THE THEORY OF DETERMINANTS in the HISTORICAL ORDER OF DEVELOPMENT

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IN THE

HISTORICAL ORDER

OF DEVELOPMENT

BY

THOMAS MUIR

FOUR VOLUMES BOUND AS TWO

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THE THEORY OF DETERMINANTS in the HISTORICAL ORDER OF DEVELOPMENT

VOLUME ONE

PART I. GENERAL DETERMINANTS UP TO 1841.

PART II. SPECIAL DETERMINANTS UP TO 1841.

PREFACE.

The main object of this work and the contents of it will be found specified in the Introductory Chapter. It is intended for the student who aims at acquiring such a knowledge as can only be got by a study of the subject in the historical order of its development, for the investigator who is specially interested in this branch of mathematics and wishes to become acquainted with the various lines of attack opened up by previous workers, and for the general working mathematician who requires guide-books and books of reference concerning special domains.

T. M.

Capetown, South Africa,
19*th July*, 1905.

CONTENTS.

PART I.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER XVI.

CHAPTER XVII.

CHAPTER I.

INTRODUCTION.

THE way in which the material for a history of the theory of Determinants has been accumulated is quite similar to that which has been observed in the case of other branches of science.

In the middle of the eighteenth century one of the independent discoverers of the fundamental idea, viz., CRAMER, was fortunate enough to attract attention to it, and in time it became the common property of mathematicians in France and elsewhere. As it slowly spread it naturally also received accretions and developments, and of the dozen or so of writers who thus handled it in the sixty years that followed Cramer's publication there were of course a few who by a more or less casual reference kept alive the memory of some of their predecessors. It was then taken up by CAUCHY, and, thanks to the prestige of his name and to the inherent excellence of his extensive monograph, its position as a theory of importance became more firmly assured. The thirty years that followed Cauchy's memoir resembled the sixty that preceded it, save that the number of contributors was considerably larger. Then another great analyst, JACOBI, the most noteworthy of those contributors, produced in Germany a monograph similar in extent and value to Cauchy's, and the importance of the subject in the eyes of mathematicians became still more enhanced. As a consequence, the single decade following gave rise to quite as many new contributions as the preceding three decades had done, and closed with the appearance of the first separately published elementary treatise on the subject, viz., SPOTTISWOODE'S. The preface to this contains the first notable historical sketch of the theory, and includes references to the writings of twelve outstanding mathematicians, beginning with Cramer (1750) and ending with the author's own contemporaries, Cayley, Sylvester and Hermite. In the same year (1850) there also occurred something out of the ordinary, for the correspondence between Leibnitz and the Marquis de l'Hôpital having been published from manuscripts in the Royal Library at Hanover, the striking discovery was made that more than half-a-century before Cramer's time the fundamental idea of determinants had been clear to LEIBNITZ, and had been expounded with considerable fulness by him in a letter to his friend. So strongly attractive had the subject now become to mathematicians that in the single year succeeding the publication of Spottiswoode's short treatise a greater number of separate contributions to the theory made their appearance than in the whole sixty-year period from Cramer to Cauchy. The wants of students everywhere had to be attended to: a second edition of Spottiswoode was consequently prepared for *Crelle's Journal* in 1853; a text-book by Brioschi was published at Pavia in 1854; French and German translations of Brioschi in 1856; and an elementary exposition by Bellavitis in 1857. So far as historical material is concerned, the last-mentioned work was of little account; that of Brioschi resembled Spottiswoode's, the number of references, however, being greater. Of quite a different character was the text-book by BALTZER, which was published at Leipzig the year after the German translation of Brioschi had appeared at Berlin, an important part of the new author's plan being to deal methodically with the history of the subject by means of footnotes. On the enunciation of almost every theorem a note with historical references was added at the foot of the page, the result being that in the portion (thirty-four pages) devoted expressly to the pure theory of determinants about as many separate writings are referred to as there are pages. This was a marked advance, and although during the next twenty years the publication of text-books became more frequent—in fact, if we include those of every language and of every scope, we shall find an average of about one per year—Baltzer's dominated the field; enlarged editions of it appeared in 1864, 1870, and 1875, and the historical notes grew correspondingly in number. Of the other text-books only one, Günther's, which was published in 1875, sought to follow the historical line taken by Baltzer and to add to the supply of material. Then in 1876 another new departure took place, this being the year in which the first writings were published which dealt with the history alone, the one being an academic thesis by E. J. Mellberg printed at Helsingfors, and the other a memoir presented by F. J. Studnička to the Bohemian Society of Sciences.

About this time, while engaged in writing my own so-called "Treatise on the Theory of Determinants," I had occasion to look into the question of the authorship and history of the various theorems, and I was reluctantly forced to the conclusion that much inaccurate statement prevailed in regard to such matters and that the whole subject was worthy of serious investigation. A resolution was accordingly taken to set about collecting the titles of all the writings which had appeared on the theory up to the end of 1880. The task was not an easy one, as will readily be understood by those who know how scanty and defective are the bibliographical aids at the disposal of mathematicians, and how often the titles given by investigators to their memoirs are imperfect and even misleading in regard to the nature of the contents. The outcome of the search was published in 1881 in the October number of the *Quarterly Journal of Mathematics* (vol. xviii. pp. 110–149) under the title of "A List of Writings on Determinants." It contained 589 entries arranged in chronological order. Some three or four years afterwards, when there had been time to test the completeness of the earlier portion of the list, the writings included in it were taken up in historical succession and suitable abstracts or reviews of them made for publication in the *Proceedings of the Royal Society of Edinburgh*; the first contribution of this kind was presented to the Society in the beginning of the year 1886. At the same time there was being prepared an additional list of writings containing omitted titles, 84 in number, belonging to the period of the first list, and 176 titles belonging to the further period 1881–1885. This second list appeared in 1886 in the June number of the *Quarterly Journal of Mathematics* (vol. xxi. pp. 299–320). In 1890 a collection was made of the contributions, just mentioned, which had up to that date been printed in the Edinburgh *Proceedings*, and with the consent of the Society was published separately. Unfortunately in that year all this train of work had to be laid aside on account of the pressure of official duties, and ten years elapsed before it could be resumed. It was thus not until March 1900 that a second series of analytic abstracts began to appear in the Edinburgh *Proceedings*, and that the preparation of a third list of writings was methodically undertaken. The period to be covered by this list was the fifteen years 1886–1900; and as the number of writers interested in the subject had in these years continued to increase, and as closer examination of the literature of the previous periods had led to new finds, the resulting compilation was more extensive than the first two put together. It was presented to the South African Association for the Advancement of Science at its inaugural meeting in April 1903 and was published in the Report; it is also to be found in the *Quarterly Journal of Mathematics* for December 1904 and February 1905 (vol. xxxvi. pp. 171–267). The number of titles in the three lists is about 1740; they furnish, it is hoped, an almost complete guide to the literature of the theory of determinants from the earliest times to the close of the nineteenth century.

From these later labours it became manifest that it was undesirable in the way of separate publication to issue merely another volume as a continuation of, and similar to, that of the year 1900. The better course clearly was to reproduce the material of that volume along with the intercalations necessitated in it by the existence of subsequently discovered papers, and to follow this up in such a way as to give finally within the compass of a reasonably sized volume a full history of the subject in all its branches up to about the middle of the nineteenth century. This is what is here attempted.

The plan followed is not to give one connected history of determinants as a whole, but to give separately the history of each of the sections into which the subject has been divided, viz., to deal with determinants in general, and thereafter in order with the various special forms. This will not only tend to smoothness in the narrative by doing away with the necessity of frequent harkings back, but it will also be of material importance to investigators who may wish to find out what has already been done in advancing any particular department of the subject. To this end, also, each new result as it appears will be numbered in Roman figures; and if the same result be obtained in a different way, or be generalised, by a subsequent worker, it will be marked among the contributions of the latter with the same Roman figures, followed by an Arabic numeral. Thus the theorem regarding the effect of the transposition of two rows of a determinant will be found under Vandermonde, marked with the number xi., and the information intended thus to be conveyed is that in the order of discovery the said theorem was the *eleventh* noteworthy result obtained: while the mark XI. 2, which occurs under Laplace, is meant to show that the theorem was not then heard of for the first time, but that Laplace contributed something additional to our knowledge of it. In this way any reader who will take the trouble to look up the sequence XI., XI. 2, XI. 3, &c., may be certain, it is hoped, of obtaining the full history of the theorem in question.

The early foreshadowings of a new domain of science, and tentative gropings at a theory of it, are so difficult for the historian to represent without either conveying too much or too little, that the only satisfactory way of dealing with a subject in its earliest stages seems to be to reproduce the exact words of the authors where essential parts of the theory are concerned. This I have resolved to do, although to some it may have the effect of rendering the account at the commencement somewhat dry and forbidding.

CHAPTER II.

DETERMINANTS IN GENERAL, FROM THE YEAR 1693 TO 1779.

THE writers here to be dealt with are seven in number, viz., Leibnitz, Fontaine, Cramer, Bézout, Vandermonde, Laplace, Lagrange. Of these the first two exercised no influence on the development of the theory; the real moving spirit was Cramer; Lagrange alone of the others may have been unaffected by this particular part of Cramer's work.

LEIBNITZ (1693).

[Leibnizens mathematische Schriften, herausg. v. C. I. Gerhardt. 1 Abth. ii. pp. 229, 238–240, 245. Berlin, 1850.]

In the fourth letter of the published correspondence between Leibnitz and De L'Hospital, the former incidentally mentions that in his algebraical investigations he occasionally uses numbers instead of letters, treating the numbers however as if they were letters. De L'Hospital, in his reply, refers to this, stating that he has some difficulty in believing that numbers can be as convenient or give as general results as letters. Thereupon Leibnitz, in his next letter (28th April 1693), proceeds with an explanation:—

"Puisque vous dites que vous avés de la peine à croire qu'il soit aussi general et aussi commode de se servir des nombres que des lettres, il faut que je ne me sois pas bien expliqué. On ne sçauroit douter de la generalité en considerant qu'il est permis de se servir de 2, 3, etc., comme d'*a* ou de *b*, pour veu qu'on considere que ce ne sont pas de nombres veritables. Ainsi 2.3 ne signifie point 6 mais autant qu'*ab*. Pour ce qui est de la commodité, il y en a des tres grandes, ce qui fait que je m'en sers souvent, sur tout dans les calculs longs et difficiles ou il est aisé de se tromper. Car outre la commodité de l'épreuve par des nombres, et même par l'abjection du novenaire, j'y trouve un tres grand avantage même pour l'avancement de l'Analyse. Comme c'est une ouverture assez extraordinaire, je n'en ay pas encor parlé à d'autres, mais voicy ce que c'est. Lorsqu'on a besoin de beaucoup de lettres, n'est il pas vray que ces lettres n'expriment point les rapports qu'il y a entre les grandeurs qu'elles signifient, au lieu qu'en me servant des nombres je puis exprimer ce rapport. Par exemple soyent proposées trois equations simples pour deux inconnues à dessein d'oster ces deux inconnues, et cela par un canon general. Je suppose

$$\begin{aligned} & 10+11x+12y = 0 \qquad (1) \\ \text{et} \quad & 20+21x+22y = 0 \qquad (2) \\ \text{et} \quad & 30+31x+32y = 0 \qquad (3) \end{aligned}$$

ou le nombre feint estant de deux characteres, le premier me marque de quelle equation il est, le second me marque à quelle lettre il appartient. Ainsi en calculant on trouve par tout des harmonies qui non seulement nous servent de garans, mais encor nous font entrevoir d'abord des regles ou theoremes. Par exemple ostant premierement y par la premiere et la seconde equation, nous aurons:

$$\begin{matrix} +10\,.\,22+11\,.\,22x \\ -12\,.\,20-12\,.\,21\,.\,. \end{matrix} = 0 \qquad (4)^*$$

et par la premiere et troisieme nous aurons:

$$\begin{matrix} +10\,.\,32+11\,.\,32x \\ -12\,.\,30-12\,.\,31\,.\,. \end{matrix} = 0 \qquad (5)$$

ou il est aise de connoistre que ces deux equations ne different qu'en ce que le charactere antecedent 2 est changé au charactere antecedent 3. Du reste, dans un même terme d'une même equation les characteres antecedens sont les mêmes, et les characteres posterieurs font une même somme. Il reste maintenant d'oster la lettre x par la quatrieme et cinquieme equation, et pour cet effect nous aurons †

$$\begin{matrix} 1_0\,.\,2_1\,.\,3_2 \\ 1_1\,.\,2_2\,.\,3_0 \\ 1_2\,.\,2_0\,.\,3_1 \end{matrix} \quad = \quad \begin{matrix} 1_0\,.\,2_2\,.\,3_1 \\ 1_1\,.\,2_0\,.\,3_2 \\ 1_2\,.\,2_1\,.\,3_0 \end{matrix}$$

qui est la derniere equation delivrée des deux inconnues qu'on vouloit oster, et qui porte sa preuve avec soy par les harmonies qui se remarquent par tout, et qu'on auroit bien de la peine à decouvrir en

* This is written shortly for $\left.\begin{matrix} +10\,.\,22+11\,.\,22x=0 \\ -12\,.\,20-12\,.\,21x=0 \end{matrix}\right\}$.

† The author here slightly changes his notation. What is meant to be indicated is

$$10.21.32 + 11.22.30 + 12.20.31 = 10.22.31 + 11.20.32 + 12.21.30.$$

employant des lettres *a*, *b*, *c*, sur tout lors que le nombre des lettres et des equations est grand. Une partic du secret de l'analyse consiste dans la caracteristique, c'est à dire dans l'art de bien employer les notes dont on se sert, et vous voyés, Monsieur, par ce petit echantillon, que Viete et des Cartes n'en ont pas encor connu tous les mysteres. En poursuivant tant soit peu ce calcul on viendra à un *theoreme general* pour quelque nombre de lettres et d'equations simples qu'on puisse prendre. Le voicy comme je l'ay trouvé autres fois :

"*Datis aequationibus quotcunque sufficientibus ad tollendas quantitates, quae simplicem gradum non egrediuntur, pro aequatione prodeunte, primo sumendae sunt omnes combinationes possibiles, quas ingreditur una tantum coefficiens uniuscujusque aequationis : secundo, eae combinationes opposita habent signa, si in eodem aequationis prodeuntis latere ponantur, quae habent tot coefficientes communes, quot sunt unitates in numero quantitatum tollendarum unitate minuto : caeterae habent eadem signa.*

"J'avoue que dans ce cas des degrés simples on auroit peut estre decouvert le même theoreme en ne se servant que de lettres à l'ordinaire, mais non pas si aisement, et ces adresses sont encor bien plus necessaires pour decouvrir des theoremes qui servent à oster les inconnues montées à des degrés plus hauts. Par exemple, ... "

It will be seen that what this amounts to is *the formation of a rule for writing out the resultant of a set of linear equations.* When the problem is presented of eliminating x and y from the equations

$$a+bx+cy = 0, \qquad d+ex+fy = 0, \qquad g+hx+ky = 0,$$

Leibnitz in effect says that first of all he prefers to write 10 for a, 11 for b, and so on; that, having done this, he can all the more readily take the next step, viz., forming every possible product whose factors are one coefficient from each equation,* the result being

$$\begin{array}{ccc} 10.21.32, & 10.22.31, & 11.20.32, \\ 11.22.30, & 12.20.31, & 12.21.30; \end{array}$$

and that; then, *one* being the number which is less by one than the number of unknowns, he makes those terms different in sign which have only *one* factor in common.

The contributions, therefore, which Leibnitz here makes to algebra may be looked upon as three in number:—

(1) A *new notation*, numerical in character and appearance, for individual members of an arranged group of magnitudes; the two members which constitute the notation being like the

* Of course, this is not exactly what Leibnitz meant to say.

Cartesian co-ordinates of a point in that they denote any one of the said magnitudes by indicating its position in the group. (I.)

(2) A rule for *forming the terms* of the expression which equated to zero is the result of eliminating the unknowns from a set of simple equations. (II.)

(3) A rule for *determining the signs* of the terms in the said result. (III.)

The last of these is manifestly the least satisfactory. In the first place, part of it is awkwardly stated. Making those terms different in sign *which have only as many factors alike as is indicated by the number which is less by one than the number of unknown quantities* is exactly the same as making those terms different in sign *which have only two factors different.* Secondly, in form it is very unpractical. The only methodical way of putting it in use is to select a term and make it positive; then seek out a second term, having all its factors except two the same as those of the first term, and make this second term negative; then seek out a third term, having all its factors except two the same as those of the second term, and make this third term positive; and so on.

Although there is evidence that Leibnitz continued, in his analytical work, to use his new notation for the coefficients of an equation (see Letters xi., xii., xiii. of the said correspondence), and that he thought highly of it (see Letter viii. "chez moi c'est une des meilleures ouvertures en Analyse"), it does not appear that by using it in connection with sets of linear equations, or by any other means, he went further on the way towards the subject with which we are concerned. Moreover, it must be remembered that the little he did effect had no influence on succeeding workers. So far as is known, the passage above quoted from his correspondence with De L'Hospital was not published until 1850. Even for some little time after the date of Gerhardt's publication it escaped observation, Lejeune Dirichlet being the first to note its historical importance. It is true that during his own lifetime, Leibnitz's *use of numbers in place of letters* was made known to the world in the *Acta Eruditorum* of Leipzig for the year 1700 (*Responsio ad Dn. Nic. Fatii Duillerii imputationes*, pp. 189–208); but the particular application of the new symbols which brings them into connection with determinants was not there given.

In a subsequent volume of *Leibnizens mathematische Schriften*,—the third volume of the second Abtheilung,—published at Halle in 1863, the following equivalent of the above 'théorème général' appears (pp. 5–6):—

"Inveni Canonem pro tollendis incognitis quotcunque aequationes non nisi simplici gradu ingredientibus, ponendo aequationum numerum excedere unitate numerum incognitarum. Id ita habet.

Fiant omnes combinationes possibiles literarum coefficientium ita ut nunquam concurrant plures coefficientes ejusdem incognitae et ejusdem aequationis. Hae combinationes affectae signis, ut mox sequetur, componuntur simul, compositumque aequatum nihilo dabit aequationem omnibus incognitis carentem.

Lex signorum haec ist. Uni ex combinationibus assignetur signum pro arbitrio, et caeterae combinationes quae ab hac differunt coefficientibus duabus, quatuor, sex etc. habebunt signum oppositum ipsius signo: quae vero ab hac differunt coefficientibus tribus, quinque, septem etc. habebunt signum idem cum ipsius signo. Ex. gr. sit

$$10+11x+12y = 0, \qquad 20+21x+22y = 0, \qquad 30+31x+32y = 0;$$

fiet

$$+10.21.32-10.22.31-11.20.32$$
$$+11.22.30+12.20.31-12.21.30 = 0.$$

Coefficientibus eas literas computo, quae sunt nullius incognitorum, ut 10, 20, 30."

Although Gerhardt, the editor, states that the original manuscript of Leibnitz, from which this is taken, bears no date, it is very probable to date farther back than 1693, and not impossible to belong to 1678.*

FONTAINE (1748).

[Mémoires donnés à l'Académie Royale des Sciences, non imprimés dans leurs temps. Par M. Fontaine † de cette Académie. 588 pp. Paris, 1764.]

These memoirs of Fontaine's, sixteen in number, cover a considerable variety of mathematical subjects: it is the seventh of

* See also GERHARDT, K. I., Leibniz über die Determinanten, *Sitzungsb.* ..... *Akad. d. Wiss.* (Berlin), 1891, pp. 407–423.

† The full name is *Alexis Fontaine des Bertins.* The very same collection was issued in 1770 under the less appropriate title *Traité de calcul différentiel et intégral.* Vandermonde is said to have been a pupil of Fontaine's (*v. Nouv. Annales de Math.*, v. p. 155).

the series which indirectly concerns determinants. There is not, however, even the most distant connection between it and the work of Leibnitz. The heading is "Le calcul intégral. Seconde méthode," the sixth memoir having given the first method. The date is indicated in the margin.

The matter which concerns us appears as a lemma near the beginning of the memoir (p. 94). The passage is as follows:—

"Soient quatre nombres quelconques

$$a1,\ a2,\ a3,\ a4,$$

et quatre autres nombres aussi quelconques

$$\alpha1,\ \alpha2,\ \alpha3,\ \alpha4\,;$$

faites

$$\begin{aligned} \alpha1\ a2 - a1\ \alpha2 &= a^{1}1, \\ \alpha2\ a3 - a2\ \alpha3 &= a^{1}2, \\ \alpha3\ a4 - a3\ \alpha4 &= a^{1}3, \\ \alpha1\ a3 - a1\ \alpha3 &= a^{2}1, \\ \alpha2\ a4 - a2\ \alpha4 &= a^{2}2, \\ \alpha1\ a4 - a1\ \alpha4 &= a^{3}1, \end{aligned}$$

vous aurez

$$a^{3}1\ a^{1}2 - a^{2}1\ a^{2}2 + a^{1}1\ a^{1}3 = 0."$$

Manifestly this is the identity which in later times came to be written

$$|a_1b_2| \cdot |a_3b_4| - |a_1b_3| \cdot |a_2b_4| + |a_1b_4| \cdot |a_2b_3| = 0,$$

and which, so far as we know, appeared first in its proper connection in the writings of Bézout. (XXIII.)

It is curious to note that Fontaine was not satisfied with the lemma in this form, but proceeded to take "autant de nombres quelconques que l'on voudra, $a1$, $a2$, ... , $a10$, . . . ," and wrote the identity one hundred and twenty-six times before he appended "et cetera," the 126th being

$$a^{3}6\ a^{1}7 - a^{2}6\ a^{2}7 + a^{1}6\ a^{1}8 = 0.$$

CRAMER (1750).

[Introduction a l'Analyse des Lignes Courbes algébriques. (Pp. 59, 60, 656–659.) Genève, 1750.]

The third chapter of Cramer's famous treatise deals with the different *orders* (degrees) of curves, and one of the earliest theorems of the chapter is the well-known one that the equation of a curve of the nth degree is determinable when $\frac{1}{2}n(n+3)$ points of the curve are known. In illustration of this theorem he deals (p. 59) with the case of finding the equation of the curve of the *second* degree which passes through *five* given points. The equation is taken in the form

$$A + By + Cx + Dyy + Exy + xx = 0;$$

the five equations for the determination of A, B, C, D, E are written down; and it is pointed out that all that is necessary is the solution of the set of five equations, and the substitution of the values of A, B, C, D, E thus found, "Le calcul véritablement en seroit assez long," he says; but in a footnote there is the remark that it is to algebra we must look for the means of shortening the process, and we are directed to the appendix for a convenient general rule which he had discovered for obtaining the solution of a set of equations of this kind. The following is the essential part of the passage in which the rule occurs:—

"Soient plusieurs inconnues z, y, x, v, &c., et autant d'équations

$$\begin{aligned} A^1 &= Z^1z + Y^1y + X^1x + V^1v + \&c. \\ A^2 &= Z^2z + Y^2y + X^2x + V^2v + \&c. \\ A^3 &= Z^3z + Y^3y + X^3x + V^3v + \&c. \\ A^4 &= Z^4z + Y^4y + X^4x + V^4v + \&c. \\ &\qquad\qquad \&c. \end{aligned}$$

où les lettres A^1, A^2, A^3, A^4, &c., ne marquent pas, comme à l'ordinaire, les puissances d'A, mais le prémier membre, supposé connu, de la prémiére, seconde, troisiéme, quatriéme, &c. équation."

[Here the solutions of the cases of 1, 2, and 3 unknowns are given, and he then proceeds.]

"L'examen de ces Formules fournit cette Règle générale. Le nombre des équations et des inconnues étant n, on trouvera la valeur de chaque inconnue en formant n fractions dont le dénominateur commun a autant de termes qu'il y a de divers arrangements de n choses différentes. Chaque terme est composé des lettres ZYXV, &c., toujours écrites dans le même ordre, mais auxquelles on distribue, comme exposants, les n prémiers chiffres rangés en toutes les maniéres possibles. Ainsi, lorsqu'on a trois inconnues, le dénominateur a [$1 \times 2 \times 3 =$] 6 termes, composés des trois lettres ZYX, qui recoivent successivement les exposants 123, 132, 213, 231, 312, 321. On donne à ces termes les signes + ou −, selon la Règle suivante. Quand un exposant est suivi dans le même terme, médiatement ou immédiatement, d'un exposant plus petit que lui, j'appellerai cela un *dérangement*.

Qu'on compte, pour chaque terme, le nombre des dérangements: s'il est pair ou nul, le terme aura le signe $+$; s'il est impair, le terme aura le signe $-$. Par ex. dans le terme $Z^1Y^2X^3$ il n'y a aucun dérangement; ce terme aura donc le signe $+$. Le terme $Z^3Y^1X^2$ a aussi le signe $+$, parce qu'il a deux dérangements, 3 avant 1 et 3 avant 2. Mais le terme $Z^3Y^2X^1$, qui a trois dérangements, 3 avant 2, 3 avant 1, et 2 avant 1, aura le signe $-$.

"Le dénominateur commun étant ainsi formé, on aura la valeur de z en donnant à ce dénominateur le numérateur qui se forme en changeant, dans tous ces termes, Z en A. Et la valeur d'y est la fraction qui a le même dénominateur et pour numérateur la quantité qui résulte quand on change Y en A, dans tous les termes du dénominateur. Et on trouve d'une maniére semblable la valeur des autres inconnues."

It is evident at once that the new results here given are—

(1) A rule for *forming the terms* of the common denominator of the fractions which express the values of the unknowns in a set of linear equations. (IV.)

(2) A rule for *determining the sign* of any individual term in the said common denominator (and, included in the rule, the notion of a "dérangement"). (III. 2)

(3) A rule for *obtaining the numerators* from the expression for the common denominator. (V.)

The problem which Cramer set himself at this point in his book was exactly that which Leibnitz had solved, viz., the elimination of n quantities from a set of $n+1$ linear equations. The solution which Cramer obtained, and which, be it remarked, was the solution best adapted for his purpose, was quite distinct in character from that of Leibnitz. Leibnitz gave a rule for writing out the final result of the elimination; what Cramer gives is a rule for writing out the values of the n unknowns as determined from n of the $n+1$ equations, after which we have got to substitute these values in the remaining $(n+1)$th equation. The notable point in regard to the two solutions is, that Cramer's rule for writing the *common denominator* of the values of the n unknowns (an expression of the nth degree in the coefficients) is exactly Leibnitz's rule for writing the *final result*, which is an expression of the $(n+1)$th degree. Had either discoverer been aware that the same rule sufficed for obtaining both of these expressions, he could not have failed, one would think, to note the *recurrent* law of formation of them. The result of eliminating w, x, y, z from the equations,

$$a_r w + b_r x + c_r y + d_r z = e_r \quad (r = 1, 2, 3, 4, 5)$$

is, according to Leibnitz, if we embody his rule in a later symbolism,

$$|a_1 b_2 c_3 d_4 e_5| = 0;$$

whereas, according to Cramer, it is—

$$a_1 \frac{|e_2 b_3 c_4 d_5|}{|a_2 b_3 c_4 d_5|} + b_1 \frac{|a_2 e_3 c_4 d_5|}{|a_2 b_3 c_4 d_5|} + c_1 \frac{|a_2 b_3 e_4 d_5|}{|a_2 b_3 c_4 d_5|} + d_1 \frac{|a_2 b_3 c_4 e_5|}{|a_2 b_3 c_4 d_5|} = e_1,$$

and from the collocation of these the one natural step is to the identity

$$-|a_1 b_2 c_3 d_4 e_5| = a_1 |e_2 b_3 c_4 d_5| + b_1 |a_2 e_3 c_4 d_5| + \ldots - e_1 |a_2 b_3 c_4 d_5|.$$

The fate of Cramer's rule was very different from that of Leibnitz'. It was soon taken up, and after a time found its way into the schools, where it continued for many years to be taught as the nutshell form of the theory of the solution of simultaneous linear equations. Indeed Gergonne is reported* to have said, "Cette méthode était tellement en faveur, que les examens aux écoles des services publics ne roulaient, pour ainsi dire, que sur elle; on était admis ou rejeté suivant qu'on la possedait bien ou mal."

Finally, the exact difference between Cramer's notation for the coefficients of the unknowns and the notation of Leibnitz should be noted, and in connection therewith the fact that when dealing with the subject of elimination between two equations of the mth and nth degrees in x Cramer uses a notation closely resembling that which Leibnitz employed, viz., $[1^2][1^3]$, &c.

BÉZOUT (1764).

[Recherches sur le degré des équations résultantes de l'évanouissement des inconnues, et sur les moyens qu'il convient d'employer pour trouver ces équations.—*Hist. de l'Acad. Roy. des Sciences,* Ann. 1764 (pp. 288–338), pp. 291–295.]

The object of Bézout's memoir is sufficiently apparent from the title; we may therefore at once give those portions of it

* By Studnička. But see Klügel's *Wörterbuch d. reinen Math.*, Suppl. II., p. 67.

which directly concern our subject. On p. 291 is the commencement of the following passage :—

"M. Cramer a donné une règle générale pour les exprimer toutes débarrassées de ce facteur : j'aurois pu m'en tenir à cette règle ; mais l'usage m'a fait connoître que quoiqu'elle soit assez simple, quant aux lettres, elle ne l'est pas de même à l'égard des signes lorsqu'on a au-delà d'un certain nombre d'inconnues à calculer ; . . . .

Lemme I.

"Si l'on a un nombre n d'équations du premier degré qui renferment chacune un pareil nombre d'inconnues, sans aucun terme absolument connu, on trouvera par la règle suivante la relation que doivent avoir les coëfficiens de ces inconnues pour que toutes ces équations aient lieu.

"Soient a, b, c, d, &c., les coëfficiens de ces inconnues dans la première équation.

a', b', c', d', &c., les coëfficiens des mêmes inconnues dans la seconde équation.

a'', b'', c'', d'', &c., ceux de la troisième & ainsi de suite.

"Formez les deux permutations ab & ba & écrivez $ab - ba$; avec ces deux permutations & la lettre c formez toutes les permutations possibles, en observant de changez de signe toutes les fois que c changera de place dans ab & la même chose à l'égard de ba ; vous aurez

$$abc - acb + cab - bac + bca - cba.$$

Avec ces six permutations & la lettre d, formez toutes les permutations possibles, en observant de changer de signe à chaque fois que d changera de place dans un même terme ; vous aurez

$$\begin{aligned} &abcd - abdc + adbc - dabc - acbd + acdb - adcb + dacb \\ &+ cabd - cadb + cdab - dcab - bacd + badc - bdac + dbac \\ &+ bcad - bcda + bdca - dbca - cbad + cbda - cdba + dcba \end{aligned}$$

& ainsi de suite jusqu'à ce que vous ayez épuisé tous les coëfficiens de la première equation.

"Alors conservez les lettres qui occupent la première place ; donnez à celles qui occupent la seconde, la même marque qu'elles ont dans la seconde équation ; à celles qui occupent la troisième, la même marque qu'elles ont dans la troisième équation, & ainsi de suite ; égalez enfin le tout à zéro et vous aurez l'équation de condition cherchée.

"Ainsi si vous avez deux équations et deux inconnues comme

$$\begin{aligned} ax + by &= 0 \\ a'x + b'y &= 0 \end{aligned}$$

l'équation de condition sera $ab' - ba' = 0$ ou $ab' - a'b = 0$ . . . ."

In the same way the next two cases are given ; then—

". . . . mais comme ces équations de condition doivent servir de formules pour l'élimination dans les équations de différens degrés, il convient de leur donner une forme qui rende les substitutions le moins pénibles qu'il se pourra ; pour cet effet, je les mets sous cette forme :

$$\begin{aligned}
&ab' - a'b = 0\\
&(ab' - a'b)c'' + (a''b - ab'')c' + (a'b'' - a''b')c = 0\\
&[(ab' - a'b)c'' + (a''b - ab'')c' + (a'b'' - a''b')c\,]d'''\\
+&[(a'b - ab')c''' + (ab''' - a'''b)c' + (a'''b' - a'b''')c\,]d''\\
+&[(a'''b - ab''')c'' + (ab'' - a''b)c''' + (a''b''' - a'''b'')c\,]d'\\
+&[(a'b''' - a'''b')c'' + (a'''b'' - a''b''')c' + (a''b' - a'b'')c''']d = 0.
\end{aligned}$$

Cette nouvelle forme a deux avantages: le premier, de rendre les substitutions à venir, plus commodes; le deuxième, c'est d'offrir une règle encore plus simple pour la formation de ces formules.

"En effet, il est facile de remarquer 1°, que le premier terme de l'une quelconque de ces équations, est formé du premier membre de l'équation précédente, multiplié par la première des lettres qu'elle ne renferme point, cette lettre étant affectée de la marque qui suit immédiatement la plus haute de celles qui entrent dans ce même membre.

"2°. Le deuxième terme se forme du premier, en changeant dans celui-ci la plus haute marque en celle qui est immédiatement au-dessous & réciproquement, & de plus en changeant les signes.

"3°. Le troisième, se forme du premier, en changeant dans celui-ci la plus haute marque en celle de deux numéros au-dessous & réciproquement, & de plus en changeant les signes.

"4°. Le quatrième, se forme du premier, en changeant dans celui-ci la plus haute marque en celle de trois numéros au-dessous & réciproquement, & changeant les signes, & toujours de même pour les suivans.

"Par exemple, ...

"D'après ces observations, il sera facile de voir que l'équation de condition pour cinq inconnues et cinq équations, sera

. . . ."

The latter part of this we are drawn to at once, as it enunciates quite clearly the Recurrent Law of Formation to which attention has above been directed. It has to be observed, however, that the three 'equations of condition' are not in the form got by merely following the 'rule,' and that by deriving each 'terme,' not from the first but from the preceding 'terme' we should obtain, viz.:

$$\begin{aligned}
&ab' - a'b = 0,\\
&(ab' - a'b)c'' - (ab'' - a''b)c' + (a'b'' - a''b')c = 0,\\
&[(ab' - a'b)c'' - (ab'' - a''b)c' + (a'b'' - a''b')c]d'''\\
-&[(ab' - a'b)c''' - (ab''' - a'''b)c' + (a'b''' - a'''b')c]d''\\
+&[(ab'' - a''b)c''' - (ab''' - a'''b)c'' + (a''b''' - a'''b'')c]d'\\
-&[(a'b'' - a''b')c''' - (a'b''' - a'''b')c'' + (a''b''' - a'''b'')c']d = 0.
\end{aligned}$$

The notable point in regard to the earlier portion is, that Bézout throws his rule of term-formation and his rule of signs into one. In the case of finding the resultant of

$$a_r x + b_r y + c_r z = 0 \qquad (r = 1, 2, 3)$$

his process consists of four steps, viz.:—

(1) a,

(2) $a\,b \;\vdots\; -b\,a$,

(3) $a\,b\,c \;-a\,c\,b \;+c\,a\,b \;-b\,a\,c \;+b\,c\,a \;-c\,b\,a$,

(4) $a_1 b_2 c_3 - a_1 c_2 b_3 + c_1 a_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 - c_1 b_2 a_3$.

The first term of (2) is got from (1) by affixing b, and the second is got from the first by advancing the b one place and changing the sign. The first term of (3) is got from the first term of (2) by affixing c, the second term is got from the first by advancing c a place and changing the sign, and the third is got from the second by advancing c a place and changing the sign; the last three are got from the second term of (2) in the same way as the first three are got from the first term of (2).

It will thus be seen that while Leibnitz and Cramer direct us to find the permutations in any way whatever, and thereafter to fix the sign of each in accordance with a rule, Bézout requires the permutations to be found by a particular process, and attention given to the question of sign throughout all this process, so that when the terms have been found their signs have likewise been determined.

Bézout's contributions to the subject thus are—

(1) A combined rule of term-formation and rule of signs. (II. 2)+(III. 3)

(2) The recurrent law of formation of the new functions. (VI.)

VANDERMONDE (1771).

[Mémoire sur l'élimination. *Hist. de l'Acad. Roy. des Sciences* (Paris), Ann. 1772, 2e partie (pp. 516–532).]

This important memoir of Vandermonde and that of Laplace, which is dealt with immediately afterwards, both appear in the *History of the French Academy of Sciences* for 1772, Laplace's memoir occupying pp. 267–376, and Vandermonde's pp. 516–532. There is, however, a footnote to the latter, which states that it was read for the first time to the Academy on 12th January 1771.

The part of it which concerns us is the first article, which treats of elimination in the case of equations of the first degree. Vandermonde here writes:—

"Je suppose que l'on représente par $\overset{1}{1,}\ \overset{2}{1,}\ \overset{3}{1,}$ &c., $\overset{1}{2,}\ \overset{2}{2,}\ \overset{3}{2,}$ &c., $\overset{1}{3,}\ \overset{2}{3,}\ \overset{3}{3,}$ &c., &c., autant de différentes quantités générales, dont l'une quelconque soit $\overset{\alpha}{a,}$ une autre quelconque soit $\overset{\beta}{b,}$ &c., & que le produit des deux soit désigné à l'ordinaire par $\overset{\alpha}{a}\,.\,\overset{\beta}{b}.$

"Des deux nombres ordinaux α & a, le premier, par exemple, désignera de quelle équation est pris le coëfficient $\overset{\alpha}{\underset{a,}{-}}$ & le second désignera le rang que tient ce coëfficient dans l'équation, comme on le verra ci-après.

"Je suppose encore le système suivant d'abréviations, & que l'on fasse

$$\frac{\alpha|\beta}{a|b}=\overset{\alpha}{a}\,.\,\overset{\beta}{b}-\overset{\alpha}{b}\,.\,\overset{\beta}{a},$$

$$\frac{\alpha|\beta|\gamma}{a|b|c}=\overset{\alpha}{a}\,.\,\frac{\beta|\gamma}{b|c}+\overset{\alpha}{b}\,.\,\frac{\beta|\gamma}{c|a}+\overset{\alpha}{c}\,.\,\frac{\beta|\gamma}{a|b},$$

$$\frac{\alpha|\beta|\gamma|\delta}{a|b|c|d}=\overset{\alpha}{a}\,.\,\frac{\beta|\gamma|\delta}{b|c|d}-\overset{\alpha}{b}\,.\,\frac{\beta|\gamma|\delta}{c|d|a}+\overset{\alpha}{c}\,.\,\frac{\beta|\gamma|\delta}{d|a|b}-\overset{\alpha}{d}\,.\,\frac{\beta|\gamma|\delta}{a|b|c},$$

$$\frac{\alpha|\beta|\gamma|\delta|\epsilon|}{a|b|c|d|e|}=\overset{\alpha}{a}\,.\,\frac{\beta|\gamma|\delta|\epsilon}{b|c|d|e}+\ \cdots\cdot$$

$$\cdots\cdots\cdots\cdots$$

"Le symbole $\frac{\,|\,|\,}{\,|\,|\,}$ sert ici de caractéristique. Les seules choses à observer sont l'ordre des signes, et la loi des permutations entre les lettres a, b, c, d, &c., qui me paroissent suffisamment indiquées ci-dessus.

"Au lieu de transposer les lettres a, b, c, d, &c., on pouvoit les laisser dans l'ordre alphabétique, & transposer au contraire les lettres α, β, γ, δ, &c., les résultats auroient été parfaitement les mêmes; ce qui a lieu aussi par rapport aux conclusions suivantes.

"Premièrement, il est clair que $\frac{\alpha|\beta}{a|b}$ représente deux termes differens, l'un positif, & l'autre négatif, résultans d'autant de permutations possibles de a & b; que $\frac{\alpha|\beta|\gamma}{a|b|c}$ en représente six, trois positifs & trois négatifs, résultans d'autant de permutations possibles de a, b, & c; que $\frac{\alpha|\beta|\gamma|\delta}{a|b|c|d}$ . . . .

"Mais de plus, la formation de ces quantités est telle que l'unique changement que puisse résulter d'une permutation, quelle qu'elle soit, faite entre les lettres du même alphabet, dans l'une de ces abréviations, sera un changement dans le signe de la première valeur.

"La démonstration de cette vérité & la recherche du signe résultant d'une permutation déterminée, dépendent généralement de deux propositions qui peuvent être énoncées ainsi qu'il suit, en se servant de nombres pour indiquer le rang des lettres.

"La première est que

$$\frac{1|2|3|\ldots|m|m+1|\ldots n}{1|2|3|\ldots|m|m+1|\ldots n}$$

$$= \pm\frac{1|\;2\;|\;3\;|\ldots|n-m+1|n-m+2|n-m+3|\ldots|\;n}{m|m+1|m+2|\ldots|\;n\;|\;1\;|\;2\;|\ldots|m-1}$$

le signe − n'ayant lieu que dans le cas où n & m sont l'un & l'autre des nombres pairs.

"La seconde est que

$$\frac{1|2|3|\ldots|m|m+1|\ldots|n}{1|2|3|\ldots|m|m+1|\ldots|n}$$

$$= -\frac{1|2|3|\ldots|m-1|\;m\;|m+1|m+2|\ldots|n}{1|2|3|\ldots|m-1|m+1|\;m\;|m+2|\ldots|n}.$$

"Il sera facile de voir que, la première équation supposée, celle-ci n'a besoin d'être prouvée que pour un seul cas, comme, par exemple, celui de $m=n-1$, c'est-à-dire, celui où les deux lettres transposées sont les deux dernières.

"Au lieu de démontrer généralement ces deux équations, ce qui exigeroit un calcul embarrassant plutôt que difficile, je me contenterai de développer les exemples les plus simples: cela suffira pour saisir l'esprit de la démonstration.

. . . ."

($2\frac{1}{2}$ pages are occupied with verifications for the case of

$$\frac{\alpha|\beta}{a|b}, \text{ of } \frac{\alpha|\beta|\gamma}{a|b|c}, \text{ and of } \frac{\alpha|\beta|\gamma|\delta}{a|b|c|d}.)$$

"On verra qu'en général la démonstration de notre seconde équation pour le cas $n=a$, dépend de cette même équation pour le cas $n=a-1$, quel que soit a: d'où il suit que puisque $\frac{1|2}{1|2} = -\frac{1|2}{2|1}$, elle est généralement vraie.

. . . .

"De ce que nous avons dit jusqu'ici *il suit que*

$$\frac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta\,|\ \ldots\ldots}{a\,|\,b\,|\,c\,|\,d\,|\ \ldots\ldots}=0,$$

si deux lettres quelconques du même alphabet sont égales entr'elles; car quelque part que soient les deux lettres égales, on peut les transposer aux deux dernières places de leur rang, ce qui ne fera au plus que changer le signe de la valeur; alors, de leur permutation particulière, il ne peut, d'une part, résulter aucun changement, puisqu'elles sont égales; d'autre part, selon notre seconde équation ci-dessus, il doit en résulter un changement de signe; cette contradiction ne peut être levée qu'en supposant la valeur zéro . . . .

"Tout cela posé; puisque l'on a identiquement,

$$\frac{1\,|\,1\,|\,2}{1\,|\,2\,|\,3}=\frac{1}{1}\,.\,\frac{1\,|\,2}{2\,|\,3}+\frac{1}{2}\,.\,\frac{1\,|\,2}{3\,|\,1}+\frac{1}{3}\,.\,\frac{1\,|\,2}{1\,|\,2}=0,$$

$$\frac{2\,|\,1\,|\,2}{1\,|\,2\,|\,3}=\frac{2}{1}\,.\,\frac{1\,|\,2}{2\,|\,3}+\frac{2}{2}\,.\,\frac{1\,|\,2}{3\,|\,1}+\frac{2}{3}\,.\,\frac{1\,|\,2}{1\,|\,2}=0,$$

si l'on propose de trouver les valeurs de $\xi 1$ et de $\xi 2$ qui satisfont aux deux équations

$$\overset{1}{1}\,.\,\xi 1+\overset{1}{2}\,.\,\xi 2+\overset{1}{3}=0$$

$$\overset{2}{1}\,.\,\xi 1+\overset{2}{2}\,.\,\xi 2+\overset{2}{3}=0,$$

on pourra comparer, & l'on aura

$$\xi 1=\frac{\dfrac{1\,|\,2}{2\,|\,3}}{\dfrac{1\,|\,2}{1\,|\,2}},\quad \xi 2=\frac{\dfrac{1\,|\,2}{3\,|\,1}}{\dfrac{1\,|\,2}{1\,|\,2}}.$$

. . . ."

(Three equations with three unknowns are similarly dealt with.)

"Il est clair que ces valeurs n'ont point de facteurs inutiles: mais pour les rendre aussi commodes qu'il est possible dans les applications, & particulièrement dans celles où l'on veut faire usage des logarithmes, il sera bon d'y employer le plus qu'il se pourra, la multiplication des facteurs complexes. J'observe donc 1° que si l'on substitue dans le développement de $\frac{\alpha|\beta|\gamma|\delta}{a|b|c|d}$, les valeurs des $\frac{\alpha|\beta|\gamma}{a|b|c}$ en $\frac{\alpha|\beta}{a|b}$, on aura, en réduisant & ordonnant, d'après les observations ci-dessus,

$$\frac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta}{a\,|\,b\,|\,c\,|\,d}=\left\{\begin{array}{l}\dfrac{\alpha\,|\,\beta}{a\,|\,b}\cdot\dfrac{\gamma\,|\,\delta}{c\,|\,d}-\dfrac{\alpha\,|\,\beta}{a\,|\,c}\cdot\dfrac{\gamma\,|\,\beta}{b\,|\,d}+\dfrac{\alpha\,|\,\beta}{a\,|\,d}\cdot\dfrac{\gamma\,|\,\delta}{b\,|\,c}\\[2ex] +\dfrac{\alpha\,|\,\beta}{b\,|\,c}\cdot\dfrac{\gamma\,|\,\delta}{a\,|\,d}-\dfrac{\alpha\,|\,\beta}{b\,|\,d}\cdot\dfrac{\gamma\,|\,\delta}{a\,|\,c}\\[2ex] +\dfrac{\alpha\,|\,\beta}{c\,|\,d}\cdot\dfrac{\gamma\,|\,\delta}{a\,|\,b}\end{array}\right.$$

si de même on substitue dans le développement des $\frac{\alpha|\beta|\gamma|\delta|\epsilon|\zeta}{a|b|c|d|e|f}$, les valeurs des $\frac{\alpha|\beta|\gamma|\delta|\epsilon}{a|b|c|d|e}$ en $\frac{\alpha|\beta|\gamma|\delta}{a|b|c|d}$, on aura, en réduisant & ordonnant, d'après les observations ci-dessus,

$$\frac{\alpha|\beta|\gamma|\delta|\epsilon|\zeta}{a|b|c|d|e|f} = \left\{ \begin{array}{l} \frac{\alpha|\beta}{a|b}\cdot\frac{\gamma|\delta|\epsilon|\zeta}{c|d|e|f} - \frac{\alpha|\beta}{a|c}\cdot\frac{\gamma|\delta|\epsilon|\zeta}{b|d|e|f} + \frac{\ |\ }{a|d}\cdot\frac{\ |\ |\ |\ }{b|c|e|f} \\ + \frac{\alpha|\beta}{b|c}\cdot\frac{\gamma|\delta|\epsilon|\zeta}{a|d|e|f} - \frac{\ |\ }{b|d}\cdot\frac{\ |\ |\ |\ }{a|c|e|f} + \frac{\ |\ }{b|e}\cdot\frac{\ |\ |\ |\ }{a|c|d|f} \\ + \frac{\ |\ }{c|d}\cdot\frac{\ |\ |\ |\ }{a|b|e|f} - \frac{\ |\ }{c|e}\cdot\frac{\ |\ |\ |\ }{a|b|d|f} + \frac{\ |\ }{c|f}\cdot\frac{\ |\ |\ |\ }{a|c|d|e} \\ + \frac{\ |\ }{d|e}\cdot\frac{\ |\ |\ |\ }{a|b|c|f} - \frac{\ |\ }{d|f}\cdot\frac{\ |\ |\ |\ }{a|b|c|e} \\ + \frac{\ |\ }{e|f}\cdot\frac{\ |\ |\ |\ }{a|b|c|d} \\ \qquad\qquad - \frac{\ |\ }{a|e}\cdot\frac{\ |\ |\ |\ }{b|c|d|f} + \frac{\ |\ }{a|f}\cdot\frac{\ |\ |\ |\ }{b|c|d|e} \\ \qquad\qquad - \frac{\ |\ }{b|f}\cdot\frac{\ |\ |\ |\ }{a|c|d|e} \end{array} \right.$$

"La loi des permutations & des signes est assez manifeste dans ces exemples, pour qu'on en puisse conclure des développemens pareils pour les cas de huit & dix lettres, &c., du même alphabet; alors, en employant les premiers développemens pour les cas d'un nombre impair de ces lettres, on aura les formules d'élimination du premier degré, sous la forme la plus concise qu'il soit possible.

"Si l'on veut exprimer ces formules, généralement pour un nombre n d'équations

$$\overset{1}{1}\,.\,\xi 1 + \overset{1}{2}\,.\,\xi 2 + \overset{1}{3}\,.\,\xi 3 + \;.\;.\;.\; + \overset{1}{m}\,.\,\xi m + \;.\;.\;.\; + \overset{1}{n}\,.\,\xi n + (n \overset{1}{+} 1) = 0$$

$$\overset{2}{1}\,.\,\xi 1 + \overset{2}{2}\,.\,\xi 2 + \overset{2}{3}\,.\,\xi 3 + \;.\;.\;.\; + \overset{2}{m}\,.\,\xi m + \;.\;.\;.\; + \overset{2}{n}\,.\,\xi n + (n \overset{2}{+} 1) = 0$$

&c.

la valeur de l'inconnue quelconque ξm, sera renfermée dans l'équation suivante, à une seule inconnue

$$\frac{1|2|3|\ldots|n}{1|2|3|\ldots|n}\cdot\xi m$$

$$\pm \frac{1\;|\;2\;|\;3\;|\ldots\ldots|n-m|n-m+1|n-m+2|n-m+3|\ldots|\;n}{m+1|m+2|m+3|\ldots\ldots|\;n\;|\;n+1\;|\;1\;|\;2\;|\ldots|m-1} = 0$$

le signe + ayant lieu seulement dans le cas où m & n sont impairs l'un & l'autre."

Taking this up in order, we observe first that Vandermonde proposes for coefficients a positional notation essentially the same as that of Leibnitz, writing $\frac{1}{2}$ where Leibnitz wrote 12 or 1_2.

Then he defines a certain class of functions by means of their recurrent law of formation—a law and class of functions at once seen to be identical with those of Bézout. A special symbolism is used for the first time to denote the functions; thus, the expression

$$1_0.2_1.3_2 + 1_1.2_2.3_0 + 1_2.2_0.3_1 - 1_0.2_2.3_1 - 1_1.2_0.3_2 - 1_2.2_1.3_0,$$

which occurs in Leibnitz's letter, Vandermonde would have denoted by

$$\frac{1 \mid 2 \mid 3}{1 \mid 2 \mid 3},$$

and the result of eliminating x, y, z, w from the set of equations

$$1_r x + 2_r y + 3_r z + 4_r w = 0 \qquad (r = 1, 2, 3, 4)$$

by

$$\frac{1 \mid 2 \mid 3 \mid 4}{1 \mid 2 \mid 3 \mid 4}.$$

It is next pointed out that permutation of the under row of indices produces the same result as permutation of the upper row, that the number of terms is the same as the number of permutations of either row of indices, and that half of the terms are positive and half negative.

The part which follows this is a little curious. The proposition is brought forward that if in the symbolism for one of the functions a transposition of indices takes place in either row, the same function is still denoted, the only change thereby possible being a change of sign. The demonstration is affirmed to be dependent on two theorems, neither of which is proved, as the proofs are said to be troublesome to set forth. Now it will be seen that the second of these theorems is to the effect that the transposition of any two consecutive indices causes a change of sign, and that consequently this alone is sufficient for the required demonstration. The first of the auxiliary theorems, in fact, is an immediate deduction from the second, the particular permutation which it concerns being produced by $(n-m+1)(m-1)$ transpositions of pairs of consecutive indices.

Passing over the illustrations of these propositions, we come next to the theorem that if any two indices of either row be equal the function vanishes identically, and we note particularly that the basis of the proof is that the interchange of the two indices in question changes the sign of the function, and yet leaves the function unaltered.

Upon this theorem the solution of a set of simultaneous linear equations is then with much neatness made to depend. In more modern notation Vandermonde's process is as follows:—It is known that

$$a_1|b_1c_2| + b_1|c_1a_2| + c_1|a_1b_2| = |a_1b_1c_2| = 0,$$

and
$$a_2|b_1c_2| + b_2|c_1a_2| + c_2|a_1b_2| = |a_2b_1c_2| = 0,$$

$$\left.\begin{aligned} \therefore\quad a_1\frac{|b_1c_2|}{|a_1b_2|} + b_1\frac{|c_1a_2|}{|a_1b_2|} + c_1 &= 0 \\ a_2\frac{|b_1c_2|}{|a_1b_2|} + b_2\frac{|c_1a_2|}{|a_1b_2|} + c_2 &= 0 \end{aligned}\right\}$$

and

hence, if the equations

$$\left.\begin{aligned} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{aligned}\right\}$$

be given us, we know that

$$x=\frac{|b_1c_2|}{|a_1b_2|},\quad y=\frac{|c_1a_2|}{|a_1b_2|}$$

is a solution.

This result, moreover, is generalised; the solution of

$$r_1x_1+r_2x_2+\ldots+r_nx_n+r_{n+1}=0 \qquad (r=1, 2, \ldots n)$$

being fully and accurately expressed in symbols, although the numerators of the values of $x_1, x_2, \ldots, x_n$ are not in so simple a form as Cramer's rule for obtaining the numerator from the denominator might have suggested.

Lastly, and almost incidentally, Vandermonde makes known a case of the widely general theorem nowadays described as the theorem for expressing a determinant as an aggregate of products of complementary minors. His case is that in which the given determinant is of the order $2m$, and one factor of each of the products is of order 2.

Summing up, therefore, we must put the statement of our indebtedness to Vandermonde as follows:—

(1) A simple and appropriate notation for the new functions, *e.g.*, $\dfrac{1|2|3}{1|2|3}$. (VII.)

(2) A new mode of defining the functions, viz., using substantially Bézout's recurring law of formation. (VIII.)

(3) The remark that the ordinary algebraical expression of any of the functions is obtainable by permutation of *either* series of indices. (IX.)

(4) The remark that the positive and negative terms are equal in number. (X.)

(5) The theorem regarding the effect of interchanging two consecutive indices. (XI.)

(6) The theorem (with proof) regarding the effect of equality of two indices belonging to the same series. (XII.)

(7) A reasoned-out solution of a set of n simultaneous linear equations, by means of the new functions as above defined. (XIII.)

(8) Expression of any of the new functions of order $2m$ as an aggregate of products of like functions of orders 2 and $2m-2$. (XIV.)

In addition to this, we must view Vandermonde's work as a whole, and note that he is the first to give a connected exposition of the theory, defining the functions apart from their connections with other matter, assigning them a notation, and thereafter logically developing their properties. After Vandermonde there could be no absolute necessity for a renovation or reconstruction on a new basis: his successors had only to extend what he had done, and, it might be, to perfect certain points of detail. Of the mathematicians whose work has thus far been passed in review, the only one fit to be viewed as the founder of the theory of determinants is Vandermonde.

LAPLACE (1772).

[Recherches sur le calcul intégral et sur le système du monde. *Hist. de l'Acad. Roy. des Sciences* (Paris), Ann. 1772, 2e partie (pp. 267–376) pp. 294–304. *Œuvres*, viii. pp. 365–406.]

In the course of his work Laplace arrives at a set of linear equations from which n quantities have to be eliminated.

This he says can be accomplished by means of rules which mathematicians have given :—

"Mais comme elles ne me paroissent avoir été jusqu'ici démontrées que par induction, et que d'ailleurs elles sont impracticables, pour peu que le nombre des équations soit considérable ; je vais reprendre de nouveau cette matière, et donner quelques procédés plus simples que ceux qui sont déjà connus, pour éliminer entre un nombre quelconque d'équations du premier degré."

Taking n homogeneous linear equations with the coefficients

$$\begin{array}{l} {}^1a,\quad {}^1b,\quad {}^1c,\ \ldots\ldots \\ {}^2a,\quad {}^2b,\quad {}^2c,\ \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

he first gives Cramer's rule for writing out what he, Laplace, calls the *Resultant*, using in the course of the rule the term *variation* instead of Cramer's term "*dérangement*." Then he gives the "perhaps simpler" rule of Bézout, and shows that of necessity it will lead to the same result as Cramer's.

The theorem in regard to the effect of transposing two letters is next enunciated, and the blank left by Vandermonde is filled, for a proof of the theorem is given. The exact words of the enunciation and proof are—

"Si au lieu de combiner d'abord la lettre a avec la lettre b, ensuite ces deux-ci avec la lettre c, et ainsi de suite ; c'est-à-dire, si au lieu de combiner les lettres a, b, c, d, e, &c., dans l'ordre a, b, c, d, e, &c., on les eût combinées dans l'ordre a, c, b, d, e, &c., ou a, d, b, c, e, &c., ou a, e, b, c, d, &c., ou &c., je dis qu'on auroit toujours eu la même quantité à la différence des signes près.

"Pour démontrer ce Théorème nommons en général, *résultante*, la quantité qui résulte de l'une quelconque de ces combinaisons, en sorte que la *première résultante* soit celle qui vient de la combinaison suivant l'ordre a, b, c, d, e, &c., que la *seconde résultante* soit celle qui vient de la combinaison suivant l'ordre a, c, b, d, e, &c., que la *troisième résultante* soit celle qui vient de la combinaison suivant l'ordre a, d, b, c, e, &c., et ainsi de suite ; cela posé, il est clair que toutes ces résultantes renferment le même nombre de termes, et précisément les mêmes, puisqu'elles renferment tous les termes qui peuvent résulter de la combinaison des n lettres a, b, c, d, e, &c., disposées entre elles de toutes les manières possibles ; il ne peut donc y avoir de différence entre deux résultantes, que dans les signes de chacun de leurs termes ; or, il est visible que la première résultante donne la seconde, si l'on change dans la première b en c, et réciproquement c en b ; mais ce changement augmente ou diminue d'une unité le nombre des variations de chaque terme; d'où il suit que dans la seconde résultante, tous les termes dont le nombre des variations est impair, auront le signe $+$, et les autres le signe $-$; partant, cette seconde résultante n'est que la première, prise négativement.

"Il est visible pareillement que . . . ." &c.

The proof is thus seen to consist in establishing (1) that the terms of the one "resultant" must, apart from sign, be the same as those of the other; and (2) that the terms of the one resultant are either all affected with the same sign as the like terms of the other, or are all affected with the opposite sign, the comparison of sign being made by comparing the number of variations.

After this, the theorem that when two letters are alike the resultant vanishes is established in a way different from Vandermonde's, but not more satisfactory, viz., by considering what Bézout's rule would lead to in that case.

Application is then made to the problem of elimination, and to the solution of a set of linear simultaneous equations, the mode of treatment being again different from Vandermonde's, but this time with better cause. He says—

"Je suppose maintenant que l'on ait les trois équations

$$0 = {}^1a.\mu + {}^1b.\mu' + {}^1c.\mu'',$$
$$0 = {}^2a.\mu + {}^2b.\mu' + {}^2c.\mu'',$$
$$0 = {}^3a.\mu + {}^3b.\mu' + {}^3c.\mu'',$$

je forme d'abord la résultante des trois lettres a, b, c, suivant l'ordre a, b, c, ce qui donne,

$${}^1a.{}^2b.{}^3c - {}^1a.{}^2c.{}^3b + {}^1c.{}^2a.{}^3b - {}^1b.{}^2a.{}^3c + {}^1b.{}^2c.{}^3a - {}^1c.{}^2b.{}^3a$$

ou $${}^1a.[{}^2b.{}^3c - {}^2c.{}^3b] + {}^2a.[{}^1c.{}^3b - {}^1b.{}^3c] + {}^3a.[{}^1b.{}^2c - {}^1c.{}^2b];$$

je multiplie ensuite la première des équations précédentes par ${}^2b.{}^3c - {}^2c.{}^3b$, la seconde par ${}^1c.{}^3b - {}^1b.{}^3c$, la troisième par ${}^1b.{}^2c - {}^1c.{}^2b$, et je les ajoute ensemble, ce qui donne,

$$\begin{aligned} 0 = \quad & \mu.[{}^1a.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2a.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3a({}^1b.{}^2c - {}^1c.{}^2b)] \\ & + \mu'.[{}^1b.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2b.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3b({}^1b.{}^2c - {}^1c.{}^2b)] \\ & + \mu''.[{}^1c.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2c.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3c({}^1b.{}^2c - {}^1c.{}^2b)]; \end{aligned}$$

or, il suit de ce que nous venons de voir, que les coëfficiens de μ' et μ'', sont identiquement nuls, puisqu'ils ne sont que la résultante des trois lettres a, b, c, dans laquelle on écrit b, ou c, par-tout où est a; donc, on aura pour l'équation de condition demandée,

$$0 = {}^1a.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2a.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3a.({}^1b.{}^2c - {}^1c.{}^2b);$$

c'est-à-dire, la résultante de la combinaison des trois lettres a, b, c égalée à zéro. On démontreroit la même chose, quel que soit le nombre des équations.

"Pour montrer l'analogie de cette matière, avec l'élimination des équations du premier degré, je suppose que l'on ait les trois équations,

$$\begin{aligned} {}^1p &= {}^1a.\mu + {}^1b.\mu' + {}^1c.\mu'', \\ {}^2p &= {}^2a.\mu + {}^2b.\mu' + {}^2c.\mu'', \\ {}^3p &= {}^3a.\mu + {}^3b.\mu' + {}^3c.\mu''. \end{aligned}$$

Je multiplie, comme ci-devant, la première par $({}^2b.{}^3c - {}^2c.{}^3b)$, la seconde par $({}^1c.{}^3b - {}^1b.{}^3c)$, et la troisième par $({}^1b.{}^2c - {}^1c.{}^2b)$, je les ajoute ensemble, et j'observe que les coëfficiens de μ' et de μ'', sont indentiquement nuls dans l'équation qui en resulte; d'où je conclus,

$$\mu = \frac{{}^1p.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2p.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3p.({}^1b.{}^2c - {}^1c.{}^2b)}{{}^1a({}^2b.{}^3c - {}^2c.{}^3b) + {}^2a({}^1c.{}^3b - {}^1b.{}^3c) + {}^3a({}^1b.{}^2c - {}^1c.{}^2b)};$$

on voit donc que le numérateur de l'expression de μ, se forme du dénominateur, en y changeant a en p; on aura ensuite μ' ou μ'', en changeant dans l'expression de μ," &c.

This mode of treatment leaves nothing to be desired. It is that which is most commonly employed in the text-books of the present day.

The next point taken up is the most important in the memoir, and requires special attention. It is introduced as "a very simple process for considerably abridging the calculation of the equation of condition between a, b, c," &c.—that is to say, the calculation of a resultant. It is, however, something of much more value than this, involving as it does a widely general expansion-theorem to which Laplace's name has been attached, but of which we have already seen special cases stated by Vandermonde. The theorem may be described as giving an expansion of a resultant in the form of an aggregate of terms each of which is a product of resultants of lower degree. Laplace's exposition is as follows:—

"Je suppose que vous ayez deux équations,

$$0 = {}^1a.\mu + {}^1b.\mu'; \quad 0 = {}^2a.\mu + {}^2b.\mu';$$

écrivez $+ab$, et donnez l'indice 1 à la première lettre, et l'indice 2 à la seconde; l'équation de condition demandée scra $+ {}^1a.{}^2b - {}^1b.{}^2a = 0$.

"Je suppose que vous ayez trois équations; écrivez $+ab$, combinez ce terme avec la lettre c de toutes les manières possibles, en changeant le signe de chaque terme chaque fois que c change de place, vous aurez ainsi $+abc-acb+cab$; donnez dans chaque terme l'indice 1 à la première lettre, l'indice 2 à la seconde, l'indice 3 à la troisième, et vous aurez $+{}^1a.{}^2b.{}^3c - {}^1a.{}^2c.{}^3b + {}^1c.{}^2a.{}^3b$; cela posé, au lieu de $+{}^1a.{}^2b.{}^3c$ écrivez $+({}^1a.{}^2b - {}^1b.{}^2a).{}^3c$; au lieu de $-{}^1a.{}^2c.{}^3b$ écrivez $-({}^1a.{}^3b - {}^1b.{}^3a).{}^2c$; et au lieu de $+{}^1c.{}^2a.{}^3b$ écrivez $+({}^2a.{}^3b - {}^2b\,{}^3a).{}^1c$; l'equation de condition demandée sera

$$0 = ({}^1a.{}^2b - {}^1b.{}^2a).{}^3c - ({}^1a.{}^3b - {}^1b.{}^3a).{}^2c + ({}^2a.{}^3b - {}^2b.{}^3a).{}^1c.$$

"Je suppose que vous ayez quatre équations, écrivez $+abc-acb+cab$, et combinez ces trois termes avec la lettre d, en observant 1° de n'admettre que les termes dans lesquels c précède d; 2° de changer de signe dans chaque terme toutes les fois que d change de place, et vous aurez

$$+abcd-acbd+acdb+cabd-cadb+cdab;$$

donnez ensuite l'indice 1 à la première lettre, l'indice 2 à la seconde, &c., et vous aurez

$$+ {}^1a.{}^2b.{}^3c.{}^4d - {}^1a.{}^2c.{}^3b.{}^4d + {}^1a.{}^2c.{}^3d.{}^4b$$
$$+ {}^1c.{}^2a.{}^3b.{}^4d - {}^1c.{}^2a.{}^3d.{}^4b + {}^1c.{}^2d.{}^3a.{}^4b;$$

cela posé, au lieu de $+{}^1a.{}^2b.{}^3c.{}^4d$ écrivez

$$+({}^1a.{}^2b - {}^1b.{}^2a).({}^3c.{}^4d - {}^3d.{}^4c),$$

et ainsi des autres termes, et l'équation de condition sera

$$\begin{aligned} 0 = {} & ({}^1a.{}^2b - {}^1b.{}^2a).({}^3c.{}^4d - {}^3d.{}^4c) - ({}^1a.{}^3b - {}^1b.{}^3a).({}^2c.{}^4d - {}^2d.{}^4c) \\ & + ({}^1a.{}^4b - {}^1b.{}^4a).({}^2c.{}^3d - {}^2d.{}^3c) + ({}^2a.{}^3b - {}^2b.{}^3a).({}^1c.{}^4d - {}^1d.{}^4c) \\ & - ({}^2a.{}^4b - {}^2b.{}^4a).({}^1c.{}^3d - {}^1d.{}^3c) + ({}^3a.{}^4b - {}^3b.{}^4a).({}^1c.{}^2d - {}^1d.{}^2c). \end{aligned}$$

"Je suppose que vous ayez cinq équations, écrivez les six termes $+abcd-acbd+\ldots$ relatifs à quatre équations, et combinez-les avec la lettre e de toutes les manières possibles, en observant de changer de signe chaque fois que e change de place; donnez ensuite l'indice 1, &c., &c., ...; au lieu du terme $+{}^1a.{}^2c.{}^3b.{}^4e.{}^5d$ écrivez

$$({}^1a.{}^3b - {}^1b.{}^3a).({}^2c.{}^5d - {}^2d.{}^5c).{}^4e, \quad \&c. \ldots .$$

"Lorsqu'on aura six équations, on combinera les termes

$$+abcde-abced+\&c.,$$

relatifs à cinq équations avec la lettre f, en observant 1° de n'admettre que les termes dans lesquels e précède f; 2° de changer de signe lorsque f change de place: on transformera ensuite, par la règle précédente, ..."

Notwithstanding the multiplicity of instances, the rule here illustrated is not made altogether clear. This is due to two causes,—first, the linking of one case to the case before it; and, second, the want of explicit notification that the letters $b, d, f \ldots$ are combined in one way, and the intervening letters $c, e, \ldots$ in another. For the sake of additional clearness, let us see all the steps necessary in the case of the resultant of the five equations

$$a_r x_1 + b_r x_2 + c_r x_3 + d_r x_4 + e_r x_5 = 0 \quad (r = 1, 2, 3, 4, 5),$$

and supposing, as we ought to do, that the case of four equations has not been already dealt with. These steps are—

1°. Combining b with a subject to the condition that a precede b: result—

$$ab.$$

2°. Combining c with this *in every possible way*, the sign being &c.: result—

$$abc - acb + cab.$$

3°. Combining d with each of these terms subject to the condition that c precede d: result—

$$abcd - acbd + acdb + cabd - cadb + cdab.$$

4°. Combining e with each of these terms *in every possible way*: result—

$$\begin{aligned} &abcde - abced + abecd - aebcd + eabcd \\ &- acbde + acbed - \ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

5°. Appending indices: result—

$$\begin{aligned} &a_1 b_2 c_3 d_4 e_5 - a_1 b_2 c_3 e_4 d_5 + \ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

6°. Changing $a_m b_n$ into $(a_m b_n - b_m a_n)$, $c_r d_s$ into $(c_r d_s - d_r c_s)$, &c.: result—

$$(a_1 b_2 - b_1 a_2)(c_3 d_4 - d_3 c_4) e_5 - (a_1 b_2 - b_1 a_2)(c_3 d_5 - d_3 c_5) e_4 + \ldots .$$

This is the required resultant in the required form.

It is of the utmost importance to notice what is accomplished in 1°, 2°, 3°, 4° is simply (a) *the finding of the arrangements of* a, b, c, d, e *subject to the conditions that* a *precede* b, *and* c *precede* d, *and obtaining each arrangement with the sign which it ought to have in accordance with Cramer's rule.* The number of necessary directions might thus be reduced to three, viz., (a), (5), (6), in which case (1), (2), (3), (4) would take their proper places as successive steps of a methodic and expeditious way of accomplishing (a).

Laplace appends a demonstration of the accuracy of this development of the resultant of the nth degree, the line taken being that if the multiplications were performed the terms found would be exactly the $1.2.3 \dots n$ terms of the resultant, and would bear the signs proper to them as such.

He then goes on to deal with a rule for obtaining a like development in which as many as possible of the factors of the terms are resultants of the *third* degree.

To do so succinctly he is obliged to introduce a *notation* for resultants. On this point his words are—

"Je désigne par (abc) la quantité

$$abc - acb + cab - bac + bca - cba,$$

et par (ab) la quantité $ab - ba$, et ainsi de suite; par $({}^1a.{}^2b.{}^3c)$ j'indiquerai la quantité (abc), dans les termes de laquelle on donne 1 pour indice à la première lettre, 2 à la seconde, et 3 à la troisième; par $({}^1a.{}^2b)$, je désignerai la quantité (ab) dans les termes de laquelle on donne 1 pour indice à la première lettre, et 2 à la seconde; et ainsi de suite."

We can but remark that here again he leaves little room for improvement: his symbolism is essentially that which is still in common use.

The exposition of the rule is as follows:—

"Je suppose maintenant que vous ayez trois équations, l'équation de condition sera

$$0 = ({}^1a.{}^2b.{}^3c).$$

"Je suppose que vous ayez quatre équations; écrivez $+abc$, et combinez ce terme de toutes les manières possibles avec la lettre d, en observant de changer de signe lorsque d change de place, ce qui donne $+abcd - abdc + adbc - dabc$; donnez l'indice 1 à la première lettre, l'indice 2 à la seconde, &c., et vous aurez

$$+{}^1a.{}^2b.{}^3c.{}^4d - {}^1a.{}^2b.{}^3d.{}^4c + {}^1a.{}^2d.{}^3b.{}^4c - {}^1d.{}^2a.{}^3b.{}^4c;$$

au lieu du terme $+{}^1a.{}^2b.{}^3c.{}^4d$, écrivez $+({}^1a.{}^2b.{}^3c).{}^4d$; au lieu de $-{}^1a.{}^2b.{}^3d.{}^4c$, écrivez $-({}^1a.{}^2b.{}^4c).{}^3d$, et ainsi de suite, et vous formerez l'équation de condition

$$0 = ({}^1a.{}^2b.{}^3c).{}^4d - ({}^1a.{}^2b.{}^4c).{}^3d + ({}^1a.{}^3b.{}^4c).{}^2d - ({}^2a.{}^3b.{}^4c).{}^1d.$$

"Je suppose que vous ayez cinq équations, combinez les termes $+abcd-abdc+$&c., relatifs à quatre équations avec la lettre e en observant 1° de n'admettre que les termes dans lesquels d précède e; 2° de changer de signe lorsque e change de place, et vous aurez

$$+abcde-abdce+abdec+\&c.$$

donnez l'indice 1 à la première lettre, l'indice 2 à la seconde, &c., et vous aurez

$$+\ {}^1a.{}^2b.{}^3c.{}^4d.{}^5e\ -\ {}^1a.{}^2b.{}^3d.{}^4c.{}^5e\ +\ {}^1a.{}^2b^3.d.{}^4e.{}^5c\ +\ \&c.;$$

ensuite, au lieu de $+\ {}^1a.{}^2b.{}^3c.{}^4d.{}^5e$, écrivez $+\ ({}^1a.{}^2b.{}^3c).({}^4d.{}^5e)$; au lieu de $-\ {}^1a.{}^2b.{}^3d.{}^4c.{}^5e$, écrivez $-\ ({}^1a.{}^2b.{}^4c).({}^3d.{}^5e)$, et ainsi de suite; et en égalant à zero la somme de tous ces termes, vous formerez l'équation de condition demandée.

"Je suppose que vous ayez six équations, combinez les termes $+abcde-$&c., relatifs à cinq équations avec la lettre f, en observant 1° de n'admettre que les termes où e précède f; 2° de changer de signe lorsque f change de place: donnez ensuite 1 pour indice à la première lettre, ...

"Si vous avez sept équations, combinez les termes $+abcdef-$&c. relatifs à six équations avec la lettre g de toutes les manières possibles; pour huit équations, combinez les termes relatifs à sept avec la lettre h, en n'admettant que les termes dans lesquels g précède h, et ainsi du reste."

The really important point in all this is in regard to the manner in which the letters are brought into combination. It will be seen that the set begun with is abc, consequently a precedes b, and b precedes c throughout: then d is combined in every possible way with this: e is combined subject to the condition that d precede e; f is combined subject to the condition that e precede f: g is combined in every way possible: h is combined subject to the condition that g precede h: and so on. It would appear therefore that the lettres are to be combined in every possible way are d and *every third one afterwards*, and that each of the other letters is conditioned to be preceded by the letter which immediately precedes it in the original arrangement $abcdefghi$ . . . . Condensing these directions after the manner of the former case, we should draft the rule as follows:—

(a) Find every possible arrangement of $abcdefghi$ . . . subject to the conditions that in each arrangement we must have a, b, c in their natural order; d, e, f in their natural order; g, h, i in their natural order; and so on.

(*b*) Prefix to each arrangement its proper sign in accordance with Cramer's rule.

(*c*) Append in order the indices 1, 2, 3, . . . to the letters of each arrangement.

(*d*) Change $a_m b_n c_r$ into $(a_m.b_n.c_r)$, $d_s e_x f_y$ into $(d_r e_x.f_y)$, &c.

Without saying anything as to the verification of the developments thus obtained, Laplace concludes as follows:—

"On décomposeroit de la même manière l'équation R en termes composés de facteurs de 4, de 5, &c., dimensions."

To show how this could be effected would have been a tedious matter, if the method of exposition used in the previous cases had been followed, viz., multiplying instances with wearisome iteration of language until the laws for the combination of the letters could with tolerable certainty be guessed. On the other hand, had Laplace condensed his directions in the way we have indicated, the rule for the case in which as many as possible of the factors are of the 4th degree could have been stated as simply as that for either of the two cases he has dealt with. The only changes necessary, in fact, are in parts (1) and (4), and merely amount to writing the letters in consecutive sets of *four* instead of *two* or *three*.

Further, when the rule is condensed in this way, the problem of finding the number of terms in any one of the new developments—a problem which Laplace solves in one case by considering how many terms of the final development each such term gives rise to—is transformed into finding the number of possible arrangements referred to in part (1) of the rule. Where the highest degree of the factors of each term is 2 and the resultant which we wish to develop is of the nth degree (which is the case Laplace takes), the number of such arrangements is evidently $(1.2.3....n)/(1.2)^s$, s being the highest integer in $n/2$; if the highest degree of the factors is 3, the number of arrangements is

$$\frac{1.2.3....n}{(1.2.3)^s\,(1.2)^t},$$

where s is the highest integer in $n/3$ and t the highest integer in $(n-3s)/2$; and so on.

The facts in reduction of the claim which Laplace has to the expansion-theorem now bearing his name are thus seen to be (1) that the case in which as many as possible of the factors of the terms of the expansion are of the 2nd degree had already been given by Vandermonde; (2) that Laplace did not give a statement of his rule in a form suitable for application to all possible cases, and, indeed, was not sufficiently explicit in the statement of it for the first two cases to enable one readily to see what change would be necessary in applying it to the next case. Notwithstanding these drawbacks, however, there can be no doubt that if any *one* name is to be attached to the theorem it should be that of Laplace.

The sum of his contributions may be put as follows:—

(1) A proof of the theorem regarding the effect of the transposition of two adjacent letters in any of the new functions. (XI. 2)

(2) A proof of the theorem regarding the effect of equalizing two of the letters. (XII. 2)

(3) A mode of arriving at the known solution of a set of simultaneous linear equations. (XIII. 2)

(4) The name *resultant* for the new functions. (XV.)

(5) A notation for a resultant, *e.g.* $({}^1a.{}^2c.{}^3b)$. (VII. 2)

(6) A rule for expressing a resultant as an aggregate of terms composed of factors which are themselves resultants. (XIV. 2)

(7) A mode of finding the number of terms in this aggregate. (XVI.)

LAGRANGE (1773).

[Nouvelle solution du problème du mouvement de rotation d'un corps de figure quelconque qui n'est animé par aucune force accélératrice. *Nouv. Mém. de l'Acad. Roy. . . . (de Berlin).* Ann. 1773 (pp. 85–128). *Œuvres*, iii. pp. 577–616].

The position of Lagrange in regard to the advancement of the subject is quite different from that of any of the preceding mathematicians. All of those were explicitly dealing with the problem of elimination, and therefore directly with the functions afterwards known as determinants. Lagrange's work, on the other hand, consists of a number of incidentally obtained algebraical identities which we nowadays with more or less readiness recognise as relations between functions of the kind referred to, but which unfortunately Lagrange himself did not view in this light, and consequently left behind him as isolated instances. With him x, y, z and x', y', z' and x'', y'', z'' occur primarily as co-ordinates of points in space, and not as coefficients in a triad of linear equations; so that

$$(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x''),$$

when it does make its appearance, comes as representing six times the bulk of a triangular pyramid and not as the result of an elimination. In days when space of four dimensions was less attempted to be thought about than at present, this circumstance might possibly account for no advance being made to like identities involving four sets of four letters x, y, z, w; x', y', z', w'; &c.

In this first memoir the algebraical identities are brought together and stated at the outset as follows:—

"Lemme.

"1. Soient neuf quantités quelconques

$$x,\ y,\ z,\ x',\ y',\ z',\ x'',\ y'',\ z''$$

je dis qu'on aura cette équation identique

$$\begin{aligned}(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'')^2\\ = (x^2+y^2+z^2)(x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2)\\ +2(xx'+yy'+zz')(xx''+yy''+zz'')(x'x''+y'y''+z'z'')\\ -(x^2+y^2+z^2)(x'x''+y'y''+z'z'')^2\\ -(x'^2+y'^2+z'^2)(xx''+yy''+zz'')^2\\ -(x''^2+y''^2+z''^2)(xx'+yy'+zz')^2.\end{aligned}$$

"Corollaire 1.

"2. Donc si l'on a entre les neuf quantités précédentes ces six équations

$$\begin{aligned}x^2+y^2+z^2 &= a, & x'x''+y'y''+z'z'' &= b,\\ x'^2+y'^2+z'^2 &= a', & xx''+yy''+zz'' &= b',\\ x''^2+y''^2+z''^2 &= a'', & xx'+yy'+zz' &= b'',\end{aligned}$$

et qu'on fasse pour abréger

$$\xi = y'z'' - z'y'', \quad \eta = z'x'' - x'z'', \quad \zeta = x'y'' - y'x'',$$
$$\beta = \surd(aa'a'' + 2bb'b'' - ab^2 - a'b'^2 - a''b''^2)\,;$$

on aura

$$x\xi + y\eta + z\zeta = \beta.$$

On aura de plus les équations identiques suivantes

$$x'\xi + y'\eta + z'\zeta = 0, \quad x''\xi + y''\eta + z''\zeta = 0,$$
$$\xi^2 + \eta^2 + \zeta^2 = a'a'' - b^2,$$
$$y'\zeta - z'\eta = bx' - a'x'', \quad y''\zeta - z''\eta = a''x' - bx'',$$
$$z'\xi - x'\zeta = by' - a'y'', \quad z''\xi - x''\zeta = a''y' - by'',$$
$$x'\eta - y'\xi = bz' - a'z'', \quad x''\eta - y''\xi = a''z' - bz'',$$

qui sont très faciles à vérifier par le calcul.

"Corollaire 2.

"3. Si on prend les trois équations

$$\begin{aligned} x\xi + y\eta + z\zeta &= \beta, \\ xx' + yy' + zz' &= b'', \\ xx'' + yy'' + zz'' &= b', \end{aligned}$$

et qu'on en tire les valeurs des quantités x, y, z, on aura par les formules connues

$$x = \frac{\beta(y'z'' - z'y'') + b'(\eta z' - \zeta y') + b''(\zeta y'' - \eta z'')}{\xi(y'z'' - z'y'') + \eta(z'x'' - x'z'') + \zeta(x'y'' - y'x'')},$$
$$y = \frac{\beta(z'x'' - x'z'') + b'(\zeta x' - \xi z') + b''(\xi z'' - \zeta x'')}{\xi(y'z'' - z'y'') + \eta(z'x'' - x'z'') + \zeta(x'y'' - y'x'')},$$
$$z = \frac{\beta(x'y'' - y'x'') + b'(\xi y' - \eta x') + b''(\eta x'' - \xi y'')}{\xi(y'z'' - z'y'') + \eta(z'x'' - x'z'') + \zeta(x'y'' - y'x'')}\,;$$

donc faisant les substitutions de l'Art. préc. et supposant pour abréger

$$\alpha = a'a'' - b^2$$

on aura

$$x = \frac{\beta\xi + (a''b'' - bb')x' + (a'b' - bb'')x''}{\alpha},$$
$$y = \frac{\beta\eta + (a''b'' - bb')y' + (a'b' - bb'')y''}{\alpha},$$
$$z = \frac{\beta\zeta + (a''b'' - bb')z' + (a'b' - bb'')z''}{\alpha}."$$

In regard to the first identity here (the so-called lemma), the important and notable point is that the right-hand member is the same kind of function of the nine quantities $x^2+y^2+z^2$,

$xx'+yy'+zz', \quad xx''+yy''+zz'', \quad xx'+yy'+zz', \quad x'^2+y'^2+z'^2,$
$x'x''+y'y''+z'z'', \quad xx''+yy''+zz'', \quad x'x''+y'y''+z'z'', \quad x''^2+y''^2+z''^2$
as the left-hand member is of the nine $x, y, z, \; x', y', z', \; x'', y'', z''$. Indeed, without this distinguishing characteristic, the identity would have been to us of comparatively little moment. Possibly Lagrange was aware of it; but, if so, it is remarkable that he did not draw attention to the fact. It is quite true that Lagrange's identity and the modern-looking identity

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}^2 = \begin{vmatrix} x^2+y^2+z^2 & xx'+yy'+zz' & xx''+yy''+zz'' \\ xx'+yy'+zz' & x'^2+y'^2+z'^2 & x'x''+y'y''+z'z'' \\ xx''+yy''+zz'' & x'x''+y'y''+z'z'' & x''^2+y''^2+z''^2 \end{vmatrix}$$

are essentially the same; but no one can deny that the latter contains on the face of it an all-important fact which is hid in the former, and which in Lagrange's time could be made known only by an additional statement in words.

The second identity

$$x'\xi+y'\eta+z'\zeta=0$$

is a simple case of one of Vandermonde's, viz., that regarding the vanishing of his functions when two of the letters involved were the same.

The third identity

$$\xi^2+\eta^2+\zeta^2=a'a''-b^2$$

is in modern notation

$$\begin{vmatrix} y' & y'' \\ z' & z'' \end{vmatrix}^2 + \begin{vmatrix} z' & z'' \\ x' & x'' \end{vmatrix}^2 + \begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}^2 = \begin{vmatrix} x'^2+y'^2+z'^2 & x'x''+y'y''+z'z'' \\ x'x''+y'y''+z'z'' & x''^2+y''^2+z''^2 \end{vmatrix},$$

and is thus seen to be a simple special instance of a very important theorem afterwards discovered.

