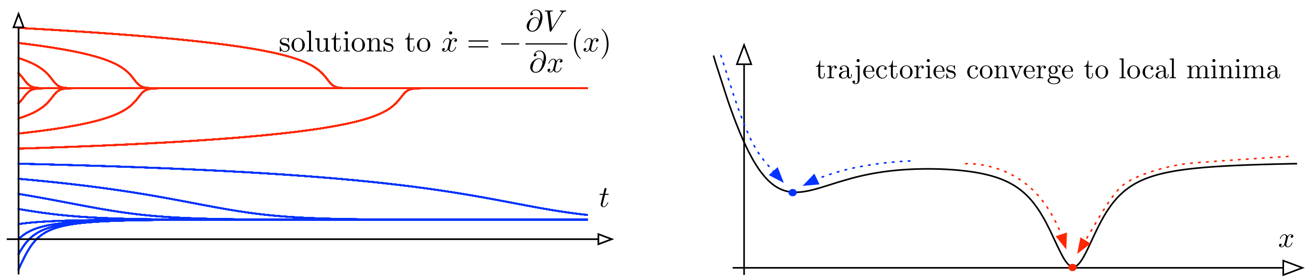
We are now ready to provide a critical tool in the study of the stability and convergence properties of a dynamical system. Roughly speaking, Lyapunov's idea is to use the concept of an energy function with a local/global minimum that is non-increasing along the system's solution.

Before proceeding, we require one final useful notion. The *Lie derivative* (also called the *directional derivative*) of a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is the function $\mathscr{L}_f V : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathscr{L}_f V(x) = \frac{\partial V}{\partial x}(x) f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x) f_i(x). \tag{A.4}$$

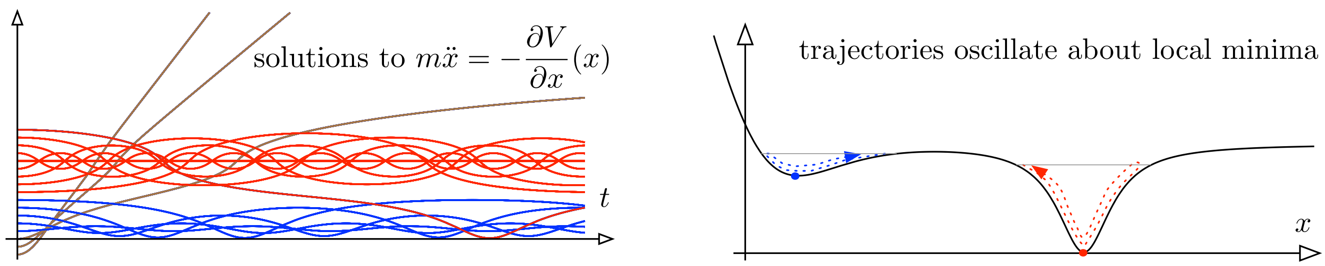
Along the flow of a dynamical system (X, f), we have

$$\frac{d}{dt} V(x(t)) = \dot{V}(x(t)) = \mathscr{L}_f V(x(t)). \tag{A.5}$$



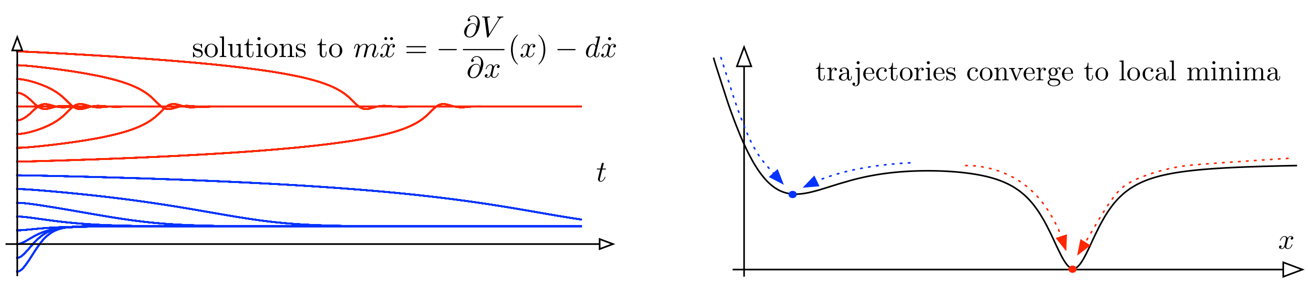
(a) Conjecture: each solution converges to one of the two local minima.

(b) Sketch of the motion on the potential energy surface.



(c) Conjecture: each solution oscillates around a local minimum or diverge.

(d) Sketch of the motion on the potential energy surface.



(e) Conjecture: each solution converges to one of the two local minima.

(f) Sketch of the motion on the potential energy surface.

Figure A.5: Numerically computed solutions (left) and graphical visualization of the solutions (right) for the three example systems with potential energy function V. Parameters are $x(0) \in \{-2, -1, \dots, 14\}$ and $m = d = 1$.

With this notation we note that $V : \mathbb{R}^n \to \mathbb{R}$ is non-increasing along every trajectory $x : \mathbb{R}_{\geq 0} \to X$ of (X, f) if

$$\dot{V}(x(t)) = \mathscr{L}_f V(x(t)) \leq 0,$$

or, equivalently, if each point $x \in X$ satisfies $\mathscr{L}_f V(x) \leq 0$. Because of this last inequality, when the vector field f is clear from the context, it is customary to adopt a slight abuse of notation and write $\dot{V}(x) = \mathscr{L}_f V(x)$.

We are now ready to present the main result of this section.

Theorem A.4 (Lyapunov Stability Criteria). *Consider a dynamical system $(\mathbb{R}^n, f)$ with differentiable vector field f and with an equilibrium point $x^* \in \mathbb{R}^n$. The equilibrium point x^* is*

stable *if there exists a continuously-differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, called a weak Lyapunov function, satisfying*

(L1) *V is locally positive-definite about x^*,* (L2) $\mathscr{L}_f V$ *is locally negative-semidefinite about* x^**;*

locally asymptotically stable *if there exists a continuously-differentiable function* $V : \mathbb{R}^n \to \mathbb{R}$*, called a local Lyapunov function, satisfying Assumption (L1) and*

(L3) $\mathscr{L}_f V$ *is locally negative-definite about* x^**;*

globally asymptotically stable *if there exists a continuously-differentiable function* $V : \mathbb{R}^n \to \mathbb{R}$*, called a global Lyapunov function, satisfying*

(L4) V *is globally positive-definite about* x^**,*

(L5) $\mathscr{L}_f V$ *is globally negative-definite about* x^**,*

(L6) V *is proper.*

Note the immediate implications: (L4) $\Longrightarrow$ (L1) and (L5) $\Longrightarrow$ (L3) $\Longrightarrow$ (L2).

Note: Theorem A.4 assumes the existence of a Lyapunov function with certain properties, but does not provide constructive methods to design or compute one. In what follows we will see that Lyapunov functions can be designed for certain classes of systems. But, in general, the design of Lyapunov function is challenging. A common procedure is based on trial-and-error: one selects a so-called *candidate Lyapunov function* and verifies which, if any, of the properties (L1)–(L6) is satisfied.

Example A.5 (Gradient and mechanical systems: Example A.3 continued). We now apply the Lyapunov Stability Criteria in Theorem A.4 to the example dynamical systems in Example A.1. Based on the properties of the function V in Figure A.2 with local minimum points x_2 and x_4, we establish most of the conjectures from Example A.3. Note that the vector fields and the Lyapunov functions we adopt in what follows are all continuously differentiable.

Negative gradient systems: For the dynamics $\dot{x} = -\partial V/\partial x$, we select the function $V(x) - V(x_2)$ as candidate Lyapunov function about x_2. We compute

$$\dot{V}(x) = -\|\partial V/\partial x\|^2 \leq 0.$$

Note that $V - V(x_2)$ is locally positive definite about x_2 (Assumption (L1)) and $\dot{V}$ is locally negative definite about x_2 (Assumption (L3)); hence $V - V(x_2)$ is a local Lyapunov function for the equilibrium point x_2. An identical argument applies to x_4. Hence, both local minima x_2 and x_4 are locally asymptotically stable;

Conservative and dissipative mechanical systems: Given an inertia coefficient $m > 0$ and a damping coefficient $d \geq 0$, we write the conservative and the dissipative mechanical systems in first order form as:

$$\dot{x} = v, \quad m\dot{v} = -dv - \frac{\partial V}{\partial x}(x),$$

where $(x, v) \in \mathbb{R}^2$ are the position and velocity coordinates. As candidate Lyapunov function about the equilibrium point $(x_2, 0)$, we consider the *mechanical energy* $\mathrm{E} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by the sum of kinetic and potential energy:

$$\mathrm{E}(x, v) = \frac{1}{2}mv^2 + V(x).$$

We compute its derivative along trajectories of the considered mechanical system as follows:

$$\dot{\mathrm{E}}(x, v) = mv\dot{v} + \frac{\partial V}{\partial x}(x)\dot{x} = v\Big(- dv - \frac{\partial V}{\partial x}(x)\Big) + \frac{\partial V}{\partial x}(x)v = -dv^2 \leq 0\,.$$

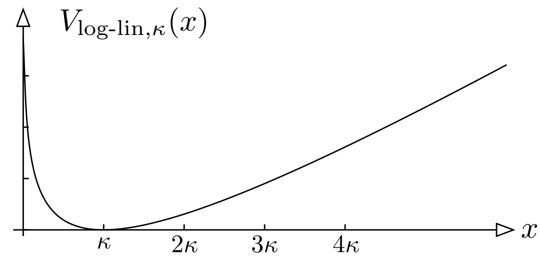


Figure A.6: The function $V_{\text{log-lin},\kappa}(x) = x - \kappa - \kappa \log(x/\kappa)$, with $\kappa > 0$.

This calculation, and x_2 being a local minimum of V, together establish that, for $d \geq 0$, the function $\mathrm{E} - V(x_2)$ is locally positive definite about x_2 (Assumption (L1)) and $\dot{\mathrm{E}}$ is locally negative semidefinite about $(x_2, 0)$ (Assumption (L2)). Hence, the function $\mathrm{E} - V(x_2)$ is a weak Lyapunov function for the equilibrium point $(x_2, 0)$ and, therefore, the point $(x_2, 0)$ is stable for both the conservative and the dissipative mechanical system. An identical argument applies to the point $(x_4, 0)$.

Note that we obtain the correct properties, i.e., consistent with the simulations in the previous exercise, for negative gradient system and for the conservative mechanical system. But more work is required to show that the local minima are locally asymptotically stable for the dissipative mechanical system. •

Example A.6 (The logistic equation). As second example, we consider the logistic equation:

$$\dot{x}(t) = rx(t)\Big(1 - \frac{x(t)}{\kappa}\Big) =: f_{\text{logistic}}(x),$$

with growth rate r and carrying capacity κ. We neglect the possible initial condition $x(0) = 0$ (with subsequent equilibrium solution $x(t) = 0$ for all $t \geq 0$) and restrict out attention to solutions in $X = \mathbb{R}_{>0}$.

For $\kappa > 0$, define the *logarithmic-linear function* $V_{\text{log-lin},\kappa} \colon \mathbb{R}_{>0} \to \mathbb{R}$, illustrated in Figure A.6, by

$$V_{\text{log-lin},\kappa}(x) = x - \kappa - \kappa \log\Big(\frac{x}{\kappa}\Big).$$

One can verify that

(i) $V_{\text{log-lin},\kappa}$ is continuously differentiable with $\frac{d}{dx} V_{\text{log-lin},\kappa}(x) = (x - \kappa)/x$,

(ii) $V_{\text{log-lin},\kappa}(x) \geq 0$ for all $x > 0$ and $V_{\text{log-lin},\kappa}(x) = 0$ if and only if $x = \kappa$, and

(iii) $\lim_{x \to 0^+} V_{\text{log-lin},\kappa}(x) = \lim_{x \to \infty} V_{\text{log-lin},\kappa}(x) = +\infty$.

Next we compute

$$\mathscr{L}_{f_{\text{logistic}}} V_{\text{log-lin},\kappa}(x) = \frac{x - \kappa}{x} \cdot rx\Big(1 - \frac{x}{\kappa}\Big) = -\frac{r}{\kappa}(x - \kappa)^2.$$

In summary, we have established that f_{logistic} is a differentiable vector field, $x^* = \kappa$ is an equilibrium point, $V_{\text{log-lin},\kappa}$ is globally positive definite about κ, $\mathscr{L}_{f_{\text{logistic}}} V_{\text{log-lin},\kappa}$ is globally negative definite about κ, and $V_{\text{log-lin},\kappa}$ is proper. Hence, $V_{\text{log-lin},\kappa}$ is a global Lyapunov function and $x^* = \kappa$ is globally asymptotically stable. (This result is consistent with the behavior characterized in Exercise E4.2.) •

A.4 The Krasovskiĭ-LaSalle Invariance Principle

While the Lyapunov Stability Criteria are very useful, it is sometimes difficult to find a Lyapunov function with a negative-definite Lie derivative. To overcome this obstacle, in this section we introduce a powerful tool for the convergence analysis, namely the Krasovskiĭ-LaSalle Invariance Principle.

Before stating the main result, we introduce two useful concepts:

(i) A curve $t \mapsto x(t)$ *approaches* a set $S \subset \mathbb{R}^n$ as $t \to +\infty$ if the distance[1] from $x(t)$ to the set S converges to 0 as $t \to +\infty$.

If the set S consists of a single point s and $t \mapsto x(t)$ approaches S, then $t \mapsto x(t)$ converges to s in the usual sense: $\lim_{t\to+\infty} x(t) = s$. If the set S consists of multiple disconnected components and $t \mapsto x(t)$ approaches S, then $t \mapsto x(t)$ must approach one of the disconnected components of S. Specifically, if the set S is composed of a finite number of points, then $t \mapsto x(t)$ must converge to one of the points.

(ii) Given a dynamical system (X, f), a set $W \subset X$ is *invariant* (or *f-invariant*) if each solution starting in W remains in W, that is, if $x(0) \in W$ implies $x(t) \in W$ for all $t \geq 0$.

For example, any sublevel set of a function is invariant for the corresponding negative gradient flow.

We are now ready to present the main result of this section.

Theorem A.7 (Krasovskiĭ-LaSalle Invariance Principle). *For a dynamical system (X, f) with differentiable f, assume that*

(KL1) *all trajectories of (X, f) are bounded,*

(KL2) *there exists a closed invariant set $W \subset X$, and*

(KL3) *there exists a continuously-differentiable function $V : W \to \mathbb{R}$ satisfying $\mathscr{L}_f V(x) \leq 0$ for all $x \in W$.*

Then for each solution $t \mapsto x(t)$ starting in W there exists $c \in \mathbb{R}$ such that x converges to the largest invariant set contained in

$$\{x \in W \;:\; \mathscr{L}_f V(x) = 0\} \cap V^{-1}(c).$$

Note: if the closed invariant set $W \subset X$ in Assumption (KL2) is also bounded, then Assumption (KL1) is automatically satisfied.

Note: unlike in the Lyapunov Stability Criteria, the Krasovskiĭ-LaSalle Invariance Principle does not require the function V to be locally positive definite and establishes certain asymptotic convergence properties without requiring the Lie derivative of V to be locally negative definite.

Note: in some examples it is sufficient for one's purposes to show that $x(t) \to \{x \in W \;:\; \mathscr{L}_f V(x) = 0\}$. In other cases, however, one really needs to analyze the largest invariant set inside $\{x \in W \;:\; \mathscr{L}_f V(x) = 0\}$.

Note: If the largest invariant set is the union of multiple disjoint non-empty sets, then the solution to the negative gradient flow must converge to one of these disjoint sets.

Example A.8 (Gradient and mechanical systems: Example A.5 continued). We continue the analysis of the example dynamical systems in Examples A.1 and A.5. Specifically, we sharpen here our results about the dissipative mechanical system about a local minimum point x_2 (or x_4) based on the Krasovskiĭ-LaSalle Invariance Principle.

First, we note that the assumptions of the Krasovskiĭ-LaSalle Invariance Principle in Theorem A.7 are satisfied:

(i) the function E and the vector field (the right-hand side of the mechanical system) are continuously differentiable;

[1] Here we define the distance from a point y to a set Z to be $\inf_{z\in Z} \|y - z\|$.

(ii) the derivative $\dot{\mathrm{E}}$ is locally negative semidefinite; and

(iii) for any initial condition $(x_0, v_0) \in \mathbb{R}^2$ sufficiently close to $(x_2, 0)$ the sublevel set $\{(x, v) \in \mathbb{R}^2 : \mathrm{E}(x, v) \leq \mathrm{E}(x_0, v_0)\}$ is compact due to the local positive definiteness of V at x_2.

It follows that $(x(t), v(t))$ converges to largest invariant set contained in

$$C = \{(x, v) \in \mathbb{R}^2 : \mathrm{E}(x, v) \leq \mathrm{E}(x_0, v_0), v = 0\} = \{(x, 0) \in \mathbb{R}^2 : \mathrm{E}(x, 0) \leq \mathrm{E}(x_0, v_0)\}.$$

A subset of C is invariant if any trajectory initiating in the subset remains in it. But this is only true if the starting position $\bar{x}$ satisfies $\frac{\partial}{\partial x} V(\bar{x}) = 0$, because otherwise the resulting trajectory would experience a strictly non-zero $\dot{v}(0)$ and hence leave C. In other words, the largest invariant set inside C is $\{(x, 0) \in \mathbb{R}^2 : \mathrm{E}(x, 0) \leq \mathrm{E}(x_0, v_0), \frac{\partial}{\partial x} V(x) = 0\}$. But the local minimum point x_2 is the unique critical point in the sublevel set and, therefore,

$$\lim_{t \to +\infty} (x(t), v(t)) = (x_2, 0). \qquad \bullet$$

A.5 Application #1: Linear and linearized systems

It is interesting to study the convergence properties of a linear system. Recall that a symmetric matrix is positive definite if all its eigenvalues are strictly positive.

Theorem A.9 (Convergence of linear systems). *For a matrix $A \in \mathbb{R}^{n \times n}$, the following properties are equivalent:*

(i) each solution to the differential equation $\dot{x} = Ax$ satisfies $\lim_{t \to +\infty} x(t) = \mathbb{0}_n$,

(ii) A is Hurwitz, i.e., all the eigenvalues of A have strictly-negative real parts, and

(iii) for every positive-definite matrix Q, there exists a unique solution positive-definite matrix P to the so-called Lyapunov matrix equation:

$$A^\top P + PA = -Q.$$

Note: one can show that statement (iii) implies statement (i) using the Lyapunov Stability Criteria with function $V(x) = x^\top Px$, whose Lie derivative along the systems solutions is $\dot{V} = x^\top(A^\top P + PA)x = -x^\top Qx \leq 0$.

Next, we show a very useful way to apply linear stability methods to analyze the local stability of a nonlinear system.