The fourth identity

$$y'\zeta-z'\eta = bx'-a'x'',$$

may be expressed in modern notation as follows:—

$$\begin{vmatrix} y' & z' \\ |z'x''| & |x'y''| \end{vmatrix} = \begin{vmatrix} x'x''+y'y''+z'z'' & x'' \\ x'^2+y'^2+z'^2 & x' \end{vmatrix},$$

and, quite probably, has also ere this been generalised in the like notation.

The fifth identity

$$x = \frac{\beta\xi+(a''b''-bb')x'+(a'b'-bb'')x''}{a},$$

is not so readily transformable, the determinantal theorem which it involves being indeed completely buried. Multiplying both sides by a; then doing away with a, which seems perversely introduced "pour abréger" when no like symbol of abridgment takes the place of $a''b''-bb'$ or of $a'b'-bb''$; and transposing, we have

$$\beta\xi = x(a'a''-b^2) - x'(a''b''-bb') + x''(bb''-a'b'),$$
$$= \begin{vmatrix} x & b'' & b' \\ x' & a' & b \\ x'' & b & a'' \end{vmatrix};$$

that is, finally,

$$|xy'z''|.|y'z''| = \begin{vmatrix} x & xx'+yy'+zz' & xx''+yy''+zz'' \\ x' & x'^2+y'^2+z'^2 & x'x''+y'y''+z'z'' \\ x'' & x'x''+y'y''+z'z'' & x''^2+y''^2+z''^2 \end{vmatrix},$$

which we recognise as an instance of the multiplication-theorem on putting

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} \times \begin{vmatrix} 1 & x' & x'' \\ 0 & y' & y'' \\ 0 & z' & z'' \end{vmatrix}$$

for the left-hand member.

LAGRANGE (1773).

[Solutions analytiques de quelques problèmes sur les pyramides triangulaires. *Nouv. Mém. de l'Acad. Roy. . . . (de Berlin).* Ann. 1773 (pp. 149–176). *Œuvres,* iii. pp. 659–692.]

In this memoir also there is a preparatory algebraical portion, the subject being the same as before, and the author's standpoint unchanged. Indeed the two introductions differ only in that the second is a rounding off and slight natural development of the first.

In addition to ξ, η, ζ, we have now $\xi', \eta', \zeta', \xi'', \eta'', \zeta''$ used as abbreviations for $zy''-yz''$, $xz''-zx''$, ...; in addition to a, we have $a', a'', \beta, \beta', \beta''$, standing for $aa''-b'^2$, $aa'-b''^2$, $b'b''-ab$, $bb''-a'b'$, $bb'-a''b''$; and X, Y, Z, X′, Y′, ..., A, A′, ... are introduced, having the same relation to $\xi, \eta, \zeta, \xi' \eta'$, ... a, a' . . . . as these latter have to x, y, z, x', y', ..., a, a', ... Lagrange then proceeds:—

"3. Or en substituant les valeurs de ξ, ξ', &c., en x, x', &c., et faisant pour abréger

$$\Delta = xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'',$$

on trouve

$$\begin{array}{lll} X = \Delta x, & Y = \Delta y, & Z = \Delta z, \\ X' = \Delta x', & Y' = \Delta y', & Z' = \Delta z', \\ X'' = \Delta x'', & Y'' = \Delta y'', & Z'' = \Delta z'', \end{array}$$

donc mettant ces valeurs dans les dernières équations ci-dessus, on aura en vertu des six équations supposées dans l'Art. 1.

$$\begin{array}{ll} A = \Delta^2 a, & B = \Delta^2 b, \\ A' = \Delta^2 a', & B' = \Delta^2 b', \\ A'' = \Delta^2 a'', & B'' = \Delta^2 b'', \end{array}$$

et de là il est facile de tirer la valeur de Δ^2 en a, a', a'', b, &c.; car on aura d'abord

$$\Delta^2 = \frac{A}{a} = \frac{\alpha'\alpha'' - \beta^2}{a},$$

et substituant les valeurs de α', α'' et β en a, a', &c. (Art. 1)

$$\Delta^2 = aa'a'' + 2bb'b'' - ab^2 - a'b'^2 - a''b''^2;$$

on trouvera la même valeur de Δ^2 par les autres équations. Si on remet dans cette équation les quantités x, y, z, x', &c., on aura la même équation identique que nous avons donnée dans le Lemme ci-dessus (p. 86).

"4. Il est bon de remarquer que la valeur de Δ^2 peut aussi se mettre sous cette forme

$$\Delta^2 = \frac{a\alpha + a'\alpha' + a''\alpha'' + 2(\beta b + \beta' b' + \beta'' b'')}{3};$$

or si on multiplie cette équation par Δ^2 et qu'on y substitue ensuite A à la place de $\Delta^2 a$, A' à la place de $\Delta^2 a'$ et ainsi de suite (Art. préc.) on aura

$$\Delta^4 = \frac{A\alpha + A'\alpha' + A''\alpha'' + 2(B\beta + B'\beta' + B''\beta'')}{3};$$

ou bien en mettant pour A, A', &c., leurs valeurs en α, α', &c. (Art. 2)

$$\Delta^4 = \alpha\alpha'\alpha'' + 2\beta\beta'\beta'' - \alpha\beta^2 - \alpha'\beta'^2 - \alpha''\beta''^2;$$

d'où l'on voit que la quantité Δ^2 et son carré Δ^4 sont des fonctions semblables, l'une de a, a', a'', b, b', b'', l'autre de α, α', α'', β, β', β''.

"5. De plus, comme l'on a (Art. 3)

$$\begin{aligned} &xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'' \\ &\qquad = \sqrt{(aa'a'' + 2bb'b'' - ab^2 - a'b'^2 - a''b''^2)} = \Delta, \end{aligned}$$

et qu'il y a entre les quantités x, y, z, x', &c., et a, a', a'', b, &c., les mêmes relations qu'entre les quantités ξ, η, ζ, ξ', &c., et α, α' α'', β, &c. (Art. 1), on aura donc aussi

$$\xi\eta'\zeta''+\eta\zeta'\xi''+\zeta\xi'\eta''-\xi\zeta'\eta''-\eta\xi'\zeta''-\zeta\eta'\xi''$$
$$=\sqrt{(\alpha\alpha'\alpha''+2\beta\beta'\beta''-\alpha\beta^2-\alpha'\beta'^2-\alpha''\beta''^2)}=\Delta^2.$$

Donc on aura cette équation identique et très remarquable

$$\xi\eta'\zeta''+\eta\zeta'\xi''+\zeta\xi'\eta''-\xi\zeta'\eta''-\eta\xi'\zeta''-\zeta\eta'\xi''$$
$$=(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'')^2."$$

The remaining portion is of little importance; its main contents are four sets of nine identities each, viz.:—

1. $x\xi+x'\xi'+x''\xi''=\Delta, \quad y\xi+y'\xi'+y''\xi''=0,$ &c.
2. $x\xi+y\eta\ +z\zeta\ \ =\Delta, \quad x'\xi+y'\eta+z'\zeta\ \ =0,$ &c.
3. $\xi=\dfrac{\alpha x+\beta''x'+\beta'x''}{\Delta},$ &c.
4. $x=\dfrac{a\xi+b''\xi'+b'\xi''}{\Delta},$ &c.

Besides the fact that Art. 3 contains a proof of the Lemma of the previous memoir, we have to note the new identity

$$\mathrm{X}=\Delta x,$$

which in modern determinantal notation is

$$\begin{Vmatrix} |xz''| & |yx''| \\ |zx'| & |xy'| \end{Vmatrix} = x\,|xy'z''|,$$

—a simple special instance of the theorem regarding what is nowadays known as "a minor of the determinant adjugate to another determinant."

The last two lines of Art. 4 by implication make it almost certain that Lagrange did not look upon

$$xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x''$$

and $$aa'a''+2bb'b''-ab^2\ \ -a'b'^2\ -a''b''^2$$

as functions of the same kind.

The new theorem in Art. 5, which Lagrange justly characterises as "very remarkable," is in modern determinantal notation

$$\begin{Vmatrix} |y'z''| & |z'x''| & |x'y''| \\ |zy''| & |xz''| & |yx''| \\ |yz'| & |zx'| & |xy'| \end{Vmatrix} = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}^2,$$

—a simple instance of the theorem which gives the relation, as we now say, "between a determinant and its adjugate."

In regard to the remaining identities which we have numbered (1), (2), (3), (4), we note that (1) and (3) are not new, although (3) is here given almost in the form desiderated above (pp. 36–37); (2) involves the fact that Δ is the same function of $x, x', x'', y, y', y'', z, z', z''$, as it is of $x, y, z, x', y', z', x'', y'', z''$; and (4) may be transformed as follows:—

$$\begin{aligned} x\Delta &= a\xi + b''\xi' + b'\xi'', \\ &= \begin{vmatrix} a & y & z \\ b'' & y' & z' \\ b' & y'' & z'' \end{vmatrix}, \\ &= \begin{vmatrix} x^2+y^2+z^2 & y & z \\ xx'+yy'+zz' & y' & z' \\ xx''+yy''+zz'' & y'' & z'' \end{vmatrix}; \end{aligned}$$

so that it may be considered as another disguised instance of the multiplication-theorem, the determinant just reached being equal to

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} \times \begin{vmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix}.$$

LAGRANGE (1773).

[Recherches d'Arithmétique. *Nouv. Mém. de l'Acad. Roy. . . . (de Berlin).* Ann. 1773 (pp. 265–312).]

This is an extensive memoir on the numbers "qui peuvent être représentées par la formule $Bt^2 + Ctu + Du^2$." At p. 285 the expression

$$py^2 + 2qyz + rz^2$$

is transformed into $\quad Ps^2 + 2Qsx + Rx^2$

by putting $\quad y = Ms + Nx,$

and $\quad z = ms + nx,$

and Lagrange says—

". . . . je substitue dans la quantité $PR - Q^2$ les valeurs de P, Q et R, et je trouve en effaçant ce qui se détruit

$$PR - Q^2 = (pr - q^2)(Mn - Nm)^2; \; . \; . \; . \; ."$$

which we at once recognise as the simplest case of the theorem connecting (as we now say) the discriminant of any quantic with the discriminant of the result of transforming the quantic by a linear substitution.

Putting now in compact form all the identities obtained from the three preceding memoirs of Lagrange, we have—

(1) $(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'')^2$
$$=aa'a''+2bb'b''-ab^2-a'b'^2-a''b''^2, \qquad \text{(XVII.)}$$
where $\quad a=x^2+y^2+z^2,\ a'= \ .\ .\ .\ .$

(2) $\xi^2+\eta^2+\zeta^2=a'a''-b^2$, where $\xi=y'z''-z'y''$, $\eta=.\ .\ .\ .$ (XVIII.)

(3) $y'\zeta-z\eta=bx'-a'x''$. (XIX.)

(4) $\xi\Delta=\alpha x+\beta''x'+\beta'x''$, where $\alpha=a'a''-b^2$, $\beta''=\ .\ .\ .\ .\ ,$
and $\Delta=xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x''$. (XVII. 2)

(5) $\mathrm{X}=\Delta x$, where $\mathrm{X}=\eta'\zeta''-\zeta'\eta''$. (XX.)

(6) $(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'')^2$
$$=\xi\eta'\zeta''+\eta\zeta'\xi''+\zeta\xi'\eta''-\xi\zeta'\eta''-\eta\xi'\zeta''-\zeta\eta'\xi''. \qquad \text{(XXI.)}$$

(7) $\mathrm{PR}-\mathrm{Q}^2=(pr-q^2)(\mathrm{M}n-\mathrm{N}m)^2$, (XXII.)
if $\quad p(\mathrm{M}s+\mathrm{N}x)^2+2q(\mathrm{M}s+\mathrm{N}x)(ms+nx)+r(ms+nx)^2$
$$=\mathrm{P}s^2+2\mathrm{Q}sx+\mathrm{R}x^2 \text{ identically.}$$

BÉZOUT (1779).

[Théorie Générale des Equations Algébriques, §§ 195–223, pp. 171–187; §§ 252–270, pp. 208–223. Paris.]

In his extensive treatise on algebraical equations Bézout was bound, as a matter of course, to take up the question of elimination; and, as he had dealt with the subject in a separate memoir in 1764, one might not unreasonably expect to find the treatise giving merely a reproduction of the contents of the memoir in a form suited to a didactic work. Such, however, is far from being the case. He merely mentions the necessary references to the work of Cramer, himself, Vandermonde, and Laplace; and then adds—

"Mais lorsqu'il a été question d'appliquer ces différentes méthodes au problème de l'élimination, envisagé dans toute son étendue, je me suis bientôt apperçu qu'ils laissoient tous encore beaucoup à desirer du côté de la pratique."

His main objection to the said methods is that when one has to deal with a set of equations of no great generality, with coefficients, it may be, expressed in figures—

"Il faut construire ces formules dans toute la généralité dont les équations sont susceptibles, et faire par conséquent le même travail que si les équations avoient toute cette généralité.

. . . .

(197). Au lieu donc de nous proposer pour but seulement, de donner des formules générales d'élimination dans les équations du premier degré, nous nous proposons de donner une règle qui soit indifférement et également applicable aux équations prises dans toute leur généralité, et aux équations considerées avec les simplifications qu'elles pourront offrir : une règle dont la marche soit la même pour les unes que pour les autres, mais qui ne fasse calculer que ce qui est absolument indispensable pour avoir la valeur des inconnues que l'on cherche : une règle qui s'applique indifféremment aux équations numériques et aux équations littérales, sans obliger de recourir à aucune formule. Telle est, si je ne me trompe, la règle suivante.

"*Règle générale pour calculer, toutes à la fois, ou séparément, les valeurs des inconnues dans les équations du premier degré, soit littérales soit numériques.*

"(198). Soient u, x, y, z. &c., des inconnues dont le nombre soit n, ainsi qui celui des équations.

"Soient a, b, c, d, &c., les coëfficiens respectifs de ces inconnues dans la première équation.

"a', b', c', d', &c., les coëfficiens des mêmes inconnues dans la seconde équation.

"a'', b'', c'', d'', &c., les coëfficiens des mêmes inconnues dans la troisième équation : et ainsi de suite.

"Supposez tacitement que le terme tout connu de chaque équation soit affecté aussi d'une inconnue que je représente par t.

"Formez le produit $uxyzt$ de toutes ces inconnues écrites dans tel ordre que vous voudrez d'abord ; mais cet ordre une fois admis, conservez-le jusqu'à la fin de l'opération.

"Echangez successivement, chaque inconnue contre son coëfficient dans la première équation, en observant de changer le signe à chaque échange pair : ce résultat sera, ce que j'appelle, une *première ligne*.

"Echangez dans cette *première ligne*, chaque inconnue, contre son coëfficient dans la seconde équation, en observant, comme ci-devant, de changer le signe à chaque échange pair: et vous aurez une *seconde ligne*.

"Echangez dans cette *seconde ligne*, chaque inconnue, contre son coëfficient dans la troisième équation, en observant de changer le signe à chaque échange pair : et vous aurez une *troisième ligne*.

"Continuez de la même manière jusqu'à la dernière équation inclusivement ; et la dernière *ligne* que vous obtiendrez, vous donnera les valeurs des inconnues de la manière suivante.

"Chaque inconnue aura pour valeur une fraction dont le numérateur sera le coëfficient de cette même inconnue dans la dernière ou n^e *ligne*, et qui aura constamment pour dénominateur le coëfficient que l'inconnue introduite t se trouvera avoir dans cette même n^e *ligne*."

The application of this very curious rule is illustrated by a considerable number of varied examples, of which we select the second—

"(200). Soient les trois équations suivantes

$$\begin{aligned} ax + by + cz + d &= 0, \\ a'x + b'y + c'z + d' &= 0, \\ a''x + b''y + c''z + d'' &= 0. \end{aligned}$$

"Je les écris ainsi

$$\begin{aligned} ax + by + cz + dt &= 0, \\ a'x + b'y + c'z + d't &= 0, \\ a''x + b''y + c''z + d''t &= 0. \end{aligned}$$

Je forme le produit $xyzt$.
Je change successivement x en a, y en b, z en c, t en d, et observant la règle des signes, j'ai pour première ligne

$$ayzt - bxzt + cxyt - dxyz.$$

Je change successivement x en a', y en b', z en c', t en d', et observant la règle des signes, j'ai pour seconde ligne

$$\begin{aligned} &(ab' - a'b)zt - (ac' - a'c)yt + (ad' - a'd)yz \\ &+ (bc' - b'c)xt - (bd' - b'd)xz + (cd' - c'd)xy. \end{aligned}$$

Je change successivement x en a'', y en b'', z en c'', t en d'', et observant la règle des signes j'ai pour troisième ligne

$$\begin{aligned} &[(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a'']t \\ &- [(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a'']z \\ &+ [(ac' - a'c)d'' - (ad' - a'd)c'' + (cd' - c'd)a'']y \\ &- [(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b'']x. \end{aligned}$$

D'où (198) je tire

$$x = \frac{-[(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b'']}{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a''},$$

$$y = \frac{+[(ac' - a'c)d'' - (ad' - a'd)c'' + (cd' - c'd)a'']}{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a''},$$

$$z = \frac{-[(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a'']}{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a''}."$$

Among the other examples are included (1) one in which the coefficients in the set of equations are given in figures; (2) one in which some of the coefficients are zero; (3) one showing the simplification possible when the value of only one unknown is wanted; (4) one showing the signification of the vanishing of one of the "*lignes*"; (5) one showing the signification of the absence of one of the unknowns from the last "*ligne*"; and (6) one or two concerned with the allied problem of elimination.

Bézout nowhere gives any reason for his rule; it is used throughout as a pure rule-of-thumb: its effectiveness being manifest, he leaves on the reader the full burden of its arbitrariness. The unreal product $xyzt$ at the very outset must have been a sore puzzle to students, and none the less so because of the certainty which many of them must have felt that a real entity underlay it.

To throw light upon the process, let us compare the above solution of a set of three linear equations with the following solution, which from one point of view may be looked upon as an improvement on the ordinary determinantal modes of solution as presented to modern readers.

The set of equations being

$$\left.\begin{array}{l} ax+by+cz+d=0 \\ a'x+b'y+c'z+d'=0 \\ a''x+b''y+c''z+d''=0 \end{array}\right\}$$

we know that the numerators of the values of x, y, z, and the common denominator are

$$-\begin{vmatrix} b & c & d \\ b' & c' & d' \\ b'' & c'' & d'' \end{vmatrix}, \quad +\begin{vmatrix} a & c & d \\ a' & c' & d' \\ a'' & c'' & d'' \end{vmatrix}, \quad -\begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix}, \quad +\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

They are therefore the coefficients of x, y, z, t in the determinant

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ x & y & z & t \end{vmatrix}, \text{ or } \Delta \text{ say.}$$

Thus the problem of solving the set of equations is transformed into finding the development of this determinant. In doing so let us use $[xyz]$ to stand for the determinant of which x, y, z is the last row, and whose other rows are the two rows immediately above x, y, z in Δ: similarly let $[zt]$ stand for the determinant of which z, t is the last row, and its other row the row c'', d'' immediately above z, t in Δ; and so on in all possible cases, including even $[xyzt]$, which of course is Δ itself.

Then clearly we have

$$[xyzt]=a[yzt]-b[xzt]+c[xyt]-d[xyz] \qquad (1)$$

Developing in the same way the four determinants here on the right side, we have as our next step

$$\begin{aligned}[xyzt]= &\ \ a(b'[zt]-c'[yt]+d'[yz])\\ &-b(a'[zt]-c'[xt]+d'[xz])\\ &+c(a'[yt]-b'[xt]+d'[xy])\\ &-d(a'[yz]-b'[xz]+c'[xy]),\\ = &\ \ (ab'-a'b)[zt]-(ac'-a'c)[yt]+(ad'-a'd)[yz]\\ &+(bc'-b'c)[xt]-(bd'-b'd)[xz]+(cd'-c'd)[xy].\end{aligned}$$

Again, developing the six determinants $[zt]$, $[yt]$, ... in the same way, and rearranging the terms, we have finally

$$\begin{aligned}[xyzt]= &\ \ \{(ab'-a'b)c''-(ac'-a'c)b''+(bc'-b'c)a''\}t\\ &-\{(ab'-a'b)d''-(ad'-a'd)b''+(bd'-b'd)a''\}z\\ &+\{(ac'-a'c)d''-(ad'-a'd)c''+(cd'-c'd)a''\}y\\ &-\{(bc'-b'c)d''-(bd'-b'd)c''+(cd'-c'd)b''\}x.\end{aligned}$$

But the coefficients of x, y, z, t in $[x\,y\,z\,t]$ were seen on starting to be the numerators and the common denominator of the values of x, y, z in the given set of equations: hence

$$x=\frac{-\{(bc'-b'c)d''-(bd'-b'd)c''+(cd'-c'd)b''\}}{\{(ab'-a'b)c''-(ac'-a'c)b''+(bc'-b'c)a''\}},$$

$$y= \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$

$$z= \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$

Now it is at once manifest that the successive developments here obtained of the determinant $[xyzt]$ are letter by letter identical with the successive "*lignes*" obtained by Bézout from the unreal product $xyzt$; but that instead of having one arbitrary step succeeding another, as in the application of Bézout's rule, there is here a fluent reasonableness characterising the whole process.* As for the peculiarities requiring elucidation in the series of special examples above referred to, they are seen, when looked at in this light, to be but matters of course.

Not only so, but it will be found that the translation of xy into $[xy]$, &c., is an unfailing key to much that follows in Bézout in connection with the subject. For example, let us take the wide extension of the rule which is expounded later on in the treatise, in a section headed

Considérations utiles pour abréger considérablement le calcul des coëfficients qui servent à l'élimination.

There are in all fifteen pages (pp. 208–223, §§ 252–270) devoted to the subject. The contents of three paragraphs will give a sufficiently clear idea of the nature of the whole. The notation used is identical with that of Laplace, *e.g.*,

$$(ab') = ab' - a'b,$$
$$(ab'c'') = (ab'-a'b)c'' - (ab''-a''b)c' + (a'b''-a''b')c,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Two of the three selected paragraphs stand as follows:—

"(264.) Cette manière de procéder au calcul des inconnues, en les grouppant, n'est pas applicable seulement à notre objet; elle peut en général être appliquée dans toutes les équations du premier degré.

---

* If the fact at the basis of the process were made use of nowadays, it would be advantageous, of course, in the first instance to simplify the determinant as much as possible. For example, the equations being (Bézout, p. 178)

$$\left.\begin{array}{l} 2x+4y+5z=22 \\ 3x+5y+2z=30 \\ 5x+6y+4z=43 \end{array}\right\},$$

we might proceed as follows:—

$$\begin{vmatrix} 2 & 4 & 5 & -22 \\ 3 & 5 & 2 & -30 \\ 5 & 6 & 4 & -43 \\ x & y & z & t \end{vmatrix} = \begin{vmatrix} 0 & 2 & 11 & -6 \\ 1 & 1 & -3 & -8 \\ 0 & -3 & -3 & 9 \\ x & y & z & t \end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & 0 & 9 & 0 \\ 1 & 0 & -4 & -5 \\ 0 & -1 & -1 & 3 \\ x & y & z & t \end{vmatrix} = 27\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 3 \\ x & y & z & t \end{vmatrix}$$

$$= 27\{-t+0z-3y-5x\};$$

whence $x=5$, $y=3$, $z=0$.

"Si l'on avoit, par exemple, les quatre équations suivantes

$$\begin{aligned} ax + by + cz + dt + e &= 0, \\ a'x + b'y + c'z + d't + e' &= 0, \\ a''x + b''y + c''z + d''t + e'' &= 0, \\ a'''x + b'''y + c'''z + d'''t + e''' &= 0. \end{aligned}$$

En se rappellant que chaque inconnue a pour valeur le coëfficient qu'elle se trouve avoir dans la dernière *ligne*, divisé constamment par celui que l'inconnue introduite aura dans cette même *ligne*, on verra bientôt qu'on peut réduire le calcul à chercher le coëfficient de l'une quelconque des inconnues dans la dernière ligne; parce que de la même manière qu'on en aura calculé un, on calculera de même tous les autres: ou même, lorsqu'on en aura calculé un, on pourra en déduire tous les autres, lorsque les équations auront toute la généralité possible. Or pour avoir la valeur du coëfficient d'une des inconnues dans la dernière ligne, la question se réduit à calculer la valeur du produit des autres inconnues. Mais pour ne pas se tromper sur les signes, il faudra toujours ne pas perdre de vue, la place que cette inconnue est censée occuper dans le produit de toutes les inconnues. Ainsi, dans le cas présent, au lieu de calculer généralement la dernière *ligne* pour avoir $xyztu$, je calcule seulement cette dernière ligne pour $yztu$: et pour l'avoir de la manière la plus commode, je grouppe en cette manière $yz.tu$, et je procède comme il suit, au calcul des lignes, observant que y est censé à la seconde place.

Première ligne. $-bz.tu - yz.du,$

Seconde ligne. $+(bc').tu - bz.d'u + b'z.du + yz.(de'),$

Troisième ligne. $-(bc').d''u + (bc'').d'u - bz.(d'e'') - (b'c'').du + b'z.(de'') - b''z.(de'),$

Quatrième ligne. $+(bc').(d''e''') - (bc'').(d'e''') + (bc''').(d'e'') + (b'c'').(de''')$
$\qquad - (b'c''').(de'') + (b''c''').(de');$

c'est le coëfficient de x dans la dernière ligne.

"Pour avoir celui de u, je calculerois de même la valeur de $xyzt$, en le grouppant ainsi, $xy.zt$, et je trouverois pour valeur du coëfficient de u dans la dernière ligne, la quantité

$$(ab').(c''d''') - (ab'').(c'd''') + (ab''').(c'd'') + (a'b'').(cd''') \\ - (a'b''').(cd'') + (a''b''').(cd');$$

D'où je conclus

$$x = \frac{+(bc').(d''e''') - (bc'').(d'e''') + (bc''').(d'e'') + (b'c'').(de''') - (b'c''').(de'') + (b''c''').(de')}{(ab').(c''d''') - (ab'').(c'd''') + (ab''').(c'd'') + (a'b'').(cd''') - (a'b''').(cd'') + (a''b''').(cd')}$$

et ainsi de suite.

(265.) Si j'avois les cinq équations suivantes—

$$\begin{aligned} ax + by + cz + dr + et + f &= 0, \\ a'x + b'y + c'z + d'r + e't + f' &= 0, \\ a''x + b''y + c''z + d''r + e''t + f'' &= 0, \\ a'''x + b'''y + c'''z + d'''r + e'''t + f''' &= 0, \\ a^{iv}x + b^{iv}y + c^{iv}z + d^{iv}r + e^{iv}t + f^{iv} &= 0. \end{aligned}$$

Je calculerois, par exemple, le coëfficient de x dans la dernière ligne, en calculant $yzr \,.\, tu$, ou $yz \,.\, rtu$, ou $yz \,.\, rt \,.\, u$.

Si j'avois six équations dont les inconnues fussent x, y, z, r, s et t, je calculerois, par exemple, le coëfficient de x, en calculant ou $yz \,.\, rs \,.\, tu$, ou $yzrs \,.\, tu$, ou $yzr \,.\, stu$, et ainsi de suite.

The next paragraph deals with an illustrative example. The twelve equations—

$$\left.\begin{array}{llll}
\mathrm{A}a+\mathrm{A}'a'+\mathrm{A}''a'' & & & =0\\
\mathrm{A}b+\mathrm{A}'b'+\mathrm{A}''b'' & & & =0\\
\mathrm{A}c+\mathrm{A}'c'+\mathrm{A}''c'' & +\mathrm{B}a+\mathrm{B}'a'+\mathrm{B}''a'' & & =0\\
 & +\mathrm{B}b+\mathrm{B}'b'+\mathrm{B}''b'' & & =0\\
 & +\mathrm{B}c+\mathrm{B}'c'+\mathrm{B}''c'' & & =0\\
 & +\mathrm{B}d+\mathrm{B}'d'+\mathrm{B}''d'' & +\mathrm{C}a+\mathrm{C}'a'+\mathrm{C}''a'' & =0\\
 & & +\mathrm{C}b+\mathrm{C}'b'+\mathrm{C}''b'' & =0\\
 & & +\mathrm{C}c+\mathrm{C}'c'+\mathrm{C}''c'' & =0\\
 & & +\mathrm{C}d+\mathrm{C}'d'+\mathrm{C}''d''+\mathrm{D}a+\mathrm{D}'a'+\mathrm{D}''a'' & =0\\
 & & \qquad +\mathrm{D}b+\mathrm{D}'b'+\mathrm{D}''b'' & =0\\
 & & \qquad +\mathrm{D}c+\mathrm{D}'c'+\mathrm{D}''c'' & =0\\
\mathrm{A}d+\mathrm{A}'d'+\mathrm{A}''d'' & & \qquad +\mathrm{D}a+\mathrm{D}'a'+\mathrm{D}''a'' & =0
\end{array}\right\}$$

are given, and what is required is the result of the elimination (*équation de condition*) of the twelve quantities—A, A′, A″, B, B′, B″, C, C′, C″, D, D′, D″. This is found (the a's in the last equation being misprints for d's) to be—

$$(ab'c'').[(bc'd'')^3-(ab'c'')^2(ab'd'')]=0.$$

The two paragraphs quoted (§§ 264, 265) show that Bézout could obtain with considerably increased ease and certitude any one of Laplace's expansions of numerator and denominator. What it accomplished in the illustrative example is virtually, in modern symbolism, the reduction of

$$\left|\begin{array}{cccccccccccc}
a & a' & a'' & . & . & . & . & . & . & . & . & .\\
b & b' & b'' & . & . & . & . & . & . & . & . & .\\
c & c' & c'' & a & a' & a'' & . & . & . & . & . & .\\
. & . & . & b & b' & b'' & . & . & . & . & . & .\\
. & . & . & c & c' & c'' & . & . & . & . & . & .\\
. & . & . & d & d' & d'' & a & a' & a'' & . & . & .\\
. & . & . & . & . & . & b & b' & b'' & . & . & .\\
. & . & . & . & . & . & c & c' & c'' & . & . & .\\
. & . & . & . & . & . & d & d' & d'' & a & a' & a''\\
. & . & . & . & . & . & . & . & . & b & b' & b''\\
. & . & . & . & . & . & . & . & . & c & c' & c''\\
d & d' & d'' & . & . & . & . & . & . & d & d' & d''
\end{array}\right|$$

to the form $\quad |ab'c''|.|bc'd''|^3 - |ab'c''|^3.|ab'd''|.$

Although this can be done nowadays with ease by means of Laplace's expansion-theorem in its modern garb, it may be safely affirmed that Laplace himself, using his own process, would not have succeeded in making the reduction. Considerable importance thus attaches from more than one point of view to Bézout's curious "rule."

The only other section with which we are concerned bears the heading

Méthode pour trouver des fonctions d'un nombre quelconque de quantités, qui soient zéro par elles-mêmes.

In the second paragraph of the section the principle is explained as follows:—

"(216) Concevons un nombre n d'équations du premier degré renfermant un nombre $n+1$ d'inconnues, et sans aucun terme absolument connu.

"Imaginons que l'on augmente le nombre de ces équations de l'une d'entr'elles; alors il est clair que ce que nous appellons la dernière ligne sera non seulement l'équation de condition nécessaire pour que ce nombre $n+1$ d'équations ait lieu; mais encore que cette équation de condition aura lieu; en sorte qu'elle sera une fonction des coëfficiens de ces équations, laquelle sera zéro par elle-même.

"Voilà donc un moyen très-simple pour trouver un nombre $n+1$* de fonctions d'un nombre $n+1$ de quantités, lesquelles fonctions soient zéro par elles-mêmes."

For example, the pair of equations

$$\left.\begin{aligned} ax+by+cz&=0\\ a'x+b'y+c'z&=0 \end{aligned}\right\}$$

is taken, the first equation is repeated, and for this set of three equations the *équation de condition* is found to be

$$(ab'-a'b)c - (ac'-a'c)b + (bc'-b'c)a = 0.$$

"Or il est clair que la troisième équation n'exprimant rien de différent de la première, cette dernière quantité doit être zero par elle-même: donc si on a ces deux suites de quantités

$$\begin{matrix} a, & b, & c \\ a', & b', & c' \end{matrix}$$

on peut être assuré qu'on aura toujours

$$(ab'-a'b)c - (ac'-a'c)b + (bc'-b'c)a = 0.$$

* Should be n.

"Et si au lieu de joindre la première équation, c'eût été la seconde, nous aurions trouvé de même

$$(ab' - a'b)c' - (ac' - a'c)b' + (bc' - b'c)a' = 0."$$

Similarly in regard to the quantities

$$\begin{matrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{matrix}$$

the identity

$$\begin{aligned} &[(ab'-a'b)c'' -(ac' -a'c)b'' + (bc' -b'c) a'']d \\ -&[(ab'-a'b)d''-(ad'-a'd)b''+(bd'-b'd)a'']c \\ +&[(ac'-a'c) d''-(ad'-a'd)c'' +(cd' -c'd)a'']b \\ -&[(bc' -b'c)d'' -(bd' -b'd)c'' +(cd' -c'd)b'']a = 0 \end{aligned}$$

and two others are established, the general theorem of course being merely referred to as easily obtainable.

Thus far there is in substance nothing new. What we have obtained is simply a different aspect of Vandermonde's theorem, that *when two indices of either set are alike the function vanishes*, or, as we should now say, *a determinant with two rows identical is equal to zero.* Indeed the identities are used by Vandermonde in Bézout's form when solving a set of simultaneous equations. But what follows is important.

By taking two of these identities

$$(ab'-a'b)c -(ac'-a'c)b +(bc'-b'c)a =0$$
$$(ab'-a'b) c'-(ac'-a'c)b'+(bc'-b'c)a'=0,$$

multiplying both sides of the first by d', both sides of the second by d, and subtracting, there is obtained in regard to the quantities

$$\begin{matrix} a, & b, & c, & d \\ a', & b', & c', & d' \end{matrix}$$

the identity

$$(ab'-a'b)(cd'-c'd) - (ac'-a'c)(bd'-b'd) + (bc'-b'c)(ad'-a'd) = 0$$

Similarly by taking the three next identities before obtained, which for shortness we may write in modern notation,

$$\begin{aligned} |ab'c''|d - |ab'd''|c + |ac'd''|b - |bc'd''|a &= 0, \\ |ab'c''|d' - |ab'd''|c' + |ac'd''|b' - |bc'd''|a' &= 0, \\ |ab'c''|d'' - |ab'd''|c'' + |ac'd''|b'' - |bc'd''|a'' &= 0, \end{aligned}$$

there is deduced in regard to the quantities

$$\begin{matrix} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \end{matrix}$$

the identities

$$|ab'c''|.|de'| - |ab'd''|.|ce'| + |ac'd''|.|be'| - |bc'd''|.|ae'| = 0,$$
$$|ab'c''|.|de''| - |ab'd''|.|ce''| + |ac'd''|.|be''| - |bc'd''|.|ae''| = 0,$$
$$|ab'c''|.|d'e''| - |ab'd''|.|c'e''| + |ac'd''|.|b'e''| - |bc'd''|.|a'e''| = 0.$$

Finally these last three identities are taken, both sides of the first multiplied by f'', both sides of the second by $-f'$, both sides of the third by f, and then by addition there is obtained in regard to the quantities

$$\begin{matrix} a, & b, & c, & d, & e, & f \\ a', & b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'', & f'' \end{matrix}$$

the identity

$$|ab'c''|.|de'f''| - |ab'd''|.|ce'f''| + |ac'd''|.|be'f''| - |bc'd''|.|ae'f''| = 0.$$

The subject of what may appropriately be called *vanishing aggregates of determinant-products* is not pursued farther, the concluding paragraph being

"(223) En voilà assez pour faire connoître la route qu'on doit tenir, pour trouver ces sortes de théorèmes. On voit qu'il y a une infinité d'autres combinaisons à faire, et qui donneront chacune de nouvelles fonctions, qui seront zéro par elles-mêmes : mais cela est facile à trouver actuellement."*

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*It is very curious to observe, in passing, that although Bézout does not obtain all his vanishing aggregates directly by means of the principle which he so carefully states at the commencement, nevertheless every one of them can be so obtained. He does not extend the principle beyond the case where only *one* of the original equations is repeated. If, however, we take the equations

$$ax+by+cz+dw=0,$$
$$a'x+b'y+c'z+d'w=0,$$

repeat *both* of them so as to have a set of four, and then proceed by the *méthode pour abréger* to find the *équation de condition*, we obtain

$$|ab'|.|cd'| - |ac'|.|bd'| + |ad'|.|bc'| + |bc'|.|ad'| - |bd'|.|ac'| + |cd'|.|ab'| = 0,$$
$$\text{i.e.}\quad 2\{|ab'|.|cd'| - |ac'|.|bd'| + |ad'|.|bc'|\} = 0.$$

This is the identity near the foot of p. 51, and others are readily seen to be obtainable in the same way.

Our second list of Bézout's contributions thus is :—

(1) An unexplained artificial process for finding the numerators and denominators of fractions which express the values of the unknowns in a set of linear equations, or for finding the resultant of the elimination of n quantities from $n+1$ linear equations,—a process especially useful when the coefficients have particular values. (II. 3 + III. 4 + IV. 2)

(2) An improved mode of finding Laplace's expansions, especially (but not exclusively) useful when the coefficients have particular values. (XIV. 3)

(3) A proof of Vandermonde's theorem regarding the effect of the equality of two indices belonging to the same set. (XII. 3)

(4) A series of identities regarding vanishing aggregates of products. (XXIII. 2)

CHAPTER III.

DETERMINANTS IN GENERAL, FROM THE YEAR 1784 TO 1812.

THE writers of this period are eight in number, viz., Hindenburg, Rothe, Gauss, Monge, Hirsch, Binet, Prasse, Wronski. Of these the first two and Prasse, belonging as they did to the so-called Combinatorial School, were not independent of one another; Hirsch was a mere expositor; and the others were authors who had not specially studied the subject, but who had attained results in it in the course of other investigations.

HINDENBURG, C. F. (1784).

[*Specimen analyticum de lineis curvis secundi ordinis, in delucidationem Analyseos Finitorum Kaestnerianæ. Auctore Christiano Friderico Rüdigero. Cum praefatione Caroli Friderici Hindenburgii, professoris Lipsiensis.* (pp. xiv–xlviii.) xlviii + 74 pp. *Lipsiæ.*]

One of the problems dealt with by Rüdiger being the finding of the equation of the conic passing through five given points ("*coefficientium determinatio Traiectoriae secundi ordinis per data quinque puncta*"), Hindenburg, in his preface, takes occasion to show how the generalised problem for $\frac{1}{2}n(n+3)$ points has been treated, pointing out that it is, of course, immediately dependent on the solution of a set of simultaneous linear equations. He directs attention to the labours of Cramer and Bezout, specially lauding the method of the latter given in the treatise of 1779. Then he says—"*Haec de Opere Bezoldino in universam, quod plurimis adhuc Lectoribus nostris ignotum* *erit, dicta sufficiant. Nunc Regulam ipsam proponam."* . . . . The seventeen pages which follow, contain a tolerably close Latin translation of the *Règle générale pour calculer* . . . ., and the *Méthode pour trouver* . . . ., pp. 172–187, §§ 198–223, which have been expounded above. Cramer's rule is next given, the second mode of putting it being in words, and the first as follows:—

"Sint plures Incognitæ z, y, x, w, &c. totidemque Aequationes simplices indeterminatæ

$$\begin{array}{l} A^1 = Z^1z + Y^1y + X^1x + W^1w + \&c. \\ A^2 = Z^2z + Y^2y + X^2x + W^2w + \&c. \\ A^3 = Z^3z + Y^3y + X^3x + W^3w + \&c. \\ A^4 = Z^4z + Y^4y + X^4x + W^4w + \&c. \\ \&c. \quad \&c. \quad \&c. \quad \&c. \quad \&c. \quad \&c. \end{array}$$

Erit, ..., positis terminorum signis, ut praecipitur in fine Tabulæ, pag. seq.

$$z = \frac{\begin{array}{c}\mathrm{A\ Y\ X\ W\ V\ U\ T} \ldots \\ \text{Permut } (1, 2, 3, 4, 5, 6, 7, \ldots .)\end{array}}{\begin{array}{c}\text{Permut } (1, 2, 3, 4, 5, 6, 7, \ldots .) \\ \mathrm{Z\ Y\ X\ W\ V\ U\ T} \ldots\end{array}} \qquad (\text{VII. } 3)$$"

The similar expressions for y, x, w, v, u, t are given, and then the "*regula signorum.*" After an illustrative example, the question of the *sequence* of the signs is taken up.

"Quod si itaque $+sg(1, 2, 3, \ldots, n)$ denotet signorum vicissitudines, quibus hic afficiuntur Permutationum a numeris $1, 2, 3, \ldots n$ singulæ species, et $-sg(1, 2, 3, \ldots n)$ signa *contraria* vel *opposita*: appatet fore

$$\begin{array}{ll} sg(1, 2) & = +sg(1) \quad -sg(1) \\ sg(1, 2, 3) & = +sg(1, 2) \quad -sg(1, 2) \quad +sg(1, 2) \\ sg(1, 2, 3, 4) & = +sg(1, 2, 3) - sg(1, 2, 3) + sg(1, 2, 3) - sg(1, 2, 3) \\ \ldots & \ldots \end{array}$$

$$\begin{array}{llll} \text{unde, quia} & sg(1) & \text{est} & +\text{, facile eruitur} \\ & sg(1, 2) & \text{esse} & + - \\ & sg(1, 2, 3) & \ldots\ldots & + - - + + - \\ & sg(1, 2, 3, 4) & \ldots\ldots & + - - + + - - + + - - + \\ & & & + - - + + - - + + - - + \\ & \ldots & \ldots & \ldots \end{array}$$"

and it is pointed out that the first sign is always $+$, and the last $+$ or $-$ according as the number $1+2+3+ \ldots + (n-1)$ is even or odd.

Bearing in mind that Hindenburg wrote his permutations in a definite order, this remark regarding the sequence of signs entitles us to view him as the author of a combined rule of term-formation and rule of signs, which may be formulated as follows:—

Write the permutations of 1, 2, 3, . . ., n *in ascending order of magnitude as if they were numbers; make the first sign* +, *the second* −, *the next pair contrary in sign to the first pair, the third pair contrary in sign to the second pair, the next six* (1 . 2 . 3) *contrary in sign to the first six, the third six contrary in sign to the second six, the fourth six contrary in sign to the third six, the next twenty-four* (1 . 2 . 3 . 4) *contrary in sign to the first twenty-four, and so on.* (II. 4 + III. 5)

ROTHE, H. A. (1800).

[Ueber Permutationen, in Beziehung auf die Stellen ihrer Elemente. Anwendung der daraus abgeleiteten Sätze auf das Eliminationsproblem. *Sammlung combinatorisch-analytischer Abhandlungen, herausg. v. C. F. Hindenburg,* ii. pp. 263–305.]

Rothe was a follower of Hindenburg, knew Hindenburg's preface to Rüdiger's Specimen Analyticum, and was familiar with what had been done by Cramer and Bezout (see his words at p. 305). His memoir is very explicit and formal, proposition following definition, and corollary following proposition, in the most methodical manner.

The idea which is made the basis of it, that of *place-index* ("Stellenexponent"), is an ill-advised and purposeless modification of Cramer's idea of a "dérangement." The definition is as follows:—In any permutation of the first n integers, the *place-index* of any integer is got *by counting the integer itself and all the elements after it which are less than it.* For example, in the permutation

6, 4, 3, 9, 8, 10, 1, 7, 2, 5

of the first ten integers, the place-index of 9 is 6, and that of 7 is 3. The counting of the integer itself makes the place-index always *one more* than the number of "dérangements" connected with the integer. This necessitates the introduction of a corresponding modification of Cramer's "rule of signs," viz.

"3. Willkührlicher Satz. Jede Permutation der Elemente 1, 2, 3, . . . , r, werde mit dem Zeichen + versehen, wenn entweder gar keine, oder eine gerade Menge gerader Zahlen, unter ihren Stellenexponenten vorkommt; mit dem Zeichen − hingegen, wenn die Menge der geraden Zahlen, unter den Stellenexponenten ungerade ist." (III. 6)

It is difficult to suggest any justification for the changes here introduced. The author himself refers to none. Indeed, in the very next paragraph he points out that to ascertain whether there be an even number of even integers among the place-indices is the same as to diminish each of the place-indices by 1, and ascertain whether there be an even number of odd integers, that is, whether the *sum* of the odd integers be even. He then concludes—

"Man kann also auch die Regel so ausdrücken: Jede Permutation bekommt das Zeichen + wenn die Summe der um 1 verminderten Stellenexponenten gerade, − hingegen, wenn sie ungerade ist."

This is simply Cramer's rule, and it is the only rule of signs employed henceforward in the memoir, the expression "die Summe der um 1 verminderten Stellenexponenten," occurring over and over again as a periphrasis for "the number of derangements."

The next four pages are occupied with a very lengthy but thorough investigation of the theorem that *two permutations differ in sign if they be so related that either is got from the other by the interchange of two of the elements of the latter.* Strictly speaking, however, the proposition proved is something more definite than this, viz.—

If in a permutation of the integers 1.2, . . . r *there be* d *integers intermediate in place and value between any two,* A *and* B, *of the integers, the interchanging of the said two would increase or diminish the number of inversions of order by* 2d+1. (III. 7)

The proof consists in finding the sum of the place-indices for the given permutation in terms of d as just defined, c the number of elements less than both A and B and situated between them, f the number of such elements situated to the right of B, and

e the number of elements between A and B in value and situated to the right of B; then finding in like manner the sum of the place-indices for the new permutation; and finally comparing the two sums. The concluding sentence is as follows:—

"Denn da . . . . , so ist die Summe der Stellenexponenten der zweyten Permutation um $d+e+1-e+d$ oder um $2d+1$ grösser als bey der ersten Permutation; folglich gilt das auch bey der Summe der um 1 verminderten Stellenexponenten, da bey beyden Permutationen r einerley ist. Also ist die eine Summe gerade, die andere ungerade, folglich haben nach (4) beyde Permutationen verschiedene Zeichen."

As immediate deductions from this, it is pointed out that

The sign of any one permutation may be determined when the sign of any other is known, by counting the number of interchanges necessary to transform the one permutation into the other; (III. 8)

and that

If one element of a permutation be made to take up a new place, by being, as it were, passed over m *other elements, the sign of the new permutation is the same as, or different from, that of the original according as* m *is even or odd.* (III. 9)

A third corollary is given, but it is, strictly speaking, a self-evident corollary to the second corollary, and is quite unimportant.

Rothe's next theorem is—

The permutations of 1, 2, 3, ... , n *being arranged after the manner in which numbers are arranged in ascending order of magnitude, any two consecutive permutations will have the same sign, if the first place in which they differ be the* $(4n+3)^{th}$ *or* $(4n+4)^{th}$ *from the end, and will be of opposite sign if the said place be the* $(4n+1)^{th}$ *or* $(4n+2)^{th}$ *from the end.* (III. 10)

Thus if the permutations of 1, 2, 3, ... , 10 be taken, and arranged as specified, two which will occur consecutively are

8, 4, 9, 3, 10, 7, 6, 5, 2, 1

8, 4, 9, 5, 1, 2, 3, 6, 7, 10;

and as the first place in which these differ is the 7^{th} from the end, it is affirmed that the signs preceding them must be alike. The mode of proving the theorem will be readily understood by seeing it applied to this illustrative example. Taking the permutation

$$8, 4, 9, 3, 10, 7, 6, 5, 2, 1,$$

and interchanging 3 and 5 we have the permutation

$$8, 4, 9, 5, 10, 7, 6, 3, 2, 1,$$

and thence by cyclical changes the permutation

$$8, 4, 9, 5, 1, 2, 3, 6, 7, 10,$$

the number of alterations of sign thus being

$$1+(5+4+3+2+1)$$
$$\textit{i.e.}\quad 1+\tfrac{1}{2}(5\times 6),$$

—an even number.

Annexed to the theorem is the following corollary, which is not essentially different from Hindenburg's proposition regarding the sequence of signs,—

If the permutations of 1, 2, 3, . . ., $n-1$ *be arranged after the manner in which numbers are arranged in ascending order of magnitude, and also in like manner the permutations of* 1, 2, 3, ..., $n-1$, n, *then those permutations of the latter arranged set which begin with* r, *say, have in order the same signs as the permutations of the former arranged set, or different signs, according as* r *is odd or even.* (III. 11)

For example, arranging the permutations of 1, 2, 3, each with its proper sign in front, we have

$$\left.\begin{array}{l} +1, 2, 3 \\ -1, 3, 2 \\ -2, 1, 3 \\ +2, 3, 1 \\ +3, 1, 2 \\ -3, 2, 1; \end{array}\right. \qquad (\mathrm{A})$$

then arranging those permutations of 1, 2, 3, 4 which begin with 3 say, each with its proper sign, we have

$$\left.\begin{array}{l} +3, 1, 2, 4 \\ -3, 1, 4, 2 \\ -3, 2, 1, 4 \\ +3, 2, 4, 1 \\ +3, 4, 1, 2 \\ -3, 4, 2, 1; \end{array}\right. \qquad (\mathrm{B})$$

and the two series of signs are seen to be identical, 3 being an odd number. Viewing this quite independently of the theorem to which it is annexed, it is evident that a change of sign at any point in the series (A) implies a change at the corresponding point in the other series, and consequently attention need only be paid to the first sign of (B) as compared with the first sign of (A). Now the first sign of (A) must necessarily be always plus, there being no inversions; and the first sign of (B) depends on the changes necessary for the transformation of the natural order 1, 2, 3, 4, into 3, 1, 2, 4. The truth of the corollary is thus apparent.

A second corollary is given, but it is of still less consequence, the difference between it and the first being that in the arranged set (B) the place whose occupant remains unchanged may be any one of the n places. (III. 12)

The next few paragraphs concern the subject of "conjugate permutations" (*verwandte Permutationen*),—apparently a fresh conception. The definition is—

Two permutations of the numbers 1, 2, 3, . . ., n *are called* CONJUGATE *when each number and the number of the place which it occupies in the one permutation are interchanged in the case of the other permutation.* (XXIV.)

For example, the permutations

$$3,\ 8,\ 5,\ 10,\ 9,\ 4,\ 6,\ 1,\ 7,\ 2 \qquad (A)$$
$$8,\ 10,\ 1,\ 6,\ 3,\ 7,\ 9,\ 2,\ 5,\ 4 \qquad (B)$$

are conjugate, because 3 is in the 1st place of (A) and 1 is in the 3rd place of (B), 8 is in the 2nd place of (A), and 2 is in the 8th place of B, and so on in every case.

The first theorem obtained is—

Conjugate permutations have the same sign. (III. 13)

This is proved in a curious and interesting way, a special conjugate pair being considered, viz., the pair just given as an example. To commence with, a square divided into 10×10 equal squares is drawn, the vertical rows of small squares being numbered 1, 2, 3, &c. from left to right, and the horizontal rows 1, 2, 3, &c. from the top downwards. The permutation

$$3,\ 8,\ 5,\ 10,\ 9,\ 4,\ 6,\ 1,\ 7,\ 2$$

is then represented by putting a dot in each of the horizontal rows, in the first under 3, in the second under 8, and so on; so that if the rows be taken in order, and the number above each dot read, the given permutation is obtained. For the representation of the conjugate permutation nothing further is necessary: we obtain it at once if we only turn the paper round clockwise until the vertical rows are horizontal, and read off in order the numbers above the dots. In the next place the number of "dérangements" belonging to the permutation 3, 8, 5, ... is indicated by inserting a cross in every small square which is to the left of one dot and above another; thus the two crosses in the first horizontal row correspond to the two "dérangements" 32, 31; the six crosses in the second horizontal row to the six "dérangements" 85, 84, 86, 81, 87, 82; and so on. Then it is observed that if we turn the paper and try to indicate the "dérangements" of the conjugate permutation by inserting a cross in every small square which is to the right of one dot and above another, we obtain exactly the same crosses as before. The signs of the two permutations must thus be alike.

	1	2	3	4	5	6	7	8	9	10
1	×	×	·							
2	×	×		×	×	×	×	·		
3	×	×		×	·					
4	×	×		×		×	×		×	·
5	×	×		×		×	×		·	
6	×	×		·						
7	×	×				·				
8	·									
9		×					·			
10		·								

Immediately following this, the 24 permutations of 1, 2, 3, 4 are given in a column, each one having opposite it, in a parallel column, its conjugate permutation. The existence of *self-conjugate* permutations, *e.g.*, the permutation 3, 4, 1 2, is thus brought to notice, and the substance of the following theorem in regard to them is given :—

If U_n *be the number of self-conjugate permutations of the first* n *integers, then*

$$U_n = U_{n-1} + (n-1)U_{n-2} \qquad \text{(xxv.)}$$

where $U_1 = 1$ *and* $U_2 = 2$.

This, however, is the only one of his results which Rothe does not attempt to prove.

In the second part of the memoir, which contains the application of the theorems of the first part to the solution of a set of linear equations, there is not so much that is noteworthy. Methods previously known are followed, the new features being formality and rigour of demonstration.

The coefficients of the equations being

$$\begin{array}{l} 11,\ 12,\ 13,\ \ldots\ldots,\ 1r \\ 21,\ 22,\ 23,\ \ldots\ldots,\ 2r \\ \ldots\ldots\ldots\ldots\ldots \\ r1,\ r2,\ r3,\ \ldots\ldots,\ rr \end{array}$$

it is noted, as Vandermonde had remarked, that the common denominator of the values of the unknown may be got in two ways, viz., by permuting either all the second integers of the couples, 11, 22, 33, ... , rr, or all the first integers: but this is supplemented by a proof, that *if any term be taken, e.g.,*

$$16\,.\,24\,.\,33\,.\,47\,.\,51\,.\,68\,.\,79\,.\,82\,.\,95$$

with the couples so arranged that the first integers are in ascending order, and the sign be determined from the number of inversions in the series of second integers, then the sign obtained will be the same as would be got by arranging the couples so as to have the second integers in ascending order, and determining the sign from the inversions in the series of first integers. The proof rests entirely on the previous theorem, that conjugate permutations have the same sign; indeed the new proposition is little else than another form of this theorem. (III. 14)

The desirability of an appropriate notation for the cofactor, which any one of the coefficients has in the common denominator, is recognised,* and the want supplied by prefixing f to the coefficient in question; for example, the cofactor of 32 is denoted by

$$\mathrm{f}32.$$

It is thus at once seen that the denominator itself is equal to

$$1n\,.\,\mathrm{f}1n + 2n\,.\,\mathrm{f}2n + \ldots\ldots + rn\,.\,\mathrm{f}rn,$$

or

$$n1\,.\,\mathrm{f}n1 + n2\,.\,\mathrm{f}n2 + \ldots\ldots + nr\,.\,\mathrm{f}nr. \qquad \text{(VI. 2)}$$

Also by this means one of Bezout's (or Vandermonde's) general theorems becomes easily expressible in symbols, viz.,

$$1n\,.\,\mathrm{f}1m + 2n\,.\,\mathrm{f}2m + \ldots\ldots + rn\,.\,\mathrm{f}rm = 0, \qquad \text{(XII. 4)}$$

---

* Lagrange's use of a corresponding letter from a different alphabet must not be forgotten.

the proof of which is given as follows. In all the terms of f $1m$, every one of the integers except one occurs as the first integer of a couple, and every one of the integers except m occurs as the second integer of a couple: consequently, in every term of $1n$.f$1m$ the first places of the couples are occupied by the integers from 1 to r inclusive, while in the second places m is still the only integer awanting and n occurs twice. Suppose then all the terms of

$$1n\,.\,\mathrm{f}1m + 2n\,.\,\mathrm{f}2m + \ldots\ldots + rn\,.\,\mathrm{f}rm$$

so written that the first integers of the couples are in ascending order of magnitude, and let us attend to a single term

$$\ldots\ldots \cdot pn \cdot \ldots\ldots\ldots \cdot qn \cdot \ldots\ldots\ldots$$

in which the two couples, having n for second integer, are the p^{th} and q^{th}. If we inquire from which of the expressions $1n$.f$1m$, $2n$.f$2m$, ... this term comes, we see that it is a term of both pn.fpm and qn.fqm, and must, therefore, occur twice. Further, we see that in pn.fqm it has the sign of the term

$$\ldots\ldots \cdot pm \cdot \ldots\ldots\ldots \cdot qn \cdot \ldots\ldots\ldots$$

of the common denominator, and that in qn.fpm, it has the sign of the term

$$\ldots\ldots \cdot pn \cdot \ldots\ldots\ldots \cdot qm \cdot \ldots\ldots\ldots$$

of the common denominator. But these two terms of the common denominator have different signs: consequently

$$1n\,.\,\mathrm{f}1m + 2n\,.\,\mathrm{f}2m + \ldots\ldots + rn\,.\,\mathrm{f}rm$$

consists of pairs of equal terms with unlike signs, and thus vanishes identically. (XII. 4)

These preparations having been attended to, the set of r equations with r unknowns is solved by Laplace's method; and a verification made after the manner of Vandermonde. It is also pointed out, that if the solution of a set of equations, say the four

$$\left.\begin{aligned} ax_1 + bx_2 + cx_3 + dx_4 &= s_1 \\ ex_1 + fx_2 + gx_3 + hx_4 &= s_2 \\ ix_1 + kx_2 + lx_3 + mx_4 &= s_3 \\ nx_1 + ox_2 + px_3 + qx_4 &= s_4 \end{aligned}\right\}$$

be
$$\left.\begin{array}{l} x_1 = \mathrm{A}s_1 + \mathrm{B}s_2 + \mathrm{C}s_3 + \mathrm{D}s_4 \\ x_2 = \mathrm{E}s_1 + \mathrm{F}s_2 + \mathrm{G}s_3 + \mathrm{H}s_4 \\ x_3 = \mathrm{I}s_1 + \mathrm{K}s_2 + \mathrm{L}s_3 + \mathrm{M}s_4 \\ x_4 = \mathrm{N}s_1 + \mathrm{O}s_2 + \mathrm{P}s_3 + \mathrm{Q}s_4 \end{array}\right\} ,$$
then the solution of the set
$$\left.\begin{array}{l} ay_1 + ey_2 + iy_3 + ny_4 = v_1 \\ by_1 + fy_2 + ky_3 + oy_4 = v_2 \\ cy_1 + gy_2 + ly_3 + py_4 = v_3 \\ dy_1 + hy_2 + my_3 + qy_4 = v_4 \end{array}\right\} ,$$
which has the same coefficients differently disposed, will be
$$\left.\begin{array}{l} y_1 = \mathrm{A}v_1 + \mathrm{E}v_2 + \mathrm{I}v_3 + \mathrm{N}v_4 \\ y_2 = \mathrm{B}v_1 + \mathrm{F}v_2 + \mathrm{K}v_3 + \mathrm{O}v_4 \\ y_3 = \mathrm{C}v_1 + \mathrm{G}v_2 + \mathrm{L}v_3 + \mathrm{P}v_4 \\ y_4 = \mathrm{D}v_1 + \mathrm{H}v_2 + \mathrm{M}v_3 + \mathrm{Q}v_4 \end{array}\right\} ; \qquad \text{(XXVI.)}$$
and hence, that the solution of a set having the special form
$$\left.\begin{array}{l} ax_1 + bx_2 + cx_3 + dx_4 = s_1 \\ bx_1 + ex_2 + fx_3 + gx_4 = s_2 \\ cx_1 + fx_2 + hx_3 + ix_4 = s_3 \\ dx_1 + gx_2 + ix_3 + jx_4 = s_4 \end{array}\right\}$$
will itself take the form, viz.
$$\left.\begin{array}{l} \mathrm{A}s_1 + \mathrm{B}s_2 + \mathrm{C}s_3 + \mathrm{D}s_4 = x_1 \\ \mathrm{B}s_1 + \mathrm{E}s_2 + \mathrm{F}s_3 + \mathrm{G}s_4 = x_2 \\ \mathrm{C}s_1 + \mathrm{F}s_2 + \mathrm{H}s_3 + \mathrm{I}s_4 = x_3 \\ \mathrm{D}s_1 + \mathrm{G}s_2 + \mathrm{I}s_3 + \mathrm{J}s_4 = x_4 \end{array}\right\} . \qquad \text{(XXVI. 2)}$$

GAUSS (1801).

[*Disquisitiones Arithmeticæ. Auctore D. Carolo Friderico Gauss.* 167 pp. Lips. *Werke*, I. (1863) Göttingen.]

The connection of Gauss with our theory was very similar to that of Lagrange, and doubtless was due to the fact that Lagrange had preceded him. The fifth chapter of his famous work, which is the only chapter we are concerned with, bears the title "*De formis æquationibusque indeterminatis secundi gradus*," and its subject may be described in exactly the same words as Lagrange used in regard to his memoir *Recherches* *d'Arithmétique* (1773: see above), viz. "les nombres qui peuvent être représentés par la formule $Bt^2+Ctu+Du^2$."

Gauss writes his form of the second degree thus—

$$axx+2bxy+cyy;$$

and for shortness speaks of it as the form (a, b, c). The function of the coefficients a, b, c, which was found by Lagrange to be of notable importance in the discussion of the form, Gauss calls the "*determinant* of the form," the exact words of his definition being

> "Numerum $bb-ac$, a cuius indole proprietates formæ (a, b, c) imprimis pendere in sequentibus docebimus, *determinantem* huius formæ uocabimus." (XV. 2)

Here then we have the first use of the term which with an extended signification has in our day come to be so familiar. It must be carefully noted that the more general functions, to which the name came afterwards to be given, also repeatedly occur in the course of Gauss' work, *e.g.*, the function $\alpha\delta-\beta\gamma$ in his statement of Lagrange's theorem (XXII.)

$$b'b'-a'c'=(bb-ac)(\alpha\delta-\beta\gamma)^2.$$

But such functions are not spoken of as belonging to the same category as $bb-ac$. In fact the new term introduced by Gauss was not "determinant" but "determinant of a form," being thus perfectly identical in meaning and usage with the modern term "discriminant."

Notwithstanding the title of the chapter Gauss did not confine himself to forms of two variables. A digression is made for the purpose of considering the ternary quadratic form ("formam ternariam secundi gradus"),

$$axx+a'x'x'+a''x''x''+2bx'x''+2b'xx''+2b''xx',$$

or as he shortly denotes it

$$\begin{pmatrix} a & a', & a'' \\ b, & b', & b'' \end{pmatrix}.$$

In the matter of nomenclature the following paragraph of this digression is interesting,—

"Ponendo $bb-a'a''=\mathrm{A}$, $b'b'-aa''=\mathrm{A}'$, $b''b''-aa'=\mathrm{A}''$,
$ab-b'b''=\mathrm{B}$, $a'b'-bb''=\mathrm{B}'$, $a''b''-bb'=\mathrm{B}''$,

oritur alia forma

$$\begin{pmatrix} \mathrm{A} & \mathrm{A}' & \mathrm{A}'' \\ \mathrm{B} & \mathrm{B}' & \mathrm{B}'' \end{pmatrix} \quad \ldots . \mathrm{F}$$

quam formæ

$$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix} \quad \ldots . f$$

adjunctam dicemus. Hinc rursus inuenitur, (XXVII.)
denotando breuitatis caussa numerum

$$abb + a'b'b' + a''b''b'' - aa'a'' - 2bb'b'' \text{ per D},$$

$$\mathrm{BB} - \mathrm{A'A''} = a\mathrm{D}, \quad \mathrm{B'B'} - \mathrm{AA''} = a'\mathrm{D}, \quad \mathrm{B''B''} - \mathrm{AA'} = a''\mathrm{D},$$
$$\mathrm{AB} - \mathrm{B'B''} = b\mathrm{D}, \quad \mathrm{A'B'} - \mathrm{BB''} = b'\mathrm{D}, \quad \mathrm{A''B''} - \mathrm{BB'} = b''\mathrm{D},$$

unde patet, formæ F adjunctam esse formam

$$\begin{pmatrix} a\mathrm{D}, & a'\mathrm{D}, & a''\mathrm{D} \\ b\mathrm{D}, & b'\mathrm{D}, & b''\mathrm{D} \end{pmatrix}.$$

Numerum D, a cuius indole proprietates formæ ternariæ f imprimis pendent, *determinantem* huius formæ uocabimus; (XV. 2)
hoc modo determinans formæ F sit $=$ DD, sive æqualis quadrato determinantis formæ f, cui adjuncta est."

In this there is no advance so far as the theory of modern determinants is concerned, the identities given being those numbered (xx) and (xxi) under Lagrange. On the same page, however, an extension is given of Lagrange's theorem (xxii), regarding the determinant of the new form obtained by effecting a linear substitution on a given form. Gauss' words in regard to this are—

"Si forma aliqua ternaria f determinantis D, cuius indeterminatæ sunt x, x', x'' (puta prima $= x$, &c.) in formam ternariam g determinantis E, cuius indeterminatæ sunt y, y', y'', transmutatur per substitutionem talem

$$\begin{aligned} x &= \alpha y + \beta y' + \gamma y'', \\ x' &= \alpha' y + \beta' y' + \gamma' y'', \\ x'' &= \alpha'' y + \beta'' y' + \gamma'' y'', \end{aligned}$$

ubi nouem coefficientes α, β, &c. omnes supponuntur esse numeri integri, breuitatis caussa neglectis indeterminatis simpliciter dicemus, f transire in g per substitutionem (S)

$$\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix}$$

atque f *implicare* ipsam g, siue *sub f contentam* esse. Ex tali itaque suppositione sponte sequuntur sex equationes pro sex coëfficientibus in g, quas apponere non erit necessarium: hinc autem per calculum facilem sequentes conclusiones euoluuntur:

"I. Designato breuitatis caussa numero

$$\alpha\beta'\gamma''+\beta\gamma'\alpha''+\gamma\alpha'\beta''-\gamma\beta'\alpha''-\alpha\gamma'\beta''-\beta\alpha'\gamma''$$

per k inuenitur post debitas reductiones

$$E=kkD, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{XXII. } 2)$$

. . . ."

When freed from its connection with ternary quadratic forms the theorem in determinants here involved is

$$\begin{aligned}
\text{If}\quad A_0&=a_0\alpha_0^2+a_1\alpha_1^2+a_2\alpha_2^2+2b_0\alpha_1\alpha_2+2b_1\alpha_0\alpha_2+2b_2\alpha_0\alpha_1,\\
A_1&=a_0\beta_0^2+a_1\beta_1^2+a_2\beta_2^2+2b_0\beta_1\beta_2+2b_1\beta_0\beta_2+2b_2\beta_0\beta_1,\\
A_2&=a_0\gamma_0^2+a_1\gamma_1^2+a_2\gamma_2^2+2b_0\gamma_1\gamma_2+2b_1\gamma_0\gamma_2+2b_2\gamma_0\gamma_1,\\
B_0&=a_0\beta_0\gamma_0+a_1\beta_1\gamma_1+a_2\beta_2\gamma_2+b_0(\beta_1\gamma_2+\beta_2\gamma_1)+b_1(\beta_0\gamma_2+\beta_2\gamma_0)+b_2(\beta_0\gamma_1+\beta_1\gamma_0),\\
B_1&=a_0\gamma_0\alpha_0+a_1\gamma_1\alpha_1+a_2\gamma_2\alpha_2+b_0(\gamma_1\alpha_2+\gamma_2\alpha_1)+b_1(\gamma_0\alpha_2+\gamma_2\alpha_0)+b_2(\gamma_0\alpha_1+\gamma_1\alpha_0),\\
B_2&=a_0\alpha_0\beta_0+a_1\alpha_1\beta_1+a_2\alpha_2\beta_2+b_0(\alpha_1\beta_2+\alpha_2\beta_1)+b_1(\alpha_0\beta_2+\alpha_2\beta_0)+b_2(\alpha_0\beta_1+\alpha_1\beta_0),
\end{aligned}$$

then

$$\begin{aligned}
&A_0B_0^2+A_1B_1^2+A_2B_2^2-A_0A_1A_2-2B_0B_1B_2\\
&=(a_0b_0^2+a_1b_1^2+a_2b_2^2-a_0a_1a_2-2b_0b_1b_2)\\
&\qquad\times(\alpha_0\beta_1\gamma_2+\beta_0\gamma_1\alpha_2+\gamma_0\alpha_1\beta_2-\gamma_0\beta_1\alpha_2-\alpha_0\gamma_1\beta_2-\beta_0\alpha_1\gamma_2)^2.
\end{aligned}$$

As thus viewed it is an instance of the multiplication-theorem, the product of three determinants (in the modern sense) being expressed as a single determinant.

The multiplication-theorem is also not very distantly connected with the following other statement of Gauss:—

"Si forma ternaria f formam ternariam f' implicat atque haec formam f'': implicabit etiam f ipsam f''. Facillime enim perspicietur, si transeat

f in f' per substitutionem	f' in f'' per substitutionem
$\alpha,\ \beta,\ \gamma$	$\delta,\ \epsilon,\ \zeta$
$\alpha',\ \beta',\ \gamma'$	$\delta',\ \epsilon',\ \zeta'$
$\alpha'',\ \beta'',\ \gamma''$	$\delta'',\ \epsilon'',\ \zeta''$

f transmutatum iri per substitutionem

$$\begin{array}{lll}
\alpha\delta+\beta\delta'+\gamma\delta'', & \alpha\epsilon+\beta\epsilon'+\gamma\epsilon'', & \alpha\zeta+\beta\zeta'+\gamma\zeta''\\
\alpha'\delta+\beta'\delta'+\gamma'\delta'', & \alpha'\epsilon+\beta'\epsilon'+\gamma'\epsilon'', & \alpha'\zeta+\beta'\zeta'+\gamma'\zeta''\\
\alpha''\delta+\beta''\delta'+\gamma''\delta'', & \alpha''\epsilon+\beta''\epsilon'+\gamma''\epsilon'', & \alpha''\zeta+\beta''\zeta'+\gamma''\zeta''.\text{"}
\end{array} \qquad (\text{XVII. } 3)$$

MONGE (1809).

[Essai d'application de l'analyse a quelques parties de la géométrie élémentaire. *Journ. de l'Éc. Polyt.*, viii. pp. 107–109.]

Lagrange, as we have already seen, was led to certain identities regarding the expression

$$xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x''$$

in the course of investigations on the subject of triangular pyramids. The position of Monge is that of Lagrange reversed. From the theory of equations he derives identities connecting such expressions, and translates them into geometrical theorems.

The simpler of these identities, as being already chronicled, we pass over. At p. 107 he takes the three equations

$$a_1u+b_1x+c_1y+d_1z+e_1=0$$
$$a_2u+b_2x+c_2y+d_2z+e_2=0$$
$$a_3u+b_3x+c_3y+d_3z+e_3=0,$$

and eliminating every pair of the letters u, x, y, z, obtains the six equations

$$\beta u+\ \alpha x+\mathrm{P}=0 \qquad (1)$$
$$\gamma x+\beta y+\mathrm{Q}=0 \qquad (2)$$
$$\delta y+\ \gamma z+\mathrm{M}=0 \qquad (3)$$
$$\alpha z+\ \delta u+\mathrm{N}=0 \qquad (4)$$
$$\gamma u-\ \alpha y+\mathrm{S}=0 \qquad (5)$$
$$\beta z-\ \delta x+\mathrm{R}=0 \qquad (6);$$

the ten letters

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \mathrm{M},\ \mathrm{N},\ \mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S}$$

being used to stand for the lengthy expressions which we nowadays denote by

$$|b_1c_2d_3|,\ |a_1c_2d_3|,\ |a_1b_2d_3|,\ |a_1b_2c_3|,$$
$$|a_1b_2e_3|,\ |b_1c_2e_3\ |,\ |c_1d_2e_3|,\ -|a_1d_2e_3|,\ |a_1c_2e_3|,\ |b_1d_2e_3|.$$

Then, taking triads of these six equations, *e.g.*, the triads (1), (2), (5) he derives the identities

$$\left.\begin{array}{r} \alpha Q+\beta S-\gamma P=0 \\ \delta P+\alpha R-\beta N=0 \\ -\gamma N+\delta S+\alpha M=0 \\ -\beta M+\gamma R+\delta Q=0 \end{array}\right\},$$

or

$$\left.\begin{array}{r} -|b_1c_2d_3|.|a_1d_2e_3|+|a_1c_2d_3|.|b_1d_2e_3|-|a_1b_2d_3|.|c_1d_2e_3|=0 \\ |a_1b_2c_3|.|c_1d_2e_3|+|b_1c_2d_3|.|a_1c_2e_3|-|a_1c_2d_3|.|b_1c_2e_3|=0 \\ -|a_1b_2d_3|.|b_1c_2e_3|+|a_1b_2c_3|.|b_1d_2e_3|+|b_1c_2d_3|.|a_1b_2e_3|=0 \\ -|a_1c_2d_3|.|a_1b_2e_3|+|a_1b_2d_3|.|a_1c_2e_3|-|a_1b_2c_3|.|a_1d_2e_3|=0 \end{array}\right\} \text{(XXIII. 3)}$$

which in their turn, he says, by processes of elimination, may be the source of many others. For example, each of the four being linear and homogeneous in α, β, γ, δ, these letters may all be eliminated with the result

$$RS+QN-PM=0,$$

or

$$|a_1c_2e_3|.|b_1d_2e_3|-|a_1d_2e_3|.|b_1c_2e_3|-|c_1d_2e_3|.|a_1b_2e_3|=0.$$

Also, eliminating P from the first and second, S from the first and third, Q from the first and fourth, and so on, we have

$$\begin{array}{r} -\beta\gamma N+\delta\alpha Q+\beta\delta S+\alpha\gamma R=0, \\ \alpha\beta M+\gamma\delta P-\beta\gamma N-\delta\alpha Q=0, \\ \alpha\beta M-\gamma\delta P+\beta\delta S-\alpha\gamma R=0, \\ \&c. \qquad \&c. \end{array}$$

i.e.

$$\left.\begin{array}{l} -|a_1c_2d_3|.|a_1b_2d_3|.|b_1c_2e_3|-|a_1b_2c_3|.|b_1c_2d_3|.|a_1d_2e_3| \\ +|a_1c_2d_3|.|a_1b_2c_3|.|b_1d_2e_3|+|b_1c_2d_3|.|a_1b_2d_3|.|a_1c_2e_3| \end{array}\right\}=0,$$

$$\&c. \qquad \&c. \qquad \text{(XXVIII.)}$$

Monge does not pursue the subject further. His method, however, is seen to be quite general; and we can readily believe that he possessed numerous other identities of the same kind. This is borne out by a statement in Binet's important memoir of 1812. Binet, who was familiar with what had been done by Vandermonde, Laplace, and Gauss, says (p. 286):—"M. Monge m'a communiqué, depuis la lecture de ce mémoire, d'autres théorèmes très-remarquables sur ces résultantes; mais ils ne sont pas du genre de ceux que nous nous proposons de donner ici."

HIRSCH (1809).

[Sammlung von Aufgaben aus der algebraischen Gleichungen, von Meier Hirsch (pp. 103–107). xvi+360 pp. Berlin.]

The 4th Chapter *Von der Elimination u. s. w.*, contains five pages on the subject of the solution of simultaneous linear equations. These embrace nothing more noteworthy than a statement, without proof, of Cramer's rule, separated into three parts (iv., iii. 2, v.), and carefully worded.

BINET (May 1811).

[Mémoire sur la théorie des axes conjugués et des momens d'inertie des corps. *Journ. de l'École Polytechnique*, ix. (pp. 41–67), pp. 45, 46.]*

In this well-known memoir, in which the conception of the *moment of inertia of a body with respect to a plane* was first made known, there repeatedly occur expressions, which at the present day would appear in the notation of determinants. There is only one paragraph, however, containing anything new in regard to these functions. It stands as follows:—

"Le moment d'inertie minimum pris par rapport au plan (C) a pour valeur

$$\Sigma mk^2 = f^2 \times \frac{\mathrm{ABC-AF^2-BE^2-CD^2+2DEF}}{g^2(\mathrm{BC-F^2})+h^2(\mathrm{AC-E^2})+i^2(\mathrm{AB-D^2})+2gh(\mathrm{EF-CD})+2gi(\mathrm{DF-BE})+2hi(\mathrm{DE-AF})}$$

Si, dans le numérateur,

$$\mathrm{ABC-AF^2-BE^2-CD^2+2DEF}$$

on remplace A, B, C, &c. par Σmx^2, Σmy^2, &c. que ces lettres représentent, on a

$$\Sigma mx^2\Sigma my^2\Sigma mz^2 - \Sigma mx^2(\Sigma myz)^2 - \Sigma my^2(\Sigma mxz)^2$$
$$- \Sigma mz^2(\Sigma mxy)^2 + 2\Sigma mxy\Sigma mxz\Sigma myz,$$

et l'on peut s'assurer que cette expression est identique à

$$\Sigma mm'm''(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'')^2;$$

---

* An abstract of this is given in the *Nouv. Bull. des Sciences par la Société Philomatique*, ii. pp. 312–316.

par une transformation analogue, on peut ramener la quantité

$$g^2(\mathrm{BC}-\mathrm{F}^2) + h^2 (\mathrm{AC}-\mathrm{E}^2) + i^2 (\mathrm{AB}-\mathrm{D}^2)$$
$$+ 2gh(\mathrm{EF}-\mathrm{CD}) + 2gi(\mathrm{DF}-\mathrm{BE}) + 2hi(\mathrm{DE}-\mathrm{AF}),$$

à celle-ci

$$\Sigma mm'[g(yz'-zy') + h(zx'-xz') + i(xy'-yx')]^2."$$

Now the numerator referred to would at the present day be written

$$\begin{vmatrix} \mathrm{A} & \mathrm{D} & \mathrm{E} \\ \mathrm{D} & \mathrm{B} & \mathrm{F} \\ \mathrm{E} & \mathrm{F} & \mathrm{C} \end{vmatrix},$$

and since Σmx^2, &c. stand for $mx^2+m_1x_1{}^2+m_2x_2{}^2+\ldots$, &c., the first identity may be put in the form

$$\begin{vmatrix} mx^2+m_1x_1{}^2+m_2x_2{}^2+\ldots & mxy+m_1x_1y_1+m_2x_2y_2+\ldots & mxz+m_1x_1z_1+m_2x_2z_2+\ldots \\ mxy+m_1x_1y_1+m_2x_2y_2+\ldots & my^2+m_1y_1{}^2+m_2y_2{}^2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots \\ mxz+m_1x_1z_1+m_2x_2z_2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots & mz^2+m_1z_1{}^2+m_2z_2{}^2+\ldots \end{vmatrix}$$

$$= mm_1m_2\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix}^2 + mm_1m_3\begin{vmatrix} x & x_1 & x_3 \\ y & y_1 & y_3 \\ z & z_1 & z_3 \end{vmatrix}^2 + \ldots \qquad \text{(XVIII. 2)}$$

where x_1, y_2, . . . are for convenience written instead of x', y'', . . . It will been seen that this is an important extension of a theorem of Lagrange, the latter theorem being the very special case of the present obtained by putting $m=m_1=m_2=1$, and $m_3=m_4=\ldots=0$,—a fact which is brought still more clearly into evidence if, instead of the left-hand member of the identity, we write the modern contraction for it, viz.

$$\begin{vmatrix} mx & m_1x_1 & m_2x_2 & m_3x_3 & \ldots \\ my & m_1y_1 & m_2y_2 & m_3y_3 & \ldots \\ mz & m_1z_1 & m_2z_2 & m_3z_3 & \ldots \end{vmatrix} \times \begin{vmatrix} x & x_1 & x_2 & x_3 & \ldots \\ y & y_1 & y_2 & y_3 & \ldots \\ z & z_1 & z_2 & z_3 & \ldots \end{vmatrix}.$$

Again the denominator

$$g^2(\mathrm{BC}-\mathrm{F}^2) + h^2\ \mathrm{AC}-\mathrm{E}^2) + i^2 (\mathrm{AB}-\mathrm{D}^2)$$
$$+ 2gh(\mathrm{EF}-\mathrm{CD}) + 2gi(\mathrm{DF}-\mathrm{BE}) + 2hi(\mathrm{DE}-\mathrm{AF})$$

being in modern notation

$$\begin{vmatrix} \cdot & g & h & i \\ g & \mathrm{A} & \mathrm{D} & \mathrm{E} \\ h & \mathrm{D} & \mathrm{B} & \mathrm{F} \\ i & \mathrm{E} & \mathrm{F} & \mathrm{C} \end{vmatrix}$$

the second identity may be written

$$\begin{vmatrix} \cdot & g & h & i \\ g & mx^2+m_1x_1^{\ 2}+\ldots & mxy+m_1x_1y_1+\ldots & mxz+m_1x_1z_1+\ldots \\ h & mxy+m_1x_1y_1+\ldots & my^2+m_1y_1^{\ 2}+\ldots & myz+m_1y_1z_1+\ldots \\ i & mxz+m_1x_1z_1+\ldots & myz+m_1y_1z_1+\ldots & mz^2+m_1z_1^{\ 2}+\ldots \end{vmatrix}$$

$$=mm_1\begin{vmatrix} g & x & x_1 \\ h & y & y_1 \\ i & z & z_1 \end{vmatrix}^2 + mm_2\begin{vmatrix} g & x & x_2 \\ h & y & y_2 \\ i & z & z_2 \end{vmatrix}^2 + m_1m_2\begin{vmatrix} g & x_1 & x_2 \\ h & y_1 & y_2 \\ i & z_1 & z_2 \end{vmatrix} + \ldots \quad \text{(XXIX.)}$$

This is also an important theorem, and is not so much an extension of previous work as a breaking of fresh ground.

BINET (November 1811).

[Sur quelques formules d'algèbre, et sur leur application à des expressions qui ont rapport aux axes conjugués des corps. *Nouv. Bull. des Sciences par la Société Philomatique,* ii. pp. 389–392.]

In this paper Binet returns to the consideration of the first of the two identities which have just been referred to, writing it now in the form

$$\begin{aligned} &\Sigma(xy'z''-xz'y''+yz'x''-yx'z''+zx'y''-zy'x'')^2 \\ &=\Sigma x^2\Sigma y^2\Sigma z^2-\Sigma x^2(\Sigma yz)^2-\Sigma y^2(\Sigma xz)^2-\Sigma z^2(\Sigma xy)^2+2\Sigma xy\Sigma xz\Sigma yz. \end{aligned}$$

He puts it in the same category as the identity

$$\Sigma(y'z-zy')^2=\Sigma y^2\Sigma z^2-(\Sigma yz)^2,$$

which he speaks of as being then known. Further, he says

"Ces deux formules sont du même genre que la suivante

$$\Sigma\left\{\begin{aligned} &ux'y''z'''-ux'z''y'''+uy'z''x'''-uy'x''z'''+uz'x''y'''-uz'y''x'''+xy'u''z'''-xy'z''u''' \\ &+xz'y''u'''-xz'u''y'''+xu'z''y'''-xu'y''z'''+yz'u''x'''-yz'x''u'''+yu'x''z'''-yu'z''x''' \\ &+yx'z''u'''-yx'u''z'''+zu'y''x'''-zu'x''y'''+zx'y''u'''-zx'u''y'''+zy'x''u'''-zy'u''x''' \end{aligned}\right\}^2$$

$$\begin{aligned} &=\Sigma u^2\Sigma x^2\Sigma y^2\Sigma z^2-\Sigma u^2\Sigma x^2(\Sigma yz)^2-\Sigma u^2\Sigma y^2(\Sigma xz)^2-\Sigma u^2\Sigma z^2(\Sigma xy)^2 \\ &\quad-\Sigma x^2\Sigma y^2(\Sigma uz)^2-\Sigma x^2\Sigma z^2(\Sigma uy)^2-\Sigma y^2\Sigma z^2(\Sigma ux)^2 \\ &\quad+2\Sigma u^2\Sigma xy\Sigma xz\Sigma yz+2\Sigma x^2\Sigma uy\Sigma uz\Sigma yz+2\Sigma y^2\Sigma ux\Sigma uz\Sigma xz \\ &\quad+2\Sigma z^2\Sigma ux\Sigma uy\Sigma xy+(\Sigma ux)^2(\Sigma yz)^2+(\Sigma uy)^2(\Sigma xz)^2+(\Sigma uz)^2(\Sigma xy)^2 \\ &\quad-2\Sigma ux\Sigma xy\Sigma yz\Sigma zu-2\Sigma uy\Sigma yz\Sigma zx\Sigma xu-2\Sigma uy\Sigma yx\Sigma xz\Sigma zu," \end{aligned}$$

—a result which in modern notation would take the form

$$\begin{vmatrix} u & u_1 & u_2 & u_3 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \end{vmatrix}^2 + \begin{vmatrix} u & u_1 & u_2 & u_4 \\ x & x_1 & x_2 & x_4 \\ y & y_1 & y_2 & y_4 \\ z & z_1 & z_2 & z_4 \end{vmatrix}^2 + \cdots$$

$$= \begin{vmatrix} u^2+u_1^2+.. & ux+u_1x_1+.. & uy+u_1y_1+.. & uz+u_1z_1+.. \\ ux+u_1x_1+.. & x^2+x_1^2+.. & xy+x_1y_1+.. & xz+x_1z_1+.. \\ uy+u_1y_1+.. & xy+x_1y_1+.. & y^2+y_1^2+.. & yz+y_1z_1+.. \\ uz+u_1z_1+.. & xz+x_1z_1+.. & yz+y_1z_1+.. & z^2+z_1^2+.. \end{vmatrix} \qquad \text{(XVIII. 3)}$$

It is thus clear that, in November 1811, Binet was well on the way towards a great generalisation. He even says that the three identities may be looked upon

"comme les trois premières d'une suite de formules construites d'après une même loi facile à saisir."

He merely indicates, however, the mode of proof he would adopt for the results obtained, and refers to possible applications of them in investigations regarding the Method of Least Squares (Laplace, *Connaissance des Tems,* 1813) and the Centre of Gravity (Lagrange, *Mém. de Berlin,* 1783). The mode of proof need not be given here, as it turns up again in the far more important memoir in which the theorem in all its generality falls to be considered.

PRASSE (1811).

[Commentationes Mathematicæ. Auctore Mauricio de Prasse. 120 pp. Lips., 1804, 1812. (Pp. 89–102; Commentatio vii.*: Demonstratio eliminationis Cramerianæ.)]

Of previous writings the one which Prasse's most resembles is Rothe's. There is less of it, and it shows less freshness; but there is the same stiff formality of arrangement, and the same effort at rigour of demonstration.

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* Separate copies of the *Demonstratio eliminationis Cramerianæ* are also to be found, bearing the invitation title-page:

Ad memoriam Kregelio-Sternbachianam in auditorio philosophorum die xviii *Julii* MDCCCXI. *h.* ix *celebrandam invitant ordinum Academiæ Lips. Decani seniores cæterique adsessores . . . . Demonstratio eliminationis Cramerianæ.*

It is these copies which fix the date. See *Nature,* xxxvii. pp. 246, 247.

The definition of a permutation (*variatio*) being given, the first problem (which, however, is called a theorem) is propounded, viz., to tabulate the permutations of $\alpha, \beta, \gamma, \delta, \ldots$ ("*Variationum ex elementis* α, β, γ, . . . *constructarum et in Classes combinatorias digestarum Tabulam parare*"). The result is

α	β	γ	δ
	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$
	$\beta\alpha$	$\beta\gamma$	$\beta\delta$
	$\gamma\alpha$	$\gamma\beta$	$\gamma\delta$
	$\delta\alpha$	$\delta\beta$	$\delta\gamma$
		$\alpha\beta\gamma$	$\alpha\beta\delta$
		$\alpha\gamma\beta$	$\alpha\gamma\delta$
		$\alpha\delta\beta$	$\alpha\delta\gamma$
		$\beta\alpha\gamma$	$\beta\alpha\delta$
		$\beta\gamma\alpha$	$\beta\gamma\delta$
		$\beta\delta\alpha$	$\beta\delta\gamma$
		$\gamma\alpha\beta$	$\gamma\alpha\delta$
		$\gamma\beta\alpha$	$\gamma\beta\delta$
		$\gamma\delta\alpha$	$\gamma\delta\beta$
		$\delta\alpha\beta$	$\delta\alpha\gamma$
		$\delta\beta\alpha$	$\delta\beta\gamma$
		$\delta\gamma\alpha$	$\delta\gamma\beta$
			$\alpha\beta\gamma\delta$
			$\alpha\beta\delta\gamma$
			$\alpha\gamma\beta\delta$
			$\alpha\gamma\delta\beta$
			$\alpha\delta\beta\gamma$
			$\alpha\delta\gamma\beta$
			$\beta\alpha\gamma\delta$
			$\beta\alpha\delta\gamma$
			$\beta\gamma\alpha\delta$
			$\beta\gamma\delta\alpha$
			$\beta\delta\alpha\gamma$
			$\beta\delta\gamma\alpha$
			$\gamma\alpha\beta\delta$
			$\gamma\alpha\delta\beta$
			$\gamma\beta\alpha\delta$
			$\gamma\beta\delta\alpha$
			$\gamma\delta\alpha\beta$
			$\gamma\delta\beta\alpha$
			$\delta\alpha\beta\gamma$
			$\delta\alpha\gamma\beta$
			$\delta\beta\alpha\gamma$
			$\delta\beta\gamma\alpha$
			$\delta\gamma\alpha\beta$
			$\delta\gamma\beta\alpha$

The first row of the permutations involving two letters is got by taking the first letter of the previous row and annexing each of the others to it in succession and in the order of their occurrence; the second row is got in like manner from the second letter; and so on. Similarly the first row of permutations involving three letters is got from $\alpha\beta$ the first obtained permutation of two letters, the second row from $\alpha\gamma$ the next obtained permutation of two letters, and so on.*

The second problem (and on this occasion actually so designated) is somewhat quaint in its indefiniteness, viz., to prefix to each permutation the sign $+$ or the sign $-$, so that the sum of all the permutations involving the same number of letters (>1) may vanish ("*Singulis Variationibus, omissis repetitionibus, signa + et − ita praefigere, ut summa secundæ et cujuslibet classis insequentis evanescat*"). There is no indefiniteness or multiplicity about the solution, which in substance is:—Make the permutations in every row of the preceding table alternately $+$ and $-$, the first sign of all being $+$, and the first permutation of every other row having the same sign as the permutation from which it was derived. In this way the table becomes

$$\left.\begin{array}{cccc} +\alpha, & -\beta, & +\gamma, & -\delta \end{array}\right\}$$

$$\left.\begin{array}{ccc} +\alpha\beta, & -\alpha\gamma, & +\alpha\delta \\ -\beta\alpha, & +\beta\gamma, & -\beta\delta \\ +\gamma\alpha, & -\gamma\beta, & +\gamma\delta \\ -\delta\alpha, & +\delta\beta, & -\delta\gamma \end{array}\right\}$$

$$\left.\begin{array}{cc} +\alpha\beta\gamma, & -\alpha\beta\delta \\ -\alpha\gamma\beta, & +\alpha\gamma\delta \\ +\alpha\delta\beta, & -\alpha\delta\gamma \\ -\beta\alpha\gamma, & +\beta\alpha\delta \\ +\beta\gamma\alpha, & -\beta\gamma\delta \\ -\beta\delta\alpha, & +\beta\delta\gamma \\ +\gamma\alpha\beta, & -\gamma\alpha\delta \\ -\gamma\beta\alpha, & +\gamma\beta\delta \\ +\gamma\delta\alpha, & -\gamma\delta\beta \\ -\delta\alpha\beta, & +\delta\alpha\gamma \\ +\delta\beta\alpha, & -\delta\beta\gamma \\ -\delta\gamma\alpha, & +\delta\gamma\beta \end{array}\right\}$$

* It will be seen that the order in which the permutations come to hand in this process of tabulation is the order in which they would be arranged according

$$
\begin{array}{l}
+\alpha\beta\gamma\delta \\
-\alpha\beta\delta\gamma \\
-\alpha\gamma\beta\delta \\
+\alpha\gamma\delta\beta \\
+\alpha\delta\beta\gamma \\
-\alpha\delta\gamma\beta \\[1ex]
-\beta\alpha\gamma\delta \\
+\beta\alpha\delta\gamma \\
+\beta\gamma\alpha\delta \\
-\beta\gamma\delta\alpha \\
-\beta\delta\alpha\gamma \\
+\beta\delta\gamma\alpha \\[1ex]
+\gamma\alpha\beta\delta \\
-\gamma\alpha\delta\beta \\
-\gamma\beta\alpha\delta \\
+\gamma\beta\delta\alpha \\
+\gamma\delta\alpha\beta \\
-\gamma\delta\beta\alpha \\[1ex]
-\delta\alpha\beta\gamma \\
+\delta\alpha\gamma\beta \\
+\delta\beta\alpha\gamma \\
-\delta\beta\gamma\alpha \\
-\delta\gamma\alpha\beta \\
+\delta\gamma\beta\alpha
\end{array}
$$

A proof by the method of mathematical induction (so-called) is given that with these signs the sum of all the permutations of any group vanishes.

Up to this point the essence of what has been furnished is a combined rule of term-formation and rule of signs. (II. 5 + III. 15) In connection with it Bezout's rule of the year 1764 may be recalled.

The third problem is to determine the sign of any single permutation from consideration of the permutation itself. The solution is:—Under each letter of the given permutation put all the letters which precede it in the natural arrangement and which are not found to precede it in the given permutation; and make the sum + or − according as the total number of such letters is even or odd.

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to magnitude if each permutation were viewed as a number of which α, β, γ, δ were the digits, α being $<\beta<\gamma<\delta$ ("ordo lexicographicus," "lexicographische Anordnung" of Hindenburg).

"EXEMP. Datæ complexiones sint hæ:

$$\epsilon\gamma\delta\beta, \quad \delta a\epsilon\gamma, \quad \epsilon\delta\gamma a, \quad \delta\beta\epsilon\gamma.$$

Literæ secundum I subjiciantur

$$\begin{array}{llll} a\,a\,a\,a & a\,.\,\beta\beta & a\,a\,a\,. & a\,a\,a\,a \\ \beta\beta\beta & \beta\gamma & \beta\beta\beta & \beta\,.\,\gamma \\ \gamma & \gamma & \gamma\gamma & \gamma \\ \delta & & \delta & \end{array}$$

quarum numeri sunt

$$9 \qquad 6 \qquad 9 \qquad 7$$

qui complexionibus datis præfigi jubent signa

$$- \qquad + \qquad - \qquad - \,.\text{"}$$

The proof that this rule of signs, which is manifestly nothing else than Cramer's, leads to the same results as the previous rule, is quite easily understood if a particular permutation be first considered. For example, let the sign of the particular permutation $\delta\beta a\gamma$ be wanted. Following the first rule, we should require to note four different members, viz.,

(1) the no. of the column in which $\delta\beta a\gamma$ occurs in the 4th group,						
(2)	"	"	$\delta\beta a$	"	3rd	"
(3)	"	"	$\delta\beta$	"	2nd	"
(4)	"	"	δ	"	1st	"

The first of these numbers being 1, we should infer that in fixing the sign of $\delta\beta a\gamma$ in the fourth group there had been no change from the sign of $\delta\beta a$ in the third group; the second number being also 1, we should make a like inference; the third number being 2, we should infer that in fixing the sign of $\delta\beta$ in the second group there had been 1 change from the sign of δ in the first group; and finally, the fourth number being 4, we should infer that in fixing the sign of δ in the first group there had been 3 changes from the sign of a in that group. The total number of changes from the sign of a in the first group being thus $3+1+0+0$, *i.e.*, 4, the sign would be made $+$. Now the 3 in this aggregate is simply the number of letters in the first group which precede δ, the 1 is simply the number of letters taken along with δ before β comes to be taken along with it to form $\delta\beta$ in the second group, and the two zeros correspond to the fact that $\delta\beta a$ on the third group and $\delta\beta a\gamma$ on the fourth group have no permutation standing to the left of them. Consequently to count the number of changes $(3+1+0+0)$ from the sign of a in accordance with the first rule is the same as to count the number of letters placed under the given permutation, thus,

$$\begin{array}{l} \underline{\delta\beta a\gamma} \\ a\,a\,.\,. \\ \beta \\ \gamma \end{array}$$

in accordance with the second rule.

Another point of resemblance between Rothe and Prasse is thus made manifest, viz., that they both refused to accept Cramer's rule of signs as fundamental, preferring to base their work on a rule equally arbitrary, and then to deduce Cramer's from it.

In case it may have escaped the reader, attention may likewise be drawn to the fact that Prasse prefixes a sign not only to permutations involving all the letters dealt with, but also to any permutation whatever involving a less number; so that in reckoning the sign of $a\delta\beta$, say, the full number of letters from which a, δ, β are chosen must be known.

A theorem like Hindenburg's is next given, viz., *If the permutations of any group be separated into sub-groups* (1) *those which begin with* a, (2) *those which begin with* β, *and so on, then the series of signs of the* 3rd, 5th, *and other odd sub-groups is identical with the series of signs of the* 1st *sub-group, and the signs of any one of the even sub-groups is got by changing each sign of the first sub-group into the opposite sign.* (III. 16)
It is more extensive than Hindenburg's in that it is true of permutations which involve less than all the letters, provided such permutations have had their signs fixed in accordance with Prasse's rule. The proof depends, of course, on the first rule of signs, and consists in showing that if the theorem be true for any group it must, by the said rule, be true for the next group. It will be remembered that Hindenburg gave no proof.

Following this is Rothe's theorem regarding the interchange of two elements of a permutation, or rather an extension of the theorem to signed permutations involving less than the whole number of letters. The proof is as lengthy as Rothe's, even more unnecessary letters than Rothe's *c*, *f*, *e* being introduced. (III. 17)

The last theorem is Vandermonde's (XII.); and this is followed by two pages of application to the solution of simultaneous linear equations.

No reference is made by Prasse to Hindenburg, Rothe, or Vandermonde.

WRONSKI (1812).

[Réfutation de la Théorie des Fonctions Analytiques de Lagrange. Par Höené Wronski. (pp. 14, 15,..., 132, 133.) 136 pp. Paris.]

In 1810 Wronski presented to the Institute of France a memoir on the so-called *Technie de l'Algorithmie*, which with his usual sanguine enthusiasm he viewed as the essential part of a new branch of Mathematics. It contained a very general theorem, now known as "Wronski's theorem," for the expansion of functions,—a theorem requiring for its expression the use of a notation for what Wronski styled *combinatory sums*. The memoir consisted merely of a statement of results, and probably on this account, although favourably reported on by Lagrange and Lacroix, was not printed. The subject of it, however, turns up repeatedly in the *Réfutation* printed two years later; and from the indications there given we can so far form an idea of the grasp which Wronski had of the theory of the said *sums*.

At page 14 the following passage occurs:—

"Soient X_1, X_2, X_3, &c. plusieurs fonctions d'une quantité variable. Nommons *somme combinatoire*, et désignons par la lettre hébraïque *sin*, de la manière que voici

$$ש[\Delta^a X_1 . \Delta^b X_2 . \Delta^c X_3 \ . \ . \ . \ \Delta^p X_\pi], \qquad \text{(XV. 3) (VII. 4)}$$

la somme des produits des différences de ces fonctions, composés de la manière suivante: Formez, avec les exposans a, b, c, . . . , p des différences dont il est question, toutes les permutations possibles; donnez ces exposans, dans chaque ordre de leurs permutations, aux différences consécutives qui composent le produit

$$\Delta X_1 . \Delta X_2 . \Delta X_3 \ . \ . \ . \ \Delta X_\pi ;$$

donnez de plus, aux produits séparés, formés de cette manière, le signe positif lorsque le nombre de variations des exposans a, b, c, etc., considérés dans leur ordre alphabétique, est nul ou pair, et le signe négatif lorsque ce nombre de variations est impair; enfin, prenez la somme de tous ces produits séparés.—Vous aurez ainsi, par exemple,

$$\text{ש}[\Delta^a X_1] = \Delta^a X_1,$$
$$\text{ש}[\Delta^a X_1 . \Delta^b X_2] = \Delta^a X_1 . \Delta^b X_2 - \Delta^b X_1 . \Delta^a X_2,$$
$$\cdots\cdots\cdots\cdots\cdots\text{''}$$

The new name, *combinatory sum*, and the new notation, did not originate in ignorance of the work of previous investigators, for memoirs of Vandermonde and Laplace are referred to. The only fresh and real point of interest lies in the fact that the first index of every pair of indices is not attached to the same letter as the second index, but belongs to an operational symbol preceding this letter, and is used for the purpose of denoting repetition of the operation. This and the allied fact that the elements are not all independent of each other, $\Delta^1 X_1$ and $\Delta^2 X_1$, for example, being connected by the equation

$$\Delta^2 X_1 = \Delta(\Delta^1 X_1),$$

indicate that Wronski's combinatory sums form a special class with properties peculiar to themselves.

CHAPTER IV.

DETERMINANTS IN GENERAL IN THE YEAR 1812.

HERE we have the record of only one year and of only two authors to deal with; but the authors, Binet and Cauchy, are of supreme importance, and the product of the year probably exceeded that of all the years that had gone before.

BINET (November 1812).

[Mémoire sur un système de formules analytiques, et leur application à des considérations géométriques. *Journ. de l'Éc. Polyt.*, ix. cah. 16, pp. 280–302, . . .]

It would seem as if the above-noted frequent recurrence of functions of the same kind had led Binet to a special study of them. In the memoir we have now come to, his standpoint towards them is changed. They are viewed as functions having a history: for information regarding them, the writings of Vandermonde, Laplace, Lagrange, and Gauss are referred to: they are spoken of by Laplace's name for them, *résultantes à deux lettres, à trois lettres, à quatre lettres*, &c.; and the first twenty-three pages of the memoir are devoted expressly to establishing new theorems regarding them.

Of these the fundamental, and by far the most notable, is the afterwards well-known *multiplication-theorem*. It is enunciated at the outset as follows:—

"Lorsqu'on a deux systèmes de n lettres chacun, et nous supposerons chaque système écrit avec une seule lettre portant divers accens, qui serviront à ranger dans le même ordre les deux systèmes; on peut former avec ces lettres un nombre $n\frac{n-1}{2}$ de résultantes à deux lettres, en ne prenant dans le second terme de chacune que des lettres portant les mêmes accens que celles du premier. Si, avec deux autres systèmes de lettres, on forme encore des résultantes à deux lettres, et qu'on les multiplie chacune par sa correspondante obtenue des deux premiers systèmes, c'est-à-dire, par celle dont les lettres portent les mêmes accens; la somme des produits de toutes ces résultantes correspondantes sera elle-même une résultante à deux lettres, dont les termes ou lettres seront des sommes de produits des élémens des deux systèmes portant les mêmes accens. Avec deux groupes de trois systèmes de n lettres chacun, on peut former semblablement deux séries de résultantes à trois lettres; faisant ensuite la somme des produits de celles qui se correspondent par les accens de leurs lettres, on aura encore une résultante à trois lettres. Pareille chose ayant lieu pour des résultantes à quatre lettres, &c., on peut conclure ce théorème: Le produit d'un nombre quelconque de sommes de produits * de deux résultantes correspondantes de même ordre, est encore une résultante de cet ordre."

(XVII. 4 + XVIII. 4)

The mode of proof adopted is lengthy, laborious, and not very satisfactory, except as affording a verification of the theorem for the cases of "résultantes" of low orders. It rests too on certain identities, the demonstration of which is open to similar criticism. All that Binet says regarding these absolutely essential identities is (p. 284)—

"Je représenterai par Σa la somme $a'+a''+a'''+$&c., des quantités a', a'', a''', &c.; par Σab la somme des produits $ab+a'b'+a''b''+$&c., dans chacun desquels les lettres a et b ont le même accent; par $\Sigma ab'$ la somme $a'b''+b'a''+a'b'''+$&c., là tous les produits d'un des a par un des b, portent un accent différent de celui de a; par $\Sigma ab'c''$ la somme $a'b''c'''+b'c''a'''+c'a''b'''+$&c., et ainsi de suite. Cela posé, on vérifie aisément les formules suivantes:

$$\begin{aligned}
\Sigma ab' &= \Sigma a\Sigma b - \Sigma ab,\\
\Sigma ab'c'' &= \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma c\Sigma ab,\\
\Sigma ab'c''d''' &= \Sigma a\Sigma b\Sigma c\Sigma d \ - 6\Sigma abcd\\
&\quad - \Sigma a\Sigma b\Sigma cd - \Sigma a\Sigma c\Sigma bd - \Sigma a\Sigma d\Sigma bc\\
&\quad - \Sigma c\Sigma d\Sigma ab - \Sigma b\Sigma d\Sigma ac - \Sigma b\Sigma c\Sigma ad\\
&\quad + \Sigma ab\Sigma cd \ + \Sigma ac\Sigma bd \ + \Sigma ab\dagger\Sigma bc\\
&\quad + 2\Sigma a\Sigma bcd + 2\Sigma b\Sigma cda + 2\Sigma c\Sigma dab + 2\Sigma d\Sigma abc,\\
\Sigma ab'c''d'''e^{\mathrm{iv}} &= \Sigma a\Sigma b\Sigma c\Sigma d\Sigma e + \&\mathrm{c}.,\\
&\quad \&\mathrm{c}."
\end{aligned}$$

---

* There is an extension here which one is scarcely prepared for, viz., "*le produit d'un nombre quelconque de sommes de produits,*" instead of *la somme d'un nombre de produits.*

† Meant for Σad.

It is thus seen that not only is no general proof of the identities given, but that even the law of formation of the right-hand members of the identities themselves is left undivulged. The exact words employed in the demonstration of the first case of the multiplication-theorem are (p. 286)—

"Avec un nombre n de lettres y', y'', y''', &c. et un même nombre de z', z'', z''', &c. on peut former $n\frac{n-1}{2}$ résultantes à deux lettres (y', z''), (y', z'''), &c. (y'', z''') &c.; ayant formé pareillement avec les lettres, v', v'', v''', &c., $\zeta', \zeta'', \zeta'''$, &c., les résultantes (v', ζ''), (v', ζ'''), &c., (v'', ζ'''), &c., considérons la somme $\Sigma(y, z')(v, \zeta')$ des produits des résultantes qui se correspondent par les accens dans les deux systèmes. On voit, en développant, par la multiplication, chacun des termes de cette somme, qu'elle revient à

$$\Sigma yv \,.\, z'\zeta' - \Sigma zv \,.\, y'\zeta'.$$

A ces deux dernières intégrales, on peut appliquer la transformation indiquée par la première des formules de l'art. 1 : on parvient ainsi à

$$\Sigma(y, z')(v, \zeta') = \Sigma yv \Sigma z\zeta - \Sigma zv \Sigma y\zeta.$$

Ce dernier membre pouvant être assimilé a là forme (y, z'), il en résulte que le produit d'un nombre quelconque de fonctions, telles que $\Sigma(y, z')(v, \zeta')$, est lui-même de la forme (y, z')."

The application here of the identity

$$\Sigma ab' = \Sigma a \Sigma b - \Sigma ab$$

requires a little attention. The result of multiplication and classification of the terms is

$$\Sigma yv \,.\, z'\zeta' - \Sigma zv \,.\, y'\zeta',$$

or, as it might preferably be written,

$$\Sigma\{\overline{yv} \,.\, \overline{z\zeta}'\} - \{\Sigma\overline{zv} \,.\, \overline{y\zeta}'\};$$

and this we know from the said identity

$$= [\Sigma\overline{yv} \,.\, \Sigma\overline{z\zeta} - \Sigma(\overline{yv} \,.\, \overline{z\zeta})] - [\Sigma\overline{zv} \,.\, \Sigma\overline{y\zeta} - \Sigma(\overline{zv} \,.\, \overline{y\zeta})],$$

which, because of the equality of $\Sigma(\overline{yv} \,.\, \overline{z\zeta})$ and $\Sigma(\overline{zv} \,.\, \overline{y\zeta})$, becomes

$$\Sigma\overline{yv} \,.\, \Sigma\overline{z\zeta} - \Sigma\overline{zv} \,.\, \Sigma\overline{y\zeta}.$$

The inherent weak points, however, of the mode of demonstration stand out more clearly when the next case comes to be considered, viz., the case for resultants of the third order. From the three sets of n letters

$$\begin{array}{l} x,\ x',\ x'',\ \ldots\ . \\ y,\ y',\ y'',\ \ldots\ . \\ z,\ z',\ z'',\ \ldots\ . \end{array}$$

all possible "résultantes à trois lettres" are formed, and each resultant is multiplied by the corresponding resultant formed from other three sets of n letters,

$$\begin{array}{l} \xi,\ \xi',\ \xi'',\ \ldots\ . \\ \upsilon,\ \upsilon',\ \upsilon'',\ \ldots\ . \\ \zeta,\ \zeta',\ \zeta'',\ \ldots\ . \end{array}$$

Each of these $\frac{1}{6}n(n-1)(n-2)$ products consists of 36 terms, there being thus $6n(n-1)(n-2)$ terms in all. But these $6n(n-1)(n-2)$ terms are found to be separable into six groups, viz.

$$+\Sigma\{x\xi\,.\,y'\upsilon'\,.\,z''\zeta''\},\quad +\Sigma\{y\xi\,.\,z'\upsilon'\,.\,x''\zeta''\},\quad \ldots\ .$$

so that the result which we are able to register at this point is

$$\begin{aligned} \Sigma(x,\,y',\,z'')(\xi,\,\upsilon',\,\zeta'') = \;& \Sigma x\xi\,.\,y'\upsilon'\,.\,z''\zeta'' + \Sigma y\xi\,.\,z'\upsilon'\,.\,x''\zeta'' \\ & + \Sigma z\xi\,.\,x'\upsilon'\,.\,y''\zeta'' - \Sigma x\xi\,.\,z'\upsilon'\,.\,y''\zeta'' \\ & - \Sigma y\xi\,.\,x'\upsilon'\,.\,z''\zeta'' - \Sigma z\xi\,.\,y'\upsilon'\,.\,x''\zeta''. \end{aligned}$$

To the right-hand member of this the substitution

$$\Sigma ab'c'' = \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma c\Sigma ab$$

is now applied six times in succession; that is to say, for

$$\Sigma x\xi\,.\,y'\upsilon'\,.\,z''\zeta''$$

and the five other term-aggregates which follow, we substitute

$$\begin{aligned} & \Sigma x\xi\Sigma y\upsilon\Sigma z\zeta + 2\Sigma(x\xi\,.\,y\upsilon\,.\,z\zeta) \\ & \qquad - \Sigma x\xi\Sigma(y\upsilon\,.\,z\zeta) - \Sigma y\upsilon\Sigma(z\zeta\,.\,x\xi) - \Sigma z\zeta\Sigma(x\xi\,.\,y\upsilon) \end{aligned}$$

and five other like expressions. By this means we arrive, "toute réduction faite," at

$$\begin{aligned} \Sigma(x,\,y',\,z'')(\xi,\,\upsilon',\,\zeta'') = \;& \Sigma x\xi\Sigma y\upsilon\Sigma z\zeta + \Sigma y\xi\Sigma z\upsilon\Sigma x\zeta + \Sigma z\xi\Sigma x\upsilon\Sigma y\zeta \\ & - \Sigma x\xi\Sigma z\upsilon\Sigma y\zeta - \Sigma y\xi\Sigma x\upsilon\Sigma z\zeta - \Sigma z\xi\Sigma y\upsilon\Sigma x\zeta, \end{aligned}$$

which is the result desired.

It is easy to imagine the troubles in store for any one who might have the hardihood to attempt to establish the next case in the same manner.

If Binet's multiplication-theorem be described as expressing *a sum of products of resultants as a single resultant,* his next theorem may be said to give *a sum of products of sums of resultants as a sum of resultants.* The paragraph in regard to it is a little too much condensed to be perfectly clear, and must therefore be given verbatim. It is (p. 288)—

"Désignons par $S(y', z'')$ une somme de résultantes, telle que

$$(y_{\prime}', z_{\prime}'') + (y_{\prime\prime}', z_{\prime\prime}'') + (y_{\prime\prime\prime}', z_{\prime\prime\prime}'') + \&c.;$$

c'est-à-dire,

$$y_{\prime}'z_{\prime}'' - z_{\prime}'y_{\prime}'' + y_{\prime\prime}'z_{\prime\prime}'' - z_{\prime\prime}'y_{\prime\prime}'' + y_{\prime\prime\prime}'z_{\prime\prime\prime}'' - z_{\prime\prime\prime}'y_{\prime\prime\prime}'' + \&c.;$$

et continuons d'employer la caractéristique Σ pour les intégrales relatives aux accens supérieurs des lettres. L'expression

$$\Sigma[S(y, z') \,.\, S(v, \zeta')]$$

devient par le développement de chacun de ses termes, et en vertu de la première formule de l'art. 1 ou de celle du no. 4,

$$\begin{aligned} &\Sigma y_{\prime}v_{\prime}\,\Sigma z_{\prime}\zeta_{\prime} - \Sigma z_{\prime}v_{\prime}\,\Sigma y_{\prime}\zeta_{\prime} + \Sigma y_{\prime\prime}v_{\prime}\,\Sigma z_{\prime\prime}\zeta_{\prime} - \Sigma z_{\prime\prime}v_{\prime}\Sigma y_{\prime\prime}\zeta_{\prime} + \&c. \\ + &\Sigma y_{\prime}v_{\prime\prime}\Sigma z_{\prime}\zeta_{\prime\prime} - \Sigma z_{\prime}v_{\prime\prime}\Sigma y_{\prime}\zeta_{\prime\prime} + \Sigma y_{\prime\prime}v_{\prime\prime}\Sigma z_{\prime\prime}\zeta_{\prime\prime} - \Sigma z_{\prime\prime}v_{\prime\prime}\Sigma y_{\prime\prime}\zeta_{\prime\prime} + \&c. \\ + &\&c. \end{aligned}$$

En indiquant donc par S_1 des intégrales qui supposent, dans chaque terme, les mêmes accens inférieurs aux lettres du même alphabet, ces accens pouvant être ou non les mêmes pour celles des alphabets différens, on pourra écrire la précédente suite, en faisant usage de ce signe, ce qui donne

$$\Sigma[S(y, z')S(v, \zeta')] = S_1[\Sigma yv\Sigma z\zeta - \Sigma zv\Sigma y\zeta].$$

Cette nouvelle quantité est encore de la forme $S(y', z'')$, en sorte qu'on peut dire que le produit de fonctions, telles que

$$\Sigma\{S(y, z')S(v, \zeta')\},$$

sera lui-même de la forme $S(y', z'')$."

This, if I understand it correctly, may be paraphrased and expanded as follows:—

Take the product of two sums of s resultants, viz.

$$\begin{aligned} &\{|y_1{}^1z_1{}^2| + |y_2{}^1z_2{}^2| + |y_3{}^1z_3{}^2| + \ldots + |y_s{}^1z_s{}^2|\} \\ \times &\{|v_1{}^1\zeta_1{}^2| + |v_2{}^1\zeta_2{}^2| + |v_3{}^1\zeta_3{}^2| + \ldots + |v_s{}^1\zeta_s{}^2|\} \end{aligned}$$

or

$$\sum_{s=1}^{s=s}|y_s{}^1z_s{}^2| \,.\, \sum_{s=1}^{s=s}|v_s{}^1\zeta_s{}^2|,$$

where, it will be observed, all the resultants in the first factor are obtained from the first resultant $|y_1{}^1z_1{}^2|$ by merely changing the lower indices into 2, 3, . . . , s in succession, and that the second factor is got from the first by writing v for y and ζ for z. Then form all the like products whose first factors are

$$|y_1^1 z_1^3|,\ |y_1^1 z_1^4|,\ \ldots,\ |y_1^{n-1} z_1^n|;$$

these being along with $|y_1^1 z_1^2|$ the $\frac{1}{2}n(n-1)$ resultants derivable from the two sets of n quantities

$$y_1^1,\ y_1^2,\ y_1^3,\ \ldots,\ y_1^n$$
$$z_1^1,\ z_1^2,\ z_1^3,\ \ldots,\ z_1^n.$$

The sum of these $\frac{1}{2}n(n-1)$ products may be represented, if we choose, by

$$\sum_{\substack{n=2\\ m<n}}^{n=n}\left[\sum_{s=1}^{s=s}|y_s^m z_s^n|\cdot\sum_{s=1}^{s=s}|v_s^m \zeta_s^n|\right].$$

Now if the multiplications be performed, there will be s^2 terms in each product, and the theorem we are concerned with has its origin in the fact that the sum of all the first terms of the products is expressible as a resultant by applying the multiplication-theorem, likewise the sum of all the second terms, and so on, the result being an aggregate of s^2 resultants. For if we fix upon a particular term of the first product, say the term

$$|y_h^1 z_h^2|\cdot|v_k^1 \zeta_k^2|$$

which arises from the multiplication of the h^{th} term of the first factor by the k^{th} term of the second factor, then take the corresponding term of the other products, and write down their sum

$$|y_h^1 z_h^2|\cdot|v_k^1 \zeta_k^2| + |y_h^1 z_h^3|\cdot|v_k^1 \zeta_k^3| + \ldots + |y_h^{n-1} \zeta_h^n|\cdot|v_k^{n-1} \zeta_k^n|,$$

it is manifest that this sum is by the multiplication-theorem

$$= \begin{vmatrix} y_h^1 v_k^1 + y_h^2 v_k^2 + \ldots + y_h^n v_k^n & z_h^1 v_k^1 + z_h^2 v_k^2 + \ldots + z_h^n v_k^n \\ y_h^1 \zeta_k^1 + y_h^2 \zeta_k^2 + \ldots + y_h^n \zeta_k^n & z_h^1 \zeta_k^1 + z_h^2 \zeta_k^2 + \ldots + z_h^n \zeta_k^n \end{vmatrix}.$$

Consequently since h may be any integer from 1 to s, and k likewise any integer from 1 to s, the theorem arrived at is accurately expressed in modern notation as follows:—

$$\sum_{\substack{n=2\\ m<n}}^{n=n}\left[\sum_{s=1}^{s=s}|y_s^m z_s^n|\cdot\sum_{s=1}^{s=s}|v_s^m \zeta_s^n|\right]$$

$$= \sum_{k=1}^{k=s}\sum_{h=1}^{h=s}\begin{vmatrix} y_h^1 v_k^1 + y_h^2 v_k^2 + \ldots + y_h^n v_k^n & z_h^1 v_k^1 + z_h^2 v_k^2 + \ldots + z_h^n v_k^n \\ y_h^1 \zeta_k^1 + y_h^2 \zeta_k^2 + \ldots + y_h^n \zeta_k^n & z_h^1 \zeta_k^1 + z_h^2 \zeta_k^2 + \ldots + z_h^n \zeta_k^n \end{vmatrix},$$

or $$\sum_{k=1}^{k=s}\sum_{h=1}^{h=s}\begin{vmatrix} y_h^1 & y_h^2 & \ldots & y_h^n \\ z_h^1 & z_h^2 & \ldots & z_h^n \end{vmatrix}\cdot\begin{vmatrix} v_k^1 & v_k^2 & \ldots & v_k^n \\ \zeta_k^1 & \zeta_k^2 & \ldots & \zeta_k^n \end{vmatrix}.$$

It is easily seen to be true of resultants of any order, as Binet himself points out. (XXX.)

When s is put equal to 1, it degenerates into the extended multiplication-theorem.

The theorem which follows upon this, but which is quite unconnected with it, may be at once stated in modern notation. It is—

If $\Sigma|x_1y_2z_3|$ denote the sum of the resultants obtainable from the three sets of n quantities

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & \ldots\ldots & x_n \\ y_1 & y_2 & y_3 & \ldots\ldots & y_n \\ z_1 & z_2 & z_3 & \ldots\ldots & z_n, \end{array}$$

and $\Sigma|x_1y_2|$ denote the like sum obtainable from the first two sets, then

$$\Sigma|x_1y_2z_3| = \Sigma x \,.\, \Sigma|y_1z_2| + \Sigma y \,.\, \Sigma|z_1x_2| + \Sigma z \,.\, \Sigma|x_1y_2|. \quad \text{(XXXI.)}$$

This is arrived at by writing out the terms of $\Sigma|y_1z_2|$, of $\Sigma|z_1x_2|$, and of $\Sigma|x_1y_2|$ in parallel columns, thus

$$\begin{array}{ccc} |y_1\ z_2| & |z_1\ x_2| & |x_1\ y_2| \\ |y_1\ z_3| & |z_1\ x_3| & |x_1\ y_3| \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ |y_{n-1}z_n| & |z_{n-1}x_n| & |x_{n-1}y_n|; \end{array}$$

then deriving n results from the members of the first row by multiplying by x_1, y_1, z_1 respectively and adding, multiplying by x_2, y_2, z_2, and adding, and so on; then treating the second and remaining rows in the same way; and then finally adding all the $n \,.\, \frac{1}{2}n(n-1)$ results together. Each of these results is a vanishing or non-vanishing resultant of the 3rd order, and it will be found that each non-vanishing resultant occurs twice with the sign $+$ and once with the sign $-$.

This process is readily seen to be simply the same as performing the multiplications indicated in the right-hand member of (XXXI.), *i.e.*,

$$\begin{aligned} &(x_1+x_2+\ldots+x_n)\,(|y_1z_2|+|y_1z_3|+\ldots+|y_{n-1}z_n|) \\ +&(y_1+y_2+\ldots+y_n)\,(|z_1x_2|+|z_1x_3|+\ldots+|z_{n-1}x_n|) \\ +&(z_1+z_2+\ldots+z_n)\,(|x_1y_2|+|x_1y_3|+\ldots+|x_{n-1}y_n|), \end{aligned}$$

summing every three corresponding terms in the products, and writing the sum as a vanishing or non-vanishing resultant. There would be $n \,.\, \frac{1}{2}n(n-1)$ resultants in all; but as each suffix occurs $n-1$ times in the second factors and once in the first factors, there must be in each product $n-1$ terms having the said suffix occurring twice: consequently there must be $n-1$ resultants vanishing on account of this recurrence, and therefore altogether $n(n-1)$ vanishing resultants. Of the non-vanishing resultants,—in number equal to $n \,.\, \frac{1}{2}n(n-1)-n(n-1)$, or $\frac{1}{2}n(n-1)(n-2)$,—each one of the form

$$|x_h y_k z_l| \qquad \text{where } h<k<l$$

must be accompanied by two others,

$$|x_k y_h z_l| \text{ and } |x_l y_h z_k|,$$

and the sum of these is

$$|x_h y_k z_l| - |x_h y_k z_l| + |x_h y_k z_l|,$$

i.e.,

$$|x_h y_k z_l|.$$

The final result is thus the sum of the resultants of the form

$$|x_h y_k z_l| \text{ where } h<k<l, \text{ and } l=3, 4, \ldots, n,$$

the number of them, as we may see from two different standpoints, being

$$\tfrac{1}{6}n(n-1)(n-2).$$

Returning to the series of identities,

$$\begin{gathered} x_3|y_1z_2| + y_3|z_1x_2| + z_3|x_1y_2| = |x_1y_2z_3|, \\ x_4|y_1z_2| + y_4|z_1x_2| + z_4|x_1y_2| = |x_1y_2z_4|, \\ \&\text{c.} \qquad\qquad \&\text{c.} \end{gathered}$$

which by addition give the result

$$\Sigma x\Sigma|y_1z_2| + \Sigma y\Sigma|z_1x_2| + \Sigma z\Sigma|x_1y_2| = \Sigma|x_1y_2z_3|,$$

Binet next raises both sides of all of them to the second power, and obtains

$$\left.\begin{aligned} 3\Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2 \\ + 2\Sigma yz\Sigma(|z_1x_2|.|x_1y_2|) + 2\Sigma zx\Sigma(|x_1y_2|.|y_1z_2|) \\ + 2\Sigma xy\Sigma(|y_1z_2|.|z_1x_2|). \end{aligned}\right\}(\text{XXXII.})$$

Substituting for $\Sigma|y_1z_2|^2$, $\Sigma|z_1x_2|^2$, ... , their equivalents as given by the multiplication-theorem, he then deduces

$$\left.\begin{aligned}\Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma y^2\Sigma z^2 + 2\Sigma yz\Sigma zx\Sigma xy - \Sigma x^2(\Sigma yz)^2 \\ -\Sigma y^2(\Sigma zx)^2 - \Sigma z^2(\Sigma xy)^2,\end{aligned}\right\}$$

not failing to note that this is not a fresh result, but merely a case of the multiplication-theorem in which the factors are equal.

By putting the right-hand member here into the form

$$\begin{aligned}&\Sigma y^2\{\Sigma z^2\Sigma x^2-(\Sigma yz)^2\} + \Sigma z^2\{\Sigma x^2\Sigma y^2-(\Sigma xy)^2\}\\ &\qquad - \Sigma x^2\{\Sigma y^2\Sigma z^2-(\Sigma yz)^2\} + 2\Sigma yz\{\Sigma zx\Sigma xy-\Sigma yz\Sigma x^2\},\end{aligned}$$

there is next arrived at the first identity of the set

$$\Sigma|x_1y_2z_3|^2$$
$$\left.\begin{aligned}&=\Sigma y^2\Sigma|z_1x_2|^2+\Sigma z^2\Sigma|x_1y_2|^2-\Sigma x^2\Sigma|y_1z_2|^2+2\Sigma yz\Sigma|z_1x_2||x_1y_2|,\\ &=\Sigma z^2\Sigma|x_1y_2|^2+\Sigma x^2\Sigma|y_1z_2|^2-\Sigma y^2\Sigma|z_1x_2|^2+2\Sigma zx\Sigma|x_1y_2||y_1z_2|,\\ &=\Sigma x^2\Sigma|y_1z_2|^2+\Sigma y^2\Sigma|z_1x_2|^2-\Sigma z^2\Sigma|x_1y_2|^2+2\Sigma xy\Sigma|y_1z_2||z_1x_2|,\end{aligned}\right\}\text{(XXXIII.)}$$

and immediately from these the set

$$\left.\begin{aligned}\Sigma|x_1y_2z_3|^2&=\Sigma x^2\Sigma|y_1z_2|^2+\Sigma zx\Sigma|x_1y_2|.|y_1z_2|+\Sigma xy\Sigma|y_1z_2|.|z_1x_2|,\\ &=\Sigma y^2\Sigma|z_1x_2|^2+\Sigma xy\Sigma|y_1z_2|.|z_1x_2|+\Sigma yz\Sigma|z_1x_2|.|x_1y_2|,\\ &=\Sigma z^2\Sigma|x_1y_2|^2+\Sigma yz\Sigma|z_1x_2|.|x_1y_2|+\Sigma zx\Sigma|x_1y_2|.|y_1z_2|.\end{aligned}\right\}\text{(XXXIV.)}$$

We may note in passing that either of these sets leads at once to the initial theorem

$$\begin{aligned}3\Sigma|x_1y_2z_3|^2 = \quad&\Sigma x^2\Sigma|y_1z_2|^2 + \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2\\ &+2\Sigma yz\Sigma|z_1x_2|.|x_1y_2| + 2\Sigma zx\Sigma|x_1y_2|.|y_1z_2|\\ &+ 2\Sigma xy\Sigma|y_1z_2|.|z_1x_2|,\end{aligned}$$

and that with the multiplication-theorem already established this reverse order would be the more natural.

The next step taken is the formation of resultants of the 2nd order from elements which are themselves resultants of the 2nd order; viz., just as from the three rows of n quantities

$$\begin{array}{cccccc}x_1 & x_2 & x_3 & \dots\dots & x_n\\ y_1 & y_2 & y_3 & \dots\dots & y_n\\ z_1 & z_2 & z_3 & \dots\dots & z_n\end{array}$$

there were formed the three other rows of $\frac{1}{2}n(n-1)$ quantities

$$\begin{array}{l}
|y_1z_2|, \quad |y_1z_3|, \quad \ldots\ldots, \quad |y_1z_n|, \quad |y_2z_3|, \quad \ldots\ldots, \quad |y_{n-1}z_n|, \\
|z_1x_2|, \quad |z_1x_3|, \quad \ldots\ldots, \quad |z_1x_n|, \quad |z_2x_3|, \quad \ldots\ldots, \quad |z_{n-1}x_n|, \\
|x_1y_2|, \quad |x_1y_3|, \quad \ldots\ldots, \quad |x_1y_n|, \quad |x_2y_3|, \quad \ldots\ldots, \quad |x_{n-1}y_n|,
\end{array}$$

so from the latter three other rows of quantities

$$\begin{array}{l}
\begin{Vmatrix} |z_1x_2| & |z_1x_3| \\ |x_1y_2| & |x_1y_3| \end{Vmatrix}, \ldots\ldots\ldots\ldots\ldots\ldots, \begin{Vmatrix} |z_{n-2}x_n| & |z_{n-1}x_n| \\ |x_{n-2}y_n| & |x_{n-1}y_n| \end{Vmatrix}, \\
\begin{Vmatrix} |x_1y_2| & |x_1y_3| \\ |y_1z_2| & |y_1z_3| \end{Vmatrix}, \ldots\ldots\ldots\ldots\ldots\ldots, \begin{Vmatrix} |x_{n-2}y_n| & |x_{n-1}y_n| \\ |y_{n-2}z_n| & |y_{n-1}z_n| \end{Vmatrix}, \\
\begin{Vmatrix} |y_1z_2| & |y_1z_3| \\ |z_1x_2| & |z_1x_3| \end{Vmatrix}, \ldots\ldots\ldots\ldots\ldots\ldots, \begin{Vmatrix} |y_{n-2}z_n| & |y_{n-1}z_n| \\ |z_{n-2}x_n| & |z_{n-1}x_n| \end{Vmatrix},
\end{array}$$

are formed, the number in each new row being clearly

$$\tfrac{1}{2}\{\tfrac{1}{2}n(n-1)\}\{\tfrac{1}{2}n(n-1)-1\}$$

i.e.,

$$\tfrac{1}{8}(n+1)n(n-1)(n-2).$$

The new quantities are, of course, not written by Binet in the form

$$\begin{Vmatrix} | & | & | & | \\ | & | & | & | \end{Vmatrix},$$

but the fact that they are resultants of the 2nd order is carefully noted. Each of them is shown to be transformable, by a theorem which may be viewed as an extension of a result given by Lagrange, so as to have two of the elements resultants of the 3rd order, and the other resultants of the 1st order. This is done by taking, for example, the identities

$$x_h|y_iz_j| + y_h|z_ix_j| + z_h|x_iy_j| = |x_hy_iz_j|,$$
$$x_k|y_iz_j| + y_k|z_ix_j| + z_k|x_iy_j| = |x_ky_iz_j|,$$

multiplying both sides of the first by x_k, and both sides of the second by x_h, subtracting, and writing the result in the form

$$\begin{aligned} |x_ky_h||z_ix_j| + |x_kz_h||x_iy_j| &= x_k|x_hy_iz_j| - x_h|x_ky_iz_j|, \\ &= \begin{vmatrix} x_k & x_h \\ |x_ky_iz_j| & |x_hy_iz_j| \end{vmatrix}, \end{aligned}$$

where of course it has to be noted that in many cases one of the resultants of the 3rd order will vanish. The quantities, therefore, to be dealt with, are

$$
\begin{array}{l}
x_1\,|x_1y_2z_3|, \;\ldots,\; x_k\,|x_hy_iz_j| - x_h\,|x_ky_iz_j|, \;\ldots,\; x_n\,|x_{n-2}y_{n-1}z_n|;\\
y_1\,|x_1y_2z_3|, \;\ldots,\; y_k\,|y_hz_ix_j| - y_h\,|y_kz_ix_j|, \;\ldots,\; y_n\,|x_{n-2}y_{n-1}z_n|;\\
z_1\,|x_1y_2z_3|, \;\ldots,\; z_k\,|z_hx_iy_j| - z_h\,|z_kx_iy_j|, \;\ldots,\; z_n\,|x_{n-2}y_{n-1}z_n|.
\end{array}
$$

By raising each of the elements of the first row to the second power, taking the sum and simplifying, we could, we are told, show that the result would be

$$\Sigma x_1^2\Sigma|x_1y_2z_3|^2.$$

Very prudently, however, another process is chosen. It is recalled that the quantities in the third triad of rows are related to those in the second as those in the second are related to those in the first, and that consequently the required sum of squares of resultants is, by the multiplication-theorem itself, expressible as a resultant, viz.,

$$\Sigma\Big\|\,z_1x_2\,|,|\,x_1y_3\,|\Big\|^2 = \Sigma|\,z_1x_2\,|^2\,.\,\Sigma|\,x_1y_2\,|^2 - (\Sigma|\,z_1x_2\,|\,|\,x_1y_2\,|)^2,$$

where the elements of the resultant on the right are sums of products of quantities in the second triad of rows. Then the same theorem is used to make a further step backwards, viz., to express each of these three sums of products of resultants as a resultant whose elements are sums of products of the quantities in the first triad of rows, the effect of the substitution being

$$
\begin{aligned}
\Sigma\Big\|\,z_1x_2\,|,|\,x_1y_3\,|\Big\|^2 = \{\Sigma z_1^2\Sigma x_1^2-(\Sigma z_1x_1)^2\}\{\Sigma x_1^2\Sigma y_1^2-(\Sigma x_1y_1)^2\}&\\
-\{\Sigma z_1x_1\Sigma x_1y_1-\Sigma y_1z_1\Sigma x_1^2\}^2.&
\end{aligned}
$$

Simple multiplication transforms this into

$$\Sigma x_1^2\left\{\begin{array}{r}\Sigma x_1^2\Sigma y_1^2\Sigma z_1^2 - \Sigma y_1^2(\Sigma z_1x_1)^2 - \Sigma z_1^2(\Sigma x_1y_1)^2\\ +2\Sigma y_1z_1\Sigma z_1x_1\Sigma x_1y_1 - \Sigma x_1^2(\Sigma y_1z_1)^2\end{array}\right\},$$

which, by still another use of the multiplication-theorem, we know is equal to

$$\Sigma x_1^2\,\Sigma|x_1y_2z_3|^2.$$

The set of six results of which this is one, is

$$\left.\begin{aligned}
\Sigma X_1^2 &= \Sigma x_1^2\ \Sigma|x_1y_2z_3|^2,\\
\Sigma Y_1^2 &= \Sigma y_1^2\ \Sigma|x_1y_2z_3|^2,\\
\Sigma Z_1^2 &= \Sigma z_1^2\ \Sigma|x_1y_2z_3|^2,\\
\Sigma Y_1Z_1 &= \Sigma y_1z_1\Sigma|x_1y_2z_3|^2,\\
\Sigma Z_1X_1 &= \Sigma z_1x_1\Sigma|x_1y_2z_3|^2,\\
\Sigma X_1Y_1 &= \Sigma x_1y_1\Sigma|x_1y_2z_3|^2,
\end{aligned}\right\} \qquad \text{(XXXV.)}$$

if, for shortness, we denote the quantities of the third triad of rows by

$$\begin{matrix} X_1, & X_2, & \dots .\\ Y_1, & Y_2, & \dots .\\ Z_1, & Z_2, & \dots . \end{matrix}$$

Following these, and deduced by means of them, is an equally noteworthy theorem regarding the sums of squares of all the resultants of the third order, which can be formed from the quantities of the second triad of rows. Denoting these quantities temporarily by

$$\begin{matrix} \xi_1, & \xi_2, & \dots .\\ \eta_1, & \eta_2, & \dots .\\ \zeta_1, & \zeta_2, & \dots . \end{matrix}$$

we know (XXXII.) that

$$\begin{aligned}
3\Sigma\,|\,\xi_1\eta_2\zeta_3\,|^2 = \Sigma X_1^2\Sigma\xi_1^2 + \Sigma Y_1^2\Sigma\eta_1^2 + \Sigma Z_1^2\Sigma\zeta_1^2\\
+ 2\Sigma Y_1Z_1.\Sigma\eta_1\zeta_1 + 2\Sigma Z_1X_1.\Sigma\zeta_1\xi_1\\
+ 2\Sigma X_1Y_1.\Sigma\xi_1\eta_1\,;
\end{aligned}$$

whence, by using the set of six results just obtained, we have

$$\begin{aligned}
&3\Sigma\,|\,\xi_1\eta_2\zeta_3\,|^2\\
&= \Sigma\,|x_1y_2z_3|^2 \begin{Bmatrix} \Sigma\xi_1^2\Sigma x_1^2 + \Sigma\eta_1^2\Sigma y_1^2 + \Sigma\zeta_1^2\Sigma z_1^2 \\ + 2\Sigma\eta_1\zeta_1.\Sigma y_1z_1 + 2\Sigma\zeta_1\xi_1.\Sigma z_1x_1 + 2\Sigma\xi_1\eta_1.\Sigma x_1y_1 \end{Bmatrix}
\end{aligned}$$

and therefore, again by (XXXIII.)

$$\Sigma\,|\,\xi_1\eta_2\zeta_3\,|^2 = \{\Sigma\,|\,x_1y_2z_3\,|^2\}^2. \qquad \text{(XXXVI.)}$$

It is finally pointed out that from the third triad of rows there might, in like manner, be formed a fourth triad, and analogous identities obtained; also that, instead of starting with three rows, we might start with *four*,

$$\begin{array}{llll} t_1, & t_2, & t_3, & \ldots\ldots, t_n \\ x_1, & x_2, & x_3, & \ldots\ldots, x_n \\ y_1, & y_2, & y_3, & \ldots\ldots, y_n \\ z_1, & z_2, & z_3, & \ldots\ldots, z_n, \end{array}$$

form from them other four

$$\begin{array}{l} |x_1y_2z_3|, \ldots\ldots\ldots\ldots \\ |y_1z_2t_3|, \ldots\ldots\ldots\ldots \\ |z_1t_2x_3|, \ldots\ldots\ldots\ldots \\ |t_1x_2y_3|, \ldots\ldots\ldots\ldots, \end{array}$$

thence in the same way a third four, and in connection therewith establish the identity (XXXI. 2)

$$\Sigma t_1 \Sigma |x_1y_2z_3| - \Sigma x_1 \Sigma |y_1z_2t_3| + \Sigma y_1 \Sigma |z_1t_2x_3| - \Sigma z_1 \Sigma |t_1x_2y_3| = 0$$

and other analogues. (XXXII. 2+XXXV. 2)

The rest of the memoir, 52 pages, consists of geometrical applications of the series of theorems thus obtained.

CAUCHY (1812).

[Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment. *Journ. de l'Ec. Polyt.*, x. Cah. 17, pp. 29–112. *Œuvres* (2) i.]

This masterly memoir of 84 pages was read to the Institute on the same day (30th November) as Binet's memoir, of which we have just given an account. The coincidence of date has to be carefully noted, because the memoirs have in part a common ground, and because there is a presumption that the authors, knowing this beforehand, had, in a friendly way, arranged for simultaneous publicity. Binet's words on the matter are (IX. p. 281)—

"Ayant eu dernièrement occasion de parler à M. Cauchy, ingénieur des ponts et chaussées, du théorème général que j'ai énoncé ci-dessus, il me dit être parvenu, dans des recherches analogues à celles de M. Gauss, à des théorèmes d'analyse qui devaient avoir rapport aux miens. Je m'en suis assuré, en jetant les yeux sur ces formules: mais j'ignore si elles ont la même généralité que les miennes: nous y sommes arrivés, je crois, par des voies très-différentes."

And Cauchy's corroboration is (p. 111)—

"J'avais rencontré l'été dernier, à Cherbourg, où j'étais fixé par les travaux de mon état, ce théorème et quelques autres du même genre, en cherchant à généraliser les formules de M. Gauss. M. Binet, dont je me félicite d'être l'ami, avait été conduit aux mêmes résultats par des recherches différentes. De retour à Paris, j'étais occupé de poursuivre mon travail, lorsque j'allai le voir. Il me montra son théorème qui était semblable au mien. Seulement il désignait sous le nom de *résultante* ce que j'avais appelé *déterminant*."

Cauchy prefaces his memoir by another, entitled

Sur le nombre des valeurs qu'une fonction peut acquérir lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme.

This latter must to a certain extent be taken into account, because it serves to show the point of view which he considered most natural for examining the subject, and also the exact position held by the functions now called determinants, when functions in general come to be classified according to the number of values they are able to assume in certain circumstances.

At the outset of it the writings of Lagrange, Vandermonde, and Ruffini are referred to; the fact is recalled that the maximum number of values which a function can acquire by interchanges among its n variables is $1.2.3 \ldots . n$; also that when the maximum is not obtained, the actual number must be a factor of the maximum; and then proof is given of the very notable theorem that *the number of values cannot be less than the greatest prime contained in* n *without being equal to* 2. It is pointed out likewise that functions capable of having only two values are known from Vandermonde to be constructible for any number of variables. For example, the number of variables being three, a_1, a_2, a_3, all that is needed is to form their difference-product

$$(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$$

or $$a_3^2 a_2 + a_2^2 a_1 + a_1^2 a_3 - (a_3^2 a_1 + a_2^2 a_3 + a_1^2 a_2),$$

when it is found that either of the parts

$$a_3^2 a_2 + a_2^2 a_1 + a_1^2 a_3,$$

or $$a_3^2 a_1 + a_2^2 a_3 + a_1^2 a_2,$$

is an instance of a function capable of only two values by permutation of the variables; the result indeed of any permutation being merely that the one function passes into the other. Further, the whole expression

$$a_3^2 a_2 + a_2^2 a_1 + a_1^2 a_3 - (a_3^2 a_1 + a_2^2 a_3 + a_1^2 a_2)$$

is another example, the difference between the two values which it can assume being however a difference of sign merely. As a reference to the title of the memoir of November 1812 will show, it is functions of this latter class which Cauchy there considers.

At the commencement he contrasts them with functions which suffer no change whatever by permutation of variables, that is to say, *symmetric* functions: and, noting the fact, afterwards ascertained, that the new functions consist of terms alternately + and −, and that were it not for this alternation of sign they would be symmetric functions, he decides to extend the term "symmetric" to them, and having done so, seeks to distinguish them from ordinary symmetric functions by calling them "fonctions symétriques alternées," and calling the other "fonctions symétriques permanentes." Cauchy's view of determinants may therefore now be described by saying that he considered them as *a special class of alternating symmetric functions.*

To include them, however, either the adoption of a convention is necessary, or an extension of the definition must be made. For example, $a_1b_2 - a_2b_1$ is not an alternating function, unless the elements be so related that the interchange of a_1 and a_2 necessitates the interchange of b_1 and b_2 at the same time; or unless the definition be so worded that interchange shall refer to *suffixes*, not to letters. Cauchy selects the former course his words being (p. 30)

"... . concevons les diverses suites de quantités

$$\begin{array}{llll} a_1, & a_2, & \ldots\ldots, & a_n \\ b_1, & b_2, & \ldots\ldots, & b_n \\ c_1, & c_2, & \ldots\ldots, & c_n \\ \ldots & & \ldots & \ldots \end{array}$$

tellement liées entre elles, que la transposition de deux indices pris dans l'une des suites, nécessite la même transposition dans toutes les autres; alors, les quantités

$$b_1,\ c_1,\ \ldots,\ b_2,\ c_2,\ \ldots,\ b_3,\ c_3,\ \ldots\ldots$$

pourront être considérées comme des fonctions semblables de

$$a_1,\ a_2,\ a_3,\ \ldots\ldots;$$

et par suite, les fonctions de

$$a_1,\ b_1,\ c_1,\ \ldots,\ a_2,\ b_2,\ c_2,\ \ldots,\ a_n,\ b_n,\ c_n,\ \ldots\ldots$$

qui ne changeront pas de valeur, mais tout au plus de signe, en vertu de transpositions opérées entre les indices 1, 2, 3, ... n, devront être rangées parmi les fonctions symétriques de $a_1, a_2, \ldots, a_n$, ou, ce qui revient au même, des indices $1, 2, 3, \ldots, n$. Ainsi

$$\begin{aligned} &a_1^2 + a_2^2 + 4a_1a_2, \\ &a_1b_1 + a_2b_2 + a_3b_3 + 2c_1c_2c_3, \\ &a_1b_2 + a_2b_3 + a_3b_1 + a_2b_1 + a_3b_2 + a_1b_3, \\ &\cos(a_1 - a_2)\cos(a_1 - a_3)\cos(a_2 - a_3), \end{aligned}$$

seront des fonctions symétriques permanentes, la première du second ordre et les autres du troisième; et au contraire,

$$\begin{aligned} &a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_1b_3 - a_3b_2, \\ &\sin(a_1 - a_2)\sin(a_1 - a_3)\sin(a_2 - a_3) \end{aligned}$$

seront des fonctions symétriques alternées du troisième ordre."

The question of nomenclature being settled there next arises the question of notation. This also is decided on the ground of the resemblance of the functions to symmetric functions. It being known that any symmetric function is representable by a typical term preceded by a symbol indicating permutation of the variables, *e.g.*

$$\mathrm{S}(a_1b_2) \text{ or } \mathrm{S}^2(a_1b_2) \text{ standing for } a_1b_2 + a_2b_1$$

and $\mathrm{S}^3(a_1b_2)$ standing for $a_1b_2 + a_2b_3 + a_3b_1 + a_2b_1 + a_3b_2 + a_1b_3$;

also, that any non-symmetric function may be taken as the typical term of a symmetric function, the question arises whether the like may not be true of alternating functions. A lengthy examination of the latter point leads to the conclusion that any non-symmetric function K *cannot* be the originating or typical term of an alternating function unless it satisfies a certain condition, viz., that it be such that any value of it obtained by an even number of interchanges of indices will be different from any other value obtained by an odd number of interchanges. Should, however, this condition be satisfied, and K_α, K_β, K_γ, ... be all the values of the former kind, and K_λ, K_μ, K_ν, ... all the values of the latter kind, then

$$(K_\alpha+K_\beta+K_\gamma+ \ldots .) - (K_\lambda+K_\mu+K_\nu+ \ldots .)$$

is an alternating function and is appropriately representable by

$$\mathbf{S}(\pm K)$$

if the indices appearing in K alone are to be permuted, and by

$$\mathbf{S}^n(\pm K)$$

if the indices to be permuted be 1, 2, 3, . . . , n. For example, taking the typical term a_1b_2 we have

$$S(\pm a_1b_2) = a_1b_2-a_2b_1,$$

and
$$S^3(\pm a_1b_2) = a_1b_2+a_2b_3+a_3b_1-a_2b_1-a_3b_2-a_1b_3,$$
$$= S^3(\mp a_2b_1) = S^3(\mp a_1b_3) = \ldots .$$

$S^4(\pm a_1b_2)$ is an impossibility, as when there are four indices a_1b_2 does not satisfy the condition required of a typical term; indeed, Cauchy notes that the number of indices in any term must either be the total number or 1 less.

The number of permutations being even, it is clear that *the number of + terms* K_α, K_β, ... *is the same as the number of negative terms* K_λ, K_μ, (x. 2)
a generalisation of a remark of Vandermonde's.

Further, since K_α, K_β, ... are all the terms that arise from an even number of transpositions, and K_λ, K_μ, ... all those that arise from an odd number of transpositions, it is plain that any single transposition performed upon each of the terms of the function

$$(K_\alpha+K_\beta+K_\gamma+\ldots\ldots)-(K_\lambda+K_\mu+K_\nu+\ldots\ldots)$$

must change it into

$$(K_\lambda+K_\mu+K_\nu+\ldots\ldots)-(K_\alpha+K\ +K_\gamma+\ldots\ldots)$$

—this is, in fact, the proof that it is an alternating function—consequently each of the parts

$$K_\alpha+K_\beta+K_\gamma+\ldots\ldots,$$
$$K_\lambda+K\ +K_\nu+\ldots\ldots,$$

belongs to the class of functions which have only two different values.

Also it is evident that *if throughout the function any particular index be changed into another and no further alteration made, the resulting expression must be equal to zero,* (XII. 5) a theorem regarding alternating functions which is the generalisation of a theorem of Vandermonde's.

We have lastly to note, that the criterion which determines whether a particular K belongs to the class K_α, K_β, ... or to the class K_λ, K_μ, ... is incidentally shown to be reducible to a more practical form. For example, if the term be K_θ, and it be derivable from K, say, by the change of the suffixes 1, 2, 3, 4, 5, 6, 7 into 3, 2, 6, 5, 4, 1, 7, that is to say, in Cauchy's language by means of the substitution

$$\begin{pmatrix}1, 2, 3, 4, 5, 6, 7\\3, 2, 6, 5, 4, 1, 7\end{pmatrix},$$

we transform this substitution into a "product" of "circular" substitutions, viz., into

$$\begin{pmatrix}1, 3, 6\\3, 6, 1\end{pmatrix}\cdot\begin{pmatrix}4\ 5\\5\ 4\end{pmatrix}\cdot\begin{pmatrix}2\\2\end{pmatrix}\cdot\begin{pmatrix}7\\7\end{pmatrix}$$

and subtracting the number of "factors," 4, from the total number of suffixes 7, make the sign + or − according as this difference is even or odd.

Here the subject of general alternating functions may be left for the present. What remains of the first part of the memoir, refers to special cases, which naturally fall to be considered in another chapter of our history. At the close of the part Cauchy says (p. 51)—

"Je vais maintenant examiner particulièrement une certaine espèce de fonctions symétriques alternées qui s'offrent d'elles-mêmes dans un grand nombre de recherches analytiques. C'est au moyen de ces fonctiòns qu'on exprime les valeurs générales des inconnues que renferment plusieurs équations du premier degré. Elles se représentent toutes les fois qu'on a des équations à former, ainsi que dans la théorie générale de l'élimination."

The writings of Laplace, Vandermonde, Bezout, and Gauss are referred to, and from the latter the name "déterminant" is adopted.

The second part bears the title—

Des fonctions symétriques alternées désignees sous le nom de déterminans. (xv. 4)

and opens with the following explanatory definition (p. 51)—

"Soient $a_1, a_2, \ldots, a_n$ plusieurs quantités différentes en nombre égal à n. On a fait voir ci-dessus qu'en multipliant le produit de ces quantités, ou

$$a_1 a_2 a_3 \ldots . a_n,$$

par le produit de leurs différences respectives, ou par

$$(a_2 - a_1)(a_3 - a_1) \ldots (a_n - a_1)(a_3 - a_2) \ldots (a_n - a_2) \ldots (a_n - a_{n-1}),$$

on obtenait pour résultat la fonction symétrique alternée

$$S(\pm a_1{}^1 a_2{}^2 a_3{}^3 \ldots . a_n{}^n),$$

qui par conséquent se trouve toujours égale au produit

$$a_1 a_2 a_3 \ldots . a_n$$
$$\times (a_2 - a_1)(a_3 - a_1) \ldots (a_n - a_1)(a_3 - a_2) \ldots (a_n - a_2) \ldots (a_n - a_{n-1}).$$

Supposons maintenant que l'on développe ce dernier produit, et que dans chaque terme du développement on remplace l'exposant de chaque lettre par un second indice égal à l'exposant dont il s'agit, en écrivant par exemple $a_{r,s}$ au lieu de $a_r{}^s$, et $a_{s,r}$ au lieu de $a_s{}^r$, on obtiendra pour résultat une nouvelle fonction symétrique alternée, qui, au lieu d'être représentée par

$$S(\pm a_1{}^1 a_2{}^2 a_3{}^3 \ldots . a_n{}^n)$$

sera représentée par

$$S(\pm a_{1\cdot1} a_{2\cdot2} a_{3\cdot3} \ldots . a_{n\cdot n}),$$

le signe S étant relatif aux premiers indices de chaque lettre. Telle est la forme la plus générale des fonctions que je désignerai dans la suite sous le nom de *déterminans.* Si l'on suppose successivement *

$$n=1,\ n=2,\ \&c.\ \ldots\ldots$$

on trouvera

$$\begin{aligned} S(\pm a_{1\cdot1}a_{2\cdot2}) &= a_{1\cdot1}a_{2\cdot2} - a_{2\cdot1}a_{1\cdot2},\\ S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}) &= a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} + a_{2\cdot1}a_{3\cdot2}a_{1\cdot3} + a_{3\cdot1}a_{1\cdot2}a_{2\cdot3}\\ &\quad - a_{1\cdot1}a_{3\cdot2}a_{2\cdot3} - a_{3\cdot1}a_{2\cdot2}a_{1\cdot3} - a_{2\cdot1}a_{1\cdot2}a_{3\cdot3},\\ \&c.\ \ldots\ldots\ldots \end{aligned}$$

pour les déterminans du second, du troisième ordre, &c. . . . . "

In regard to this it is important to notice that there are really two definitions given us. The latter, viz., that involved in the symbolism of alternating functions,

$$S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}\ldots\ldots a_{n\cdot n})$$

contains nothing more than Leibnitz's rule of formation and an improved rule of signs. The former is new and may be paraphrased as follows:—

If the multiplications indicated in the expression

$$a_1a_2a_3\ \ldots\ a_n$$
$$\times(a_2-a_1)(a_3-a_1)\ldots(a_n-a_1)(a_3-a_2)\ldots(a_n-a_2)\ldots(a_n-a_{n-1})$$

be performed, and in the result every index of a power be changed into a second suffix, e.g., a_r^s into $a_{r,s}$, the expression so obtained is called a determinant, (III. 18), (VIII. 2)
and is denoted by $S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}\ \ldots\ldots\ a_{n\cdot n})$. (VII. 5)

In this definition the rule of signs and the rule of term-formation are inseparable—a peculiarity already observed in the case of Bezout's rule of 1764.

After the definitions various technical terms are introduced. The n^2 different quantities involved in

$$S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}\ \ldots\ldots\ a_{n\cdot n})$$

are arranged thus

$$\left\{\begin{array}{l} a_{1\cdot1},\ a_{1\cdot2},\ a_{1\cdot3},\ \ldots\ldots\ a_{1\cdot n}\\ a_{2\cdot1},\ a_{2\cdot2},\ a_{2\cdot3},\ \ldots\ldots\ a_{2\cdot n}\\ a_{3\cdot1},\ a_{3\cdot2},\ a_{3\cdot3},\ \ldots\ldots\ a_{3\cdot n}\\ \&c.\ \ldots\ldots\\ a_{n\cdot1},\ a_{n\cdot2},\ a_{n\cdot3},\ \ldots\ldots\ a_{n\cdot n} \end{array}\right.$$

* $n=2$, $n=3$, &c. is meant.

"sur un nombre égal à n de lignes horizontales et sur autant de colonnes verticales," and as thus arranged are said to form a *symmetric system* of order n. The individual quantities $a_{1\cdot1}$, &c., are called the *terms* of the system, and the letter a when free of suffixes the *characteristic.* The "terms" in a horizontal line are said to form a *suite horizontale,* in a vertical column a *suite verticale. Conjugate* terms are defined as those whose suffixes ("indices") differ in order, *e.g.*, $a_{2\cdot3}$ and $a_{3\cdot2}$; and terms which are self-conjugate, *e.g.*, $a_{1\cdot1}$, $a_{2\cdot2}$, . . . are called *principal* terms. The determinant is said to *belong* to the system, or to be the determinant *of* the system. The parts of the expanded determinant which are connected by the signs + and − are called *symmetric products,* and the product

$$a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}\ldots a_{n\cdot n}$$

of the principal "terms" is called the *principal* product. The "principal product," however, is also called the *terme indicatif* of the determinant, and thus an awkward double use of the word "terme" is brought into prominence. The system

$$\left\{\begin{matrix} a_{1\cdot1} & a_{2\cdot1} & a_{3\cdot1} & \ldots & a_{n\cdot1} \\ a_{1\cdot2} & a_{2\cdot2} & a_{3\cdot2} & \ldots & a_{n\cdot2} \\ a_{1\cdot3} & a_{2\cdot3} & a_{3\cdot3} & \ldots & a_{n\cdot3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1\cdot n} & a_{2\cdot n} & a_{3\cdot n} & \ldots & a_{n\cdot n} \end{matrix}\right.$$

derived from the previous system by interchanging the suffixes of each "terme" is said to be *conjugate* to the previous system. A symbol for each of these systems is got by taking the last "terme" of its first "suite horizontale," and enclosing the "terme" in brackets: in this way we are enabled to say that $(a_{1\cdot n})$ and $(a_{n\cdot1})$ *are conjugate systems.*

In the course of these explanations a modification of the rule of term-formation is incidentally noted, the form taken being specially applicable when the quantities of the system have been disposed in a square. Cauchy's wording of this now familiar rule is (p. 55)—

. . . . "pour former chacun des termes dont il s'agit, il suffira de multiplier entre elles n quantités différentes prises respectivement dans les différentes colonnes verticales du système, et situées en même temps dans les diverses lignes horizontales de ce système." (II. 6)

Here we may note in passing that the disposal of the "termes" in a square might have enabled Cauchy to point out (which he did not do) the difference between Gauss' use of the word "determinant" and his own, by saying that the "determinant of a form" had its conjugate "termes" equal.

The rule of signs applicable to alternating functions in general is modified for the special case of determinants, and takes the following form (p. 56):—

"Étant donné un produit symétrique quelconque, pour obtenir le signe dont il est affecté dans le déterminant

$$S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}\ldots a_{n\cdot n})$$

il suffira d'appliquer la règle qui sert à déterminer le signe d'un terme pris à volonté dans une fonction symétrique alternée. Soit

$$a_{\alpha\cdot1}\,a_{\beta\cdot2}\ldots a_{\zeta\cdot n}$$

le produit symétrique dont il s'agit, et désignons par g le nombre des substitutions circulaires équivalentes à la substitution

$$\begin{pmatrix}1 & 2 & 3 & \ldots\ldots & n\\ \alpha & \beta & \gamma & \ldots\ldots & \zeta\end{pmatrix}.$$

Ce produit devra être affecté du signe + si $n-g$ est un nombre pair, et du signe − dans le cas contraire." (III. 19)

Thus if the sign of the term

$$a_{6\cdot1}\,a_{8\cdot2}\,a_{3\cdot3}\,a_{1\cdot4}\,a_{9\cdot5}\,a_{2\cdot6}\,a_{5\cdot7}\,a_{4\cdot8}\,a_{7\cdot9}$$

in the determinant

$$S(\pm a_{1\cdot1}\,a_{2\cdot2}\,a_{3\cdot3}\ldots a_{9\cdot9}),$$

be wanted, we write the series of first suffixes 6, 8, . . . under the corresponding suffixes of the "principal product," that is to say, under the series 1, 2, 3, . . . , 9, obtaining the substitution

$$\begin{pmatrix}1&2&3&4&5&6&7&8&9\\6&8&3&1&9&2&5&4&7\end{pmatrix};$$

this we separate into circular substitutions, finding them three in number, viz.,

$$\begin{pmatrix}3\\3\end{pmatrix},\ \begin{pmatrix}5&7&9\\9&5&7\end{pmatrix},\ \begin{pmatrix}1&2&4&6&8\\6&8&1&2&4\end{pmatrix};$$

and the determinant being of the 9th order, we thence conclude that the desired sign is $(-)^{9-3}$, *i.e.*, $+$. In connection with this subject a modification of Cramer's rule is given, no reference being made to "dérangements" at all. Put into the fewest possible words it is—*The sign of the term* $a_{\alpha\cdot1}\,a_{\beta\cdot2}\ldots a_{\zeta\cdot n}$ *is the same as the sign of the difference-product of the first suffixes, that is, the sign of*

$$(\beta-\alpha)(\gamma-\alpha)\ldots\ldots(\zeta-\alpha)(\gamma-\beta)\ldots\ldots \qquad \text{(III. 20)}$$

For example, the sign of

$$a_{6\cdot1}\,a_{8\cdot2}\,a_{3\cdot3}\,a_{1\cdot4}\,a_{9\cdot5}\,a_{2\cdot6}\,a_{5\cdot7}\,a_{4\cdot8}\,a_{7\cdot9}\,,$$

above sought, is the sign of the difference-product of

$$6,\ 8,\ 3,\ 1,\ 9,\ 2,\ 5,\ 4,\ 7$$

i.e., the sign of

$$\begin{array}{r}(7-4)(7-5)(7-2)(7-9)(7-1)(7-3)(7-8)(7-6)\\ \times(4-5)(4-2)\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (4-6)\\ \times(5-2)\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (5-6)\\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\ \times(8-6)\end{array}$$

The object which Cauchy had in view in stating the rule in this unnecessarily complex form was doubtless to show its essential identity with the rule implied in his new definition. He says (p. 58)—

"On démontre facilement cette règle par ce qui précède, attendu qu'une transposition opérée entre deux indices change toujours, comme on l'a fait voir, le signe du produit

$$(a_\beta-a_\alpha)(a_\gamma-a_\alpha)\ldots(a_\zeta-a_\alpha)(a_\gamma-a_\beta)\ldots\ldots,$$

et par conséquent celui du produit

$$(\beta-\alpha)(\gamma-\alpha)\ldots(\zeta-\alpha)(\gamma-\beta)\ldots\ldots"$$

The way having thus been prepared, the propositions of determinants are entered on. Those known to his predecessors we may dispose of rapidly, giving little, if anything, more than the enunciation of them, in order that the new garb in which they appear may be seen.

. . . . "le déterminant du système $(a_{n\cdot 1})$ est égal à celui du système $(a_{1\cdot n})$ . . . . En conséquence, dans l'expression

$$S(\pm a_{1\cdot 1}a_{2\cdot 2} \ldots . a_{n\cdot n})$$

on peut supposer indifféremment, ou que le signe S se rapporte aux premiers indices, ou qu'il se rapporte aux seconds. (IX. 2)

Si l'on échange entre elles deux suites horizontales ou deux suites verticales du système $(a_{1\cdot n})$ de manière à faire passer dans une des suites tous les termes de l'autre et réciproquement on obtiendra un nouveau système symétrique, dont le déterminant sera évidemment égal mais de signe contraire à celui du système $(a_{1\cdot n})$. Si l'on répète la même opération plusieurs fois de suite, on obtiendra divers systèmes symétriques dont les déterminans seront égaux entre eux, mais alternativement positifs et négatifs. On peut faire la même remarque à l'égard du systéme $(a_{n\cdot 1})$. (XI. 3)

. . . . . si l'on développe la fonction symétrique alternée

$$S[\pm a_{n\cdot n}S(\pm a_{1\cdot 1}\, a_{2\cdot 2} \ldots . a_{n-1\cdot n-1})]$$

tous les termes du développement seront des produits symétriques de l'ordre n, qui auront l'unité pour coefficient. Ces termes seront donc respectivement égaux à ceux qu'on obtient en développant le déterminant

$$D_n = S(\pm a_{1\cdot 1}\, a_{2\cdot 2} \ldots . a_{n\cdot n});$$

et comme le produit principal $a_{1\cdot 1}\, a_{2\cdot 2} \ldots a_{n\cdot n}$ est positif de part et d'autre, on aura nécessairement

$$\begin{aligned} D_n &= S[\pm a_{n\cdot n}S(\pm a_{1\cdot 1}\, a_{2\cdot 2} \ldots . a_{n-1\cdot n-1})] \qquad \text{(VI. 3)} \\ &= a_{n\cdot n}b_{n\cdot n} + a_{n-1\cdot n}b_{n-1\cdot n} + \ldots . + a_{1\cdot n}b_{1\cdot n}. \end{aligned}$$

En général, si l'on désigne par μ l'un des indices 1, 2, 3, . . . , n on trouvera de la même manière

$$D_n = S[\pm a_{\mu\cdot\mu}S(\pm a_{1\cdot 1}\, a_{2\cdot 2} \ldots . a_{\mu-1\cdot\mu-1}a_{\mu+1\cdot\mu+1} \ldots . a_{n\cdot n})]. \qquad \text{(VI. 4)}$$

. . . . . Cette dernière équation

$$0 = a_{1\cdot\nu}b_{1\cdot\mu} + a_{2\cdot\nu}b_{2\cdot\mu} + \ldots + a_{n\cdot\nu}b_{n\cdot\mu} \qquad \text{(XII. 6)}$$

sera satisfaite toutes les fois que ν et μ seront deux nombres différens l'un de l'autre.

. . . . on aura donc aussi

$$D_n = a_{\mu\cdot 1}b_{\mu\cdot 1} + a_{\mu\cdot 2}b_{\mu\cdot 2} + \ldots . + a_{\mu\cdot n}b_{\mu\cdot n} \qquad \text{(VI. 4)}$$

$$0 = a_{\nu\cdot 1}b_{\mu\cdot 1} + a_{\nu\cdot 2}b_{\mu\cdot 2} + \ldots . + a_{\nu\cdot n}b_{\mu\cdot n} \qquad \text{(XII. 6)}$$

les indices μ et ν étant censés inégaux."

The expressions here denoted by $b_{1\cdot1}, b_{1\cdot2}, \ldots$ are spoken of as *adjugate* ("adjointes") to $a_{1\cdot1}, a_{1\cdot2}, \ldots$; and the system

$$\begin{cases} b_{1\cdot1} & b_{1\cdot2} & \ldots & b_{1\cdot n} \\ b_{2\cdot1} & b_{2\cdot2} & \ldots & b_{2\cdot n} \\ & \ldots\ldots & & \\ b_{n\cdot1} & b_{n\cdot2} & \ldots & b_{n\cdot n} \end{cases}$$

as adjugate to the system $(a_{1\cdot n})$. Similarly the system $(b_{n\cdot1})$ is said to be adjugate to the system $(a_{n\cdot1})$; and, on the other hand, it is said to be *adjugate and conjugate* to the system $(a_{1\cdot n})$. (XXVII. 2)

Up to this point no new property has been brought forward. The following paragraph (p. 68), however, opens new ground, the formula given in it being of some considerable importance in the after development of the theory.

"Si dans le système de quantités $(a_{1\cdot n})$ on supprime la dernière suite horizontale et la dernière suite verticale, on aura le système suivant,

$$\begin{cases} a_{1\cdot1}, & a_{2\cdot1} \ldots\ldots & a_{1\cdot n-1}, \\ a_{2\cdot1}, & a_{2\cdot2} & a_{2\cdot n-1}, \\ \&c. \ldots\ldots & & \\ a_{n-1\cdot1}, & a_{n-1\cdot2} & a_{n-1\cdot n-1}, \end{cases}$$

que je désignerai à l'ordinaire par $(a_{1\cdot n-1})$.

"Soit maintenant $(e_{1\cdot n-1})$ le système adjoint au précédent. Si dans l'équation (13) on change b en e et n en $n-1$, on aura en général

$$D_{n-1} = b_{n\cdot n} = a_{\mu\cdot1}e_{\ \cdot1} + a_{\mu\cdot2}e_{\mu\cdot2} + \ldots + a_{\mu\cdot n-1}e_{\mu\cdot n-1}.$$

Pour déduire de cette dernière équation la valeur de $b_{\mu\cdot n}$, il suffira en vertu des règles établies, de changer $a_{\mu\cdot\nu}$ en $a_{n\cdot\nu}$ dans l'expression précédente de $b_{n\cdot n}$, et de changer en outre le signe du second membre: ou aura donc généralement

$$b_{\mu\cdot n} = -(a_{n\cdot1}e_{\mu\cdot1} + a_{n\cdot2}e_{\mu\cdot2} + \ldots\ldots + a_{n\cdot n-1}e_{\mu\cdot n-1}).$$

Si dans cette équation on donne successivement à μ toutes les valeurs entières depuis 1 jusqu'à $n-1$, et que l'on substitue les valeurs qui en résulteront pour $b_{1\cdot n}, b_{2\cdot n}, \ldots, b_{n-1\cdot n}$ dans l'équation

$$D_n = a_{1\cdot n}b_{1\cdot n} + a_{2\cdot n}b_{2\cdot n} + \ldots\ldots + a_{n\cdot n}b_{n\cdot n},$$

on obtiendra la formule suivante,

$$D_n = a_{n\cdot n}b_{n\cdot n} - \begin{cases} a_{1\cdot n}a_{n\cdot1}e_{1\cdot1} & + a_{2\cdot n}a_{n\cdot2}e_{2\cdot2} + \ldots + a_{n-1\cdot n}a_{n\cdot n-1}e_{n-1\cdot n-1} \\ + a_{1\cdot n}(a_{n\cdot2}e_{1\cdot2} & + a_{n\cdot3}e_{1\cdot3} \quad + \ldots + a_{n\cdot n-1}e_{1\cdot n-1}) \\ + a_{2\cdot n}(a_{n\cdot1}e_{2\cdot1} & + a_{n\cdot3}e_{2\cdot3} \quad + \ldots + a_{n\cdot n-1}e_{2\cdot n-1}) \\ + \&c. \ldots\ldots & \\ + a_{n-1\cdot n}(a_{n\cdot1}e_{n-1\cdot1} & + a_{n\cdot2}e_{n-1\cdot2} \quad + \ldots + a_{n\cdot n-2}e_{n-1\cdot n-2}). \end{cases}$$

Cette équation peut être mise sous la forme

$$D_n = a_{n\cdot n}D^*_{n\cdot 1} - S^{n-1}S^{n-1}(a_{\nu\cdot n}a_{n\cdot\mu}\epsilon_{\nu\cdot\mu}), \qquad \text{(XXXVII.)}$$

les deux signes S étant relatifs le premier à l'indice μ et le second à l'indice ν."

This is the well-known formula nowadays described as giving the development of a determinant according to binary products of a row and column. The special row here used is the n^{th} and the special column the n^{th} likewise.