The *linearization at the equilibrium point x^** of the dynamical system (X, f) is the linear dynamical system defined by the differential equation $\dot{y} = Ay$, where

$$A = \frac{\partial f}{\partial x}(x^*).$$

Theorem A.10 (Convergence of nonlinear systems via linearization). *Consider a dynamical system (X, f) with an equilibrium point x^*, with twice differentiable vector field f, and with linearization A at x^*. The following statements hold:*

(i) the equilibrium point x^* is locally exponentially stable if all the eigenvalues of A have strictly-negative real parts; and

(ii) the equilibrium point x^* is unstable if at least one eigenvalue of A has strictly-positive real part.

Example A.11 (Two coupled oscillators). For $\theta \in \mathbb{R}$, consider the dynamical system arising from two coupled oscillators:

$$\dot{\theta} = f(\theta) = \omega - \sin(\theta).$$

If $\omega \in [0, 1[$, then there are two equilibrium points inside the range $\theta \in [0, 2\pi[$:

$$\theta_1^* = \arcsin(\omega) \in [0, \pi/2[, \qquad \text{and} \qquad \theta_2^* = \pi - \arcsin(\omega) \in]\pi/2, +\pi].$$

Moreover, for $\theta \in \mathbb{R}$, the 2π-periodic set of equilibria are $\{\theta_1^* + 2k\pi \,:\, k \in \mathbb{Z}\}$ and $\{\theta_2^* + 2k\pi \,:\, k \in \mathbb{Z}\}$. The linearization matrix $A(\theta_i^*) = \frac{\partial f}{\partial \theta}(\theta_i^*) = -\cos(\theta_i^*)$ for $i \in \{1, 2\}$ shows that θ_1^* is locally exponentially stable and θ_2^* is unstable. •

Example A.12 (A third order scalar system). Pick a scalar c and, for $x \in \mathbb{R}$, consider the dynamical system

$$\dot{x} = f(x) = c \cdot x^3.$$

The linearization at the equilibrium $x^* = 0$ is indefinite: $A(x^*) = 0$. Thus, Theorem A.10 offers no conclusions other than the equilibrium cannot be exponentially stable. On the other hand, the Krasovskiĭ-LaSalle Invariance Principle shows that for $c < 0$ every trajectory converges to $x^* = 0$. Here, a non-increasing and differentiable function is given by $V(x) = x^2$ with Lie derivative $\mathscr{L}_f V(x) = -2cx^4 \leq 0$. Since $V(x(t))$ is non-increasing along the solution to the dynamical system, a compact invariant set is then readily given by any sublevel set $\{x \,:\, V(x) \leq \ell\}$ for $\ell \geq 0$. •

A.6 Application #2: Negative gradient systems

We now summarize and extend the analysis given in Example A.3 of the stability properties of negative gradient systems. Recall for convenience that, given a differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$, the *negative gradient flow* defined by V is the dynamical system

$$\dot{x}(t) = -\frac{\partial V}{\partial x}(x(t)). \tag{A.6}$$

We start by noting that, as in the Exercise, the Lie derivative of V along the negative gradient flow is

$$\mathscr{L}_{-\frac{\partial V}{\partial x}} V(x) = -\left\| \frac{\partial V}{\partial x}(x) \right\|^2 \leq 0,$$

and that, therefore, each sublevel set $V_{\leq}^{-1}(\ell)$, for $\ell \in \mathbb{R}$ is invariant (provided it is non-empty).

Given a twice differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$ and a point $x \in \mathbb{R}^n$, the *Hessian matrix of V*, denoted by $\operatorname{Hess} V(x) \in \mathbb{R}^{n\times n}$, is the symmetric matrix of second order partial derivatives at x: $(\operatorname{Hess} V)_{ij}(x) = \partial^2 V/\partial x_i \partial x_j(s)$. Given a critical point x^* of V, if the Hessian matrix $\operatorname{Hess} V(x^*)$ is positive definite, then x^* is an isolated local minimum point of V. The converse is not true; as a counterexample, consider the function $V(x) = x^4$ and the critical point $x^* = 0$.

Theorem A.13 (Convergence of negative gradient flow). *Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be twice-differentiable and assume its sublevel set $V_{\leq}^{-1}(\ell) = \{x \in \mathbb{R}^n \,:\, V(x) \leq \ell\}$ is compact for some $\ell \in \mathbb{R}$. Then the negative gradient flow (A.6) has the following properties:*

(i) each solution $t \mapsto x(t)$ starting in $V_{\leq}^{-1}(\ell)$ satisfies $\lim_{t\to+\infty} V(x(t)) = c$, for some $c \leq \ell$, and approaches the set of critical points of V:

$$\Big\{x \in \mathbb{R}^n \,:\, \frac{\partial V}{\partial x}(x) = \mathbb{0}_n\Big\},$$

(ii) each local minimum point x^* is locally asymptotically stable and it is locally exponentially stable if and only if $\operatorname{Hess} V(x^*)$ is positive definite,

(iii) a critical point x^* is unstable if at least one eigenvalue of $\operatorname{Hess} V(x^*)$ is strictly negative,

(iv) if the function V is analytic, then every solution starting in a compact sublevel set has finite length (as a curve in $\mathbb{R}^n$) and converges to a single equilibrium point.

Proof. To show statement (i), we verify that the assumptions of the Krasovskiĭ-LaSalle Invariance Principle are satisfied as follows. First, as set W we adopt the sublevel set $V_{\leq}^{-1}(\ell)$ which is compact by assumption and is invariant. Second we know the Lie derivative of V along the vector field is non-positive. Statement (i) is now an immediate consequence of the Krasovskiĭ-LaSalle Invariance Principle.

The statements (ii) and (iii) follow from observing that the linearization of the negative gradient system at the equilibrium x^* is the negative Hessian matrix evaluated at x^* and from applying Theorem A.10.

Regarding statement (iv), we refer to the original source (Łojasiewicz, 1984) and to the review in (Absil et al., 2005, Section 2). ∎

Note: If the function V has isolated critical points, then the negative gradient flow evolving in a compact set must converge to a single critical point.

A.7 Application #3: Continuous-time averaging systems and Laplacian matrices

Following up on the treatment of semicontraction theory in Chapter 5, in this section we revisit the continuous-time averaging system, i.e., the Laplacian flow,

$$\dot{x} = -Lx.$$

We recall the max-min seminorm $V_{\text{max-min}} \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by

$$V_{\text{max-min}}(x) = \max_{i \in \{1,...,n\}} x_i - \min_{i \in \{1,...,n\}} x_i.$$

We will refer to this as the max-min function. We recall that $V_{\text{max-min}}(x) \geq 0$, and $V_{\text{max-min}}(x) = 0$ if an only if $x = \alpha \mathbb{1}_n$ for some $\alpha \in \mathbb{R}$.

Lemma A.14 (The max-min function along the Laplacian flow). *Let $L \in \mathbb{R}^{n \times n}$ be the Laplacian matrix of a weighted digraph G. Let $x(t)$ be the solution to the Laplacian flow $\dot{x} = -Lx$. Then*

(i) $t \mapsto V_{\text{max-min}}(x(t))$ is non-increasing,

(ii) if G has a globally reachable node, then, for some $\alpha \in \mathbb{R}$,

$$\lim_{t \to \infty} V_{\text{max-min}}(x(t)) = 0 \quad \textit{and} \quad \lim_{t \to \infty} x(t) = \alpha \mathbb{1}_n.$$

Numerous proofs for these results are possible (e.g., statement (ii) is established in Theorem B.16). A second approach is to use the properties of the row-stochastic matrix $\exp(-Lt)$, $t \in \mathbb{R}_{\geq 0}$, as established in Theorem B.15.

Here we pursue a strategy based on adopting $V_{\text{max-min}}$ as a weak Lyapunov function and, because $V_{\text{max-min}}$ is not continuously-differentiable, applying an appropriate generalization of the Krasovskiĭ-LaSalle Invariance Principle in Theorem A.7. For our purposes here, it suffices to present the following concepts.

Definition A.15. *The upper right Dini derivative and upper left Dini derivative of a continuous function $f\colon]a,b[\to\mathbb{R}$ at a point $t\in]a,b[$ are defined by, respectively,*

$$D^+f(t)=\limsup_{\Delta t>0,\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t},\quad \textit{and}\quad D^-f(t)=\limsup_{\Delta t<0,\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}.$$

The *limit superior* of a real sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined by $\limsup_{n\to\infty}a_n=\lim_{n\to\infty}\sup_{m\geq n}a_m$. Since the sup operator is always well defined (possibly equal to $+\infty$), so are the Dini derivatives.

Lemma A.16 (Properties of the upper Dini derivatives). *Given a continuous function $f\colon]a,b[\to\mathbb{R}$,*

(i) if f is differentiable at $t\in]a,b[$, then $D^+f(t)=D^-f(t)=\frac{d}{dt}f(t)$ is the usual derivative of f at t, and

(ii) if $D^+f(t)\leq 0$ or $D^-f(t)\leq 0$ for all $t\in]a,b[$, then f is non-increasing on $]a,b[$.

Morever, given differentiable functions $f_1,\dots,f_m\colon]a,b[\to\mathbb{R}$, the max function $f_{\max}(t)=\max\{f_i(t)\,:\,i\in\{1,\dots,m\}\}$ satisfies

(iii) $D^+f_{\max}(t)=\max\Big\{\dfrac{d}{dt}f_i(t)\,:\,i\in\operatorname{argmax}(f_{\max}(t))\Big\}$, *and*
$D^-f_{\max}(t)=\min\Big\{\dfrac{d}{dt}f_i(t)\,:\,i\in\operatorname{argmax}(f_{\max}(t))\Big\}$,

(iv) if $D^+f_{\max}(t)\leq 0$ for all $t\in]a,b[$, then $f_{\max}$ is non-increasing on $]a,b[$.

Note: statement (i) follows from the definition of derivative of a differentiable function. Statement (ii) is a consequence of Theorem 1.14 in (Giorgi and Komlósi, 1992), to which we refer for all proofs. Statement (iii) is known as Danskin's Lemma.

Proof of Lemma A.14. Define the quantities $x_{\max}(t)=\max(x(t))$ and $x_{\min}(t)=\min(x(t))$ as well as $\operatorname{argmax}(x(t))=\{i\in\{1,\dots,n\}\,:\,x_i(t)=x_{\max}(t)\}$ and $\operatorname{argmin}(x(t))=\{i\in\{1,\dots,n\}\,:\,x_i(t)=x_{\min}(t)\}$. Along the Laplacian flow $\dot{x}_i=\sum_{j=1}^n a_{ij}(x_j-x_i)$, Lemma A.16(iii) (Danskin's Lemma) implies

$$\begin{aligned}D^+V_{\text{max-min}}(x(t))&=\max\{\dot{x}_i(t)\,:\,i\in\operatorname{argmax}(x(t))\}-\min\{\dot{x}_i(t)\,:\,i\in\operatorname{argmin}(x(t))\}\\&=\max\Big\{\sum_{j=1}^n a_{ij}(x_j-x_{\max})\,:\,i\in\operatorname{argmax}(x(t))\Big\}\\&\qquad-\min\Big\{\sum_{j=1}^n a_{ij}(x_j-x_{\min})\,:\,i\in\operatorname{argmin}(x(t))\Big\},\end{aligned}$$

where we have used $-\min(x)=\max(-x)$. Because $x_j-x_{\max}\leq 0$ and $x_j-x_{\min}\geq 0$ for all $j\in\{1,\dots,n\}$, we have established that $D^+V_{\text{max-min}}(x(t))$ is the sum of two non-positive terms. This property, combined with Lemma A.16(iv), implies that $t\mapsto V_{\text{max-min}}(x(t))$ is non-increasing, thereby completing the proof of statement (i).

To establish statement (ii) we invoke a generalized version of the Krasovskiĭ-LaSalle Invariance Principle A.7. First, we note that statement (i) implies that any solution is bounded inside $[x_{\min}(0),x_{\max}(0)]^n$; this is a sufficient property (in lieu of the compactness of the set W). Second, we know the continuous function $V_{\text{max-min}}$ along the Laplacian flow is non-increasing (in lieu of the same property for a Lie derivative of a continuously-differentiable function). Therefore, we now know that there exists c such that the solution starting from $x(0)$ converges to the largest invariant set C contained in

$$\big\{x\in[x_{\min}(0),x_{\max}(0)]^n\,:\,D^+V_{\text{max-min}}(x)\big|_{\dot{x}=-Lx}=0\big\}\cap V_{\text{max-min}}^{-1}(c).$$

Because $V_{\text{max-min}}$ is non-negative, we know $c \geq 0$. We now assume by absurd that $c > 0$, we let $y(t)$ be a trajectory originating in C, and we aim to show that $V_{\text{max-min}}(y(t))$ decreases along time (which is a contradiction because C is invariant).

Let k be a globally reachable node. Let i (resp. j) be an arbitrary index in $\operatorname{argmax}(y(0))$ (resp. $\operatorname{argmin}(y(0))$) so that $y_i(0) - y_j(0) = c > 0$. Without loss of generality we assume $y_k(0) < y_i(0)$. (Otherwise it would need to be $y_k(0) > y_j(0)$ and we would proceed similarly.) Recall we know $\dot{y}_i(0) \leq 0$. We now note that, if $\dot{y}_i(t) = 0$ for all $t \in (0, \varepsilon)$ for a positive ε, then the equation $\dot{y}_i = \sum_j a_{ij}(y_j - y_i)$ and the property $y_i(0) = \max y(0)$ together imply that $y_j(t) = y_i(t)$ for all $t \in (0, \varepsilon)$ and for all j such that $a_{ij} > 0$. Iterating this argument along the directed path from i to k, we get the contradiction that $y_k(t) = y_i(t)$ for all $t \in (0, \varepsilon)$. Therefore, we know that $\dot{y}_i(t) < 0$ for small times. Because i is an arbitrary index in $\operatorname{argmax}(y(0))$, we have proved that $t \mapsto \max y(t)$ is strictly decreasing for small times. This establishes that C is not invariant if $c > 0$ and completes the proof of statement (ii). ■

A.8 Historical notes and further reading

Classic historical works on stability properties of physical systems include (Lagrange, 1788; Maxwell, 1868; Thomson and Tait, 1867). Modern stability theory started with the work by Lyapunov (1892), who proposed the key ideas towards a general treatment of stability notions and tests for nonlinear dynamical systems. Lyapunov's ideas were extended by Barbashin and Krasovskiĭ (1952); Krasovskiĭ (1963) and LaSalle (1960, 1968, 1976) through their work on invariance principles. Other influential works include (Chetaev, 1961; Hahn, 1967).

For comprehensive treatments, we refer the reader to the numerous excellent texts in this area, e.g., including the classic control texts (Sontag, 1998; Khalil, 2002; Vidyasagar, 1978a), the classic dynamical systems texts (Hirsch and Smale, 1974; Arnol'd, 1992; Guckenheimer and Holmes, 1990), and the more recent works (Haddad and Chellaboina, 2008; Goebel et al., 2012; Blanchini and Miani, 2015).

This chapter has treated systems evolving in continuous time. Naturally, it is possible to develop a Lyapunov theory for discrete-time systems, even though remarkably there are only few references on this topic; see (LaSalle, 1976, Chapter 1).

We refer to (Clarke et al., 1998; Cortés, 2008) for a comprehensive review of stability theory for nonsmooth systems and Lyapunov functions. Properties of the Dini derivatives are reviewed by Giorgi and Komlósi (1992). The usefulness of Dini derivatives in continuous-time averaging systems is highlighted for example by Lin et al. (2007); see also (Danskin, 1966) for Danskin's Lemma.

Appendix **B**

Appendix: Linear Network Systems

This chapter is a highly-abbreviated version of material from the first ten chapters in the textbook (Bullo, 2022).

Recent years have witnessed the emergence of a discipline of study focused on modeling, analyzing, and designing dynamic phenomena over networks. We refer to such systems as network systems; they are also equivalently referred to as multi-agent or distributed systems. This emerging discipline, rooted in graph theory, control theory, and matrix analysis, is increasingly relevant because of its broad set of application domains. Network systems appear naturally in (i) social networks and mathematical sociology, (ii) electric, mechanical and physical networks, and (iii) animal behavior, population dynamics, and ecosystems. Network systems are designed in the context of networked control systems, robotic networks, power grids, parallel and scientific computation, and transmission and traffic networks, to name a few.

Textbook treatments of network systems and their applications are given in the recent books (Ren and Beard, 2008; Bullo et al., 2009; Mesbahi and Egerstedt, 2010; Bai et al., 2011; Cristiani et al., 2014; Francis and Maggiore, 2016; Arcak et al., 2016) and recent related surveys include (Martínez et al., 2007; Ren et al., 2007; Garin and Schenato, 2010; Cao et al., 2013; Oh et al., 2015).

Chapter organization and related literature Sections B.1 and B.2 review Perron Frobenius and algebraic graph theory. Classic references on this material include (Gantmacher, 1959; Seneta, 1981; Horn and Johnson, 2012).

Section B.3 describes models of opinion dynamics. This classic field initiated with the seminal papers by French Jr. (1956), Harary (1959), Abelson (1964), and DeGroot (1974). The classic discrete-time linear averaging model is well known as the DeGroot model, but a more accurate historic name would be the French-Harary-DeGroot model since modeling concepts were contained in (French Jr., 1956) and analysis results in (Harary, 1959).

B.1 Perron–Frobenius theory

We let $\mathbb{1}_n \in \mathbb{R}^n$ (respectively $\mathbb{0}_n \in \mathbb{R}^n$) be the column vector with all entries equal to $+1$ (respectively 0).

The square matrix $A \in \mathbb{R}^{n\times n}$ is

(i) *non-negative* (respectively *positive*) if $a_{ij} \geq 0$ (respectively $a_{ij} > 0$) for all i and j in $\{1, \dots, n\}$;

(ii) *row-stochastic* if non-negative and $A\mathbb{1}_n = \mathbb{1}_n$;

(iii) *column-stochastic* if non-negative and $A^\mathsf{T}\mathbb{1}_n = \mathbb{1}_n$; and

(iv) *doubly-stochastic* if it is row- and column-stochastic.

In the following, we write $A > 0$ and $v > 0$ (respectively $A \geq 0$ and $v \geq 0$) for a positive (respectively non-negative) matrix A and vector v.

We classify non-negative matrices in terms of their zero/nonzero pattern and of the asymptotic behavior of their powers.

Definition B.1 (Irreducible and primitive matrices). *A square $n \times n$ non-negative matrix A, for $n \geq 2$, is*

(i) irreducible if $\sum_{k=0}^{n-1} A^k$ is positive,

(ii) primitive if there exists $k \in \mathbb{N}$ such that A^k is positive.

A matrix that is not irreducible is said to be reducible.