The four pages regarding the application of determinants to the solution of a set of simultaneous equations may be passed over with the remark that they give evidence of the importance attached by Cauchy to his new definition of determinants, the solution in the case of the example

$$\left.\begin{array}{l} a_1x_1+b_1x_2 = m_1 \\ a_2x_1+b_2x_2 = m_2 \end{array}\right\}$$

being first put in the form

$$x = \frac{mb(b-m)}{ab(b-a)}, \quad y = \frac{am(m-a)}{ab(b-a)};$$

and similarly in the case of the example

$$a_rx_1+b_rx_2+c_rx_3 = m_r \qquad (r=1, 2, 3).$$

The determinant solution of a set of simultaneous equations is put to good use by Cauchy to obtain new properties of the functions. Taking the set of equations

$$(20)\ \begin{cases} a_{1\cdot1}x_1 + a_{1\cdot2}x_2 + \ldots\ldots + a_{1\cdot n}x_n = m_1 \\ a_{2\cdot1}x_1 + a_{2\cdot2}x_2 + \ldots\ldots + a_{2\cdot n}x_n = m_2 \\ \&c.\ \ldots\ldots \\ a_{n\cdot1}x_1 + a_{n\cdot2}x_2 + \ldots\ldots + a_{n\cdot n}x_n = m_n \end{cases}$$

and solving for $x_1, x_2, \ldots$ he obtains of course the set

$$\left.\begin{array}{l} m_1b_{1\cdot1} + m_2b_{2\cdot1} + \ldots\ldots + m_nb_{n\cdot1} = D_nx_1, \\ m_1b_{1\cdot2} + m_2b_{2\cdot2} + \ldots\ldots + m_nb_{n\cdot2} = D_nx_2, \\ \&c.\ \ldots\ldots\ldots \\ m_1b_{1\cdot n} + m_2b_{2\cdot n} + \ldots\ldots + m_nb_{n\cdot n} = D_nx_n, \end{array}\right\}$$

* Misprint in original, for D_{n-1}.

where $b_{1\cdot1}, b_{2\cdot1}, \ldots\ldots$ have the signification above indicated, and D_n stands for $S(\pm a_{1\cdot1}a_{2\cdot2} \ldots a_{n\cdot n})$. This second set may be treated in the same way as the first set, the quantities $m_1, m_2, \ldots, m_n$ being viewed as the unknowns. To express the result the system of quantities adjugate to $(b_{1\cdot n})$ is denoted by $(c_{1\cdot n})$, and the determinant of the system $(b_{1\cdot n})$ is denoted by B_n, the new set thus being

$$(27)\quad \begin{cases} c_{1\cdot1}D_nx_1 + c_{1\cdot2}D_nx_2 + \ldots\ldots + c_{1\cdot n}D_nx_n = B_nm_1, \\ c_{2\cdot1}D_nx_1 + c_{2\cdot2}D_nx_2 + \ldots\ldots + c_{2\cdot n}D_nx_n = B_nm_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{n\cdot1}D_nx_1 + c_{n\cdot2}D_nx_2 + \ldots\ldots + c_{n\cdot n}D_nx_n = B_nm_n, \end{cases}$$

Cauchy then proceeds (p. 77)—

"Les équations (27) peuvent encore être mises sous la forme suivante,

$$\begin{cases} c_{1\cdot1}\dfrac{D_n}{B_n}x_1 + c_{1\cdot2}\dfrac{D_n}{B_n}x_2 + \ldots\ldots + c_{1\cdot n}\dfrac{D_n}{B_n}x_n = m_1, \\ c_{2\cdot1}\dfrac{D_n}{B_n}x_1 + c_{2\cdot2}\dfrac{D_n}{B_n}x_2 + \ldots\ldots + c_{2\cdot n}\dfrac{D_n}{B_n}x_n = m_2, \\ \&c. \ldots\ldots\ldots \\ c_{n\cdot1}\dfrac{D_n}{B_n}x_1 + c_{n\cdot2}\dfrac{D_n}{B_n}x_2 + \ldots\ldots + c_{n\cdot n}\dfrac{D_n}{B_n}x_n = m_n; \end{cases}$$

et comme celles-ci doivent avoir lieu en même temps que les équations (20), sans que l'on suppose d'ailleurs entre les termes de la suite $x_1, x_2, \ldots, x_n$ et ceux du système $(a_{1\cdot n})$ aucune relation particulière, il faudra nécessairement que l'on ait, quels que soient μ et ν,

$$c_{\mu\cdot\nu}\frac{D_n}{B_n} = a_{\mu\cdot\nu},$$

ou

$$c_{\mu\cdot\nu} = \frac{B_n}{D_n}a_{\mu\cdot\nu}. \qquad \text{(XXXVIII.)}$$

Cette équation établit un rapport constant entre les termes du système $(a_{1\cdot n})$ et les termes du système adjoint du second ordre $(c_{1\cdot n})$."

More definitely, and in more modern nomenclature, the theorem is

The ratio of any element of a determinant to the corresponding element of the second adjugate determinant is equal to the ratio of the determinant itself to its first adjugate. (XXXVIII.)

Attention is next directed to the group of equations—

$$\left.\begin{array}{lll}
\alpha_{1\cdot1}a_{1\cdot1}+\alpha_{1\cdot2}a_{1\cdot2}+\,.\,.\,+\alpha_{1\cdot n}a_{1\cdot n}=m_{1\cdot1} & \alpha_{2\cdot1}a_{1\cdot1}+\alpha_{2\cdot2}a_{1\cdot2}+\,.\,.\,+\alpha_{2\cdot n}a_{1\cdot n}=m_{1\cdot2}, \quad .\;.\;.\;. & \alpha_{n\cdot1}a_{1\cdot1}+\alpha_{n\cdot2}a_{1\cdot2}+\,.\,.\,+\alpha_{n\cdot n}a_{1\cdot n}=m_{1\cdot n} \\
\alpha_{1\cdot1}a_{2\cdot1}+\alpha_{1\cdot2}a_{2\cdot2}+\,.\,.\,+\alpha_{1\cdot n}a_{2\cdot n}=m_{2\cdot1} & \alpha_{2\cdot1}a_{2\cdot1}+\alpha_{2\cdot2}a_{2\cdot2}+\,.\,.\,+\alpha_{2\cdot n}a_{2\,n}=m_{2\cdot2}, \quad .\;.\;.\;. & \alpha_{n\cdot1}a_{2\cdot1}+\alpha_{n\cdot2}a_{2\,2}+\,.\,.\,+\alpha_{n\cdot n}a_{2\cdot n}=m_{2\cdot n} \\
.\quad.\quad.\quad.\quad.\quad. & .\quad.\quad.\quad.\quad.\quad.\quad.\quad. & .\quad.\quad.\quad.\quad.\quad.\quad. \\
\alpha_{1\cdot1}a_{n\cdot1}+\alpha_{1\cdot2}a_{n\cdot2}+\,.\,.\,+\alpha_{1\cdot n}a_{n\cdot n}=m_{n\cdot1} & \alpha_{2\cdot1}a_{n\cdot1}+\alpha_{2\cdot2}a_{n\cdot2}+\,.\,.\,+\alpha_{2\cdot n}a_{n\cdot n}=m_{n\cdot2}, \quad .\;.\;.\;. & \alpha_{n\cdot1}a_{n\cdot1}+\alpha_{n\cdot2}a_{n\,2}+\,.\,.\,+\alpha_{n\cdot n}a_{n\cdot n}=m_{n\cdot n}
\end{array}\right\}$$

Here there are three symmetric systems of quantities

$$(a_{1\cdot n}),\ (\alpha_{1\cdot n}),\ (m_{1\cdot n}),$$

the first appearing in every column of equations, the second in every row, and the third only once. The determinants of these systems are denoted by

$$\mathrm{D}_n,\ \delta_n,\ \mathrm{M}_n,$$

respectively: that is to say

$$\begin{aligned}\mathrm{D}_n &= \mathrm{S}(\pm\ a_{1\cdot 1}\, a_{2\cdot 2}\ .\ .\ .\ a_{n\cdot n})\\ \delta_n &= \mathrm{S}(\pm\ \alpha_{1\cdot 1}\, \alpha_{2\cdot 2}\ .\ .\ .\ \alpha_{n\cdot n})\\ \mathrm{M}_n &= \mathrm{S}(\pm m_{1\cdot 1} m_{2\cdot 2}\ .\ .\ .\ m_{n\cdot n}).\end{aligned}$$

If now in

$$\mathrm{S}(\pm m_{1\cdot 1} m_{2\cdot 2}\ .\ .\ .\ m_{n\cdot n})$$

there be substituted for $m_{1\cdot 1}, m_{1\cdot 2}, \ldots$ their values as given by the group of equations, there will be obtained a function of all the a's and α's, which must be an alternating function with respect to the first indices of the a's and also with respect to the first indices of the α's. Further, since each of the m's is of the first degree in the a's and of the first degree also in the α's, each term of the development of $\mathrm{S}(\pm m_{1\cdot 1} m_{2\cdot 2} \ldots m_{n\cdot n})$ must evidently be of the form

$$\pm\, a_{1\cdot\mu} a_{2\cdot\nu} \ldots a_{n\cdot\pi} \alpha_{1\cdot\mu} \alpha_{2\cdot\nu} \ldots \alpha_{n\cdot\pi}.$$

But the development by reason of its double alternating character cannot contain such a term without containing all the terms of the product

$$\pm\,\mathrm{S}(\pm\, a_{1\cdot\mu} a_{2\cdot\nu} \ldots a_{n\cdot\pi})\mathrm{S}(\pm\, \alpha_{1\cdot\mu} \alpha_{2\cdot\nu} \ldots \alpha_{n\cdot\pi}).$$

Consequently it must equal one or more products of this kind. But again the indices $\mu, \nu, \ldots, \pi$ are either all different or not. If they be different, we have

$$\mathrm{S}(\pm\, a_{1\cdot\mu} a_{2\cdot\nu} \ldots a_{n\cdot\pi}) = \pm\,\mathrm{S}(\pm\, a_{1\cdot 1} a_{2\cdot 2} \ldots a_{n\cdot n}) = \pm\,\delta;$$

and if any two of them be equal

$$\mathrm{S}(\pm\, a_{1\cdot\mu} a_{2\cdot} \ldots a_{n\cdot\pi}) = 0.$$

The like is true in regard to $\mathrm{S}\pm(\alpha_{1\cdot\mu} \alpha_{2\cdot\nu} \ldots \alpha_{n\cdot\pi})$. This enables us to conclude that the development of M_n is equal to one or more products of the form

$$\pm D_n \delta_n ;$$

in other words, that

$$M_n = cD_n \delta_n ,$$

where c is a constant. But if we take the very special case where

$$a_{\mu \cdot \mu} = 1, \ \ \alpha_{\mu \cdot \mu} = 1, \ \ a_{\mu \cdot \nu} = 0, \ \ \alpha_{\mu \cdot \nu} = 0,$$

and where consequently

$$m_{\mu \cdot \mu} = 1, \ \ m_{\mu \cdot \nu} = 0,$$

we see that

$$M_n = 1, \ \ D_n = 1, \ \ \delta_n = 1,$$

and that therefore

$$c = 1.$$

Hence the final result is

$$M_n = D_n \delta_n . \qquad \text{(XVII. 5)}$$

This, the now well-known multiplication-theorem of determinants, Cauchy puts in words as follows (p. 82):—

> *Lorsqu'un système de quantités est déterminé symétriquement au moyen de deux autres systèmes, le déterminant du système résultant est toujours égal au produit des déterminans des deux systèmes composans.* (XVII. 5)

It is quite clear, from what has been said above, that it was discovered independently, and about the same time, by Binet and Cauchy, and ought to bear the names of both. Binet has the further merit of having reached a theorem of which Cauchy's is a special case, and then made an additional generalisation in a different direction; and Cauchy has the advantage over Binet of having produced, along with his special case, a satisfactory proof of it.

From the theorem Cauchy goes on to deduce several results equally important. Substituting for the system $(a_{1 \cdot n})$ the system $(b_{1 \cdot n})$ adjugate to $(a_{1 \cdot n})$ so that

$$\delta_n = S(\pm b_{1 \cdot 1} b_{2 \cdot 2} \ldots b_{n \cdot n}) = B_n ,$$

we know that then

$$m_{\mu \cdot \mu} = D_n \ \text{ and } \ m_{\mu \cdot \nu} = 0 ;$$

that consequently M_n consists of but a single term, viz.

$$m_{1\cdot 1}m_{2\cdot 2}\ldots m_{n\cdot n}, \quad \textit{i.e.} \quad D_n^n;$$

and that therefore by the theorem

$$D_n^n = B_n D_n,$$

whence $$B_n = D_n^{n-1}. \qquad \text{(XXI. 2)}$$

This result, afterwards so well known, Cauchy translates into words as follows (p. 82):—

> . . . . *le déterminant du système* $(b_{1\cdot n})$ *adjoint au système* $(a_{1\cdot n})$ *est égal à la* $(n-1)^{me}$ *puissance du déterminant de ce dernier système.* (XXI. 2)

Again, by returning to the identity

$$c_{\mu\cdot\nu} = \frac{B_n}{D_n} a_{\mu\cdot\nu},$$

and substituting the value of B_n just obtained, there is deduced the result

$$c_{\mu\cdot\nu} = D_n^{n-2} a_{\mu\cdot\nu}; \qquad \text{(XXXVIII. 2)}$$

or, in words,

> . . . *étant donné un terme quelconque* $a_{\mu\cdot\nu}$ *du système* $(a_{1\cdot n})$, *pour obtenir le terme correspondant du système adjoint du second ordre* $(c_{1\cdot n})$ *il suffira de multiplier le terme donné par la* $(n-2)^{me}$ *puissance du déterminant du premier système.*

A considerable amount of space (pp. 82–92) is devoted to the consideration of the adjugate systems of

$$(a_{1\cdot n}), \ (a_{1\cdot n}), \ (m_{1\cdot n}),$$

and the adjugates of these adjugates; but nothing new is elicited. The section closes with the manifest identity

$$\begin{aligned}
& (a_{1\cdot 1} + a_{2\cdot 1} + \ldots + a_{n\cdot 1})(a_{1\cdot 1} + a_{2\cdot 1} + \ldots + a_{n\cdot 1}) \\
& + (a_{1\cdot 2} + a_{2\cdot 2} + \ldots + a_{n\cdot 2})(a_{1\cdot 2} + a_{2\cdot 2} + \ldots + a_{n\cdot 2}) \\
& + \&c. \ldots\ldots. \\
& + (a_{1\cdot n} + a_{2\cdot n} + \ldots + a_{n\cdot n})(a_{1\cdot n} + a_{2\cdot n} + \ldots + a_{n\cdot n}) \\
= \ & m_{1\cdot 1} + m_{2\cdot 1} + \ldots + m_{n\cdot 1} \\
& + m_{1\cdot 2} + m_{2\cdot 2} + \ldots + m_{n\cdot 2} \\
& + \ldots.. \\
& + m_{1\cdot n} + m_{2\cdot n} + \ldots + m_{n\cdot n},
\end{aligned}$$

which, using later technical terms, we may express as follows:—

If there be two determinants, and the sum of the elements of one first column be multiplied by the sum of the elements of the other first column, the sum of the elements of one second column by the sum of the elements of the other second column, and so on, then the sum of these products is equal to the sum of the elements of the product of the two determinants. (XXXIX.)

The third section breaks entirely fresh ground, its heading being

Des Systèmes de Quantités dérivées et de leurs Déterminans.

Of the integers 1, 2, 3, . . . , n all the possible sets of p integers are supposed to be taken, and arranged in order on the principle that any one has precedence of any other if the product of the members of the former be less than the product of the members of the latter. The number $n(n-1)\ldots.(n-p+1)/1.2.3\ldots.p$ of the said sets being denoted by P, the P$^{\text{th}}$ and last set would thus be

$$n-p+1,\ n-p+2,\ \ldots\ldots,\ n-1,\ n.$$

Now, any two of the sets being fixed upon, say the μ^{th} and ν^{th}, the system of quantities $(a_{1\cdot n})$ is returned to, and from it are deleted (1) all the "termes" whose first index is not found in the μ^{th} set, and (2) all the "termes" whose second index is not found in the ν^{th} set. What is left after this action is clearly "un système de quantités symétriques de l'ordre p," the determinant of which may be denoted by $a^{(p)}_{\mu,\nu}$. For example, if $\mu=\nu=1$, all the a's would be deleted whose first or second index was not included in the set 1, 2, 3, . . . , p, and there would be left the system

$$\begin{cases} a_{1\cdot1} & a_{1\cdot2} & \ldots. & a_{1\cdot p} \\ a_{2\cdot1} & a_{2\cdot2} & \ldots. & a_{2\cdot p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p\cdot1} & a_{p\cdot2} & \ldots. & a_{p\cdot p} \end{cases}$$

of which the determinant would be denoted by

$$a^{(p)}_{1\cdot1}.$$

As any one of the P sets could be taken along with any other, preparatory to forming such a determinant, there would necessarily be in all P × P possible determinants. Arranged in a square as follows:—

$$\begin{cases} a_{1\cdot1}^{(p)} & a_{1\cdot2}^{(p)} & \cdots\cdots & a_{1\cdot\mathrm{P}}^{(p)} \\ a_{2\cdot1}^{(p)} & a_{2\cdot2}^{(p)} & \cdots\cdots & a_{2\cdot\mathrm{P}}^{(p)} \\ \cdots\cdots & \cdots\cdots & & \\ a_{\mathrm{P}\cdot1}^{(p)} & a_{\mathrm{P}\cdot2}^{(p} & \cdots\cdots & a_{\mathrm{P}\cdot\mathrm{P}}^{(p)} \end{cases}$$

they manifestly form "un système symétrique de l'ordre P," which, in strict accordance with previous convention, is denoted by

$$\left(a_{1\cdot\mathrm{P}}^{(p)}\right).$$

Cauchy then proceeds (p. 96)—

Si l'on donne successivement à p toutes les valeurs

$$1, 2, 3, \ldots\ldots, n-3, n-2, n-1$$

P prendra les valeurs suivantes,

$$n,\ \frac{n(n-1)}{1\,.\,2},\ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3},\ \ldots,\ \frac{n(n-1)}{1\,.\,2},\ n,$$

et l'on obtiendra par suite un nombre égal à $n-1$ de systèmes symétriques différens les uns des autres, dont le premier sera le système donné $(a_{1\cdot n})$. Ces différens systèmes seront désignés respectivement par

$$(a_{1\cdot n}),\quad \left[a_{1\cdot\frac{n(n-1)}{1\,.\,2}}^{(2)}\right],\quad \left[a_{1\cdot\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}}^{(3)}\right],\ \ldots\ldots$$
$$\left[a_{1\cdot\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}}^{(n-3)}\right],\quad \left[a_{1\cdot\frac{n(n-1)}{1\,.\,2}}^{(n-2)}\right],\quad \left(a_{1.n}^{(n-1)}\right); \qquad \text{(XL.)}$$

je les appellerai *systèmes dérivés* de $(a_{1\cdot n})$. Parmi ces systèmes, ceux qui correspondent à des valeurs de p dont la somme est égale à n sont toujours de même ordre; je les appellerai *systèmes dérivés complémentaires.* Ainsi en général

$$\left(a_{1.\mathrm{P}}^{(p)}\right) \quad \text{et} \quad \left(a_{1.\mathrm{P}}^{n-p}\right)$$

sont deux systèmes dérivés complémentaires l'un de l'autre, dont l'ordre est égal a

$$\mathrm{P} = \frac{n(n-1)\ldots\ldots(n-p+1)}{1\,.\,2\,.\,3\ldots p}.$$

Up to this point a thorough understanding of the notation

$$\left(a_{1\cdot\mathrm{P}}^{(p)}\right)$$

is the one essential. Taking the particular instance

$$\left(a_{1\cdot10}^{(2)}\right)$$

we first call to mind that it is an abbreviation for the "système symétrique" whose first row has for its last "terme" the determinant

$$a^{(2)}_{1\cdot 10}$$

—that is to say, an abbreviation for the system whose determinant we should nowadays write in the form

$$\begin{vmatrix} a^{(2)}_{1.1} & a^{(2)}_{1.2} & \ldots & a^{(2)}_{1.10} \\ a^{(2)}_{2.1} & a^{(2)}_{2.2} & \ldots & a^{(2)}_{2.10} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a^{(2)}_{10.1} & a^{(2)}_{10.2} & \ldots & a^{(2)}_{10.10} \end{vmatrix}.$$

The next point is to realise what determinants are denoted by

$$a^{(2)}_{1.1},\quad a^{(2)}_{1.2},\quad \ldots\ldots\ldots$$

Now the number 10 being of necessity a combinatorial, and, as the figure in brackets above it indicates, of the form

$$\frac{n(n-1)}{1\,.\,2},$$

we see that n must be 5, and that the said determinants are all derived from

$$\begin{vmatrix} a_{1\cdot 1} & a_{1\cdot 2} & a_{1\cdot 3} & a_{1\cdot 4} & a_{1\cdot 5} \\ a_{2\cdot 1} & a_{2\cdot 2} & a_{2\cdot 3} & a_{2\cdot 4} & a_{2\cdot 5} \\ a_{3\cdot 1} & a_{3\cdot 2} & a_{3\cdot 3} & a_{3\cdot 4} & a_{3\cdot 5} \\ a_{4\cdot 1} & a_{4\cdot 2} & a_{4\cdot 3} & a_{4\cdot 4} & a_{4\cdot 5} \\ a_{5\cdot 1} & a_{5\cdot 2} & a_{5\cdot 3} & a_{5\cdot 4} & a_{5\cdot 5} \end{vmatrix}.$$

The details of the process of derivation are recalled in connection with the interpretation of the pairs of suffixes. A requisite preliminary is to form all the different pairs of the numbers 1, 2, 3, 4, 5; arrange them in the order

12, 13, 14, 15, 23, 24, 25, 34, 35, 45;

and then number them

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

These last are the numbers from which the suffixes are taken, and what each one as a suffix refers to, is the combination under which it is here placed. For example, the first suffix in $a^{(2)}_{1.1}$ refers to the combination 1 2, and implies the deletion of all the rows of the above determinant of the fifth order except the 1st and

2nd; the second suffix refers to the same combination, and implies the deletion of all the columns except the 1st and 2nd; and the symbol as a whole thus comes to stand for

$$\begin{vmatrix} a_{1\cdot1} & a_{1\cdot2} \\ a_{2\cdot1} & a_{2\cdot2} \end{vmatrix}. \qquad \text{(XL. 2)}$$

Interpreting $a^{(2)}_{1.2}$, $a^{(2)}_{1.3}$, ... in the same way, we see that

$$(a^{(2)}_{1.10})$$

is a compact notation for the system of which the determinant is

$$\begin{vmatrix} |a_{1\cdot1}a_{2\cdot2}|, & . & . & . & . & . & . & . & . & . & . & . & . & |a_{1\cdot4}a_{2\cdot5}| \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & |a_{3\cdot4}a_{4\cdot5}| \\ |a_{3\cdot1}a_{5\cdot2}|, & . & . & . & , & |a_{3\cdot3}a_{5\cdot4}|, & & |a_{3\cdot3}a_{5\cdot5}|, & & & & & & |a_{3\cdot4}a_{5\cdot5}| \\ |a_{4\cdot1}a_{5\cdot2}|, & . & . & . & , & |a_{4\cdot3}a_{5\cdot4}|, & & |a_{4\cdot3}a_{5\cdot5}|, & & & & & & |a_{4\cdot4}a_{5\cdot5}| \end{vmatrix}.$$

Similarly $$(a^{(3)}_{1.10})$$

stands for the system of which the determinant is

$$\begin{vmatrix} |a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}|, & . & . & . & . & . & . & . & . & . & . & . & |a_{1\cdot3}a_{2\cdot4}a_{3\cdot5}| \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & |a_{2\cdot3}a_{3\cdot4}a_{5\cdot5}| \\ |a_{2\cdot1}a_{4\cdot2}a_{5\cdot3}|, & . & . & . & . & . & . & , & |a_{2\cdot2}a_{4\cdot4}a_{5\cdot5}|, & & & & |a_{2\cdot3}a_{4\cdot4}a_{5\cdot5}| \\ |a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}|, & . & . & . & . & . & . & , & |a_{3\cdot2}a_{4\cdot4}a_{5\cdot5}|, & & & & |a_{3\cdot3}a_{4\cdot4}a_{5\cdot5}| \end{vmatrix}.$$

and which is called the "complementary derived system." (XL. 3) To every "terme" of the latter there corresponds a "terme" of the former, the one "terme" consisting exactly of those a's of the original determinant which are awanting in the other. This relationship Cauchy goes on to mark by means of a name and a notation. He calls two such "termes," $|a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}|$ and $|a_{1\cdot4}a_{2\cdot5}|$ for example, "termes complémentaires des deux systèmes;" (XL. 4) and if the symbol for the one be by previous agreement

$$a^{(p)}_{\mu.\pi}$$

the symbol for the other is made *

$$a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}. \qquad \text{(XL. 5)}$$

* If Cauchy had adopted a slightly different principle for determining the order of combinations, the μ^{th} combination of p things and the $(\mathrm{P}-\mu+1)^{\text{th}}$ combination of $n-p$ things would have been mutually exclusive, and the convention here made in regard to notation would have been unnecessary.

As for the signs of the "termes" in "derived systems," Cauchy's words are (p. 98)—

"En général, il est facile de voir que le produit de deux termes complémentaires pris à volonté est toujours, au signe près, une portion de ce même déterminant (D_n). Cela posé, étant donné le signe de l'un de ces deux termes, on déterminera celui de l'autre par la condition que leur produit soit affecté de même signe que la portion correspondante du déterminant D_n."

All these preliminaries having been settled, the weighty matters of the section are entered on. The first of these is a complete and perfectly accurate statement of the expansion-theorem, known by the name of Laplace, but which, as we have seen, Laplace and even Bezout who followed him were very far from fully formulating. The passage is of the greatest interest. No better example could be chosen to illustrate the powerful grasp which Cauchy had of the subject. What Laplace and Bezout laboured at, lengthily expounding one special case after another, Cauchy sets forth with ease and in all its generality in the space of a page. His words are (p. 99)—

"On a fait voir dans le § 3^e que la fonction symétrique alternée

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3}\ldots a_{n\cdot n}) = D_n$$

était équivalente à celle-ci

$$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n-1\cdot n-1})\,.\,a_{n\cdot n}].$$

On fera voir de même qu'elle est encore équivalente à

$$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p})\,.\,S(\pm a_{p+1\cdot p+1}\ldots a_{n-1\cdot n-1}a_{n\cdot n}],$$

les opérations indiquées par le signe S pouvant être considérées comme relatives, soit aux premiers, soit aux seconds indices. On a d'ailleurs par ce qui précède

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) = \pm a_{1.1}^{(p)},$$

$$S(\pm a_{p+1\cdot p+1}\ldots a_{n\cdot n}) = \pm a_{\mathrm{P.P}}^{(n-p)}.$$

Enfin les signes des quantités de la forme

$$a_{1.1}^{(p)}\quad,\quad a_{\mathrm{P.P}}^{(n-p)}$$

doivent être tels que les produits semblables à

$$a_{1.1}^{(p)}a_{\mathrm{P.P}}^{(n-p)}$$

soient dans le déterminant D_n affectés du signe +. Cela posé, il résulte de l'équation

$$D_n = S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) \,.\, S(\pm_{\,p+1\,\cdot\,p+1}\ldots a_{n\cdot n})],$$

que D_n est la somme de plusieurs produits de la forme

$$a_{1.1}^{(p)}\,a_{\mathrm{P.P}}^{(n-p)}.$$

Selon que pour obtenir ces différens produits on échangera entre eux les premiers ou les seconds indices du système $(a_{1\cdot n})$, on trouvera ou l'équation

$$D_n = a_{1.1}^{p}\,a_{\mathrm{P.P}}^{(n-p)} + a_{2.1}^{(p)}\,a_{\mathrm{P-1.P}}^{(n-p)} + \ldots + a_{\mathrm{P.1}}^{(p)}\,a_{\mathrm{1.P}}^{(n-p)},$$

ou celle-ci

$$D_n = a_{1.1}^{p}\,a_{\mathrm{P.P}}^{(n-p)} + a_{1.2}^{(p)}\,a_{\mathrm{P.P-1}}^{(n-p)} + \ldots + a_{\mathrm{1.P}}^{(p)}\,a_{\mathrm{P.1}}^{(n-p)}.$$

On aura de même en général les deux équations

$$D_n = a_{1.\pi}^{(p)}\,a_{\mathrm{P.P}-\pi+1}^{(n-p)} + a_{2.\pi}^{(p)}\,a_{\mathrm{P-1.P}-\pi+1}^{(n-p)} + \ldots + a_{\mathrm{P}.\pi}^{(p)}\,a_{\mathrm{1.P}-\pi+1}^{(n-p)},$$

$$D_n = a_{\mu.1}^{(p)}\,a_{\mathrm{P}-\mu+1.\mathrm{P}}^{(n-p)} + a_{\mu.2}^{(p)}\,a_{\mathrm{P}-\mu+1.\mathrm{P}}^{(n-p)} + \ldots + a_{\mu.\mathrm{P}}^{(p)}\,a_{\mathrm{P}-\mu+1.1}^{(n-p)}.$$

Ces deux équations sont comprises dans la suivante

$$D_n = S^{\mathrm{P}}\left(a_{\mu.\pi}^{(p)}\,a_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}^{(n-p)}\right), \qquad \text{(XIV. 4)}$$

qui a lieu également, soit que l'on considère le signe S comme relatif à l'indice μ, soit qu'on le considère comme relatif à l'indice π."

Taking as an illustration the case where $n=5$, $p=2$, and $\pi=7$ (that is, the ordinal number corresponding to the pair 2 5, of the suffixes 1, 2, 3, 4, 5), and translating literally from Cauchy's notation into our own, we have

$$|a_{11}a_{22}a_{33}a_{44}a_{55}| = |a_{12}a_{25}|.|a_{31}a_{43}a_{54}| - |a_{12}a_{35}|.|a_{21}a_{43}a_{54}| + \ldots$$
$$\ldots\ldots + |a_{42}a_{55}|.|a_{11}a_{23}a_{34}|.$$

With the same certainty of touch and with still greater conciseness, all the identities directly obtainable by Bezout's *Méthode pour trouver des fonctions . . . . qui soient zéro par elles-mêmes*, are formulated as one general identity, and established on a proper basis. The paragraph is (p. 100)—

"D_n étant une fonction symétrique alternée des indices du système $(a_{1\cdot n})$ doit se réduire à zéro, lorsqu'on y remplace un de ces indices par un autre. Si l'on opère de semblables remplacemens à l'égard des indices qui occupent la première place dans le système $(a_{1\cdot n})$, et qui entrent dans la combinaison (μ), cette même combinaison se trouvera transformée en une autre que je désignerai par (ν), et $a_{\mu.\pi}^{(p)}$ sera changé en $a_{\nu.\pi}^{(p)}$. D'ailleurs, en supposant le signe S relatif à π, on a

$$D_n = S^P\left(a_{\mu.\pi}^{(p)}\, a_{P-\mu+1.P-\pi+1}^{(n-p)}\right);$$

on aura donc par suite

$$0 = S^P\left(a_{\nu.\pi}^{(p)}\, a_{P-\mu+1.P-\pi+1}^{(n-p)}\right). \qquad \text{(XII. 7; XXIII. 4)}$$

On aurait de même, en supposant le signe S relatif à l'indice μ, et en désignant par (τ) une nouvelle combinaison différente de (π)

$$0 = S^P\left(a_{\mu.\tau}^{(p)}\, a_{P-\mu+1.P-\pi+1}^{(n-p)}\right)." \qquad \text{(XII. 7; XXIII. 4)}$$

As this theorem is twin with the preceding, it is best to illustrate it by the same special case. By so doing, indeed, both theorems become more readily grasped and their details better understood. Taking then as before $n=5$, $p=2$ and $\pi=7$, we first form the determinants which Cauchy would have denoted by

$$a_{1.7}^{(2)}\ ,\ a_{2.7}^{(2)}\ ,\ \ldots\ldots\ ,\ a_{10.7}^{(2)},$$

and which we denote by

$$|a_{12}a_{25}|\ ,\ |a_{12}a_{35}|\ ,\ \ldots\ldots\ ,\ |a_{42}a_{55}|.$$

Next, for cofactors, we form the determinants which are complementary, not of these, as in the preceding theorem, but of the members of one of the nine other groups corresponding to the values 1, 2, 3, 4, 5, 6, 8, 9, 10 of π,—say the group

$$a_{1.6}^{(2)}\ ,\ a_{2.6}^{(2)}\ ,\ \ldots\ldots\ ,\ a_{10.6}^{(2)}.$$

These complementaries being

$$|a_{31}a_{43}a_{55}|\ ,\ |a_{21}a_{43}a_{55}|\ ,\ \ldots\ldots\ ,\ |a_{11}a_{23}a_{35}|,$$

we have the desired identity

$$0 = |a_{12}a_{25}|\cdot|a_{31}a_{43}a_{55}| - |a_{12}a_{35}|\cdot|a_{21}a_{43}a_{55}| \ldots\ldots + |a_{42}a_{55}|\cdot|_{11}a_{23}a_{35}|,$$

the right-hand side of which is nothing more than an expansion of the zero determinant which arises from the determinant $|a_{11}a_{22}a_{33}a_{44}a_{55}|$ "lorsqu'on y remplace un des indices par un autre," viz., the second 4 by 5.

With the help of these two theorems a third theorem of almost equal importance is derived, viz., regarding the product of the determinants of two complementary systems. Denoting the determinant of the system

$$\left(a_{1.P}^{(p)}\right) \text{ by } D_P^{(p)},$$

and that of the complementary system

$$\left(a_{1.\mathrm{P}}^{(n-p)}\right) \text{ by } \mathrm{D}_{\mathrm{P}}^{(n-p)},$$

and multiplying the two determinants together, we see with Cauchy that by (XIV. 4) the principal "termes" of the resulting determinant are each equal to

$$\mathrm{D}_n,$$

and by (XII. 7) all the other "termes" are equal to zero. Consequently

$$\mathrm{D}_{\mathrm{P}}^{(p)} . \mathrm{D}_{\mathrm{P}}^{(n-p)} = (\mathrm{D}_n)^{\mathrm{P}} \qquad \text{(XLI.)}$$

As an example of this theorem, it may be added that the product of the two determinants printed above (p. 114) to illustrate the notation

$$\left(a_{1.\mathrm{P}}^{(p)}\right),$$

that is to say, the determinants of the systems

$$\left(a_{1.10}^{(2)}\right) , \left(a_{1.10}^{(3)}\right),$$

is equal to

$$| a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}a_{4\cdot4}a_{5\cdot5} |^{10}.$$

In connection with all the three theorems, the special case $p=1$, is given, so that their relation to previously well-known theorems (VI., XII., XXI.) may be noted. It is also pointed out, that when in the third theorem n is even and $p=\frac{1}{2}n$, the result takes the interesting form

$$\mathrm{D}_{\mathrm{P}}^{\left(\frac{n}{2}\right)} = (\mathrm{D}_n)^{\frac{\mathrm{P}}{2}}, \qquad \text{(XLI. 2)}$$

This brings us to the last section of the memoir, the fourth, bearing the heading

Des Systèmes d'Équations dérivées et de leur Déterminans.

What it is concerned with is the relations subsisting between a "derived system" of the product-determinant

$$\begin{vmatrix} m_{1\cdot1} & m_{1\cdot2} & \cdots & m_{1\cdot n} \\ m_{2\cdot1} & m_{2\cdot2} & \cdots & m_{2\cdot n} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ m_{n\cdot1} & m_{n\cdot2} & \cdots & m_{n\cdot n} \end{vmatrix},$$

and the corresponding "derived systems" of the factors

$$\left|\begin{array}{cccc} a_{1\cdot1} & a_{1\cdot2} & \ldots & a_{1\cdot n} \\ a_{2\cdot1} & a_{2\cdot2} & \ldots & a_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n\cdot1} & a_{n\cdot2} & \ldots & a_{n\cdot n} \end{array}\right| , \quad \left|\begin{array}{cccc} a_{1\cdot1} & a_{1\cdot2} & \ldots & a_{1\cdot n} \\ a_{2\cdot1} & a_{2\cdot2} & \ldots & a_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n\cdot1} & a_{n\cdot2} & \ldots & a_{n\cdot n} \end{array}\right| ;$$

in other words, the relations which must connect the systems

$$\left(a^{(p)}_{1.\mathrm{P}}\right) , \ \left(a^{(p)}_{1.\mathrm{P}}\right) , \ \left(m^{(p)}_{1.\mathrm{P}}\right)$$

by reason of the relations

$$\Sigma[\mathrm{S}^n(a_{\nu\cdot1}\, a_{\mu\cdot1}) = m_{\mu\cdot\nu}]$$

(given in full above on p. 107) which connect the systems

$$(a_{1\cdot n}) , \ (a_{1\cdot n}) , \ (m_{1\cdot n}).$$

First of all, attention is concentrated on a single "terme" of the system

$$\left(m^{(p)}_{1.\mathrm{P}}\right),$$

or, as we should nowadays say, on a minor of the product-determinant. The process of reasoning, which occupies about four quarto pages, is exactly analogous to that previously followed in dealing with the product-determinant itself; and the result obtained is

$$m^{(p)}_{\mu.\nu} = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.1}\, a^{(p)}_{\mu.1}\right), \qquad \text{(XVIII. 5)}$$

where S^{P} is meant to indicate that the terms on the right-hand side are got by changing the second suffixes into 2, 3, 4, . . . , P in succession. Speaking roughly and in modern phraseology, we may say that this means that

Any minor of a product-determinant is expressible as a sum of products of minors of the two factors. (XVIII. 5)

Cauchy then proceeds (p. 107)—

"Si dans cette équation [XVIII. 5] on donne successivement à μ et à ν toutes les valeurs entières depuis 1 jusqu'à P, on aura un système d'équations symétriques de l'ordre P, que l'on pourra représenter par le symbole

(63) $$\Sigma\left\{\mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.1}\, a^{(p)}_{\mu.1}\right) = m^{(p)}_{\mu.\nu}\right\},$$

P étant toujours égal à

$$\frac{n(n-1) \ldots\ldots (n-p+1)}{1.2.3 \ldots p}.$$

Pour déduire des équations

$$\Sigma[S^n(a_{\nu.1} a_{\mu.1}) = m_{\mu.\nu}]$$

les équations (63), il suffit évidemment de remplacer les trois systèmes de quantités

$$(a_{1\cdot n}) \quad , \quad (a_{1\cdot n}) \quad , \quad (m_{1\cdot n})$$

par les systèmes dérivés de même ordre

$$\left(a_{1\cdot P}^{(p)}\right) \;,\; \left(a_{1.P}^{(p)}\right) \;,\; \left(m_{1.P}^{(p)}\right).$$

Je dirai pour cette raison que le second système d'équations est dérivé du premier." (XL. 6)

The close outward resemblance here noted between the original and the derived system of connecting equations is due of course to the choice of the notation

$$a_{1.P}^{(p)}$$

for the minors of the determinant

$$S \pm (a_{1\cdot 1} a_{2\cdot 2} \ldots a_{n\cdot n}),$$

and is so far a recommendation of that notation.

From the system of equations (63) two deductions follow immediately. In regard to the first Cauchy's words are (p. 108)—

"Désignons par

$$\delta_P^{(p)} \quad , \quad D_P^{(p)} \quad , \quad M_P^{(p)}$$

les déterminans des trois systèmes

$$\left(a_{1.P}^{(p)}\right) \;,\; \left(a_{1.P}^{(p)}\right) \;,\; \left(m_{1.P}^{(p)}\right);$$

on aura en vertu des équations (63)

$$(65) \qquad M_P^{(p)} = D_P^{(p)} \delta_P^{(p)}.\text{"} \qquad \text{(XLII.)}$$

The enunciation of this in modern phraseology would be—

Any compound of a product-determinant is equal to the product of the corresponding compounds of the two factors. (XLII.)

The next deduction is stated equally succinctly (p. 109)—

"Si l'on ajoute entre elles les équations (63) on aura la suivante,

$$(66) \qquad S^p\{S^p(a^{(p)}_{\mu.\nu})S^p(a^{(p)}_{\mu.\nu})\} = S^pS^p(m^{(p)}_{\mu.\nu}), \qquad (\text{XXX. } 2)$$

le premier signe S, c'est-à-dire le signe extérieur, étant relatif à l'indice ν, et les autres, c'est-à-dire les signes intérieurs, étant relatifs à l'indice μ."

This (66) corresponds to (XXXIX.) as (65) corresponds to the multiplication-theorem

$$M_n = D_n\delta_n,$$

the transition from the general to the particular being effected in both cases by putting $p=1$.

With these deductions, the 4th Section practically comes to an end; but one or two results, intentionally omitted in the account of the 2nd Section because they seemed to belong naturally to the 4th, fall now to be noted.

The first is very simple. It arises (p. 91) from observing that

$$(D_n)^{n-1}\times(\delta_n)^{n-1} = (D_n\delta_n)^{n-1},$$

$$\text{and } \therefore \qquad = (M_n)^{n-1}$$

by the multiplication-theorem. The result (XXI. 2) above (p. 110), is then thrice applied, and a theorem at once takes shape, which in later times we find enunciated as follows:—

The adjugate of the product-determinant is equal to the product of the adjugates of the two factors. (XLII. 2)

It is not noted, however, by Cauchy that this is but a case of XLIII., viz., where $p=n-1$.

The next is

$$\Sigma[S^n(m_{1\cdot\nu}\, b_{1\cdot\mu}) = D_n a_{\nu\cdot\mu}],$$

$$\text{or} \qquad \Sigma[S^n(m_{\mu\cdot1}\, \beta_{1\cdot\nu}) = \delta_n a_{\mu\cdot\nu}]. \qquad (\text{XLIII.})$$

It is nothing more than the result of solving the $n.n$ equations

$$(33) \quad \Sigma[S^n(a_{\nu\cdot1}\, a_{\mu\cdot1}) = m_{\mu\cdot\nu}]$$

first, in columns, for all the a's, and secondly, in rows, for all the α's.

The last is

$$\Sigma[S^n(a_{1\cdot\mu}\ r_{\nu\cdot 1}) = \delta_n b_{\nu\cdot\mu}],$$

or $$\Sigma[S^n(a_{1\cdot\nu}\ r_{1\cdot\mu}) = D_n\beta_{\mu\cdot\nu}] \qquad \text{(XLIII. 2)}$$

where $(r_{1\cdot n})$ is the system adjugate to $(m_{1\cdot n})$. It is obtained from the $n.n$ equations (XLIII.) just as they were obtained from the $n.n$ equations (33), use being made of the theorem

$$M_n = D_n\delta_n.$$

In concluding, Cauchy refers to Binet's researches on similar matters. Most of what he says in regard to them has already been given (see p. 93 above). The rest of it is as follows (p. 111):—

"Il [Binet] me dit en outre qu'il avait généralisé le théorème dont il s'agit [$M_n = D_n\delta_n$], en substituant au produit de deux résultantes des sommes de produits de même espèce. J'avais dès lors déjà démontré le théorème suivant :

D'un système quelconque d'équations symétriques on peut déduire cinq autres systèmes du même ordre ; mais on n'en saurait déduire un plus grand nombre.

J'ai démontré depuis à l'aide des méthodes précédentes cet autre théorème :

D'un système quelconque d'équations symétriques de l'ordre n, *on peut toujours déduire deux systèmes d'équations symétriques de l'ordre*

$$\frac{n(n-1)}{2},$$

deux systèmes d'équations symétriques de l'ordre

$$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}, \text{ \&c.} \quad . \quad . \quad . \quad . \quad .$$

En ajoutant entre elles les équations symétriques comprises dans un même système, on obtient, comme on l'a vu, les formules (50), (51) et (70) qui me paraissent devoir être semblables à celles dont M. Binet m'a parlé."

The last sentence here raises an important question for the historian to settle, viz., whether Cauchy is to share with Binet the credit of the generalisation of the multiplication-theorem. The identities on which the claim is based are—

$$S^n\{S^n(a_{\mu.\nu})\ S^n(a_{\mu.\nu})\} = S^nS^n(m_{\mu.\nu}) \qquad (50)$$

$$S^n\{S^n(\beta_{\mu.\nu})\ S^n(b_{\mu.\nu})\} = S^nS^n(r_{\mu.\nu}) \qquad (51)$$

$$S^p\{S^p(a^{(p)}_{\mu.\nu})S^p(a^{(p)}_{\mu.\nu})\} = S^pS^p(m^{(p)}_{\mu.\nu}) \qquad (70)$$

The first of these, given formerly (p. 110) in the uncontracted form

$$\begin{aligned}
&(a_{1\cdot1}+a_{2\cdot1}+\ldots+a_{n\cdot1})(a_{1\cdot1}+a_{2\cdot1}+\ldots+a_{n\cdot1})\\
&+(a_{1\cdot2}+a_{2\cdot2}+\ldots+a_{n\cdot2})(a_{1\cdot2}+a_{2\cdot2}+\ldots+a_{n\cdot2})\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+(a_{1\cdot n}+a_{2\cdot n}+\ldots+a_{n\cdot n})(a_{1\cdot n}+a_{2\cdot n}+\ldots+a_{n\cdot n})\\
=\;&m_{1\cdot1}+m_{2\cdot1}+\ldots+m_{n\cdot1}\\
&+m_{1\cdot2}+m_{2\cdot2}+\ldots+m_{n\cdot2}\\
&+\ldots\ldots\ldots\\
&+m_{1\cdot n}+m_{2\cdot n}+\ldots+m_{n\cdot n}
\end{aligned}$$

where $\quad m_{\mu.\nu}=a_{\mu\cdot1}a_{\nu\cdot1}+a_{\mu\cdot2}a_{\nu\cdot2}+\ldots+a_{\mu\cdot n}a_{\nu\cdot n},$

may be at once left out of consideration; it is not even a case of the multiplication-theorem. Cauchy, we may be sure, mentioned it only because it is the first of the series to which (51) and (70) belong. The next concerns the systems

$$(\beta_{1\cdot n})\;,\;(b_{1\cdot n})\;,\;(r_{1\cdot n})$$

adjugate to the systems

$$(a_{1\cdot n})\;,\;(a_{1\cdot n})\;,\;(m_{1\cdot n})$$

dealt with in (50). It indeed is comparable with Binet's theorem; but as it is only a case of (70),—the minors in (70) being of any order whatever, whereas in (51) they are the principal minors,—we may without loss pass it over. Directing our attention, then, to (70) let us for the sake of greater definiteness take the case where $n=5$ and $p=2$, and where consequently $\mathrm{P}=\frac{5\,.\,4}{1\,.\,2}=10$. The theorem then becomes

$$\mathrm{S}^{10}\big\{\mathrm{S}^{10}\big(a^{(2)}_{\mu.\nu}\big)\mathrm{S}^{10}\big(a^{(2)}_{\mu.\nu}\big)\big\}=\mathrm{S}^{10}\mathrm{S}^{10}\big(m^{(2)}_{\mu.\nu}\big).$$

For the purpose of comparison with Binet's result, it is absolutely necessary, however, to depart from this exceedingly condensed mode of statement. Remembering that the inner S's refer always to the first suffix, and the outer to the second suffix, we obtain the more developed form

$$
\begin{aligned}
&\left(\alpha^{(2)}_{1.1}+\alpha^{(2)}_{2.1}+\ .\ .\ .\ +\alpha^{(2)}_{10.1}\right)\left(a^{(2)}_{1.1}+a^{(2)}_{2.1}+\ .\ .\ .\ +a^{(2)}_{10.1}\right)\\
&+\left(\alpha^{(2)}_{1.2}+\alpha^{(2)}_{2.2}+\ .\ .\ .\ +\alpha^{(2)}_{10.2}\right)\left(a^{(2)}_{1.2}+a^{(2)}_{2.2}+\ .\ .\ .\ +a^{(2)}_{10.2}\right)\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
&+\left(\alpha^{(2)}_{1.10}+\alpha^{(2)}_{2.10}+\ .\ .\ .\ +\alpha^{(2)}_{10.10}\right)\left(a^{(2)}_{1.10}+a^{(2)}_{2.10}+\ .\ .\ .\ +a^{(2)}_{10.10}\right)\\
=\ &\ m^{(2)}_{1.1}+m^{(2)}_{2.1}+\ .\ .\ .\ +m^{(2)}_{10.1}\\
&+m^{(2)}_{1.2}+m^{(2)}_{2.2}+\ .\ .\ .\ +m^{(2)}_{10.2}\\
&+\ .\ .\ .\ .\ .\ .\\
&+m^{(2)}_{1.10}+m^{(2)}_{2.10}+\ .\ .\ .\ +m^{(2)}_{10.10}.
\end{aligned}
$$

Interpreting now the suffixes and superfixes of the α's, a's, and m's, after the manner already described,—any suffix r signifying along with the superfix (2) the r^{th} combination of two numbers taken from 1, 2, 3, 4, 5,—we finally reach the suitable form

$$
\begin{aligned}
&\left\{|\alpha_{1\cdot1}\alpha_{2\cdot2}| + |\alpha_{1\cdot1}\alpha_{3\cdot2}| + \cdot\cdot\cdot\cdot\cdot + |\alpha_{3\cdot1}\alpha_{4\cdot2}| + |\alpha_{3\cdot1}\alpha_{5\cdot2}| + |\alpha_{4\cdot1}\alpha_{5\cdot2}|\right\}\\
&\cdot\left\{|a_{1\cdot1}a_{2\cdot2}| + |a_{1\cdot1}a_{3\cdot2}| + \cdot\cdot\cdot\cdot\cdot + |a_{3\cdot1}a_{4\cdot2}| + |a_{3\cdot1}a_{5\cdot2}| + |a_{4\cdot1}a_{5\cdot2}|\right\}\\
&+\ .\\
&+\left\{|\alpha_{1\cdot3}\alpha_{2\cdot5}| + |\alpha_{1\cdot3}\alpha_{3\cdot5}| + \cdot\cdot\cdot\cdot\cdot + |\alpha_{3\cdot3}\alpha_{4\cdot5}| + |\alpha_{3\cdot3}\alpha_{5\cdot5}| + |\alpha_{4\cdot3}\alpha_{5\cdot5}|\right\}\\
&\cdot\left\{|a_{1\cdot3}a_{2\cdot5}| + |a_{1\cdot3}a_{3\cdot5}| + \cdot\cdot\cdot\cdot\cdot + |a_{3\cdot3}a_{4\cdot5}| + |a_{3\cdot3}a_{5\cdot5}| + |a_{4\cdot3}a_{5\cdot5}|\right\}\\
&+\left\{|\alpha_{1\cdot4}\alpha_{2\cdot5}| + |\alpha_{1\cdot4}\alpha_{3\cdot5}| + \cdot\cdot\cdot\cdot\cdot + |\alpha_{3\cdot4}\alpha_{4\cdot5}| + |\alpha_{3\cdot4}\alpha_{5\cdot5}| + |\alpha_{4\cdot4}\alpha_{5\cdot5}|\right\}\\
&\cdot\left\{|a_{1\cdot4}a_{2\cdot5}| + |a_{1\cdot4}a_{3\cdot5}| + \cdot\cdot\cdot\cdot\cdot + |a_{3\cdot4}a_{4\cdot5}| + |a_{3\cdot4}a_{5\cdot5}| + |a_{4\cdot4}a_{5\cdot5}|\right\}\\
=\ &\ |m_{1\cdot1}m_{2\cdot2}| + |m_{1\cdot1}m_{3\cdot2}| + \cdot\cdot\cdot + |m_{3\cdot1}m_{4\cdot2}| + |m_{3\cdot1}m_{5\cdot2}| + |m_{4\cdot1}m_{5\cdot2}|\\
&+\ .\\
&+|m_{1\cdot3}m_{2\cdot5}| + |m_{1\cdot3}m_{3\cdot5}| + \cdot\cdot\cdot + |m_{3\cdot3}m_{4\cdot5}| + |m_{3\cdot3}m_{5\cdot5}| + |m_{4\cdot3}m_{5\cdot5}|\\
&+|m_{1\cdot4}m_{2\cdot5}| + |m_{1\cdot4}m_{3\cdot5}| + \cdot\cdot\ + |m_{3\cdot4}m_{4\cdot5}| + |m_{3\cdot4}m_{5\cdot5}| + |m_{4\cdot4}m_{4\cdot5}|,
\end{aligned}
$$

where $\qquad m_{\mu\cdot\nu}=\alpha_{\mu\cdot1}a_{\nu\cdot1}+\alpha_{\mu\cdot2}a_{\nu\cdot2}+\ .\ .\ .\ +\alpha_{\mu\cdot5}a_{\nu\cdot5}.$

The series of suffixes for the α's, a's, and m's are seen to be the same, the series of pairs of first suffixes in every row and the series of pairs of second suffixes in every column being

12, 13, 14, 15, 23, 24, 25, 34, 35, 45;

that is to say, the combinations arranged in ascending order, of the numbers 1, 2, 3, 4, 5, taken two at a time. On the first side of the identity are 10 products, and as both factors of each product contain 10 terms, the result of the multiplication would be to produce 1000 terms of the form

$$|a_{rp}a_{sq}| . |a_{mp}a_{nq}|,$$

the whole expansion in fact being

$$\sum_{\substack{q=2\\p<q}}^{q=5} \sum_{\substack{s=2\\r<s}}^{s=5} \sum_{\substack{n=2\\m<n}}^{n=5} |a_{rp}a_{sq}| . |a_{mp}a_{nq}|.$$

On the right-hand side are 100 terms of the form

$$|m_{rp}m_{sq}|,$$

and if a proof of the identity were wanted, we should only have to show that each of the 100 terms of the latter kind gives rise to a particular 10 terms of the former kind. This, too, it is interesting to note, Cauchy himself could have done. For example, the last of the 100 terms,

$$|m_{44}m_{55}|$$

$$= \begin{vmatrix} a_{41}a_{41}+a_{42}a_{42}+\ldots+a_{45}a_{45} & a_{41}a_{51}+a_{42}a_{52}+\ldots+a_{45}a_{55} \\ a_{51}a_{41}+a_{52}a_{42}+\ldots+a_{55}a_{45} & a_{51}a_{51}+a_{52}a_{52}+\ldots+a_{55}a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \times \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} . \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} + \begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix} . \begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix} + \ldots\ldots + \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} . \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix},$$

which is nothing more than Cauchy's formula (62)

$$m^{(p)}_{\mu.\nu} = S^P\left(a^{(p)}_{\nu.1} a^{(p)}_{\mu.1}\right),$$

when we put $\mu = 10 = \nu$, and $p = 2$. Instead of 1000 terms on the left-hand side and 100 on the right, we should clearly have for the general theorem P^3 terms on the left and P^2 terms on the right, P be it remembered being the combinatorial

$$\frac{n(n-1)(n-2)\ \ldots\ (n-p+1)}{1.2.3\ \ldots\ p}.$$

Leaving Cauchy, let us now return to Binet, and in order that the comparison between the two may be complete, let us formally enunciate in all its generality the latter's theorem also. Binet himself did not do this. After dealing with the case in which the determinants involved are of the 2nd order, he merely added (p. 289)—

"On aura encore pour les intégrales

$$\Sigma\{S(x,y',z'')\ S(\xi,v',\zeta'')\},\quad \Sigma\{S(t,x',y'',z''')\ S(\tau,\xi',v'',\zeta''')\},\ \&c.$$

des résultats semblables, savoir,

$$\Sigma\{S(x,y'z'')\ S(\xi,v',\zeta'')\}$$
$$= S_1\left\{\begin{array}{l} \quad\Sigma x\xi\Sigma yv\Sigma z\zeta + \Sigma y\xi\Sigma zv\Sigma x\zeta + \Sigma z\xi\Sigma xv\Sigma y\zeta \\ -\Sigma x\xi\Sigma zv\Sigma y\zeta - \Sigma y\xi\Sigma xv\Sigma z\zeta - \Sigma z\xi\Sigma yv\Sigma x\zeta, \end{array}\right.$$

$$\Sigma\{S(t,x',y'',z''')\ S(\tau,\xi',v'',\zeta''')\}$$
$$= S_1\{\Sigma t\tau\Sigma x\xi\Sigma yv\Sigma z\zeta + \Sigma x\tau\Sigma y\xi\Sigma tv\Sigma z\zeta + \&c.\}$$

&c."

With the help of modern phraseology, the general theorem thus intended to be indicated can be made sufficiently clear. Binet in effect says:—

Take s rectangular arrays each with m elements in the row and n elements in the column, m being greater than n, viz.—

$$\begin{array}{llll} (a_1)_{11}(a_1)_{12}\dots(a_1)_{1m} & (a_1)_{21}(a_1)_{22}\dots(a_1)_{2m} & \cdot\ \cdot & (a_1)_{s1}(a_1)_{s2}\dots(a_1)_{sm} \\ (a_2)_{11}(a_2)_{12}\dots(a_2)_{1m} & (a_2)_{21}(a_2)_{22}\dots(a_2)_{2m} & \cdot\ \cdot & (a_2)_{s1}(a_2)_{s2}\dots(a_1)_{sm} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (a_n)_{11}(a_n)_{12}\dots(a_n)_{1m}, & (a_n)_{21}(a_n)_{22}\dots(a_n)_{2m}, & \cdot\ \cdot & (a_n)_{s1}(a_n)_{s2}\dots(a_n)_{sm} \end{array}$$

and other s rectangular arrays of the same kind, viz.—

$$\begin{array}{llll} (b_1)_{11}(b_1)_{12}\dots(b_1)_{1m} & (b_1)_{21}(b_1)_{22}\dots(b_1)_{2m} & \cdot\ \cdot & (b_1)_{s1}(b_2)_{s2}\dots(b_1)_{sm} \\ (b_2)_{11}(b_1)_{12}\dots(b_2)_{1m} & (b_2)_{21}(b_1)_{22}\dots(b_2)_{2m} & \cdot\ \cdot & (b_2)_{s1}(b_2)_{s2}\dots(b_2)_{sm} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (b_n)_{11}(b_n)_{12}\dots(b_n)_{1m} & (b_n)_{21}(b_n)_{22}\dots(b_n)_{2m} & \cdot\ \cdot & (b_n)_{s1}(b_n)_{s2}\dots(b_n)_{sm} \end{array}$$

From each array, by taking every set of n columns, form $C_{m,n}$ determinants, arranging them in any order, provided it be the same for all the arrays. Add together all the 1st determinants formed from the first s arrays, and multiply the sum by the corresponding sum for the second s arrays; obtain the like product involving all the 2nd determinants, the like product involving all the 3rd determinants, and so on. Then, the sum of these products is equal to the sum of the products obtained by multiplying each array of the first set by each array of the second set.

Or we may put it alternatively as a formal proposition, thus:—

If s *rectangular arrays be taken, each with* m *elements in the row and* n *elements in the column,* m *being greater than* n, viz.

$$X_1,\ X_2,\ \ldots\ldots,\ X_s$$

and other s *rectangular arrays of the same kind,* viz.,

$$\Xi_1,\ \Xi_2,\ \ldots\ldots,\ \Xi_s;$$

and if the minor determinants of the n*th* *order formed from* X_1, X_2, ... , Ξ_{s-1}, Ξ_s *be*

$$\begin{array}{llll} x_{11} & x_{12} & \ldots & x_{1C} \\ x_{21} & x_{22} & \ldots & x_{2C} \\ \cdot & \cdot & \cdot & \cdot \\ x_{s1} & x_{s2} & \ldots & x_{sC} \end{array} \qquad \begin{array}{llll} \xi_{11} & \xi_{12} & \ldots & \xi_{1C} \\ \xi_{21} & \xi_{22} & \ldots & \xi_{2C} \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{s1} & \xi_{s} & \ldots & \xi_{sC} \end{array}$$

then

$$\begin{aligned} & (x_{11}+x_{21}+\ldots+x_{s1})(\xi_{11}+\xi_{21}+\ldots+\xi_{s1}) \\ & +(x_{12}+x_{22}+\ldots+x_{s2})(\xi_{12}+\xi_{22}+\ldots+\xi_{s2}) \\ & +\ \ldots\ldots\ldots\ldots\ldots\ldots \\ & +(x_{1C}+x_{2C}+\ldots+x_{sC})(\xi_{1C}+\xi_{2C}\ldots+\xi_{sC}) \\ = \ & (X_1+X_2+\ldots+X_s)(\Xi_1+\Xi_2+\ldots+\Xi_s), \end{aligned}$$

where C stands for $C_{m,n}$ *i.e.*, $m(m-1)\ldots\ldots(m-n+1)/1.2.3\ldots\ldots n.$

Now, counting the terms here as we did in the case of Cauchy's theorem, we see that on the left-hand side there are C multiplications to be performed, each giving rise to $s \times s$ terms, and that therefore the full number of terms in the development of this side is

$$s^2C;$$

also that on the right-hand side the number is

$$s^2.$$

In Cauchy's theorem the corresponding numbers were found to be P^3 and P^2, P being not any whole number as s is, but like C a combinatorial. Without further investigation, we might consequently assert that, supposing the two theorems to be alike in other respects, Binet's must be the more general, the passage from it to Cauchy's being effected by taking $s = C$. A closer examination, however, will show that this is not the full measure of the difference between the two theorems as to generality. Not only must we specialise by putting $s = C$, but s must become C in a very special way. In order to make this clear, let us take the particular case of Binet's theorem which approximates as nearly as possible to the particular case of Cauchy's given above. In the latter the determinants were of the 2nd order; therefore to get the comparable case of Binet's theorem we must put $n = 2$. Again, since P in the particular case of Cauchy's theorem was 10, we must for the same purpose put

$$s = 10$$

$$\text{and } C_{m.n} = 10, \text{ and } \therefore\ m = 5.$$

The result is

$$\left\{\begin{vmatrix} a_{11} a_{12} \\ b_{11} b_{12} \end{vmatrix} + \begin{vmatrix} a_{21} a_{22} \\ b_{21} b_{22} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,1} a_{10,2} \\ b_{10,1} b_{10,2} \end{vmatrix}\right\} \left\{\begin{vmatrix} a_{11} a_{12} \\ \beta_{11} \beta_{12} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,1} a_{10,2} \\ \beta_{10,1} \beta_{10,2} \end{vmatrix}\right\}$$
$$+ \left\{\begin{vmatrix} a_{11} a_{13} \\ b_{11} b_{13} \end{vmatrix} + \begin{vmatrix} a_{21} a_{23} \\ b_{21} b_{23} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,1} a_{10,3} \\ b_{10,1} b_{10,3} \end{vmatrix}\right\} \left\{\begin{vmatrix} a_{11} a_{13} \\ \beta_{11} \beta_{13} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,1} a_{10,3} \\ \beta_{10,1} \beta_{10,3} \end{vmatrix}\right\}$$
$$+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$
$$+ \left\{\begin{vmatrix} a_{14} a_{15} \\ b_{14} b_{15} \end{vmatrix} + \begin{vmatrix} a_{24} a_{25} \\ b_{24} b_{25} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,4} a_{10,5} \\ b_{10,4} b_{10,5} \end{vmatrix}\right\} \left\{\begin{vmatrix} a_{14} a_{15} \\ \beta_{14} \beta_{15} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,4} a_{10,5} \\ \beta_{10,4} \beta_{10,5} \end{vmatrix}\right\}$$
$$= \left\{\begin{vmatrix} a_{11} a_{12} \dots a_{15} \\ b_{11} b_{12} \dots b_{15} \end{vmatrix} + \begin{vmatrix} a_{21} a_{22} \dots a_{25} \\ b_{21} b_{22} \dots b_{25} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,1} a_{10,2} \dots a_{10,5} \\ b_{10,1} b_{10,2} \dots b_{10,5} \end{vmatrix}\right\} \times$$
$$\left\{\begin{vmatrix} a_{11} a_{12} \dots a_{15} \\ \beta_{11} \beta_{12} \dots \beta_{15} \end{vmatrix} + \begin{vmatrix} a_{21} a_{22} \dots a_{25} \\ \beta_{21} \beta_{22} \dots \beta_{25} \end{vmatrix} + \dots + \begin{vmatrix} a_{10,1} a_{10,2} \dots a_{10,5} \\ \beta_{10,1} \beta_{10,2} \dots \beta_{10,5} \end{vmatrix}\right\},$$

the elements involved being 200 in number, and disposable in two sets of arrays—

$$\begin{matrix} a_{11}\, a_{12} \dots a_{15} & a_{21}\, a_{22} \dots a_{25} & \dots\dots & a_{10,1}\, a_{10,2} \dots a_{10,5} \\ b_{11}\, b_{12} \dots b_{15}, & b_{21}\, b_{22} \dots b_{25}, & \dots\dots & b_{10,1}\, b_{10,2} \dots b_{10,5}, \end{matrix}$$

and

$$\begin{matrix} \alpha_{11}\,\alpha_{12}\,\ldots\,\alpha_{15} \\ \beta_{11}\,\beta_{12}\,\ldots\,\beta_{15}, \end{matrix} \quad \begin{matrix} \alpha_{21}\,\alpha_{22}\,\ldots\,\alpha_{25} \\ \beta_{21}\,\beta_{22}\,\ldots\,\beta_{25}, \end{matrix} \quad \ldots\ldots \quad \begin{matrix} \alpha_{10,1}\,\alpha_{10,2}\,\ldots\,\alpha_{10,5} \\ \beta_{10,1}\,\beta_{10,2}\,\ldots\,\beta_{10,5}. \end{matrix}$$

In the corresponding identity of Cauchy there are only 50 different elements, viz., the elements of the two square arrays—

$$\begin{matrix} a_{11}\,a_{12}\,\ldots\,a_{15} \\ a_{21}\,a_{22}\,\ldots\,a_{25} \\ \ldots\ldots\ldots \\ a_{51}\,a_{52}\,\ldots\,a_{55}, \end{matrix} \qquad \begin{matrix} \alpha_{11}\,\alpha_{12}\,\ldots\,\alpha_{15} \\ \alpha_{21}\,\alpha_{22}\,\ldots\,\alpha_{25} \\ \ldots\ldots\ldots \\ \alpha_{51}\,\alpha_{52}\,\ldots\,\alpha_{55}. \end{matrix}$$

Indeed,—and it is this which brings the comparison to a point,—if from the first of these square arrays we form 10 rectangular arrays by taking every possible pair of rows, thus using each row 4 times over, viz.,

$$\begin{matrix} a_{11}\,a_{12}\,\ldots\,a_{15} \\ a_{21}\,a_{22}\,\ldots\,a_{25}, \end{matrix} \quad \begin{matrix} a_{11}\,a_{12}\,\ldots\,a_{15} \\ a_{31}\,a_{32}\,\ldots\,a_{35}, \end{matrix} \quad \ldots\ldots, \quad \begin{matrix} a_{41}\,a_{42}\,\ldots\,a_{45} \\ a_{51}\,a_{52}\,\ldots\,a_{55}, \end{matrix}$$

and similarly from the α's form a second set of 10 arrays, viz.,

$$\begin{matrix} \alpha_{11}\,\alpha_{12}\,\ldots\,\alpha_{15} \\ \alpha_{21}\,\alpha_{22}\,\ldots\,\alpha_{25}, \end{matrix} \quad \begin{matrix} \alpha_{11}\,\alpha_{12}\,\ldots\,\alpha_{15} \\ \alpha_{31}\,\alpha_{32}\,\ldots\,\alpha_{35}, \end{matrix} \quad \ldots\ldots, \quad \begin{matrix} \alpha_{41}\,\alpha_{42}\,\ldots\,\alpha_{45} \\ \alpha_{51}\,\alpha_{52}\,\ldots\,\alpha_{55}; \end{matrix}$$

and then to these two special sets of arrays apply Binet's theorem, we obtain Cauchy's theorem. Regarding the two theorems in all their generality, the decision we have reached may therefore be expressed by saying that Binet's is a theorem concerning $2smn$ quantities, where s, m, n are any positive integers, and Cauchy's is a case of it in which

$$s = m(m-1)\ldots(m-n+1)/1.2.3\ldots n,$$

and in which, further, the number of different quantities involved is not

$$2.\frac{m(m-1)\ldots(m-n+1)}{1.2\quad\ldots\quad n}\times mn,$$

but by reason of repetitions is only

$$2m^2.$$

Although this decision is against Cauchy's claim as put by himself, it deserves to be noticed that, apparently by oversight, he failed to make his case as strong as he might have done. It will be remembered that Binet made two advances in the generalisation of the multiplication-theorem. In the first place, he gave the generalisation from which the multiplication-theorem is got by putting $m=n$, or, as we nowadays say, by substituting two square matrices for two rectangular matrices, and then he gave the theorem which we have been comparing with Cauchy's and which degenerates into his own first theorem when s is put equal to 1. Now the first of these generalisations Cauchy could justly have laid claim to. His identity (XVIII. 5) is not indeed stated or viewed as a generalisation of the multiplication-theorem, but it is unquestionably so in reality. Ostensibly the identity concerns any minor of a product-determinant, but every such minor is obtained by multiplying together two rectangular matrices, and, conversely, every determinant which is the product of two rectangular matrices may be viewed as a minor of the product of two determinants.

On looking back, however, at Cauchy's memoir as a whole, one cannot but be struck with admiration both at the quality and the quantity of its contents. Supposing that none of its theorems had been new, and that it had not even presented a single old theorem in a fresh light, the memoir would have been most valuable, furnishing, as it did, to the mathematicians of the time an almost exhaustive treatise on the theory of general determinants. It is not too much to say, although it may come to many as a surprise, that the ordinary text-books of determinants supplied to university students of the present day do not contain much more of the general theory than is to be found in Cauchy's memoir of about eighty years ago. One apparently trivial instrument, which Cauchy had not received from his predecessors and which he did not make for himself, viz., a notation for determinants whose elements had special values, is at the foundation of the whole difference between his treatise and those at present employed. When this want came to be supplied later on, the functions crept steadily into everyday use, and a fresh impetus was consequently given to the study of them. But if from the work of the said eighty years all researches regarding special forms of determinants be left out, and all investigations which ended in mere rediscoveries or in rehabilitations of old ideas, there is a surprisingly small proportion left. If one bears this in mind, and recalls the fact, temporarily set aside, that Cauchy, instead of being a compiler, presented the entire subject from a perfectly new point of view, added many results previously unthought of, and opened up a whole avenue of fresh investigation, one cannot but assign to him the place of highest honour among all the workers from 1693 to 1812. It is, no doubt, impossible to call him, as some have done, the formal founder of the theory. This honour is certainly due to Vandermonde, who, however, erected on the foundation comparatively little of a superstructure. Those who followed Vandermonde contributed, knowingly or unknowingly, only a stone or two, larger or smaller, to the building. Cauchy relaid the foundation, rebuilt the whole, and initiated new enlargements; the result being an edifice which the architects of to-day may still admire and find worthy of study.

CHAPTER V.

DETERMINANTS IN GENERAL, FROM 1693 TO 1812; A RETROSPECT.

From what has just been said by way of estimate of Cauchy's memoir, it will readily appear that a suitable opportunity has now presented itself for taking a general retrospect of the work done from the date at which the history commences. The system which has been pursued, of numbering the new advances made by each writer, enables us to do this very conveniently, and with a tolerable approximation to accuracy by means of a tabular form. The table, herewith annexed, so far explains itself. The authors' names, it will be seen, are arranged both vertically and horizontally in chronological order; and vertical and horizontal lines of separation are drawn so as to apportion to each name a gnomon-shaped space. The crediting of any entirely new result to an author is done by giving its number in Roman figures after his name in the vertical list. On the other hand, any mere modification, fresh presentment, or extension of a previously known result, is notified to the right of the original number of the result, and under the new writer's name in the horizontal series. Instead of the Arabic figures placed in the gnomon-shaped spaces, a cross or other uniform mark would have sufficed, but in order to increase the usefulness of the table, a number has been inserted, telling the page at which the result is to be found. For example, if we look to the space allotted to Bezout (1779), we find him credited with one entirely new result, numbered XXIII., and with some contribution to each of five previously known results, whose numbers are II., III., IV., XII., XIV.; and we likewise see that information regarding them

TABLE—SHOWING THE ADVANCE OF THE

		Leibnitz, 1693.	Fontaine, 1748.	Cramer, 1750.	Bézout, 1764.	Vandermonde, 1771.	Laplace, 1772.	Lagrange, 1773.	Bézout, 1779.	Hindenburg, 1784.	
1693. Leibnitz,	I.	9									
	II.	9			17				52	55	
	III.	9		13	17				52	55	56(2),
1748. Fontaine,											
1750. Cramer,	IV.			13					52		
	V.			13							
1764. Bézout,	VI.				17						61
1771. Vandermonde,	VII.					24	33			54	
	VIII.					24					
	IX.					24					
	X.					24					
	XI.					24	33				
	XII.					24	33		52		61-2
	XIII.					24	33				
	XIV.					24	33		52		
1772. Laplace,	XV.						33				
	XVI.						33				
1773. Lagrange,	XVII.							41, 41			
	XVIII.							41			
	XIX.							41			
	XX.							41			
	XXI.							41			
	XXII.							41			
1779. Bézout,	XXIII.								52		
1784. Hindenburg,											
1800. Rothe,	XXIV.										59
	XXV.										60
	XXVI.										63, 63
1801. Gauss,	XXVII.										
1809. Monge,	XXVIII.										
Hirsch,											
1811. Binet,	XXIX.										
Prasse,											
1812. Wronski,											
Binet,	XXX.										
	XXXI.										
	XXXII.										
	XXXIII.										
	XXXIV.										
	XXXV.										
	XXXVI.										
Cauchy,	XXXVII.										
	XXXVIII.										
	XXXIX.										
	XL.										
	XLI.										
	XLII.										
	XLIII.										

Rothe, 1800.	Gauss, 1801.	Monge, 1809.	Hirsch,	Binet, 1811.	Prasse,	Wronski, 1812.	Binet,	Cauchy.
					75			101
57(3), 58, 59(2), 61					75, 77, 78			99, 101, 102
								103, 103
						78		99
								99
								103
								96
								103
								97, 103, 117
								116
	64-5					78		93
	66						81	109
				70, 72			81	119
								110
	66							
		68						117
	65							104
		68						
				71				
							86	121
							86, 92	
							87, 92	
							88	
							88	
							91, 92	
							91	
								105
								106, 110
								111
								112, 114, 114, 114, 114, 120
								118, 118
								120, 121
								121, 122

all will be got at p. 52 of the History.* Speaking generally, more importance ought to be attached to the existence of numbers at the corner of a gnomon than elsewhere, because these indicate fresh departures in the theory. Sometimes, however, a fresh departure may have been very trivial, the real advance being indicated by a number well removed from the corner of a subsequent gnomon. Thus if we examine the history of the multiplication-theorem (Nos. XVII., XVIII.), we find the first step in the direction of it credited by the table to Lagrange, and subsequent steps to Gauss, Binet, and Cauchy; whereas careful investigation at the pages mentioned shows that what Lagrange accomplished was of exceedingly little moment, in comparison with the magnificent generalisation of Binet and Cauchy. Again, it must be borne in mind that all the results numbered in Roman figures are not of equal importance, it being well known that one theorem in any mathematical subject may have vastly more influence on the after development of the subject than half a dozen others. Such imperfections, however, being allowed for, the table will be found to afford a very ready means of estimating with considerable accuracy the proportionate importance to be assigned to the various early investigators of the theory.

If we look for a moment, in conclusion, at the nationality of the authors, one outstanding fact immediately arrests attention, viz., that almost every important advance is due to the mathematicians of France. Were the contributions of Bezout, Vandermonde, Laplace, Lagrange, Monge, Binet, and Cauchy left out, there would be exceedingly little left to any one else, and even that little would be of minor interest.

* As regards the newness of XXIII. the table is not quite in accord with the text, an earlier writer's work having been duly noted in the latter (p. 11).

CHAPTER VI.

DETERMINANTS IN GENERAL, FROM THE YEAR 1813 TO 1825.

THE writers of this period are seven in number, viz., Gergonne, Garnier, Wronski, Desnanot, Cauchy, Scherk, Schweins. Of these Gergonne, Garnier and Cauchy are merely expository; Wronski only recalls an earlier communication; Desnanot is a follower of Bezout; Scherk is a follower of Hindenburg; Schweins alone stands prominently forward as being well read in the subject, fit to give a full exposition, and fruitful in new results.

GERGONNE (1813).

[Développement de la théorie donnée par M. Laplace pour l'élimination au premier degré. *Annales de Mathématiques*, iv. pp. 148–155.]

This is such an exposition of the primary elements of the theory of determinants and their application to the solution of a set of simultaneous linear equations as might be given in the course of an hour's lecture. It is confessedly founded on Laplace's memoir of 1772; but, though the matter of it is thus not original, it is nevertheless noteworthy on account of its brevity, clearness, and elegance.

The word "inversion" is introduced to denote (III. 21) what Cramer called a "dérangement," and then by easy steps the reader is led up to the theorem regarding the interchange of two non-contiguous letters.

"(9) Donc, si l'on permute entre elles deux lettres non consécutives, on changera nécessairement l'espèce du nombre des inversions. Soit en effet n le nombre des lettres intermédiaires à ces deux-là; on pourra d'abord porter la lettre la plus à gauche immédiatement à gauche de l'autre, ce qui lui fera parcourir n places; puis remettre cette dernière à la place de la première; et, comme elle sera obligée de passer par-dessus celle-ci, elle se trouvera avoir parcouru $n+1$ places. Le nombre total des places parcourues par les deux lettres sera donc $2n+1$, et conséquemment l'espèce du nombre des inversions se trouvera changée." (III. 22)

This, it must be noted, is not identical with Rothe's proposition on the same subject, Gergonne's n being different from Rothe's d.

The proof, that a determinant vanishes if two of the letters bearing suffixes be the same, proceeds on the same lines as Rothe's, but is put very shortly and not less convincingly as follows:—

"Supposons, en effet, que l'on change h en g, sans toucher à g ni aux indices. Soient, pour un terme pris au hasard dans le polynôme, p et q les indices respectifs de g et h; ce polynôme, renfermant toutes les permutations, doit avoir un autre terme ne différant uniquement de celui-là qu'en ce que c'est h qui y porte l'indice p et g l'indice q; et de plus (9) ces deux termes doivent être affectés de signes contraires; ils se détruiront donc, lorsqu'on changera h en g; et il en sera de même de tous les autres termes pris deux à deux." (XII. 8)

On putting "le polynôme D," *i.e.* the determinant $|a_1b_2c_3\ldots|$, in the form

$$\mathrm{A}_1a_1+\mathrm{A}_2a_2+\mathrm{A}_3a_3+\ldots+\mathrm{A}_ma_m,$$

this theorem of course leads at once to the identities

$$\left.\begin{array}{l} \mathrm{A}_1b_1+\mathrm{A}_2b_2+\mathrm{A}_3b_3+\ldots+\mathrm{A}_mb_m=0 \\ \mathrm{A}_1c_1+\mathrm{A}_2c_2+\mathrm{A}_3c_3+\ldots+\mathrm{A}_mc_m=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\},$$

and these to the solution of m linear equations in m unknowns.

GARNIER (1814).

[Analyse Algébrique, faisant suite à la première section de l'algèbre. 2e édition, revue et considérablement augmentée. xvi+668 pp. Paris.]

The title of Garnier's chapter xxvii. (pp. 541–555) is "*Développement de la théorie donnée par M. Laplace pour l'élimination au premier degré.*" It consists, however, of nothing but a simple exposition, confessedly borrowed from Gergonne's paper of 1813, and six pages of extracts from

Laplace's original memoir of 1772. As forming part of a popular text-book, it probably did more service in bringing the theory to the notice of mathematicians than a memoir in a recondite serial publication could have done; and we certainly know that Sylvester, who afterwards did so much to advance the theory, expresses himself indebted to it.

WRONSKI (1815).

[Philosophie de la Technie Algorithmique. Première Section, contenant la loi suprême et universelle des Mathématiques. Par Hoëné Wronski. (pp. 175–181, &c.) Paris.]

Here as in the *Réfutation* of 1812 "combinatory sums" make their appearance, as being necessary for the expression of the "loi suprême." Wronski's point of view is unaltered toward them. He now, however, calls them

Schin functions, (xv. 5)

from the letter formerly introduced to denote them, "et pour ne pas introduire de noms nouveaux"! Two or three pages are occupied with the statement of the recurrent law of formation (Bezout, 1764).

DESNANOT, P. (1819).

[Complément de la Théorie des Équations du Premier Degré, contenant . . . . . . Par P. Desnanot, Censeur au Collége Royal de Nancy, ... Paris.]

As far as can be gathered, Desnanot was acquainted with the writings of very few of his predecessors in the investigation of determinants. The only one he himself mentions is Bezout, and the first part of his work is in direct continuation of a topic which the latter had begun. His book is a marvel of laboured detail. No expositor could take more pains with his reader, space being held of no moment if clearness had to be secured. As might be expected, therefore, all that is really worth preserving of his work is but a small fraction of the 264 pages which he occupies in exposition.

The first chapter bears the heading

Recherche des Relations qui ont lieu entre le dénominateur et les numérateurs des valeurs générales des inconnues dans chaque système d'équations du premier degré;

and, after a reference to the impossibility of obtaining any result in the case of one equation with one unknown, proceeds as follows:—

"Si l'on a les deux équations

$$ax+by=c, \quad a'x+b'y=c',$$

elles donnent

$$x=\frac{cb'-bc'}{ab'-ba'}, \quad y=\frac{ac'-ca'}{ab'-ba'};$$

nommant D le dénominateur commun, N et N′ les numérateurs des valeurs de x et de y, nous aurons

$$\mathrm{D}=ab'-ba', \quad \mathrm{N}=cb'-bc', \quad \mathrm{N}'=ac'-ca'.$$

Multiplions N par a, N′ par b et ajoutons, nous trouverons

$$a\mathrm{N}+b\mathrm{N}'=c(ab'-ba')=c\mathrm{D};$$

donc

$$a\mathrm{N}+b\mathrm{N}'=c\mathrm{D}.$$

Nous aurions de même, en multipliant N par a' et N′ par b', cette autre équation

$$a'\mathrm{N}+b'\mathrm{N}'=c'\mathrm{D}."$$

With this may be compared Bezout's *Méthode pour trouver des fonctions . . . . qui soient zéro par elles-mêmes* (see p. 49).

Exactly the same method is followed with the set of equations

$$\left.\begin{aligned} ax+by+cz&=d \\ a'x+b'y+c'z&=d' \\ a''x+b''y+c''z&=d'' \end{aligned}\right\}.$$

Here fifteen relations are obtained, only seven of which are viewed as necessary, viz.,

$$\left.\begin{aligned} (ab'-ba')\mathrm{N}'+(ac'-ca')\mathrm{N}''&=(ad'-da')\mathrm{D} \\ (ab''-ba'')\mathrm{N}'+(ac''-ca'')\mathrm{N}''&=(ad''-da'')\mathrm{D} \end{aligned}\right\},$$

$$\left.\begin{aligned} (da'-ad')\mathrm{N}+(db'-bd')\mathrm{N}'+(dc'-cd')\mathrm{N}''&=0 \\ (da''-ad'')\mathrm{N}+(db''-bd'')\mathrm{N}'+(dc''-cd'')\mathrm{N}''&=0 \end{aligned}\right\},$$

$$\left.\begin{aligned} a\,\mathrm{N}+b\,\mathrm{N}'+c\,\mathrm{N}''&=d\,\mathrm{D} \\ a'\mathrm{N}+b'\mathrm{N}'+c'\mathrm{N}''&=d'\mathrm{D} \\ a''\mathrm{N}+b''\mathrm{N}'+c''\mathrm{N}''&=d''\mathrm{D} \end{aligned}\right\}.$$

From a modern point of view there are but *two* which are really different, viz.,

$$|ab'|.|ac'd''| - |ac'|.|ab'd''| + |ad'|.|ab'c''| = 0$$

and $$a|bc'd''| - b|ac'd''| + c|ab'd''| - d|ab'c''| = 0,$$

the twelve quantities concerned being

$$\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d''. \end{matrix}$$

The former is obtainable from Bezout's identity

$$|ab'c''|.|de'f''| - |ab'd''|.|ce'f''| + |ac'd''|.|be'f''| - |bc'd''|.|ae'f''| = 0$$

by putting

$$f, f', f'' = 0, 0, 1$$

and $$e, e', e'' = a, a', a''.$$

The other, as is well known, comes from Vandermonde.

Before proceeding to the case of four unknowns, a notation is introduced in the following words (p. 6):—

"Soient a, b, c, d, f, g, h, etc. des lettres représentant des quantités quelconques; k, l, m, p, q, r, etc. des indices d'accens qui doivent être placés à la droite des lettres. Au lieu de mettre ces indices comme des exposans, plaçons-les au-dessus des lettres qu'ils doivent affecter, de manière que $\overset{k}{a}$ désigne a affecté du nombre k d'accens; que $\overset{k}{a}\overset{l}{b}$ indique le produit de $\overset{k}{a}$ par $\overset{l}{b}$; ainsi de suite. Représentons la quantité $\overset{k}{a}\overset{l}{b} - \overset{k}{b}\overset{l}{a}$ par $\left(\begin{smallmatrix} k & l \\ a & b \end{smallmatrix}\right)$ de sorte que nous ayons cette équation

$$\begin{pmatrix} k & l \\ a & b \end{pmatrix} = \overset{k}{a}\overset{l}{b} - \overset{l}{b}\overset{k}{a}."$$

This being settled, the similar quantities of higher orders are defined by the equations

$$\begin{pmatrix} k & l & m \\ a & b & c \end{pmatrix} = \overset{m}{c}\begin{pmatrix} k & l \\ a & b \end{pmatrix} - \overset{l}{c}\begin{pmatrix} k & m \\ a & b \end{pmatrix} + \overset{k}{c}\begin{pmatrix} l & m \\ a & b \end{pmatrix},$$

$$\begin{pmatrix} k & l & m & p \\ a & b & c & d \end{pmatrix} = \overset{p}{d}\begin{pmatrix} k & l & m \\ a & b & c \end{pmatrix} - \overset{m}{d}\begin{pmatrix} k & l & p \\ a & b & c \end{pmatrix} + \overset{l}{d}\begin{pmatrix} k & m & p \\ a & b & c \end{pmatrix} - \overset{k}{d}\begin{pmatrix} l & m & p \\ a & b & c \end{pmatrix},$$

&c. &c. &c.

It is thus seen that Desnanot's definition is almost exactly the same as Vandermonde's, and his notation essentially the same as Laplace's. To this definition and the proof of the theorem regarding the effect of the interchange of two indices or two letters seven pages are devoted, and then a fresh step is taken. The exact words of the original (pp. 13, 14) must be given, as they distinctly foreshadow a great theorem of later times.

"14. Si nous développons cette expression

$$\binom{k\,l}{a\,b}\binom{m\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{m\,l}{a\,b}$$

le résultat sera

$$\binom{k\,m}{a\,b}\binom{l\,p}{a\,b};$$

donc nous avons cette équation

$$\text{(A)}\qquad \binom{k\,l}{a\,b}\binom{m\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{m\,l}{a\,b} = \binom{k\,m}{a\,b}\binom{l\,p}{a\,b}.$$

15. De cette formule je vais en déduire d'autres. Je dis que si j'introduis la lettre c dans les seconds facteurs de chaque terme et en même temps l'indice k, l'équation subsistera encore, et que j'aurai

$$\text{(B)}\qquad \binom{k\,l}{a\,b}\binom{k\,m\,p}{a\,b\,c} - \binom{k\,p}{a\,b}\binom{k\,m\,l}{a\,b\,c} = \binom{k\,m}{a\,b}\binom{k\,l\,p}{a\,b\,c}. \qquad \text{(XLIV.)}$$

L'égalité serait prouvée si en développant les deux membres, les quantités multipliées par la même lettre c, affectée d'indices égaux, étaient égales dans chaque membre ; or j'ai

$$\left.\begin{array}{l} \overset{p}{c}\binom{k\,l}{a\,b}\binom{k\,m}{a\,b} \\ -\overset{m}{c}\left(\binom{k\,l}{a\,b}\binom{k\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{k\,l}{a\,b}\right) \\ +\overset{k}{c}\left(\binom{k\,l}{a\,b}\binom{m\,p}{a\,b} - \binom{k\,p}{a\,b}\binom{m\,l}{a\,b}\right) \\ -\overset{l}{c}\binom{k\,p}{a\,b}\binom{k\,m}{a\,b} \end{array}\right\} = \left\{\begin{array}{l} +\overset{p}{c}\binom{k\,m}{a\,b}\binom{k\,l}{a\,b} \\ \\ +\overset{k}{c}\binom{k\,m}{a\,b}\binom{l\,p}{a\,b} \\ -\overset{l}{c}\binom{k\,m}{a\,b}\binom{k\,p}{a\,b}. \end{array}\right.$$

Les quantités multipliées par $\overset{p}{c}$, $\overset{m}{c}$ et $\overset{l}{c}$ dans chaque membre sont égales entr'elles, c'est évident ; et la formule (A) rend les coefficiens de $\overset{k}{c}$ égaux ; donc puisque dans (B), il n'y a que des termes multipliés par $\overset{p}{c}$, $\overset{m}{c}$, $\overset{k}{c}$, et $\overset{l}{c}$, je conclus que l'équation (B) est exacte."

Having thus shown that if in each of the second factors of the identity

$$|a_1b_2||a_3b_4| - |a_1b_3||a_2b_4| + |a_1b_4||a_2b_3| = 0 \qquad \text{(A)},$$

a new letter c be added and the index 1 be prefixed, the sign of equality may still be retained, so that we have a new identity

$$|a_1b_2\|a_1b_3c_4| - |a_1b_3\|a_1b_2c_4| + |a_1b_4\|a_1b_2c_3| = 0 \qquad \text{(B)};$$

he then goes on to prove in the same fashion that the first factors of this derived identity may be treated in a similar way with impunity, viz., that they may be extended by the appending of the letter c with a new index 5, so that we have a further derived identity

$$|a_1b_2c_5\|a_1b_3c_4| - |a_1b_3c_5\|a_1b_2c_4| + |a_1b_4c_5\|a_1b_2c_3| = 0 \qquad \text{(C)},$$

already known to us from Monge.

And this is not all, for the next paragraph shows that these two extensions may be repeated in order as often as we please, the opening of the paragraph being as follows (p. 15):—

"17. Généralisons et prouvons que si la formule

$$\binom{k\ l\ .\,.\ q}{a\ b\ .\,.\ c}\binom{k\ m\ .\,.\ p}{a\ b\ .\,.\ c} - \binom{k\ m\ .\,.\ l}{a\ b\ .\,.\ c}\binom{k\ p\ .\,.\ q}{a\ b\ .\,.\ c} = \binom{k\ l\ .\,.\ p}{a\ b\ .\,.\ c}\binom{k\ m\ .\,.\ q}{a\ b\ .\,.\ c}$$

est vraie dans le cas où il y aurait n lettres comprises dans chaque facteur, elle sera encore vraie en ajoutant une nouvelle lettre d dans les seconds facteurs de chaque terme avec l'indice l qui n'y entre pas; et qu'ensuite, si l'on ajoute la même lettre d dans les premiers facteurs de chaque terme avec un nouvel indice r, l'égalité ne sera pas troublée.

Il s'agit donc de démontrer que ces deux formules sont exactes:

$$\left.\begin{aligned}
&\binom{k\ .\,.\ q}{a\ b\ .\,.\ c}\binom{k\ m\ .\,.\ l\ p}{a\ b\ .\,.\ c\ d} - \binom{k\ m\ .\,.\ l}{a\ b\ .\,.\ c}\binom{k\ p\ .\,.\ l\ q}{a\ b\ .\,.\ c\ d} = \binom{k\ l\ .\,.\ p}{a\ b\ .\,.\ c}\binom{k\ m\ .\,.\ l\ q}{a\ b\ .\,.\ c\ d},\\
&\binom{k\ l\ .\,.\ q\ r}{a\ b\ .\,.\ c\ d}\binom{k\ m\ .\,.\ l\ p}{a\ b\ .\,.\ c\ d} - \binom{k\ m\ .\,.\ l\ r}{a\ b\ .\,.\ c\ d}\binom{k\ p\ .\,.\ l\ q}{a\ b\ .\,.\ c\ d} = \binom{k\ l\ .\,.\ p\ r}{a\ b\ .\,.\ c\ d}\binom{k\ m\ .\,.\ l\ q}{a\ b\ .\,.\ c\ d}.\text{''}
\end{aligned}\right\}\begin{matrix}\text{(XLIV. 2)}\\ \text{(XXIII. 5)}\end{matrix}$$

The line of proof is still the same, and may be shortly indicated by treating the case

$$\text{(D)}\quad |a_1b_2c_5\|a_1b_2c_3d_4| - |a_1b_2c_4\|a_1b_2c_3d_5| + |a_1b_2c_3\|a_1b_2c_4d_5| = 0,$$

which comes immediately after (C), and is derived from it by extending the factors *in which* a_1b_2 *does not occur.* Since by definition

$$|a_1b_2c_3d_4| = d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4|,$$

and $|a_1b_2c_3d_5| = d_5|a_1b_2c_3| - d_3|a_1b_2c_5| + d_2|a_1b_3c_5| - d_1|a_2b_3c_5|$, it follows that

$$\begin{aligned}|a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| \\ = \left.\begin{aligned} &\{d_4|a_1b_2c_5| - d_5|a_1b_2c_4|\}|a_1b_2c_3| \\ +&\{|a_1b_2c_5||a_1b_3c_4| - |a_1b_2c_4||a_1b_3c_5|\}d_2 \\ -&\{|a_1b_2c_5||a_2b_3c_4| - |a_1b_2c_4||a_2b_3c_5|\}d_1.\end{aligned}\right\}\end{aligned}$$

But the cofactor here of d_2 is by (C) equal to

$$-|a_1b_4c_5||a_1b_2c_3|;$$

and the cofactor of d_1

$$= |a_2b_1c_5||a_2b_3c_4| - |a_2b_1c_4||a_2b_3c_5|,$$

and therefore by (C)

$$\begin{aligned} &= -|a_2b_1c_3||a_2b_4c_5|, \\ &= \quad |a_1b_2c_3||a_2b_4c_5|.\end{aligned}$$

Making these substitutions, we have

$$\begin{aligned} &|a_1b_2c_5|a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| \\ &= -|a_1b_2c_3|\{d_5|a_1b_2c_4| - d_4|a_1b_2c_5| + d_2|a_1b_4c_5| - d_1|a_2b_4c_5|\} \\ &= -|a_1b_2c_3||a_1b_2c_4d_5|,\end{aligned}$$

as was to be shown.

The next three cases are

$$|a_1b_2c_5d_6||a_1b_2c_3d_4| - |a_1b_2c_4d_6||a_1b_2c_3d_5| + |a_1b_2c_3d_6||a_1b_2c_4d_5| = 0 \quad \text{(E)}$$

$$|a_1b_2c_3d_4||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6||a_1b_2c_3d_4e_5| = 0 \quad \text{(F)}$$

$$|a_1b_2c_3d_4e_7||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5e_7||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6e_7||a_1b_2c_3d_4e_5| = 0 \quad \text{(G).}$$

When the factors of each product are of the same order, as in (C), (E), (G), the identity is, in modern phraseology, an "extensional" of (A); that is to say, there is a part common to every factor of the identity, *e.g.*, a_1 in (C), a_1b_2 in (E), $a_1b_2c_3$ in (G), and this common part being deleted, the result is simply the identity (A). When the factors of each product are of different orders, as in (B), (D), (F), the identity is an "extensional" of something still simpler than (A), viz.,

$$a_1|a_2b_3| - a_2|a_1b_3| + a_3|a_1b_2| = 0.$$

In exactly the same manner and at quite as great length the identity

$$\binom{k\ l}{a\,f}\binom{k\ r}{a\,g} - \binom{k\ l}{a\,g}\binom{k\ r}{a\,f} = \binom{k\ l\ r}{a\,f\,g}\binom{k}{a}$$

—already known to us from Lagrange—is made the source of a numerous progeny. By putting figures for $k, l, \ldots$ and at the same time writing them as suffixes, these identities, original and derived, take the form

$$|a_1f_2||a_1g_6| - |a_1g_2||a_1f_6| = |a_1f_2g_6||a_1|, \quad (\text{A}')$$
$$|a_1f_2||a_1b_2g_6| - |a_1g_2||a_1b_2f_6| = |a_1f_2g_6||a_1b_2|, \quad (\text{B}')$$
$$|a_1b_2f_3||a_1b_2g_6| - |a_1b_2g_3||a_1b_2f_6| = |a_1b_2f_3g_6||a_1b_2|, \quad (\text{C}')$$
$$|a_1b_2f_3||a_1b_2c_3g_6| - |a_1b_2g_3||a_1b_2c_3f_6| = |a_1b_2f_3g_6||a_1b_2c_3|, \quad (\text{D}')$$
$$|a_1b_2c_3f_4||a_1b_2c_3g_6| - |a_1b_2c_3g_4||a_1b_2c_3f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3|, \quad (\text{E}')$$
$$|a_1b_2c_3f_4||a_1b_2c_3d_4g_6| - |a_1b_2c_3g_4||a_1b_2c_3d_4f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3d_4|, \quad (\text{F}')$$
$$|a_1b_2c_3d_4f_5||a_1b_2c_3d_4g_6| - |a_1b_2c_3d_4g_5||a_1b_2c_3d_4f_6| = |a_1b_2c_3d_4f_5g_6||a_1b_2c_3d_4|. \quad (\text{G}')$$

Of these (C′), (E′), (G′) deserve to be noted, being along with the original (A′) extensionals of the manifest identity

$$f_2g_6 - g_2f_6 = |f_2g_6|. \qquad (\text{XLIV. } 3), (\text{XXIII. } 6)$$

On the other hand (B′), (D′), (F′) are essentially the same as (B), (D), (F) already obtained—a fact which Desnanot overlooks.

As the source of a third series of results, obtained in still the same way, the identity

$$\binom{k\ l}{a\,h}\binom{k\ l}{f\,g} - \binom{k\ l}{a\,g}\binom{k\ l}{f\,h} = \binom{k\ l}{a\,f}\binom{k\ l}{h\,g} \qquad (\text{A}'')$$

is next taken. In reality, however, this does not differ from the first identity so treated, viz.,

$$\binom{k\ l}{a\,b}\binom{m\ p}{a\,b} - \binom{k\ p}{a\,b}\binom{m\ l}{a\,b} = \binom{k\ m}{a\,b}\binom{l\ p}{a\,b} \qquad (\text{A}).$$

In (A) the letters ab remain unchanged throughout, and the indices vary; while in (A″) the indices remain the same, and the letters vary. As we should now say, the difference is a mere matter of rows and columns. The derived identities (B″), (C″), and (D″), . . . are consequently found to be quite the same as (B), (C), (D), ...

The fourth and last source made use of is the well-known theorem regarding the aggregate of products whose first factors constitute what Cauchy would have called a "suite verticale," and whose second factors are the cofactors, in the determinant of the system, of another "suite verticale." Desnanot however, viewing the theorem from a different stand-point, enunciates it as follows (p. 26):—

"*Si l'on a* n *lettres* ab .... cdf, *et qu'on les combine* n − 1 *à* n − 1, *on aura* n *arrangemens* ab .... cd, ab .... cf, ab .... df, ... , a .... cdf, b .... cdf; *qu'on applique dans chaque arrangement les* n − 1 *indices* kl .... mp, *ce qui donnera ces quantités*

$$\binom{k\,l\,..\,m\,p}{a\,b\,..\,c\,d},\ \binom{k\,l\,..\,m\,p}{a\,b\,..\,c\,f},\ \binom{k\,l\,..\,m\,p}{a\,b\,..\,d\,f},\cdots\cdots\binom{k\,l\,..\,m\,p}{a\,..\,c\,d\,f},\ \binom{k\,l\,..\,m\,p}{b\,..\,c\,d\,f};$$

et qu'ensuite on les multiplie chacune par la lettre qui n'entre pas dans l'arrangement en l'affectant d'un même indice et donnant au produit le signe plus *ou le signe* moins, *suivant que la lettre multiplicateur occupe un rang impair ou pair dans les* n *lettres, en partant de la droite, la somme des produits sera zéro.*" (XII. 9)

Before proceeding to deduce others from it, he gives a proof of it for the case

$$(\mathrm{B}''')\quad \overset{p}{f}\binom{k\,l\,..\,m\,p}{a\,b\,..\,c\,d}-\overset{p}{d}\binom{k\,l\,..\,m\,p}{a\,b\,..\,c\,f}+\overset{p}{c}\binom{k\,l\,..\,m\,p}{a\,b\,..\,d\,f}-\cdots\cdot$$
$$\mp\overset{p}{b}\binom{k\,l\,..\,m\,p}{a\,..\,c\,d\,f}\pm\overset{p}{a}\binom{k\,l\,..\,m\,p}{b\,..\,c\,d\,f}=0.$$

The method of proof is interesting, because it depends almost entirely on the definition which Desnanot follows Vandermonde in using. It will be readily understood by seeing it applied to the simple case

$$b_1|b_2c_3d_4| - b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4|b_1c_2d_3| = 0.$$

Expanding each of the determinants $|b_2c_3d_4|$, $|b_1c_3d_4|$, ... in terms of the b's and their cofactors, we have

$$\begin{aligned}
&\quad b_1|b_2c_3d_4| - b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4|b_1c_2d_3|\\
&= \left.\begin{aligned}
&b_1\{\quad b_2|c_3d_4| - b_3|c_2d_4| + b_4|c_2d_3|\}\\
-&b_2\{b_1|c_3d_4| \quad - b_3|c_1d_4| + b_4|c_1d_3|\}\\
+&b_3\{b_1|c_2d_4| - b_2|c_1d_4| \quad + b_4|c_1d_2|\}\\
-&b_4\{b_1|c_2d_3| - b_2|c_1d_3| + b_3|c_1d_2| \quad\}
\end{aligned}\right\},\\
&= 0,
\end{aligned}$$

for the terms in the expanded form destroy each other in pairs.

The derived identities are obtained exactly in the manner followed by Bezout in 1779 (see pp. 51, 52). The fundamental identity is taken, say in the form

$$f_5|a_1b_2c_3d_4e_5| - e_5|a_1b_2c_3d_4f_5| + d_5|a_1b_2c_3e_4f_5| - c_5|a_1b_2d_3e_4f_5| \\ + b_5|a_1c_2d_3e_4f_5| - a_5|b_1c_2d_3e_4f_5| = 0,$$

and another instance is put alongside of it, in which the same letters and suffixes are involved, say

$$f_1|a_1b_2c_3d_4e_5| - e_1|a_1b_2c_3d_4f_5| + d_1|a_1b_2c_3e_4f_5| - c_1|a_1b_2d_3e_4f_5| \\ + b_1|a_1c_2d_3e_4f_5| - a_1|b_1c_2d_3e_4f_5| = 0.$$

One of the constituent determinants, say the last, $|b_1c_2d_3e_4f_5|$ is then eliminated by equalisation of coefficients and subtraction, the result being

$$|a_1f_5|.|a_1b_2c_3d_4e_5| - |a_1e_5||a_1b_2c_3d_4f_5| + |a_1d_5||a_1b_2c_3e_4f_5| \\ - |a_1c_5||a_1b_2d_3e_4f_5| + |a_1b_5||a_1c_2d_3e_4f_5| = 0 \quad (\mathrm{C}''')$$

In the next place, two additional instances of this derived identity are taken along with it, the first differing from it in having a 2 instead of a 5 in all the first factors, and the second in having a 2 instead of a 1; viz.,

$$|a_1f_2||a_1b_2c_3d_4e_5| - |a_1e_2||a_1b_2c_3d_4f_5| + |a_1d_2||a_1b_2c_3e_4f_5| \\ - |a_1c_2||a_1b_2d_3e_4f_5| + |a_1b_2||a_1c_2d_3e_4f_5| = 0,$$

and

$$|a_2f_5||a_1b_2c_3d_4e_5| - |a_2e_5||a_1b_2c_3d_4f_5| + |a_2d_5||a_1b_2c_3e_4f_5| \\ - |a_2c_5||a_1b_2d_3e_4f_5| + |a_2b_5||a_1c_2d_3e_4f_5| = 0.$$

Multiplication by $b_2, -b_5, -b_1$ is then effected and addition performed, when by reason of such identities as

$$b_2|a_1f_5| - b_5|a_1f_2| - b_1|a_2f_5| = |a_1b_2f_5|,$$

and

$$b_2|a_1b_5| - b_5|a_1b_2| - b_1|a_2b_5| = 0,$$

elimination of $|a_1c_2d_3e_4f_5|$ is produced, and the result takes the form

$$|a_1b_2f_5||a_1b_2c_3d_4e_5| - |a_1b_2e_5||a_1b_2c_3d_4f_5| + |a_1b_2d_5||a_1b_2c_3e_4f_5| \\ (\mathrm{D}''') \qquad - |a_1b_2c_5||a_1b_2d_3e_4f_5| = 0.$$

The process of derivation may be pursued further, giving next an identity in which the first factors are all of the fourth order. Desnanot says (pp. 31, 32)—

"Pour ne pas nous répéter constamment, nous dirons que cette formule s'étendrait à un nombre quelconque de lettres placées dans les premiers facteurs, et que

$$(\mathrm{H}''') \quad \begin{aligned}&\binom{k\, l\, \ldots\, p}{a\, b\, \ldots\, f}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, c\, d} - \binom{k\, l\, \ldots\, p}{a\, b\, \ldots\, d}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, c\, f} \\ &\qquad + \binom{k\, l\, \ldots\, p}{a\, b\, \ldots\, c}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, d\, f} - \;\ldots\ldots = 0.\end{aligned}$$

Les termes sont alternativement positifs et négatifs, les indices sont les mêmes dans les premiers facteurs de chaque terme, ils font partie des indices qui se trouvent dans les autres facteurs et sont placés dans le même ordre ; quant aux lettres, il y a ou une, ou deux, ou trois, etc. lettres communes aux seconds facteurs écrites toujours dans le même ordre et suivies de la $n^{\text{ième}}$ lettre qui n'entre pas dans les seconds facteurs ; de sorte que s'il y a n' lettres communes à tous les facteurs, le nombre des termes de (H''') sera $n - n'$." (XXIII. 7) (XLIV. 4)

The general result (H''') is simply what would now be called the 'extensional' of the identity of Vandermonde from which Desnanot derives it.

Co-ordinate, in a sense, with the said identity, is that other which Desnanot uses as a definition; and this latter is the next of which the extensional is found. The process, so far as indicated, is exactly similar to that employed in the preceding case. The results obtained are

$$(\mathrm{B}'''') \quad \begin{aligned}&\binom{p\ \ r}{a\ f}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, c\, d} - \binom{p\ \ r}{a\ d}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, c\, f} + \binom{p\ \ r}{a\ c}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, d\, f} - \\ &\qquad \ldots\ldots \mp \binom{k\ \ r}{a\ b}\binom{k l\, \ldots\, m\, p}{a\, \ldots\, c\, d\, f} = \binom{p}{a}\binom{k\, l\, \ldots\, m\, p\, r}{a\, b\, \ldots\, c\, d\, f},\end{aligned}$$

and

$$(\mathrm{C}'''') \quad \begin{aligned}&\binom{k\, p\, r}{a\, b\, f}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, c\, d} - \binom{k\, p\, r}{a\, b\, d}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, c\, f} \\ &\qquad + \binom{k\, p\, r}{a\, b\, c}\binom{k\, l\, \ldots\, m\, p}{a\, b\, \ldots\, d\, f} - \;\ldots\ldots = \binom{k\ p}{a\ b}\binom{k\, l\, \ldots\, m\, p\, r}{a\, b\, \ldots\, c\, d\, f};\end{aligned}$$

and the general result including them is referred to. (VI. 4) (XLV.)

That they are extensionals of the definition is evident from the fact that the index p may be moved to the left so as to make $\overset{p}{a}$ common to every factor of (B''''), and $\overset{k}{a}\overset{p}{b}$ common to every factor of (C'''').

Still another series of results is obtained, but they are essentially the same as the foregoing, the difference again being merely a matter of rows and columns.

All these preparations having been made, Desnanot returns to the subject of the relations between the numerators and denominators of the values of the unknowns in a set of linear equations. Thirteen pages are occupied with the case of four unknowns, the number of relations found being 74, of which, after scrutiny, 14 are retained. The case of five unknowns, and the case of six unknowns are gone into with about as much detail, and then, lastly, the general set of n equations with n unknowns is dealt with. None of the relations obtained need be given, as they are all included in the identities which have been spoken of above as extensionals.

The second chapter (p. 94) bears the heading

Simplification des formules générales qui donnent les valeurs des inconnues dans les équations du premier degré, lorsqu'on veut les évaluer en nombres.

Here again the cases of three, four, five, six unknowns are dwelt upon with equal fulness in succession. The consideration of one of them will suffice to show the nature of the method, and will enable the reader to judge of the amount of labour saved by employing it. Choosing the case of four unknowns, we find at the outset the equations stated and the solution condensed as follows (p. 104):—

"Equations données.

$$\begin{aligned}
ax + by + cz + dt &= f \\
a'x + b'y + c'z + d't &= f' \\
a''x + b''y + c''z + d''t &= f'' \\
a'''x + b'''y + c'''z + d'''t &= f'''.
\end{aligned}$$

CALCUL.

$$ab' - ba' = a,$$
$$ab'' - b''a = \beta,$$
$$a'b'' - b''a' = \gamma,$$
$$ab''' - ba''' = \delta,$$
$$a'b''' - b'a''' = \epsilon;$$

$$m = c''\,a - c'\beta + c\gamma,$$
$$n = c'''\,a - c'\delta + c\epsilon,$$
$$m' = f''\,a - f'\beta + f\gamma,$$
$$n' = f'''\,a - f'\delta + f\epsilon;$$

$$D = \frac{1}{a}\left\{m(ad''' - \delta d' + \epsilon d) - n(ad'' - \beta d' + \gamma d)\right\},$$

$$N''' = \frac{1}{a}\left(mn' - nm'\right),$$

$$N'' = \frac{1}{a}\left\{m'(ad''' - \delta d' + \epsilon d) - n'(ad'' - \beta d' + \gamma d)\right\};$$

$$fD - cN'' - dN''' = S,$$
$$f'D - c'N'' - d'N''' = S';$$

$$N' = \frac{aS' - Sa'}{a},$$

$$N = \frac{Sb' - bS'}{a};$$

$$x = \frac{N}{D}, \quad y = \frac{N'}{D}, \quad z = \frac{N''}{D}, \quad t = \frac{N'''}{D}.\text{"}$$

The explanation of the mode of procedure is not difficult to see:—

(1) The determinants $|ab'|, |ab''|, |a'b''|, |ab'''|, |a'b'''|$ are calculated.

(2) With the help of these are next got four of a higher order, viz. $|ab'c''|, |ab'c'''|, |ab'f''|, |ab'f'''|$.

(3) Two others of the same order, viz.

$$ad''' - \delta d' + \epsilon d, \quad ad'' - \beta d' + \gamma d,$$

i.e.

$$|ab'd'''|, \quad |ab'd''|,$$

having been calculated, the identity

$$|ab'|\,.\,\mathrm{D} = |ab'c''|\,.\,|ab'd'''| - |ab'c'''|\,.\,|ab'd''|$$

is used to find D.

(4) A similar identity

$$|ab'|\,.\,\mathrm{N}''' = |ab'c''|\,.\,|ab'f'''| - |ab'c'''|\,.\,|ab'f''|$$

is used to find N‴.

(5) A similar identity

$$|ab'|\,.\,\mathrm{N}'' = |ab'f''|\,.\,|ab'd'''| - |ab'f'''|\,.\,|ab'd''|$$

is used to find N″.

(6) Two subsidiary quantities S, S′ are calculated, the first being

$$= f|ab'c''d'''| - c|ab'f''d'''| - d|ab'c''f'''|,$$

and the second

$$= f'|ab'c''d'''| - c'|ab'f''d'''| - d'|ab'c''f'''|.$$

(7) From these N′ and N are readily got. For evidently

$$a\mathrm{S}' - \mathrm{S}a'$$

$$= |af'|\,.\,|ab'c''d'''| - |ac'|\,.\,|ab'f''d'''| - |ad'|\,.\,|ab'c''f'''|$$

and this by a previous theorem

$$= |ab'|\,.\,|af'c''d'''|,$$

$$= |ab'|\,.\,\mathrm{N}'. \qquad \text{(XIII. 3)}$$

The third chapter consists of a lengthy examination (pp. 157-264) of the singular cases met with in the solution of linear equations, and does not concern us.

CAUCHY (1821).

[Cours d'Analyse de l'École Royale Polytechnique. I. Analyse Algébrique.* xvi+576 pp. Paris. *Œuvres,* 2[e] sér. iii. pp. 73–82, 426–428.]

When Cauchy came to write his *Course of Analysis,* afterwards so well known, he did not fail to assign a position in it to the subject of his memoir of 1812. The third chapter bears the heading, "*Des Fonctions Symétriques et des Fonctions Alternées.*"

*No more published.

It occupies, however, only fifteen pages (pp. 70–84), and of these only nine are devoted to alternating functions and the solution of simultaneous linear equations. Of course, in so limited a space, the merest sketch of a theory is all that is possible. An alternating function is first defined, the word "alternée" being now set in contrast with "symétrique," and not, as formerly, with "permanente." Functions other than those that are rational and integral being left aside, the latter, if alternating, are shown (1) to consist of as many positive as negative terms, in each of which all the variables occur with different indices, and (2) to be divisible by the simplest of all alternating functions of the variables, viz., the difference-product. The set of equations

$$a_r x + b_r y + c_r z + \ldots + g_r u + h_r v = k_r \qquad (r = 0, 1, \ldots, n-1)$$

is then attacked, the method being—to take the difference-product of $a, b, \ldots, h$,—denote by D what the expansion of this becomes when exponents are changed into suffixes,—denote by A_r the co-factor of a_r in D,—then obtain the equations

$$\begin{aligned} A_0a_0 + A_1a_1 + A_2a_2 + \ldots + A_{n-1}a_{n-1} &= D, \\ A_0b_0 + A_1b_1 + A_2b_2 + \ldots + A_{n-1}b_{n-1} &= 0, \\ A_0c_0 + A_1c_1 + A_2c_2 + \ldots + A_{n-1}c_{n-1} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_0h_0 + A_1h_1 + A_2h_2 + \ldots + A_{n-1}h_{n-1} &= 0, \end{aligned}$$

—and thereafter proceed as Laplace had taught. As in the memoir of 1812, the "symbolic" form of the values of $x, y, \ldots$ is unfailingly given.

A note is added (pp. 521–524) on the development of the difference-product, showing how all the terms may be got from one by interchanging one exponent with another, how the signs depend on the number of said interchanges, and how, by counting the number of cycles (here called *groups*), it may be ascertained whether any two given terms have like or unlike signs.

It will thus be seen that not only is the name "determinant" never mentioned in the chapter, and the notation $S \pm a_0b_1c_2 \ldots h_{n-1}$ never used, but that the subject is scarcely so much as touched upon. Although, therefore, Cauchy's text-book went through a considerable number of editions, and had a widespread influence, it gave no such impulse as it might have done to the study of the theory of determinants.

SCHERK (1825).

[Mathematische Abhandlungen. Von Dr. Heinrich Ferdinand Scherk, ... iv+116 pp. Berlin. (Pp. 31–66. Zweite Abhandlung: Allgemeine Auflösung der Gleichungen des ersten Grades mit jeder beliebigen Anzahl von unbekannten Grössen, und einige dahin gehörige analytische Untersuchungen.)]

The only previous writings of importance known to Scherk were, according to his own statement, those of Cramer, Bezout (1764), Vandermonde, Bezout (1779), Hindenburg, and Rothe. His style bears most resemblance to Rothe's, whose paper, however, he does not speak of with unmixed eulogy, characterising it as containing "eine strenge aber ziemlich weitläufige Auflösung der Aufgabe."

The main part of the memoir consists of a lengthy demonstration, extending, indeed, to 17 pages quarto, of Cramer's rule, or rather of Cramer's set of three rules (iv., v., iii. 2), by the method of so-called mathematical induction. The peculiarity of the demonstration is that it is entered upon without any previous examination of the properties of Cramer's functions (determinants); and it is noteworthy on two grounds—(1) as being new, and (2) because the properties, which it really if not explicitly employs, had also not been previously referred to.

The cases of one equation with one unknown, two equations with two unknowns, three equations with three unknowns, are dealt with in succession, the solution of one case being used in obtaining the solution of the next. All three solutions are noted as being in accordance with Cramer's rules, and the said rules being formulated, and upposed to hold for n equations with n unknowns, it is sought to establish their validity for $n+1$ equations with $n+1$ unknowns. In other words, the set of n equations being

$$\left.\begin{array}{l}
\overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{1}{a}}\overset{n-2}{x} + \ldots\ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{2}{a}}\overset{n-2}{x} + \ldots\ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{n}{a}}\overset{n-2}{x} + \ldots\ldots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s}
\end{array}\right\}$$

and the corresponding values of

$$\overset{1}{x},\ \overset{2}{x},\ \ldots\ldots,\ \overset{n}{x},$$

being

$$\frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{1}{\underset{h}{a}},\overset{1}{\underset{h}{a}}\right)},\quad \frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\overset{2}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{2}{\underset{h}{a}},\overset{2}{\underset{h}{a}}\right)},\quad \ldots\ldots,\quad \frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\overset{n}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{n}{\underset{h}{a}},\overset{n}{\underset{h}{a}}\right)},$$

it is required to show that the solution of the set of $n+1$ equations

$$\left.\begin{array}{l}
\overset{n+1}{\underset{1}{a}}\overset{n+1}{x} + \overset{n}{\underset{1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n+1}{\underset{2}{a}}\overset{n+1}{x} + \overset{n}{\underset{2}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\overset{n+1}{\underset{n+1}{a}}\overset{n+1}{x} + \overset{n}{\underset{n+1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{n+1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{n+1}{a}}\overset{1}{x} = \underset{n+1}{s}
\end{array}\right\}$$

is

$$\overset{1}{x} = \frac{P\left(\overset{n+1}{\underset{n+1}{a}};\ \underset{h}{s},\overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n+1}{\underset{n+1}{a}};\overset{1}{\underset{h}{a}},\overset{1}{\underset{h}{a}}\right)},\quad \ldots\ldots,\quad \overset{n+1}{x} = \frac{\left(\overset{n+1}{\underset{n+1}{a}};\ \underset{h}{s},\overset{n+1}{\underset{h}{a}}\right)}{\left(\overset{n+1}{\underset{n+1}{a}};\overset{n+1}{\underset{h}{a}},\overset{n+1}{\underset{h}{a}}\right)}. \qquad \text{(XIII. 4)}$$

Before proceeding, the notation

$$P\left(\overset{n}{\underset{n}{a}};\underset{h}{s},\overset{1}{\underset{h}{a}}\right)$$

requires attention. It is meant to be an epitome of Cramer's rules; the first half of the group of symbols, viz. $P\left(\overset{n}{\underset{n}{a}}\right.$ implying permutation of the under-indices of the product $\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}}\ldots\overset{n}{\underset{n}{a}}$ and aggregation of the different products thus obtained, each taken with its proper sign: and the second half implying that in every term of this aggregate $\underset{h}{s}$ is to be substituted for $\overset{1}{\underset{h}{a}}$. A modern writer would denote the same thing by

$$\begin{vmatrix} \underset{1}{s} & \overset{2}{\underset{1}{a}} & \overset{3}{\underset{1}{a}} & \dots & \overset{n}{\underset{1}{a}} \\ \underset{2}{s} & \overset{2}{\underset{2}{a}} & \overset{3}{\underset{2}{a}} & \dots & \overset{n}{\underset{2}{a}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \underset{n}{s} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \dots & \overset{n}{\underset{n}{a}} \end{vmatrix},$$

only it must be noted that in using $\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\underset{h}{s}\,,\overset{1}{\underset{h}{a}}\right)$ at this stage, we leave out of account the signs of the terms composing it, the rule of signs being the subject of a separate investigation. Any one of the forms

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\overset{1}{\underset{h}{a}}\,,\overset{1}{\underset{h}{a}}\right),\quad \mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\overset{2}{\underset{h}{a}}\,,\overset{2}{\underset{h}{a}}\right),\quad \dots\dots,$$

it need scarcely be added, will thus stand for the common denominator.

Of the $n+1$ equations the first n are taken, written in the form

$$\left.\begin{array}{l} \overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\overset{n-1}{x} + \dots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \dots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} - \overset{n+1}{\underset{1}{a}}\overset{n+1}{x} \\ \overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\overset{n-1}{x} + \dots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \dots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} - \overset{n+1}{\underset{2}{a}}\overset{n+1}{x} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\overset{n-1}{x} + \dots + \overset{k}{\underset{n}{a}}\overset{k}{x} + \dots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s} - \overset{n+1}{\underset{n}{a}}\overset{n+1}{x} \end{array}\right\},$$

and solved, the results being by hypothesis

$$\overset{1}{x} = \frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\,,\overset{1}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\overset{1}{\underset{h}{a}}\,,\overset{1}{\underset{h}{a}}\right)},$$

$$\dots\dots\dots$$

$$\overset{k}{x} = \frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \overset{2}{\underset{h}{a}} , \overset{2}{\underset{h}{a}}\right)},$$

$$\cdots\cdots\cdots\cdots$$

$$\overset{n}{x} = \frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{n}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \overset{n}{\underset{h}{a}} , \overset{n}{\underset{h}{a}}\right)}.$$

These values are then of course substituted in the $(n+1)^{\text{th}}$ equation, which thus becomes

$$\overset{n+1}{\underset{n+1}{a}}\,\overset{n+1}{x} + \overset{n}{\underset{n+1}{a}}\frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{1}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \overset{1}{\underset{h}{a}} , \overset{1}{\underset{h}{a}}\right)} + \ldots + \overset{k}{\underset{n+1}{a}}\frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \overset{k}{\underset{h}{a}} , \overset{k}{\underset{h}{a}}\right)}$$

$$+ \ldots + \overset{1}{\underset{n+1}{a}}\frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{1}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \overset{n}{\underset{h}{a}} , \overset{n}{\underset{h}{a}}\right)} = \underset{n+1}{s};$$

and as this manifestly involves none of the unknowns but $\overset{n+1}{x}$, the object must now be to solve for $\overset{n+1}{x}$, and then show what the value obtained is transformable into. The way in which this is effected is well worthy of attention. Scherk's own words in regard to the first steps are (p. 40)—

"Da aber $\underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x}$ in jeder einzelnen Permutationsform nur Einmal, nämlich in der ersten Potenz vorkommt, so bedeutet das Zeichen

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right)$$

dass in jede der in I. beschriebenen Permutationsformen für $\overset{k}{\underset{h}{a}}$ erst $\underset{h}{s}$, dann $\overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x}$ gesetzt, und beide Resultate von einander abgezogen werden sollen: folglich ist

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right) = \mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{k}{\underset{h}{a}}\right) - \mathrm{P}\left(\overset{n}{\underset{n}{a}} ; \overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right).$$

In dem letzten Gliede dieser Gleichung kommt aber in jeder Form $\overset{n+1}{x}$, und zwar zur ersten Potenz, vor; $\overset{n+1}{x}$ ist also gemeinschaftlicher Factor aller Formen, und folglich ist

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\,,\,\overset{k}{\underset{h}{a}}\right) = \mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{k}{\underset{h}{a}}\right) - \overset{n+1}{x}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right).$$

Macht man diese Substitution für $k=1, 2, \ldots, n$, in der letzten Gleichung, und bemerkt, dass

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{1}{\underset{h}{a}}\,,\,\overset{1}{\underset{h}{a}}\right) = \mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{2}{\underset{h}{a}}\,,\,\overset{2}{\underset{h}{a}}\right) = \ldots\ldots = \mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right),$$

so geht diese in folgende Gleichung über

$$\begin{aligned}
&\overset{n+1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)\overset{n+1}{x}\\
&+\left\{\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{n}{\underset{h}{a}}\right)+\ldots+\overset{k}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{k}{\underset{h}{a}}\right)+\ldots+\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{\underset{h}{a}}\right)\right\}\\
&-\left\{\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{n}{\underset{h}{a}}\right)+\ldots+\overset{k}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)+\ldots+\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{1}{\underset{h}{a}}\right)\right\}\overset{n+1}{x}\\
&=\underset{n+1}{s}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right);
\end{aligned}$$

folglich

$$\overset{n+1}{x} = \frac{-\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{\underset{h}{a}}\right)-\overset{2}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{2}{\underset{h}{a}}\right)-\ldots-\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{n}{\underset{h}{a}}\right)+\underset{n+1}{s}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)}{-\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{1}{\underset{h}{a}}\right)-\overset{2}{\underset{n+1}{a}}\,\mathrm{P}\left(\underset{n}{a}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{2}{\underset{h}{a}}\right)-\ldots-\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{n}{\underset{h}{a}}\right)+\overset{n+1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)}."$$

The first theorem here made use of and formulated, viz.,

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\,,\,\overset{k}{\underset{h}{a}}\right) = \mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\underset{h}{a}\right) - \mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\,,\,\overset{k}{\underset{h}{a}}\right) \qquad \text{(XLVI.)}$$

is the now familiar rule for the partition of a determinant with a row or column of binomial elements into two determinants, or for the addition of two determinants which are identical except in one row or one column. The second theorem, viz.,

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\;\overset{k}{a}\right) = \overset{n+1}{x}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right) \qquad \text{XLVII.)}$$

is the now equally familiar theorem regarding the multiplication of a determinant by means of the multiplication of all the elements of a row or column. That these two very elementary theorems should not have been noted until the time of Scherk is rather remarkable.

The consideration of the constitution of

$$\mathrm{P}\left(\overset{n+1}{\underset{n+1}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)$$

is next entered upon, with the object of showing that the terms are exactly the terms of the denominator

$$-\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\overset{1}{\underset{h}{a}}\right)-\overset{2}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\overset{2}{\underset{h}{a}}\right)-\ldots\ldots+\overset{n+1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right).$$

More than two pages are occupied with this part proof of Bezout's recurrent law of formation. The identity of the terms of

$$\mathrm{P}\left(\overset{n+1}{\underset{n+1}{a}}\,;\,\underset{h}{s}\,,\,\overset{n+1}{\underset{h}{a}}\right)$$

with the terms of the numerator then follows at once; and the desired form for the value of $\overset{n+1}{x}$, so far as the *magnitude* of the terms is concerned, is thus obtained. The corresponding forms for $x_1, x_2, \ldots$ are of course immediately deducible.

The rules for obtaining the terms of the numerator and denominator having been thus established in all their generality, the rule of signs is next dealt with. The treatment is cumbersome, but fresh and interesting. It is pointed out, to start with, that the counting of the inversions of order of a permutation, is equivalent to subtracting separately from each element all the elements which follow it, reckoning $+1$ as a sign-factor when the difference is positive, and -1 when the difference is negative, and then taking the product of all the said factors. This, it will be recalled, is essentially identical with an observation of Cauchy's. Scherk, however, goes on to remark that these sign-factors may be viewed as functions of the differences which give rise to them, and may be so represented. Whether there actually be a function which equals $+1$ for all positive values of the argument and equals -1 for all negative values is left for future consideration. Cramer's rule of signs is thus made to take the following form (p. 45):—

"Wenn $\phi(\beta)$ eine solche Function der ganzen Zahl β ist, welche $= +1$ ist wenn β positiv, und -1 wenn β negativ ist, so ist das Vorzeichen Z irgend eines in dem Aggregate

$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\ \overset{h}{\underset{k}{a}}\,,\ \overset{h}{\underset{k}{a}}\right)$ enthaltenen Gliedes

$$\overset{1}{\underset{\alpha}{a}}\ \overset{2}{\underset{\alpha}{a}}\ \overset{3}{\underset{\alpha'''}{a}}\ \ldots\ \overset{k-1}{\underset{\alpha^{(k-1)}}{a}}\ \overset{k}{\underset{\alpha^{(k)}}{a}}\ \overset{k--1}{\underset{\alpha^{(k+1)}}{a}}\ \ldots\ \overset{n}{\underset{\alpha^{(n)}}{a}}$$

folgendes:

$$\begin{aligned}
Z = {} & \phi(\alpha''-\alpha').\phi(\alpha'''-\alpha')\ldots\phi(\alpha^{(k-1)}-\alpha').\phi(\alpha^{(k)}-\alpha').\phi(\alpha^{(k+1)}-\alpha')\ldots\phi(\alpha^{(n)}-\alpha') \\
& \phi(\alpha'''-\alpha'').\phi(\alpha''''-\alpha'')\ldots\phi(\alpha^{(k)}-\alpha'').\phi(\alpha^{(k+1)}-\alpha'').\phi(\alpha^{(k+2)}-\alpha'')\ldots\phi(\alpha^{(n)}-\alpha'') \\
& \ldots\ldots\ldots\ldots\ldots\ldots \\
& \phi(\alpha^{(k-1)}-\alpha^{(k-2)}).\phi(\alpha^{(k}-\alpha^{(k-2)})\ldots\phi(\alpha^{(n)}-\alpha^{(k-2)}) \\
& \phi(\alpha^{(k)}-\alpha^{(k-1)}).\phi(\alpha^{(k+1)}-\alpha^{(k-1)})\ldots\phi(\alpha^{(n)}-\alpha^{(k-1)}) \\
& \phi(\alpha^{(k+1)}-\alpha^{(k)}).\phi(\alpha^{(k+2)}-\alpha^{(k)})\ldots\phi(\alpha^{(n)}-\alpha^{(k)}) \\
& \ldots\ldots \\
& \phi(\alpha^{(n)}-\alpha^{(n-1}).\text{"} \qquad\qquad (\text{III. } 23)
\end{aligned}$$

And it is this form which Scherk seeks to establish. The mode of proof is again the so-called inductive mode. In the case of *two* permutable indices the law is readily seen to hold. We thus have, preparatory for the next case,

$$\begin{aligned}
\mathrm{P}\left(\overset{2}{\underset{2}{a}};\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right) &= \phi(2-1)\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}, \\
\mathrm{P}\left(\overset{2}{\underset{2}{a}};\ \overset{3}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right) &= \phi(2-1)\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{3}{\underset{2}{a}}\overset{2}{\underset{1}{a}}, \\
\mathrm{P}\left(\overset{2}{\underset{2}{a}};\ \overset{3}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right) &= \phi(2-1)\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}.
\end{aligned}$$

But "nach dem Obigem"

$$\mathrm{P}\left(\overset{3}{\underset{3}{a}};\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right) = -\overset{1}{\underset{3}{a}}\mathrm{P}\left(\overset{2}{\underset{2}{a}};\ \overset{3}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right) - \overset{2}{\underset{3}{a}}\mathrm{P}\left(\overset{2}{\underset{2}{a}};\ \overset{3}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right) + \overset{3}{\underset{3}{a}}\mathrm{P}\left(\overset{2}{\underset{2}{a}};\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right).$$

Consequently

$$\begin{aligned}
\mathrm{P}\left(\overset{3}{\underset{3}{a}};\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right) = & -\overset{1}{\underset{3}{a}}\left\{\phi(2-1)\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\right\} \\
& -\overset{2}{\underset{3}{a}}\left\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}\right\} \\
& +\overset{3}{\underset{3}{a}}\left\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\right\}.
\end{aligned}$$

But as

$$-\phi(2-1) = \phi(1-2)$$
$$\text{and} \quad -1 = \phi(2-3)$$
$$\text{and} \quad -1 = \phi(1-3),$$

we may substitute

$$\phi(1-2)\phi(2-3)\phi(1-3) \text{ for } -\phi(2-1).$$

This and five other similar substitutions give us

$$\begin{aligned} \mathrm{P}\Big(\overset{3}{\underset{3}{a}}\,;\overset{k}{\underset{h}{a}}\,,\overset{k}{\underset{h}{a}}\Big) = \;& \phi(2-1)\phi(3-1)\phi(3-2)\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}} + \phi(3-1)\phi(2-1)\phi(2-3)\overset{1}{\underset{1}{a}}\overset{2}{\underset{3}{a}}\overset{3}{\underset{2}{a}} \\ &+\phi(1-2)\phi(3-2)\phi(3-1)\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\overset{3}{\underset{3}{a}} + \phi(3-2)\phi(1-2)\phi(1-3)\overset{1}{\underset{2}{a}}\overset{2}{\underset{3}{a}}\overset{3}{\underset{1}{a}} \\ &+\phi(1-3)\phi(2-3)\phi(2-1)\overset{1}{\underset{3}{a}}\overset{2}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(2-3)\phi(1-3)\phi(1-2)\overset{1}{\underset{3}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{1}{a}}; \end{aligned}$$

so that the law is seen to hold also for the case of *three* permutable indices. The completion of the proof, giving the transition from n to $n+1$ permutable indices, occupies three pages.

This is followed by two pages devoted to the subjects temporarily set aside at the outset, viz., the possible existence of functions having the peculiar properties of ϕ. Two amusing instances of such functions are given,—

$$(1) \quad \phi(\beta) = \left.\begin{array}{l} \mathrm{P}_0^{\beta-1}+\mathrm{P}_0^{\beta-2}+\mathrm{P}_0^{\beta-3}+\dots \\ -\mathrm{P}_0^{\beta-1}-\mathrm{P}_0^{\beta-2}-\mathrm{P}_0^{\beta-3}-\dots \end{array}\right\}$$

$$(2) \quad \phi(\beta) = \left.\begin{array}{l} \dfrac{\sin 2\beta\pi}{(\beta-1)2\pi}+\dfrac{\sin 2\beta\pi}{(\beta-2)2\pi}+\dfrac{\sin 2\beta\pi}{(\beta-3)2\pi}+\dots \\ -\dfrac{\sin 2\beta\pi}{(\beta+1)2\pi}-\dfrac{\sin 2\beta\pi}{(\beta+2)2\pi}-\dfrac{\sin 2\beta\pi}{(\beta+3)2\pi}-\dots \end{array}\right\}$$

$$= \Big(\frac{1}{\beta^2-1}+\frac{2}{\beta^2-4}+\frac{3}{\beta^2-9}+\dots\Big)\frac{\sin 2\beta\pi}{\pi},$$

where P_0^k stands for the k^{th} coefficient in the expansion of $(a+b)^0$. Success, however far from brilliant it may be, in thus expressing the rule of signs by means of the symbols of analysis, led Scherk to try to do the same for the rule of formation of the terms. Nothing came of the attempt, however. "Bald aber," he says, "zeigte es sich dass Permutationen niemals durch andere analytische Zeichen ersetzt werden kõnnten."

Such speculations are not altogether uninteresting when later work like Hankel's comes to be considered.

In an Appendix dealing (1) with the case of a set of linear equations which are not all independent, (2) with the solution of particular sets of equations, there is given at the outset a proof of the theorem regarding the sign of a permutation which is got from another permutation by the interchange of two elements. If the under-indices of the one term whose sign is z be

$$a'a''a''' \ldots . a^{(i-1)}a^{(i)}a^{(i+1)} \ldots . a^{(k-1)}a^{(k)}a^{(k+1)} \ldots . a^{(n)},$$

and of the other whose sign is Z* be

$$a'a''a''' \ldots . a^{(i-1)}a^{(k)}a^{(i+1)} \ldots . a^{(k-1)}a^{i}a^{(k+1)} \ldots . a^{(n)}$$

it is shown that

$$\begin{aligned} \frac{z}{Z} &= \frac{\phi(a^{i+1}-a^{i})}{\phi(a^{i}-a^{i+1})} \cdot \frac{\phi(a^{i+2}-a^{i})}{\phi(a^{i}-a^{i+2})} \cdots \frac{\phi(a^{k}-a^{i})}{\phi(a^{i}-a^{k})} \\ &\times \frac{\phi(a^{k}-a^{i})}{\phi(a^{i}-a^{k})} \cdot \frac{\phi(a^{k}-a^{i+1})}{\phi(a^{i+1}-a^{k})} \cdots \frac{\phi(a^{k}-a^{k-1})}{\phi(a^{k-1}-a^{k})}; \end{aligned}$$

and there being here $2k-2i-1$ quotients each $=-1$, the result arrived at is

$$\frac{z}{Z} = -1 \qquad \text{or} \quad z = -Z,$$

as was to be proved. (III. 24)

The body of the Appendix contains, along with other matter which falls to be considered later, the statement and proof of propositions identical in essence but not in form with the following:—

$$(1) \qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & \overset{2}{\underset{1}{a}} & \ldots . & \overset{n-1}{\underset{1}{a}} & \overset{n}{\underset{1}{a}} \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & \ldots . & \overset{n-1}{\underset{2}{a}} & \overset{n}{\underset{2}{a}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \overset{1}{\underset{n-1}{a}} & \overset{2}{\underset{n-1}{a}} & \ldots . & \overset{n-1}{\underset{n-1}{a}} & \overset{n}{\underset{n-1}{a}} \\ \overset{1}{T} & \overset{2}{T} & \ldots . & \overset{n-1}{T} & \overset{n}{T} \end{vmatrix} = 0, \qquad \text{(XLVIII.)}$$

* More than a page is occupied in writing the expressions for z and Z.

where
$$\overset{1}{\mathrm{T}} = \underset{1}{m}\,\overset{1}{\underset{1}{a}} + \underset{2}{m}\,\overset{1}{\underset{2}{a}} + \ldots + \underset{n-1}{m}\;\overset{1}{\underset{n-1}{a}}$$
$$\overset{2}{\mathrm{T}} = \underset{1}{m}\,\overset{2}{\underset{1}{a}} + \underset{2}{m}\,\overset{2}{\underset{2}{a}} + \ldots + \underset{n-1}{m}\;\overset{2}{\underset{n-1}{a}}$$
$$\cdots\cdots\cdots\cdots\cdots$$

$$(2)\qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & & & & \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & & & \\ \overset{1}{\underset{3}{a}} & \overset{2}{\underset{3}{a}} & \overset{3}{\underset{3}{a}} & & \\ \cdot & \cdot & \cdot\ \cdot & \cdot & \\ \cdot & \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot & \\ \overset{1}{\underset{n}{a}} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \ldots\ldots & \overset{n}{\underset{n}{a}} \end{vmatrix} = \overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}}\ldots\ldots\overset{n}{\underset{n}{a}}.$$

The first of these is proved from first principles, and not by the immediate use of theorems XLVI., XLVII. above. The second is proved by noting that any other term is got from the first,

$$\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}}\ldots\ldots\overset{n}{\underset{n}{a}},$$

by permutation of the under-indices, that any such permutation will introduce one or more elements whose upper-index exceeds the lower, and that such are all zero. (VI. 6)

SCHWEINS (1825).

[Theorie der Differenzen und Differentiale, u.s.w. Von Ferd. Schweins. vi + 666 pp. Heidelberg. (Pp. 317–431; Theorie der Producte mit Versetzungen.)]

With much of the preceding literature Schweins, our next author, was thoroughly familiar. Cramer, Bezout, Hindenburg, Rothe, Laplace, Desnanot, and Wronski he refers to by name. The one notable investigator left out of his list is Cauchy, whose important memoir bearing date 1812 might have been known, one would think, to a writer who knew Desnanot's book of 1819 and Wronski's memoirs of 1810, 1811, &c. Still more curious is the omission of Vandermonde's name, whose memoir, as we have seen, is to be found in the very same volume as that of Laplace.

Schweins' portly volume consists of seven separate treatises. It is the third, headed *Theorie der Producte mit Versetzungen*, which deals expressly and exclusively with the subject of determinants. The treatise is logically arranged and carefully written. It opens with an introduction of 4 pp., the main part of which serves as a table of contents and as a guide to the theorems which the author considered his own. It consists of four Sections (Abtheilungen), subdivided into portions which we may call chapters, the first section containing five chapters, the second also five, the third one, and the fourth four.

Schweins' name for the function is

Producte mit Versetzungen; (XV. 6)

his notation is a modification of Laplace's, viz., he uses

$$\Big\| \qquad \Big) \qquad \text{(VII. 6)}$$

where Laplace used simply

$$\Big(\qquad \Big)$$

and his definition is essentially the same as Vandermonde's; that is to say, he employs Bezout's law of recurring formation. His words at the outset are—

"Die Bildungsweise der Producte, welche hier untersucht werden sollen, geben folgende [illegible]en an :—

$$\left\| \overset{a_1}{A_1} \right) = \overset{a_1}{A_1},$$

$$\left\| \overset{a_1}{A_1}\,\overset{a_2}{A_2} \right) = \left\| \overset{a_1}{A_1} \right).\overset{a_2}{A_2} - \left\| \overset{a_2}{A_1} \right).\overset{a_1}{A_2},$$

$$\left\| \overset{a_1}{A_1}\,\overset{a_2}{A_2}\,\overset{a_3}{A_3} \right) = \left\| \overset{a_1}{A_1}\,\overset{a_2}{A_2} \right).\overset{a_3}{A_3} - \left\| \overset{a_1}{A_1}\,\overset{a_3}{A_2} \right).\overset{a_2}{A_3} + \left\| \overset{a_2}{A_1}\,\overset{a_3}{A_2} \right).\overset{a_1}{A_3},$$

$$\begin{aligned}\left\| \overset{a_1}{A_1}\,\overset{a_2}{A_2}\,\overset{a_3}{A_3}\,\overset{a_4}{A_4} \right) = {} & \left\| \overset{a_1}{A_1}\,\overset{a_2}{A_2}\,\overset{a_3}{A_3} \right).\overset{a_4}{A_4} - \left\| \overset{a_1}{A_1}\,\overset{a_2}{A_2}\,\overset{a_4}{A_3} \right).\overset{a_3}{A_4} \\ & + \left\| \overset{a_1}{A_1}\,\overset{a_3}{A_2}\,\overset{a_4}{A_3} \right).\overset{a_2}{A_4} - \left\| \overset{a_2}{A_1}\,\overset{a_3}{A_2}\,\overset{a_4}{A_3} \right).\overset{a_1}{A_4},\end{aligned}$$

und allgemein

$$\begin{aligned}\left\| \overset{a_1 \cdots\cdots a_n}{A_1 \cdots\cdots A_n} \right) = {} & (-)\left\| \overset{a_1 \cdots\cdots a_{n-1}}{A_1 \cdots\cdots A_{n-1}} \right).\overset{a_n}{A_n} + (-)^1 \left\| \overset{a_1 \cdots a_{n-2}\, a_n}{A_1 \cdots\cdots A_{n-1}} \right)\overset{a_{n-1}}{A_n} + \cdots \\ & + (-)^x \left\| \overset{a_1 \cdots a_{n-x-1}\, a_{n-x+1} \cdots a_n}{A_1 \cdots\cdots\cdots\cdots A_{n-1}} \right)\overset{a_{n-x}}{A_n} + \cdots + (-)^{n-1} \left\| \overset{a_1 \cdots\cdots a_n}{A_1 \cdots\cdots A_{n-1}} \right)\end{aligned}$$

oder

$$\left\| \overset{a_1 \cdots\cdots a_n}{A_1 \cdots\cdots A_n} \right) = \sum (-)^x \left\| \overset{a_1 \cdots a_{n-x-1}\, a_{n-x+1} \cdots a_n}{A_1 \cdots\cdots\cdots\cdots A_{n-1}} \right).\overset{a_{n-x}}{A_n} \qquad x = 0,\ 1,\ \ldots,\ n-$$

The sequence of propositions as might be expected is not unlike that found in Vandermonde. The first six propositions are—

1. The under elements ($A_1, A_2, \ldots$) being allowed to remain unchanged, the upper elements ($a_1, a_2, \ldots$) are interchanged in every possible way to obtain the full development.

2. The sign preceding each term is dependent upon the number of interchanges of elements necessary to arrive at the term.

3. If two adjacent upper elements be interchanged, the sign of the *determinant* is altered.

4. If an upper element be moved a number of places to the right or left, the sign of the determinant is changed or not according as the number of places is odd or even.

5. If several upper elements change places, the sign of the determinant is altered or not according as the number is odd or even which indicates how many cases there are of an element following one which in the original order it preceded.

6. If in any *term* the said number of pairs of elements occurring in reversed order be even, the sign preceding the term must be positive; and if the number be odd, the sign must be negative.

The proof of the 3rd of these, which gave trouble to Vandermonde, is easily effected in what after all is Vandermonde's way, viz., by showing that the case for n elements follows with the help of the definition from the case for $n-1$ elements. (XI. 4)

Schweins' 7th proposition is that there is an alternative recurring law of formation in which the under elements play the part of the upper elements in the original law, and *vice versa*. This amounts to saying, in modern phraseology, that if a determinant has been shown to be developable in terms of the elements of a row and their complementary minors, it is also developable in terms of the elements of a column and their complementary minors. The proof is affected by the so-called method of induction, and is interesting both on its own account and from the fact that Cauchy's development in terms of binary products of a row and column turns up in the course of it. The character of the proof will be understood by the following illustrative example in the modern notation :—

By the original law of formation we have

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - a_2|b_1c_3d_4| + a_3|b_1c_2d_4| - a_4|b_1c_2d_3|;$$

and, as the new law is supposed to have been proved for determinants of the 3rd order, it follows that

$$\begin{aligned}|a_1b_2c_3d_4| = a_1|b_2c_3d_4| &- a_2\{b_1|c_3d_4| - c_1|b_3d_4| + d_1|b_3c_4|\}\\ &+ a_3\{b_1|c_2d_4| - c_1|b_2d_4| + d_1|b_2c_4|\}\\ &- a_4\{b_1|c_2d_3| - c_1|b_2d_3| + d_1|b_2c_3|\}.\end{aligned}$$

Combining now by the original law the terms involving b_1 as a factor, the terms involving c_1, and those involving d_1, we obtain

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - b_1|a_2c_3d_4| + c_1|a_2b_3d_4| - d_1|a_2b_3c_4|,$$

and thus prove that the new law holds for determinants of the 4th order. (VI. 7) (IX. 3)

Cauchy's development above referred to appears in the penultimate identity in the convenient form of one term $a_1|b_2c_3d_4|$ followed by a square array of 9 terms. The form in Schweins is—

$$\left.\begin{aligned}&\left\|\begin{matrix}a_1 & \cdots\cdots & a_n\\ A_1 & \ldots & A_n\end{matrix}\right) = \left\|\begin{matrix}a_1 & \cdots\cdots & a_{n-1}\\ A_1 & \ldots & A_{n-1}\end{matrix}\right)\cdot\begin{matrix}a_n\\ A_n\end{matrix}\\ &+\textstyle\sum_x\sum_y(-)^{x+y-1}\left\|\begin{matrix}a_1 & \cdots\cdots & a_{n-x-1} & a_{n-x+1} & \cdots\cdots & a_{n-1}\\ A_1 & \ldots & A_{n-y-1} & A_{n-y+1} & \ldots & A_{n-1}\end{matrix}\right)\cdot\begin{matrix}a_{n-x}\\ A_n\end{matrix}\cdot\begin{matrix}a_n\\ A_{n-y}\end{matrix}\end{aligned}\right\}. \quad (\text{XXXVII. } 2)$$

Laplace's expansion-theorem is next taken up. To prepare the way a theorem in permutations is first given, the enunciation being as follows: *If from* n *different elements every permutation of* q *elements be formed, and every permutation of* n − q *elements; and if each of the latter be appended to all such of the former as have no elements in common with it, all the permutations of the whole* n *elements will be obtained.* Thus, if the permutations of 1 2 3 4 5, or say P(1 2 3 4 5), be wanted, we first take the permutations three at a time, viz.,

P(1 2 3), P(1 2 4), P(1 2 5), ... , P(3 4 5)

where 1 2 3, 1 2 4, 1 2 5, . . . , 3 4 5 are the orderly arranged combinations of three elements; secondly, we take the permutations two at a time, viz.,

P(1 2), P(1 3), P(1 4), ... , P(4 5);

and, thirdly, we append each of the two permutations included in P(4 5) to each of the six included in P(1 2 3), each of the two in P(3 5) to each of the six in P(1 2 4), and so on. The identity here involved Schweins writes as follows, the only difference being that P is put instead of V (*Versetzungen*):—

$$\begin{aligned} P(1\,2\,3\,4\,5) = \ & P(1\,2\,3)\times P(4\,5) \\ & + P(1\,2\,4)\times P(3\,5) \\ & + P(1\,2\,5)\times P(3\,4) \\ & + P(1\,3\,4)\times P(2\,5) \\ & + P(1\,3\,5)\times P(2\,4) \\ & + P(1\,4\,5)\times P(2\,3) \\ & + P(2\,3\,4)\times P(1\,5) \\ & + P(2\,3\,5)\times P(1\,4) \\ & + P(2\,4\,5)\times P(1\,3) \\ & + P(3\,4\,5)\times P(1\,2). \end{aligned}$$

Another example is—

$$\begin{aligned} P(1\,2\,3\,4\,5\,6) = \ & P(1\,2\,3)\,.\,P(4\,5\,6) \\ & + P(1\,2\,4)\,.\,P(3\,5\,6) \\ & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ & + P(3\,5\,6)\,.\,P(1\,2\,4) \\ & + P(4\,5\,6)\,.\,P(1\,2\,3)\,. \end{aligned}$$

The proof consists in the assertion that no permutation can occur twice on the right-hand side, and in showing that the number of permutations which occur is the full number.

From this lemma Laplace's expansion-theorem is given as an immediate deduction. The passage (p. 335) is interesting, as the mode of enunciating the theorem approximates closely to that of modern writers, and has a certain advantage over Cauchy's, perfectly accurate, more general and more compact though the latter be.

"Nach dieser Weise, alle Versetzungen zu bilden, welche wir hier zuerst bekannt machen, können auch die Summen der Producte mit Versetzungen und mit veränderlichen Zeichen in niedrigere Summen zerlegt werden, wenn bei jeder Versetzung nach der oben gefundenen Vorschrift das zugehörige Zeichen bestimmt wird; z. B.

$$
\begin{array}{rll}
\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 & A_4 & A_5 \end{matrix}\right) = & \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix}\right) & = \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix}\right) \\
& -\left\|\begin{matrix} a_1 & a_2 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_3 & a_5 \\ A_4 & A_5 \end{matrix}\right) & -\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_4 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_3 & A_5 \end{matrix}\right) \\
& +\left\|\begin{matrix} a_1 & a_2 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_3 & a_4 \\ A_4 & A_5 \end{matrix}\right) & +\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_3 & A_4 \end{matrix}\right) \\
& +\left\|\begin{matrix} a_1 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_2 & a_5 \\ A_4 & A_5 \end{matrix}\right) & +\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_4 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_5 \end{matrix}\right) \\
& -\left\|\begin{matrix} a_1 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_2 & a_4 \\ A_4 & A_5 \end{matrix}\right) & -\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_4 \end{matrix}\right) \\
& +\left\|\begin{matrix} a_1 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_2 & a_3 \\ A_4 & A_5 \end{matrix}\right) & +\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_4 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} u_4 & a_5 \\ A_2 & A_3 \end{matrix}\right) \\
& -\left\|\begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_5 \\ A_4 & A_5 \end{matrix}\right) & -\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_4 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_5 \end{matrix}\right) \\
& +\left\|\begin{matrix} a_2 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_4 \\ A_4 & A_5 \end{matrix}\right) & +\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_4 \end{matrix}\right) \\
& -\left\|\begin{matrix} a_2 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_3 \\ A_4 & A_5 \end{matrix}\right) & -\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_4 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_3 \end{matrix}\right) \\
& +\left\|\begin{matrix} a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_2 \\ A_4 & A_5 \end{matrix}\right), & +\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_3 & A_4 & A_5 \end{matrix}\right)\cdot\left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_2 \end{matrix}\right).
\end{array}
$$

In der ersten Scheitelreihe sind die oberen und in der zweiten die unteren Elemente veränderlich; die Zeichen + und − befolgen das Gesetz in § 140. Eben so ist

(Another example is given.)

Wir wollen für diese Bildungsweise folgende allgemeine Zeichen wählen:

$$\left\|\begin{matrix} a_1 \cdots\cdots & a_n \\ A_1 \;\ldots & A_n \end{matrix}\right) = \sum(-)^* \left\|\begin{matrix} (a_1\, a_2 \cdots\cdots a_n)^{(q)} \\ A_1 \ldots\ldots A_q \end{matrix}\right)\cdot\left\|\begin{matrix} (a_1 \quad a_2 \quad \cdots\cdots a_n)^{(n-q)} \\ A_{q+1}\; A_{q+2} \ldots A_n \end{matrix}\right)$$

und

$$= \sum(-)^* \left\|\begin{matrix} a_1 \quad a_2 \;\cdots\cdots\cdots\; a_q \quad (q) \\ (A_1, A_2, \ldots, A_n) \end{matrix}\right)\cdot\left\|\begin{matrix} a_{q+1} \quad a_{p+2} \cdots\cdots a_n \quad (n-q) \\ (A_1, A_2, \ldots, A_n) \end{matrix}\right),$$

wo * nach dem Gesetze bestimmt werden muss, welches in § 140 gefunden ist." (xiv. 5)

The one imperfection in this is in regard to the question of sign. It is implied that the sign to precede any product, say the product

$$\left\|\begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right)\cdot\left\|\begin{matrix} a_1 & a_5 \\ A_4 & A_5 \end{matrix}\right)$$

is fixed by making it the same as the sign of the *term*

$$\overset{a_2}{A_1}\,\overset{a_3}{A_2}\,\overset{a_4}{A_3}\,\overset{a_1}{A_4}\,\overset{a_5}{A_5};$$

but nothing is said as to how this ensures that the 11 other terms of the product shall have their proper sign.

Considerably less interest attaches to the next theorem dealt with,—Vandermonde's theorem regarding the effect of the equality of two upper or two lower elements. All that is fresh is the lengthy demonstration by the method of so-called induction. The identities immediately following from it by expansion Schweins expresses as follows:—

$$\sum(-)^x \left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{q-1}\, a_q\, a_{q+1} \ldots a_{n-1} \\ A_1 \ldots A_{x-1}\, A_{x+1} \ldots\ldots\ldots A_n \end{matrix} \right) \cdot \overset{a_q}{A_x} = 0,$$

$$\sum(-)^x \left\| \begin{matrix} a_1 \ldots a_{x-1}\, a_{x+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_n \\ A_1 \ldots\ldots A_{q-1}\, A_q\, A_{q+1} \ldots A_{n-1} \end{matrix} \right) \cdot \overset{a_x}{A_q} = 0,$$

$$\text{where } x = 1, 2, \ldots\ldots, n. \qquad \text{(XII. 10)}$$

This concludes the first chapter of the first section.

The second chapter deals with a most notable generalisation and is worthy of being reproduced with little or no abridgment. The subject may be described as the transformation of an aggregate of products of pairs of determinants into another aggregate of similar kind. A special example of the transformation is taken to open the chapter with, the initial aggregate of products being in this case

$$|a_1b_2c_3d_4|\cdot|e_5f_6g_7| - |a_1b_2c_3e_4|\cdot|d_5f_6g_7| \\ + |a_1b_2c_3f_4|\cdot|d_5e_6g_7| - |a_1b_2c_3g_4|\cdot|d_5e_6f_7|.$$

Expanding the first factor of each product Schweins obtains

$$\begin{aligned} &\Big\{d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4|\Big\}\cdot|e_5f_6g_7| \\ -&\Big\{e_4|a_1b_2c_3| - e_3|a_1b_2c_4| + e_2|a_1b_3c_4| - e_1|a_2b_3c_4|\Big\}\cdot|d_5f_6g_7| \\ +&\Big\{f_4|a_1b_2c_3| - f_3|a_1b_2c_4| + f_2|a_1b_3c_4| - f_1|a_2b_3c_4|\Big\}\cdot|d_5e_6g_7| \\ -&\Big\{g_4|a_1b_2c_3| - g_3|a_1b_2c_4| + g_2|a_1b_3c_4| - g_1|a_2b_3c_4|\Big\}\cdot|d_5e_6f_7|. \end{aligned}$$

He then combines the terms which contain $|a_1b_2c_3|$ as a factor, the terms which contain $|a_1b_2c_4|$ as a factor, and so forth, the result being by the law of formation,

$$|a_1b_2c_3|.|d_4e_5f_6g_7| - |a_1b_2c_4|.|d_3e_5f_6g_7| \\ + |a_1b_3c_4|.|d_2e_5f_6g_7| - |a_2b_3c_4|.|d_1e_5f_6g_7|.$$

The identity of this aggregate with the similar original aggregate constitutes the theorem.

The only point left in want of explanation in connection with it is the construction of the aggregate of products presented at the outset, it being, of course, impossible that any aggregate taken at will can be so transformable. A moment's examination suffices to show that when once the first product of all

$$|a_1b_2c_3d_4|.|e_5f_6g_7|$$

is chosen, the others are derivable from it in accordance with a simple law,—the requirements being (1) no change of suffixes, (2) the last letter of the first factor to be replaced by the letters of the second factor in succession, (3) the signs of the products to be + and − alternately. As for the first product of all, it is not difficult to see that the orders of the determinants composing it are quite immaterial. Instead of taking determinants of the 4^{th} and 3^{rd} orders, and producing by transformation an aggregate of products of determinants of the 3^{rd} and 4^{th} orders, we might have taken determinants of the $(n+1)^{\text{th}}$ and m^{th} orders, applied the transformation, and obtained an aggregate of products of determinants of the n^{th} and $(m+1)^{\text{th}}$ orders. This is the essence of Schweins' first generalisation. His own statement and proof of it leave little to be desired, and are worthy of examination in order that his firm grasp of the subject and his command of the notation may be known. He says (p. 345)—

"Die Reihe, welche in eine andere übertragen werden soll, sei

$$Q = \sum_x (-)^{x-1} \Big\| \overset{a_1}{A_1} \ldots\ldots \overset{}{A_n} \overset{a_{n+1}}{B_x} \Big) \cdot \Big\| \overset{b_1}{B_1} \ldots B_{x-1} B_{x+1} \ldots \overset{b_m}{B_{m+1}} \Big)$$

wo $x = 1, 2, \ldots\ldots, m+1.$

Der erste Factor wird nach 515 in niedere Summen aufgelöst

$$\Big\| \overset{a_1}{A_1} \ldots\ldots \overset{}{A_n} \overset{a_{n+1}}{B_x} \Big) = \sum_y (-)^{n-y+1} \Big\| \overset{a_1 \ldots a_{y-1}\, a_{y+1} \ldots}{A_1 \ldots\ldots\ldots} \overset{a_{n+1}}{A_n} \Big) \cdot \overset{a_y}{B_x}$$

wo $y = 1, 2, \ldots, n+1$

wodurch die vorgegebene Reihe in folgende übergeht:

$$Q = \sum_x \sum_y (-)^{n+x-y} \left\| \begin{matrix} a_1 \dots a_{y-1}\; a_{y+1} \dots\dots a_{n+1} \\ A_1 \dots\dots\dots\dots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \dots\dots\dots\dots\dots\dots b_m \\ B_1 \dots B_{x-1}\; B_{x+1} \dots B_{m+1} \end{matrix} \right) \cdot \overset{a_y}{B_x}.$$

Es ist aber nach 522

$$\begin{aligned} \sum_x (-)^{m+1-x} \left\| \begin{matrix} b_1 \dots\dots\dots\dots\dots\dots b_m \\ B_1 \dots B_{x-1}\; B_{x+1} \dots B_{m+1} \end{matrix} \right) \cdot \overset{a_y}{B_x} &= \left\| \begin{matrix} b_1 \dots\dots b_m\; a_y \\ B_1 \dots\dots B_{m+1} \end{matrix} \right) \\ &= (-)^m \left\| \begin{matrix} a_y\; b_1\; b_2 \dots\dots b_m \\ B_1 \dots\dots B_{m+1} \end{matrix} \right); \end{aligned}$$

folglich

$$Q = \sum_y (-)^{n-y+1} \left\| \begin{matrix} a_1 \dots a_{y-1}\; a_{y+1} \dots a_{n+1} \\ A_1 \dots\dots\dots\dots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_y\; b_1\; b_2 \dots b_m \\ B_1 \dots\dots B_{m+1} \end{matrix} \right),$$

oder es ist

$$\begin{aligned} &\sum_x (-)^{x-1} \left\| \begin{matrix} a_1 \dots\dots\dots a_{n+1} \\ A_1 \dots A_n\; B_x \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \dots\dots\dots\dots\dots\dots b_m \\ B_1 \dots B_{x-1}\; B_{x+1} \dots B_{m+1} \end{matrix} \right) \\ = &\sum_y (-)^{n-y+1} \left\| \begin{matrix} a_1 \dots a_{y-1}\; a_{y+1} \dots a_{n+1} \\ A_1 \dots\dots\dots\dots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_y\; b_1\; b_2 \dots\dots b_m \\ B_1 \dots\dots B_{m+1} \end{matrix} \right) \end{aligned}$$

wo

$$x = 1, 2, \dots\dots, m+1,$$
$$y = 1, 2, \dots\dots, n+1;$$

oder es ist

$$\begin{aligned} &\left. \begin{array}{l} \left\| \begin{matrix} a_1 \dots\dots a_{n+1} \\ A_1 \dots A_n\; B_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \dots\dots\dots b_m \\ B_2 \dots\dots B_{m+1} \end{matrix} \right) \\ - \left\| \begin{matrix} a_1 \dots\dots a_{n+1} \\ A_1 \dots A_n\; B_2 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \dots\dots\dots b_m \\ B_1\; B_3 \dots B_{m+1} \end{matrix} \right) \\ + \left\| \begin{matrix} a_1 \dots\dots a_{n+1} \\ A_1 \dots A_n\; B_3 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \dots\dots\dots b_m \\ B_1\; B_2\; B_4 \dots B_{m+1} \end{matrix} \right) \\ - \quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ (-)^m \left\| \begin{matrix} a_1 \dots\dots a_{n+1} \\ A_1 \dots A_n\; B_{m+1} \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \dots\dots b_m \\ B_1 \dots\dots B_m \end{matrix} \right), \end{array} \right\} \\ = &\left. \begin{array}{l} \left\| \begin{matrix} a_1 \dots\dots a_n \\ A_1 \dots\dots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_{n+1}\; b_1 \dots\dots b_m \\ B_1 \dots\dots B_{m+1} \end{matrix} \right) \\ - \left\| \begin{matrix} a_1 \dots a_{n-1}\; a_{n+1} \\ A_1 \dots\dots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_n\; b_1 \dots\dots b_m \\ B_1 \dots\dots B_{m+1} \end{matrix} \right) \\ + \left\| \begin{matrix} a_1 \dots a_{n-2}\; a_n\; a_{n+1} \\ A_1 \dots\dots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_{n-1}\; b_1 \dots\dots b_m \\ B_1 \dots\dots B_{m+1} \end{matrix} \right) \\ - \quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ (-)^n \left\| \begin{matrix} a_2 \dots\dots a_{n+1} \\ A_1 \dots\dots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_1\; b_1 \dots\dots b_m \\ B_1 \dots\dots B_{m+1} \end{matrix} \right) \end{array} \right\}. \end{aligned} \qquad \text{(XLIX.)}$$

The further generalisation of which this is possible, and which Schweins effects, depends on the fact that the law of formation twice used in proving the identity is but the simplest case of Laplace's expansion-theorem, and that the said theorem can be similarly used in all its generality. In other words, instead of taking only *one* of the B's at a time to go along with the A's to form the first factors of the left-hand aggregate, we may take any fixed number of them. For example, out of six B's we may take every set of *three* to go along with two A's, and we shall have the aggregate

$$\begin{aligned}
&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_3}\big)\cdot\big|\overset{b_1}{B_4}\overset{b_2}{B_5}\overset{b_3}{B_6}\big) - \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_4}\big)\cdot\big|\overset{b_1}{B_3}\overset{b_2}{B_5}\overset{b_3}{B_6}\big)\\
+&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_5}\big)\cdot\big|\overset{b_1}{B_3}\overset{b_2}{B_4}\overset{b_3}{B_6}\big) - \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_6}\big)\ \big|\overset{b_1}{B_3}\overset{b_2}{B_4}\overset{b_3}{B_5}\big)\\
+&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_4}\big)\cdot\big|\overset{b_1}{B_2}\overset{b_2}{B_5}\overset{b_3}{B_6}\big) - \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_5}\big)\cdot\big|\overset{b_1}{B_2}\overset{b_2}{B_4}\overset{b_3}{B_6}\big)\\
+&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_2}\overset{b_2}{B_4}\overset{b_3}{B_5}\big) + \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_4}\overset{a_5}{B_5}\big)\cdot\big|\overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_6}\big)\\
-&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_4}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_5}\big) + \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_5}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_4}\big)\\
-&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_4}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_5}\overset{b_3}{B_6}\big) + \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_5}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_4}\overset{b_3}{B_6}\big)\\
-&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_4}\overset{b_3}{B_5}\big) - \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_4}\overset{a_5}{B_5}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_6}\big)\\
+&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_4}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_5}\big) - \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_5}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_4}\big)\\
+&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_4}\overset{a_5}{B_5}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_6}\big) - \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_4}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_5}\big)\\
+&\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_5}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_4}\big) - \big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_4}\overset{a_4}{B_5}\overset{a_5}{B_6}\big)\cdot\big|\overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_3}\big),
\end{aligned}$$

—the sign of any term being determined by the number of inversions of order among the suffixes of all the B's of the term. In this particular case the first use of Laplace's expansion-theorem is to transform

$$\big|\overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_3}\big)$$

and the other similar determinants each into an aggregate of ten products, the two factors of any product in the expansion of

$$\left\| \overset{a_1}{A_1} \overset{a_2}{A_2} \overset{a_3}{B_1} \overset{a_4}{B_2} \overset{a_5}{B_3} \right)$$

being, as we should nowadays say, a minor formed from the first two rows and the complementary minor. In this way would arise 20 rows of 10 terms each, and these being combined by a second use of Laplace's expansion-theorem in columns of 20 terms each, the outcome would be an aggregate of 10 products, viz., the aggregate

$$\begin{aligned}
&\left\| \overset{a_1}{A_1} \overset{a_2}{A_2} \right) \cdot \left\| \overset{a_3}{B_1} \overset{a_4}{B_2} \overset{a_5}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) - \left\| \overset{a_1}{A_1} \overset{a_3}{A_2} \right) \cdot \left\| \overset{a_2}{B_1} \overset{a_4}{B_2} \overset{a_5}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) \\
+ &\left\| \overset{a_1}{A_1} \overset{a_4}{A_2} \right) \cdot \left\| \overset{a_2}{B_1} \overset{a_3}{B_2} \overset{a_5}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) - \left\| \overset{a_1}{A_1} \overset{a_5}{A_2} \right) \cdot \left\| \overset{a_2}{B_1} \overset{a_3}{B_2} \overset{a_4}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) \\
+ &\left\| \overset{a_2}{A_1} \overset{a_3}{A_2} \right) \cdot \left\| \overset{a_1}{B_1} \overset{a_4}{B_2} \overset{a_5}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) - \left\| \overset{a_2}{A_1} \overset{a_4}{A_2} \right) \cdot \left\| \overset{a_1}{B_1} \overset{a_3}{B_2} \overset{a_5}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) \\
+ &\left\| \overset{a_2}{A_1} \overset{a_5}{A_2} \right) \cdot \left\| \overset{a_1}{B_1} \overset{a_3}{B_2} \overset{a_4}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) + \left\| \overset{a_3}{A_1} \overset{a_4}{A_2} \right) \cdot \left\| \overset{a_1}{B_1} \overset{a_2}{B_2} \overset{a_5}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) \\
- &\left\| \overset{a_3}{A_1} \overset{a_5}{A_2} \right) \cdot \left\| \overset{a_1}{B_1} \overset{a_2}{B_2} \overset{a_4}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right) + \left\| \overset{a_4}{A_1} \overset{a_5}{A_2} \right) \cdot \left\| \overset{a_1}{B_1} \overset{a_2}{B_2} \overset{a_3}{B_3} \overset{b_1}{B_4} \overset{b_2}{B_5} \overset{b_3}{B_6} \right).
\end{aligned}$$

The following is Schweins' statement of this most general theorem :—

$$\begin{aligned}
&\sum(-)^* \left\| \overset{a_1}{A_1} \ldots\ldots A_{n-q} B'_1 \ldots\ldots \overset{a_n}{B'} \right) \cdot \left\| \overset{b_1}{B'_{q+1}} \ldots\ldots \overset{b_{m-q}}{B'_m} \right) \qquad \text{(XLIX. 2)} \\
= &\sum(-)^* \left\| \overset{a'_1}{A_1} \ldots\ldots\ldots\ldots \overset{a'_{n-q}}{A_{n-q}} \right) \cdot \left\| \overset{a'_{n-q+1} \ldots a'_n \, b_1 \ldots}{B_1} \ldots\ldots \overset{b_{m-q}}{B_m} \right).
\end{aligned}$$

The only points about it requiring explanation are the exact effect to be given to the symbol Σ, and the meaning of the dashes affixed to certain of the letters. The two symbols are connected with each other, the dashes not being permanently attached to the letters, but merely put in to assist in explaining the duty of the Σ. On the left-hand member of the identity, the two symbols indicate that the first term is got by dropping the dashes, and that from this first term another term is got, if we substitute for $B_1 \ldots . B_q$, some other set of q B's chosen from

$B_1 \ldots . B_m$, and take the remaining B's to form the B's of the second determinant,—the two sets of B's being in both cases first arranged in ascending order of their suffixes. On the other side of the identity, the use of the symbols is exactly similar, $n-q$ of the n upper elements $a_1, \ldots, a_n$ being taken for the first determinant of any term of the series, and the remainder for the second determinant. The number of terms in the series on the one side is evidently $m!/q!(m-q)!$ and on the other $n!/q!(n-q)!$

In the demonstration of the theorem greater fulness is evidently necessary than in the case of the previous theorem, the rule of signs in particular requiring attention. This Schweins does not give. He merely tells the character of the first transformation, symbolising the expansion obtainable, and then says that a recombination is possible, giving the result.