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	: $\operatorname{spec}(A_1) = \{1, 1\}$, the zero/nonzero pattern in A_1^k is constant, and $\lim_{k\to\infty} A_1^k = I_2$,
$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	: $\operatorname{spec}(A_2) = \{1, -1\}$, the zero/nonzero pattern in A_2^k is periodic, and $\lim_{k\to\infty} A_2^k$ does not exist,
$A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	: $\operatorname{spec}(A_3) = \{0, 0\}$, the zero/nonzero pattern is $A_3^k = 0$ for all $k \geq 2$, and $\lim_{k\to\infty} A_3^k = 0$,
$A_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$	: $\operatorname{spec}(A_4) = \{1, -1/2\}$, the zero/nonzero pattern is $A_4^k > 0$ for all $k \geq 2$, and $\lim_{k\to\infty} A_4^k = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$, and
$A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	: $\operatorname{spec}(A_5) = \{1, 1\}$, the zero/nonzero pattern in A_5^k is constant and $\lim_{k\to\infty} A_5^k$ is unbounded.

Table B.1: Example 2-dimensional non-negative matrices and their properties

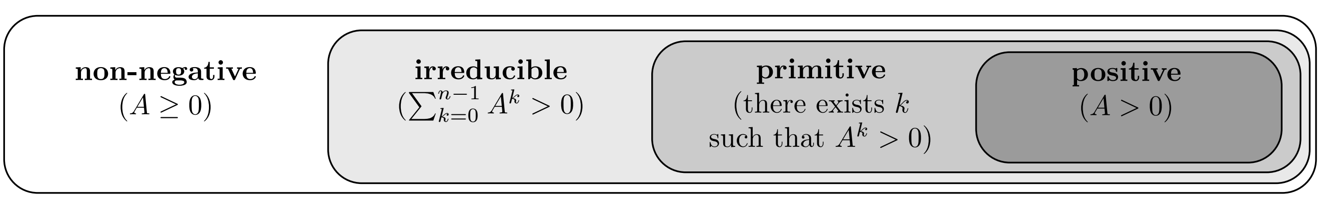


Figure B.1: The set of non-negative square matrices and its increasingly smaller subsets of irreducible, primitive and positive matrices.

Theorem B.2 (Perron–Frobenius Theorem). *Let $A \in \mathbb{R}^{n\times n}$, $n \geq 2$. If A is non-negative, then*

(i) there exists a real eigenvalue $\lambda \geq |\mu| \geq 0$ for all other eigenvalues μ,

(ii) the right and left eigenvectors v and w of λ can be selected non-negative.

If additionally A is irreducible, then

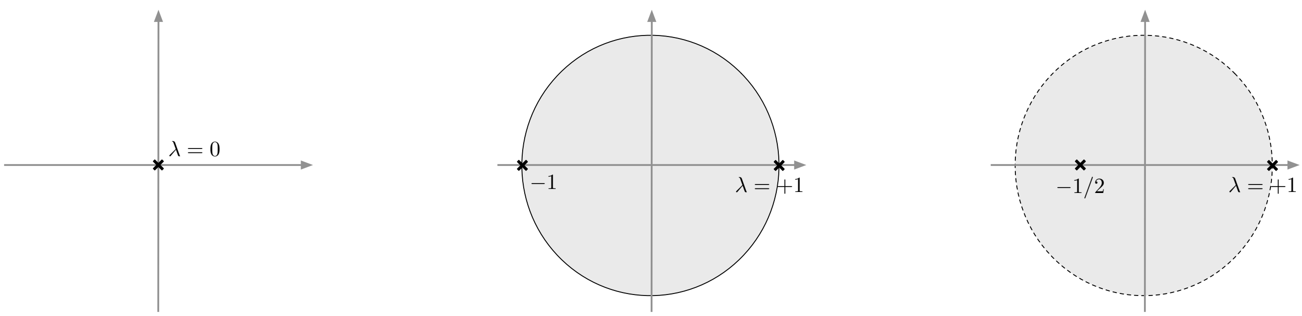
(iii) the eigenvalue λ is strictly positive and simple,

(iv) the right and left eigenvectors v and w of λ are unique and positive, up to rescaling.

If additionally A is primitive, then

(v) *the eigenvalue λ satisfies $\lambda > |\mu|$ for all other eigenvalues μ.*

The real non-negative eigenvalue λ is the spectral radius $\rho(A)$ of A and it is usually referred to as the *dominant or Perron eigenvalue* of A. The right and left eigenvectors w and v (unique up to rescaling and selected non-negative) of the dominant eigenvalue λ are called the *right and left dominant eigenvectors*, respectively.



(a) The matrix A_3 is reducible: its dominant eigenvalue is 0 and so is its other eigenvalue.

(b) The matrix A_2 is irreducible but not primitive: its dominant eigenvalues $+1$ is not strictly larger, in magnitude, than the other eigenvalue -1.

(c) The matrix A_4 is primitive: its dominant eigenvalue $+1$ is strictly larger than the other eigenvalue $-1/2$.

Figure B.2: Spectra of non-negative matrices consistent with the Perron–Frobenius Theorem

Finally, the Perron–Frobenius Theorem for primitive matrices has immediate consequences for the asymptotic behavior of the discrete time dynamical system $x(k+1) = Ax(k)$, that is, for the powers A^k as $k \to \infty$.

Proposition B.3 (Powers of primitive matrices). *Consider a square $n \times n$ non-negative matrix A, for $n \geq 2$. Let λ be the dominant eigenvalue and let w and v be the right and left dominant eigenvectors of A normalized so that they are both positive and satisfy $w^\top v = 1$. Then*

$$\lim_{k\to\infty} \frac{A^k}{\lambda^k} = wv^\top.$$

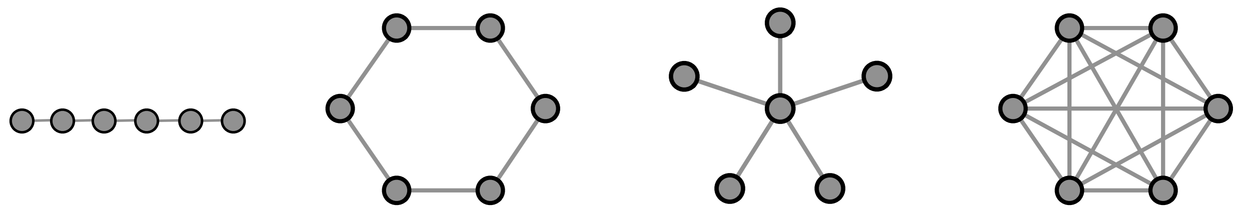
B.2 Graph theory

In this section we review some basic and prototypical results that involve correspondences between graphs and adjacency matrices. We let G denote a weighted digraph and A its weighted adjacency matrix or, equivalently, we let A be a non-negative matrix and we let G be its *associated weighted digraph* (i.e., the digraph with nodes $\{1, \dots, n\}$ and with weighted adjacency matrix A).

The *weighted out-degree matrix* D_{out} and the *weighted in-degree matrix* D_{in} of a weighted digraph are the diagonal matrices defined by

$$D_{\text{out}} = \operatorname{diag}(A\mathbb{1}_n) = \begin{bmatrix} d_{\text{out}}(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_{\text{out}}(n) \end{bmatrix}, \quad \text{and} \quad D_{\text{in}} = \operatorname{diag}(A^\top \mathbb{1}_n),$$

where $\operatorname{diag}(z_1, \dots, z_n)$ is the diagonal matrix with diagonal entries equal to $z_1, \dots, z_n$.

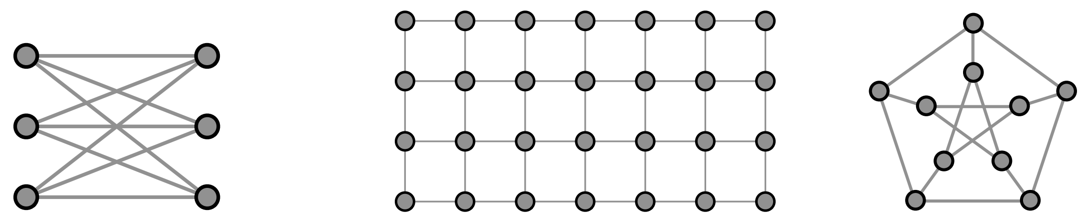


(a) *Path graph* with 6 nodes, denoted by P_6

(b) *Cycle (ring) graph* with 6 nodes, denoted by C_6

(c) *Star graph* with 6 nodes, denoted by S_6

(d) *Complete graph* with 6 nodes, denoted by K_6

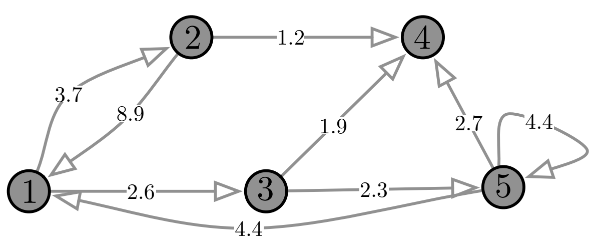


(e) *Complete bipartite graph* with $3+3$ nodes, denoted by $K_{3,3}$

(f) *Two-dimensional grid graph* with 4×7 nodes, denoted by $G_{4,7}$

(g) *Petersen graph*

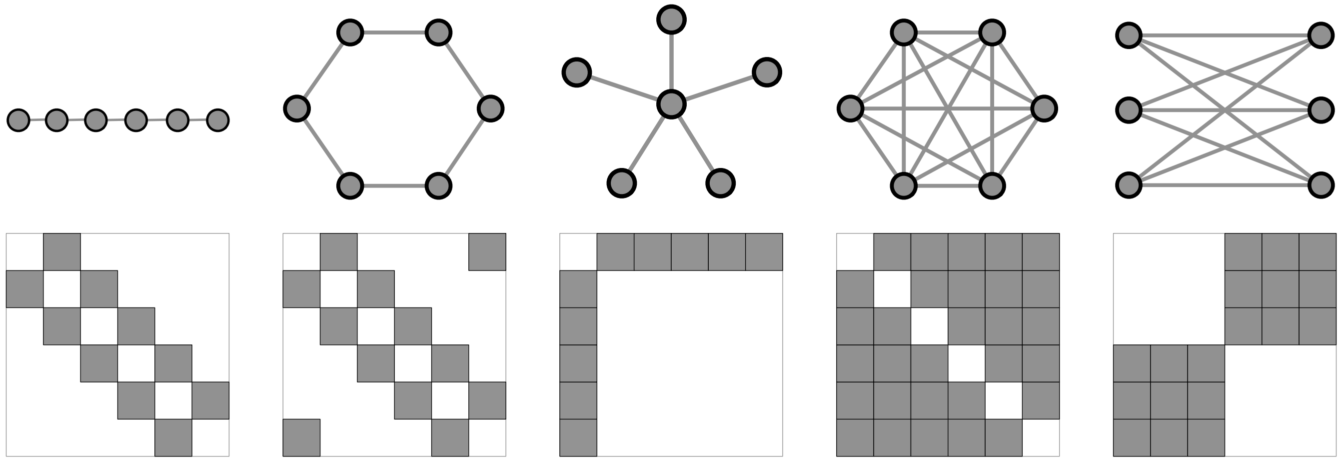
Figure B.3: Example graphs.



$$A = \begin{bmatrix} 0 & 3.7 & 2.6 & 0 & 0 \\ 8.9 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.9 & 2.3 \\ 0 & 0 & 0 & 0 & 0 \\ 4.4 & 0 & 0 & 2.7 & 4.4 \end{bmatrix}.$$

Figure B.4: A weighted digraph and its adjacency matrix

Digraph G	Non-negative matrix A (adjacency of G)
G is undirected	$A = A^\top$
G is weight-balanced	$A\mathbb{1}_n = A^\top\mathbb{1}_n$, that is, $D_{\text{out}} = D_{\text{in}}$
(no self-loops) node i is a sink	(zero diagonal) ith row-sum of A is zero
(no self-loops) node i is a source	(zero diagonal) ith column-sum of A is zero
each node has weighted out-degree equal to 1 ($D_{\text{out}} = I_n$)	A is row-stochastic
each node has weighted out- and in-degree equal to 1 ($D_{\text{out}} = D_{\text{in}} = I_n$)	A is doubly-stochastic

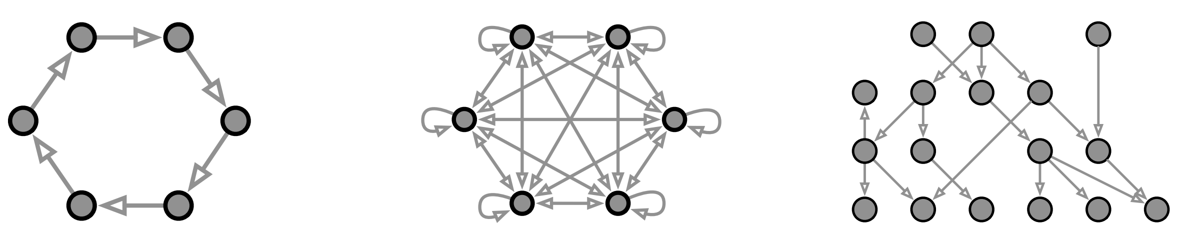


P_6 C_6 S_6 K_6 $K_{3,3}$

Figure B.5: Path, cycle, star, complete and complete bipartite graph (from Figure B.3) and their binary adjacency matrices depicted in their respective *pixel picture* representation.

Graph	Adjacency Matrix	Adjacency Spectrum
path graph P_n	Toeplitz tridiagonal	$\{2\cos(\pi i/(n+1)) \,:\, i \in \{1,\dots,n\}\}$
cycle graph C_n	circulant	$\{2\cos(2\pi i/n)) \,:\, i \in \{1,\dots,n\}\}$
star graph S_n	$\mathbb{e}_1\mathbb{e}_{-1}^\top + \mathbb{e}_{-1}\mathbb{e}_1^\top$	$\{\sqrt{n-1}, 0, \dots, 0, -\sqrt{n-1}\}$
complete graph K_n	$\mathbb{1}_n\mathbb{1}_n^\top - I_n$	$\{(n-1), -1, \dots, -1\}$
complete bipartite $K_{n,m}$	$\begin{bmatrix} \mathbb{0}_{n\times n} & \mathbb{1}_{n\times m} \\ \mathbb{1}_{m\times n} & \mathbb{0}_{m\times m} \end{bmatrix}$	$\{\sqrt{nm}, 0, \dots, 0, -\sqrt{nm}\}$

Table B.2: Adjacency spectrum for basic graphs (we adopt the notation $\mathbb{e}_{-i} = \mathbb{1}_n - \mathbb{e}_i$)

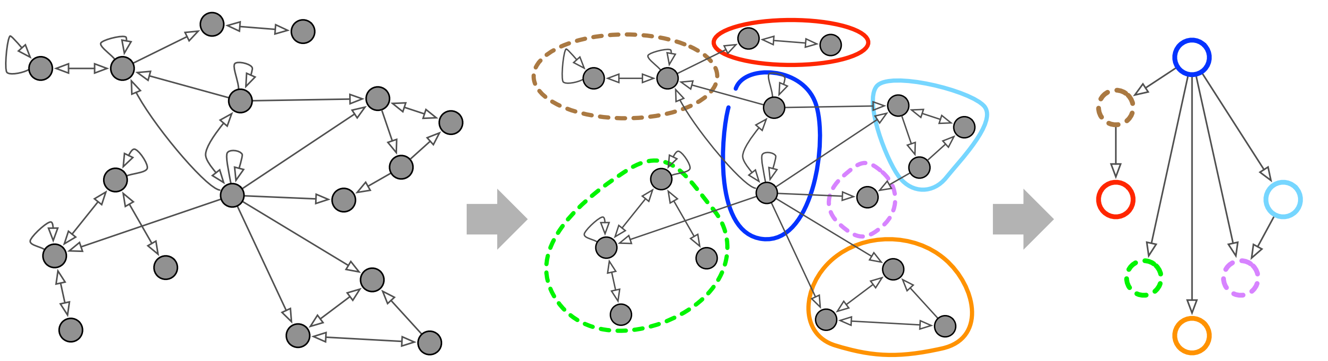


(a) *Cycle digraph* with 6 nodes (b) *Complete digraph* with 6 nodes (c) A digraph with no directed cycles

Figure B.6: Example digraphs

We start with some basic definitions about a directed graph G. A node i is *globally reachable* if, for every other node j, there exists a directed walk in G from node j to node i. A directed graph is *strongly connected* if each node is globally reachable. A *subgraph* of G is a subset of nodes and edges of G. A subgraph H is a *strongly connected component* of G if H is strongly connected and any other subgraph of G containing H is not strongly connected. A directed graph G is *aperiodic* if there exists no integer greater than 1 that divides the length of each cycle of G.

We will also need the notion of condensation of a digraph. Given a directed graph G, the *condensation digraph* of G is formed by contracting each strongly connected component into a single node and letting an arc exist from one component to another if and only if at least one arc exists from a member of one component to a member of the other in G. The condensation digraph is acyclic and, therefore, contains at least one sink.



(a) An example digraph G

(b) The strongly connected components of the digraph G

(c) The condensation digraph $C(G)$

Figure B.7: An example digraph, its strongly connected components and its condensation digraph.

B.2.1 Algebraic graph theory

The first result we present relate the powers of the adjacency matrix with directed walks on the graph.

Lemma B.4 (Directed paths and powers of the adjacency matrix). *Let G be a weighted digraph with n nodes, with adjacency matrix A and binary adjacency matrix $A_{0,1} \in \{0,1\}^{n\times n}$. For all $i, j \in \{1, \dots, n\}$ and $k \in \mathbb{N}$*

(i) the (i, j) entry of $A_{0,1}^k$ equals the number of paths of length k from node i to node j; and

(ii) the (i, j) entry of A^k is positive if and only if there exists a path of length k from node i to node j.

(Paths here are directed paths that possibly include self-loops.)

Theorem B.5 (Connectivity properties of the digraph and positive powers of the adjacency matrix). *Let G be a weighted digraph with $n \geq 2$ nodes and weighted adjacency matrix A. The following statements are equivalent:*

(i) A is irreducible, that is, $\sum_{k=0}^{n-1} A^k > 0$;

(ii) there exists no permutation matrix P such that $P^\top AP$ is block triangular;

(iii) G is strongly connected;

(iv) for all partitions $\{\mathcal{I}, \mathcal{J}\}$ of the index set $\{1, \dots, n\}$, there exists $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that $\{i, j\}$ is an edge in G.

Let us remark that, instead of the order in which we presented matters here, most references define an irreducible matrix through property (ii) or, possibly, through property (iv).