The succeeding five pages (pp. 350–355) are devoted to evolving and stating special cases. This is by no means unnecessary work, as in the case of a theorem of so great generality it is often a matter of some trouble to ascertain whether a particular given result be really included in it or not. To students of the history of the subject the special cases are doubly interesting, because it is in them we may expect to find links of connection with the work of previous investigators.

The first descent from generality is made by putting some of the B's equal to A's, the theorem then being

$$\sum(-)^* \Big\| \begin{matrix} a_1 & \ldots\ldots\ldots\ldots & a_{p+s+q} \\ A_1 \ldots A_{p+s} & B'_1 \ldots & B'_q \end{matrix} \Big) \cdot \Big\| \begin{matrix} b_1 & \ldots\ldots\ldots\ldots & b_{k+p} \\ B'_{q+1} \ldots & B'_{q+k} A_1 \ldots & A_p \end{matrix} \Big) \qquad \text{(XLIX. 3)}$$

$$= \sum(-)^* \Big\| \begin{matrix} a'_1 & \ldots\ldots & a'_{p+s} \\ A_1 & \ldots\ldots & A_{p+s} \end{matrix} \Big) \cdot \Big\| \begin{matrix} a'_{p+s+1} & \ldots\ldots & a'_{p+s+q} & b_1 & \ldots\ldots & b_{k+p} \\ B_1 & \ldots\ldots & B_{q+k} & A_1 & \ldots\ldots & A_p \end{matrix} \Big).$$

If in addition to this specialisation, some of the b's be put equal to the a's, the result is

$$\sum(-)^* \Big\| \begin{matrix} b_1 \ldots b_h & a'_1 \ldots & a'_{p+s-h} \\ A_1 & \ldots\ldots & A_{p+s} \end{matrix} \Big) \cdot \Big\| \begin{matrix} a'_{p+s-h+1} & \ldots\ldots & a'_{p+s-h+q}, & b_1 \ldots & b_{h+k} \\ B_1 & \ldots & B_{h+k-p+q} & A_1 \ldots & A_p \end{matrix} \Big) \qquad \text{(XLIX. 4)}$$

$$= \sum(-)^* \Big\| \begin{matrix} b_1 \ldots b_h & a_1 \ldots\ldots & a_{p+s+q-h} \\ A_1 \ldots A_{p+s} & B'_1 \ldots & B'_q \end{matrix} \Big) \cdot \Big\| \begin{matrix} b_1 & \ldots\ldots\ldots\ldots & b_{h+k} \\ B'_{q+1} \ldots & B'_{q+h+k-p} A_1 \ldots & A_p \end{matrix} \Big)$$

—a notable theorem, which it would not be inappropriate to consider rather as a generalisation than as a special case of the theorem from which it is derived. Returning, however, to the preceding case, and putting $k=0$, we obtain

$$\begin{aligned} &\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots a_{p+s+q} \\ A_1 \ldots A_{p+s}B_1 \ldots B_q \end{matrix}\right) \cdot \left\|\begin{matrix} b_1 \ldots\ldots\ldots b_p \\ A_1 \ldots\ldots A_p \end{matrix}\right) \qquad \text{(XLIX. 5)} \\ = \sum(-)^* &\left\|\begin{matrix} a'_1 \ldots\ldots a'_{p+s} \\ A_1 \ldots\ldots A_{p+s} \end{matrix}\right) \cdot \left\|\begin{matrix} a'_{p+s+1} \ldots\ldots\ldots a'_{p+s+q}\, b_1 \ldots\ldots b_p \\ B_1 \ldots\ldots B_q\, A_1 \ldots\ldots\ldots\ldots A_p \end{matrix}\right). \end{aligned}$$

This may be viewed as an extension of Laplace's expansion-theorem to which it degenerates when p is put equal to 0. Though comparatively a very special identity it is considerably beyond anything attained by Schweins' predecessors. In fact, we only come upon something like known ground, when in descending further, we put in it $q=1$. The result thus obtained is

$$\begin{aligned} &\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots a_{p+s+1} \\ A_1 \;\ldots\ldots\; A_{p+s}\; B_1 \end{matrix}\right) \cdot \left\|\begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_p \\ A_1 \ldots\ldots\ldots\ldots A_p \end{matrix}\right) \\ = \sum(-)^* &\left\|\begin{matrix} a'_1 \ldots\ldots\ldots\ldots a'_{p+s} \\ A_1 \ldots\ldots\ldots A_{p+s} \end{matrix}\right) \cdot \left\|\begin{matrix} a'_{p+s+1}\; b_1 \ldots\ldots\ldots b_p \\ B_1 \quad A_1 \ldots\ldots A_p \end{matrix}\right), \qquad \text{(XLV. 2)} \end{aligned}$$

which closely resembles a theorem of Desnanot's. The difference between them consists in the fact that here the second factor on the left-hand side is *any* determinant of a lower order than the cofactor, whereas in Desnanot the second factor is a *minor* of the cofactor. A further specialisation, viz. putting $B_1 = A_{p+1}$, brings us to the result

$$\left.\begin{aligned} &\sum(-)^* \left\|\begin{matrix} a'_1 \ldots\ldots\ldots a'_{p+s} \\ A_1 \ldots\ldots A_{p+s} \end{matrix}\right) \cdot \left\|\begin{matrix} a'_{p+s+1}\; b_1 \ldots\ldots b_p \\ A_1 \ldots\ldots\ldots A_{p+1} \end{matrix}\right) = 0, \\ &\text{or} \\ &\sum(-)^* \left\|\begin{matrix} b_1 \ldots\ldots\ldots b_{p+1} \\ A_1 \ldots\ldots A_p B_1 \end{matrix}\right) \cdot \left\|\begin{matrix} b_1 \quad b_2 \ldots\ldots b_{p+2} \\ B'_2\, B'_3 \ldots\ldots B'_{p+3} \end{matrix}\right) = 0. \end{aligned}\right\} \quad \text{(XXIII. 8)}$$

The form here is that of a vanishing aggregate of products of pairs of determinants, and identities of this form we have already had to consider in dealing with Bezout, Monge, Cauchy, and Desnanot. To the last of these only does Schweins refer. His words are (p. 352)—

"Wird in dieser Gleichung $s = 2$ gesetzt, so entsteht folgende :—

$$\sum(-)^* \left\| \overset{b_1}{A_1} \ldots\ldots A_p \overset{b_{p+1}}{B'_1} \right) \cdot \left\| \overset{b_1}{B'_2} \overset{b_2}{B'_3} \ldots\ldots \overset{b_{p+2}}{B'_{p+3}} \right) = 0,$$

wovon Desnanot einige ganz specielle Fälle gefunden hat, oder vielmehr der ganze Inhalt seiner Untersuchung ist in folgenden dreien Gleichungen begriffen

$$\sum(-)^* \left\| \overset{b_1}{A_1} \overset{b_2}{A_2} \overset{b_3}{B'_1} \right) \cdot \left\| \overset{b_1}{B'_2} \overset{b_2}{B'_3} \overset{b_3}{B'_4} \overset{b_4}{B'_5} \right) = 0,$$

$$\sum(-)^* \left\| \overset{b_1}{A_1} \overset{b_2}{B'_1} \right) \cdot \left\| \overset{b_1}{B'_2} \overset{b_2}{B'_3} \overset{b_3}{B'_4} \right) = 0,$$

$$\sum(-)^* \overset{b_1}{B'_1} \cdot \left\| \overset{b_1}{B'_2} \overset{b_2}{B'_3} \overset{b_3}{B'_4} \overset{b_4}{B'_5} \right) = 0,$$

welche mit ermüdender Weitläufigkeit bewiesen sind . . . . . ."

This statement is unfortunately not by any means accurate. As for the "ermüdende Weitläufigkeit," there can be no doubt about it, and to assert its existence is fair criticism; but to say that the whole of Desnanot's results are to be found in the three identities specified is a misrepresentation of the actual facts, and therefore quite unfair. The reader has only to turn back for a moment to our account of Desnanot's work, to verify the fact that the two most important general results attained by the latter (XXIII. 7 and XLV.) are ignored by Schweins altogether.

The remaining paragraphs of the chapter are taken up with the very elementary case in which the products are three in number, and the theorem itself nothing more than one of the extensionals so lengthily dwelt upon by Desnanot, viz. the extensional of

$$a_1|b_1c_2| - b_1|a_1c_2| + c_1|a_1b_2| = 0.$$

It is written in several forms, *e.g.*—

$$\begin{aligned}
&\left\| \overset{\mathfrak{a}_1}{A_1} \ldots\ldots\ldots \overset{\mathfrak{a}_{n+m}}{A_{n+m}} \right) \cdot \left\| \overset{\mathfrak{a}_1}{A_1} \ldots A_{n-1} A_{n+1} \ldots A_{n+m+1} \overset{\mathfrak{a}_{n+m+1}}{B} \right) \\
-&\left\| \overset{\mathfrak{a}_1}{A_1} \ldots A_{n-1} A_{n+1} \ldots \overset{\mathfrak{a}_{n+m}}{A_{n+m+1}} \right) \cdot \left\| \overset{\mathfrak{a}_1}{A_1} \ldots\ldots\ldots A_{n+m} \overset{\mathfrak{a}_{n+m+1}}{B} \right) \\
+&\left\| \overset{\mathfrak{a}_1}{A_1} \ldots A_{n-1} A_{n+1} \ldots A_{n+m} \overset{\mathfrak{a}_{n+m}}{B} \right) \cdot \left\| \overset{\mathfrak{a}_1}{A_1} \ldots\ldots\ldots \overset{\mathfrak{a}_{n+m+1}}{A_{n+m+1}} \right) = 0.
\end{aligned}$$

The next chapter, the third, concerns the solution of a set of linear equations, although according to the title its subject is the transformation of determinants into other determinants when the elements are connected by linear equations. It presents no new feature.

The fourth chapter deals with a special form of determinant, the consideration of which must therefore be deferred. Suffice it for the present to say, as an evidence of Schweins' grasp of the subject, that the solution of the problem attempted is complete and the result very interesting.

The fifth gives the solution of a problem on which the general Theory of Series is said to depend, the problem being the transformation of

$$\frac{\left\| \overset{\mathfrak{a}_1}{\mathrm{B}}\,\overset{\mathfrak{a}_2}{\mathrm{A}} \ldots \mathrm{A}_{n-1}\mathrm{A}_{n+1} \ldots \overset{\mathfrak{a}_\infty}{\mathrm{A}_\infty} \right)}{\left\| \overset{\mathfrak{a}_1}{\mathrm{A}_1}\,\overset{\mathfrak{a}_2}{\mathrm{A}_2} \ldots\ldots \overset{\mathfrak{a}_\infty}{\mathrm{A}_\infty} \right)}$$

into an unending series. The numerator, it will be observed, is of the order ∞: the denominator is of the same order: and all the rows of the former except one occur in the latter. Indeed, if the first row of the numerator were deleted, and the n^{th} row of the denominator, there would be nothing to distinguish the one from the other. The subject is best illustrated by a special example in more modern notation. Recurring to the extensional above referred to as the concluding theorem of the second chapter, and taking the case where the factors are of the 4th and 5th orders, we manifestly have

$$|a_1b_2c_3e_4|\cdot|a_1b_2c_3d_4f_5| - |a_1b_2c_3d_4|\cdot|a_1b_2c_3e_4f_5| + |a_1b_2c_3f_4|\cdot|a_1b_2c_3e_4d_5| = 0,$$

from which, on dividing by $|a_1b_2c_3e_4|\cdot|a_1b_2c_3e_4d_5|$, we obtain

$$\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3e_4d_5|} - \frac{|a_1b_2c_3d_4|}{|a_1b_2c_3e_4|}\cdot\frac{|a_1b_2c_3e_4f_5|}{|a_1b_2c_3e_4d_5|} + \frac{|a_1b_2c_3f_4|}{|a_1b_2c_3e_4|} = 0.$$

Similarly

$$\frac{|a_1b_2c_3f_4|}{|a_1b_2e_3c_4|} - \frac{|a_1b_2c_3|}{|a_1b_2e_3|}\cdot\frac{|a_1b_2e_3f_4|}{|a_1b_2e_3c_4|} + \frac{|a_1b_2f_3|}{|a_1b_2e_3|} = 0,$$

$$\frac{|a_1b_2f_3|}{|a_1e_2b_3|} - \frac{|a_1b_2|}{|a_1e_2|}\cdot\frac{|a_1e_2f_3|}{|a_1e_2b_3|} + \frac{|a_1f_2|}{|a_1e_2|} = 0,$$

and

$$\frac{|a_1f_2|}{|e_1a_2|} - \frac{a_1}{e_1}\cdot\frac{|e_1f_2|}{|e_1a_2|} + \frac{f_1}{e_1} = 0,$$

the last fraction of each identity, be it observed, being the same as the first of the next with its sign changed. From the four by addition we have

$$\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3d_4e_5|} = \frac{|a_1b_2c_3d_4|}{|a_1b_2c_3e_4|}\cdot\frac{|a_1b_2c_3e_4f_5|}{|a_1b_2c_3d_4e_5|}$$
$$+\frac{|a_1b_2c_3|}{|a_1b_2e_3|}\cdot\frac{|a_1b_2e_3f_4|}{|a_1b_2c_3e_4|}$$
$$+\frac{|a_1b_2|}{|a_1e_2|}\cdot\frac{|a_1e_2f_3|}{|a_1b_2e_3|}$$
$$+\frac{a_1}{e_1}\cdot\frac{|e_1f_2|}{|a_1e_2|}$$
$$+\frac{f_1}{e_1}.$$

The general result, as stated by Schweins, is that

$$(-)^{n+1}\frac{\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{n+m+1} \\ \mathrm{B}\,\mathrm{A}_1\ \ldots\ \mathrm{A}_{n-1}\mathrm{A}_{n+1}\ \ldots\ \mathrm{A}_{n+m+1}\end{matrix}\right)}{\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{n+m+1} \\ \mathrm{A}_1\ \ldots\ldots\ldots\ldots\ldots\ \mathrm{A}_{n+m+1}\end{matrix}\right)}$$

$$= \mathrm{L}_0^{(n)}\,.\,\mathrm{V}^{(n)} - \mathrm{L}_1^{(n)}\,.\,\mathrm{V}^{(n+1)} + \mathrm{L}_2^{(n)}\,.\,\mathrm{V}^{(n+2)} - \ldots.(-)^{m+1}\mathrm{L}_{m+1}^{(n)}\,.\,\mathrm{V}^{(n+m+1)},$$

$$\text{where}\qquad \mathrm{L}_m^{(n)} = \frac{\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{n+m-1} \\ \mathrm{A}_1\ \ldots\ \mathrm{A}_{n-1}\mathrm{A}_{n+1}\ \ldots\ldots\ \mathrm{A}_{n+m}\end{matrix}\right)}{\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{n+m-1} \\ \mathrm{A}_1\ \ldots\ldots\ldots\ldots\ldots\ \mathrm{A}_{n+m-1}\end{matrix}\right)},$$

$$\text{and}\qquad \mathrm{V}^{(n+m)} = \frac{\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{n+m} \\ \mathrm{A}_1\ \ldots\ldots\ldots\ldots\ \mathrm{A}_{n+m-1}\mathrm{B}\end{matrix}\right)}{\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{n+m} \\ \mathrm{A}_1\ \ldots\ldots\ldots\ldots\ldots\ldots\ \mathrm{A}\end{matrix}\right)}. \qquad (\mathrm{L}.)$$

Since the expression thus expanded is itself one of the L's, viz., $\mathrm{L}_{m+2}^{(n)}$—as is readily seen by transferring the B from the beginning to the end, and denoting it by A_{n+m+2},—and since $\mathrm{L}_0^{(n)}=1$, the identity may equally appropriately be written with $\mathrm{L}_{m+2}^{(n)}$ at the end of the right-hand member, and looked upon as the recurring law of formation of the L's in terms of the V's. This Schweins does, giving indeed the result of solving for $\mathrm{L}_1^{(n)}$, $\mathrm{L}_2^{(n)}$, . . .

The Second Section (pp. 369–398), consisting of five chapters, and extending to 30 pp., is devoted to a special form of determinants, viz., those already partly investigated by Cauchy, and afterwards known as alternants.

The Third Section (pp. 399–403), extending only to 4 pp., deals with another special form, whose elements are finite differences of a set of functions.

The Fourth Section, (pp. 404–431), consisting of four chapters, and extending to 27 pp., has for its subject a third special form, foreshadowed by Wronski, the characteristic of which is that one of the indices denotes repetition of an operation involving differentiation.

When these Sections come to be considered in their proper places, it will be seen that very great credit is due to Schweins for his labours, and that he has been most undeservedly neglected. The fact that he had ever written on determinants was only brought to light in 1884:* and, so far as can be gathered, his treatise had no influence whatever either on the work of succeeding investigators, or in diffusing a knowledge of the subject.

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*v. *Philos. Magazine* for November: *An overlooked Discoverer in the Theory of Determinants.*

CHAPTER VII.

DETERMINANTS IN GENERAL, FROM THE YEAR 1827 TO 1835.

THE writers of this period are six in number, viz., Jacobi, Reiss, Minding, Cauchy, Drinkwater, Mainardi. Of these by far the most important both as regards quality and quantity is Jacobi; Cauchy contributes an investigation in which determinants are used; Minding makes some little use of the functions without knowing it; all the others are unimportant expositors.

JACOBI (1827).

[Ueber die Hauptaxen der Flächen der zweiten Ordnung. *Crelle's Journal*, ii. pp. 227–233; or *Werke*, iii. pp. 45–53.]

[De singulari quadam duplicis Integralis transformatione. *Crelle's Journal*, ii. pp. 234–242; or *Werke*, iii. pp. 55–56.]

[Ueber die Pfaffsche Methode, eine gewöhnliche lineäre Differential-gleichung zwischen $2n$ Variabeln durch ein System von n Gleichungen zu integriren. *Crelle's Journal*, ii. pp. 347–357; or *Werke*, iv. pp. 17–29.]

We come here simultaneously on the names of a great mathematician and a great mathematical journal. Crelle's *Journal für die reine und angewandte Mathematik*, which began to appear at the end of the year 1825, and which without any of the symptoms of old age still survives, has rendered on more than one occasion important service towards the advancement of the theory of determinants. Its first contributor on the subject and one of its greatest was Jacobi. At a later date he published in the *Journal* an excellent monograph on Determinants; but even his earliest papers show that he had begun to find it a useful weapon of research.

In the first of the memoirs above noted, dealing with the subject of orthogonal substitution, constant use is, of course, made of the functions; but there is no special notation employed, nor indeed anything to indicate that the expressions used were members of a class having properties peculiar to themselves.

In the second memoir, which likewise is taken up with a transformation, but in which the sets of equations involve *four* unknowns, any special notation is still avoided. Expressions, readily seen to be determinants of the third order, are even not set down, because, as the author expressly states, they would be too lengthy. The last clause of the passage in which this statement occurs is noteworthy. The words are (p. 236)—

"Dato systemate æquationum

$$\begin{aligned} \alpha\ u + \beta\ x + \gamma\ y + \delta\ z &= m, \\ \alpha'\ u + \beta'\ x + \gamma'\ y + \delta'\ z &= m', \\ \alpha'' u + \beta'' x + \gamma'' y + \delta'' z &= m'', \\ \alpha''' u + \beta''' x + \gamma''' y + \delta''' z &= m''', \end{aligned}$$

ponamus earum resolutione erui:

$$\begin{aligned} \mathrm{A}m + \mathrm{A}'m' + \mathrm{A}''m'' + \mathrm{A}'''m''' &= u, \\ \mathrm{B}m + \mathrm{B}'m' + \mathrm{B}''m'' + \mathrm{B}'''m''' &= x, \\ \mathrm{C}m + \mathrm{C}'m' + \mathrm{C}''m'' + \mathrm{C}'''m''' &= y, \\ \mathrm{D}m + \mathrm{D}'m' + \mathrm{D}''m'' + \mathrm{D}'''m''' &= z. \end{aligned}$$

Valores sedecim quantitatum A, B, etc., supprimimus eorum prolixitatis causa; in libris algebraicis passim traduntur, et algorithmus, cuius ope formantur, hodie abunde notus est."

On the next page, in eliminating D, D′, D″, D‴ from the set of equations

$$\begin{aligned} 0 &= \mathrm{D}(a-x) + \mathrm{D}'b' + \mathrm{D}''b'' + \mathrm{D}'''b''', \\ 0 &= \mathrm{D}b' + \mathrm{D}'(a'+x) + \mathrm{D}''c''' + \mathrm{D}'''c'', \\ 0 &= \mathrm{D}b'' + \mathrm{D}'c''' + \mathrm{D}''(a''+x) + \mathrm{D}'''c', \\ 0 &= \mathrm{D}b''' + \mathrm{D}'c'' + \mathrm{D}''c' + \mathrm{D}'''(a'''+x), \end{aligned}$$

he arranges the resultant as one would now do who had expanded it from the determinant form according to products of the elements of the principal diagonal, viz., he says (p. 238)—

"Fit illa, eliminationis negotio rite instituto

$$\begin{aligned}0 = \ & (a-x)(a'+x)(a''+x)(a'''+x) \\ & -(a-x)(a'+x)c'c' - (a-x)(a''+x)c''c'' - (a-x)(a'''+x)c'''c''' \\ & -(a''+x)(a'''+x)b'b' - (a'''+x)(a'+x)b''b'' - (a'+x)(a''+x)b'''b''' \\ & +2c'c''c'''(a-x) + 2c'b''b'''(a'+x) \qquad\qquad (\text{LI.}) \\ & +2c''b'''b'(a''+x) + 2c'''b'b''(a'''+x) \\ & +b'b'c'c'+b''b''c''c''+b'''b'''c'''c''' - 2b'b''c'c'' - 2b''b'''c''c''' - 2b'''b'c'''c'."\end{aligned}$$

From the next paragraph we learn his sources of information, and infer that as yet Cauchy's memoir was unknown to him. The first sentence is (p. 239)—

"Inter sedecim quantitates α, β, etc. et sedecim, quæ ex iis derivantur, A, A′, etc. plurimæ intercedunt relationes perelegantes, quæ cum analystis ex iis, quæ Laplace, Vandermonde, in commentariis academiæ Parisiensis A. 1772 p. ii., Gauss in disquis. arithm. sectio V., J. Binet in vol. ix. diariorum instituti polytechnici Parisiensis, aliique tradiderunt, satis notæ sint, paucas tantum referam, quæ casu nostro speciali ope æquationum (IV) facile ex iis derivantur."

The third memoir is by far the most important to us. In the course of the investigation a new special form of determinants, afterwards so well known by the designation *skew* determinants, turns up; and three pages are devoted to an examination of the final expanded form of it. This examination, we cannot, of course, now enter upon; but it is of importance to note that in it Jacobi takes the step of adopting the name *determinant*,—a fact which doubtless was decisive of the fate of the word. The adoption thus made (although stated to be from Gauss), and the occurrence of the words "Horizontalreihen," "Verticalreihen," make it probable that Cauchy's memoir had now come to his notice.

REISS (1829).

[Mémoire sur les fonctions semblables de plusieurs groupes d'un certain nombre de fonctions ou élémens. *Correspondance math. et phys.*, v. pp. 201–215.]

In Reiss we have an author who starts to his subject as if it were entirely new, the only preceding mathematician whom he mentions being Lagrange. Like Cauchy he opens by explaining a mode of forming functions more general than those of which he afterwards treats, the essence of it being that an expression involving several of the $n.\nu$ quantities,

$$\begin{array}{ccccc} a^\alpha & a^\beta & a^\gamma & \ldots & a^\rho \\ b^\alpha & b^\beta & b^\gamma & \ldots & b^\rho \\ c^\alpha & c^\beta & c^\gamma & \ldots & c^\rho \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ r^\alpha & r^\beta & r^\gamma & \ldots & r^\rho \end{array}$$

is taken, and each *exponent* ("exposant") changed successively with all the other exponents, α, β, . . . , or each *base* changed with all the other bases, a, b, ... Only a line or two, however, is given to this, the special class known to us as determinants being taken up at once.

His notation for

$$a^1b^2c^3 - a^1b^3c^2 - a^2b^1c^3 + a^2b^3c^1 + a^3b^1c^2 - a^3b^2c^1$$

is

$$(abc\,,\ \overline{123}), \qquad \text{(VII. 7)}$$

a line being drawn above the exponents to indicate permutation. His rule of term-formation and rule of signs are combined after the manner of Hindenburg. Like Hindenburg he arranges the permutations as one arranges numbers in increasing order of magnitude; but, unlike Hindenburg, after the arrangement has been made he determines the sign of any *particular* term. On this point his words are (p. 202)

"Cela fait, déterminons généralement le signe du M^{me} produit (soit $\dot{M}$) de la manière suivante. Le nombre M sera renfermé entre les produits $1.2.3\ldots l$ et $1.2.3\ldots l(l+1)$; soit $M = m + \lambda \times 1.2.3\ldots l$, de sorte que $\lambda < l+1$, et $m > 0$ et $< 1 + 1.2.3\ldots l$. Cela étant, faisons $\dot{M} = \dot{m}(-1)^\lambda$." (III. 25)

This apparently means that if the sign of the 23rd term in the expansion of

$$(abcd\,,\ \overline{1234})\ *$$

* Or $(abcde\,,\ \overline{12345})$, or indeed $(a_1a_2\ldots a_n,\ \overline{123\ldots n})$.

be wanted, we divide 23 by 1.2.3, getting the quotient 3 and the remainder 5, and thence conclude that the sign wanted is got from the sign of the 5th term by multiplying the latter by $(-1)^3$. Of course 5 has then to be dealt with after the manner of 23, the quotient and remainder this time being 2 and 1, so that we conclude that the sign of the 5th term is got from the sign of the 1st term by multiplying by $(-1)^2$. And the sign of the 1st term being +, the sign of the 23rd is thus seen to be

$$(-1)^{3+2} \quad i.e. \quad -.$$

It would seem at first as if the case where M is itself a factorial were neglected. This however is not so, the condition $m < 1 + 1.2.3 \ldots l$ being corrective of the opening statement that M must lie between $1.2.3 \ldots l$ and $1.2.3 \ldots l(l+1)$. For example, the term being the 24th, we put 24 in the form $3 \times 1.2.3 + 6$, and thus learn that the sign required is different from the sign of the 6th term: then we put 6 in the form $2 \times 1.2 + 2$, and thus learn that the sign of the 6th term is the same as the sign of the 2nd term; finally, we put 2 in the form $1 \times 1 + 1$, which shows that the sign of the 2nd term differs from the sign of the 1st: the conclusion of the whole being that the signs of the 24th and 1st terms are the same, or that they are connected by the factor $(-1)^{3+2+1}$.

Though interesting in itself, a more troublesome form of the rule of signs for the purposes of demonstration it is scarcely possible to conceive, and, as might therefore be expected, it is on the score of logical development that Reiss's paper is weak. Through inability to use the rule later in the demonstration of the so-called Laplace's expansion-theorem, he is forced to supplement it by another convention. His words are (p. 203)—

"Avant d'aller plus loin, faisons encore la détermination suivante. Soit ω une fonction quelconque dans laquelle les k quantités $A, B, C, \ldots A^k$ entrent d'une manière quelconque. Supposons que ces dernières soient les k premières de l'échelle $\begin{pmatrix} A & B & C & \ldots & A^k & \ldots & S \\ 1 & 2 & 3 & \ldots & k & \ldots & s \end{pmatrix}$. Qu'on fasse avec ces s élémens toutes les combinaisons sans répétition de la classe k, et qu'on les substitue successivment au lieu de $A, B, \ldots A^k$ dans la fonction ω; c'est-à-dire le premier élément de chaque combinaison à A, le second à B, etc. Nous obtiendrons par là autant de fonctions semblables à ω qu'il y a de combinaisons de la classe k de s élémens. Or, entre toutes les combinaisons qui en précèdent une quelconque, il s'en trouvera une qui aura $k-1$ élémens communs avec elle, tandis que les deux élémens qui restent isolés dans l'une et l'autre se suivent immédiatement dans l'échelle. Donnons à la fonction qui contient la dernière de ces combinaisons le signe opposé à celui de l'autre fonction; par conséquent les signes de toutes les fonctions semblables à ω seront parfaitement déterminés. et dépendront du signe de la première fonction ($f(\mathrm{A,B,C}, \ldots \mathrm{A}^k)$). Soit, par exemple, $s=5$, $k=3$; nous aurons successivement, en remplaçant A,B,C, . . . S par 1, 2, 3, 4, 5, et en donnant le signe $(+)$ à $f(123)$,

$$+f(123),\ -f(124),\ +f(125),\ +f(134),\ -f(135)$$
$$+f(145),\ -f(234),\ +f(235),\ -f(245),\ +f(345).$$

Voici comment on déterminera le signe de chaque fonction semblable à ω d'après celui d'une autre quelconque. Qu'on cherche les nombres qui se trouvent dans l'échelle $\begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \ldots & \mathrm{A}^k & \ldots & \mathrm{S} \\ 1 & 2 & 3 & \ldots & k & \ldots & s \end{pmatrix}$ sous les élémens de l'une et de l'autre de ces fonctions. Si l'on nomme h et h' leurs sommes respectives, on trouvera le signe de l'une des fonctions $=(-1)^{h'-h} \times$ le signe de l'autre."

Four theorems he considers fundamental, viz., those known to us as (1) Bezout's recurrent law of formation, in all its generality; (2) Vandermonde's proposition that permutation of bases leads to the same result as permutation of exponents; (3) Laplace's expansion-theorem; (4) Vandermonde's proposition regarding the effect of making two bases or two exponents equal. The two most important, viz. (1) and (3), he leaves without proof, and the 4th he says he would at once deduce from the 3rd,—doubtless by choosing the expansion in which the first factor of every term would be of the form

$$(aa,\ \overline{a\beta})$$

and therefore equal to zero.

The proof of the 2nd theorem, viz.,

$$(abc\ldots r\,,\overline{a\beta\gamma\ldots\rho}) \;=\; (\overline{abc\ldots r}\,,a\beta\gamma\ldots\rho),$$

is by the method of so-called induction, and may be illustrated in a later notation by considering the case

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

From theorem (1) we have

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}, \\ &= -b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}, \\ &= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \end{aligned}$$

But by hypothesis all the determinants on the right here may have their rows changed into columns; and this being done we have by addition and the use of theorem (1)

$$3\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and thence the identity required. (IX. 4)

To this proof the following note is appended (p. 207):—

"Cette démonstration quoiqu'assez simple semble reposer cependant sur un artifice de calcul: mais en cherchant une démonstration *directe*, j'ai rencontré une difficulté d'un genre particulier. En effet, on trouve facilement que l^{me} terme de l'une des fonctions en question est aussi égal ou au même terme de l'autre, ou généralement au m^{me}, et que, dans le dernier cas, le m^{me} terme de la première est aussi égal au l^{me} de la seconde, abstraction faite des signes. (IX. 5)
Mais l'identité de ces derniers (qui est de rigueur) exige des explications très-longues et beaucoup moins élémentaires que la démonstration que je viens de donner."

The remaining six or seven pages of the paper are more interesting, and concern the subject of vanishing aggregates of products of pairs of determinants. The theorems were suggested by taking, as we now say, a determinant of even order having its last n rows identical with its first n rows, *e.g.*, the determinant

$$(abab\ ,\ \overline{1234}),$$

and using theorem (3) to expand it in terms of minors formed from the first n rows and their complementary minors. When n is even, a proof is thus obtained, as we have seen in the footnote to the account of Bezout's paper of 1779, that the first half of the expansion is equal to zero. When n is odd, the method fails, although the proposition is still true.* Reiss's enunciation is as follows (p. 209):—

"Théorème V.—Soient les échelles

$$\begin{pmatrix} a\ b\ldots r, & a\ , & b\ ,\ldots r \\ 1\ 2\ldots n, & n+1\ , & n+2,\ldots 2n \end{pmatrix} \text{ et } \begin{pmatrix} \alpha\ \beta\ \gamma\ldots\alpha^n, & \alpha^{n+1}\ ,\ldots\rho \\ 1\ 2\ 3\ldots n\ , & n+1\ ,\ldots 2n \end{pmatrix},$$

qu'on fasse avec les élémens $\beta,\gamma,\ldots,\rho$ toutes les combinaisons de la classe $(n-1)$, et qu'on les substitue successivement dans le premier facteur du produit

$$(ab\ldots r\ ,\ \alpha\beta\gamma\ldots\alpha^n)\,.\,(ab\ldots r\ ,\ \alpha^{n+1}\ldots\rho)$$

au lieu de $\beta\gamma\ldots\alpha^n$; qu'on remplace maintenant dans l'autre facteur les exposans $\alpha^{n+1}\ldots\rho$ par tous ceux qui ne se trouvent pas dans

* It is worthy of note in passing, that a common method does exist for establishing the two cases,—a method quite analogous to Reiss's, but difficult of suggestion to one who used his notation, or indeed to any one who had no notation suitable for determinants whose elements had special numerical values. All the change necessary is to make the last n elements of the first column each equal to zero. This causes no difference in the result when n is even, *e.g.*, from the identity

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \cdot & a_2 & a_3 & a_4 \\ \cdot & b_2 & b_3 & b_4 \end{vmatrix} = 0$$

we have, as before,

$$|a_1b_2|\,.\,|a_3b_4| - |a_1b_3|\,.\,|a_2b_4| + |a_1b_4|\,.\,|a_2b_3| = 0\,;$$

and when n is odd, the second half of the terms which previously gave trouble do not occur.

le premier, en ayant soin de les écrire suivant l'ordre indiqué par les échelles. Si l'on donne au premier produit le signe (+), et qu'on détermine les signes de tous les autres d'après (II), la somme algébrique en sera $=0$, que le nombre n soit pair ou impair."

(XXIII. 9)

An example of it is

$$\begin{aligned}
&(abc, 123)(abc, 456) - (abc, 124)(abc, 356)\\
+&(abc, 125)(abc, 346) - (abc, 126)(abc, 345)\\
+&(abc, 134)(abc, 256) - (abc, 135)(abc, 246)\\
+&(abc, 136)(abc, 245) + (abc, 145)(abc, 236)\\
-&(abc, 146)(abc, 235) + (abc, 156)(abc, 234) = 0,
\end{aligned}$$

the left-hand side being nothing more than the first ten terms of one of the expansions of the vanishing determinant

$$\begin{vmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6\\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6\\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6\\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6
\end{vmatrix},$$

or the other ten terms with their signs changed. Reiss's proof is lengthy and troublesome, the method being to expand each factor in terms of the a's and their complementary minors, perform the multiplications (*e.g.*, in the special case just given the multiplication of

$$a_1|b_2c_3 - a_2|b_1c_3| + a_3|b_1c_2| \quad \text{by} \quad a_4|b_5c_6| - a_5|b_4c_6| + a_6|b_4c_5|, \text{ \&c.})$$

and show that the terms of the final aggregate occur in pairs which annul themselves.

The next theorem is of still greater interest, because it is that peculiar generalisation of the preceding which in later times came to be known as the 'extensional.' The way in which it is established is also noteworthy, viz., by deducing it as a special case from the theorem of which, as we have said, it may be viewed as a generalisation. The author's words are (p. 213):—

"Ce théorème nous conduit à une relation qui existe dans le cas le plus général, savoir si $\nu - n$ est un nombre quelconque ou positif ou négatif. Supposons $\nu > n$, et $\nu - n = \mathrm{N}$; soient les échelles,

$$\begin{pmatrix} a\,b\ldots r\,, & a\,, & b\,, & \ldots r\,, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{R} \\ 1\,2\ldots \mathrm{N}\,, & \mathrm{N}+1\,, & \mathrm{N}+2\,, & \ldots 2\mathrm{N}\,, & 2\mathrm{N}+1\,, & 2\mathrm{N}+2\,, & \ldots \nu \end{pmatrix}$$

et

$$\begin{pmatrix} \alpha\,\beta\ldots \alpha^{\mathrm{N}}, & \alpha^{\mathrm{N}+1}\,, & \ldots \rho\,, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{P} \\ 1\,2\ldots \mathrm{N}\,, & \mathrm{N}+1\,, & \ldots 2\mathrm{N}\,, & 2\mathrm{N}+1\,, & 2\mathrm{N}+2\,, & \ldots \nu \end{pmatrix}$$

Qu'on fasse avec les élémens $\beta, \gamma, \ldots \alpha^{\mathrm{N}}, \alpha^{\mathrm{N}+1} \ldots \rho$ toutes les combinaisons de la classe $\mathrm{N}-1$; qu'on les substitue successivement au lieu de $\beta \ldots \alpha^{\mathrm{N}}$ dans le premier facteur du produit

$$(ab\ldots r\mathrm{AB}\ldots\mathrm{R},\ \alpha\beta\ \ldots \alpha^{\mathrm{N}}\mathrm{AB}\ldots\mathrm{P})$$
$$\times(ab\ldots r\mathrm{AB}\ldots\mathrm{R},\ \alpha^{\mathrm{N}+1}\ldots\rho\ \mathrm{AB}\ldots\mathrm{P});$$

qu'on remplace dans l'autre facteur les exposans $\alpha^{\mathrm{N}+1} \ldots \rho$ par tous ceux qui ne se trouvent pas dans le premier: qu'on détermine enfin le signe de chaque produit d'après (II): la somme algébrique en sera $= 0$. (XXIII. 10) (XLIV. 5)

"En effet, supposons les échelles

$$\begin{pmatrix} a\,b\ldots r\,, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{R}, & a, & b, & \ldots r, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{R} \\ 1\,2\ldots\mathrm{N}\,, & \mathrm{N}+1\,, & \mathrm{N}+2\,, & \ldots\nu-\mathrm{N}\,, & \nu-\mathrm{N}+1\,, & \nu-\mathrm{N}+2\,, & \ldots\nu\,, & \nu+1\,, & \nu+2\,, & \ldots 2\nu-2\mathrm{N} \end{pmatrix}$$

et

$$\begin{pmatrix} \alpha\,\beta\ldots\alpha^{\mathrm{N}}, & \alpha^{\mathrm{N}+1}\,, & \ldots\rho\,, & \mathrm{A}, & \mathrm{B}, & \ldots \mathrm{A}^{\nu-3\mathrm{N}}, & \mathrm{A}^{\nu-3\mathrm{N}+1}\,, & \ldots\mathrm{P}, & \mathrm{A}\,, & \ldots \mathrm{P} \\ 1\,2\ldots\mathrm{N}\,, & \mathrm{N}+1\,, & \ldots 2\mathrm{N}\,, & 2\mathrm{N}+1\,, & 2\mathrm{N}+2\,, & \ldots\nu-\mathrm{N}\,, & \nu-\mathrm{N}+1\,, & \ldots\nu\,, & \nu+1\,, & \ldots 2\nu-2\mathrm{N} \end{pmatrix}.$$

Formons avec ces élémens la fonction décrite dans le dernier théorème: la somme totale en sera donc $= 0$, et le premier terme aura la forme

$$(ab\ldots r\mathrm{AB}\ldots\mathrm{R},\ \alpha\beta\ldots\rho\mathrm{A}\ldots\mathrm{A}^{\nu-3\mathrm{N}})$$
$$\times(ab\ldots r\mathrm{AB}\ldots\mathrm{R},\ \mathrm{A}^{\nu-3\mathrm{N}+1}\ldots\mathrm{PA}\ldots\mathrm{P}).$$

Or, on voit facilement que tous les termes qui ne contiennent pas dans chaque facteur *tous* les exposans A, B, ... P, s'évanouiront séparément, parce qu'il y aura des exposans identiques dans l'un ou l'autre des facteurs. Il ne restera donc que les termes qui, contenant α dans le premier facteur, y épuisent successivement toutes les combinaisons de la classe $\mathrm{N}-1$ des élémens $\beta, \gamma, \ldots \rho$. Mais les signes de ces termes sont évidemment déterminés comme ils devaient l'être; partant la somme algébrique de tous les termes est $=0$, ce qu'il fallait démontrer."

This will be best understood by considering a special example. Going back to the previous theorem, and selecting its simplest case, we have

$$|a_1b_2|\cdot|a_3b_4| - |a_1b_3|\cdot|a_2b_4| + |a_1b_4|\cdot|a_2b_3| = 0.$$

Now what the new theorem asserts in regard to this is that we may with impunity *extend* each of the determinants occurring in it, provided the extension be the same throughout. For example, choosing the extension $\xi_5\,\eta_6\,\zeta_7$,* we can, in virtue of the new theorem, assert the truth of the identity

$$|a_1b_2\xi_5\eta_6\zeta_7|\cdot|a_3b_4\xi_5\eta_6\zeta_7| - |a_1b_3\xi_5\eta_6\zeta_7|\cdot|a_2b_4\xi_5\eta_6\zeta_7| \\ + |a_1b_4\xi_5\eta_6\zeta_7|\cdot|a_2b_3\xi_5\eta_6\zeta_7| = 0.$$

That the two may be viewed as cases of the same theorem will be apparent when it is pointed out that just as the first is derivable from

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \cdot & a_2 & a_3 & a_4 \\ \cdot & b_2 & b_3 & b_4 \end{vmatrix} = 0,$$

so the second is derivable in exactly the same way from a perfectly similar identity,† viz.

* In Reiss's notation the extension is $A_A\, B_B \ldots R_P$.

† It is perhaps a little more readily seen to be derivable from

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \cdot & \cdot & \cdot \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \cdot & \cdot & \cdot \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \cdot & \cdot & \cdot \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 & \cdot & \cdot & \cdot \\ \cdot & a_2 & a_3 & a_4 & \cdot & \cdot & \cdot & a_5 & a_6 & a_7 \\ \cdot & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot & b_5 & b_6 & b_7 \\ \cdot & \xi_2 & \xi_3 & \xi_4 & \cdot & \cdot & \cdot & \xi_5 & \xi_6 & \xi_7 \\ \cdot & \eta_2 & \eta_3 & \eta_4 & \cdot & \cdot & \cdot & \eta_5 & \eta_6 & \eta_7 \\ \cdot & \zeta_2 & \zeta_3 & \zeta_4 & \cdot & \cdot & \cdot & \zeta_5 & \zeta_6 & \zeta_7 \end{vmatrix} = 0.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_5 & b_6 & b_7 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_5 & \xi_6 & \xi_7 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_5 & \eta_6 & \eta_7 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 & \zeta_5 & \zeta_6 & \zeta_7 \\ \cdot & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_5 & a_6 & a_7 \\ \cdot & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_5 & b_6 & b_7 \\ \cdot & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_5 & \xi_6 & \xi_7 \\ \cdot & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_5 & \eta_6 & \eta_7 \\ \cdot & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 & \zeta_5 & \zeta_6 & \zeta_7 \end{vmatrix} = 0.$$

Many more products than three (126 in fact) arise in the latter case; but, for the reason stated by Reiss, only three of them do not vanish.

CAUCHY (1829).

[Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes. *Exercices de Math.*, iv. pp. 140–160; or *Œuvres* (2) ix. pp. 172–195.]

As the title would lead one to expect, the determinants which occur in this important memoir belong to the class afterwards distinguished by the name "axisymmetric," and thus fall to be considered along with others of that class. Since, however, the proof employed for one of the theorems therein enunciated is equally applicable to all kinds of determinants, it would be scarcely fair to omit here all mention of the said theorem. In modern phraseology its formal enunciation might stand as follows:—

S *being any axisymmetric determinant,* R *the determinant got by deleting the first row and first column of* S, Y *the determinant got by deleting the first row and second column of* S, *and* Q *the determinant got from* R *as* R *from* S, *then, if* $R = 0$,

$$SQ = -Y^2;$$

and the theorem in general determinants whose validity is warranted by the proof given is in later notation—

If $|\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4| = 0$, *then* $|\mathrm{a}_2\mathrm{c}_3\mathrm{d}_4|\cdot|\mathrm{b}_1\mathrm{c}_3\mathrm{d}_4| = -|\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4|\cdot|\mathrm{c}_3\mathrm{d}_4|$. (XX. 2)

This, it is readily seen, is not a very obscure foreshadowing of Jacobi's identity

$$|\mathrm{A}_1\mathrm{B}_2| = |a_1b_2c_3d_4|\cdot|c_3d_4|.$$

JACOBI (1829).

[Exercitatio algebraica circa discerptionem singularem fractionum, quae plures variabiles involvunt. *Crelle's Journ.*, v. pp. 344–364; or *Werke*, iii. pp. 67–90.]

In the ordinary expansion of $(ax+by+cz-t)^{-1}$ there are evidently only negative powers of x and positive powers of y and z; in the like expansion of $(b'y+c'z+a'x-t')^{-1}$ there are only negative powers of y and positive powers of z and x; and similarly for $(c''z+a''x+b''y-t'')^{-1}$. It follows from this that the ordinary expansion of

$$(ax+by+cz-t)^{-1}.(b'y+c'z+a'x-t)^{-1}.(c''z+a''x+b''y-t'')^{-1},$$

looked at from the point of view of the powers of x, y, z, contains a considerable variety of terms; for example, terms in which negative powers of x occur along with positive powers of y and z, terms in which x does not occur at all, and so forth. There is thus suggested the curious problem of partitioning the fraction

$$\frac{1}{(ax+by+cz-t)(b'y+c'z+a'x-t')(c''z+a''x+b''y-t'')}$$

into a number of fractions each of which is the equivalent of the series of terms of one of those types. This is the problem with which Jacobi is here concerned.

In the case of two variables he counts three types of terms, viz., that in which the indices of both x and y are negative, that in which the index of x only is negative, and that in which the index of y only is negative. n the case of three variables he counts seven types, viz., that in which the indices of x, y, z are all negative, the three in which the index of only one variable is negative, and the three in which the index of only one variable is not negative. These two cases are gone fully into, with the result that the expressions for the three aggregates in the former are all found to contain the factor $(a\,b')^{-1}$, and the expressions for the seven aggregates in the latter the factor $(a\,b'c'')^{-1}$. The reciprocal of each of those factors is recognised as the common denominator of the values of the unknowns in a set of linear equations, a denominator "quam quibusdam *determinantem* nuncupamus et designemus per Δ." Its persistent appearance in the problem under discussion,—a persistency, in fact, sufficient to suggest the change of the numerator of the given fraction from 1 to $(a\,b')$ in the case of two variables and from 1 to $(a\,b'c'')$ in the case of three,—is remarked upon:—"Quam determinantem in hac quaestione magnas partes agere videbimus, videlicet *omnes illas series infinitas, quas ut coëfficientes producti propositi evoluti invenimus, ex evolutione dignitatum negativarum determinantis provenire.*" Then fixing the attention on a unique term of the expansion Jacobi ventures on the generalisation that the coefficient of

$$1/(x\,x_1\,x_2\,\ldots\,x_{n-1})$$

in the expansion of

$$1/(u\,u_1\,u_2\,\ldots\,u_{n-1}),$$

that is to say, of

$$(ax+by+cz+\ldots)^{-1}(b'y+c'z+\ldots\quad)^{-1}(c''z+\ldots\quad)^{-1}\ldots\ldots$$

is the determinant

$$(a\,b'c''\,\ldots\ldots)^{-1}. \qquad \text{(LII.)}$$

No proof, however, is given, save for the cases where $n=2$ and $n=3$. The proposition is most noteworthy in that it supplies *the generating function of the reciprocal of a determinant.*

To obtain a generalisation in a different direction, viz., from

$(ax+by)^{-1}(b_1y+a_1x)^{-1}$ to $(ax+by)^{-m}(b_1y+a_1x)^{-n}$, Jacobi proceeds in a very curious and interesting way. Denoting

$$\ldots\ldots\ldots+\beta^{-3}\alpha^2+\beta^{-2}\alpha+\beta^{-1}+\alpha^{-1}+\alpha^{-2}\beta+\alpha^{-3}\beta^2+\ldots.$$

or
$$\sum_{m=+\infty}^{m=-\infty}\alpha^{-m}\beta^{m-1}$$

by *

$$\frac{1}{\beta-\alpha}+\frac{1}{\alpha-\beta},$$

since it is the sum of the infinite series for $(\beta-\alpha)^{-1}$ and $(\alpha-\beta)^{-1}$, he proves after a fashion that its product by $\beta-\alpha$ or $\alpha-\beta$ is 0, and that therefore its product by

$$\frac{1}{\gamma+m(\beta-\alpha)}\quad\text{or}\quad\frac{1}{\gamma+m(\alpha-\beta)}$$

is simply its product by $\dfrac{1}{\gamma}$. Turning then from this lemma to the product

$$\left(\frac{1}{u_0-t_0}+\frac{1}{t_0-u_0}\right)\left(\frac{1}{u_1-t_1}+\frac{1}{t_1-u_1}\right)$$

where $u_0=a_0x+b_0y$, $u_1=b_1y+a_1x$, he substitutes for the first factor of it

$$\frac{b_1}{|a_0b_1|x-|b_1t_0|+b_0(u_1-t_1)}+\frac{b_1}{|b_1t_0|-|a_0b_1|x-b_0(u_1-t_1)}$$

his justification being the fact that

$$b_1(u_0-t_0)=|a_0b_1|x-|b_1t_0|+b_0(u_1-t_1);$$

but, on account of the said lemma, he leaves the term $b_0(u_1-t_1)$ out of both denominators. For the second factor there is thereupon substituted

$$\frac{|a_0b_1|}{b_1\{|a_0b_1|y-|a_0t_1|\}+a_1\{|a_0b_1|x-|b_1t_0|\}}$$
$$+\frac{|a_0b_1|}{b_1\{|a_0t_1|-|a_0b_1|y\}+a_1\{|b_1t_0|-|a_0b_1|x\}}$$

---

* Jacobi writes it $\frac{1}{\beta-\alpha}+\frac{1}{\alpha-\beta}$ with the caution that the two parts are not to be taken as cancelling one another. Of course, also, lower down he does not write $|a_0b_1|$ but $a_0b_1-a_1b_0$ or later (a_0b_1).

on the ground that we have the identity

$$|a_0b_1|\cdot(u_1-t_1)=b_1\{|a_0b_1|y-|a_0t_1|\}+a_1\{|a_0b_1|x-|b_1t_0|\},$$

the term $a_1\{|a_0b_1|x-|b_1t_0|\}$ being subsequently left out of both denominators for the same reason as before. The result thus reached is consequently

$$\begin{aligned}|a_0b_1|\cdot\left(\frac{1}{u_0-t_0}+\frac{1}{t_0-u_0}\right)\left(\frac{1}{u_1-t_1}+\frac{1}{t_1-u_1}\right)\\ =\left(\frac{|a_0b_1|}{|a_0b_1|x-|b_1t_0|}+\frac{|a_0b_1|}{|b_1t_0|-|a_0b_1|x}\right)\\ \cdot\left(\frac{|a_0b_1|}{|a_0b_1|y-|a_0t_1|}+\frac{|a_0b_1|}{|a_0t_1|-|a_0b_1|y}\right),\end{aligned}$$

or, if we write ξ, η for the values of x, y which make $u_0-t_0=0$, $u_1-t_1=0$,

$$\begin{aligned}|a_0b_1|\cdot\left(\frac{1}{u_0-t_0}+\frac{1}{t_0-u_0}\right)\left(\frac{1}{u_1-t_1}+\frac{1}{t_1-u_1}\right)\\ =\left(\frac{1}{x-\xi}+\frac{1}{\xi-x}\right)\left(\frac{1}{y-\eta}+\frac{1}{\eta-y}\right).\end{aligned}\qquad (a)$$

Since the general terms of the four doubly-infinite series here are

$$\frac{t_0^{\,m}}{u_0^{\,m+1}},\quad\frac{t_1^{\,n}}{u_1^{\,n+1}},\quad\frac{\xi^\mu}{x^{\mu+1}},\quad\frac{\eta^\nu}{y^{\nu+1}},$$

we deduce

$$|a_0b_1|\cdot\sum\frac{t_0^{\,m}t_1^{\,n}}{u_0^{\,m+1}u_1^{\,n+1}}=\sum\frac{\xi^\mu\eta^\nu}{x^{\mu+1}y^{\nu+1}},$$

i.e.,

$$\begin{aligned}|a_0b_1|\cdot\sum\frac{t_0^{\,m}t_1^{\,n}}{(a_0x+b_0y)^{m+1}(b_1y+a_1x)^{n+1}}\\ =\sum\frac{|b_1t_0|^\mu\cdot|a_0t_1|^\nu}{|a_0b_1|^{\mu+\nu}\cdot x^{\mu+1}y^{\nu+1}},\end{aligned}$$

where m, n on the one side and μ, ν on the other are to have all integral values from $-\infty$ to $+\infty$. Since the coefficients of $t_0^m t_1^n / x^\mu y^\nu$ on the two sides must be equal, we obtain the theorem:—*The coefficient of* $\frac{1}{\mathrm{x}^\mu \mathrm{y}^\nu}$ *in the expansion of*

$$\frac{1}{(\mathrm{a}_0\mathrm{x}+\mathrm{b}_0\mathrm{y})^{\mathrm{m}+1}(\mathrm{b}_1\mathrm{y}+\mathrm{a}_1\mathrm{x})^{\mathrm{n}+1}}$$

is the same as the coefficient of $\mathrm{t}_0^{\mathrm{m}}\mathrm{t}_1^{\mathrm{n}}$ *in the expansion of*

$$\frac{(\mathrm{b}_1\mathrm{t}_0-\mathrm{b}_0\mathrm{t}_1)^{\mu-1}(\mathrm{a}_0\mathrm{t}_1-\mathrm{a}_1\mathrm{t}_0)^{\nu-1}}{|\,\mathrm{a}_0\mathrm{b}_1\,|^{\mu+\nu-1}},$$

it being remembered that m *and* n *are of the same sign as* μ *and* ν *respectively and that* $\mathrm{m}+\mathrm{n}=\mu+\nu-2$. (LII. 2)

In similar fashion the author deals with the case of three functions u_0, u_1, u_2 of three variables x, y, z, proving laboriously and not very neatly the neat result

$$|\,a_0\,b_1\,c_2\,|\cdot\left(\frac{1}{u_0-t_0}+\frac{1}{t_0-u_0}\right)\left(\frac{1}{u_1-t_1}+\frac{1}{t_1-u_1}\right)\left(\frac{1}{u_2-t_2}+\frac{1}{t_2-u_2}\right)$$

$$=\left(\frac{1}{x-\xi}+\frac{1}{\xi-x}\right)\left(\frac{1}{y-\eta}+\frac{1}{\eta-y}\right)\left(\frac{1}{z-\zeta}+\frac{1}{\zeta-z}\right) \qquad (\beta)$$

thence deriving

$$|\,a_0\,b_1\,c_2\,|\cdot\sum\frac{t_0^{\,m}t_1^{\,n}t_2^{\,r}}{u_0^{\,m+1}u_1^{\,n+1}u_2^{\,r+1}}=\sum\frac{\xi^\mu\eta^\nu\zeta^\rho}{x^{\mu+1}y^{\nu+1}z^{\rho+1}},$$

and ending with the theorem:—

The coefficient of $\frac{1}{\mathrm{x}^\mu\,\mathrm{y}^\nu\,\mathrm{z}^\rho}$ *in the expansion of*

$$\frac{1}{(\mathrm{a}_0\mathrm{x}+\mathrm{b}_0\mathrm{y}+\mathrm{c}_0\mathrm{z})^{\mathrm{m}+1}(\mathrm{b}_1\mathrm{y}+\mathrm{c}_1\mathrm{z}+\mathrm{a}_1\mathrm{x})^{\mathrm{n}+1}(\mathrm{c}_2\mathrm{z}+\mathrm{a}_2\mathrm{x}+\mathrm{b}_2\mathrm{y})^{\mathrm{r}+1}}$$

is the same as the coefficient of $\mathrm{t}_0^{\mathrm{m}}\mathrm{t}_1^{\mathrm{n}}\mathrm{t}_2^{\mathrm{r}}$ *in the expansion of*

$$\frac{\{|\mathrm{b}_1\mathrm{c}_2|\mathrm{t}_0+|\mathrm{b}_2\mathrm{c}_0|\mathrm{t}_1+|\mathrm{b}_0\mathrm{c}_1|\mathrm{t}_2\}^{\mu-1}\{|\mathrm{c}_2\mathrm{a}_0|\mathrm{t}_1+|\mathrm{c}_0\mathrm{a}_1|\mathrm{t}_2+|\mathrm{c}_1\mathrm{a}_2|\mathrm{t}_0\}^{\nu-1}\{|\mathrm{a}_0\mathrm{b}_1|\mathrm{t}_2+|\mathrm{a}_1\mathrm{b}_2|\mathrm{t}_0+|\mathrm{a}_2\mathrm{b}_0|\mathrm{t}_1\}^{\rho}}{|\mathrm{a}_0\mathrm{b}_1\mathrm{c}_2|^{\mu+\nu+\rho-2}}$$

it being understood that m, n, r *are of the same sign as* μ, ν, ρ *respectively and that* $\mathrm{m}+\mathrm{n}+\mathrm{r}=\mu+\nu+\rho-3$. (LII. 2)

The corresponding results for n functions of n variables are evident. They had already been enunciated in the introductory section of the paper, and Jacobi now merely adds "Omnino similia theoremata de numero quolibet variabilium, quae § 1 proposuimus, eruuntur." It has to be noted, however, that belief in the general fundamental theorem, viz., that which includes (α) and (β) above, is more strongly induced by the elegance of the form of the theorem than by the mode of proof. In § 1 it stands approximately thus—

$$\left(\frac{1}{u_0-t_0}+\frac{1}{t_0-u_0}\right)\left(\frac{1}{u_1-t_1}+\frac{1}{t_1-u_1}\right)\cdots\cdots\left(\frac{1}{u_{n-1}-t_{n-1}}+\frac{1}{t_{n-1}-u_{n-1}}\right)$$

$$=\frac{1}{\Delta}\left(\frac{1}{x_0-p_0}+\frac{1}{p_0-x_0}\right)\left(\frac{1}{x_1-p_1}+\frac{1}{p_1-x_1}\right)\cdots\left(\frac{1}{x_{n-1}-p_{n-1}}+\frac{1}{p_{n-1}-x_{n-1}}\right)$$

and then follows the passage containing the two deductions, viz.,

"quam aequationem etiam hunc in modum repraesentare licet:

$$\sum\frac{t_0^{\alpha_0}\,t_1^{\alpha_1}\ldots t_{n-1}^{\alpha_{n-1}}}{u_0^{\alpha_0+1}\,u_1^{\alpha_1+1}\ldots u_{n-1}^{\alpha_{n-1}+1}}=\frac{1}{\Delta}\cdot\sum\frac{p_0^{\beta_0}\,p_1^{\beta_1}\ldots p_{n-1}^{\beta_{n-1}}}{x_0^{\beta_0+1}x_1^{\beta_1+1}\ldots x_{n-1}^{\beta_{n-1}+1}},$$

designantibus α_0, α_1, etc. β_0, β_1, etc. numeros omnes et positivos et negativos a $-\infty$ ad $+\infty$. E quo theoremate videmus, coefficientem termini

$$\frac{1}{x_0^{\beta_0+1}\,x_1^{\beta_1+1}\ldots x_{n-1}^{\beta_{n-1}+1}}$$

in expressione

$$\frac{1}{u_0^{\alpha_0+1}\,u_1^{\alpha_1+1}\ldots u_{n-1}^{\alpha_{n-1}+1}}$$

aequalem fore coëfficienti termini $t_0^{\alpha_1}\,t_1^{\alpha_1}\ldots t_{n-1}^{\alpha_{n-1}}$
in expressione

$$\frac{1}{\Delta}p_0^{\beta_0}\,p_1^{\beta_1}\ldots p_{n-1}^{\beta_{n-1}}." \tag{LII. 3}$$

The use here of $\beta_0+1, \beta_1+1, \ldots.$ rather than the change made in the two special cases to the less natural $\beta_0, \beta_1, \ldots$ is worth noting.

The theorems of the remaining four pages of the paper have a less direct bearing on our subject.

MINDING (1829).

[Auflösung einiger Aufgaben der analytischen Geometrie vermittelst des barycentrischen Calculs. *Crelle's Journal*, v. pp. 397–401.]

Unlike Jacobi, Minding was unaware, apparently, of the existence of a theory of determinants. The functions occur at every step of his investigation, yet he makes no use of their known properties to obtain his results.

He deals with four problems in his memoir, the second two being the analogues, in space, of the first two. Nothing noteworthy occurs in connection with the latter save that use is made of the identity,

$$\frac{\beta'\gamma''-\beta''\gamma'}{a'} = a(b'c''-b''c') + a'(b''c-bc'') + a''(bc'-b'c),$$

where
$$\beta' = ba' - b'a, \qquad \beta'' = b'a'' - b''a',$$
$$\gamma' = ca' - c'a, \qquad \gamma'' = c'a'' - c''a'.$$

This identity, it may be remembered, we have noted under Lagrange as an elementary case of the theorem afterwards well known regarding a minor of the adjugate determinant. Strange to say, it makes only its second appearance here fifty-six years afterwards. In the interim, too, no other special case of the theorem seems to have been established.

The third is that if P, P′ P″, P‴ be four points in space, given by the equations,

$$\begin{aligned} q\,\mathrm{P} &= a\,\mathrm{A} + b\,\mathrm{B} + c\,\mathrm{C} + d\,\mathrm{D},\\ q'\,\mathrm{P}' &= a'\,\mathrm{A} + b'\,\mathrm{B} + c'\,\mathrm{C} + d'\,\mathrm{D},\\ q''\,\mathrm{P}'' &= a''\,\mathrm{A} + b''\,\mathrm{B} + c''\,\mathrm{C} + d''\,\mathrm{D},\\ q'''\mathrm{P}''' &= a'''\mathrm{A} + b'''\mathrm{B} + c'''\mathrm{C} + d'''\mathrm{D}; \end{aligned}$$

then for the bulk of the tetrahedron PP′P″P‴, we have

$$\frac{\mathrm{PP'P''P'''}}{\mathrm{ABC\,D}} = \frac{\mathrm{A}+\mathrm{A}'+\mathrm{A}''}{qq'q''q'''},$$

where

$$\mathrm{A} = \delta'(\beta''\gamma'''-\beta'''\gamma''),\ \ \mathrm{A}' = \delta''(\beta'''\gamma'-\beta'\gamma'''),\ \ \mathrm{A}'' = \delta'''(\beta'\gamma''-\beta''\gamma'),$$

and

$$\begin{aligned}
\beta' &= a' b - a b' , & \gamma' &= a' c - a c' , & \delta' &= a' d - a d' , \\
\beta'' &= a'' b' - a' b'' , & \gamma'' &= a'' c' - a' c'' , & \delta'' &= a'' d' - a' d'' , \\
\beta''' &= a''' b'' - a'' b''' , & \gamma''' &= a''' c'' - a'' c''' , & \delta''' &= a''' d'' - a'' d''' .
\end{aligned}$$

The transformation of $A+A'+A''$ into the form

$$a'a''|ab'c''d'''|$$

—a transformation all-important for Minding's purpose—is not made: but in the remark,

"Man kann den Ausdruck $A+A'+A''$ leicht entwickeln, und wird ihn dann durch $a'a''$ theilbar finden,"

there is evidently a foreshadowing of the identity

$$\begin{vmatrix} |a' b|, & |a' c|, & |a' d| \\ |a'' b'|, & |a'' c'|, & |a'' d'| \\ |a''' b''|, & |a''' c''|, & |a''' d''| \end{vmatrix} = - a'a''|ab'c''d'''| .$$

The fourth theorem, concerning the tetrahedron enclosed by four given planes,

$$\begin{aligned}
&A+xB+yC+(a +b x+c y)C , \\
&A+xB+yC+(a' +b' x+c' y)C , \\
&A+xB+yC+(a'' +b'' x+c'' y)C , \\
&A+xB+yC+(a'''+b'''x+c'''y)C ,
\end{aligned}$$

is made dependent on the third. The intersections Π, Π', Π'', Π''' of the four triads of planes are found to be given by

$$\begin{aligned}
q\ \Pi &= (b\ c')A + (c\ a')B + (a\ b')C + (a\ b\ c)D , \\
q'\ \Pi' &= (b'\ c'')A + (c'\ a'')B + (a'\ b'')C + (a'\ b'\ c')D , \\
q''\ \Pi'' &= (b''\ c''')A + (c''\ a''')B + (a''\ b''')C + (a''\ b''\ c'')D , \\
q'''\Pi''' &= (b'''c)A + (c'''a)B + (a'''b)C + (a'''b'''c''')D ,
\end{aligned}$$

where

$$\begin{aligned}
(bc') &= b(c'-c'') + b'(c''-c) + b''(c-c') , \\
(ca') &= c(a'-a'') + c'(a''-a) + c''(a-a') , \\
(ab') &= a(b'-b'') + a'(b''-b) + a''(b-b') ,
\end{aligned}$$

and

$$\begin{aligned}
(abc) &= a(bc') + b(ca') + c(ab') , \\
&= a(b'c''-b''c') + a'(b''c-bc'') + a''(bc'-b'c) .
\end{aligned}$$

Hence, by the third theorem,

$$\frac{\Pi\Pi'\Pi''\Pi'''}{\mathrm{A\ B\ C\ D}} = \frac{\mathrm{A}+\mathrm{A}'+\mathrm{A}''}{qq'q''q'''(b'c'')(b''c''')},$$

where now

$$\mathrm{A}=\delta'(\beta''\gamma'''-\beta'''\gamma''),\quad \mathrm{A}'=\delta''(\beta'''\gamma'-\beta'\gamma'''),\quad \mathrm{A}''=\delta'''(\beta'\gamma''-\beta''\gamma'),$$

and

$$\begin{array}{lll} \beta' = (b'c'')(ca') - (b\,c')(c'a''), & \beta'' = \ldots, & \beta''' = \ldots, \\ \gamma' = (b'c'')(ab') - (b\,c')(a'b''), & \gamma'' = \ldots, & \gamma''' = \ldots, \\ \delta' = (b'c'')(abc) - (b\,c')(a'b'c'). & \delta'' = \ldots, & \delta''' = \ldots, \end{array}$$

Minding then continues (pp. 399, 400):—

"Man setze

$$a'''(bc') - a(b'c'') + a'(b''c''') - a''(b'''c) = \mathrm{M}.$$

Nach den nöthigen Reductionen erhält man:

$$\begin{array}{lll} \beta' = -(c''-c')\mathrm{M}, & \gamma' = -(b'-b'')\mathrm{M}, & \delta' = -(b'c''-b''c')\mathrm{M}, \\ \beta'' = +(c'''-c'')\mathrm{M}, & \gamma'' = +(b''-b''')\mathrm{M}, & \delta'' = +(b''c'''-b'''c'')\mathrm{M}, \\ \beta''' = -(c-c''')\mathrm{M}, & \gamma''' = -(b'''-b)\mathrm{M}, & \delta''' = -(b'''c-b\,c''')\mathrm{M}. \end{array}$$

Hieraus erhält man weiter:

$$\begin{aligned} \mathrm{A} &= -\mathrm{M}^3(b''c'-b'c'').(b''c'''), \\ \mathrm{A}' &= -\mathrm{M}^3(b'''c''-b''c''').\{(b''c''')-(b'''c)\}, \\ \mathrm{A}'' &= -\mathrm{M}^3(b\,c'''-b'''c).(b'c''). \end{aligned}$$

Eine weitere Reduction ergiebt:

$$(bc'''-b'''c)(b'c'') - (b'''c)(b'''c''-b''c''') = (c'''b'-c'b''')(b''c''').$$

Hieraus folgt $\mathrm{A}+\mathrm{A}'+\mathrm{A}''=\mathrm{M}^3(b'c'')(b''c''')$, und als Resultat:

$$\frac{\Pi\Pi'\Pi''\Pi'''}{\mathrm{A\ B\ C\ D}} = \frac{\mathrm{M}^3}{qq'q''q'''}."$$

The first point to be noted here is, that since

$$(bc'),\quad (ca'),\quad (ab'),$$

are in later notation

$$\begin{vmatrix} b & b' & b'' \\ c & c' & c'' \\ 1 & 1 & 1 \end{vmatrix},\quad \begin{vmatrix} c & c' & c'' \\ a & a' & a'' \\ 1 & 1 & 1 \end{vmatrix},\quad \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ 1 & 1 & 1 \end{vmatrix}$$

the identity

$$a(bc') + b(ca') + c(ab') = a(b'c''-b''c') + a'(b''c - bc'') + a''(bc' - b'c)$$

is the same as

$$a\begin{vmatrix} b & b' & b'' \\ c & c' & c'' \\ 1 & 1 & 1 \end{vmatrix} + b\begin{vmatrix} c & c' & c'' \\ a & a' & a'' \\ 1 & 1 & 1 \end{vmatrix} + c\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix},$$

—a disguised special case of Vandermonde's theorem (XII.), the four elements of one row being each unity. (XII. 11)

The next point is, that since the expression denoted by M, viz.,

$$a'''(bc') - a(b'c'') + a'(b''c''') - a''(b'''c)$$

is in modern notation

$$-\begin{vmatrix} a & a' & a'' & a''' \\ b & b' & b'' & b''' \\ c & c' & c'' & c''' \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

the identity

$$\delta' = -(b'c''-b''c')\mathrm{M}$$

is the same as

$$\left|\begin{matrix} \begin{vmatrix} b' & b'' & b''' \\ c' & c'' & c''' \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} b & b' & b'' \\ c & c' & c'' \\ 1 & 1 & 1 \end{vmatrix} \\ \begin{vmatrix} a' & a'' & a''' \\ b' & b'' & b''' \\ c' & c'' & c''' \end{vmatrix} & \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} \end{matrix}\right| = \begin{vmatrix} b' & b'' \\ c' & c'' \end{vmatrix} \cdot \begin{vmatrix} a & a' & a'' & a''' \\ b & b' & b'' & b''' \\ c & c' & c'' & c''' \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

and therefore is, like its eight companions, a fresh case of the theorem regarding a minor of the adjugate.* (XX. 3)

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* Instead of following Minding's lengthy process, a mathematician of the present time would of course observe that the coefficients of A, B, C, D on p. 195 are the principal minors of M, and using Cauchy's theorem would at once reach the desired conclusion, viz., that the determinant of them $= \mathrm{M}^3$.

DRINKWATER, J. E. (1831).

[On Simple Elimination. *Philosophical Magazine*, x. pp. 24–28.]

Up to this date, almost 140 years after the publication of Leibnitz's letter to De L'Hôpital, no English mathematician's name occurs in connection with the subject of determinants,—a fact most significant of the comparative neglect of mathematical studies in Britain during the 18th century. Apart from the contents, therefore, some little interest attaches to Drinkwater's short paper, as being the first sign to us of that revival which, as is well known otherwise, had taken place some few years before.

Drinkwater knew of the investigations of Cramer, Bezout, and Laplace; and professed only to put the elements of the subject "in a more convenient form." His rule of signs is stated and illustrated as follows (p. 25):—

"Write down the series of natural numbers 1 2 3 4 . . . n, and underneath it all the permutations of these n numbers, prefixing to each a positive or negative sign according to the following condition:—

"Any permutation may be derived from the first by considering a requisite number of figures to move from left to right by a certain number of single steps or descents of a single place. If the whole number of such single steps necessary to derive any permutation from the first be even, that permutation has a positive sign prefixed to it; the others are negative. For instance, 4 2 1 3 . . . n may be derived from 1 2 3 4 . . . . n, by first causing the 3 to descend below the 4, requiring one single step: then the 2 below the new place of the 4, another single step; lastly, the 1 below the new place of the 2, requiring two more steps, making in all 4. Therefore this permutation requires the positive sign."

In this there is essentially nothing new: it at once recalls a theorem of Rothe's (III. 8). In the following paragraph, however, we find the discussion of a point not previously dealt with. The words are (p. 25):—

"The same permutation may be derived in various ways, and it is necessary, therefore, to show that this rule is not inconsistent with itself: thus the same permutation 4 2 1 3 . . . n might have been obtained by first marching 1 through three places, then 2 through two; and, lastly, 3 through one [?], making six [?] in all, an even number as before. Without accumulating instances, it is plain, if q be the smallest number of steps by which any number p reaches the place it is intended finally to occupy in that permutation, that if p should advance in the first instance m places beyond this, it must subsequently return through m places: or, which is the same thing, it must at a later period of the march, allow m of those which it has passed to repass it, so that it will regain its proper place after the number of steps has been increased from q to $q+2m$, which, by the rule, require the same sign as q. The same reasoning applies to every other figure; and hence the consistency of the rule is evident." (III. 26)

He then establishes four properties of the functions, viz. (1) Vandermonde's theorem regarding the effect produced on the *function* by transposition of a pair of letters; (2) Bezout's recurrent law of formation; (3) Scherk's theorem regarding the partition of one of the functions into two; and (4) Scherk's theorem regarding the removal of a constant factor from one of the functions. The two latter theorems, which, as we have seen, had been stated for the first time only six years before, are given by Drinkwater in the following form (p. 27):—

"(8) If any factor in $f\{XYZT \ldots (n)\}$, as X, be divided into two parts, $X = V + W$, the function may be similarly divided, so that

$$f\{(V+W)YZT \ldots (n)\} = f\{VYZT \ldots (n)\} + f(WYZT \ldots (n)\},$$

placing each part of X in the same relative position (which in this example is the first) which X itself occupied before the division. (XLVI. 2)

(9) If any quantity which does not vary from one equation to the other, and which, therefore, is not liable to be affected with an index, is found under the symbol, it may be considered a constant coefficient of every term of the developed function; and written as such on the outside of the symbol: of this nature are the unknown quantities themselves, so that for instance,

$$f\{XYxZT \ldots. (n)\} = xf\{XYZT \ldots (n)\},$$

and so of like quantities." (XLVII. 2)

After these preliminaries the problem of the solution of n linear equations in n unknowns is taken up. The method followed is essentially the same as Scherk's.

MAINARDI (1832).

[Trasformazioni di alcune funzioni algebraiche, e loro uso nella geometria e nella meccanica. Memoria di Gaspare Mainardi. 44 pp. Pavia, 1832.]

In his preface Mainardi explains that the algebraical functions referred to in the title are "*funzioni risultanti* o *determinanti*." But although he thus speaks of them as if they were known to mathematicians by name, and mentions the researches of Monge, Lagrange, Cauchy, and Binet in regard to them, he does not take for granted that his reader has a knowledge of any of their properties. The one theorem on determinants,—the multiplication-theorem,—which forms the basis of the whole memoir, is consequently sought to be established without the use of any previously proved theorem. The attempt, as might be expected, is interesting.

The first two sections (pp. 9–29) of the three into which the memoir is divided may be passed over without much comment. The first deals with the multiplication-theorem for two determinants of the 2nd order, and with those applications of it to geometry which arise on making the elements of each determinant the Cartesian co-ordinates of two points in a plane. No proof is considered necessary for this simple case, the opening paragraph of the memoir being;—

"Rappresentate con x_m, x_n, x_a, x_b; y_m, y_n, y_a, y_b otto quantità qualsivogliano, ed indicati per brevità il binomio

$$x_m . x_a + y_m . y_a \text{ col simbolo } (x_m x_a),$$

il binomio

$$x_n . x_b + y_n . y_b \quad \text{con} \quad (x_n x_b)$$

e simili, si proverà facilmente essere

$$\text{(a)} \qquad \begin{aligned} &(x_m y_n - x_n y_m)(x_a y_b - x_b y_a) \\ &= (x_m x_a)(x_n x_b) - (x_m x_b)(x_n x_a)." \end{aligned}$$

All the seven other paragraphs are geometrical.

The second section in like manner opens with an algebraical theorem, viz. (p. 13)—

$$\begin{aligned} &\{x_m(y_p - y_n)\}\{x_a(y_c - y_b)\} \\ +\ &\{x_m(z_p - z_n)\}\{x_a(z_c - z_b)\} \\ +\ &\{y_m(z_p - z_n)\}\{y_a(z_c - z_b)\} \end{aligned}$$

$$
\begin{aligned}
= \quad & (x_m x_a)(x_p x_c) - (x_m x_c)(x_p x_a) + (x_n x_a)(x_m x_c) \\
& - (x_n x_c)(x_m x_a) + (x_p x_a)(x_n x_c) - (x_p x_c)(x_n x_a) \\
& + (x_m x_b)(x_p x_a) - (x_m x_a)(x_p x_b) + (x_n x_b)(x_m x_a) \\
& - (x_n x_a)(x_m x_b) + (x_p x_b)(x_n x_a) - (x_p x_a)(x_n x_b) \\
& + (x_m x_c)(x_p x_b) - (x_m x_b)(x_p x_c) + (x_n x_c)(x_m x_b) \\
& - (x_n x_b)(x_m x_c) + (x_p x_c)(x_n x_b) - (x_p x_b)(x_n x_c), \quad \text{(XXIX. 2)}
\end{aligned}
$$

where $\{x_m(y_p - y_n)\}$ and $(x_m x_a)$ stand for

$$(x_m y_p - x_p y_m) + (x_n y_m - x_m y_n) + (x_p y_n - x_n y_p)$$

and
$$x_m x_a + y_m y_a + z_m z_a$$

respectively; and the remainder is occupied with the applications of the theorem to geometry and dynamics. Each factor of the left-hand side of the identity is evidently a determinant of the third order, and the three pairs of lines on the right-hand side are each the expansion of a determinant of the same order; so that in the notation of the present day the identity may be written

$$
\begin{aligned}
&\begin{vmatrix} x_m & y_m & 1 \\ x_n & y_n & 1 \\ x_p & y_p & 1 \end{vmatrix} \cdot \begin{vmatrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix} + \begin{vmatrix} x_m & z_m & 1 \\ x_n & z_n & 1 \\ x_p & z_p & 1 \end{vmatrix} \cdot \begin{vmatrix} x_a & z_a & 1 \\ x_b & z_b & 1 \\ x_c & z_c & 1 \end{vmatrix} \\
&+ \begin{vmatrix} y_m & z_m & 1 \\ y_n & z_n & 1 \\ y_p & z_p & 1 \end{vmatrix} \cdot \begin{vmatrix} y_a & z_a & 1 \\ y_b & z_b & 1 \\ y_c & z_c & 1 \end{vmatrix} = \begin{vmatrix} (x_m x_c) & (x_m x_a) & 1 \\ (x_n x_c) & (x_n x_a) & 1 \\ (x_p x_c) & (x_p x_a) & 1 \end{vmatrix} \\
&\qquad + \begin{vmatrix} (x_m x_a) & (x_m x_b) & 1 \\ (x_n x_a) & (x_n x_b) & 1 \\ (x_p x_a) & (x_p x_b) & 1 \end{vmatrix} + \begin{vmatrix} (x_m x_b) & (x_m x_c) & 1 \\ (x_n x_b) & (x_n x_c) & 1 \\ (x_p x_b) & (x_p x_c) & 1 \end{vmatrix}
\end{aligned}
$$

There has been no previous instance of an identity perfectly similar to this; the nearest approach to such being, as the numbering shows, a result obtained by Binet in 1811. The exact character of the affinity between the two, and the general theorem which both foreshadow, will be most readily brought into evidence by a little additional transformation. Taking first the right-hand side of the identity, we observe that the three determinants have only twelve elements among them, being obtainable in fact from a single array of three rows and four columns. Their sum may consequently be put in the form

$$\begin{vmatrix} 1 & (x_m x_a) & (x_m x_b) & (x_m x_c) \\ 1 & (x_n x_a) & (x_n x_b) & (x_n x_c) \\ 1 & (x_p x_a) & (x_p x_b) & (x_p x_c) \\ 0 & 1 & 1 & 1 \end{vmatrix}.$$

Secondly, we observe that the first factors on the left-hand side are similarly obtainable from

$$\begin{matrix} x_m & y_m & z_m & 1 \\ x_n & y_n & z_n & 1 \\ x_p & y_p & z_p & 1\,; \end{matrix}$$

and the second factors from

$$\begin{matrix} x_a & y_a & z_a & 1 \\ x_b & y_b & z_b & 1 \\ x_c & y_c & z_c & 1\,; \end{matrix}$$

and as the determinant which is the so-called product of these arrays is equal to the said left-hand member diminished by

$$\begin{vmatrix} x_m & y_m & z_m \\ x_n & y_n & z_n \\ x_p & y_p & z_p \end{vmatrix} \cdot \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix},$$

Mainardi's theorem may be put in the much altered form—

$$\begin{vmatrix} 1 & (x_m x_a) & (x_m x_b) & (x_m x_c) \\ 1 & (x_n x_a) & (x_n x_b) & (x_n x_c) \\ 1 & (x_p x_a) & (x_p x_b) & (x_p x_c) \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x_m & y_m & z_m & 1 \\ x_n & y_n & z_n & 1 \\ x_p & y_p & z_p & 1 \end{vmatrix} \cdot \begin{vmatrix} x_a & y_a & z_a & 1 \\ x_b & y_b & z_b & 1 \\ x_c & y_c & z_c & 1 \end{vmatrix}$$

$$- \begin{vmatrix} x_m & y_m & z_m \\ x_n & y_n & z_n \\ x_p & y_p & z_p \end{vmatrix} \cdot \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}.$$

The constitution of the 3rd section is quite like that of the others, the first paragraph dealing with the multiplication-theorem for the case of determinants of the 3rd order, the second paragraph with the same theorem for determinants of the 4th order, and the remaining eight paragraphs with geometrical applications. The mode of proof of the multiplication-theorem is partly indicated by saying that any particular case is made dependent on the case immediately preceding it; but its exact character can only be understood by a somewhat minute examination. The investigation for the case of determinants of the 3rd order stands as follows (p. 29):—

"Si considerino i due polinomj

$$(l)\qquad \begin{aligned} x_m(y_nz_p - y_pz_n) + x_n(z_my_p - y_mz_p) + x_p(y_mz_n - y_nz_m) &= \{x_m, y_n, z_p\}, \\ x_a(y_bz_c - y_cz_b) + x_b(z_ay_c - z_cy_a) + x_c(y_az_b - y_bz_a) &= \{x_a, y_b, z_c\}. \end{aligned}$$

Se ne effettui il prodotto, il quale, mediante l'equazione (a) del primo articolo, si potrà disporre sotto la forma seguente

$$(h)\qquad \begin{aligned} & x_mx_a(y_ny_b)(y_py_c) - x_mx_a(y_ny_c)(y_py_b) \\ + & x_nx_a(y_my_c)(y_py_b) - x_nx_a(y_my_b)(y_py_c) \\ + & x_px_a(y_my_b)(y_ny_c) - x_px_a(y_my_c)(y_ny_b) \\ + & x_mx_b(y_ny_c)(y_py_a) - x_mx_b(y_ny_a)(y_py_c) \\ + & x_nx_b(y_my_a)(y_py_c) - x_nx_b(y_my_c)(y_py_a) \\ + & x_px_b(y_my_c)(y_ny_a) - x_px_b(y_my_a)(y_ny_c) \\ + & x_mx_c(y_ny_a)(y_py_b) - x_mx_c(y_ny_b)(y_py_a) \\ + & x_nx_c(y_my_b)(y_py_a) - x_nx_c(y_my_a)(y_py_b) \\ + & x_px_c(y_my_a)(y_ny_b) - x_px_c(y_my_b)(y_ny_a). \end{aligned}$$

Esaminando ora la quantità

$$\begin{aligned} & x_mx_a\{x_nx_b(y_py_c) + x_px_c(y_ny_b) + x_nx_bx_px_c \\ & \qquad - x_nx_c(y_py_b) - x_px_b(y_ny_c) - x_nx_px_bx_c\} \\ + & x_nx_a\{x_mx_c(y_py_b) + x_px_b(y_my_c) + x_mx_cx_px_b \\ & \qquad - x_mx_b(y_py_c) - x_px_c(y_my_b) - x_mx_bx_px_c\} \\ + & x_px_a\{x_mx_b(y_ny_c) + x_nx_c(y_my_b) + x_mx_bx_nx_c \\ & \qquad - x_mx_c(y_ny_b) - x_nx_b(y_my_c) - x_mx_cx_nx_b\}, \end{aligned}$$

e le due espressioni che si traggono da questa, cambiando, prima a in b, b in c, c in a; poscia a in c, c in b, b in a; con facilità si scorge che la somma di questi polinomj è nulla identicamente, per cui si potrà aggiungere al prodotto (h) senza punto alterarlo. Fatta quest' addizione, l'aggregato altro non sarà che lo stesso polinomio (h), ove si supponga che i simboli $(y_n y_b)$, $(y_p y_c)$, ecc. rappresentino rispettivamente i trinomj seguenti

$$x_n x_b + y_n y_b + z_n z_b, \quad x_p x_c + y_p y_c + z_p z_c, \quad \text{ecc.}$$

Se ora si ordineranno le espressioni (l) portando fuori dalle parentesi y ovvero z in luogo di x, formeremo il prodotto delle medesime così scritte, ed opereremo come sopra, il risultato sarà il polinomio che si desume da (h) cambiando le x che sono fuori dalle parentesi in y ovvero in z egualmente accentate. Se faremo per ultimo la somma di queste tre espressioni, tal somma si caverà dal polinomio (h) scrivendo $(x_m x_a)$ ovvero $(y_m y_a)$ invece di $x_m x_a$; $(x_p x_a)$ in luogo di $x_p x_a$ ec. ec. e sarà eguale al triplo prodotto delle expressioni (l).