Lemma B.6 (Global reachability and powers of the adjacency matrix). *Let G be a weighted digraph with $n \geq 2$ nodes and weighted adjacency matrix A. For any $j \in \{1, \dots, n\}$, the following statements are equivalent:*

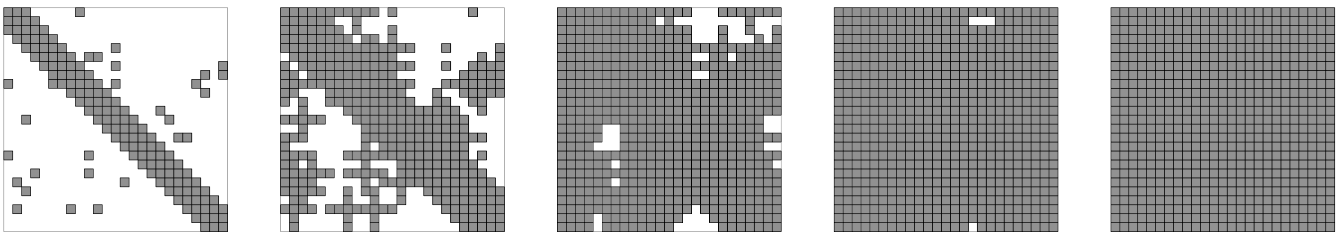
(i) the jth node of G is globally reachable, and

(ii) the jth column of $\sum_{k=0}^{n-1} A^k$ is positive.

Theorem B.7 (Strongly connected and aperiodic digraphs and primitive adjacency matrices). *Let G be a weighted digraph with $n \geq 2$ nodes and with weighted adjacency matrix A. The following two statements are equivalent:*

(i) G is strongly connected and aperiodic; and

(ii) A is primitive, that is, there exists $k \in \mathbb{N}$ such that A^k is positive.



A A^2 A^3 A^4 A^5

Figure B.8: These five images depict increasing powers of a non-negative matrix $A \in \mathbb{R}^{25\times 25}$. The digraph associated to A is strongly connected and has self-loops at each node so that, by Theorem B.7, there exists k (in this case $k = 5$) such that $A^k > 0$.

B.3 Linear averaging dynamics over networks

This section reviews some classic models for linear averaging dynamics over networks.

We start by presenting some convergence results for systems of the form

$$x(k+1) = Ax(k), \qquad \text{where } A \text{ is row-stochastic.} \tag{B.1}$$

Recall that the non-negative square matrix A is said to be row-stochastic if all its row-sums are equal to one, that is, if $A\mathbb{1}_n = \mathbb{1}_n$. Therefore, the right eigenvector of the eigenvalue 1 can be selected as $\mathbb{1}_n$.

The discrete-time averaging model (B.1) is well known as the DeGroot opinion dynamics model, but a more accurate historic name would be the French-Harary-DeGroot model, as discussed in the introduction. The matrix A is sometimes referred to as an *interpersonal influence network*.

Theorem B.8 (Consensus for row-stochastic matrices with a globally-reachable aperiodic strongly-connected component). *Let A be a row-stochastic matrix and let G be its associated digraph. The following statements are equivalent:*

(A1) *the eigenvalue 1 is simple, $\rho(A) = 1$, and all other eigenvalues have magnitude strictly smaller than 1,*

(A2) *A is semi-convergent (i.e., $\lim_{k\to\infty} A^k$ exists and is finite) and $\lim_{k\to\infty} A^k = \mathbb{1}_n v^\top$, for some $v \in \mathbb{R}^n$, $v \geq 0$, and $\mathbb{1}_n^\top v = 1$,*

(A3) *the digraph associated to A contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic.*

If any, and therefore all, of the previous conditions are satisfied, then the matrix A is said to be indecomposable and the following properties hold:

(i) $v \geq 0$ is the left dominant eigenvector of A and $(v)_i > 0$ if and only if node i is globally reachable;

(ii) *the solution to the averaging model $x(k+1) = Ax(k)$ in equation (B.1) satisfies*

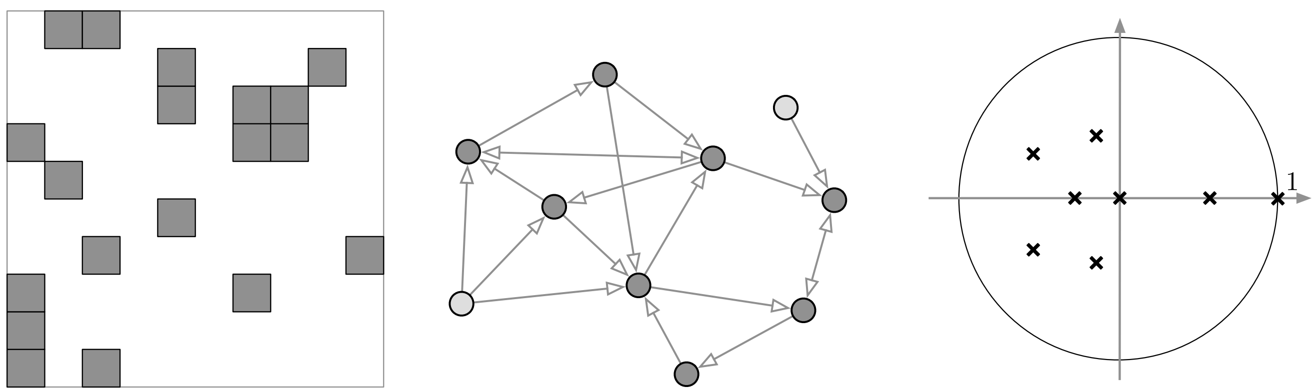
$$\lim_{k\to\infty} x(k) = \big(v^{\mathsf{T}} x(0)\big)\mathbb{1}_n;$$

In this case we say that the dynamical system achieves consensus;

(iii) *if additionally A is doubly-stochastic, then $v = \frac{1}{n}\mathbb{1}_n$ (because $A^{\mathsf{T}}\mathbb{1}_n = \mathbb{1}_n$ and $\frac{1}{n}\mathbb{1}_n^{\mathsf{T}}\mathbb{1}_n = 1$) so that*

$$\lim_{k\to\infty} x(k) = \frac{\mathbb{1}_n^{\mathsf{T}} x(0)}{n}\mathbb{1}_n = \operatorname{average}\big(x(0)\big)\mathbb{1}_n.$$

In this case we say that the dynamical system achieves average consensus.



(a) A rows-stochastic matrix; in each row, nonzero entries are equal and sum to 1.

(b) The corresponding digraph has an aperiodic subgraph of globally reachable nodes.

(c) The spectrum of the adjacency matrix includes a dominant eigenvalue.

Figure B.9: An example indecomposible row-stochastic matrix, its associated digraph consistent with Theorem B.8(A2), and its spectrum consistent with Theorem B.8(A1)

Theorem B.9 (Convergence to dissensus for row-stochastic matrices with multiple aperiodic sinks). *Let A be a row-stochastic matrix, let G be its associated digraph, and let $M \geq 2$ be the number of sinks in the condensation digraph $C(G)$. If each of the M sinks is aperiodic, then*

(i) *the semi-simple eigenvalue $\rho(A) = 1$ has multiplicity equal M and is strictly larger than the magnitude of all other eigenvalues, hence A is semi-convergent,*

(ii) *there exist M left eigenvectors of A, denoted by $v^m \in \mathbb{R}^n$, for $m \in \{1,\dots,M\}$, with the properties that: $v^m \geq 0$, $\mathbb{1}_n^{\mathsf{T}} v^m = 1$ and $(v^m)_i$ is positive if and only if node i belongs to the m-th sink,*

(iii) *the solution to the averaging model $x(k+1) = Ax(k)$ with initial condition $x(0)$ satisfies*

$$\lim_{k\to\infty} x_i(k) = \begin{cases} (v^m)^{\mathsf{T}} x(0), & \text{if node } i \text{ belongs to the } m\text{-th sink,} \\ (v^m)^{\mathsf{T}} x(0), & \text{if node } i \text{ is connected only with the } m\text{-th sink,} \\ \displaystyle\sum_{m=1}^{M} z_{i,m}\big((v^m)^{\mathsf{T}} x(0)\big), & \text{if node } i \text{ is connected to more than one sink,} \end{cases}$$

where, for each node i connected to more than one sink, the coefficients $z_{i,m}$, $m \in \{1,\dots,S\}$, are convex combination coefficients and are strictly positive if and only if there exists a directed walk from node i to the sink m.

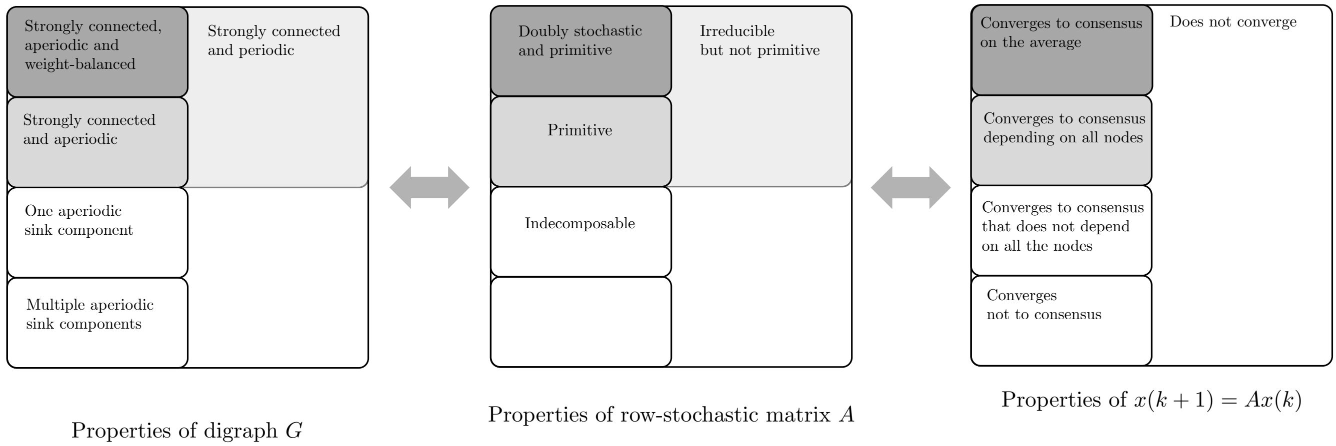


Figure B.10: Corresponding properties for the discrete-time averaging dynamical system $x(k+1) = Ax(k)$, the row-stochastic matrix A and the associated weighted digraph.

B.4 Laplacian matrices

Definition B.10 (Laplacian matrix of a digraph). *Given a weighted digraph G with adjacency matrix A and out-degree matrix $D_{\text{out}} = \operatorname{diag}(A\mathbb{1}_n)$, the Laplacian matrix of G is*

$$L = D_{\text{out}} - A.$$

In components $L = (\ell_{ij})_{i,j\in\{1,...,n\}}$

$$\ell_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j, \\ \displaystyle\sum_{h=1,h\neq i}^{n} a_{ih}, & \text{if } i = j, \end{cases}$$

or, for an unweighted undirected graph, $\ell_{ij} = \begin{cases} -1, & \text{if } \{i,j\} \text{ is an edge and not self-loop,} \\ d(i), & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$

The Laplacian matrix satisfies two useful equalities. First, for $x \in \mathbb{R}^n$,

$$(Lx)_i = \sum_{j=1}^{n} \ell_{ij} x_j = \sum_{j=1,j\neq i}^{n} a_{ij}(x_i - x_j) = \sum_{j\in\mathcal{N}^{\text{out}}(i)} a_{ij}(x_i - x_j). \tag{B.2}$$

Second, if $L = L^\top$ (i.e., $a_{ij} = a_{ji}$), then

$$x^\top L x = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2 = \sum_{\{i,j\}\in E} a_{ij}(x_i - x_j)^2. \tag{B.3}$$

Lemma B.11 (Zero row-sums). *Let G be a weighted digraph with Laplacian L and n nodes. Then*

$$L\mathbb{1}_n = \mathbb{0}_n.$$

Moreover, the following statements are equivalent:

(i) G *is weight-balanced,i.e.,* $D_{\text{out}} = D_{\text{in}} = \operatorname{diag}(A^\top \mathbb{1}_n)$*; and*

(ii) $\mathbb{1}_n^\top L = \mathbb{0}_n^\top$.

In equivalent words, 0 is an eigenvalue of L with eigenvector $\mathbb{1}_n$.

Motivated by this lemma, we say a matrix $L \in \mathbb{R}^{n\times n}$, $n \geq 2$, is *Laplacian* if

(i) its row-sums are zero,

(ii) its diagonal entries are non-negative, and

(iii) its non-diagonal entries are non-positive.

Theorem B.12 (Spectrum and rank of the Laplacian matrix). *Consider a weighted digraph* G *with Laplacian* L*. Then*

(i) the eigenvalues of L *different from* 0 *have strictly-positive real part and*

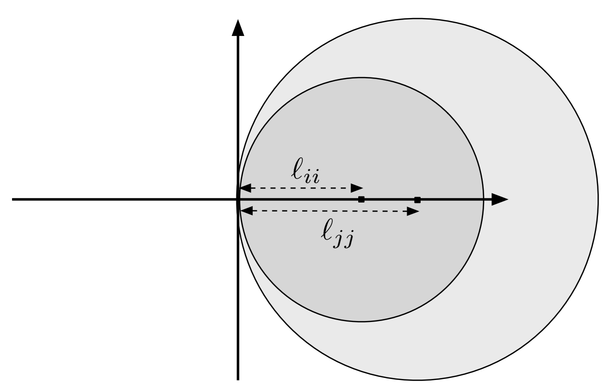
(ii) $\operatorname{rank}(L) = n - d$*, where* $d \geq 1$ *is the number of sinks in the condensation digraph of* G*.*

Moreover, the following statements are equivalent:

(iii) G *contains a globally reachable node,*

(iv) $\operatorname{rank}(L) = n - 1$*, and*

(v) the 0 *eigenvalue of* L *is simple.*



Next, Assume that A and, therefore, L is symmetric. Therefore, we know that all eigenvalues of L are real, that at least one is zero by Lemma B.11, and that all others are non-negative. By convention, we write these eigenvalues as

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

Definition B.13 (Algebraic connectivity). *The second smallest eigenvalue* λ_2 *of a symmetric Laplacian* L *of a weighted digraph* G *is called the algebraic connectivity of* G*.*

(The algebraic connectivity and its associated eigenvector are also referred to as the Fiedler eigenvalue and Fiedler vector, in recognition of the early work by Fiedler (1973).)

We can now transcribe the results above. For a weighted graph G with symmetric Laplacian L:

(i) G is connected if and only if $\lambda_2 > 0$; and

(ii) the multiplicity of 0 as an eigenvalue of L is equal to the number of connected components of G.

Lemma B.14 (Properties of the algebraic connectivity). *Consider a weighted digraph with symmetric adjacency matrix A, Laplacian matrix L, and algebraic connectivity λ_2.*

(i) The algebraic connectivity satisfies the following so-called variational description:

$$\lambda_2 = \min_{x \in \mathbb{R}^n, \|x\|_2 = 1, x \perp \mathbb{1}_n} x^\top L x. \tag{B.4}$$

(ii) Consider a second digraph with adjacency matrix A' and algebraic connectivity λ_2'. Then

$$A \leq A' \implies \lambda_2 \leq \lambda_2'.$$

Graph	Algebraic Connectivity	Laplacian spectrum
path graph P_n	$2(1-\cos(\pi/n)) \sim \pi^2/n^2$	$\{0\} \cup \{2(1-\cos(\pi i/n)) : i \in \{1,\dots,n-1\}\}$
cycle graph C_n	$2(1-\cos(2\pi/n)) \sim 4\pi^2/n^2$	$\{0\} \cup \{2(1-\cos(2\pi i/n)) : i \in \{1,\dots,n-1\}\}$
star graph S_n	1	$\{0, 1, \dots, 1, n\}$
complete graph K_n	n	$\{0, n, \dots, n\}$
complete bipartite $K_{n,m}$	$\min(n, m)$	$\{0, m, \dots, m, n, \dots, n, m+n\}$, where m has multiplicity $n-1$ and n has multiplicity $m-1$

B.5 The Laplacian flow

Let G be a weighted directed graph with n nodes and Laplacian matrix L. The *Laplacian flow* on $\mathbb{R}^n$ is the dynamical system

$$\dot{x} = -Lx, \tag{B.5}$$

or, equivalently in components,

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i) = \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij}(x_j - x_i).$$

One can show that, if G contains a globally reachable node, then the set of equilibrium points of the Laplacian flow (B.5) is $\{\alpha \mathbb{1}_n : \alpha \in \mathbb{R}\}$.

B.5.1 Matrix exponential of a Laplacian matrix

Theorem B.15 (The matrix exponential of a Laplacian matrix). *Let $L \in \mathbb{R}^{n \times n}$ be a Laplacian matrix with associated weighted digraph G and with maximum diagonal entry $\ell_{\max} = \max\{\ell_{11}, \dots, \ell_{nn}\}$. Then*

(i) $\exp(-L) \geq \mathrm{e}^{-\ell_{\max}} I_n \geq 0$, *for any digraph G,*

(ii) $\exp(-L)\mathbb{1}_n = \mathbb{1}_n$, *for any digraph G,*

(iii) $\mathbb{1}_n^\top \exp(-L) = \mathbb{1}_n^\top$, *for a weight-balanced G (i.e., $\mathbb{1}_n^\top L = \mathbb{0}_n^\top$),*

(iv) $\exp(-L)\mathbb{e}_j > 0$, *for a digraph G whose j-th node is globally reachable, and*

(v) $\exp(-L) > 0$, *for a strongly connected digraph G (i.e., for an irreducible L).*

Note that properties (i) and (ii) together imply that $\exp(-L)$ is row-stochastic.

B.5.2 Consensus with Laplacian flow with globally reachable node

Theorem B.16 (Consensus for Laplacian matrices with globally reachable node). *If a Laplacian matrix L has associated digraph G with a globally reachable node, then*

(i) the eigenvalue 0 of $-L$ is simple and all other eigenvalues of $-L$ have negative real part,

(ii) $\lim_{t\to\infty} \exp(-Lt) = \mathbb{1}_n w^\top$, where w is the left eigenvector of L with eigenvalue 0 satisfying $\mathbb{1}_n^\top w = 1$,

(iii) $w_i \geq 0$ for all nodes i and $w_i > 0$ if and only if node i is globally reachable,

(iv) the solution to $\frac{d}{dt}x(t) = -Lx(t)$ satisfies

$$\lim_{t\to\infty} x(t) = \big(w^\top x(0)\big)\mathbb{1}_n,$$

(v) if additionally G is weight-balanced, then G is strongly connected, $\mathbb{1}_n^\top L = \mathbb{0}_n^\top$, $w = \frac{1}{n}\mathbb{1}_n$, and

$$\lim_{t\to\infty} x(t) = \frac{\mathbb{1}_n^\top x(0)}{n}\mathbb{1}_n = \text{average}\big(x(0)\big)\mathbb{1}_n.$$

Note: as a corollary to the statement (iii), the left eigenvector $w \in \mathbb{R}^n$ associated to the 0 eigenvalue has strictly positive entries if and only if G is strongly connected.

B.6 Metzler matrices and linear compartmental systems

Definition B.17 (Metzler matrix). *For a matrix $M \in \mathbb{R}^{n\times n}$, $n \geq 2$,*

(i) M is Metzler if all its off-diagonal elements are non-negative;

(ii) if M is Metzler, its associated digraph is a weighted digraph defined as follows: $\{1, \dots, n\}$ are the nodes, there are no self-loops, (i, j), $i \neq j$ is an edge with weight a_{ij} if and only if $a_{ij} > 0$; and

(iii) if M is Metzler, M is irreducible if its induced digraph is strongly connected.

Theorem B.18 (Positive affine systems and Metzler matrices). *For the affine system $\dot{x}(t) = Mx(t) + b$, the following statements are equivalent:*

(i) the system is positive, that is, $x(t) \geq \mathbb{0}_n$ for all t and all $x(0) \geq \mathbb{0}_n$,

(ii) M is Metzler and $b \geq \mathbb{0}_n$.

Theorem B.19 (Perron-Frobenius Theorem for Metzler matrices). *If $M \in \mathbb{R}^{n\times n}$, $n \geq 2$, is Metzler, then*

(i) there exists a real eigenvalue λ such that $\lambda \geq \Re(\mu)$ for all other eigenvalues μ, and

(ii) the right and left eigenvectors of λ can be selected non-negative.

If additionally M is irreducible, then

(iii) there exists a real simple eigenvalue λ such that $\lambda \geq \Re(\mu)$ for all other eigenvalues μ, and

(iv) the right and left eigenvectors of λ are unique and positive (up to rescaling).

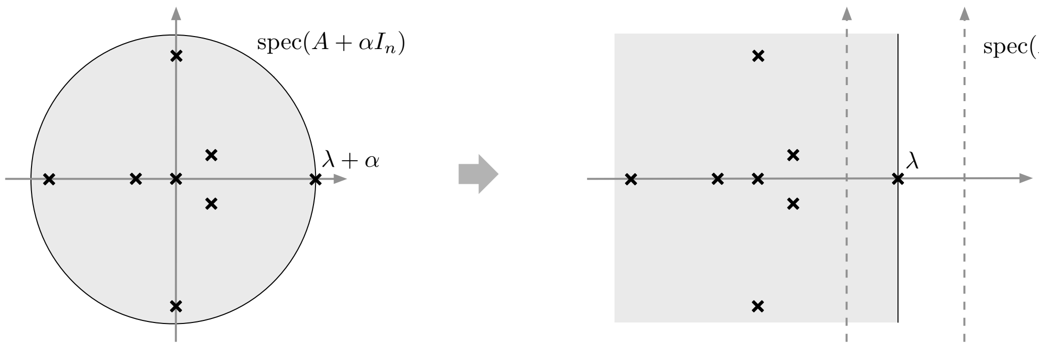


Figure B.11: Illustrating the Perron–Frobenius Theorem B.19 for a Metzler matrix M. Left image: for sufficiently large γ, $M+\gamma I_n$ is non-negative and $\lambda+\gamma$ is its dominant Perron eigenvalue. Right image: the spectrum of M is equal to that of $M+\gamma I_n$ translated by $-\gamma$; λ is dominant in the sense that $\lambda \geq \Re(\mu)$ for all other eigenvalues μ; it is not determined whether $\lambda < 0$ (the imaginary axis is to the right of λ) or $\lambda > 0$ (the imaginary axis is to the left of λ).

Theorem B.20 (Properties of Hurwitz Metzler matrices). *For a Metzler matrix M, the following statements are equivalent:*

(i) M is Hurwitz,

(ii) M is invertible and $-M^{-1} \geq 0$, and

(iii) for all $b \geq \mathbb{0}_n$, there exists $x^* \geq \mathbb{0}_n$ solving $Mx^* + b = \mathbb{0}_n$.

Moreover, if M is Metzler, Hurwitz and irreducible, then $-M^{-1} > 0$.

Corollary B.21 (Existence, positivity and stability of equilibria for positive affine systems). *Consider a continuous-time positive affine system $\dot{x} = Mx + b$, where M is Metzler and b is non-negative. If the matrix M is Hurwitz, then*

(i) the system has a unique equilibrium point $x^* \in \mathbb{R}^n$, that is, a unique solution to $Mx^* + b = \mathbb{0}_n$,

(ii) the equilibrium point x^* is non-negative, and

(iii) all trajectories converge asymptotically to x^*.

B.6.1 Compartmental systems

A *compartmental system* is a dynamical system in which material is stored at individual locations and is transferred along the edges of directed graph, called the *compartmental digraph*; see Figure B.12b.

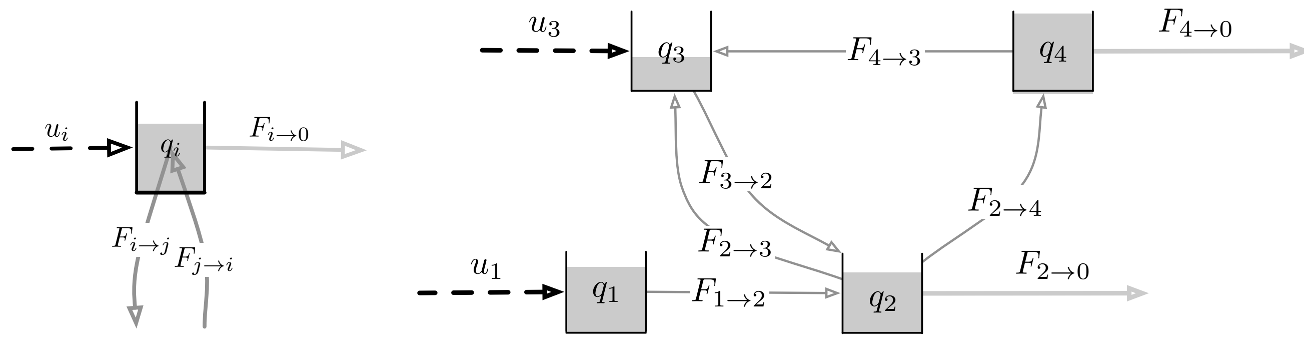


Figure B.12: A compartment and a compartmental system

The "storage" nodes are referred to as *compartments*; each compartment contains a time-varying quantity $q_i(t)$. Each directed arc (i, j) represents a *mass flow* (or *flux*), denoted $F_{i\to j}$, from compartment i to compartment j.

In summary, a compartmental system is described by a directed graph G_F, by maps $F_{i\to j}$ for all edges (i, j) of G_F, and by inflow and outflow maps. The dynamic equations of the compartmental system are obtained by the *instantaneous flow balance* at each compartment, i.e., asking that the rate of accumulation at each compartment equals the net inflow rate we obtain:

$$\dot{q}_i(t) = \sum_{j=1,j\neq i}^{n} (F_{j\to i} - F_{i\to j}) - F_{i\to 0} + u_i. \tag{B.6}$$

In general, the flow along (i, j) is a function of the entire system state $q = (q_1, \dots, q_n)$ and of time t, so that $F_{i\to j} = F_{i\to j}(q, t)$.

B.6.2 Linear compartmental systems

Definition B.22 (Linear compartmental systems). *A linear compartmental system with n compartments is a triplet (F, f_0, u) consisting of*

(i) a non-negative $n \times n$ matrix $F = (f_{ij})_{i,j\in\{1,...,n\}}$ with zero diagonal, called the flow rate matrix,

(ii) a vector $f_0 \geq \mathbb{0}_n$, called the outflow rates vector, and

(iii) a vector $u \geq \mathbb{0}_n$, called the inflow vector.

The flow rate matrix F is the adjacency matrix of the compartmental digraph G_F (a weighted digraph without self-loops).

The flow rate matrix F encodes the following information: the nodes are the compartments $\{1, \dots, n\}$, there is an edge (i, j) if there is a flow from compartment i to compartment j, and the weight f_{ij} of the (i, j) edge is the corresponding flow rate constant. In a linear compartmental system,

$$\begin{aligned} F_{i\to j}(q, t) &= f_{ij} q_i, \quad &&\text{for } j \in \{1, \dots, n\}, \\ F_{i\to 0}(q, t) &= f_{0i} q_i, \quad &&\text{and} \\ u_i(q, t) &= u_i. \end{aligned}$$

The affine dynamics describing a linear compartmental system is

$$\dot{q}_i(t) = -\Big(f_{0i} + \sum_{j=1, j\neq i}^{n} f_{ij}\Big) q_i(t) + \sum_{j=1, j\neq i}^{n} f_{ji} q_j(t) + u_i. \tag{B.7}$$

Definition B.23 (Compartmental matrix). *The compartmental matrix $C = (c_{ij})_{i,j\in\{1,...,n\}}$ of a compartmental system (F, f_0, u) is defined by*

$$c_{ij} = \begin{cases} f_{ji}, & \text{if } i \neq j, \\ -f_{0i} - \sum_{h=1,h\neq i}^{n} f_{ih}, & \text{if } i = j. \end{cases}$$

Equivalently,

$$C = F^{\top} - \operatorname{diag}(F\mathbb{1}_n + f_0). \tag{B.8}$$

In what follows it is convenient to call *compartmental* any matrix C with the following properties:

(i) C is Metzler, that is, $c_{ij} \geq 0$, for $i \neq j$,

(ii) C has non-positive diagonal entries, that is, $c_{ii} \leq 0$ for all i, and

(iii) C is *column diagonally dominant*, that is, $|c_{ii}| \geq \sum_{h=1,h\neq i}^{n} c_{hi}$ for all i.

With the notion of compartmental matrix, the dynamics of the linear compartmental system (B.7) can be written as

$$\dot{q}(t) = Cq(t) + u. \tag{B.9}$$

Lemma B.24 (Spectral properties of compartmental matrices). *For a compartmental system (F, f_0, u) with compartmental matrix C,*

(i) if $\lambda \in \operatorname{spec}(C)$, then either $\lambda = 0$ or $\Re(\lambda) < 0$, and

(ii) C is invertible if and only if C is Hurwitz (i.e., $\Re(\lambda) < 0$ for all $\lambda \in \operatorname{spec}(C)$).

B.6.3 The algebraic graph theory of compartmental systems

Next, we introduce some useful graph-theoretical notions. In the compartmental digraph, a set of compartments S is

(i) *outflow-connected* if there exists a directed walk from every compartment in S to the environment, that is, to a compartment j with a positive flow rate constant $f_{0j} > 0$,

(ii) *inflow-connected* if there exists a directed walk from the environment to every compartment in S, that is, from a compartment i with a positive inflow $u_i > 0$,

(iii) a *trap* if there is no directed walk from any of the compartments in S to the environment or to any compartment outside S, and

(iv) a *simple trap* is a trap that has no traps inside it.

Theorem B.25 (Algebraic graph theory of compartmental systems). *Consider the linear compartmental system (F, f_0, u) with dynamics (B.9) with compartmental matrix C and compartmental digraph G_F. The following statements are equivalent:*

(i) the system is outflow-connected,

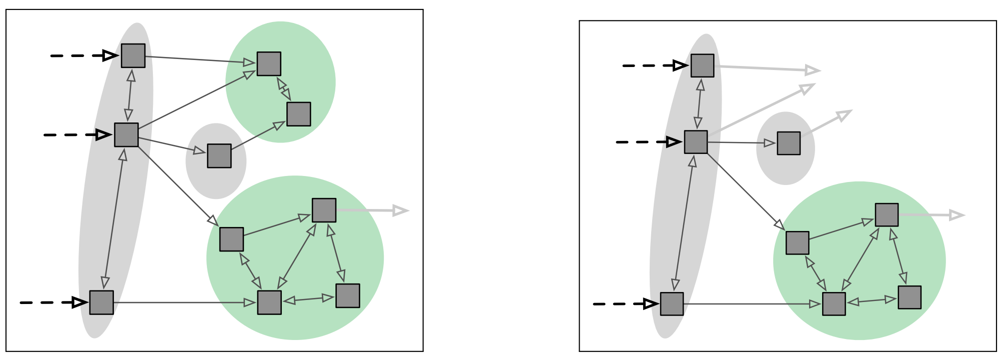
(ii) each sink of the condensation of G_F is outflow-connected, and

(iii) the compartmental matrix C is Hurwitz.

Moreover, the sinks of the condensation of G_F that are not outflow-connected are precisely the simple traps of the system and their number equals the multiplicity of 0 as a semi-simple eigenvalue of C.

B.6.4 Dynamic properties of linear compartmental systems

Consider a linear compartmental system (F, f_0, u) with compartmental matrix C and compartmental digraph G_F. Assuming the system has at least one trap, we define the *reduced compartmental system* $(F_{\text{rd}}, f_{0,\text{rd}}, u_{\text{rd}})$ as follows: remove all traps from G_F and regard the edges into the trapping compartments as outflow edges into the environment, e.g., see Figure B.13.



(a) A compartmental system that is not outflow-connected

(b) The corresponding reduced compartmental system

Figure B.13: An example reduced compartmental system

We now state our main result about the asymptotic behavior of linear compartmental systems.

Theorem B.26 (Asymptotic behavior of compartmental systems). *The linear compartmental system (F, f_0, u) with compartmental matrix C and compartmental digraph G_F has the following possible asymptotic behaviors:*

(i) if the system is outflow-connected, then the compartmental matrix C is invertible, every solution tends exponentially to the unique equilibrium $q^* = -C^{-1}u \geq \mathbb{0}_n$, and in the ith compartment $q_i^* > 0$ if and only if the ith compartment is inflow-connected to a positive inflow;

(ii) if the system contains one or more simple traps, then:

a) the reduced compartmental system $(F_{\text{rd}}, f_{0,\text{rd}}, u_{\text{rd}})$ is outflow-connected and all its solutions converge exponentially fast to the unique non-negative equilibrium $-C_{\text{rd}}^{-1}u_{\text{rd}}$, for $C_{\text{rd}} = F_{\text{rd}}^{\top} - \operatorname{diag}(F_{\text{rd}}\mathbb{1}_n + f_{0,\text{rd}})$;

b) any simple trap H contains non-decreasing mass along time. If H is inflow-connected to a positive inflow, then the mass inside H goes to infinity. Otherwise, the mass inside H converges to a scalar multiple of the right eigenvector corresponding to the eigenvalue 0 of the compartmental submatrix for H.

B.7 Historical notes and further reading

This field comprises a set of models, tools, and results that have been developed over the decades by scientists from multiple distinct disciplines. It would appear that (1) scientists in sociometrics and mathematical sociology were early to study network systems and that (2) mathematical biologists were close seconds.

Here are some historical comments:

- discrete-time averaging: this model is now typically referred to as the DeGroot model from (DeGroot, 1974), but it was introduced by French Jr. (1956), analyzed by Harary (1959) and extended to continuous time by Abelson (1964) (who calls it the "French-Harary model"). (Surprisingly, DeGroot's work did not appear in a sociological venue and did not cite the previous literature.)
- average consensus: the concept of average consensus was popularized by (Olfati-Saber and Murray, 2004), but an earlier reference is (Harary, 1959, Theorem 14, page 180):

  > A strong group attains unanimity at the arithmetic mean of the initial opinions if and only if its matrix M is doubly stochastic.

- time-varying discrete-time averaging: the early work (Tsitsiklis et al., 1986) was rediscovered in the award-winning (Jadbabaie et al., 2003), which however provides much novel graph theoretical insight. (Surprisingly, both papers appeared in IEEE Transactions on Automatic Control.)
- network SIS model for epidemic propagation: Van Mieghem et al. (2009) refer to it as the "N-intertwined SIS epidemic network model;" model and comprehensive analysis was given 33 years earlier by Lajmanovich and Yorke (1976). (Surprisingly, Yorke and Hethcote even wrote a book on the topic (Hethcote and Yorke, 1984).)
- Pagerank score was introduced by Brin and Page (1998) and, as the story goes, the corresponding algorithm led to the establishment of the Google search engine. Similar ideas appeared in (Friedkin, 1991).
- The rank of the Laplacian was studied as early as in (Fife, 1972; Foster and Jacquez, 1975). A mathematical approach is given in (Agaev and Chebotarev, 2000) which features the first necessary and sufficient characterization for the maximal rank case. We refer to the more recent (Lin et al., 2005; Ren and Beard, 2005) for the specific case of $\operatorname{rank}(L) = n - 1$.

Bibliography

R. P. Abelson. Mathematical models of the distribution of attitudes under controversy. In N. Frederiksen and H. Gulliksen, editors, *Contributions to Mathematical Psychology*, volume 14, pages 142–160. Holt, Rinehart, & Winston, 1964. ISBN 0030430100.

R. Abraham, J. E. Marsden, and T. S. Ratiu. *Manifolds, Tensor Analysis, and Applications*, volume 75 of *Applied Mathematical Sciences*. Springer, 2 edition, 1988. ISBN 0387967907.

P.-A. Absil, R. Mahony, and B. Andrews. Convergence of the iterates of descent methods for analytic cost functions. *SIAM Journal on Control and Optimization*, 6(2):531–547, 2005. doi:10.1137/040605266.

R. P. Agaev and P. Y. Chebotarev. The matrix of maximum out forests of a digraph and its applications. *Automation and Remote Control*, 61(9):1424–1450, 2000. URL https://arxiv.org/pdf/math/0602059.

R. P. Agarwal and V. Lakshmikantham. *Uniqueness and nonuniqueness criteria for ordinary differential equations*. World Scientific, 1993.

R. P. Agarwal, M. Meehan, and D. O'Regan. *Fixed Point Theory and Applications*. Cambridge University Press, 2001. ISBN 0-521-80250-4.

A. A. Ahmadi, E. de Klerk, and G. Hall. Polynomial norms. *SIAM Journal on Optimization*, 29(1):399–422, 2019. doi:10.1137/18M1172843.

M. A. Al-Radhawi and D. Angeli. New approach to the stability of chemical reaction networks: Piecewise linear in rates Lyapunov functions. *IEEE Transactions on Automatic Control*, 61(1):76–89, 2016. doi:10.1109/TAC.2015.2427691.

J. Albrecht. Minimal norms of nonnegative irreducible matrices. *Linear Algebra and its Applications*, 249(1):255–258, 1996. doi:10.1016/0024-3795(95)00360-6.

Z. Aminzare. *On Synchronous Behavior in Complex Nonlinear Dynamical Systems*. PhD thesis, Rutgers, 2015.

Z. Aminzare. Stochastic logarithmic Lipschitz constants: A tool to analyze contractivity of stochastic differential equations. *IEEE Control Systems Letters*, 6:2311–2316, 2022. doi:10.1109/LCSYS.2022.3148945.

Z. Aminzare and E. D. Sontag. Logarithmic Lipschitz norms and diffusion-induced instability. *Nonlinear Analysis: Theory, Methods & Applications*, 83:31–49, 2013. doi:10.1016/j.na.2013.01.001.