Essendo poi quella somma divisibile per tre, effettuata la divisione per questo numero, avremo

$$\begin{aligned}\{x_m, y_n, z_p\} . \{x_a, y_b, z_c\} = \quad & (x_m x_a)(x_n x_b)(x_p x_c) + (x_n x_a)(x_p x_b)(x_m x_c) \\ & \qquad + (x_p x_a)(x_m x_b)(x_n x_c) \\ & - (x_m x_a)(x_p x_b)(x_n x_c) - (x_n x_a)(x_m x_b)(x_p x_c) \\ & \qquad - (x_p x_a)(x_n x_b)(x_m x_c).\text{"}\end{aligned}$$

(XVII. 6)

That the essential points of this method of demonstration may be seen, let us apply it as it would be applied if adopted at the present day :—

The given determinants being

$$|a_1 b_2 c_3| \text{ and } |\alpha_1 \beta_2 \gamma_3|,$$

we should say

$$|a_1 b_2 c_3| = a_1 |b_2 c_3| - a_2 |b_1 c_3| + a_3 |b_1 c_2|,$$

and

$$|\alpha_1 \beta_2 \gamma_3| = \alpha_1 |\beta_2 \gamma_3| - \alpha_2 |\beta_1 \gamma_3| + \alpha_3 |\beta_1 \gamma_2|;$$

hence, using the multiplication-theorem as established for determinants of the 2nd order, and (to save on the breadth of the page) denoting

$$a\alpha + b\beta + c\gamma + \dots \text{ by } \frac{a, b, c, \dots}{\alpha, \beta, \gamma, \dots}.$$

we should have

$$|a_1 b_2 c_3| . |\alpha_1 \beta_2 \gamma_3|$$

$$
\begin{aligned}
= \; & a_1\alpha_1 \begin{vmatrix} \frac{b_2,\, c_2}{\beta_2, \gamma_2} & \frac{b_2,\, c_2}{\beta_3, \gamma_3} \\ \frac{b_3,\, c_3}{\beta_2, \gamma_2} & \frac{b_3,\, c_3}{\beta_3, \gamma_3} \end{vmatrix} - a_2\alpha_1 \begin{vmatrix} \frac{b_1,\, c_1}{\beta_2, \gamma_2} & \frac{b_1,\, c_1}{\beta_3, \gamma_3} \\ \frac{b_3,\, c_3}{\beta_2, \gamma_2} & \frac{b_3,\, c_3}{\beta_3, \gamma_3} \end{vmatrix} + a_3\alpha_1 \begin{vmatrix} \frac{b_1,\, c_1}{\beta_2, \gamma_2} & \frac{b_1,\, c_1}{\beta_3, \gamma_3} \\ \frac{b_2,\, c_2}{\beta_2, \gamma_2} & \frac{b_2,\, c_2}{\beta_3, \gamma_3} \end{vmatrix} \\
- \; & a_1\alpha_2 \begin{vmatrix} \frac{b_2,\, c_2}{\beta_1, \gamma_1} & \frac{b_2,\, c_2}{\beta_3, \gamma_3} \\ \frac{b_3,\, c_3}{\beta_1, \gamma_1} & \frac{b_3,\, c_3}{\beta_3, \gamma_3} \end{vmatrix} + a_2\alpha_2 \begin{vmatrix} \frac{b_1,\, c_1}{\beta_1, \gamma_1} & \frac{b_1,\, c_1}{\beta_3, \gamma_3} \\ \frac{b_3,\, c_3}{\beta_1, \gamma_1} & \frac{b_3,\, c_3}{\beta_3, \gamma_3} \end{vmatrix} - a_3\alpha_2 \begin{vmatrix} \frac{b_1,\, c_1}{\beta_1, \gamma_1} & \frac{b_1,\, c_1}{\beta_3, \gamma_3} \\ \frac{b_2,\, c_2}{\beta_1, \gamma_1} & \frac{b_2,\, c_2}{\beta_3, \gamma_3} \end{vmatrix} \\
+ \; & a_1\alpha_3 \begin{vmatrix} \frac{b_2,\, c_2}{\beta_1, \gamma_1} & \frac{b_2,\, c_2}{\beta_2, \gamma_2} \\ \frac{b_3,\, c_3}{\beta_1, \gamma_1} & \frac{b_3,\, c_3}{\beta_2, \gamma_2} \end{vmatrix} - a_2\alpha_3 \begin{vmatrix} \frac{b_1,\, c_1}{\beta_1, \gamma_1} & \frac{b_1,\, c_1}{\beta_2, \gamma_2} \\ \frac{b_3,\, c_3}{\beta_1, \gamma_1} & \frac{b_3,\, c_3}{\beta_2, \gamma_2} \end{vmatrix} + a_3\alpha_3 \begin{vmatrix} \frac{b_1,\, c_1}{\beta_1, \gamma_1} & \frac{b_1,\, c_1}{\beta_2, \gamma_2} \\ \frac{b_2,\, c_2}{\beta_1, \gamma_1} & \frac{b_2,\, c_2}{\beta_2, \gamma_2} \end{vmatrix}.
\end{aligned}
$$

That each line of this result is not altered in substance by writing

$$
\frac{a_2, b_2, c_2}{\alpha_2, \beta_2, \gamma_2} \text{ for } \frac{b_2,\, c_2}{\beta_2, \gamma_2}, \quad \frac{a_2, b_2, c_2}{\alpha_3, \beta_3, \gamma_3} \text{ for } \frac{b_2,\, c_2}{\beta_3, \gamma_3}, \quad \&c.,
$$

would probably be shown by expressing the line in the form of a determinant of the 3rd order, *e.g.*, the first line in the form

$$
\alpha_1 \begin{vmatrix} a_1 & \frac{b_1,\, c_1}{\beta_2, \gamma_2} & \frac{b_1,\, c_1}{\beta_3, \gamma_3} \\ a_2 & \frac{b_2,\, c_2}{\beta_2, \gamma_2} & \frac{b_2,\, c_2}{\beta_3, \gamma_3} \\ a_3 & \frac{b_3,\, c_3}{\beta_2, \gamma_2} & \frac{b_3,\, c_3}{\beta_3, \gamma_3} \end{vmatrix};
$$

and increasing each element of the second column by α_2 times the corresponding element of the first, and each element of the third column by α_3 times the corresponding element of the first. The whole result would in this way be transformed into

$$
\begin{vmatrix} a_1\alpha_1 & \frac{a_1, b_1, c_1}{\alpha_2, \beta_2, \gamma_2} & \frac{a_1, b_1, c_1}{\alpha_3, \beta_3, \gamma_3} \\ a_2\alpha_1 & \frac{a_2, b_2, c_2}{\alpha_2, \beta_2, \gamma_2} & \frac{a_2, b_2, c_2}{\alpha_3, \beta_3, \gamma_3} \\ a_3\alpha_1 & \frac{a_3, b_3, c_3}{\alpha_2, \beta_2, \gamma_2} & \frac{a_3, b_3, c_3}{\alpha_3, \beta_3, \gamma_3} \end{vmatrix} - \begin{vmatrix} a_1\alpha_2 & \frac{a_1, b_1, c_1}{\alpha_1, \beta_1, \gamma_1} & \frac{a_1, b_1, c_1}{\alpha_3, \beta_3, \gamma_3} \\ a_2\alpha_2 & \frac{a_2, b_2, c_2}{\alpha_1, \beta_1, \gamma_1} & \frac{a_2, b_2, c_2}{\alpha_3, \beta_3, \gamma_3} \\ a_3\alpha_2 & \frac{a_3, b_3, c_3}{\alpha_1, \beta_1, \gamma_1} & \frac{a_3, b_3, c_3}{\alpha_3, \beta_3, \gamma_3} \end{vmatrix}
$$

$$+\begin{vmatrix} a_1a_3 & \frac{a_1,\,b_1,\,c_1}{\alpha_1,\beta_1,\gamma_1} & \frac{a_1,\,b_1,\,c_1}{\alpha_2,\beta_2,\gamma_2} \\ a_2a_3 & \frac{a_2,\,b_2,\,c_2}{\alpha_1,\beta_1,\gamma_1} & \frac{a_2,\,b_2,\,c_2}{\alpha_2,\beta_2,\gamma_2} \\ a_3a_3 & \frac{a_3,\,b_3,\,c_3}{\alpha_1,\beta_1,\gamma_1} & \frac{a_3,\,b_3,\,c_3}{\alpha_2,\beta_2,\gamma_2} \end{vmatrix}.$$

Now by either of the interchanges

$$\begin{pmatrix} a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3 \\ b_1, b_2, b_3, \beta_1, \beta_2, \beta_3 \end{pmatrix}, \begin{pmatrix} a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3 \\ c_1, c_2, c_3, \gamma_1, \gamma_2, \gamma_3 \end{pmatrix}$$

the first columns of this,—and the first columns only,—would be affected, the a's and α's becoming b's and β's respectively in the one case, and c's and γ's in the other; and as neither interchange could affect the left-hand side of our identity, we should consequently note that thus three different expressions would be at once obtained for $|a_1b_2c_3|\cdot|\alpha_1\beta_2\gamma_3|$. Adding these together, and combining the nine determinants of the sum in sets of three by means of the addition-theorem (XLVI.), we should have finally

$$3|a_1b_2c_3|\cdot|\alpha_1\beta_2\gamma_3| = 3\begin{vmatrix} \frac{a_1,\,b_1,\,c_1}{\alpha_1,\beta_1,\gamma_1} & \frac{a_1,\,b_1,\,c_1}{\alpha_2,\beta_2,\gamma_2} & \frac{a_1,\,b_1,\,c_1}{\alpha_3,\beta_3,\gamma_3} \\ \frac{a_2,\,b_2,\,c_2}{\alpha_1,\beta_1,\gamma_1} & \frac{a_2,\,b_2,\,c_2}{\alpha_2,\beta_2,\gamma_2} & \frac{a_2,\,b_2,\,c_2}{\alpha_3,\beta_3,\gamma_3} \\ \frac{a_3,\,b_3,\,c_3}{\alpha_1,\beta_1,\gamma_1} & \frac{a_3,\,b_3,\,c_3}{\alpha_2,\beta_2,\gamma_2} & \frac{a_3,\,b_3,\,c_3}{\alpha_3,\beta_3,\gamma_3} \end{vmatrix},$$

from which it is only necessary to delete the common factor 3.

JACOBI (1831–33).

[De transformatione integralis duplicis indefiniti

$$\int\frac{\partial\phi\partial\psi}{\mathrm{A}+\mathrm{B}\cos\phi+\mathrm{C}\sin\phi+(\mathrm{A}'+\mathrm{B}'\cos\phi+\mathrm{C}'\sin\phi)\cos\psi+(\mathrm{A}''+\mathrm{B}''\cos\phi+\mathrm{C}''\sin\phi)\sin}$$

in formam simpliciorem $\int\frac{\partial\eta\partial\theta}{\mathrm{G}-\mathrm{G}'\cos\eta\cos\theta-\mathrm{G}''\sin\eta\sin\theta}$.

Crelle's Journal, viii. pp. 253–279, 321–357; or *Werke*, iii. pp. 91–158.]

[De transformatione et determinatione integralium duplicium commentatio tertia. *Crelle's Journal*, x. pp. 101–128; or *Werke*, iii. pp. 159–189.]

[De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integralium multiplicium. *Crelle's Journal*, xii. pp. 1–69; or *Werke*, iii. pp. 191–268.]

The first two of these memoirs may be viewed as continuations of a memoir with a similar title, which appeared in the second volume of *Crelle's Journal*, and to which we have already referred. They are noted here merely in order that the thread of investigation may be preserved unbroken, for the last memoir practically swallows up, by means of its splendid generalisations, all those that had gone before.

So long as we confine ourselves, in problems of transformation, to three independent variables, the explicit employment of the theory of determinants may be dispensed with. When, however, a sufficient number of special cases have been investigated, and an alluring glimpse has thereby been got of a generalisation involving them all, he who attempts the establishment of the generalisation must have recourse to the new weapon. In this latter position Jacobi now found himself. He wished to pass from the problem of orthogonal substitution in the case of three variables to the analogous problem in which the number of variables is n, or in his own words (p. 7):—

"Investigare substitutiones lineares huiusmodi

$$\begin{aligned} y_1 &= \alpha_1{}' x_1 + \alpha_2{}' x_2 + \dots + \alpha_n{}' x_n, \\ y_2 &= \alpha_1{}'' x_1 + \alpha_2{}'' x_2 + \dots + \alpha_n{}'' x_n, \\ &\dots\dots\dots\dots\dots\dots \\ y_n &= \alpha_1{}^{(n)} x_1 + \alpha_2{}^{(n)} x_2 + \dots + \alpha_n{}^{(n)} x_n, \end{aligned}$$

quibus efficiatur

$$y_1y_1 + y_2y_2 + \dots + y_ny_n = x_1x_1 + x_2x_2 + \dots + x_nx_n,$$

simulque data functio homogenea secundi ordinis variabilium $x_1, x_2, \dots, x_n$ transformetur in aliam variabilium $y_1, y_2, \dots, y_n$, de qua binarum producta evanuerunt."

This being the case he introduces determinants at the outset, fixing upon a notation which is practically Cauchy's, and immediately using properties of them without proof. Much that is contained in the memoir falls to be considered later, as it concerns special forms of determinants,—those afterwards known as Jacobians, axisymmetric determinants, and, of course, determinants of an orthogonal substitution. Indeed, the half-page of introduction is almost all that is of interest at present, but even in this a new and important theorem is enunciated. The first sentence of it stands as follows:—

"Supponamus, designantibus $\alpha_k^{(m)}$ datas quantitates quaslibet, ex n æquationibus linearibus propositis huiusmodi

$$y_m = \alpha_1^{(m)}x_1 + \alpha_2^{(m)}x_2 + \;.\;.\;.\;. + \alpha_n^{(m)}x_n,$$

per notas regulas resolutionis algebraicæ haberi æquationes formæ:

$$\mathrm{A}x_k = \beta' y_1 + \beta_k'' y_2 + \;.\;.\;.\;. + \beta_k^{(n)} y_n.$$

Ipsum A supponimus denominatorem communem valorum incognitarum, qui per algorithmos notos formatur: sive fit

$$\mathrm{A} = \Sigma \pm \alpha_1{}'\alpha_2{}'' \;.\;.\;.\;. \alpha_n^{(n)},$$

signo summatorio amplectente terminos omnes, qui indicibus aut inferioribus aut superioribus omnimodis permutatis proveniunt; signis eorum alternantibus secundum notam regulam, quam ita enunciare licet, ut termino cuilibet per certam permutationem *indicum* orto idem signum tribuatur, quo afficitur productum sequens conflatum e differentiis numerorum 1, 2, ..., n

$$(2-1)(3-1)\;.\;.\;.\;.\;(n-1).(3-2)(4-2)\;.\;.\;.\;.\;(n-2).(4-3) \text{ etc.},$$

eadem *numerorum* permutatione facta."

It will be at once observed here that Cauchy's italic letters S, a, b are simply changed into Greek Σ, α, β.

The next sentence is:—

"Eadem notatione adhibita, sit

$$\mathrm{B} = \Sigma \pm \beta_1{}'\beta_2{}'' \;.\;.\;.\;. \beta_n^{(n)},$$

ubi ipsam B e quantitatibus $\beta_k^{(m)}$ eodem modo compositam accipimus, quo A ex ipsis $\alpha_k^{(m)}$ componitur. Quibus statutis observo fieri:

$$\mathrm{B} = \mathrm{A}^{n-1},$$

ac generalius:

$$\Sigma \pm \beta_1{}'\beta_2{}'' \;.\;.\;.\;. \beta_m^{(m)} = \mathrm{A}^{m-1}\Sigma \pm \alpha_{m+1}^{(m+1)}\alpha_{m+2}^{(m+2)} \;.\;.\;.\;. \alpha_n^{(n)}.\text{"} \qquad \text{(XX.}$$

As for the first theorem thus formulated, the credit of it is, of course, due to Cauchy: the second, however, is new, being indeed the theorem referred to above under Minding as having been foreshadowed by Lagrange, and left for over fifty years undisturbed. Jacobi evidently knew it in all its generality, for he adds—

"De qua formula generali cum pro variis valoribus ipsius m, tum indicibus et superioribus et inferioribus omnimodis permutatis, permultae aliae similes formulae profluunt."

Jacobi's mode of proving the two theorems occupies § 6 (pp. 9–11). Temporarily denoting by X_m the left-hand member of the m^{th} given equation

$$a_1^{(m)}x_1 + a_2^{(m)}x_2 + \dots + a_n^{(m)}x_n = y_m,$$

and by Y_m the left-hand member of the m^{th} derived equation

$$\beta'_m y_1 + \beta''_m y_2 + \dots + \beta_m^{(n)} y_n = Ax_m:$$

and explaining that by

$$\left[\mathrm{U}\right]_{\frac{1}{x_1x_2\dots x_n}}$$

he means the coefficient of $x_1^{-1}x_2^{-1}\dots x_n^{-1}$ in a certain specified expansion of U, he recalls his paper of the year 1829 on the "discerptio singularis," and affirms that he had there proved

"fore

$$\left[\frac{1}{X_1X_2\dots X_n}\right]_{\frac{1}{x_1x_2\dots x_n}} = \frac{1}{A}$$

sive etiam, quod idem est,

$$\left[\frac{1}{Y_1Y_2\dots Y_n}\right]_{\frac{1}{y_1y_2\dots y_n}} = \frac{1}{B}$$

ac generalius

$$\left[\frac{x_1^{s_1}x_2^{s_2}\dots x_n^{s_n}}{X_1^{r_1+1}X_2^{r_2+1}\dots X_n^{r_n+1}}\right]_{\frac{1}{x_1x_2\dots x_n}} \tag{LII. 4}$$

$$= \frac{1}{A^{r_1+r_2+\dots+r_n+1}}\left[\frac{Y_1^{r_1}Y_2^{r_2}\dots Y_n^{r_n}}{y_1^{s_1+1}y_2^{s_2+1}\dots y_n^{s_n+1}}\right]_{\frac{1}{y_1y_2\dots y_n}}$$

designantibus $r_1, r_2, \dots, r_n$ ac $s_1, s_2, \dots, s_n$ numeros quoslibet integros sive positivos sive negativos."

A glance, however, suffices to convince one that the concluding general theorem here given differs considerably from the theorem which he had previously enunciated and possibly proved. As originally stated the theorem was—

$$\left[\frac{1}{u_0^{a_0+1}u_1^{a_1+1}\ldots u_{n-1}^{a_{n-1}+1}}\right]_{\dfrac{1}{x_0^{\beta_0+1}x_1^{\beta_1+1}\ldots x_{n-1}^{\beta_{n-1}+1}}} = \frac{1}{\Delta}\left[p_0^{\beta_0}p_1^{\beta_1}\ldots p_{n-1}^{\beta_{n-1}}\right]_{t_0^{a_0}t_1^{a_1}\ldots t_{n-1}^{a_{n-1}}},$$

which being altered into the notation of his present paper by the substitutions

$$\begin{aligned} x_0,\ x_1,\ \ldots &= x_1,\ x_2,\ \ldots. \\ u_0,\ u_1,\ \ldots &= X_1,\ X_2,\ \ldots. \\ p_0,\ p_1,\ \ldots &= \frac{Y_1}{A},\ \frac{Y_2}{A},\ \ldots. \\ a_0,\ a_1,\ \ldots &= r_1,\ r_2,\ \ldots. \\ \beta_0,\ \beta_1,\ \ldots &= s_1,\ s_2,\ \ldots. \\ \Delta &= A, \end{aligned}$$

becomes

$$\left[\frac{1}{X_1^{r_1+1}X_2^{r_2+1}\ldots X_n^{r_n+1}}\right]_{\dfrac{1}{x_1^{s_1+1}x_2^{s_2+1}\ldots x_n^{s_n+1}}} = \frac{1}{A^{s_1+s_2+\ldots+s_n+1}}\left[Y_1^{s_1}Y_2^{s_2}\ldots Y_n^{s_n}\right]_{y_1^{r_1}y_2^{r_2}\ldots y_n^{r_n}}.$$

Using on both sides of this the fact that if an expanded function be multiplied by the product of certain powers of the variables, any particular coefficient in the original expansion has now for facient its original facient multiplied by the said product, we obtain

$$\left[\frac{x_1^{s_1}x_2^{s_2}\ldots x_n^{s_n}}{X_1^{r_1+1}X_2^{r_2+1}\ldots X_n^{r_n+1}}\right]_{\dfrac{1}{x_1x_2\ldots x_n}} = \frac{1}{A^{s_1+s_2+\ldots+s_n+1}}\left[\frac{Y_1^{s_1}Y_2^{s_2}\ldots Y_n^{s_n}}{y_1^{r_1+1}y_2^{r_2+1}\ldots y_n^{r_n+1}}\right]_{\dfrac{1}{y_1y_2\ldots y_n}}$$

—a statement differing from Jacobi's in having r's and s's on the right-hand side where he has s's and r's respectively. The oversight was probably not noticed by reason of the fact that in the special instances considered by him the values of any r and the corresponding s are the same.

In the first of these instances he puts

$$r_1 = r_2 = \ldots = r_n = -1$$
$$s_1 = s_2 = \ldots = s_n = -1,$$

and obtains

$$1 = A^{n-1}\left[\frac{1}{Y_1 Y_2 \ldots Y_n}\right]_{\dfrac{1}{y_1 y_2 \cdots y_n}} = \frac{A^{n-1}}{B},$$

thus arriving at Cauchy's theorem regarding the adjugate, viz.,

$$B = A^{n-1}. \qquad \text{(XXI. 3)}$$

In the second instance, he puts

$$r_1 = r_2 = \ldots = r_m = -1, \quad r_{m+1} = r_{m+2} = \ldots = r_n = 0,$$
$$s_1 = s_2 = \ldots = s_m = -1, \quad s_{m+1} = s_{m+2} = \ldots = s_n = 0,$$

and obtains

$$\left[\frac{1}{X_{m+1} X_{m+2} \ldots X_n}\right]_{\dfrac{1}{x_{m+1} x_{m+2} \ldots x_n}} = A^{m-1}\left[\frac{1}{Y_1 Y_2 \ldots Y_m}\right]_{\dfrac{1}{y_1 y_2 \cdots y_m}}.$$

He then recalls the fact that by the conditions attaching to the expansion of the expressions enclosed in rectangular brackets the powers of $x_1, x_2, \ldots x_m$ contained in the one and the powers of $y_{m+1}, y_{m+2}, \ldots y_n$ contained in the other are all positive; and argues that as we are concerned only with terms that do not involve these variables, it is quite allowable to put them all equal to 0. This being done it is seen that

$$\left[\frac{1}{X_{m+1} X_{m+2} \ldots X_n}\right]_{\dfrac{1}{x_{m+1} x_{m+2} \ldots x_n}} = \frac{1}{\Sigma \pm a_{m+1}^{(m+1)} a_{m+2}^{(m+2)} \ldots a_n^{(n)}},$$

and

$$\left[\frac{1}{Y_1 Y_2 \ldots Y_m}\right]_{\dfrac{1}{y_1 y_2 \cdots y_m}} = \frac{1}{\Sigma \pm \beta_1' \beta_2'' \ldots \beta_m^{(m)}},$$

so that there is obtained

$$\Sigma \pm \beta_1' \beta_2'' \ldots \beta_m^{(m)} = \mathrm{A}^{m-1} . \Sigma \pm a_{m+1}^{(m+1)} a_{m+2}^{(m+2)} \ldots a_n^{(n)} ,$$

as was expected.

The only other point to be noted at present is contained in the casual remark that the β's may be expressed as *differential coefficients* of A. When dealing later (p. 20), with a special form of determinant, he says—

"Data occasione observo generaliter, si $\alpha_{\kappa,\lambda}$ et $\alpha_{\lambda,\kappa}$ inter se diversi sunt, propositis n aequationibus linearibus hujusmodi:

$$\begin{array}{l} \alpha_{1,1}u_1 + \alpha_{1,2}u_2 + \ldots . + \alpha_{1,n}u_n = v_1, \\ \alpha_{2,1}u_1 + \alpha_{2,2}u_2 + \ldots . + \alpha_{2,n}u_n = v_2, \\ \ldots \ldots \ldots \ldots \ldots \\ \alpha_{n,1}u_1 + \alpha_{n,2}u_2 + \ldots . + \alpha_{n,n}u_n = v_n, \end{array}$$

statuto

$$\Gamma = \Sigma \pm \alpha_{1,1}\alpha_{2,2} \ldots . \alpha_{n,n},$$

sequi vice versa

$$\begin{array}{l} \Gamma u_1 = \dfrac{\partial \Gamma}{\partial \alpha_{1,1}} v_1 + \dfrac{\partial \Gamma}{\partial \alpha_{2,1}} v_2 + \ldots . + \dfrac{\partial \Gamma}{\partial \alpha_{n,1}} v_n, \\ \Gamma u_2 = \dfrac{\partial \Gamma}{\partial \alpha_{1,2}} v_1 + \dfrac{\partial \Gamma}{\partial \alpha_{2,2}} v_2 + \ldots . + \dfrac{\partial \Gamma}{\partial \alpha_{n,2}} v_n, \\ \ldots \ldots \ldots \ldots \ldots \\ \Gamma u_n = \dfrac{\partial \Gamma}{\partial \alpha_{1,n}} v_1 + \dfrac{\partial \Gamma}{\partial \alpha_{2,n}} v_2 + \ldots . + \dfrac{\partial \Gamma}{\partial \alpha_{n,n}} v_n ." \end{array}$$ (XIII. 5) (VI. 8)

JACOBI (1834).

[Dato systemate n aequationum linearium inter n incognitas, valores incognitarum per integralia definita $(n-1)$tuplicia exhibentur. *Crelle's Journ.*, xiv. pp. 51–55; or *Werke*, vi. pp. 79–85.]

This short paper is, as it were, a by-product of the investigation which resulted in Jacobi's long memoir of the preceding year. Its only interest for us at present lies in the fact that values which are ordinarily expressed by means of determinants are here given in the form of definite multiple integrals. Indeed, instead of viewing the result obtained as being the solution of a set of simultaneous linear equations, it might be equally appropriate to consider the investigation as belonging to the subject of definite integration. It will suffice, therefore, merely to give a statement of the theorem arrived at. In Jacobi's own words, it is,—

"Sit propositum inter n incognitas $z_1, z_2, \ldots, z_n$ systema n aequationum linearium

$$\begin{aligned} b_{11}z_1 + b_{12}z_2 + \ldots\ldots + b_{1n}z_n &= m_1, \\ b_{21}z_1 + b_{22}z_2 + \ldots\ldots + b_{2n}z_n &= m_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{n1}z_1 + b_{n2}z_2 + \ldots\ldots + b_{nn}z_n &= m_n ; \end{aligned}$$

statuamus

$$\begin{aligned} X = \quad & [b_{11}x_1 + b_{21}x_2 + \ldots + b_{n1}x_n]^2 \\ & + [b_{12}x_1 + b_{22}x_2 + \ldots + b_{n2}x_n]^2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + [b_{1n}x_1 + b_{2n}x_2 + \ldots + b_{nn}x_n]^2, \end{aligned}$$

porro

$$M = m_1x_1 + m_2x_2 + \ldots + m_nx_n$$

ubi

$$x_n = \sqrt{(1 - x_1^2 - x_2^2 - \ldots - x_{n-1}^2)}$$

radicali positive accepto; porro ponamus

$$\nabla = \pm \sum \pm b_{11}b_{22} \ldots b_{nn},$$

signo ancipiti, ante ipsum Σ posito, ita determinato, ut valor ipsius ∇ positivus prodeat. Quibus omnibus positis, erit

$$\frac{n}{2^{n-1}S} \cdot \frac{z_1}{\nabla} = \int^{n-1} \frac{M(b_{11}x_1 + b_{21}x_2 + \ldots + b_{n1}x_n)\,\delta x_1 \delta x_2 \ldots \delta x_{n-1}}{x_n X^{\frac{1}{2}(n+2)}},$$

$$\frac{n}{2^{n-1}S} \cdot \frac{z_2}{\nabla} = \int^{n-1} \frac{M(b_{12}x_1 + b_{22}x_2 + \ldots + b_{n2}x_n)\,\delta x_1 \delta x_2 \ldots \delta x_{n-1}}{x_n X^{\frac{1}{2}(n+2)}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{n}{2^{n-1}S} \cdot \frac{z_n}{\nabla} = \int^{n-1} \frac{M(b_{1n}x_1 + b_{2n}x_2 + \ldots + b_{nn}x_n)\,\delta x_1 \delta x_2 \ldots \delta x_{n-1}}{x_n X^{\frac{1}{2}(n+2)}},$$

integralibus $(n-1)$tuplicibus extensis ad omnes valores reales ipsorum $x_1, x_2, \ldots, x_{n-1}$ et positivos et negativos, pro quibus etiam x_n realis sit sive pro quibus

$$x_1^2 + x_2^2 + \ldots + x_{n-1}^2 \leqq 1;$$

et designante S aut

$$\frac{1}{2 \,.\, 4 \ldots. (n-2)}\left(\frac{\pi}{2}\right)^{\frac{n}{2}} \quad \text{aut} \quad \frac{1}{1 \,.\, 3 \,.\, 5 \ldots (n-2)}\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}}$$

prout n aut par aut impar." (LIII.) (XIII. 6)

JACOBI (1835).

[De eliminatione variabilis e duabus aequationibus algebraicis. *Crelle's Journal*, xv. pp. 101–124; or *Nouv. Annales de Math.*, vii. pp. 158–171, 287–294; or *Werke*, iii. pp. 295–320.]

In a memoir having for its subject Bezout's method of eliminating x from the equations

$$\left.\begin{aligned} a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 &= 0,\\ b_n x^n + b_{n-1}x^{n-1} + \dots + b_1 x + b_0 &= 0,\end{aligned}\right\}$$

determinants are certain to occur explicitly or implicitly; and, the author being Jacobi, one is not surprised to find them introduced near the outset and employed thenceforward. It is of course only a special form of them which appears, viz., that afterwards distinguished by the term *persymmetric;* consequently, for the present the main contents of the memoir do not concern us. Note has to be made, however, of two points —(1) that while Jacobi does not discard his former notation $\Sigma \pm a_{r_0,s_0} a_{r_1,s_1} \dots a_{r_m,s_m}$, he introduces and uses another, viz.,

$$a\begin{Bmatrix} r_0, & r_1, & r_2, & \dots, & r_m \\ s_0, & s_1, & s_2, & \dots, & s_m \end{Bmatrix}; \qquad \text{(VII. 8)}$$

(2) that a page is devoted to a fuller statement of the above-mentioned theorems regarding the adjugate determinant and a minor of the adjugate. The final sentence of this statement is all that need be reproduced. It is

"Sint igitur $r, r', r'', \dots, r^{(n-1)}$ atque $s, s', s'', \dots, s^{(n-1)}$ numeri omnes $0, 1, 2, \dots, n-1$, quocunque ordine scripti; erit

$$\mathrm{A}\begin{Bmatrix} r^{(m)}, & r^{(m+1)}, & \dots, & r^{(n-1)} \\ s^{(m)}, & s^{(m+1)}, & \dots, & s^{(n-1)} \end{Bmatrix} = \mathrm{L}^{n-(1+m)}.\,a\begin{Bmatrix} r, & r', & \dots, & r^{(m-1)} \\ s, & s', & \dots, & s^{(m-1)} \end{Bmatrix}, \text{"} \qquad \text{(XX. 5)}$$

where L stands for $\Sigma \pm a_{0,0} a_{1,1} \dots a_{n-1,n-1}$ and the adjugate of L is $\Sigma \pm \mathrm{A}_{0,0} \mathrm{A}_{1,1} \dots \mathrm{A}_{n-1,n-1}$. No proofs of the theorems are given.

CHAPTER VIII.

DETERMINANTS IN GENERAL, FROM THE YEAR 1836 TO 1840.

THE writers of this period are nine in number, viz. Grunert, Lebesgue, Reiss, Catalan, Molins, Sylvester, Richelot, Cauchy, Craufurd. Of these the most prominent is Sylvester, who apparently in ignorance of all previous work discovers the functions for himself, gives a fresh investigation of some of their properties, and in a second paper makes an afterwards widely-known application of them to the theory of elimination; Richelot, Cauchy, Craufurd contribute papers dealing with the said application; Lebesgue explains the results of another application previously made by Jacobi and Cauchy; and Grunert Reiss, Catalan, Molins give elementary expositions of the general theory.

GRUNERT (1836).

[Supplemente zu Georg Simon Klügel's Wörterbuch der reinen Mathematik. Art. *Elimination* (I. Gleichungen des ersten Grades), ii. pp. 52–60.]

With Grunert it is necessary to take a long step backward. Although the memoirs of Bezout, Vandermonde, and Laplace were known to him, in addition to those of Hindenburg, Rothe, and Scherk, he advances only a short distance into the subject; his aim, indeed, is little more than the establishment of Cramer's rule for the solution of a set of simultaneous linear equations. His mode of presenting the subject, however, is fresh and interesting, the method of "undetermined multipliers" being taken to start with.

Writing his equations in the form

$$\left.\begin{array}{l}(1)_1x_1 + (2)_1x_2 + (3)_1x_3 + \ldots\ldots + (n)_1x_n = [1]_1\\ (1)_2x_1 + (2)_2x_2 + (3)_2x_3 + \ldots\ldots + (n)_2x_n = [1]_2\\ (1)_3x_1 + (2)_3x_2 + (3)_3x_3 + \ldots\ldots + (n)_3x_n = [1]_3\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (1)_nx_1 + (2)_nx_2 + (3)_nx_3 + \ldots\ldots + (n)_nx_n = [1]_n\end{array}\right\},$$

and taking $p_1, p_2, p_3, \ldots, p_n$ as multipliers, he readily shows of course that if the multipliers can be got to satisfy the conditions

$$\left.\begin{array}{l}(2)_1p_1 + (2)_2p_2 + (2)_3p_3 + \ldots\ldots + (2)_np_n = 0\\ (3)_1p_1 + (3)_2p_2 + (3)_3p_3 + \ldots\ldots + (3)_np_n = 0\\ (4)_1p_1 + (4)_2p_2 + (4)_3p_3 + \ldots\ldots + (4)_np_n = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (n)_1p_1 + (n)_2p_2 + (n)_3p_3 + \ldots\ldots + (n)_np_n = 0\end{array}\right\},$$

the value of x_1 will be

$$\frac{[1]_1p_1 + [1]_2p_2 + [1]_3p_3 + \ldots\ldots + [1]_np_n}{(1)_1p_1 + (1)_2p_2 + (1)_3p_3 + \ldots\ldots + (1)_np_n};$$

in other words, that x_1 can be determined at once if a function

$$(1)_1p_1 + (1)_2p_2 + (1)_3p_3 + \ldots\ldots + (1)_np_n$$

can be formed of such a character that it will vanish when instead of the coefficients $(1)_1, (1)_2, (1)_3, \ldots, (1)_n$ we substitute the members of any one of the $n-1$ rows

$$\begin{array}{lllll}(2)_1 & (2)_2 & (2)_3 & \ldots\ldots & (2)_n\\ (3)_1 & (3)_2 & (3)_3 & \ldots\ldots & (3)_n\\ (4)_1 & (4)_2 & (4)_3 & \ldots\ldots & (4)_n\\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots\\ (n)_1 & (n)_2 & (n)_3 & \ldots\ldots & (n)_n;\end{array}$$

the said function itself being the denominator of the value of x_1 and the numerator being derivable from the denominator by inserting $[1]_1, [1]_2, [1]_3, \ldots, [1]_n$ in place of $(1)_1, (1)_2, (1)_3, \ldots, (1)_n$. Further, as any one of the unknowns may be made the first, the complete solution is thus put in prospect. "Alles kommt demnach auf die Entwickelung einer Function von der angegebenen Beschaffenheit an." (XIII. 7)

Two rules, Grunert says, have been given for the construction of such a function, one by Cramer, the other by Bezout. The former he states, and illustrates by constructing the desired function for the case where $n=4$. The proof of it is then attempted, and is said at the outset to consist essentially in establishing the proposition that a permutation and any other derivable from it by the simple interchange of two indices must, according to Cramer's rule, differ in sign. This proposition is therefore attacked. The permutation

$$\ldots\ldots (k)_\alpha \ldots\ldots\ldots (1)_{\alpha+\beta} \ldots\ldots \qquad \mathrm{A}$$

is taken in which the inferior indices are in their natural order 1, 2, 3, . . . , n, and k and 1 being interchanged, there arises the permutation

$$\ldots\ldots (1)_\alpha \ldots\ldots\ldots (k)_{\alpha+\beta} \ldots\ldots \qquad \mathrm{B}$$

The part preceding $(k)_\alpha$ in A is called I., which thus of course also denotes the part preceding $(1)_\alpha$ in B; the part between $(k)_\alpha$ and $(1)_{\alpha+\beta}$ in A or between $(1)_\alpha$ and $(k)_{\alpha+\beta}$ in B is called II.; and the remaining part common to both A and B is called III. The number of inversions in both, when 1 and k are left out of account, is denoted by γ; the number in both due to k and the division III. is denoted by λ; the number in A due to k and the division II. by λ'; and the number in both due to the division I. and k by λ''. The counting of the inversions then takes place for the two permutations. In the case of A there are the inversions due

(1)	to I. and k,	which are	λ'' in number.
(2)	to I. and II.		
(3)	to I. and 1,	. . . . . .	$\alpha-1$ . . . . .
(4)	to I. and III.		
(5)	to k and II.,	. . . . . .	λ' . . . . .
(6)	to k and 1,	. . . . . .	1 . . . . .
(7)	to k and III.,	. . . . . .	λ . . . . .
(8)	to II. and 1,	. . . . . .	$\beta-1$ . . . . .
(9)	to II. and III.		
(10)	to 1 and III.,	. . . . . .	0 . . . . .

and as those not counted here are γ in number, the total is seen to be
$$\alpha + \beta + \gamma + \lambda + \lambda' + \lambda'' - 1.$$
Similarly in the case of B the total is found to be
$$\alpha + \beta + \gamma + \lambda - \lambda' + \lambda'' - 2.$$
But the former total exceeds the latter by $2\lambda' + 1$, and this being an odd number, the proposition is proved. (III. 27)

Before proceeding further it is important to note that Grunert here establishes a more definite theorem than he proposed to himself, viz., the theorem of Rothe (III. 7). If he attains greater simplicity it is in part due to the fact that instead of taking *any* two indices for interchange, k and r say, he takes k and 1.

To prove now that the function constructed in accordance with Cramer's rule will satisfy the requisite conditions, it suffices to show by means of this theorem that on making any one of the $n-1$ specified sets of substitutions the function will be transformed into one consisting of pairs of terms which annul each other; in other words, to prove Vandermonde's theorem regarding the effect of making two indices alike. This is done; and then it is shown how x_κ can be got by interchanging x_κ and x_1 in all the given equations, the first step being of course to establish the fact that the denominator of x_κ and the denominator of x_1 only differ in sign.

Bezout's rule of 1764 is next taken up, and shown to be identical in effect with Cramer's. The proof, by reason of the recurring character of the former, is inductive; that is to say, it is demonstrated that, if the two rules agree in the case of n unknowns, they must also agree in the case of $n+1$. Paraphrasing the proof, but taking for shortness' sake the case where $n=4$, we say that it is agreed that both rules give in this case the signed permutations
$$1234, \quad -1243, \quad +1423, \quad -4123, \quad -1324, \quad + \dots$$
Now for the case where $n=5$ Bezout's rule directs that to the end of each of these permutations, *e.g.*, the permutation -4123, a 5 is to be put, and asserts that the result -41235 will be one of the desired permutations with its proper sign. That it is a permutation of the first five integers is manifest, and since the number of inversions in 41235 is necessarily the same as the

number in 4123, its sign is correct according to Cramer's rule. In order to obtain four other permutations, Bezout's rule then proceeds to bid us shift the 5 one place and alter the sign, shift the 5 another place and alter the sign again, and so on. The result is

$$+41253, \quad -41523, \quad +45123, \quad -54123.$$

In regard to this, it is clear as before that permutations of the first five integers have been got, and that the altering of the sign simultaneously with the shifting of the 5 is in accordance with Cramer's rule, because every time that the 5 is moved one place to the left the number of inversions is increased by unity. The only question remaining is as to whether *all* the permutations are thus obtainable; and as it is seen that each of the 24 permutations of the first four integers gives rise to 5 permutations of the first five, we have at once grounds for a satisfactory answer. (III. 28)

LEBESGUE (1837).

[Thèses de Mécanique et d'Astronomie. Première Partie: Formules pour la transformation des fonctions homogènes du second degré à plusieurs inconnues. *Journal* (*de Liouville*) *de Math.*, ii. pp. 337*–355.]

This simply-worded and clear exposition is a natural outcome of a study of Jacobi's memoirs on the subject. Like these it mainly concerns determinants of the special form afterwards individualised by the term axisymmetric; and, indeed, it is notable as being the first memoir in which a special name is given to a special form, the expression "déterminants symétriques" being repeatedly used for the particular determinants referred to.

His general definition is (p. 343):—

"Si l'on considère le système d'équations

$$\left\{\begin{array}{l} A_{1,1}t_1 + A_{1,2}t_2 + \dots\dots + A_{1,n}t_n = m_1, \\ A_{2,1}t_1 + A_{2,2}t_2 + \dots\dots + A_{2,n}t_n = m_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_{n,1}t_1 + A_{n,2}t_2 + \dots\dots + A_{n,n}t_n = m_n, \end{array}\right.$$

* *N.B.*—There are *two* pages numbered 337.

le dénominateur commun des inconnues $t_1, t_2, \ldots, t_n$ est ce que l'on nomme le déterminant du système des nombres

$$(17) \qquad \left\{ \begin{array}{llll} A_{1,1} & A_{1,2} & \ldots\ldots & A_{1,n}, \\ A_{2,1} & A_{2,2} & \ldots\ldots & A_{2,n}, \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ A_{n,1} & A_{n,2} & \ldots\ldots & A_{n,n}. \end{array} \right.$$

Comme ce dénominateur peut changer de signe, selon le mode de solution qu'on emploiera, on conviendra de le prendre de sorte que le terme $A_{1,1}A_{2,2}A_{3,3} \ldots A_{n,n}$, qui en fait partie, soit positif." (VIII. 3)

No use, however, is made of this for the purpose of establishing the properties of the functions, results being for the most part taken from previous investigators and merely restated. A notation for what are nowadays called the minors of a determinant is given in the following words (p. 344):— (XL. 7)

"Ceci rappelé, si l'on représente par D le déterminant du système (17), par $[g, i]$ le déterminant du système qui se tire du système (17) par la suppression de la série horizontale de rang g et de la série verticale de rang i, et semblablement par la notation $\begin{bmatrix} g, & i \\ h, & k \end{bmatrix}$ le déterminant du système qui résulte de l'omission des séries horizontales de rangs g et i et des séries verticales de rangs i et k dans le système (17), on pourra, . . . "

Further, the determinants thus denoted are spoken of on page 346 as "déterminants partiels." (XL. 8)

REISS (1838).

[Essai analytique et géométrique. *Correspondance math. et phys.*, x. pp. 229–290.]

Reiss's memoir, the first part of which appeared in 1829, was never completed. In the course of some remarks introductory to the present essay, he says by way of excuse:—

"Je m'aperçus bientôt, et plusieurs savans me l'ont fait remarquer, que ces recherches, fussent-elles très-fécondes en résultats élégans, étaient trop abstraites pour intéresser le public qui n'apprécie les théories que selon le degré de leur utilité. J'ai donc tâché de montrer, par un exemple, de quelle manière on peut se servir de ces fonctions dans la géométrie analytique: et j'ai choisi le *tétraèdre* qui, par le concours de plusieurs circonstances qu'on aura occasion de reconnaître plus tard, permettait une application très-facile et presque immédiate des premières conséquences auxquelles j'étais parvenu."

The analytical portion of the essay is to a considerable extent identical with the original memoir. In so far as there is a difference, the change is towards greater simplicity, less seemingly aimless plunging into widely extensive theorems, and in general a better and more attractive style of exposition. Less space too is given to it,—not even half what is occupied by the portion on the tetrahedron, the main aim now being to urge on mathematicians the capabilities of the analysis in its application to geometry.

The matters falling to be noted as not having been given in the original memoir are few in number and of little importance. In restating the theorem

$$(abc\ \ldots\ r,\quad \overline{\alpha\beta\gamma\ \ldots\ \rho}) = \overline{(abc\ \ldots\ r,}\quad \alpha\beta\gamma\ \ldots\ \rho)$$

the remark is incidentally made that the order of the terms on the one side is never the same as that on the other except when the number of bases is 1, 2, or 3; for example, the number of bases being 4, we have

$$\begin{aligned}(abcd,\ \overline{1234}) = a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4& \\ + a_1b_3c_4d_2 + \ldots\ldots,&\end{aligned}$$

whereas

$$\begin{aligned}(abcd,\ \overline{1234}) = a_1b_2c_3d_4 - a_1b_2d_3c_4 - a_1c_2b_3d_4& \\ + a_1c_2d_3b_4 + \ldots\ldots,&\end{aligned}$$

the difference first appearing at the fourth term. (IX. 6)

Bezout's recurrent law of formation, formerly merely enunciated, is now accompanied by a demonstration. This is not without its weak point, the cause of which, as might be expected, is the awkwardness of Reiss's rule of signs. The first paragraph, which will suffice to show its character, is as follows (p. 233):—

"Portons notre attention d'abord, seulement sur la fonction $(abc\ \ldots\ r,\ \overline{\alpha\beta\gamma\ \ldots\ \rho})$. Si l'on se représente la manière dont on fait les permutations des n élémens $\alpha,\beta,\gamma,\ \ldots\ \rho$, on verra qu'à partir de la première, il y aura $1.2.3\ \ldots\ (n-1)$ complexions qui commencent par α, et que, si l'on sépare cet élément par un trait vertical des autres, on aura à droite toutes les permutations des élémens $\beta,\gamma, \ldots \rho$. Les $1.2.3. \ldots (n-1)$ premiers termes de $(abc \ldots r,\ \overline{\alpha\beta\gamma \ldots \rho})$ commencent donc tous par a^α, et puisque les signes de ces termes sont déterminés d'après la manière exposée plus haut, on trouvera leur somme $= a^\alpha(bc \ldots r,\ \overline{\beta\gamma \ldots \rho})$."

Vandermonde's theorem regarding the effect, on the function, of interchanging two bases is stated generally, and a demonstration is given. The mode of demonstration, which occupies one page and a half, will be readily understood by seeing it applied in later notation to the case where there are *four* bases, that is to say, where the theorem to be proved is

$$|a_\alpha b_\beta c_\gamma d_\delta| = -\,|b_\alpha a_\beta c_\gamma d_\delta|.$$

By repeated use of the recurrent law of formation we have

$$\begin{aligned}
|a_\alpha b_\beta c_\gamma d_\delta| &= a_\alpha|b_\beta c_\gamma d_\delta| - a_\beta|b_\alpha c_\gamma d_\delta| + a_\gamma|b_\alpha c_\beta d_\delta| - a_\delta|b_\alpha c_\beta d_\gamma|,\\
&= a_\alpha\Big\{ \qquad\qquad\quad b_\beta|c_\gamma d_\delta| - b_\gamma|c_\beta d_\delta| + b_\delta|c_\beta d_\gamma|\Big\}\\
&\quad - a_\beta\Big\{b_\alpha|c_\gamma d_\delta| \qquad\qquad\quad - b_\gamma|c_\alpha d_\delta| + b_\delta|c_\alpha d_\gamma|\Big\}\\
&\quad + a_\gamma\Big\{b_\alpha|c_\beta d_\delta| - b_\beta|c_\alpha d_\delta| \qquad\qquad\quad + b_\delta|c_\alpha d_\beta|\Big\}\\
&\quad - a_\delta\Big\{b_\alpha|c_\beta d_\gamma| - b_\beta|c_\alpha d_\gamma| + b_\gamma|c_\alpha d_\beta| \qquad\qquad\quad\Big\}.
\end{aligned}$$

By collecting the terms which have b_α for a common factor, b_β for a common factor, and so on, this result becomes

$$\begin{aligned}
|a_\alpha b_\beta c_\gamma d_\delta| &= -\,b_\alpha\Big\{a_\beta|c_\gamma d_\delta| - a_\gamma|c_\beta d_\delta| + a_\delta|c_\beta d_\gamma|\Big\}\\
&\quad + b_\beta\Big\{a_\alpha|c_\gamma d_\delta| - a_\gamma|c_\alpha d_\delta| + a_\delta|c_\alpha d_\gamma|\Big\}\\
&\quad - b_\gamma\Big\{a_\alpha|c_\beta d_\delta| - a_\beta|c_\alpha d_\delta| + a_\delta|c_\alpha d_\beta|\Big\}\\
&\quad + b_\delta\Big\{a_\alpha|c_\beta d_\gamma| - a_\beta|c_\alpha d_\gamma| + a_\gamma|c_\alpha d_\beta|\Big\},\\
&= -\,b_\alpha|a_\beta c_\gamma d_\delta| + b_\beta|a_\alpha c_\gamma d_\delta| - b_\gamma|a_\alpha c_\beta d_\delta| + b_\delta|a_\alpha c_\beta d_\gamma|,\\
&= -\,|b_\alpha a_\beta c_\gamma d_\delta|,
\end{aligned}$$

as was to be proved. (XI. 5)

The suggestion readily arises that this process would be equally applicable in proving Vandermonde's theorem regarding the vanishing of a function in which two bases are identical, and the process, it may be remembered, was actually so employed by Desnanot.

One of the theorems given by Scherk, and later by Drinkwater, appears in the following form (p. 240), the peculiar notation adopted for a determinant with a row of unit elements being constantly employed throughout the remainder of the essay :—

"Si une des bases, par exemple a, est telle que la quantité qu'elle représente soit la même quel que soit l'exposant dont elle est affectée, c'est-à-dire, si $a^\alpha = a^\beta = a^\gamma = \ldots$, on aura

$$(abc \ldots r, \quad \alpha\beta\gamma \ldots \rho)$$

$$= a^\alpha[(bc\ldots r,\ \beta\gamma\ldots\rho) - (bc\ldots r,\ \alpha\gamma\ldots\rho) + (bc\ldots r,\ \alpha\beta\delta\ldots\rho) \mp \ldots].$$

La quantité qui se trouve sous la parenthèse, peut donc être représentée de la manière suivante :

$$(\mathrm{I}bc \ldots r, \quad \alpha\beta\gamma \ldots \rho)\,; \qquad (\text{XLVII. } 3)$$

en admettant une fois pour toutes que le chiffre romain I soit tel que $1 = \mathrm{I}^\alpha = \mathrm{I}^\beta = \mathrm{I}^\gamma = \ldots$ Il va sans dire que toutes les propriétés qui ont lieu pour $(abc \ldots r, \alpha\beta\gamma \ldots \rho)$ se rapportent également à

$$(\mathrm{I}bc \ldots r, \quad \alpha\beta\gamma \ldots \rho).\text{"}$$

The character of the identities used in the treatment of the tetrahedron will be learned from a glance at the following examples :—

$$a_1(\mathrm{I}bc, 123) - b_1(\mathrm{I}ac, 123) + c_1(\mathrm{I}ab, 123) = (abc, 123).$$

$$(a_1-a_2)(\mathrm{I}bc, 123) - (b_1-b_2)(\mathrm{I}ac,123) + (c_1-c_2)(\mathrm{I}ab, 123) = 0.$$

$$(ab, 12)(ac, 34) - (ab, 34)(ac, 12) = -a_1(abc, 234) + a_2(abc, 134),$$

$$= + a_3(abc, 124) - a_4(abc, 123).$$

$$(\mathrm{I}ab, 123)(\mathrm{I}ac, 124) - (\mathrm{I}ab, 124)(\mathrm{I}ac, 123) = -(a_1-a_2)(\mathrm{I}abc, 1234).$$

$$(\mathrm{I}ab, 123)(abc, 124) - (\mathrm{I}ab, 124)(abc, 123) = +(ab, 12)(\mathrm{I}abc, 1234).$$

The first of these we have already seen used by Minding; the second is nothing more than the manifest identity,

$$\begin{vmatrix} . & 1 & 1 & 1 \\ a_1-a_2 & a_1 & a_2 & a_3 \\ b_1-b_2 & b_1 & b_2 & b_3 \\ c_1-c_2 & c_1 & c_2 & c_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_1 & a_2 & a_3 \\ b_1 & b_1 & b_2 & b_3 \\ c_1 & c_1 & c_2 & c_3 \end{vmatrix} = 0\,;$$

the third is evidently the equatement of two expansions of

$$\begin{vmatrix} a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_3 & \cdot & \cdot & a_4 \\ a_3 & a_1 & a_2 & a_4 \\ b_3 & b_1 & b_2 & b_4 \\ c_3 & c_1 & c_2 & c_4 \end{vmatrix};$$

the fourth is a case of the fifth: and the fifth is itself a case of a theorem (C') of Desnanot's.

CATALAN (1839).

[Sur la transformation des variables dans les intégrales multiples. *Mémoires couronnés par l'Académie royale . . . de Bruxelles*, xiv. 2me partie, 49 pp.]

The first of the four parts into which Catalan's memoir is divided bears the title "*Valeurs générales des inconnues dans les équations du premier degré, et propriétés des dénominateurs communs*," and in the introduction it is said to contain several remarkable new properties of the functions called *resultants* by Laplace "et connues aujourd'hui sous le nom de *déterminants*."

His method of dealing with the opening problem is to derive the solution of n equations with n unknowns from the solution of $n-1$ equations with $n-1$ unknowns; or more definitely, to show that if the multipliers $\lambda_1, \lambda_2, \lambda_3$ necessary for the solution of the set of equations,

$$\left.\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= \alpha_1 \\ a_2x_1 + b_2x_2 + c_2x_3 &= \alpha_2 \\ a_3x_1 + b_3x_2 + c_3x_3 &= \alpha_3 \end{aligned}\right\},$$

be the determinants of the systems

$$\begin{matrix} a_2 & b_2 \\ a_3 & b_3, \end{matrix} \qquad \begin{matrix} a_3 & b_3 \\ a_1 & b_1, \end{matrix} \qquad \begin{matrix} a_1 & b_1 \\ a_2 & b_2, \end{matrix}$$

then the multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ necessary for the solution of the set

$$\left.\begin{array}{l} a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 = \alpha_1 \\ a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4 = \alpha_2 \\ a_3x_1 + b_3x_2 + c_3x_3 + d_3x_4 = \alpha_3 \\ a_4x_1 + b_4x_2 + c_4x_3 + d_4x_4 = \alpha_4 \end{array}\right\}$$

are the determinants of the systems

$$\begin{array}{ccc} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{array} \qquad \begin{array}{ccc} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \end{array} \qquad \begin{array}{ccc} a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array} \qquad \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3. \end{array} \qquad \text{(XIII. 8)}$$

This of course means that in the first case

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = 0,$$
$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = 0,$$
$$\text{and} \qquad x_3 = \frac{\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3}{\lambda_1c_1 + \lambda_2c_2 + \lambda_3c_3};$$

and in the other

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 0,\text{*}$$
$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_4 = 0,$$
$$c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 + c_4\lambda_4 = 0,$$
$$\text{and} \qquad x_4 = \frac{\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3 + \lambda_4\alpha_4}{\lambda_1d_1 + \lambda_2d_2 + \lambda_3d_3 + \lambda_4d_4}.$$

The proof is disappointingly weak and unsatisfactory, and, what is still more surprising, rests at one point on a manifest inaccuracy. He says (p. 9)—

"Par un calcul direct, on vérifie la formule (6) et les relations (5) pour le cas de trois équations. En même temps, l'on reconnait que

"1° Le dénominateur de la valeur de x_3, par exemple, renferme toutes les combinaisons trois à trois des coefficients, chaque combinaison ne contenant ni deux fois la même lettre, ni deux fois le même indice.

"2° Deux termes qui, dans l'expression de ce dénominateur, peuvent se déduire l'un de l'autre par une permutation tournante ont même signe.

"3° Deux termes qui ne diffèrent que par le changement d'une lettre en une autre, et réciproquement, sont de signes contraires.

* Note, however, the error in sign of λ_2 and λ_4.

"4° Par suite, le dénominateur est le même pour toutes les inconnues, pourvu que l'on prenne convenablement le signe du numérateur."

He then proceeds—

"Supposons donc que pareille vérification ait été faite pour $n-1$ équations entre $n-1$ inconnues, je dis qu'elle se fera encore dans le cas de n équations."

Now although the statement in 2° is true for the case of three equations, it is not true generally, and therefore cannot be proved.*

The theorems which follow this introductory matter concern a special determinant, viz., the determinant of the system,

$$\begin{array}{cccccc} a_1 & b_1 & c_1 & \ldots\ldots & k_1 & l_1 \\ a_2 & b_2 & c_2 & \ldots\ldots & k_2 & l_2 \\ \cdot & \cdot & \cdot & \ldots\ldots & \cdot & \cdot \\ a_n & b_n & c_n & \ldots\ldots & k_n & l_n, \end{array}$$

in which the elements are connected by the $\frac{1}{2}n(n-1)$ relations

$$\left.\begin{array}{l} a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n = 0 \\ a_1c_1 + a_2c_2 + a_3c_3 + \ldots + a_nc_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1l_1 + a_2l_2 + a_3l_3 + \ldots + a_nl_n = 0 \\ b_1c_1 + b_2c_2 + b_3c_3 + \ldots + b_nc_n = 0 \\ b_1d_1 + b_2d_2 + b_3d_3 + \ldots + b_nd_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_1l_1 + b_2l_2 + b_3l_3 + \ldots + b_nl_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ k_1l_1 + k_2l_2 + k_3l_3 + \ldots + k_nl_n = 0 \end{array}\right\}.$$

Such determinants are only a little less special than determinants of an orthogonal substitution, and thus naturally fall to be considered later along with those of the latter class.

* In the proof he is fortunate (or unfortunate) enough to use another special case in which the statement is true. He says :—"Les deux termes $a_7b_6c_1d_3e_5f_2$ et $e_7f_6a_1b_3c_5d_2$ qui entrent dans D_4, et qui se déduisent l'un de l'autre par une permutation tournante entre les lettres ont même signe."

SYLVESTER (1839).

[On Derivation of Coexistence: Part I.* Being the Theory of simultaneous simple homogeneous Equations. *Philosophical Magazine*, xvi. pp. 37–43; or *Collected Math. Papers*, i. pp. 47–53.]

Sylvester was apparently first brought into contact with determinants while investigating the subject of the elimination of x between two equations of the m^{th} and n^{th} degrees. At the close of a paper on this subject (*Phil. Mag.*, xv. p. 435) he says—"I trust to be able to present the readers of this magazine with a *direct* and *symmetrical* method of eliminating any number of unknown quantities between any number of equations of any degree, by a newly invented process of symbolical multiplication, and the use of *compound* symbols of notation." These last words, indicative of the method, exactly describe the matter dealt with in the paper we have now come to, and as will soon be seen, the functions which are the outcome of the said "compound symbol" of operations are determinants.

It would also appear that Sylvester was unacquainted with any of the important memoirs of his predecessors regarding the functions: the twenty-seventh chapter of Garnier's *Analyse Algébrique*, to which he refers, may very probably indicate the extent of his knowledge.

Premising that he is going to use such symbols as $a_1, a_2, \ldots$ he calls the letter a the "base," and the complete symbol "an argument of the base," a_1 being the first argument, a_2 the second, and so on. Taking then a number of expressions, "each of which is made up of one or more terms, consisting solely of linear arguments of different bases, *i.e.*, characters bearing indices below but none above," *e.g.*, the expressions,

$$a_1 - b_1, \quad a_1 - c_1;$$

he alters them by writing the index-numbers *above, e.g.*,

$$a^1 - b^1, \quad a^1 - c^1;$$

takes the product of these resulting expressions in its expanded form

$$a^2 - a^1b^1 - a^1c^1 + b^1c^1;$$

* Misprint for II., as an expression in the paper itself shows.

and then reverses the operation on the index-numbers, thus finally obtaining

$$a_2 - a_1b_1 - a_1c_1 + b_1c_1.$$

The full series of these operations he indicates by the letter ζ, and denotes by the name of "*zeta-ic multiplication.*" Thus, as results in zeta-ic multiplication, we have

$$\zeta(a_1 - b_1)(a_1 - c_1) = a_2 - a_1b_1 - a_1c_1 + b_1c_1,$$

and

$$\zeta(a_1 + b_1)^2 = a_2 + 2a_1b_1 + b_2.\text{*}$$

Further ζ_{+r} is used to denote that, after the operations ζ have been performed, the indices are all to be increased by r, the result of so doing being called the zeta-ic product *in its* rth *phase.*

He nexts recalls a notation previously introduced by him for the functions which came later to be known shortly as difference-products; denoting, for example,

$$(b-a)(c-a)(c-b) \text{ by } \mathrm{PD}(abc),$$
$$(b-a)(c-a)(c-b)(d-a)(d-b)(d-c) \text{ by } \mathrm{PD}(abcd),$$

and

$$\therefore\ abc(b-a)(c-a)(c-b) \text{ by } \mathrm{PD}(0abc).$$

Lastly, he combines the two notations; and any reader who remembers Cauchy's mode of solving a set of simultaneous linear equations can with certainty predict the result of the union to be determinants. A new notation and a new name for the functions thus come into being together, the determinant of the system

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}$$

being represented by

$$\zeta abc\mathrm{PD}(abc) \quad \text{or} \quad \zeta\mathrm{PD}(0abc), \qquad (\text{VII. } 9)$$

and being called a *zeta-ic product of differences.* (XV. 7)

These special zeta-ic products being reached, the rest of the paper is taken up with an account of some of their properties, and the application of them to the discussion of simultaneous

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* He would not hesitate even to extend the use of the symbol, denoting, for example,

$$1 - \frac{a_2}{1.2} + \frac{a_4}{1.2.3.4} - \ldots \text{ by } \zeta\cos(a_1).$$

linear equations. Some of the matter may be passed over as being already familiar to us, although its earlier appearances were certainly made in a less picturesque dress. The first really fresh theorem concerns the zeta-ic multiplication of a determinant $\zeta\mathrm{PD}(0abc\ldots l)$, by those symmetric functions of $a, b, c, \ldots, l$, which we should now denote by

$$\Sigma a, \quad \Sigma ab, \quad \Sigma abc, \quad \ldots .$$

but which Sylvester writes in the form

$$\mathrm{S}_1(abc\ldots l), \quad \mathrm{S}_2(abc\ldots l), \quad \mathrm{S}_3(abc\ldots l), \quad \ldots .$$

In his own words it stands as follows (p. 39):—

"Let $a, b, c, \ldots l$ denote any number of independent bases, say $(n-1)$; but let the argument of each base be periodic, and the number of terms in each period the same for every base, namely (n), so that

$$\begin{array}{ll} a_r = a_{r+n} = a_{r-n} & a_n = a_0 = a_{-n} \\ b_r = b_{r+n} = b_{r-n} & b_n = b_0 = b_{-n} \\ c_r = c_{r+n} = c_{r-n} & c_n = c_0 = c_{-n} \\ \cdots & \cdots \\ l_r = l_{r+n} = l_{r-n} & l_r = l_0 = l_{-n}, \end{array}$$

r being any number whatever. Then

$$\begin{aligned} \zeta_{-1}\mathrm{PD}(0abc\ldots l) &= \zeta\big(\mathrm{S}_1(abc\ldots l)\,.\,\zeta\mathrm{PD}(0abc\ldots l)\big) \\ \zeta_{-2}\mathrm{PD}(0abc\ldots l) &= \zeta\big(\mathrm{S}_2(abc\ldots l)\,.\,\zeta\mathrm{PD}(0abc\ldots l)\big) \\ &\cdots\cdots \\ &\cdots\cdots \\ \zeta_{-r}\mathrm{PD}(0abc\ldots l) &= \zeta\big(\mathrm{S}_r(abc\ldots l)\,.\,\zeta\mathrm{PD}(0abc\ldots l)\big)." \end{aligned}$$

The limitation made upon the arguments of the base would seem to imply that the theorem only concerned determinants of a very special kind. Such, however, is not the case. A special example in more modern notation will bring out its true character. Let the determinant chosen be

$$|\,a_1b_2c_3d_4\,|,$$

and let the symmetric function be

$$ab + ac + ad + bc + bd + cd.$$

Multiplying the two together "zeta-ically," that is to say, in accordance with the law

$$a_r \times a_s = a_{r+s},$$

we find that 120 of the total 144 terms of the product cancel each other, and that the remaining 24 terms constitute the determinant

$$|a_1 b_2 c_4 d_5|,$$

the identity thus reached being

$$\zeta(|a_1 b_2 c_3 d_4| \cdot \Sigma ab) = |a_1 b_2 c_4 d_5|.$$

Now Sylvester's ζPD notation being unequal to the representation of the determinant $|a_1 b_2 c_4 d_5|$ in which the index-numbers do not proceed by the common difference 1, he would seem to have been compelled to give a periodic character to the arguments of the bases in order to remove the difficulty. At any rate the difficulty *is* removed; for the number of terms in the period being 5 the index-numbers 4 and 5 become changeable into -1 and 0, and thus we can have

$$\begin{aligned} |a_1 b_2 c_4 d_5| &= |a_1 b_2 c_{-1} d_0|, \\ &= |a_{-1} b_0 c_1 d_2|, \end{aligned}$$

—a determinant in which the index-numbers proceed by the common difference 1, and which is obtainable from $|a_1 b_2 c_3 d_4|$ by diminishing each index-number by 2. Sylvester's form of the result thus is

$$\zeta\{S_2(abcd) \cdot \zeta\mathrm{PD}(0abcd)\} = \zeta_{-2}\mathrm{PD}(0abcd).*$$

* It is rather curious that Sylvester overlooks the fact that the legitimate equatement of two zeta-ic products implies an identity altogether independent of the existence of zeta-ic multiplication. Thus, the identity just discussed is essentially the same as the identity

$$\begin{vmatrix} a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \\ d & d^2 & d^3 & d^4 \end{vmatrix} \times (ab + ac + ad + bc + bd + cd) = \begin{vmatrix} a & a^2 & a^4 & a^5 \\ b & b^2 & b^4 & b^5 \\ c & c^2 & c^4 & c^5 \\ d & d^2 & d^4 & d^5 \end{vmatrix},$$

where the index-number denotes a power and the multiplication is performed in accordance with the ordinary algebraic laws. From this point of view the above quoted proposition of Sylvester's involves an important theorem regarding the special determinants afterwards known by the name of *alternants*.

Following this comes the application to simultaneous linear equations, or as they are called "equations of coexistence." The system is represented by the typical equation

$$a_r x + b_r y + c_r z + \ . \ . \ . + l_r t = 0,$$

in which r can take up all integer values from $-\infty$ to $+\infty$, there being really, however, only n equations, because of the periodicity imposed on the arguments of the bases. One so-called "leading theorem" is given in regard to the system, its subject being the derivation of an equation linear in $x, y, z, \ldots, t$ by a combination of the equations of the system. The theorem is enunciated as follows (p. 40):—

"Take $f, g, \ . \ . \ . \ , k$ as the *arbitrary* bases of new and absolutely independent but periodic arguments, having the same index of periodicity (n) as $a, b, c, \ . \ . \ . \ , l$, and being in number $(n-1)$, *i.e.*, one fewer than there are units in that index.

"The number of *differing* arbitrary constants thus *manufactured* is $n(n-1)$.

"Let $Ax + By + Cz + \ . \ . \ . \ + Lt = 0$ be the general *prime* derivative from the given equations, then we may make

$$\begin{aligned} A &= \zeta \mathrm{PD}(0afg \ldots k) \\ B &= \zeta \mathrm{PD}(0bfg \ldots k) \\ C &= \zeta \mathrm{PD}(0cfg \ldots k) \\ &\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ L &= \zeta \mathrm{PD}(0lfg \ldots k)." \end{aligned} \qquad \text{(XIII. 9)}$$

As in the case of the other theorems, no demonstration is vouchsafed. In order, however, that the connection between it and previous work may be more readily manifest, it is desirable to indicate how it would most probably be established now. Taking the case where the number of unknowns is *three* and the number of given equations *four*, viz.—

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \\ a_4x + b_4y + c_4z &= 0 \end{aligned}\right\},$$

we should form an array of $4(4-1)$, *i.e.* 12, arbitrary quantities,

$$\begin{matrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \\ f_4 & g_4 & h_4 , \end{matrix}$$

from which we should select the multiplier $|f_2g_3h_4|$ for the first given equation, the multiplier $|f_1g_3h_4|$ for the second equation, and so on. The multiplication then being performed we should by addition obtain

$$|a_1f_2g_3h_4|x + |b_1f_2g_3h_4|y + |c_1f_2g_3h_4|z = 0,$$

which is what Sylvester would call "the general prime derivative of the four given equations," the process being an instance of what he would similarly term the "derivation of coexistence."

By proper choice of the arbitrary quantities it may be readily shown, as Sylvester proceeds to do, that the theorem gives (1) the result of the elimination of n unknowns from n equations; (2) the *two* equations of condition in the case of $n+1$ equations connecting n unknowns; (3) the ratio of any two unknowns in the case of $n-1$ equations connecting n unknowns; and (4) the relation between any three unknowns in the case of $n-2$ equations connecting n unknowns. For example, the equations being

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}\right\} ,$$

the theorem gives the general derivative

$$\begin{vmatrix} a_1 & f_1 & g_1 \\ a_2 & f_2 & g_2 \\ a_3 & f_3 & g_3 \end{vmatrix} x + \begin{vmatrix} b_1 & f_1 & g_1 \\ b_2 & f_2 & g_2 \\ b_3 & f_3 & g_3 \end{vmatrix} y + \begin{vmatrix} c_1 & f_1 & g_1 \\ c_2 & f_2 & g_2 \\ c_3 & f_3 & g_3 \end{vmatrix} z = 0 ,$$

which is true whatever $f_1, f_2, f_3, g_1, g_2, g_3$ may be. By putting $f_1, f_2, f_3, g_1, g_2, g_3 = b_1, b_2, b_3, c_1, c_2, c_3$, this takes the form

$$|a_1b_2c_3|x + |b_1b_2c_3|y + |c_1b_2c_3|z = 0 ,$$

whence the equation of condition, or resultant of elimination,

$$|a_1b_2c_3| = 0 .$$

As a corollary to one of the deductions from the leading theorem,—the deduction numbered (3) above,—the following proposition of a different character is given (p. 42):—

"If there be any number of bases $(abc \ldots l)$, and any other, two fewer in number, $(fg \ \ldots \ k)$,

$$\begin{aligned} &\zeta\mathrm{PD}(afg \ldots k) \times \zeta\mathrm{PD}(bc \ldots l) \\ &+ \zeta\mathrm{PD}(bfg \ldots k) \times \zeta\mathrm{PD}(ac \ldots l) \\ &+ \zeta\mathrm{PD}(afg \ldots k) \times \zeta\mathrm{PD}(bc \ldots l) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ \zeta\mathrm{PD}(lfg \ldots k) \times \zeta\mathrm{PD}(abc \ldots \ ) = 0, \end{aligned}$$

a formula that from its very nature suggests and proves a wide extension of itself." (XXIII. 11)

It belongs evidently to the class of vanishing aggregates of products of pairs of determinants, of which so many instances have presented themselves. There is a manifest misprint in the third product, which should surely be

$$\zeta\mathrm{PD}(cfg \ \ldots \ k) \times \zeta\mathrm{PD}(ab \ \ldots \ l);$$

and there is an error in the signs connecting the products, which, instead of being all $+$, should be $+$ and $-$ alternately. When the determinants involved are of the third order, the theorem in the later notation is

$$|a_1f_2g_3|\cdot|b_1c_2d_3| - |b_1f_2g_3|\cdot|a_1c_2d_3| + |c_1f_2g_3|\cdot|a_1b_2d_3| - |d_1f_2g_3|\cdot|a_1b_2c_3| = 0,$$

which is readily recognised as an identity given by Bezout.

With this theorem the paper proper ends, but in a postscript an additional theorem of a curious character is given. As enunciated by the author—even his double mark of exclamation being reprinted—it is (p. 43):—

"Let there be $(n-1)$ bases $a, b, c, \ldots, l$, and let the arguments of each be "recurrents of the n^{th} order," that is to say, let

$$a_i = \phi\left(\cos.\frac{2\pi i}{n}\right), \quad b_i = \psi\left(\cos.\frac{2\pi i}{n}\right), \quad c_i = \chi\left(\cos.\frac{2\pi i}{n}\right),$$

$$\ldots\ldots, \quad l_i = \omega\left(\cos.\frac{2\pi i}{n}\right).$$

Let R_t denote that any symmetrical function of the r^{th} degree is to be taken of the quantities in a parenthesis which come after it, and let ϑ indicate any function whatever. Then the zeta-ic product,

$$\zeta\big(\zeta R_t(abc \ldots l) \times \zeta_\rho \vartheta \mathrm{PD}(0abc \ldots l)\big)$$

is equal to the product of the *number*

$$R_t\Big(\Big(\cos.\frac{2\pi}{n}+\sqrt{-1}.\sin\frac{2\pi}{n}\Big),\ \Big(\cos.\frac{4\pi}{n}+\sqrt{-1}.\sin\frac{4\pi}{n}\Big),$$
$$\Big(\cos.\frac{6\pi}{n}+\sqrt{-1}.\sin\frac{6\pi}{n}\Big),\ \ldots\ldots$$
$$\cos.\Big(\frac{(2n-1)\pi}{n}+\sqrt{-1}.\sin\frac{2(n-1)\pi}{n}\Big)\Big)$$

multiplied by the zeta-ic phase

$$\zeta_{\rho-t}\vartheta \mathrm{PD}(0abc \ldots l)\,!!"$$

Unfortunately the meaning of the proposition is seriously obscured by misprints and inaccurate use of symbols. Instead of "r^{th}" degree we should have t^{th} degree; the ζ preceding $R_t(abc \ldots l)$ is meaningless, and should be deleted; ζ preceding $\vartheta\mathrm{PD}\,(0abc \ldots l)$ in the first member of the identity is unnecessary when a ζ has already been printed at the commencement; and the subscript ρ, although giving an appearance of greater generality, serves no purpose whatever. Making the corrections thus suggested, and denoting

$$\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n},\quad \cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n},\quad \ldots\quad ,$$

which are the roots of the equation

$$x^{n-1}+x^{n-2}+x^{n-3}+\ \ldots\ +x+1=0,$$

by $\alpha, \beta, \gamma, \ldots, \lambda$, we are enabled to put the theorem in the more elegant form

$$\zeta\{R_t(a,b,c\ \ \ldots,\ l).\vartheta.\mathrm{PD}(0,a,b,c,\ldots,l)\}$$
$$=\zeta_{-t}\{R_t(\alpha,\beta,\gamma,\ldots,\lambda).\vartheta.\mathrm{PD}(0,a,b,c,\ldots,l)\}.$$

It is readily seen to be a generalisation of the first theorem of the paper, into which it degenerates when ϑ, instead of being any function of $a,b,c, \ldots l$, is a constant, and R_t, instead of being *any* symmetric function, is one of the series Σa, Σab, Σabc, ... As, however, the constant $R_t(a,\beta,\gamma, \ldots \lambda)$ on the right-hand side will then be one of the series, Σa, $\Sigma a\beta$, $\Sigma a\beta\gamma$, ... and will not therefore be $+1$ unless when t is even, there must be an inattention to sign in one or other theorem. The matter can be more appropriately inquired into when we come to the subject of alternants, because, as has been pointed out in a recent footnote, it is to this branch of the subject that identities between two zeta-ic multiplications of difference-products really belong.

This early paper, one cannot but observe, has all the characteristics afterwards so familiar to readers of Sylvester's writings, —fervid imagination, vigorous originality, bold exuberance of diction, hasty if not contemptuous disregard of historical research, the outstripping of demonstration by enunciation, and an infective enthusiasm as to the vistas opened up by his work.

MOLINS (1839).

[Démonstration de la formule générale qui donne les valeurs des inconnues dans les équations du premier degré. *Journ. (de Liouville) de Math.*, iv. pp. 509–515.]

The real object of Molins was simply to give a rigorous demonstration of Cramer's rules. His literary progenitors, so far as determinants were concerned, were apparently Cramer, Bezout, Laplace, and Gergonne, the last of whom, it may be remembered, wrote a paper which might well have borne the same title as the above. The writer, however, whose work that of Molins most closely resembles is Scherk, and very probably the two were unknown to each other. Both had the same purpose in view, and both used the method of so-called "mathematical induction." The difference between them may be most easily explained by using a special example and modern notation.

To make the solution of the set of three equations

$$\left.\begin{aligned} a_1x + a_2y + a_3z &= a_4 \\ b_1x + b_2y + b_3z &= b_4 \\ c_1x + c_2y + c_3z &= c_4 \end{aligned}\right\}$$

dependent upon the already obtained solution of two, Scherk put the first pair of equations in the form

$$\left.\begin{aligned} a_1x + a_2y &= a_4 - a_3z \\ b_1x + b_2y &= b_4 - b_3z \end{aligned}\right\} ,$$

solved for x and y, and substituted the values in the third equation.

Molins, on the other hand, having used the multipliers m_1, m_2, 1, with the equations of the given set, performed addition, solved the pair of equations

$$\left.\begin{aligned} m_1a_2 + m_2b_2 + c_2 &= 0 \\ m_1a_3 + m_2b_3 + c_3 &= 0 \end{aligned}\right\}$$

for m_1 and m_2, and substituted the obtained values in the result

$$x_1 = \frac{m_1a_4 + m_2b_4 + c_4}{m_1a_1 + m_2b_1 + c_1}. \qquad \text{(XIII. 10)}$$

His exposition is laboured and uninviting.

SYLVESTER (1840).

[A method of determining by mere inspection the derivatives from two equations of any degree. *Philosophical Magazine*, xvi. pp. 132–135; or *Collected Math. Papers*, i. pp. 54–57.]

The two equations taken are

$$\left.\begin{aligned} a_nx^n + a_{n-1}x^{n-1} + \ \ldots\ + a_1x + a_0 &= 0 \\ b_mx^m + b_{m-1}x^{m-1} + \ \ldots\ + b_1x + b_0 &= 0 \end{aligned}\right\} ,$$

and rules are given for attaining three different objects, viz. (1) a rule for absolutely eliminating x; (2) a rule for finding the prime derivative of the first degree, that is to say of the form $\mathrm{A}x-\mathrm{B}=0$; (3) a rule for finding the prime derivative of any degree. The first of these concerns the process afterwards so well known by the name "dialytic." Only part of it need be given (p. 132):—

"Form out of the a progression of coefficients m lines, and in like manner out of the b progression of coefficients form n lines in the following manner: Attach $m-1$ zeros all to the *right* of the terms in the a progression; next attach $m-2$ zeros to the right and carry 1 over to the left; next attach $m-3$ zeros to the right and carry 2 over to the left. Proceed in like manner until all the $m-1$ zeros are carried over to the left, and none remain on the right. The m lines thus formed are to be written under one another.

Proceed in like manner to form n lines out of the b progression by scattering $n-1$ zeros between the right and left.

If we write these n lines under the m lines last obtained, we shall have a solid square $m+n$ terms deep and $m+n$ terms broad." (LIV.)

The rest of the rule deals of course with the formation of the terms from this square of elements, the old and familiar method being followed of taking all possible permutations and separating the permutations into positive and negative. As applied by Sylvester in the case of the elimination of x between the equations

$$\left.\begin{aligned} ax^2 + bx \ + c &= 0 \\ lx^2 + mx + n &= 0 \end{aligned}\right\},$$

that is to say, as applied to the development of the determinant of the system

$$\begin{matrix} a & b & c & 0 \\ 0 & a & b & c \\ l & m & n & 0 \\ 0 & l & m & n, \end{matrix}$$

the method is lengthy.

No hint at an explanation of this or either of the two other rules is given. The principle at the basis of them all, however, is essentially that of the preceding paper. A single example will make this plain, and will at the same time serve to give a better idea of the two remaining rules than could be got by mere quotation.* Let the two given equations be

$$\left.\begin{aligned} ax^3 + bx^2 + cx + d &= 0 \\ \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon &= 0 \end{aligned}\right\},$$

and suppose that it is desired to obtain their "prime derivative" of the 2nd (r^{th}) degree, that is to say, the derivative of the form

$$\mathrm{A}x^2 + \mathrm{B}x + \mathrm{C} = 0.$$

* The third rule is incorrectly stated.

Taking the first equation followed by $m-r-1$ equations derived from it by repeated multiplication by x, and then the second equation followed by $n-r-1$ equations derived from it in like manner, we have $m+n-2r$ equations,

$$\left.\begin{array}{r} ax^3 + bx^2 + cx + d = 0 \\ ax^4 + bx^3 + cx^2 + dx = 0 \\ \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon = 0 \end{array}\right\} ,$$

from which we have to deduce an equation involving no power of x higher than the 2nd. To do so we employ, as just stated, exactly the same method as was used in obtaining the "leading theorem" of the preceding paper. That is to say, we form multipliers

$$\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix}, \qquad -\begin{vmatrix} . & a \\ \alpha & \beta \end{vmatrix}, \qquad \begin{vmatrix} . & a \\ a & b \end{vmatrix},$$

effect the multiplications, and add, the result being

$$\begin{vmatrix} . & a & b \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix} x^2 + \begin{vmatrix} . & a & c \\ a & b & d \\ \alpha & \beta & \delta \end{vmatrix} x + \begin{vmatrix} . & a & d \\ a & b & . \\ \alpha & \beta & \epsilon \end{vmatrix} = 0. \qquad \text{(LIV. 2)}$$

This is what Sylvester's third rule would give. His second rule is simply a case of the third, viz., where $r=1$; and his first rule is another case, viz., where $r=0$. Had he followed the order of his former paper, he would have called the third rule his "leading theorem," and given the others as corollaries from it.

RICHELOT (May 1840).

[Nota ad theoriam eliminationis pertinens. *Crelle's Journal*, xxi. pp. 226–234; or *Nouv. Annales de Math.*, ix. pp. 228–232.]

Just as Jacobi (1835) brought determinants to bear on Bezout's abridged method of eliminating x from two equations of the n^{th} degree, so did his fellow-professor Richelot, in treating of the other method of elimination, Euler's and Bezout's, discovered in the same year (1764). Euler's method, it will be remembered, consists in transforming the problem into the simpler one of eliminating a set of unknowns from a sufficient number of *linear* equations; and Richelot in a few lines (p. 227) points out that this may, of course, be done by equating to zero the determinant of the system of equations. An investigation connected therewith occupies the main portion of the paper.

Sylvester's method (1840) is described in passing, and the principle at the basis of it given. We have just seen that, when originally made known by the author, it was merely in the form of a rule without any explanation. Although no doubt exists as to the mode in which it was obtained, still this first published description of the mode by Richelot deserves to be put on record. The whole passage in regard to it is as follows (p. 226):—

"Quam æquationem * inveniendi methodi diversæ a geometris adhibentur, ex quarum numero eius, quæ a clarissimo *Sylvester* in diario *The London and Edinburgh Philosophical Magazine and Journal of Science* nuper exposita est, mentionem faciendi hanc occasionem haud prætermittere velim. Ibi illius eliminationis problema reducitur ad problema eliminationis $m+n-1$ quantitatum ex systemate $m+n$ æquationum linearium. Multiplicata enim æquatione $f_1=0$ ex ordine per y^{n-1}, y^{n-2}, ... , y^0, nec non æquatione $f_2=0$ ex ordine per y^{m-1}, y^{m-2}, ... , y^0, adipiscimur systema $m+n$ æquationum linearium inter quantitates y^{m+n-1}, y^{m+n-2}, . . . , y^0, quarum $m+n-1$ prioribus eliminatis, æquatio inter coefficientes † a' et a'' prodit. Quæ eliminatio facillime ita instituitur, ut determinantem harum $m+n$ æquationum linearium ponamus $=0$. Determinans vero, cum quantitates a' et a'' in æquationibus ipsæ tantum lineariter involvantur, et quantitates a' in n, nec non quantitates a'' in m ceteris æquationibus solis reperiantur, respectu illarum dimensiones $n$tæ est, respectuque harum $m$tæ. Unde concluditur, eam positam $=0$, esse quæsitam illam æquationem finalem $X=0$, quæ omni factore superflua careat. Notissima enim est proprietas ab *Eulero* inventa æquationis $X=0$, quod eius dimensio respectu quantitatum a' est $=n$, atque respectu quantitatum $a''=m$, ita ut quæque functio integra evanescens, inter quantitates a' et a'', has dimensiones quadrans, pro genuina æquatione finali habenda sit." (LIV. 3)

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* *I.e.*, æquationem finalem.