Z. Aminzare and E. D. Sontag. Synchronization of diffusively-connected nonlinear systems: Results based on contractions with respect to general norms. *IEEE Transactions on Network Science and Engineering*, 1(2):91–106, 2014a. doi:10.1109/TNSE.2015.2395075.

Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014b. doi:10.1109/CDC.2014.7039986.

Z. Aminzare, Y. Shafi, M. Arcak, and E. D. Sontag. Guaranteeing spatial uniformity in reaction-diffusion systems using weighted L_2 norm contractions. In *A Systems Theoretic Approach to Systems and Synthetic Biology I: Models and System Characterizations*, chapter 3, pages 73–101. Springer, 2014. ISBN 9789401790413. doi:10.1007/978-94-017-9041-3_3.

Z. Aminzare, B. Dey, E. N. Davison, and N. E. Leonard. Cluster synchronization of diffusively coupled nonlinear systems: A contraction-based approach. *Journal of Nonlinear Science*, 2018. doi:10.1007/s00332-018-9457-y.

D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3): 410–421, 2002. doi:10.1109/9.989067.

D. Angeli and E. D. Sontag. Monotone control systems. *IEEE Transactions on Automatic Control*, 48(10):1684–1698, 2003. doi:10.1109/TAC.2003.817920.

D. Angeli and E. D. Sontag. Translation-invariant monotone systems, and a global convergence result for enzymatic futile cycles. *Nonlinear Analysis: Real World Applications*, 9(1):128–140, 2008. doi:10.1016/j.nonrwa.2006.09.006.

M. T. Angulo and J.-J. E. Slotine. Qualitative stability of nonlinear networked systems. *IEEE Transactions on Automatic Control*, 62(8):4080–4085, 2017. doi:10.1109/tac.2016.2617780.

M. T. Angulo, Y.-Y. Liu, and J.-J. E. Slotine. Network motifs emerge from interconnections that favour stability. *Nature Physics*, 11(10):848–852, 2015. doi:10.1038/nphys3402.

M. Arcak, C. Meissen, and A. Packard. *Networks of Dissipative Systems: Compositional Certification of Stability, Performance, and Safety*. Springer, 2016. ISBN 978-3-319-29928-0. doi:10.1007/978-3-319-29928-0.

S. Arik. A note on the global stability of dynamical neural networks. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49(4):502–504, 2002. doi:10.1109/81.995665.

V. I. Arnol'd. *Ordinary Differential Equations*. Springer, 1992. ISBN 3-540-54813-0. Translation of the third Russian edition by R. Cooke.

Z. Askarzadeh, R. Fu, A. Halder, Y. Chen, and T. T. Georgiou. Stability theory of stochastic models in opinion dynamics. *IEEE Transactions on Automatic Control*, 65(2):522–533, 2020. doi:10.1109/TAC.2019.2912490.

E. M. Aylward, P. A. Parrilo, and J.-J. E. Slotine. Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming. *Automatica*, 44(8):2163–2170, 2008. doi:10.1016/j.automatica.2007.12.012.

H. Bai, M. Arcak, and J. Wen. *Cooperative Control Design*. Springer, 2011. ISBN 1461429072.

S. Baigent. Lotka-Volterra Dynamics — An Introduction. Unpublished Lecture Notes, University of College, London, Mar. 2010. URL `http://www.ltcc.ac.uk/media/london-taught-course-centre/documents/Bio-Mathematics-(APPLIED).pdf`.

S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922. doi:10.4064/fm-3-1-133-181.

E. A. Barbashin and N. N. Krasovskiĭ. On global stability of motion. *Doklady Akademii Nauk SSSR*, 86(3):453–456, 1952. (In Russian).

F. L. Bauer, J. Stoer, and C. Witzgall. Absolute and monotonic norms. *Numerische Mathematik*, 3:257–264, 1961. doi:10.1007/BF01386026.

M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta Numerica*, 14:1–137, 2005. doi:10.1017/S0962492904000212.

V. Berinde. *Iterative Approximation of Fixed Points*. Springer, 2007. ISBN 3-540-72233-5.

A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, 1994. ISBN 978-0898713213.

D. S. Bernstein. *Matrix Mathematics*. Princeton University Press, 2 edition, 2009. ISBN 0691140391.

D. Bertsekas, A. Nedić, and A. Ozdaglar. *Convex Analysis and Optimization*. Athena Scientific, 2003. ISBN 1-886529-45-0.

D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, 1997. ISBN 1886529019.

S. P. Bhat and D. S. Bernstein. Lyapunov analysis of semistability. In *American Control Conference*, volume 3, pages 1608–1612, 1999. doi:10.1109/ACC.1999.786101.

R. Bhatia. On the exponential metric increasing property. *Linear Algebra and its Applications*, 375:211–220, Dec. 2003. doi:10.1016/s0024-3795(03)00647-5.

G. Birkhoff. Extensions of Jentzsch's theorem. *Transactions of the American Mathematical Society*, 85(1):219–227, 1957. doi:10.2307/1992971.

F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999. doi:10.1016/S0005-1098(99)00113-2.

F. Blanchini and S. Miani. *Set-Theoretic Methods in Control*. Springer, 2015. ISBN 9783319179322.

S. Bonnabel and J.-J. E. Slotine. A contraction theory-based analysis of the stability of the deterministic extended Kalman filter. *IEEE Transactions on Automatic Control*, 60(2):565–569, 2015. doi:10.1109/tac.2014.2336991.

V. S. Borkar and K. Soumyanatha. An analog scheme for fixed point computation. I. Theory. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 44(4):351–355, 1997. doi:10.1109/81.563625.

P. Bougerol. Kalman filtering with random coefficients and contractions. *SIAM Journal on Control and Optimization*, 31(4):942–959, July 1993. doi:10.1137/0331041.

S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004. ISBN 0521833787.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994. ISBN 089871334X.

S. Brin and L. Page. The anatomy of a large-scale hypertextual Web search engine. *Computer Networks*, 30:107–117, 1998. doi:10.1016/S0169-7552(98)00110-X.

F. Bullo. *Lectures on Network Systems*. Kindle Direct Publishing, 1.6 edition, Jan. 2022. ISBN 978-1986425643. URL `http://motion.me.ucsb.edu/book-lns`.

F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Princeton University Press, 2009. ISBN 978-0-691-14195-4. URL `http://www.coordinationbook.info`.

F. Bullo, P. Cisneros-Velarde, A. Davydov, and S. Jafarpour. From contraction theory to fixed point algorithms on Riemannian and non-Euclidean spaces. In *IEEE Conf. on Decision and Control*, Dec. 2021. doi:10.1109/CDC45484.2021.9682883.

W. Bunse. A class of diagonal transformation methods for the computation of the spectral radius of a nonnegative irreducible matrix. *SIAM Journal on Numerical Analysis*, 18(4):693–704, 1981. doi:10.1137/0718046.

P. J. Bushell. Hilbert's metric and positive contraction mappings in a Banach space. *Archive for Rational Mechanics and Analysis*, 52(4):330–338, 1973. doi:10.1007/BF00247467.

R. Caccioppoli. Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale. *Rendiconti dell'Accademia Nazionale dei Lincei*, 11:794–799, 1930.

Y. Cao, W. Yu, W. Ren, and G. Chen. An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Transactions on Industrial informatics*, 9(1):427–438, 2013. doi:10.1109/TII.2012.2219061.

T. Caraballo and P. E. Kloeden. The persistence of synchronization under environmental noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2059):2257–2267, 2005. doi:10.1098/rspa.2005.1484.

V. Centorrino, F. Bullo, and G. Russo. Modelling and contractivity of neural-synaptic networks with Hebbian learning. *Automatica*, July 2022. doi:10.48550/arXiv.2204.05382. Submitted.

A. Cherukuri, B. Gharesifard, and J. Cortes. Saddle-point dynamics: Conditions for asymptotic stability of saddle points. *SIAM Journal on Control and Optimization*, 55(1):486–511, 2017. doi:10.1137/15M1026924.

N. G. Chetaev. *The Stability of Motion*. Pergamon, 1961. Translation from Russian by M. Nadler.

G. E. Cho and C. D. Meyer. Comparison of perturbation bounds for the stationary distribution of a Markov chain. *Linear Algebra and its Applications*, 335(1):137–150, 2001. doi:10.1016/S0024-3795(01)00320-2.

H. Chu, X. Liu, W. Zhang, and Y. Cai. Observer-based consensus tracking of multi-agent systems with one-sided Lipschitz nonlinearity. *Journal of the Franklin Institute*, 353(7):1594–1614, 2016. doi:10.1016/j.jfranklin.2015.10.011.

L. Chua and D. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks. *IEEE Transactions on Circuits and Systems*, 23(6):355–379, 1976. doi:10.1109/TCS.1976.1084228.

L. O. Chua and L. Yang. Cellular neural networks: Theory. *IEEE Transactions on Circuits and Systems*, 35(10):1257–1272, 1988. doi:10.1109/31.7600.

S.-J. Chung and J.-J. E. Slotine. Cooperative robot control and concurrent synchronization of Lagrangian systems. *IEEE Transactions on Robotics*, 25(3):686–700, 2009. doi:10.1109/TRO.2009.2014125.

P. Cisneros-Velarde, S. Jafarpour, and F. Bullo. Distributed and time-varying primal-dual dynamics via contraction analysis. *IEEE Transactions on Automatic Control*, 67(7):3560–3566, 2022a. doi:10.1109/TAC.2021.3103865.

P. Cisneros-Velarde, S. Jafarpour, and F. Bullo. Contraction theory for dynamical systems on Hilbert spaces. *IEEE Transactions on Automatic Control*, 67(12):6710–6715, 2022b. doi:10.1109/TAC.2021.3133270.

F. H. Clarke, Y. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, 1998. ISBN 0387983368.

G. Como. On resilient control of dynamical flow networks. *Annual Reviews in Control*, 43:80–90, 2017. doi:10.1016/j.arcontrol.2017.01.001.

G. Como, E. Lovisari, and K. Savla. Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing. *IEEE Transactions on Control of Network Systems*, 2(1):57–67, 2015. doi:10.1109/TCNS.2014.2367361.

S. Coogan. Separability of Lyapunov functions for contractive monotone systems. In *IEEE Conf. on Decision and Control*, pages 2184–2189, Las Vegas, USA, Dec. 2016. doi:10.1109/CDC.2016.7798587.

S. Coogan. A contractive approach to separable Lyapunov functions for monotone systems. *Automatica*, 106: 349–357, 2019. doi:10.1016/j.automatica.2019.05.001.

S. Coogan and M. Arcak. A note on norm-based Lyapunov functions via contraction analysis, 2013. URL `https://arxiv.org/pdf/1308.0586`.

S. Coogan and M. Arcak. A compartmental model for traffic networks and its dynamical behavior. *IEEE Transactions on Automatic Control*, 60(10):2698–2703, 2015a. doi:10.1109/TAC.2015.2411916.

S. Coogan and M. Arcak. Efficient finite abstraction of mixed monotone systems. In *Hybrid Systems: Computation and Control*, pages 58–67, Apr. 2015b. doi:10.1145/2728606.2728607.

S. Coogan and M. Arcak. Stability of traffic flow networks with a polytree topology. *Automatica*, 66:246–253, 2016. doi:10.1016/j.automatica.2015.12.015.

S. Coogan and M. Margaliot. Approximating the steady-state periodic solutions of contractive systems. *IEEE Transactions on Automatic Control*, 64(2):847–853, 2019. doi:10.1109/TAC.2018.2838054.

W. A. Coppel. *Stability and Asymptotic Behavior of Differential Equations*. Heath, 1965. ISBN 0669190187.

J. Cortés. Discontinuous dynamical systems. *IEEE Control Systems*, 28(3):36–73, 2008. doi:10.1109/MCS.2008.919306.

E. Cristiani, B. Piccoli, and A. Tosin. *Multiscale Modeling of Pedestrian Dynamics*. Springer, 2014. ISBN 978-3-319-06619-6.

C. F. Daganzo. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. *Transportation Research Part B: Methodological*, 28(4):269–287, 1994. doi:10.1016/0191-2615(94)90002-7.

G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations*. PhD thesis, (Reprinted in Trans. Royal Inst. of Technology, No. 130, Stockholm, Sweden, 1959), 1958.

G. Dahlquist. Error analysis for a class of methods for stiff non-linear initial value problems. In G. A. Watson, editor, *Numerical Analysis*, pages 60–72. Springer, 1976. doi:10.1007/BFb0080115.

G. Dahlquist. On matrix majorants and minorants, with applications to differential equations. *Linear Algebra and its Applications*, 52-53:199–216, 1983. doi:10.1016/0024-3795(83)80014-7.

G. Dahlquist. 33 years of numerical instability, Part I. *BIT Numerical Mathematics*, 25(1):188–204, 1985. doi:10.1007/bf01934997.

L. D'Alto and M. Corless. Incremental quadratic stability. *Numerical Algebra, Control and Optimization*, 3:175–201, 2013. doi:10.3934/naco.2013.3.175.

J. M. Danskin. The theory of max-min, with applications. *SIAM Journal on Applied Mathematics*, 14(4):641–664, 1966. doi:10.1137/0114053.

A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, 67(12):6667–6681, 2022a. doi:10.1109/TAC.2022.3183966.

A. Davydov, S. Jafarpour, A. V. Proskurnikov, and F. Bullo. Non-Euclidean monotone operator theory with applications to recurrent neural networks. In *IEEE Conf. on Decision and Control*, Cancún, México, Dec. 2022b. doi:10.1109/CDC51059.2022.9993197.

A. Davydov, A. V. Proskurnikov, and F. Bullo. Non-Euclidean contractivity of recurrent neural networks. In *American Control Conference*, pages 1527–1534, Atlanta, USA, May 2022c. doi:10.23919/ACC53348.2022.9867357.

G. De Pasquale, K. D. Smith, F. Bullo, and M. E. Valcher. Dual seminorms, ergodic coefficients, and semicontraction theory. *IEEE Transactions on Automatic Control*, Dec. 2022. doi:10.48550/arXiv.2201.03103.

M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974. doi:10.1080/01621459.1974.10480137.

K. Deimling. *Nonlinear Functional Analysis*. Springer, 1985. ISBN 3-540-13928-1.

K. Dekker and J. G. Verwer. *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations*. North-Holland, 1984. ISBN 0-444-87634-0.

P. DeLellis, M. Di Bernardo, and G. Russo. On QUAD, Lipschitz, and contracting vector fields for consensus and synchronization of networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 58(3):576–583, 2011. doi:10.1109/TCSI.2010.2072270.

B. P. Demidovič. Dissipativity of a nonlinear system of differential equations. *Uspekhi Matematicheskikh Nauk*, 16 (3(99)):216, 1961.

C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972. doi:10.1109/TCT.1972.1083507.

C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, 1975. ISBN 978-0-12-212050-3. doi:10.1137/1.9780898719055.

E. Deutsch. On matrix norms and logarithmic norms. *Numerische Mathematik*, 24(1):49–51, 1975. doi:10.1007/bf01437217.

M. Di Bernardo, D. Liuzza, and G. Russo. Contraction analysis for a class of nondifferentiable systems with applications to stability and network synchronization. *SIAM Journal on Control and Optimization*, 52(5):3203–3227, 2014. doi:10.1137/120883001.

M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In *Complex Systems and Networks*, pages 313–339. Springer, 2016. ISBN 978-3-662-47824-0. doi:10.1007/978-3-662-47824-0_12.

R. L. Dobrushin. Central limit theorem for nonstationary Markov chains. I. *Theory of Probability & Its Applications*, 1(1):65–80, 1956. doi:10.1137/1101006.

W. Doeblin. Le cas discontinu des probabilités en chaîne. *Publ. Faculty of Science University Masaryk (Brno)*, pages 3–13, 1937. ISSN 0371-2125.

F. Dörfler, J. W. Simpson-Porco, and F. Bullo. Electrical networks and algebraic graph theory: Models, properties, and applications. *Proceedings of the IEEE*, 106(5):977–1005, 2018. doi:10.1109/JPROC.2018.2821924.

J. C. Doyle, B. A. Francis, and A. R. Tannenbaum. *Feedback Control Theory*. MacMillan Publishing Co, 1990. ISBN 0486469336.

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic stability conditions for Metzler matrices and monotone systems. *SIAM Journal on Control and Optimization*, 59(5):3447–3471, 2021. doi:10.1137/20M131802X.

C. Fan, J. Kapinski, X. Jin, and S. Mitra. Simulation-driven reachability using matrix measures. *ACM Transactions on Embedded Computing Systems*, 17(1):1–28, 2018. doi:10.1145/3126685.

Y. Fang and T. G. Kincaid. Stability analysis of dynamical neural networks. *IEEE Transactions on Neural Networks*, 7(4):996–1006, 1996. doi:10.1109/72.508941.

Y. Fang, K. A. Loparo, and X. Feng. Sufficient conditions for the stability of interval matrices. *International Journal of Control*, 58(4):969–977, 1993. doi:10.1080/00207179308923038.

D. Feijer and F. Paganini. Stability of primal–dual gradient dynamics and applications to network optimization. *Automatica*, 46(12):1974–1981, 2010. doi:10.1016/j.automatica.2010.08.011.

M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(2):298–305, 1973. URL `http://dml.cz/dmlcz/101168`.

M. Fiedler and V. Ptak. On matrices with non-positive off-diagonal elements and positive principal minors. *Czechoslovak Mathematical Journal*, 12(3):382–400, 1962. doi:10.21136/CMJ.1962.100526.

D. Fife. Which linear compartmental systems contain traps? *Mathematical Biosciences*, 14(3):311–315, 1972. doi:10.1016/0025-5564(72)90082-X.

A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer, 1988. ISBN 902772699X.

D. Fiore, S. J. Hogan, and M. Di Bernardo. Contraction analysis of switched systems via regularization. *Automatica*, 73:279–288, 2016. doi:10.1016/j.automatica.2016.06.028.

R. FitzHugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1(6): 445–466, 1961. doi:10.1016/S0006-3495(61)86902-6.

F. Forni and R. Sepulchre. A differential Lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*, 59(3):614–628, 2014. doi:10.1109/TAC.2013.2285771.

M. Forti and A. Tesi. New conditions for global stability of neural networks with application to linear and quadratic programming problems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 42(7): 354–366, 1995. doi:10.1109/81.401145.

M. Forti, S. Manetti, and M. Marini. Necessary and sufficient condition for absolute stability of neural networks. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 41(7):491–494, 1994. doi:10.1109/81.298364.

D. M. Foster and J. A. Jacquez. Multiple zeros for eigenvalues and the multiplicity of traps of a linear compartmental system. *Mathematical Biosciences*, 26(1):89–97, 1975. doi:10.1016/0025-5564(75)90096-6.

B. A. Francis and M. Maggiore. *Flocking and Rendezvous in Distributed Robotics*. Springer, 2016. ISBN 978-3-319-24727-4.

J. R. P. French Jr. A formal theory of social power. *Psychological Review*, 63(3):181–194, 1956. doi:10.1037/h0046123.

N. E. Friedkin. Theoretical foundations for centrality measures. *American Journal of Sociology*, 96(6):1478–1504, 1991. doi:10.1086/229694.

V. Fromion, S. Monaco, and D. Normand-Cyrot. Asymptotic properties of incrementally stable systems. *IEEE Transactions on Automatic Control*, 41(5):721–723, 1996. doi:10.1109/9.489210.

F. R. Gantmacher. *The Theory of Matrices*, volume 1 and 2. Chelsea, New York, 1959. ISBN 0-8218-1376-5 and 0-8218-2664-6. Translation of German edition by K. A. Hirsch.

F. Garin and L. Schenato. A survey on distributed estimation and control applications using linear consensus algorithms. In A. Bemporad, M. Heemels, and M. Johansson, editors, *Networked Control Systems*, pages 75–107. Springer, 2010. doi:10.1007/978-0-85729-033-5_3.

R. Ghrist. *Elementary Applied Topology*. Createspace, 1.0 edition, 2014. ISBN 978-1502880857.

P. Giesl, S. Hafstein, and C. Kawan. Review on contraction analysis and computation of contraction metrics. *Journal of Computational Dynamics*, 10(1):1–47, 2023. doi:10.3934/jcd.2022018.

G. Giorgi and S. Komlósi. Dini derivatives in optimization — Part I. *Rivista di Matematica Per Le Scienze Economiche e Sociali*, 15(1):3–30, 1992. doi:10.1007/BF02086523.

R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012. ISBN 9780691153896.

B. S. Goh. Global stability in two species interactions. *Journal of Mathematical Biology*, 3(3-4):313–318, 1976. doi:10.1007/BF00275063.

B. S. Goh. Stability in models of mutualism. *American Naturalist*, pages 261–275, 1979. doi:10.1086/283384.

B.-S. Goh. *Management and Analysis of Biological Populations*. Elsevier, 1980. ISBN 978-0-444-41793-0.

G. Gomes, R. Horowitz, A. A. Kurzhanskiy, P. Varaiya, and J. Kwon. Behavior of the cell transmission model and effectiveness of ramp metering. *Transportation Research Part C: Emerging Technologies*, 16(4):485–513, 2008. doi:10.1016/j.trc.2007.10.005.

B. C. Goodwin. Oscillatory behavior in enzymatic control processes. *Advances in Enzyme Regulation*, 3:425–437, 1965. doi:10.1016/0065-2571(65)90067-1.

A. Granas and J. Dugundji. *Fixed Point Theory*. Springer, 2003. ISBN 978-1-4419-1805-5. doi:10.1007/978-0-387-21593-8.

M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.1, Mar. 2014. URL `http://cvxr.com/cvx`.

R. Gray, A. Franci, V. Srivastava, and N. E. Leonard. Multiagent decision-making dynamics inspired by honeybees. *IEEE Transactions on Control of Network Systems*, 5(2):793–806, 2018. doi:10.1109/tcns.2018.2796301.

J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer, 1990. ISBN 0387908196.

W. M. Haddad and V. Chellaboina. *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton University Press, 2008. ISBN 9780691133294.

W. Hahn. *Stability of Motion*. Springer, 1967. ISBN 978-3-642-50085-5.

E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer, 1993. doi:10.1007/978-3-540-78862-1.

A. Halanay. *Differential Equations: Stability, Oscillations, Time Lags*. Academic Press, 1966. ISBN 978-0-12-317950-0.

A. Hamadeh, E. Sontag, and D. Del Vecchio. A contraction approach to input tracking via high gain feedback. In *IEEE Conf. on Decision and Control*, pages 7689–7694, Dec. 2015. doi:10.1109/CDC.2015.7403435.

R. Hamming. You and your research. In *Bell Communications Research Colloquium Seminar*, page 483. Bell Communications Research, Morristown, NJ, 1986. URL `http://motion.me.ucsb.edu/RHamming-YouAndYourResearch-1986.pdf`. Transcripted by James F. Kaiser. (Later version of the talk at `https://www.youtube.com/watch?v=a1zDuOPkMSw`).

F. Harary. A criterion for unanimity in French's theory of social power. In D. Cartwright, editor, *Studies in Social Power*, pages 168–182. University of Michigan, 1959. ISBN 0879442301. URL `http://psycnet.apa.org/psycinfo/1960-06701-006`.

W. He and J. Cao. Exponential synchronization of chaotic neural networks: a matrix measure approach. *Nonlinear Dynamics*, 55:55–65, 2009. doi:10.1007/s11071-008-9344-4.

H. W. Hethcote and J. A. Yorke. *Gonorrhea Transmission Dynamics and Control*. Springer, 1984. ISBN 978-3-540-13870-9. doi:10.1007/978-3-662-07544-9.

I. Higueras and B. Garcia-Celayeta. Logarithmic norms for matrix pencils. *SIAM Journal on Matrix Analysis and Applications*, 20(3):646–666, 1999. doi:10.1137/s0895479897325955.

M. W. Hirsch. Systems of differential equations which are competitive or cooperative: I. Limit sets. *SIAM Journal on Mathematical Analysis*, 13(2):167–179, 1982. doi:10.1137/0513013.

M. W. Hirsch. Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere. *SIAM Journal on Mathematical Analysis*, 16(3):423–439, 1985. doi:10.1137/0516030.

M. W. Hirsch. Systems of differential equations which are competitive or cooperative: III. Competing species. *Nonlinearity*, 1(1):51–71, 1988. doi:10.1088/0951-7715/1/1/003.

M. W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems and Linear Algebra*. Academic Press, 1974. ISBN 0123495504.

M. W. Hirsch and H. L. Smith. Monotone dynamical systems. In A. Canada, P. Drabek, and A. Fonda, editors, *Handbook of Differential Equations: Ordinary Differential Equations*, volume 2, pages 239–357. Elsevier, 2005.

J. Hofbauer and K. Sigmund. *Evolutionary Games and Population Dynamics*. Cambridge University Press, 1998. ISBN 052162570X.

J. J. Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the National Academy of Sciences*, 79(8):2554–2558, 1982. doi:10.1073/pnas.79.8.2554.

J. J. Hopfield. Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings of the National Academy of Sciences*, 81(10):3088–3092, 1984. doi:10.1073/pnas.81.10.3088.

R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1994. ISBN 0521467136.

R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 2nd edition, 2012. ISBN 0521548233.

G. Hu and M. Liu. Properties of the weighted logarithmic matrix norms. *IMA Journal of Mathematical Control and Information*, 25(1):75–84, 2008. doi:10.1093/imamci/dnm006.

I. C. F. Ipsen and T. M. Selee. Ergodicity coefficients defined by vector norms. *SIAM Journal on Matrix Analysis and Applications*, 32(1):153–200, 2011. doi:10.1137/090752948.

J. A. Jacquez and C. P. Simon. Qualitative theory of compartmental systems. *SIAM Review*, 35(1):43–79, 1993. doi:10.1137/1035003.

A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003. doi:10.1109/TAC.2003.812781.

S. Jafarpour. A sign contractivity property of Metzler matrices. Personal communication, 2019.

S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. Weak and semi-contraction for network systems and diffusively-coupled oscillators. 2021a. URL `https://arxiv.org/abs/2005.09774`. Extended report with proofs.

S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021b. doi:10.48550/arXiv.2106.03194.

S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. Weak and semi-contraction for network systems and diffusively-coupled oscillators. *IEEE Transactions on Automatic Control*, 67(3):1285–1300, 2022. doi:10.1109/TAC.2021.3073096.

S. Jafarpour, A. Davydov, and F. Bullo. Non-Euclidean contraction theory for monotone and positive systems. *IEEE Transactions on Automatic Control*, 2023. doi:10.1109/TAC.2022.3224094. To appear.

C. R. Johnson. Two submatrix properties of certain induced norms. *Journal of Research of the National Bureau of Standards - B. Mathematical Sciences*, 79:97–102, 1975. URL `https://nvlpubs.nist.gov/nistpubs/jres/79B/jresv79Bn3-4p97_A1b.pdf`.

J. Jouffroy and T. I. Fossen. A tutorial on incremental stability analysis using contraction theory. *Modeling, Identification and Control*, 31(3):93–106, 2010. doi:10.4173/mic.2010.3.2.

J. Jouffroy and J.-J. E. Slotine. Methodological remarks on contraction theory. In *IEEE Conf. on Decision and Control*, volume 3, pages 2537–2543, 2004. doi:10.1109/CDC.2004.1428824.

R. I. Kachurovskii. Monotone operators and convex functionals. *Uspekhi Matematicheskikh Nauk*, 15(4):213–215, 1960.

B. Kågström. Bounds and perturbation bounds for the matrix exponential. *BIT Numerical Mathematics*, 17:39–57, 1977. doi:10.1007/BF01932398.

E. Kamke. Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II. *Acta Mathematica*, 58:57–85, 1932. doi:10.1007/BF02547774.

E. Kaszkurewicz and A. Bhaya. On a class of globally stable neural circuits. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 41(2):171–174, 1994. doi:10.1109/81.269055.

Y. Kawano, B. Besselink, and M. Cao. Contraction analysis of monotone systems via separable functions. *IEEE Transactions on Automatic Control*, 65(8):3486–3501, 2020. doi:10.1109/TAC.2019.2944923.

H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3 edition, 2002. ISBN 0130673897.

M. A. Khamsi and W. A. Kirk. *An Introduction to Metric Spaces and Fixed Point Theory*. John Wiley & Sons, 2001. ISBN 9781118031322.

H. Kiendl, J. Adamy, and P. Stelzner. Vector norms as Lyapunov functions for linear systems. *IEEE Transactions on Automatic Control*, 37(6):839–842, 1992. doi:10.1109/9.256362.

E. Kohlberg and J. W. Pratt. The contraction mapping approach to the perron-frobenius theory: Why Hilbert's metric? *Mathematics of Operations Research*, 7(2):198–210, 1982. doi:10.1287/moor.7.2.198.

A. N. Kolmogorov. Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung. *Mathematische Annalen*, 104:415–158, 1931. doi:10.1007/BF01457949.

V. V. Kolpakov. Matrix seminorms and related inequalities. *Journal of Soviet Mathematics*, 23:2094–2106, 1983. doi:10.1007/BF01093289.

L. Kozachkov, M. Lundqvist, J.-J. E. Slotine, and E. K. Miller. Achieving stable dynamics in neural circuits. *PLoS Computational Biology*, 16(8):1–15, 2020. doi:10.1371/journal.pcbi.1007659.

L. Kozachkov, M. Ennis, and J.-J. E. Slotine. RNNs of RNNs: Recursive construction of stable assemblies of recurrent neural networks, 2021. URL `https://arxiv.org/abs/2106.08928`.

M. A. Krasnoselskiĭ and S. G. Krein. Nonlocal existence theorems and uniqueness theorems for systems of ordinary differential equations. 102(1):13–16, 1955.

N. N. Krasovskiĭ. *Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*. Stanford University Press, 1963. Translation of the 1959 edition in Russian by J. L. Brenner.

U. Krause. Perron's stability theorem for non-linear mappings. *Journal of Mathematical Economics*, 15(3):275–282, 1986. doi:10.1016/0304-4068(86)90016-9.

U. Krause. Relative stability for ascending and positively homogeneous operators on Banach spaces. *Journal of Mathematical Analysis and Applications*, 188(1):182–202, 1994. doi:10.1006/jmaa.1994.1420.

U. Krause. Concave Perron-Frobenius Theory and applications. *Nonlinear Analysis, Theory, Methods & Applications*, 47(3):1457–1466, 2001. doi:10.1016/S0362-546X(01)00281-4.

J. L. Lagrange. *Mécanique Analytique*. Chez la Veuve Desaint, Paris, 1788.

A. Lajmanovich and J. A. Yorke. A deterministic model for gonorrhea in a nonhomogeneous population. *Mathematical Biosciences*, 28(3):221–236, 1976. doi:10.1016/0025-5564(76)90125-5.

P. Lancaster and H. K. Farahat. Norms on direct sums and tensor products. *Mathematics of Computation*, 26(118): 401–414, 1972. doi:10.2307/2005167.

J. P. LaSalle. Some extensions of Liapunov's second method. *IRE Transactions on Circuit Theory*, CT-7:520–527, 1960. doi:10.1109/TCT.1960.1086720.

J. P. LaSalle. Stability theory for ordinary differential equations. *Journal of Differential Equations*, 4:57–65, 1968. doi:10.1016/0022-0396(68)90048-X.

J. P. LaSalle. *The Stability of Dynamical Systems*. SIAM, 1976. ISBN 9780898710229. doi:10.1137/1.9781611970432.

J. Lawson and Y. Lim. A Birkhoff contraction formula with applications to Riccati equations. *SIAM Journal on Control and Optimization*, 46(3):930–951, 2007. doi:10.1137/050637637.

H. Lee and Y. Lim. Invariant metrics, contractions and nonlinear matrix equations. *Nonlinearity*, 21(4):857–878, 2008. doi:10.1088/0951-7715/21/4/011.

B. Lemmens and R. Nussbaum. *Nonlinear Perron-Frobenius Theory*. Cambridge Tracts in Mathematics. Cambridge University Press, 2012. ISBN 9780521898812.

A. D. Lewis. A top nine list: Most popular induced matrix norms. Technical report, Queen's University, Kingston, Ontario, Canada, 2010. URL `https://mast.queensu.ca/~andrew/notes/pdf/2010a.pdf`.

D. C. Lewis. Differential equations referred to a variable metric. *American Journal of Mathematics*, 73(1):48–58, 1951. doi:10.2307/2372159.

M. Y. Li and J. S. Muldowney. A geometric approach to global-stability problems. *SIAM Journal on Mathematical Analysis*, 27(4):1070–1083, 1996. doi:10.1137/s0036141094266449.

M. Y. Li and L. Wang. A criterion for stability of matrices. *Journal of Mathematical Analysis and Applications*, 225 (1):249–264, 1998. doi:10.1006/jmaa.1998.6020.

T. C. Lim. On fixed point stability for set-valued contractive mappings with applications to generalized differential equations. *Journal of Mathematical Analysis and Applications*, 110(2):436–441, 1985. doi:10.1016/0022-247X(85)90306-3.

Z. Lin, B. Francis, and M. Maggiore. Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Transactions on Automatic Control*, 50(1):121–127, 2005. doi:10.1109/TAC.2004.841121.

Z. Lin, B. Francis, and M. Maggiore. State agreement for continuous-time coupled nonlinear systems. *SIAM Journal on Control and Optimization*, 46(1):288–307, 2007. doi:10.1137/050626405.

J. Liu, X. Liu, W.-C. Xie, and H. Zhang. Stochastic consensus seeking with communication delays. *Automatica*, 47 (12):2689–2696, 2011a. doi:10.1016/j.automatica.2011.09.005.

J. Liu, A. S. Morse, B. D. O. Anderson, and C. Yu. Contractions for consensus processes. In *IEEE Conf. on Decision and Control and European Control Conference*, pages 1974–1979, 2011b. doi:10.1109/CDC.2011.6160989.

J. Liu, S. Mou, A. S. Morse, B. D. O. Anderson, and C. Yu. Deterministic gossiping. *Proceedings of the IEEE*, 99(9): 1505–1524, 2011c. doi:10.1109/JPROC.2011.2159689.

W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998. doi:10.1016/S0005-1098(98)00019-3.

W. Lohmiller and J.-J. E. Slotine. Control system design for mechanical systems using contraction theory. *IEEE Transactions on Automatic Control*, 45(5):984–989, 2000. doi:10.1109/9.855568.

S. Łojasiewicz. Sur les trajectoires du gradient d'une fonction analytique. *Seminari di Geometria 1982-1983*, pages 115–117, 1984. Istituto di Geometria, Dipartimento di Matematica, Università di Bologna, Italy.

B. T. Lopez and J.-J. E. Slotine. Adaptive nonlinear control with contraction metrics. *IEEE Control Systems Letters*, 5 (1):205–210, 2021. doi:10.1109/LCSYS.2020.3000190.

K. Loskot, A. Polanski, and R. Rudnicki. Further comments on "Vector norms as Lyapunov functions for linear systems". *IEEE Transactions on Automatic Control*, 43(2):289–291, 1998. doi:10.1109/9.661083.

A. J. Lotka. Analytical note on certain rhythmic relations in organic systems. *Proceedings of the National Academy of Sciences*, 6(7):410–415, 1920. doi:10.1073/pnas.6.7.410.

E. Lovisari, G. Como, and K. Savla. Stability of monotone dynamical flow networks. In *IEEE Conf. on Decision and Control*, pages 2384–2389, Los Angeles, USA, Dec. 2014. doi:10.1109/CDC.2014.7039752.

S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958. URL `http://mi.mathnet.ru/eng/ivm2980`. (in Russian).

W. Lu and T. Chen. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D: Nonlinear Phenomena*, 213(2):214–230, 2006. doi:10.1016/j.physd.2005.11.009.

W. Lu and M. Di Bernardo. Contraction and incremental stability of switched Carathéodory systems using multiple norms. *Automatica*, 70:1–8, 2016. doi:10.1016/j.automatica.2016.02.039.

A. M. Lyapunov. *Obščaya zadača ob ustoĭčivosti dviženiya*. Fakul′teta i Khar′kovskogo Matematicheskogo Obshchestva, Kharkov, 1892. Translation: (Lyapunov, 1992).

A. M. Lyapunov. *The General Problem of the Stability of Motion*. Taylor & Francis, 1992. Translation from Russian by A. T. Fuller.

H. Maeda, S. Kodama, and Y. Ohta. Asymptotic behavior of nonlinear compartmental systems: Nonoscillation and stability. *IEEE Transactions on Circuits and Systems*, 25(6):372–378, 1978. doi:10.1109/TCS.1978.1084490.

J. Maidens and M. Arcak. Reachability analysis of nonlinear systems using matrix measures. *IEEE Transactions on Automatic Control*, 60(1):265–270, 2015. doi:10.1109/TAC.2014.2325635.

I. R. Manchester and J.-J. E. Slotine. Transverse contraction criteria for existence, stability, and robustness of a limit cycle. *Systems & Control Letters*, 63:32–38, 2014. doi:10.1016/j.sysconle.2013.10.005.

I. R. Manchester and J.-J. E. Slotine. Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design. *IEEE Transactions on Automatic Control*, 62(6):3046–3053, 2017. doi:10.1109/TAC.2017.2668380.

I. R. Manchester and J.-J. E. Slotine. Robust control contraction metrics: A convex approach to nonlinear state-feedback H^∞ control. *IEEE Control Systems Letters*, 2(3):333–338, 2018. doi:10.1109/lcsys.2018.2836355.

M. Margaliot, E. D. Sontag, and T. Tuller. Entrainment to periodic initiation and transition rates in a computational model for gene translation. *PLoS One*, 9(5):e96039, 2014. doi:10.1371/journal.pone.0096039.

M. Margaliot, E. D. Sontag, and T. Tuller. Contraction after small transients. *Automatica*, 67:178–184, 2016. doi:10.1016/j.automatica.2016.01.018.

A. A. Markov. Extensions of the law of large numbers to dependent quantities. *Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete*, 15, 1906. (in Russian).

L. Markus and H. Yamabe. Global stability criteria for differential systems. *Osaka Mathematical Journal*, 12(2): 305–317, 1960. doi:10.18910/9397.

R. Marsli and F. J. Hall. Some properties of ergodicity coefficients with applications in spectral graph theory. *Linear and Multilinear Algebra*, 0(0):1–21, 2020. doi:10.1080/03081087.2020.1777251.

S. Martínez, J. Cortés, and F. Bullo. Motion coordination with distributed information. *IEEE Control Systems*, 27(4): 75–88, 2007. doi:10.1109/MCS.2007.384124.

J. C. Maxwell. On governors. *Proceedings of the Royal Society. London. Series A. Mathematical and Physical Sciences*, 16:270–283, 1868. doi:10.1098/rspl.1867.0055.

W. Mei, S. Mohagheghi, S. Zampieri, and F. Bullo. On the dynamics of deterministic epidemic propagation over networks. *Annual Reviews in Control*, 44:116–128, 2017. doi:10.1016/j.arcontrol.2017.09.002.

M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010. ISBN 9781400835355.

A. N. Michel, J. A. Farrell, and W. Porod. Qualitative analysis of neural networks. *IEEE Transactions on Circuits and Systems*, 36(2):229–243, 1989. doi:10.1109/31.20200.

J. Mierczyński. Strictly cooperative systems with a first integral. *SIAM Journal on Mathematical Analysis*, 18(3): 642–646, 1987. doi:10.1137/0518049.

K. D. Miller and F. Fumarola. Mathematical equivalence of two common forms of firing rate models of neural networks. *Neural Computation*, 24(1):25–31, 2012. doi:10.1162/NECO_a_00221.

G. J. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Mathematical Journal*, 29(3):341–346, 1962. doi:10.1215/S0012-7094-62-02933-2.

G. J. Minty. On the monotonicity of the gradient of a convex function. *Pacific Journal of Mathematics*, 14(1):243–247, 1964. doi:10.2140/pjm.1964.14.243.

A. P. Molchanov and E. S. Pyatnitsky. Lyapunov functions defining necessary and sufficient conditions for the absolute stability of nonlinear nonstationary control systems. *Automation and Remote Control*, 5:38–49, 1986. URL `http://mi.mathnet.ru/eng/at6230`. (In Russian).

C. Moler and C. V. Loan. Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Review*, 45(1):3–49, 2003. doi:10.1137/S00361445024180.

J. Monteil, G. Russo, and R. Shorten. On $\mathcal{L}_\infty$ string stability of nonlinear bidirectional asymmetric heterogeneous platoon systems. *Automatica*, 105:198–205, 2019. doi:10.1016/j.automatica.2019.03.025.

P. J. Moylan. Matrices with positive principal minors. *Linear Algebra and its Applications*, 17(1):53–58, 1977. doi:10.1016/0024-3795(77)90040-4.

J. S. Muldowney. Compound matrices and ordinary differential equations. *Rocky Mountain Journal of Mathematics*, 20(4):857–872, 1990. doi:10.1216/rmjm/1181073047.

M. Müller. Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen. *Mathematische Zeitschrift*, 26(1):619–645, 1927. doi:10.1007/BF01475477.

M. Nagumo. Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 24:551–559, 1942. doi:10.11429/ppmsj1919.24.0_551.

Y. Nesterov and A. Nemirovski. Finding the stationary states of Markov chains by iterative methods. *Applied Mathematics and Computation*, 255:58–65, 2015. doi:10.1016/j.amc.2014.04.053.

H. D. Nguyen, T. L. Vu, K. Turitsyn, and J.-J. E. Slotine. Contraction and robustness of continuous time primal-dual dynamics. *IEEE Control Systems Letters*, 2(4):755–760, 2018. doi:10.1109/LCSYS.2018.2847408.

E. P. Odum. *Fundamentals of Ecology*. Saunders Company, 1959.

R. Ofir, F. Bullo, and M. Margaliot. Minimum effort decentralized control design for contracting network systems. *IEEE Control Systems Letters*, 6:2731–2736, 2022. doi:10.1109/LCSYS.2022.3176196.

K.-K. Oh, M.-C. Park, and H.-S. Ahn. A survey of multi-agent formation control. *Automatica*, 53:424–440, 2015. doi:10.1016/j.automatica.2014.10.022.

R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004. doi:10.1109/TAC.2004.834113.

O. Pastravanu and M. H. Matcovschi. Linear time-variant systems: Lyapunov functions and invariant sets defined by Hölder norms. *Journal of the Franklin Institute*, 347(3):627–640, 2010. doi:10.1016/j.jfranklin.2010.01.002.

O. Pastravanu and M. Voicu. Generalized matrix diagonal stability and linear dynamical systems. *Linear Algebra and its Applications*, 419(2):299–310, 2006. doi:10.1016/j.laa.2006.04.021.

A. Pavlov, A. Pogromsky, N. Van de Wouw, and H. Nijmeijer. Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Systems & Control Letters*, 52(3-4):257–261, 2004. doi:10.1016/j.sysconle.2004.02.003.

Q. C. Pham and J.-J. E. Slotine. Stable concurrent synchronization in dynamic system networks. *Neural Networks*, 20(1):62–77, 2007. doi:10.1016/j.neunet.2006.07.008.

Q. C. Pham, N. Tabareau, and J.-J. E. Slotine. A contraction theory approach to stochastic incremental stability. *IEEE Transactions on Automatic Control*, 54(4):816–820, 2009. doi:10.1109/tac.2008.2009619.

A. Polanski. Lyapunov function construction by linear programming. *IEEE Transactions on Automatic Control*, 42 (7):1013–1016, 1997. doi:10.1109/9.599986.

A. Polyakov. Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, 29(3):682–701, 2018. doi:10.1002/rnc.4058.

H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001. doi:10.1109/72.914530.

G. Qu and N. Li. On the exponential stability of primal-dual gradient dynamics. *IEEE Control Systems Letters*, 3(1): 43–48, 2019. doi:10.1109/LCSYS.2018.2851375.

W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005. doi:10.1109/TAC.2005.846556.

W. Ren and R. W. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control*. Communications and Control Engineering. Springer, 2008. ISBN 978-1-84800-014-8.

W. Ren, R. W. Beard, and E. M. Atkins. Information consensus in multivehicle cooperative control. *IEEE Control Systems*, 27(2):71–82, 2007. doi:10.1109/MCS.2007.338264.

M. Revay, R. Wang, and I. R. Manchester. Lipschitz bounded equilibrium networks. 2020. URL `https://arxiv.org/abs/2010.01732`.

H. H. Rosenbrock. A Lyapunov function with applications to some nonlinear physical systems. *Automatica*, 1(1): 31–53, 1963. doi:10.1016/0005-1098(63)90005-0.

G. Russo. *Analysis, Control and Synchronization of Nonlinear Systems and Networks via Contraction Theory: Theory and Applications*. PhD thesis, Universita degli Studi di Napoli Federico II, 2010.

G. Russo and M. Di Bernardo. Contraction theory and master stability function: Linking two approaches to study synchronization of complex networks. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 56(2):177–181, 2009. doi:10.1109/TCSII.2008.2011611.

G. Russo and J. J. E. Slotine. Global convergence of quorum-sensing networks. *Physical Review E*, 82(041919), 2010. doi:10.1103/PhysRevE.82.041919.

G. Russo, M. Di Bernardo, and E. D. Sontag. Global entrainment of transcriptional systems to periodic inputs. *PLoS Computational Biology*, 6(4):e1000739, 2010. doi:10.1371/journal.pcbi.1000739.

G. Russo, M. Di Bernardo, and J.-J. E. Slotine. A graphical approach to prove contraction of nonlinear circuits and systems. *IEEE Transactions on Circuits and Systems I*, 58(2):336–348, 2011. doi:10.1109/TCSI.2010.2071810.

G. Russo, M. Di Bernardo, and E. D. Sontag. A contraction approach to the hierarchical analysis and design of networked systems. *IEEE Transactions on Automatic Control*, 58(5):1328–1331, 2013. doi:10.1109/TAC.2012.2223355.

R. A. Ryan. *Introduction to Tensor Products of Banach Spaces*. Springer, 2002. ISBN 9781852334376.

E. K. Ryu and S. Boyd. Primer on monotone operator methods. *Applied Computational Mathematics*, 15(1):3–43, 2016.

I. W. Sandberg. On the mathematical foundations of compartmental analysis in biology, medicine, and ecology. *IEEE Transactions on Circuits and Systems*, 25(5):273–279, 1978. doi:10.1109/TCS.1978.1084473.

W. H. Sandholm. *Population Games and Evolutionary Dynamics*. MIT Press, 2010. ISBN 0262195879.

L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, 2009. doi:10.1016/j.automatica.2009.07.006.

E. Seneta. *Non-negative Matrices and Markov Chains*. Springer, 2 edition, 1981. ISBN 0387297650.

E. Seneta. Perturbation of the stationary distribution measured by ergodicity coefficients. *Advances in Applied Probability*, 20(1):228–230, 1988. doi:10.2307/1427277.

E. Seneta. Markov and the creation of Markov chains. In A. N. Langville and W. J. Stewart, editors, *Markov Anniversary Meeting*, pages 1–20. C & M Online Media, 2006. ISBN 1-932482-34-2. URL `https://www.csc2.ncsu.edu/conferences/nsmc`.

K. Seo, S. J. Chung, and J.-J. E. Slotine. CPG-based control of a turtle-like underwater vehicle. *Autonomous Robots*, 28:247–269, 2010. doi:10.1007/s10514-009-9169-0.

D. D. Šiljak. *Large-Scale Dynamic Systems Stability & Structure*. North-Holland, 1978. ISBN 0486462854.

J. W. Simpson-Porco and F. Bullo. Contraction theory on Riemannian manifolds. *Systems & Control Letters*, 65: 74–80, 2014. doi:10.1016/j.sysconle.2013.12.016.

S. Singh, S. M. Richards, V. Sindhwani, J.-J. E. Slotine, and M. Pavone. Learning stabilizable nonlinear dynamics with contraction-based regularization. *International Journal of Robotics Research*, 40(10-11):1123–1150, 2020. doi:10.1177/0278364920949931.

J.-J. E. Slotine. Modular stability tools for distributed computation and control. *International Journal of Adaptive Control and Signal Processing*, 17(6):397–416, 2003. doi:10.1002/acs.754.

J.-J. E. Slotine and W. Lohmiller. Modularity, evolution, and the binding problem: a view from stability theory. *Neural Networks*, 14(2):137–145, 2001. doi:10.1016/S0893-6080(00)00089-7.

J.-J. E. Slotine and W. Wang. A study of synchronization and group cooperation using partial contraction theory. In V. Kumar, N. E. Leonard, and A. S. Morse, editors, *Cooperative Control. (Proceedings of the 2003 Block Island Workshop on Cooperative Control)*, pages 207–228. Springer, 2005. doi:10.1007/978-3-540-31595-7_12.

H. L. Smith. Systems of ordinary differential equations which generate an order preserving flow. A survey of results. *SIAM Review*, 30(1):87–113, 1988. doi:10.1137/1030003.

H. L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. American Mathematical Society, 1995. ISBN 082180393X.

K. D. Smith and F. Bullo. Contractivity of the method of successive approximations for optimal control. *IEEE Control Systems Letters*, (7):919–924, 2023. doi:10.1109/LCSYS.2022.3228723.

K. D. Smith, S. Jafarpour, and F. Bullo. Transient stability of droop-controlled inverter networks with operating constraints. *IEEE Transactions on Automatic Control*, 67(2):633–645, 2022. doi:10.1109/TAC.2021.3053552.

R. A. Smith. Some applications of Hausdorff dimension inequalities for ordinary differential equations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 104(3-4):235–259, 1986a. doi:10.1017/S030821050001920X.

R. A. Smith. Massera's convergence theorem for periodic nonlinear differential equations. *Journal of Mathematical Analysis and Applications*, 120(2):679–708, 1986b. doi:10.1016/0022-247x(86)90189-7.

G. Söderlind. The logarithmic norm. History and modern theory. *BIT Numerical Mathematics*, 46(3):631–652, 2006. doi:10.1007/s10543-006-0069-9.

E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, 2 edition, 1998. ISBN 0387984895.

E. D. Sontag. Monotone and near-monotone biochemical networks. *Systems and Synthetic Biology*, 1(2):59–87, 2007. doi:10.1007/s11693-007-9005-9.

E. D. Sontag. Contractive systems with inputs. In J. C. Willems, S. Hara, Y. Ohta, and H. Fujioka, editors, *Perspectives in Mathematical System Theory, Control, and Signal Processing*, pages 217–228. Springer, 2010. ISBN 978-3540939177.

J. Stoer and C. Witzgall. Transformations by diagonal matrices in a normed space. *Numerische Mathematik*, 4:158–171, 1962. doi:10.1007/BF01386309.

T. Ström. On logarithmic norms. *SIAM Journal on Numerical Analysis*, 12(5):741–753, 1975. doi:10.1137/0712055.

Y. Takeuchi. *Global Dynamical Properties of Lotka-Volterra Systems*. World Scientific Publishing, 1996. ISBN 9810224710.

Y. Takeuchi, N. Adachi, and H. Tokumaru. The stability of generalized Volterra equations. *Journal of Mathematical Analysis and Applications*, 62(3):453–473, 1978. doi:10.1016/0022-247X(78)90139-7.

A. C. Thompson. On certain contraction mappings in a partially ordered vector space. *Proceedings of the American Mathematical Society*, 14(3):438, June 1963. doi:10.2307/2033816.

W. Thomson and P. G. Tait. *Treatise on Natural Philosophy*. Oxford University Press, 1867.

M. J. Tsatsomeros. Generating and detecting matrices with positive principal minors. In L. Li, editor, *Focus on Computational Neurobiology*, pages 115–132. Nova Science Publishers, 2004. ISBN 1590339150.

J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 31(9):803–812, 1986. doi:10.1109/TAC.1986.1104412.

H. Tsukamoto and S.-J. Chung. Neural contraction metrics for robust estimation and control: A convex optimization approach. *IEEE Control Systems Letters*, 5(1):211–216, 2021. doi:10.1109/lcsys.2020.3001646.

H. Tsukamoto, S.-J. Chung, and J.-J. E. Slotine. Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview. *Annual Reviews in Control*, 52:135–169, 2021. doi:10.1016/j.arcontrol.2021.10.001.

A. M. Turing. The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, 237(641):37–72, 1952. doi:10.1098/rstb.1952.0012.

P. Van At. Diagonal transformation methods for computing the maximal eigenvalue and eigenvector of a nonnegative irreducible matrix. *Linear Algebra and its Applications*, 148:93–123, 1991. doi:10.1016/0024-3795(91)90089-F.

P. Van Mieghem, J. Omic, and R. Kooij. Virus spread in networks. *IEEE/ACM Transactions on Networking*, 17(1):1–14, 2009. doi:10.1109/TNET.2008.925623.

M. Vidyasagar. On matrix measures and convex Liapunov functions. *Journal of Mathematical Analysis and Applications*, 62(1):90–103, 1978a. doi:10.1016/0022-247X(78)90221-4.

M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, 1978b. ISBN 0136232809. doi:10.1137/1.9780898719185.

V. Volterra. Variations and fluctuations of the number of individuals in animal species living together. *ICES Journal of Marine Science*, 3(1):3–51, 1928. doi:10.1093/icesjms/3.1.3.

R. Vrabel. A note on uniform exponential stability of linear periodic time-varying systems. *IEEE Transactions on Automatic Control*, 65(4):1647–1651, 2020. doi:10.1109/tac.2019.2927949.

J. Wang and N. Elia. A control perspective for centralized and distributed convex optimization. In *IEEE Conf. on Decision and Control and European Control Conference*, pages 3800–3805, Orlando, USA, 2011. doi:10.1109/CDC.2011.6161503.

W. Wang and J. J. Slotine. On partial contraction analysis for coupled nonlinear oscillators. *Biological Cybernetics*, 92(1):38–53, 2005. doi:10.1007/s00422-004-0527-x.

P. M. Wensing and J.-J. E. Slotine. Beyond convexity — Contraction and global convergence of gradient descent. *PLoS One*, 15(8):1–29, 2020. doi:10.1371/journal.pone.0236661.

C. Wu, I. Kanevskiy, and M. Margaliot. k-contraction: Theory and applications. *Automatica*, 136:110048, 2022a. doi:10.1016/j.automatica.2021.110048.

C. Wu, R. Pines, M. Margaliot, and J.-J. E. Slotine. Generalization of the multiplicative and additive compounds of square matrices and contraction in the Hausdorff dimension. *IEEE Transactions on Automatic Control*, 2022b. doi:10.1109/TAC.2022.3162547.

C. W. Wu and L. O. Chua. Synchronization in an array of linearly coupled dynamical systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 42:430–447, 1995. doi:10.1109/81.404047.

D. Wu. *Analyse de contraction des systèmes non-linéaires sur des variétés Riemanniennes*. PhD thesis, Thèse de doctorat de l'université Paris-Saclay et de Harbin Institute of Technology, 2022.

S. Xie, G. Russo, and R. H. Middleton. Scalability in nonlinear network systems affected by delays and disturbances. *IEEE Transactions on Control of Network Systems*, 8(3):1128–1138, 2021. doi:10.1109/TCNS.2021.3058934.

T. Yang, X. Yi, J. Wu, Y. Yuan, D. Wu, Z. Meng, Y. Hong, H. Wang, Z. Lin, and K. H. Johansson. A survey of distributed optimization. *Annual Reviews in Control*, 47:278–305, 2019. doi:10.1016/j.arcontrol.2019.05.006.

J. A. Yorke. Periods of periodic solutions and the Lipschitz constant. *Proceedings of the American Mathematical Society*, 22(2):509–512, 1969. doi:10.2307/2037090.

T. Yoshizawa. *Stability Theory by Liapunov's Second Method*. Mathematical Society of Japan, 1966.

Z. Zahreddine. Matrix measure and application to stability of matrices and interval dynamical systems. *International Journal of Mathematics and Mathematical Sciences*, 2003. doi:10.1155/S0161171203202295. Article ID: 937084.

E. Zeidler. *Nonlinear Functional Analysis and Its Applications. I: Fixed Point Theorems*. Springer, 1986. ISBN 978-0-387-90914-1.

H. Zhang, Z. Wang, and D. Liu. A comprehensive review of stability analysis of continuous-time recurrent neural networks. *IEEE Transactions on Neural Networks and Learning Systems*, 25(7):1229–1262, 2014. doi:10.1109/TNNLS.2014.2317880.