† The equations are taken in the form

$$f_1 = a'_m y^m + a'_{m-1} y^{m-1} + \; . \; . \; . \; . \; + a'_0 = 0,$$
$$f_2 = a''_n y^n + a''_{n-1} y^{n-1} + \; . \; . \; . \; . \; + a''_0 = 0.$$

Taking Sylvester's example,

$$\left.\begin{array}{l} ax^2 + bx + c = 0 \\ \alpha x^2 + \beta x + \gamma = 0 \end{array}\right\},$$

and doing as Richelot here directs, we should first multiply both members of the first equation by x^{2-1} and by x^{1-1}, then both members of the second by x^{2-1} and by x^{1-1}, thus obtaining

$$\begin{array}{rl} ax^3 + bx^2 + \ cx \qquad & = 0, \\ ax^2 + \ bx + c & = 0, \\ \alpha x^3 + \beta x^2 + \gamma x \qquad & = 0, \\ \alpha x^2 + \beta x + \gamma & = 0, \end{array}$$

and finally eliminate from these four equations x^3, x^2, x^1, by equating to zero the determinant of the system.

The statement "Ibi illius . . . . . linearium," which seems to contradict what we have above said in regard to the absence of explanation in Sylvester's paper, is not literally true. Richelot may have meant by it that Sylvester's result *implied* that the problem had been transformed as stated.

CAUCHY (1840).

[Mémoire sur l'élimination d'une variable entre deux équations algébriques. *Exercices d'analyse et de phys. math.*, i. pp. 385–422; or *Œuvres complètes*, 2e Sér. xi.]

After the appearance of the special papers on this subject by Jacobi, Sylvester, and Richelot, a review of the whole matter could not but be a desideratum. This was supplied by Cauchy in the singularly clear and able memoir which we have now reached. After an introduction of four pages there is an account (1) of Newton's method as expounded by Euler in 1748; (2) of Euler's and Bezout's method of 1764; (3) of Bezout's abridged method; and (4) of a method* by means of the differences of the roots of the equations.

* Euler's, although not called so.

Euler and Bezout's method is shown to lead to the same determinant as Sylvester's, and the cause is made apparent. Cauchy's says (p. 389):—

"Supposons, pour fixer les idées, que les fonctions $f(x)$, $F(x)$ soient l'une du troisième degré, l'autre du second, en sorte qu'on ait

$$f(x) = ax^3 + bx^2 + cx + d,$$
$$F(x) = Ax^2 + Bx + C.$$

Alors u, v devront être de la forme

$$u = Px + Q,$$
$$v = px^2 + qx + r\,;$$

et, si l'on élimine x entre les deux équations

$$f(x) = 0, \quad F(x) = 0,$$

l'équation résultante sera précisément celle qu'on obtiendra, lorsqu'on choisera les coefficients

$$p,\ q,\ r,\ P,\ Q$$

de manière à faire disparaître x de la formule

$$uf(x) + vF(x) = 0, \tag{2}$$

par conséquent de la formule

$$(Px+Q)f(x) + (px^2+qx+r)F(x) = 0,$$

que l'on peut encore écrire comme il suit :

$$Pxf(x) + Qf(x) + px^2F(x) + qxF(x) + rF(x) = 0. \tag{3}$$

Les valeurs de

$$p,\ q,\ r,\ P,\ Q$$

qui remplissent cette condition sont celles qui vérifient les équations linéaires,

$$\left\{\begin{array}{lllllll} aP & & + Ap & & & = 0, \\ bP & + aQ & + Bp & + Aq & & = 0, \\ cP & + bQ & + Cp & + Bq & + Ar & = 0, \\ dP & + cQ & & + Cq & + Br & = 0, \\ & + dQ & & & + Cr & = 0. \end{array}\right. \tag{4}$$

Donc, pour obtenir la résultante cherchée, il suffira d'éliminer les coefficients

$$P,\ Q,\ p,\ q,\ r$$

entre les équations (4), ou, ce qui revient au même, d'égaler à zéro la fonction alternée formée avec les quantités que présente le tableau

$$(5)\qquad \left\{\begin{array}{lllll} a, & 0, & A, & 0, & 0, \\ b, & a, & B, & A, & 0, \\ c, & b, & C, & B, & A, \\ d, & c, & 0, & C, & B, \\ 0, & d, & 0, & 0, & C. \end{array}\right.$$

On arriverait encore aux mêmes conclusions en partant de la formule (3). En effet, choisir les coefficients P, Q, p, q, r, de manière à faire disparaître de cette formule les diverses puissances

$$x, \quad x^2, \quad x^3, \quad \ldots, \quad x^{m+n-1},$$

de la variable x, c'est éliminer ces puissances des cinq équations,

$$(6)\quad xf(x) = 0, \quad f(x) = 0, \quad x^2F(x) = 0, \quad xF(x) = 0, \quad F(x) = 0,$$

ou

$$(7)\qquad \left\{\begin{array}{rl} ax^4 + bx^3 + cx^2 + dx \phantom{{}+d} & = 0, \\ ax^3 + bx^2 + cx + d & = 0, \\ Ax^4 + Bx^3 + Cx^2 \phantom{{}+d+dx} & = 0, \\ Ax^3 + Bx^2 + Cx \phantom{{}+d} & = 0, \\ Ax^2 + Bx + C & = 0. \end{array}\right.$$

C'est donc égaler à zéro la fonction alternée formée avec les quantités que présente le tableau,

$$(8)\qquad \left\{\begin{array}{lllll} a, & b, & c, & d, & 0, \\ 0, & a, & b, & c, & d, \\ A, & B, & C, & 0, & 0, \\ 0, & A, & B, & C, & 0, \\ 0, & 0, & A, & B, & C. \end{array}\right.$$

Or cette fonction alternée ne différera pas de celle que nous avons déjà mentionnée, attendu que, pour passer du tableau (5) au tableau (8), il suffit de remplacer les lignes horizontales par les lignes verticales, et réciproquement." (LIV. 4)

Bezout's abridged method for the equations

$$\left.\begin{array}{l} a_0x^n + a_1x^{n-1} + \ldots . + a_{n-1}x + a_n = 0 \\ b_0x^n + b_1x^{n-1} + \ldots . + b_{n-1}x + b_n = 0 \end{array}\right\}$$

is shown to lead to the final equation

$$S = 0,$$

where S is "une fonction alternée de l'ordre n formée avec les quantités que renferme le tableau,

$$\begin{cases} A_{0,0} & A_{0,1} & \cdots & A_{0,n-2} & A_{0,n-1} \\ A_{0,1} & A_{1,1} & \cdots & A_{1,n-2} & A_{1,n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ A_{0,n-2} & A_{1,n-2} & \cdots & A_{n-2,n-2} & A_{n-2,n-1} \\ A_{0,n-1} & A_{1,n-1} & \cdots & A_{n-2,n-1} & A_{n-1,n-1} \end{cases}"$$

in which

$$A_{0,l} = a_0 b_{l+1} - b_0 a_{l+1},$$
$$A_{1,l} = a_1 b_{l+1} - b_1 a_{l+1} + A_{0,l+1},$$
$$A_{2,l} = a_2 b_{l+1} - b_2 a_{l+1} + A_{1,l+1},$$
$$\cdots\cdots\cdots$$

In connection with this, however, no reference is made to Jacobi's paper of 1835.

The fourth method, which occupies much the largest space (pp. 397–442), is not a determinant method.

SYLVESTER (January 1841).

[Examples of the dialytic method of elimination as applied to ternary systems of equations. *Cambridge Math. Journ.*, ii. pp. 232–236; or *Collected Math. Papers*, i. pp. 61–65.]

In returning to extend the method, here and generally afterwards called "dialytic," Sylvester takes occasion to say that "the principle of the rule will be found correctly stated by Professor Richelot of Königsberg in a late number of *Crelle's Journal*." It may be noted, too, that he now for the first time uses the word *determinant*.

Only the first and last of the four examples need be given, as the subject strictly belongs to the application rather than the theory of determinants. Even these, however, will suffice to show the masterly grip which Sylvester had of his own method.

"To eliminate x, y, z between the three homogeneous equations

$$Ay^2 - 2C'xy + Bx^2 = 0 \qquad (1),$$

$$Bz^2 - 2A'yz + Cy^2 = 0 \qquad (2),$$

$$Cx^2 - 2B'zx + Az^2 = 0 \qquad (3).$$

Multiply the equations in order by $-z^2$, x^2, y^2, add together, and divide out by $2xy$; we obtain

$$C'z^2 + Cxy - A'xz - B'yz = 0 \qquad (4).$$

By similar processes we obtain

$$A'x^2 + Ayz - B'yx - C'zx = 0 \qquad (5),$$

$$B'y^2 + Bzx - C'zy - A'xy = 0 \qquad (6).$$

Between these six, treated as simple equations, the six functions of x, y, z, viz, x^2, y^2, z^2, xy, xz, yz, treated as *independent* of each other, may be eliminated; the result may be seen, by mere inspection, to come out

$$ABC(ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C') = 0,$$

or rejecting the special (N.B. not *irrelevant*) factor ABC we obtain

$$ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C' = 0." \qquad \text{(LIV. 5)}$$

The example, however satisfactory as illustrating the dialytic method, cannot be passed over without a note in regard to the unaccountable blunder made in developing the determinant involved. In later notation the determinant is

$$\begin{vmatrix} . & C & B & -2A' & . & . \\ C & . & A & . & -2B' & . \\ B & A & . & . & . & -2C' \\ A' & . & . & A & -C' & -B' \\ . & B' & . & -C' & B & -A' \\ . & . & C' & -B' & -A' & C \end{vmatrix}.$$

Now neither of the factors given by Sylvester are really factors of this, the truth being that it

$$= 2(ABC + 2A'B'C' - BB'^2 - CC'^2 - AA'^2)^2.$$

The fourth example concerns the elimination of x, y, z between the three equations

$$\left.\begin{aligned} Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy &= 0 \\ Lx^2 + My^2 + Nz^2 + 2L'yz + 2M'zx + 2N'xy &= 0 \\ Px^2 + Qy^2 + Rz^2 + 2P'yz + 2Q'zx + 2R'xy &= 0 \end{aligned}\right\}.$$

Using each of the three multipliers x, y, z with each of the three equations, we obtain nine equations linear in the ten quantities,

$$x^3,\ y^3,\ z^3,\ x^2y,\ x^2z,\ y^2x,\ y^2z,\ z^2x,\ z^2y,\ xyz.$$

Another such equation is thus necessary for success. Sylvester obtains it very ingeniously by writing the given equations in the form

$$\left.\begin{aligned} (Ax+B'z+C'y)x + (By+C'x+A'z)y + (Cz+A'y+B'x)z &= 0 \\ (Lx+M'z+N'y)x + (My+N'x+L'z)y + (Nz+L'y+M'x)z &= 0 \\ (Px+Q'z+R'y)x + (Qy+R'x+P'z)y + (Rz+P'y+Q'x)z &= 0 \end{aligned}\right\},$$

and then eliminating x, y, z. The work is not continued further.

We may ourselves note, in conclusion, that the fourth example includes in a sense the three others, but that it does not follow therefrom that by giving the requisite special values to the coefficients in the result of the general example, we should obtain the results for the particular examples in the forms already reached. Indeed, it is on account of this apparent non-agreement that the dialytic method is valuable to the theory of determinants, some very remarkable identities being arrived at by its aid. An explanation is also thus afforded of the trouble we have taken to elucidate its history.

CRAUFURD, A. Q. G.* (February 1841).

[On a method of algebraic elimination. *Cambridge Math. Journal,* ii. pp. 276–278.]

In Craufurd we have an independent discoverer of the dialytic method. A full account of his paper is quite unnecessary: the

* Only the initials A. Q. G. C. are appended to the article. There can be little doubt, however, that they belong to Craufurd, whose name in full appears elsewhere in the *Journal.*

few lines dealing with his introductory example will suffice to establish the fact. He says:—

"Let it be required to eliminate x from the equations

$$x^2 + px + q = 0,$$
$$x^2 + p'x + q' = 0.$$

Multiply each of the proposed equations by x, and you obtain

$$x^3 + px^2 + qx = 0,$$
$$x^3 + p'x^2 + q'x = 0.$$

These two combined with the two given equations make a system of four equations containing three quantities to be eliminated, viz., x, x^2, x^3; and they are of the first degree with respect to each of these quantities. We may, therefore, eliminate x, x^2, x^3 by the rules for equations of the first degree. The result is . . . . "

He enunciates a general rule, and then takes up the analogous subject in Differential Equations, where successive differentiation takes the place of successive multiplication by x. In a postscript he acknowledges Sylvester's priority which the editor had pointed out to him. He knew nothing of determinants.

CHAPTER IX.

DETERMINANTS IN GENERAL IN THE YEAR 1841.

Like the year 1812 the year 1841 merits a chapter to itself; and in 1841 as in 1812 it is the work of only two authors that concerns us. Strange to say, however, the two notable years had an author in common, the writers of 1812 being Binet and Cauchy, and those of 1841 being Cauchy and Jacobi.* In 1841 Jacobi's contributions constituted a comprehensive monograph similar to that produced by Cauchy in 1812, and Cauchy's in 1841, as was to be expected, were more of the nature of an aftermath.

CAUCHY (March 8, 1841).

[Note sur la formation des fonctions alternées qui servent à résoudre le problème de l'élimination. *Comptes Rendus* . . . *Paris*, xii. pp. 414–426; or *Œuvres complètes d'Augustin Cauchy*, 1re Sér., vi. pp. 87–90.]

Recalling the fact that the final equation, resulting from the elimination of several unknowns from a set of linear equations, has for its first member "une fonction alternée," and pointing out the further fact that the same holds good in regard to the elimination of one unknown from two equations of any degree, "puisque les méthodes de Bezout et d'Euler reduisent ce dernier problème au premier," Cauchy affirms the importance of being able easily to write out the full expansion of such functions. There can be little doubt, however, that it was the second fact alone,—in other words, the discoveries of Jacobi, Sylvester, and

* Cauchy was born fifteen years before Jacobi and lived six years after him.

Richelot,—which influenced the veteran Cauchy to return to a subject practically untouched by him for thirty years.

The opening part of the paper is, of course, necessarily old matter. One thing to be noted is that Cauchy tacitly discards the term *determinant*, which he was the means of introducing, using uniformly the more general expression *fonction alternée* instead. Another is that he adopts the rule of signs which makes use of the number of *interchanges*. From this his own peculiar rule of signs is deduced, and made the starting point for the fresh investigation which forms the main portion of the paper. The exposition of his rule, which differs from that of 1812, is worthy of a little attention, both on its own account and because otherwise the matter following would be scarcely intelligible. In the case of any term ("terme" or "produit") of the determinant

$$\Sigma \pm a_{0,0}a_{1,1}a_{2,2}a_{3,3}a_{4,4}a_{5,5}a_{6,6}\,,$$

say the term

$$a_{0,1}a_{1,0}a_{2,5}a_{3,3}a_{4,6}a_{5,4}a_{6,2}\,,$$

there is an underlying separation of the indices $0, 1, \ldots, 6$ into groups ("groupes"), by reason of the system of pairing; that is to say, since an index is found paired along with one index and not with another, there arises the possibility of looking upon those which happen to be paired with one another as belonging to the same family group. Thus, attending to the first a of the term, we see that 1 and 0 belong to the same group, and as on scanning the rest of the term, we find neither of them associated with any other index, we conclude that the group is *binary* ("un groupe *binaire*"). Again, we see that 2 is paired with 5, 5 with 4, 4 with 6, and 6 with 2; this gives us the quaternary group (2, 5, 4, 6). Lastly, 3 is seen to be paired with 3, and thus forms a group by itself. Now, if we wish to find how many interchanges of the second indices are necessary in order to obtain the given term

$$a_{0,1}a_{1,0}a_{2,5}a_{3,3}a_{4,6}a_{5,4}a_{6,2}$$

from the typical term

$$a_{0,0}a_{1,1}a_{2,2}a_{3,3}a_{4,4}a_{5,5}a_{6,6}\,,$$

we may do the counting piecemeal, attending at one time to only that part of the term which corresponds to one of the groups of indices. In the case of the group (3), the number of interchanges is 0; in the case of the binary group (0, 1) it is 1; and in the case of the quaternary group it is 3—the number of interchanges being "évidemment" one less than the number of indices in the group. If, therefore, for a given term there be in all m groups, viz. f groups of one index each, g groups of two indices each, h of three, k of four, &c., the number of necessary interchanges will be

$$0.f + 1.g + 2.h + 3.k + \ \ldots,$$

which

$$\begin{aligned} &= \quad f + 2.g + 3.h + 4.k + \ \ldots, \\ &\quad -(f + \ g + \ h + \ k + \ \ldots), \\ &= \quad n - m; \end{aligned}$$

and consequently the sign of the term will be + or -1 according as $n-m$ is even or odd. (III. 29)

The first step of the new investigation is to define "termes semblables ou de même espèce." *Two terms are said to be alike or of the same species when the one may be obtained from the other by subjecting* both *sets of indices in the latter to one and the same substitution or permutation.* Thus recurring to the term above used,

$$a_{0,1}a_{1,0}a_{2,5}a_{3,3}a_{4,6}a_{5,4}a_{6,2}\,,$$

and substituting in both of its sets of indices 6, 0, 1, 4, 3, 2, 5, instead of 0, 1, 2, 3, 4, 5, 6 respectively—in order words, and with the notation of the memoir of 1812, performing the substitution

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 0 & 1 & 4 & 3 & 2 & 5 \end{pmatrix},$$

we obtain the like term

$$a_{6,0}a_{0,6}a_{1,2}a_{4,4}a_{3,5}a_{2,3}a_{5,1}\,. \qquad \text{(LV.)}$$

The groups in two like terms are evidently similar, the values of $f, g, h, \ldots$ for the one being the same as those for the other. Indeed, since it is in this matter of groups or cycles that the terms have any likeness at all, the expression "*cyclically alike*" would have been a better term for Cauchy to use.

From the definition there arises the self-evident proposition—*Terms which have similar index-cycles or are cyclically alike in their indices have the same sign.* (III. 30)

Also, *the full expansion of a determinant may be represented by writing a term of each cyclical species, and prefixing to each such typical term the symbol* Σ *with its proper sign,* $+$ *or* $-$.

(LV. 2)

To obtain a term of any given cyclical species, that is to say, corresponding to given values of $f, g, h, \ldots$, all the preparation that is necessary is to write the indices

$$0,\ 1,\ 2,\ 3,\ \ldots,\ (n-1),$$

enclose each of the first f of them in brackets, enclose in brackets each of the next g pairs, then each of the next h triads, and so on. This gives the groups of the term, and the term itself readily follows. For example, if we desire in the case of the determinant $\Sigma \pm a_{00}a_{11}a_{22}a_{33}a_{44}a_{55}a_{66}$ a term corresponding to $f=2$, $g=1$, $h=1$* we take the indices

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6;$$

bracket them thus

$$(0),\ (1),\ (2, 3),\ (4, 5, 6):$$

and with the help of this, write finally

$$a_{0,0}\, a_{1,1}\, a_{2,3}\, a_{3,2}\, a_{4,5}\, a_{5,6}\, a_{6\,4}. \qquad \text{(II. 7)}$$

The number of different cyclical species of terms in a determinant of the n^{th} order is evidently equal to the number of positive integral solutions of the equation

$$f + 2g + 3h + \ \ldots\ + nl = n. \qquad \text{(LV. 3)}$$

Cauchy's illustration of this is clearness itself. He says (p. 419):—

"Si, pour fixer les idées, on suppose $n=5$, alors, la valeur de n pouvant être présentée sous l'une quelconque des formes,

$$\begin{array}{l} 1 + 1 + 1 + 1 + 1, \\ 1 + 1 + 1 + 2, \\ 1 + 2 + 2, \\ 1 + 1 + 3, \\ 2 + 3, \\ 1 + 4, \\ 5, \end{array}$$

* It would be convenient to say, *a term whose index-cycle scheme is*

$$2(1) + 1(2) + 1(3).$$

les systèmes de valeurs de

$$f,\ g,\ h,\ k,\ l,$$

se réduiront à l'un des sept systèmes

$$\begin{array}{lllll} f=5, & g=0, & h=0, & k=0, & l=0, \\ f=3, & g=1, & h=0, & k=0, & l=0, \\ f=1, & g=2, & h=0, & k=0, & l=0, \\ f=2, & g=0, & h=1, & k=0, & l=0, \\ f=0, & g=1, & h=1, & k=0, & l=0, \\ f=1, & g=0, & h=0, & k=1, & l=0, \\ f=0, & g=0, & h=0, & k=0, & l=1; \end{array}$$

et par suite, une fonction alternée du cinquième ordre renfermera sept espèces de termes."

The next question considered is as to the number of terms of a given cyclical species which exist in any determinant of the n^{th} order. The species being characterised by f groups of one index each, g groups of two indices each, h groups of three indices each, &c., the required number of terms is denoted by

$$N_{f,\, g,\, h,\, \ldots,\, l}\,.$$

Now all the terms of the species will certainly be got if we write in succession the various permutations of the n indices $0, 1, 2, 3, \ldots, n-1$, and then in the usual way mark off each permutation into the specified groups, viz., first f groups of one index each, then g groups of two indices each, and so on. As a rule, however, each term of the species will, in this way, be obtained more than once. For, if we examine in its grouped form the particular permutation which was the first to give rise to a certain term, we shall find that changes are possible upon it without entailing any change in the term. For example, the set of groups

$$(0),\ (1),\ (2, 3),\ (4, 5, 6),$$

instanced above as corresponding to the term

$$a_{0,0}a_{1,1}a_{2,3}a_{3,2}a_{4,5}a_{5,6}a_{6,4}\,,$$

might be changed into

$$(1),\ (0),\ (2, 3),\ (4, 5, 6)$$

or $\qquad (1),\ (0),\ (3, 2),\ (6, 4, 5)$

or $\qquad \cdots\cdots\cdots$

which, while still corresponding to the term

$$a_{0,0}a_{1,1}a_{2,3}a_{3,2}a_{4,5}a_{5,6}a_{6,4},$$

are derivable from different permutations of the seven indices 0, 1, 2, 3, 4, 5, 6. In fact, the f groups of one index each may be permuted among themselves in every possible way, so may the g binary groups, the h ternary groups, &c. Further, with like immunity to the term, each separate group may be written in as many ways as there are indices in it,—the group (4, 5, 6), for example, being safely changeable into (5, 6, 4) or (6, 4, 5). The number, therefore, of different permutations of 0, 1, 2, 3, 4, 5, 6, which will give rise to any particular term, is

$$(1.2.3\ldots f\times 1.2.3\ldots g\times 1.2.3\ldots h\times\ldots\times 1.2.3\ldots l)\times(1^f2^g3^h\ldots n^l),$$

or say,

$$(f!g!h!\ldots l!)(1^f2^g3^h\ldots n^l).$$

There thus results the equation

$$(f!g!h!\ldots l!)(1^f2^g3^h\ldots n^l)\,\mathrm{N}_{f,g,h,\ldots,l}=n!,$$

whence $$\mathrm{N}_{f,g,h,\ldots,l}=\frac{n!}{(f!g!h!\ldots l!)(1^f2^g3^h\ldots n^l)}. \qquad (\text{LV. } 4)$$

Following this interesting result a few deductions and verifications are given. First of all it is pointed out that since the total number of terms of all species is $n!$ we must conclude that

$$n!=\sum\frac{n!}{(f!g!h!\ldots l!)(1^f2^g3^h\ldots n^l)},$$

where $$f+2g+3h+\ \ldots\ +nl=n.$$

Cauchy says (p. 423):—

"Cette dernière formule paraît digne d'être remarquée. Si, pour fixer les idées, on prend $n=5$ l'équation donnera

$$1.2.3.4.5=\mathrm{N}_{5,0,0,0,0}+\mathrm{N}_{3,1,0,0,0}+\mathrm{N}_{1,2,0,0,0}+\mathrm{N}_{2,0,1,0,0}$$
$$+\mathrm{N}_{0,1,1,0,0}+\mathrm{N}_{1,0,0,1,0}+\mathrm{N}_{0,0,0,0,1},$$

et par suite

$$1.2.3.4.5=1+10+15+20+20+30+24=120,$$

ce qui est exact."

Again, since the number of positive terms in a determinant is equal to the number of negative terms, and since the terms, whose number $N_{f,g,h,\ldots,l}$, has just been found, have all the sign-factor

$$(-1)^{n-(f+g+h+\ldots+l)},$$

we have on leaving out the common factor $(-1)^n$ the identity

$$0 = \sum(-1)^{f+g+h+\ldots+l}\frac{n\,!}{(f!\,g!\,h!\ldots l!)(1^f 2^g 3^h \ldots n^l)},$$

which like its companion may be illustrated by the case of $n=5$, viz.,

$$0 = 1 - 10 + 15 + 20 - 20 - 30 + 24.\text{*}$$

Lastly, attention is directed to the fact that when n is a prime, and therefore not exactly divisible by any integer less than itself, the number

$$\frac{n\,!}{(f!\,g!\,h!\ldots l!)(1^f 2^g 3^h \ldots n^l)}$$

must be exactly divisible by n, except in the case

$$f=n, \quad g=0, \quad h=0, \quad \ldots, \quad l=0,$$

when it has the value 1, and in the case

$$f=0, \quad g=0, \quad h=0, \quad \ldots, \quad l=1,$$

when it has the value $(n-1)!$ It, therefore, follows from either of the two preceding identities, that the sum of these two values must be divisible by n,—which is Wilson's theorem.

The remaining two pages are occupied with the expansion of a determinant of special form, viz., that afterwards known by the name *axisymmetric*.

JACOBI (1841).

[De formatione et proprietatibus Determinantium. *Crelle's Journal*, xxii. pp. 285–318; or *Werke*, iii. pp. 355–392.]

The value which Jacobi attached to determinants as an instrument of research has already become well known to us: we have

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* In connection with this and in illustration of a previous remark regarding a mode of expressing the full expansion of a determinant, we have

$$\begin{aligned}\Sigma \pm a_{00}a_{11}a_{22}a_{33}a_{44} = \quad & a_{00}a_{11}a_{22}a_{33}a_{44} - \Sigma a_{00}a_{11}a_{22}a_{34}a_{43} \\ &+ \Sigma a_{00}a_{12}a_{21}a_{34}a_{43} + \Sigma a_{00}a_{11}a_{23}a_{34}a_{42} \\ &- \Sigma a_{01}a_{10}a_{23}a_{34}a_{42} - \Sigma a_{00}a_{12}a_{23}a_{34}a_{41} \\ &+ \Sigma a_{01}a_{12}a_{23}a_{34}a_{40}. \qquad \text{(LV. 2)}\end{aligned}$$

found him, indeed, in almost constant employment of the functions. In the memoir now reached, however, we have still stronger evidence of his interest in the subject, and of his opinion as to its importance. Knowing of no succinct and logically arranged exposition of their properties readily accessible to mathematicians, he deliberately set himself the task of preparing a memoir to supply the want. In his few words of preface he says:—

"Sunt quidem notissimi Algorithmi, qui aequationum linearium litteralium resolutioni inserviunt. Neque tamen video eorum proprietates praecipuas, ita breviter enarratas atque in conspectum positas esse, quantum optare debemus propter earum in gravissimis quaestionibus Analyticis usum. Scilicet illae proprietates quamvis elementares non omnes ita tritae sunt, ut quas indemonstratas relinquere deceat, et valde molestum est earum demonstrationibus altiorum ratiociniorum decursum interrumpere. Cui defectui hic supplere volo quo commodius in aliis commentationibus ad hanc recurrere possim; neutiquam vero mihi propono totam illam materiam absolvere."

While Jacobi was aware, as we have already partly seen, of the labours of Cramer, Bezout, Vandermonde, Laplace, Gauss, and Binet, his main source of inspiration is Cauchy. Of all the writers since Cauchy's time, indeed, he is the first who gives evidence of having read and mastered the famous memoir of 1812. It scarcely needs be said, however, that his own individuality and powerful grasp are manifest throughout the whole exposition.

At the outset there is a reversal of former orders of things; Cramer's rule of signs for a permutation and Cauchy's rule being led up to by a series of propositions instead of one of them being made an initial convention or definition. This implies, of course, that a new definition of a signed permutation is adopted, and that conversely this definition must have appeared as a deduced theorem in any exposition having either of these rules as its starting point.

The new definition has its source in Cauchy, and rests on the well-known agreement as to a definite mode of forming the product P of the differences of an ordered series of quantities. This being settled to be

$$\begin{array}{r}(a_1-a_0)(a_2-a_0)(a_3-a_0)\ \ldots\ldots\ (a_n-a_0)\\ (a_2-a_1)(a_3-a_1)\ \ldots\ldots\ (a_n-a_1)\\ (a_3-a_2)\ \ldots\ldots\ (a_n-a_2)\\ \ldots\ldots\ldots\ldots\\ (a_n-a_{n-1})\end{array}$$

for the quantities $a_0, a_1, a_2, \ldots\ldots a_n$, while in the order here written, the definition stands as follows (pp. 285–286):—

"Vocemus eas indicum 0, 1, ... , n permutationes, pro quibus P valorem eundem servat, *positivas*; eas pro quibus P valorem oppositum induit, *negativas*; sive priores dicamus pertinere ad *classem positivam permutationum*, posteriores ad *classem negativam*."

This implies of course that the original permutation 0, 1, 2, ... , n is to be considered positive; and, such being the case, there seems to be a certain appropriateness in applying the term *negative* to a permutation whose corresponding difference-product is of the opposite sign from the difference-product corresponding to 0, 1, 2, ... , n.

The propositions which lead from the definition to Cramer's rule may be enunciated as follows:—

(*a*) One permutation performed upon another gives rise to a third, and the combined effect produced by performing the second and first in succession is the same as the effect of performing the third.

(*b*) Two given permutations belong to the same class or to opposite classes according as the permutation by means of which the one is obtained from the other belongs to the positive or negative class

(*c*) If the same permutation be performed on a number of permutations which all belong to one class, the resulting permutations will still all belong to one class, viz., the same or the opposite according as the operating permutation is positive or negative.

(*d*) The order of compounding a set of permutations is, as a rule, not immaterial.

(*e*) The permutations which arise by compounding a set of permutations in every possible order belong all to the same class. (III. 32)

(*f*) The interchange of two indices is equivalent to the performance of a negative permutation.

(*g*) The interchange of two indices causes all the positive permutations to become negative, and all the negative to become positive.

Definition.—Two permutations may be called reciprocal which being performed in succession do not alter the order existing before the operations. (XXIV. 2)

(*h*) Reciprocal permutations belong to the same class.

In the original, it must be borne in mind, these are not separated and numbered, but appear merely as consecutive sentences in a paragraph. The words "classem negativam" of the definition above given are followed in the same line by

"Binis propositis permutationibus quibuscunque, certa exstabit permutatio, qua post alteram adhibita altera prodit. Pertinebunt duæ permutationes propositæ ad classem eundem aut ad classes oppositas, prout permutatio, qua altera ex altera obtinetur, ad classem positivam aut negativam pertinet," &c.

—that is to say, by the propositions which have been paraphrased into (*a*), (*b*), &c.

The most essential point to be considered in connection with them is the probable meaning of the expression "permutationem adhibere," or the free English translation of it, "to perform a permutation." An example will make it clear. To perform the permutation 35412 would seem to be the operation of removing the 3rd member of a series of five things to the first place, the 5th member to the second place, the 4th member to the third place, and so on. With this explanation the proposition (*a*) is self-evident, an example of it being (if we may improvise a symbolism)

$$(35412)(41352) = (32541),$$

where 35412 is the operating permutation. Cauchy's usage, it may be remembered, was to speak of "applying a substitution to a permutation."*

Of the proposition (*b*) a proof is given, which may be paraphrased as follows:—Let the three permutations referred to change P, the original product of differences, into e_1P, e_2P, e_3P, respectively, the *e*'s of course being either $+1$ or -1. Then as the performance of the first two permutations in succession will result in the change of P into $e_1.e_2$P, we must have

$$e_1.e_2 = e_3\,,$$

so that e_1 and e_3 have the same or opposite signs according as e_2 is $+1$ or -1; and this is virtually the proposition to be proved. (III. 31)

A demonstration of (*d*) is also given. The two permutations being A and B, *l* the first index of A, and *m* the first index of B, the performance of A on B implies that the l^{th} index in B is to take the first place, and the performance of B on A that the m^{th} index of A is to take the first place. The resulting permutations will consequently not agree in the first index, unless the l^{th} index of B is the same as the m^{th} index of A, which manifestly need not be the case.†

To prove (*f*) is of course the same as to prove that the interchange of two indices *r* and *s*, *r* being the greater, alters the sign of the product of differences; and this is done by separating the product into three portions, viz., (1) the portion which contains neither a_r nor a_s, (2) the single factor which contains both, $a_r - a_s$, and (3) the product of all the factors having either one or the other for a term. It is then asserted that the interchange of *r* and *s* cannot alter the last of these, because it is symmetrical with respect to a_r and a_s; also, that no alteration is possible in

* He says, for example (*Journ. de l'Éc. Polyt.*, x. p. 10), "Si en appliquant successivement à la permutation A_1 les deux substitutions $\binom{A_2}{A_3}$ et $\binom{A_4}{A_5}$, on obtient pour résultat la permutation A_6; la substitution $\binom{A_1}{A_6}$ sera équivalente au produit des deux autres et j'indiquerai cette équivalence comme il suit

$$\binom{A_1}{A_6} = \binom{A_2}{A_3}\binom{A_4}{A_5}."$$

† This also is a paraphrase of Jacobi's proof.

the first, and consequently that the change in the second accounts for the validity of the proposition. (III. 33)

As for the permutations which are called reciprocal they are, exactly those whose existence we have seen noted by Rothe, and called by him "verwandte Permutationen." Jacobi's definition, however, presents them in a slightly different light, the property involved in it being readily deducible from Rothe's. The latter's illustrative example was, as may be seen on looking back,

$$\left.\begin{array}{l} 3,\ \ 8,\ 5,\ 10,\ 9,\ 4,\ 6,\ 1,\ 7,\ 2 \qquad \mathrm{A} \\ 8,\ 10,\ 1,\ \ 6,\ 3,\ 7,\ 9,\ 2,\ 5,\ 4 \qquad \mathrm{B} \end{array}\right\}.$$

Now the performance of either A on B or B on A* gives rise to

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,$$

the original arrangement: consequently A and B satisfy Jacobi's definition. The proposition (h) is also Rothe's.

After these propositions, as already intimated, the subject of other rules of signs is taken up, the first rule considered being Cramer's. Since in the product of differences corresponding to any permutation every factor in which an index is preceded by a smaller index would require the sign-factor -1 to be annexed to it in order that the said product might be transformed into the original product of differences, it is clear that the determination of the class to which the permutation belongs is reduced to counting the number of such inversions. But the pairs of indices in the product of differences corresponding to the given permutation are exactly the pairs of indices to be examined in applying Cramer's rule. The identity of the two rules is thus apparent. (III. 34)

To the demonstration Jacobi adds "quam regulam olim cel. Cramer dedit ill. Laplace demonstravit." The last assertion is notable for two reasons: first, because the rule like Jacobi's own is incapable of proof being a definition, postulate, or convention according to the mode in which it is expressed: secondly, because an examination of Laplace's memoir shows that there is no ground for the statement. The fitness of the rule for the deter-

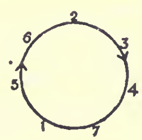
* In the compounding of reciprocal permutations the order is immaterial. This is the exception hinted at in (d).

mination of the signs of the numerators and denominators of the unknowns in a set of simultaneous linear equations may of course be demonstrated, and perhaps this was in Jacobi's mind, but prior to the statement the abstract subject of permutations had alone been discussed.

The other rule of signs dealt with is Cauchy's, in which permutation-cycles are counted instead of inversions. The existence of such cycles is the first point to be established, that is to say, it has to be shown that *any permutation of* 1 2 3 . . . n *may be obtained from any other by the performance of one or more cyclical permutations.* Let 3271654 be the permutation sought,* and 2647513 the permutation from which it is to be derived. Placing the former under the latter, thus

$$\begin{matrix} 2 & 6 & 4 & 7 & 5 & 1 & 3 \\ 3 & 2 & 7 & 1 & 6 & 5 & 4, \end{matrix}$$

we see that 2 has to be changed into 3, then seeking 3 in the upper line we see that it has to be changed into 4, similarly that 4 has to be changed into 7, 7 into 1, 1 into 5, 5 into 6, and 6 into 2, the element with which we started. Now the proof turns upon the simple fact that the elements in the two lines being exactly the same, by following a string of changes like this we are bound sooner or later to reach in the second line the element we started with in the first. It may be that as here one cycle



suffices for the second transformation; but if not, as in the case of the two permutations

$$\begin{matrix} 2 & 6 & 4 & 7 & 5 & 1 & 3 \\ 4 & 1 & 5 & 7 & 2 & 3 & 6, \end{matrix}$$

where the short cycle 245 is obtained, we turn to the remaining

* This is a paraphrase of Jacobi's demonstration, which is not so simple as it might have been. The notation of substitutions, which Jacobi did not follow Cauchy in using, is here a great help toward clearness.

elements, and knowing that those in the first line are of necessity the same as those in the second, we see that the application of the same process to them must, for the same reason as before, lead to a cycle. The possibility of arriving at any permutation by means of cyclical permutations alone is thus made manifest. The next point to be established is that *a cyclical permutation of* r *elements can be accomplished by* r − 1 *interchanges of pairs of elements.* Little more than the statement of this is necessary. For if the elements of the cycle be $a_1, a_2, a_3, \ldots, a_r$, it is clear that to change a_1 into a_2, a_2 into a_3, &c., has the same effect as to interchange a_1 and a_2, then a_1 and a_3, then a_1 and a_4 and so on, the final interchange being that of a_1 and a_r; and there are in all $r-1$ interchanges. This being proved, the final step is taken as in Cauchy's Note of 8th March. (III. 35)

This rule of Cauchy's Jacobi deservedly characterises as beautiful. It is important, however, to take note that it possesses the other quality of usefulness in as marked a degree; and such being the case one is surprised to find that it has not received the attention which was its due. Any reader who will make a comparison of it and Cramer's by actual application of them to a number of examples will soon find that Cramer's is more lengthy and requires more care to be given to it to avoid errors.*

The preliminary subject of permutations having been thus dealt with, determinants are taken up. In the first section regarding them there is little noteworthy. Cauchy's word "terme" is supplanted by the fitter word *element*, and *term* ("terminus") is put to a more appropriate use; that is to say, $a_k^{(i)}$ is called an element of the determinant $\Sigma \pm aa'_1a''_2 \ldots a_n^{(n)}$ and $a_k a'_{k'} a''_{k''} \ldots a_{kn}^{(n)}$ a term. Further, the word *degree* is employed in place of Cauchy's more suitable word *order*, "ipsum R dicam determinans $n+1^{ti}$ *gradus.*"

A section of two pages is given to considering the effect

*The best way perhaps of applying Cauchy's rule is to write the primitive permutation, 123456789 say, above the given permutation, 683192457 say, draw the pen through 1 and the figure below it, seek 6 in the upper line and draw the pen through it and the figure below it, and so on, marking down 1 on the completion of every cycle.

produced upon the aggregate of terms by the vanishing of certain of the elements. The propositions enunciated, with the exception of one made use of at an earlier date by Scherk, are as follows (pp. 291, 292):—

"I. Quoties pro indicis k valoribus $0, 1, 2, \ldots, m-1$ evanescant elementa $a_k^{(m)}$, $a_k^{m+1)}$, ... $a_k^{(n)}$, determinans

$$\Sigma \pm aa_1'a_2'' \ldots . a_n^{(n)}$$

abire in productum a duobus determinantibus

$$\Sigma \pm aa_1' \ldots . a_{m-1}^{(m-1)} \,.\, \Sigma \pm a_m^{(m)} a_{m+1}^{(m+1)} \ldots . a_n^{(n)}. \qquad \text{(XIV. 6)}$$

II. Evanescentibus elementis omnibus,

$$a_k^m, \quad a_k^{(m+1)}, \quad \ldots, \quad a_k^{(n)}$$

in quibus respective index inferior k indicibus superioribus m, $m+1$, ..., n minor est, fieri

$$\Sigma \pm aa_1'a_2'' \ldots . a_n^{(n)} = a_m^{(m)} a_{m+1}^{(m+1)} \ldots . a_n^{(n)} \,.\, \Sigma \pm aa_1' \ldots . a_{m-1}^{(m-1)}. \qquad \text{(VI. 9)}$$

IV. Evanescentibus elementis omnibus,

$$a_k^{(m)}, \quad a_k^{(m+1)}, \quad \ldots \ldots, \quad a_k^{(n)},$$

in quibus indices inferiores superioribus minores sunt, si insuper habetur,

$$a_m^{(m)} = a_{m+1}^{(m+1)} \ \ldots . = a_n^{(n)} = 1,$$

fit

$$\Sigma \pm aa_1'a_2'' \ldots . a_n^{(n)} = \Sigma \pm aa_1' \ldots . a_{m-1}^{(m-1)}." \qquad \text{(VI. 9)}$$

As immediate deductions from the definition these are somewhat out of place, the trouble of demonstrating the first of them being virtually thrown away. The trouble taken by Jacobi, too, was less than required, the question of sign, for example, being inadequately discussed.

In the course of the next section which deals with what we have called the recurrent law of formation, and with the vanishing aggregate connected with this law, Jacobi gives an expression for the complete differential of a determinant, the elements being viewed as independent variables. The passage is (p. 293):—

"Determinans R est singularum quantitatum $a_k^{(i)}$ respectu expressio linearis, atque ipsius $a_k^{(i)}$ coefficientem, qua in determinante R afficitur, vocavimus $A^{(i)}_k$; unde adhibita differentialium notatione ipsum $A^{(i)}_k$ exhibere licet per formulam,

$$3.\quad A^{(i)}_k = \frac{\partial R}{\partial a^{(i)}_k}.$$

Hinc si quantitatibus $a^{(i)}_k$ incrementa infinite parva tribuimus,

$$da^{(i)}_k,$$

simulque R incrementum dR capit, fit

$$4.\quad dR = \sum A^{(i)}_k da^{(i)}_k. \qquad \text{(LVI.)}$$

siquidem sub signo summatorio utrique indici i et k valores 0, 1, 2, . . . , n conferuntur."

The recurrent law of formation and its dependent neighbour formula he is enabled, by means of (3), to view as the partial differential equations which the determinant must satisfy. His words are (p. 295):—

"Substituendo formulas (3), inventas formulas sic quoque exhibere licet:

$$\begin{aligned} 9.\quad R &= a^{(i)}\frac{\partial R}{\partial a^{(i)}} + a^{(i)}_1\frac{\partial R}{\partial a^{(i)}_1} + \ldots + a^{(i)}_n\frac{\partial R}{\partial a^{(i)}_n}, \\ &= a_k\frac{\partial R}{\partial a_k} + a'_k\frac{\partial R}{\partial a'_k} + \ldots + a^{(n)}_k\frac{\partial R}{\partial a^{(n)}_k}, \\ 10.\quad 0 &= a^{(i')}\frac{\partial R}{\partial a^{(i)}} + a^{(i')}_1\frac{\partial R}{\partial a^{(i)}_1} + \ldots + a^{(i')}_n\frac{\partial R}{\partial a^{(i)}_n}, \\ &= a_{k'}\frac{\partial R}{\partial a_k} + a'_{k'}\frac{\partial R}{\partial a'_k} + \ldots + a^{(n)}_{k'}\frac{\partial R}{\partial a^{(n)}_k}. \end{aligned}$$

Quae sunt aequationes differentiales partiales quibus determinans R satisfacit."

Passing over a section (7) on simultaneous linear equations, and a short section (8) in which Laplace's expansion-theorem is enunciated, we come to two sections dealing with what at a later time would have been called the secondary minors. No name is given to them by Jacobi; they only appear as co-factors of the product of a pair of elements, the aggregate of the terms containing $a^{(f)}_g\, a^{(f')}_{g'}$ as a factor being denoted by

$$a^{(f)}_g\, a^{(f')}_{g'} \,.\, A^{f,f'}_{g,g'}. \qquad \text{(XL. 9)}$$

From observing that the interchange of f and f' or of g and g' alters R into $-$R and cannot alter $A^{f',f}_{g,g}$, it is concluded that

$$A^{f,f'}_{g',g} = A^{f',f}_{g,g'} = -A^{f,f'}_{g,g'},$$

and that the full co-factor of $A^{f,f'}_{g,g'}$ is $a^{(f)}_g a^{(f')}_{g'} - a^{(f)}_{g'} a^{(f')}_g$ in accordance with the expansion-theorem of the previous section. The remark that $A^{(f)}_g$ can be expressed in terms of n of the quantities $A^{f,f'}_{g,g'}$ leads up to a curious set of equations the determinant of which belongs to the special class of determinants known afterwards as zero-axial skew determinants. The passage is (pp. 300, 301):—

"Designemus br. causa per (k, k') expressionem

$$10. \qquad A^{f,f'}_{k,k'} = (k, k'),$$

ita ut sit $\qquad (k, k') = -(k', k).$

Fit e (8) ipsi g substituendo numeros $0, 1, 2, \ldots, n$:

$$11. \quad \left\{\begin{array}{l} A^{(f)} = \quad * \quad + a_1^{(f')}(0, 1) + a_2^{(f')}(0, 2) \,.\,.\, + a_n^{(f')}(0, n) \\ A_1^{(f)} = a^{(f')}(1, 0) + \quad * \quad + a_2^{(f')}(1, 2) \,.\,.\, + a_n^{(f')}(1, n) \\ A_2^{(f)} = a^{(f')}(2, 0) + a_1^{(f')}(2, 1) + \quad * \quad .\,.\, + a_n^{(f')}(2, n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n^{(f)} = a^{(f')}(n, 0) + a_1^{(f')}(n, 1) + a_2^{(f')}(n, 2) \,.\,.\, + \quad * \end{array}\right.$$

Similes formulae e (9) derivari possunt. In aequationibus (11) ipsorum $a^{(f')}$, $a_1^{(f')}$, etc. coëfficientes in diagonali positi evanescunt, bini quilibet coëfficientes diagonalis respectu symmetrice positi valoribus oppositis gaudent. Quae est species aequationum linearium memorabilis in variis quaestionibus analyticis obveniens." (LVII.)

The simple step from the expression of $A^{(f)}_g$ as a differential coefficient to the similar expression for $A^{f,f'}_{g,g'}$ is next made (p. 301):—

"Ex ipsa enim aggregati $A^{f,f'}_{g,g'}$ definitione eruimus formulas

$$1. \qquad A^{f,f'}_{g,g'} = \frac{\partial^2 R}{\partial a^{(f)}_g \partial a^{(f')}_{g'}} = -\frac{\partial^2 R}{\partial a^{(f')}_g \partial a^{(f)}_{g'}},$$

$$= \frac{\partial \,.\, A^{(f)}_g}{\partial a^{(f')}_{g'}} = \frac{\partial \,.\, A^{(f')}_{g'}}{\partial a^{(f)}_g} = -\frac{\partial \,.\, A^{(f')}_g}{\partial a^{(f)}_{g'}} = -\frac{\partial \,.\, A^{(f)}_{g'}}{\partial a^{(f')}_g}."$$

By taking the identities

$$\begin{aligned}
0 &= a\mathrm{A}_k + a'\mathrm{A}'_k + \ldots + a^{(n)}\mathrm{A}^{(n)}_k,\\
0 &= a_1\mathrm{A}_k + a'_1\mathrm{A}'_k + \ldots + a_1^{(n)}\mathrm{A}^{(n)}_k,\\
&\ldots\ldots\ldots\ldots\ldots\\
\mathrm{R} &= a_k\mathrm{A}_k + a'_k\mathrm{A}'_k + \ldots + a_k^{(n)}\mathrm{A}^{(n)}_k,\\
&\ldots\ldots\ldots\ldots\ldots\\
0 &= a_n\mathrm{A}_k + a'_n\mathrm{A}'_k + \ldots + a_n^{(n)}\mathrm{A}^{(n)}_k;
\end{aligned}$$

using the multipliers

$$\mathrm{A}^{i,i'}_{0,k'},\quad \mathrm{A}^{i,i'}_{1,k'},\quad \ldots\ldots,\quad \mathrm{A}^{i,i'}_{k,k'},\quad \ldots\ldots,\quad \mathrm{A}^{i,i'}_{n,k'},$$

and adding, there is obtained

$$4. \qquad \mathrm{R}.\mathrm{A}^{i,i'}_{k,k'} = \mathrm{A}^{(i)}_k\mathrm{A}^{(i')}_{k'} - \mathrm{A}^{(i)}_{k'}\mathrm{A}^{(i')}_k,$$

—a result at once recognisable as a case of the theorem regarding a minor of the adjugate. Next by starting with Bezout's identity connecting any eight quantities, the particular eight taken being

$$\begin{array}{cccc}
\mathrm{A}^{(i)}_k, & \mathrm{A}^{(i)}_{k'}, & \mathrm{A}^{(i)}_{k''}, & \mathrm{A}^{(i)}_{k'''},\\
\mathrm{A}^{(i')}_k, & \mathrm{A}^{(i')}_{k'}, & \mathrm{A}^{(i')}_{k''}, & \mathrm{A}^{(i')}_{k'''},
\end{array}$$

and making six substitutions of the kind

$$\mathrm{A}^{(i)}_k\mathrm{A}^{(i')}_{k'} - \mathrm{A}^{(i)}_{k'}\mathrm{A}^{(i')}_k = \mathrm{R}.\mathrm{A}^{i,i'}_{k,k'},$$

just seen to be valid, there arises the identity

$$\mathrm{A}^{i,i'}_{k,k'}\mathrm{A}^{i,i'}_{k'',k'''} + \mathrm{A}^{i,i'}_{k,k''}\mathrm{A}^{i,i'}_{k''',k'} + \mathrm{A}^{i,i'}_{k,k'''}\mathrm{A}^{i,i'}_{k',k''} = 0. \qquad \text{(XXIII. 12)}$$

This clearly belongs to the class of vanishing aggregates of products of pairs of determinants; but in order that its true character may be seen, and comparison made possible between it and others of the same class already obtained, a more lengthy notation is necessary. Taking for shortness the case where the primitive determinant is of the 8th order, but writing it in the form

$$|a_1b_2c_3d_4e_5f_6g_7h_8|$$

and making

$$i, i' = 3, 6 \quad \text{and} \quad k, k', k'', k''' = 5, 6, 7, 8,$$

we find the identity to be

$$|a_1b_2d_3e_4g_7h_8|.|a_1b_2d_3e_4g_5h_6| - |a_1b_2d_3e_4g_6h_8|.|a_1b_2d_3e_4g_5h_7| + |a_1b_2d_3e_4g_6h_7|.|a_1b_2d_3e_4g_5h_8| = 0,$$

a glance at which suffices to show that it is nothing more than the extensional of

$$|g_7h_8|.|g_5h_6| - |g_6h_8|.|g_5h_7| + |g_6h_7|.|g_5h_8| = 0,$$

the very identity of Bezout which was taken as a basis for it. As the same extensional has already been found among those of Desnanot, any new interest in it is due to the peculiar way in which Jacobi obtained it. By the same method, viz., by substituting for secondary minors an expression (4) involving primary minors and the primitive determinant, he shows that

$$\mathrm{A}_k^{(i)}\mathrm{A}_{k',k''}^{i,i'} + \mathrm{A}_{k'}^{(i)}\mathrm{A}_{k'',k}^{i,i'} + \mathrm{A}_{k''}^{(i)}\mathrm{A}_{k,k'}^{i,i'} = 0. \qquad \text{(XXIII. 13)}$$

This being translated in the same manner as the preceding, becomes

$$|a_1b_2d_3e_4f_6g_7h_8|.|a_1b_2d_3e_4g_5h_8| - |a_1b_2d_3e_4f_5g_7h_8|.|a_1b_2d_3e_4g_6h_8| + |a_1b_2d_3e_4f_5g_6h_8|.|a_1b_2d_3e_4g_7h_8| = 0,$$

and is thus seen to be another of Desnanot's results, viz., the extensional of

$$|f_6g_7|.g_5 - |f_5g_7|.g_6 + |f_5g_6|.g_7 = 0. \qquad \text{(XXIII. 13)}$$

The deduction

$$\frac{\partial\cdot\dfrac{\mathrm{A}_{k'}^{(i)}}{\mathrm{A}_k^{(i)}}}{\partial a_{k''}^{(i')}} = -\frac{\mathrm{A}_{k''}^{(i)}\mathrm{A}_{k,k'}^{i,i'}}{\mathrm{A}_k^{(i)}\,\mathrm{A}_k^{(i)}}, \quad \frac{\partial\cdot\dfrac{\mathrm{A}_k^{(i)}}{\mathrm{A}_k^{(i')}}}{\partial a_{k'}^{(i'')}} = -\frac{\mathrm{A}_k^{(i'')}\mathrm{A}_{k,k'}^{i,i'}}{\mathrm{A}_k^{(i)}\,\mathrm{A}_k^{(i)}},$$

is made from it by substituting appropriate differential coefficients for the primary and secondary minors involved in it.

(LVIII.)

The eleventh section is devoted to the establishment of the general theorem which includes the theorem

$$\mathrm{R}.\mathrm{A}_{k,k'}^{i,i'} = \mathrm{A}_k^{(i)}\mathrm{A}_{k'}^{(i')} - \mathrm{A}_{k'}^{(i)}\mathrm{A}_k^{(i')}$$

of the preceding section, and which, as we have seen, Jacobi had first enunciated in 1833. To start with it is repeated that the system of equations

$$\left.\begin{array}{l}
a\ t + a_1 t_1 + \ldots + a_k t_k + a_{k+1}t_{k+1} + \ldots + a_n t_n = u, \\
a'\ t + a_1' t_1 + \ldots + a_k' t_k + a_{k+1}' t_{k+1} + \ldots + a_n' t_n = u_1, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a^{(k)}t + a_1^{(k)}t_1 + \ldots + a_k^{(k)}t_k + a_{k+1}^{(k)}t_{k+1} + \ldots + a_n^{(k)}t_n = u_k, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a^{(n)}t + a_1^{(n)}t_1 + \ldots + a_k^{(n)}t_k + a_{k+1}^{(n)}t_{k+1} + \ldots + a_n^{(n)}t_n = u_n,
\end{array}\right\}$$

gives rise to the system

$$\left.\begin{array}{l}
A\ u + A'\, u_1 + \ldots + A^{(k)}u_k + A^{(k+1)}u_{k+1} + \ldots + A^{(n)}u_n = R.t \\
A_1 u + A_1' u_1 + \ldots + A_1^{(k)}u_k + A_1^{(k+1)}u_{k+1} + \ldots + A_1^{(n)}u_n = R.t_1 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
A_k u + A_k' u_1 + \ldots + A_k^{(k)}u_k + A_k^{(k+1)}u_{k+1} + \ldots + A_k^{(n)}u_n = R.t_k \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
A_n u + A_n' u_1 + \ldots + A_n^{(k)}u_k + A_n^{(k+1)}u_{k+1} + \ldots + A_n^{(n)}u_n = R.t_n
\end{array}\right\}$$

in which

$$R = \Sigma \pm a a_1' \ldots . a_n^{(n)}, \qquad A_n^{(n)} = \Sigma \pm a a_1' \ldots . a_{(n-1)}^{(n-1)}.$$

Then taking only the first $k+1$ equations of the first system and eliminating $t, t_1, \ldots, t_{k-1}$, there is obtained

$$C_k t_k + C_{k+1}t_{k+1} + \ldots . + C_n t_n = Du + D_1 u_1 + \ldots + D_k u_k, \quad \text{(X)}$$

where the multipliers $D, D_1, \ldots, D_k$, by which the elimination is effected, are

$$\begin{array}{c}
(-1)^k \Sigma \pm a' a_1'' \ldots . a_{k-1}^k, \\
(-1)^{k+1} \Sigma \pm a a_1' \ldots . a_{k-1}^k, \\
\ldots\ldots\ldots\ldots \\
+ \Sigma \pm a a_1' a_2'' \ldots . a_{k-1}^{(k-1)},
\end{array}$$

and consequently by $C_k, C_{k+1}, \ldots, C_n$ are denoted

$$\begin{array}{l}
\Sigma \pm a a_1' a_2'' \ldots . a_{k-1}^{(k-1)} a_k^{(k)}, \\
\Sigma \pm a a_1' a_2'' \ldots . a_{k-1}^{(k-1)} a_{k+1}^{(k)}, \\
\ldots\ldots\ldots\ldots \\
\Sigma \pm a a_1' a_2'' \ldots . a_{k-1}^{(k-1} a_n^{(k)}.
\end{array}$$

Similarly, taking only the last $n-k+1$ equations of the second system and eliminating $u_{k+1}, u_{k+2}, \ldots, u_n$ there is obtained

$$Eu + E_1u_1 + \ldots + E_ku_k = RF_kt_k + RF_{k+1}t_{k+1} + \ldots + RF_nt_n, \quad (Y)$$

where the multipliers $F_k, F_{k+1}, \ldots, F_n$ by which the elimination is effected are

$$\Sigma \pm A_{k+1}^{(k+1)}A_{k+2}^{(k+2)} \ldots\ldots A_n^{(n)},$$
$$-\Sigma \pm A_k^{(k+1)}A_{k+2}^{(k+2)} \ldots\ldots A_n^{(n)},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(-1)^{n-k}\Sigma \pm A_k^{(k+1)}A_{k+1}^{(k+2)} \ldots\ldots A_{n-1}^{(n)},$$

and consequently by $E, E_1, \ldots, E_k$ are denoted

$$\Sigma \pm A_kA_{k+1}^{(k+1)} \ldots\ldots A_n^{(n)},$$
$$\Sigma \pm A_k'A_{k+1}^{(k+1)} \ldots\ldots A_n^{(n)},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\Sigma \pm A_k^{(k)}A_{k+1}^{(k+1)} \ldots\ldots A_n^{(n)}.$$

These two derived equations (X), (Y), however, must be identical, because they may be both viewed as giving t_k in terms of $t_{k+1}, t_{k+2}, \ldots, t_n, u, u_1, \ldots, u_k$, and, as the first system of equations shows, this can only be done in one way. We thus have the deduction

$$\frac{D_k}{C_k} = \frac{E_k}{R \, . \, F_k},$$

$$i.e. \quad \frac{\Sigma \pm aa_1'a_2'' \ldots\ldots a_{k-1}^{(k-1)}}{\Sigma \pm aa_1'a_2'' \ldots\ldots a_k^{(k)}} = \frac{\Sigma \pm A_k^{(k)}A_{k+1}^{(k+1)} \ldots\ldots A_n^{(n)}}{R\Sigma \pm A_{k+1}^{(k+1)}A_{k+2}^{(k+2)} \ldots\ldots A_n^{(n)}}.$$

This is the keystone of the demonstration. The simple continuation of it may for sake of historical colour be given in Jacobi's own words (p. 304):*—

"In hac formula generali ipsi k tribuendo valores

$$n-1, \; n-2, \; n-3, \; \ldots, \; 1,$$

prodit:

* The demonstration in the original is considerably disfigured by misprints.

$$\left\{\begin{array}{l} \dfrac{\Sigma \pm aa_1' \ldots . a_{n-2}^{(n-2)}}{\Sigma \pm aa_1' \ldots . a_{n-1}^{(n-1)}} = \dfrac{\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)}}{RA_n^{(n)}}, \\[2ex] \dfrac{\Sigma \pm aa_1' \ldots . a_{n-3}^{(n-3)}}{\Sigma \pm aa_1' \ldots . a_{n-2}^{(n-2)}} = \dfrac{\Sigma \pm A_{n-2}^{(n-2)} A_{n-1}^{(n-1)} A_n^{(n)}}{R \Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)}}, \\[2ex] \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\[2ex] \qquad \dfrac{a}{\Sigma \pm aa_1'} = \dfrac{\Sigma \pm A_1' A_2'' \ldots . A_n^{(n)}}{R \Sigma \pm A_2'' A_3''' \ldots . A_n^{(n)}}. \end{array}\right.$$

"Harum aequationum prima suppeditat,

$$\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)} = R \Sigma \pm aa_1' \ldots . a_{n-2}^{(n-2)} = RA_{n-1,\,n}^{n-1,\,n},$$

quae cum formula (4) § pr. convenit. Deinde aequationum (10) duas, tres, quatuor etc. primas inter se multiplicando, prodit formularum systema hoc:

$$\left\{\begin{array}{l} \qquad \Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)} = R\, \Sigma \pm aa_1' \ldots . a_{n-2}^{(n-2)}, \\[1ex] \Sigma \pm A_{n-2}^{(n-2)} A_{n-1}^{(n-1)} A_n^{(n)} = R^2 \Sigma \pm aa_1' \ldots . a_{n-3}^{(n-3)}, \\[1ex] \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\[1ex] \Sigma \pm A_1' A_2'' \ldots . A_n^{(n)} = aR^{n-1}. \end{array}\right.$$

Quas formulas amplectitur formula generalis,

$$\Sigma \pm A_{k+1}^{(k+1)} A_{k+2}^{(k+2)} \ldots . A_n^{(n)} = R^{n-k-1} \Sigma \pm aa'_1 \ldots . a_k^{(k)}.\text{"} \qquad \text{(xx. 6)}$$

Cauchy's theorem

$$\Sigma \pm AA_1' \ldots . A_n^{(n)} = R^n,$$

which may be viewed as the ultimate case of this, Jacobi arrives at by expressing $\Sigma \pm AA_1' \ldots A_n^{(n)}$ in terms of $A, A_1, \ldots, A_n$ and their cofactors, substituting for the said cofactors their equivalents as just obtained, viz.

$$aR^{n-1}, \quad a_1R^{n-1}, \quad a_2R^{n-1}, \quad \ldots \ldots, \quad a_nR^{n-1},$$

and then using the identity

$$Aa + A_1a_1 + \ldots + A_na_n = R.$$

Passing over the twelfth section, which relates to certain special systems of equations, we come to two sections devoted to the multiplication-theorem. Of the five formally enunciated propositions which they contain, two, the second and fourth, need not be more than referred to, as their substance comes from Binet and Cauchy, and as the mode in which they are established will be sufficiently understood from the treatment of one of the others. The general problem of the two sections is the investigation of the determinant

$$\Sigma \pm cc_1 c_2'' \dots . c_n^{(n)},$$

where

$$c_k^{(i)} = a^{(i)}\alpha^{(k)} + a_1^{(i)}\alpha_1^{(k)} + \dots + a_p^{(i)}\alpha_p^{(k)}.$$

Taking a single term of the determinant, we have of course

$$\begin{aligned} cc_1'c_2'' \dots c_n^{(n)} = \quad & (a\,\alpha + a_1\alpha_1 + \dots + a_p\alpha_p) \\ & \times (a'\alpha' + a_1'\alpha_1' + \dots + a_p'\alpha_p') \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \times (a^{(n)}\alpha^{(n)} + a_1^{(n)}\alpha_1^{(n)} + \dots + a_p^{(n)}\alpha_p^{(n)}), \end{aligned}$$

and we see that if the multiplications indicated on the right be performed there must arise a series of $(p+1)^{n+1}$ terms of the type

$$a_r\alpha_r \,.\, a_s'\alpha_s' \,.\, a_t''\alpha_t'' \dots\dots a_w^{(n)}\alpha_w^{(n)},$$

or by alteration of the order of the factors

$$a_r a_s' a_t'' \dots a_w^{(n)} \,.\, \alpha_r \alpha_s' \alpha_t'' \dots \alpha_w^{(n)},$$

where each of the inferior indices $r, s, t, \dots, w$ may be any member of the series $0, 1, 2, \dots, p$. If we bear in mind the meaning which we thereby assign to the summatory symbol S we may write this in the form

$$cc_1'c_2'' \dots . c_n^{(n)} = \mathrm{S}\big(a_r a_s' a_t'' \dots a_w^{(n)} \,.\, \alpha_r \alpha_s' \alpha_t'' \dots \alpha_w^{(n)}\big).$$

The next point to consider is the transition from the single term $cc_1'c_2'' \dots c_n^{(n)}$ to the full aggregate $\Sigma \pm cc_1'c_2'' \dots c_n^{(n)}$. A glance at the sum of terms denoted by $c_k^{(i)}$ shows that by permuting the superior indices of $cc_1'c_2'' \dots c_n^{(n)}$, the superior indices of the α's are subjected to the same permutation, and that, on the other hand, when we permute the inferior indices of $cc_1'c_2'' \dots c_n^{(n)}$ it is the a's that are affected, the like permutation being given to the superior indices. Making the choice of the *superior* indices of the c's, let us permute them in every possible way, and to each term thus derived from $cc_1'c_2'' \dots c_n^{(n)}$ prefix the sign $+$ or $-$ according as its superior indices constitute a positive or negative permutation. By so doing the left-hand side of our identity becomes $\Sigma \pm cc_1'c_2'' \dots c_n^{(n)}$; and, owing to the consequent permutation of the superior indices of the a's, each term on the right-hand side gives rise to $1.2.3 \dots (n+1)$ terms whose signs are the same as the signs of the terms corresponding to them on the left-hand side;—in other words, each term $a_r a_s' a_t'' \dots a_w^{(n)} \,.\, a_r a_s' a_t'' \dots a_w^{(n)}$ gives rise to the compound term

$$a_r a_s' a_t'' \dots a_w^{(n)} . \Sigma \pm a_r a_s' a_t'' \dots . a_w^{(n)} .$$

We thus reach the result

$$\Sigma \pm cc_1'c_2'' \dots c_n^{(n)} = \mathrm{S}\big(a_r a_s' a_t'' \dots a_w^{(n)} \,.\, \Sigma \pm a_r a_s' a_t'' \dots . a_w^{(n)}\big) .$$

Although the number of terms on the right is the same as before, viz. $(p+1)^{n+1}$, arising from giving to each of the $n+1$ indices $r, s, t, \dots, w$ any one of the $p+1$ values $0, 1, 2, \dots, p$, it has now to be noticed that a goodly proportion of them must vanish because of the fact that $\Sigma \pm a_r a_s' a_t'' \dots a_w^{(n)} = 0$ when any two of its inferior indices are alike. The right-hand side will thus not be altered in substance if the summatory symbol be now taken to mean that $r, s, t, \dots, w$ are to be any $n+1$ of the $p+1$ indices $0, 1, 2, \dots, p$. If p be less than n it will be impossible to have $r, s, t, \dots, w$ all different, so that in that case the right-hand side must be 0. This is Jacobi's first proposition, and it constitutes his addition to the multiplication-theorem. His formal enunciation of it is (p. 309):—

"Sit

$$c_k^{(i)} = a^{(i)} a^{(k)} + a_1^{(i)} a_1^{(k)} + \dots + a_p^{(i)} a_p^{(k)} ,$$

quoties $p < n$ evanescit determinans

$$\Sigma \pm cc_1'c_2'' \dots . c_n^{(n)}." \qquad \text{(XVIII. 6)}$$

The consideration of the case when $p = n$ leads to his second proposition. The natural addendum is then made regarding the multiplication of more than two determinants of the same degree (p. 310):—

"Datis quotcunque eiusdem gradus determinantibus, eorum productum ut eiusdem gradus exhiberi posse determinans, cuius elementa expressiones sint rationales integrae elementorum determinantium propositorum." (XVII. 7)

The equally natural transition to the subject of the multiplication of two determinants of different degrees results in the proposition (p. 311):—

"Sit pro indicis i valoribus 0, 1, 2, ... , m,

$$c_k^{(i)} = \alpha^{(i)} a^{(k)} + \alpha_1^{(i)} a_1^{(k)} + \ . \ . \ . \ + \alpha_n^{(i)} a_n^{(k)} ,$$

pro indicis i valoribus maioribus quam m,

$$c_k^{(i)} = a_i^{(k)} + \alpha_{i+1}^{(i)} a_{i+1}^{(k)} + \alpha_{i+2}^{(i)} a_{i+2}^{(k)} + \ . \ . \ . \ . \ + \alpha_n^{(i)} a_n^{(k)} :$$

erit

$$\Sigma \pm \alpha \alpha_1' \dots . \alpha_m^{(m)} . \Sigma \pm \alpha a_1' \dots . a_n^{(n)} = \Sigma \pm c c_1' c_2'' \dots . c_n^{(n)} ." \qquad \text{(XVII. 8)}$$

Proposition IV. concerns the case where $p > n$. Proposition V. is but a corollary to the combined propositions I., II., IV., its subject being the effect of the specialisation

$$\alpha_k^{(i)} = a_k^{(i)} .$$

The enunciation is as follows (p. 312):—

"Posito

$$c_k^{(i)} = c_i^{(k)} = a^{(i)} a^{(k)} + a_1^{(i)} a_1^{(k)} + \ . \ . \ . \ + a_p^{(i)} a_p^{(k)} ,$$

sit determinans

$$\Sigma \pm c c_1' \dots . c_n^{(n)} = \mathrm{P} ;$$

ubi $p < n$ fit

$$\mathrm{P} = 0 ;$$

ubi $p = n$ fit

$$\mathrm{P} = \left\{ \Sigma \pm a a_1' \dots . a_n^{(n)} \right\}^2 ;$$

ubi $p > n$ fit

$$\mathrm{P} = \mathrm{S} \left\{ \Sigma \pm a_m a'_{m'} \dots . a_{m^{(n)}}^{(n)} \right\}^2 ,$$

siquidem pro indicibus inferioribus m, m' &c. sumuntur quilibet $n+1$ diversi e numeris 0, 1, 2 . . . . p." (XVIII. 7)

The two remaining sections (15 and 16) deal with a special system of simultaneous linear equations, interesting application being made to the theory of the Method of Least Squares—an application probably due to a suggestion of Binet's in his note of November, 1811.

It is important to note, in conclusion, that from one point of view Jacobi's memoir was but the introduction to two others of really greater importance, both treating of a special class of determinants. The first concerns determinants of the kind afterwards deservedly associated with his name, and bears the title "*De determinantibus functionalibus.*" It occupies the forty-one pages (pp. 319–359) immediately following the general memoir. The other, with the title "*De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum,*" treats of those determinants, first considered by Cauchy, in which the members of one set of indices represent powers, and to which the name *alternants* afterwards came to be assigned. It extends to twelve pages (pp. 360–371). The three memoirs together constitute an excellent treatise on the subject, and are known to have been markedly influential in spreading a knowledge of it among mathematicians.

The second and third memoirs, from the nature of their subject-matter, fall to be considered later. On the last page of the third memoir, however, where a possible simplification of a special determinant has to be effected, the general theorem on which the simplifying operation rests is enunciated; and as this theorem does not appear in the first memoir, it calls for attention now. The wording is:—

"Constat enim non mutari Determinans si singulis seriei horizontali terminis addantur earundem serierum verticalium termini multiplicati per quantitates quascunque, quae tamen pro omnibus eiusdem seriei horizontali terminis eaedem esse debent." (LIX.)

One cannot but wonder why the afterwards familiar fact regarding the effect of "increasing a row by a multiple of another row" was not formulated long before this date.

CAUCHY (1841).

[Note sur les diverses suites que l'on peut former avec des termes donnés. *Exercices d'analyse et de phys. math.*, ii. pp. 145–150; or *Œuvres complètes,* 2ᵉ Sér. xii.]

[Mémoire sur les fonctions alternées et sur les sommes alternées. *Exercices d'analyse et de phys. math.*, ii. pp. 151–159; or *Œuvres complètes,* 2ᵉ Sér. xii.]

[Mémoire sur les sommes alternées, connues sous le nom de résultantes. *Exercices d'analyse et de phys. math.*, ii. pp. 160–176; or *Œuvres complètes,* 2ᵉ Sér. xii.]

[Mémoire sur les fonctions différentielles alternées. *Exercices d'analyse et de phys. math.*, ii. pp. 176–187; or *Œuvres complètes,* 2ᵉ Sér. xii.]

From internal evidence there can be little doubt that this series of papers, containing the fundamental conceptions and salient propositions of the theory of determinants, was prompted by the appearance of Jacobi's memoirs, and by the consequent conviction that the work of 1812 had begun to bear fruit. The first paper, called a "note," is introductory, on the subject of signed permutations; the three others, called "memoirs," correspond to Jacobi's,—the first of them to Jacob's third, the second to Jacobi's first, and the third to Jacobi's second.

The note, although on so trite a subject as the division of permutations into positive and negative, is most interesting. Cauchy's original stand-point with regard to the subject is so far unaltered that the rule of signs specially known by his name is made fundamental, and all others deduced from it. The explanations preparatory for the rule are, however, on the lines of his paper of 1840, that is to say, it is *groups* and not *circular substitutions* that are spoken of. The preference is a little difficult to justify; for notwithstanding Cauchy's assertion that groups come naturally into evidence, the idea is far-fetched as compared with that of circular substitutions. He says (p. 145):—

"Si l'on compare une quelconque des nouvelles suites* à la première, on se trouvera naturellement conduit par cette comparaison à distribuer es divers termes

$$a,\ b,\ c,\ d,\ \ldots .$$

---

* *I.e.*, permutations of a, b, c, d, . . .

en plusieurs groupes, en faisant entrer deux termes dans un même groupe, toutes les fois qu'ils occuperont le même rang dans la première suite et dans la nouvelle, et en formant un groupe isolé de chaque terme qui n'aura pas changé de rang dans le passage d'une suite à l'autre."

The question of the natural order of ideas and the best mode of presentment is really, however, of small importance, for in application a *group* and a *circular substitution* are essentially the same. The difference is entirely one of stand-point, nomenclature, and notation. The permutation

$$e,\ a,\ b,\ d,\ c,\ g,\ f,$$

being in question, and comparison between it and the primitive permutation

$$a,\ b,\ c,\ d,\ e,\ f,\ g,$$

having been instituted, we are directed to form the members ("termes") of the permutation into groups, commencing to form a group with e and a, because they occupy like positions in the two permutations, putting b in the same group because it occupies the same position in the second permutation as one already in the group occupies in the first permutation, putting c in for the same reason, making d constitute a group by itself, and finally putting f and g together to form a third group. We are directed further, to write the members of each group in such an order that any member and the one following it may be found to occupy like positions in the primitive and derived permutations respectively. The result thus is

$$(a,\ e,\ c,\ b),\quad (d),\quad (f,\ g),$$

or $$(e,\ c,\ b,\ a),\quad (d),\quad (g,\ f),$$

or $$.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .,$$

it being possible to write the first group in four ways, and the last in two. Now all this is nothing more than an unreasoning way of arriving at the circular substitutions which are necessary for the derivation of the given permutation from the primitive one. Cauchy himself, indeed, in pointing out that there would only be one way of writing a group if the members were disposed in a circumference instead of in a straight line, says:—"C'est par ce motif que dans le tome x du *Journal de l'École*

Polytechnique j'ai désigné sous le nom de *substitution circulaire* l'opération qui embrasse le système entier des remplacements indiqués par un même groupe." It must be borne in mind, however, that not only the operation, but the symbol of the operation, was so denoted, and such being the case, we may then very pertinently ask, What is a group in Cauchy's usage but the symbol of a circular substitution?

The peculiarity of using the number of groups to separate the various permutations of a, b, c, d, ... into two classes makes its appearance in the following sentence (p. 147):—

"De plus, ces mêmes suites ou arrangements se partageront en deux classes bien distinctes, la comparaison de chaque nouvel arrangement au premier

$$a,\ b,\ c,\ d,\ \ldots\ldots$$

pouvant donner naissance à un nombre pair ou à un nombre impair de groupes."

Of course, the primitive permutation is looked upon as having its groups also, viz., one for every letter in the permutation.

Then comes the important proposition—*The interchange of two letters increases or diminishes the number of groups (substitution-cycles) by unity.* In proving it the two letters are first taken in different groups,

$$(a,b,c,\ \ldots,\ h,k),\quad (l,m,n,\ \ldots,\ r,s);$$

and since any member of a group may occupy the first place, the letters a and l are fixed upon. Now what the groups imply is that the letters

$$a,\ b,\ c,\ \ldots\ldots\ h,\ k,\ l,\ m,\ n,\ \ldots\ldots\ r,\ s$$

in the primitive permutation are changed into

$$b,\ c,\quad \ldots\ldots\ k,\ a,\ m,\ n,\quad \ldots\ldots\ s,\ l$$

respectively to form the given permutation. If therefore in the given permutation the letters a and l be interchanged, the new permutation so obtained will be got from the primitive by changing

$$a,\ b,\ c,\ \ldots,\ h,\ k,\ l,\ m,\ n,\ \ldots,\ r,\ s$$

into

$$b,\ c,\ \ldots\ldots\ k,\ l,\ m,\ n,\ldots\ldots,\ s,\ a;$$

that is to say, by the changes indicated by the single group

$$(a,b,c, \ldots, h,k,l,m,n, \ldots, r,s).$$

The interchange of two letters belonging to different groups is thus seen to reduce the number of groups by one. On the other hand, it is clear that had this single group belonged to the given permutation, the interchange of two letters, a and l say, would have had the effect of breaking up the group into two,

$$(a,b,c, \ldots, h,k) \text{ and } (l,m,n, \ldots, r,s).$$

The theorem is thus established. (III. 36)

It is next pointed out that the transformation of the primitive permutation into any other may be accomplished by interchanges only, because by this means any given letter may be made to occupy the first place, then any other given letter to occupy the second place, and so on. From this also it follows that any system of circular substitutions may be replaced by a system of interchanges. Should the transformation of one permutation into another be effected by interchanges, the number of these will be even or odd according as the two permutations belong to the same or different classes; for, by the above theorem, every interchange makes only one group more or one group less, and consequently the total number of interchanges, and the net increase or diminution of the number of groups, must be both even or both odd. The counting of *interchanges* may thus be substituted for the counting of cycles. (III. 37)

Finally, Cramer's rule is introduced, in which, as we know, it is neither cycles nor interchanges that are counted, but *inverted-pairs*, or, as Cauchy, like Gergonne, calls them, *inversions*. To establish the rule, it is clear that two courses were open, viz., to connect inversions directly with cycles or to connect them with interchanges. The latter course is taken, the requisite connecting theorem being that *the interchange of two elements of a permutation increases or diminishes the number of inversions by an odd number*, an odd number of interchanges thus corresponding to an odd number of inversions, and an even to an even. The proof is not direct, like Rothe's, being effected with the help of a fourth related entity, the difference-product. The order of thought in it is as follows:—

If we define the difference-product of the primitive permutation $a, b, c, d, \ldots$ to be

$$(a-b)(a-c) \ldots . (b-c) \ldots . ,$$

then it is clear that in the difference-product of any derived permutation there will be found exactly as many factors with changed sign as there are inversions of order in the permutation. A change of sign in the difference-product thus becomes a test for the existence of an odd number of inversions, and consequently, instead of the theorem just enunciated, it will suffice to show that *the interchange of two elements of a permutation alters the sign of the difference-product.* This Cauchy says must be true, for, the elements being h and k, it is manifest that the factor which involves them both,

$$h-k \quad \text{or} \quad k-h,$$

must change sign, but that the factors which involve them and any third element s constitute a partial product

$$(h-s)(k-s) \quad \text{or} \quad (h-s)(s-k),$$

the sign of which cannot change. (III. 38)

Of the three memoirs, the first and third, like Jacobi's third and second, do not at present require attention. A slight reference to the first—on alternating functions—is, however, necessary, because Cauchy, unlike Jacobi, makes determinants a special class of alternating functions, and it is therefore of importance to see the exact position he assigns to them. It will be remembered that in 1812 he partitioned symmetric functions into permanent and alternating, and made determinants a class of the latter; that is to say, his scheme of logical relationship was

$$\text{Functions}\left\{\begin{array}{l} \text{(A) Symmetric}\left\{\begin{array}{l} (a)\text{ Alternating}\left\{\begin{array}{l} (a)\text{ Determinants.} \\ \\ \end{array}\right. \\ (b)\text{ Permanent} \end{array}\right. \\ \text{(B)} \end{array}\right.$$

The memoirs we have now come to indicate a departure from this, both verbal and substantial. The change is made too without any reason being assigned; indeed, there is not even a word to imply that any change had taken place. Alternating functions are, as in his *Cours d'analyse*, put on the same level as symmetric functions; the term *permanent* is dispensed with; a new entity, *alternating aggregates*, is introduced; what were formerly called determinants are made a class of these alternating aggregates; and for the name determinant *resultant* is substituted. The scheme of relationship is thus transformed into

$$\text{Functions}\begin{cases}\text{(A) Alternating}\begin{cases}(a)\ \text{Alternating Aggregates}\begin{cases}(\alpha)\ \text{Resultants.}\\ (\beta)\end{cases}\\ (b)\end{cases}\\ \text{(B) Symmetric}\\ \text{(C)}\end{cases}$$

Neither scheme, we must at the same time remember, is really as simple as here indicated, being complicated by the fact that a function may be alternating in more than one way. This is brought out much more explicitly and clearly in the present memoirs than in that of 1812, as the following quotations will show. We have first of all (p. 151), an *alternating function of several variables.*

"Une fonction alternée de plusieurs variables x, y, z, . . . , est celle qui change de signe, en conservant, au signe près, la même valeur lorsqu'on échange deux de ces variables entre elles."

Next we have an *alternating function with respect to several indices* (p. 155):—

"Quelquefois on représente ces mêmes variables par une seule lettre affectée de divers indices

$$0,\ 1,\ 2,\ 3,\ \ldots\ldots,\ n,$$

et l'on peut dire alors que la fonction ou la somme dont il s'agit est *alternée par rapport à ces indices.* Ainsi, par exemple, le produit

$$(x_0-x_1)(x_0-x_2)(x_1-x_2)$$

est une fonction alternée par rapport aux variables

$$x_0,\ x_1,\ x_2,$$

ou, ce qui revient au même, par rapport aux indices

$$0,\ 1,\ 2."$$

This example being an alternating function according to the first definition, it would seem that here we have a mere abbreviation or variation of language. There are, however, it must be borne in mind, functions which are alternating with respect to indices, and are not alternating according to the first definition. For example, any determinant, like

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$$

is alternating with respect to all the indices involved, but is not alternating with respect to all or any other number of the variables a_1,a_2,a_3, b_1,b_2,b_3, c_1,c_2,c_3. Strange to say, Cauchy makes no mention of this, but goes on to a third definition, by means of which alternating functions are made in another way to include determinants. He says (p. 156):—

"On pourrait obtenir aussi des fonctions qui seraient *alternées par rapport à diverses suites*, c'est à dire, des fonctions qui auraient la propriété de changer de signe, en conservant, au signe près, la même valeur quand on échangerait entre eux les termes correspondants de ces mêmes suites. Considérons, par exemple, m suites différentes composées chacune de n termes qui se trouvent représentés, pour la première suite, par

$$x_0,\ x_1,\ \ldots\ldots,\ x_{n-1},$$

pour la seconde suite, par

$$y_0,\ y_1,\ \ldots\ldots,\ y_{n-1},$$

pour la troisième suite, par

$$z_0,\ z_1,\ \ldots\ldots,\ z_{n-1},$$

etc., ... ; et soit

$$\mathbf{f}(x_0,\ x_1,\ \ldots\ldots,\ x_{n-1};\ y_0,\ y_1,\ \ldots\ldots,\ y_{n-1};\ z_0,\ z_1,\ \ldots\ldots,\ z_{n-1};\ \ldots\ldots)$$

une fonction donnée de ces divers termes. Si à cette fonction l'on ajoute toutes celles que l'on peut en déduire, à l'aide d'un ou de plusieurs échanges opérés entre les lettres

$$x,\ y,\ z,\ \ldots\ldots$$

prises deux à deux, chacune des nouvelles fonctions étant prise avec le signe + ou avec le signe −, suivant qu'elle se déduit de la première par un nombre pair, ou par un nombre impair d'échanges; le résultat de cette addition sera une somme alternée par rapport aux suites dont il s'agit."

It is a little unfortunate that this definition proceeds on different lines from the others, being rather indeed a *rule for the forma-*

tion of an alternating function with respect to several sets of variables than a definition of such a function. It would have been much more appropriate and instructive to have said that a function was called *alternating with respect to two or more sets of the same number of variables* when the interchange of each member of a set with the corresponding member of another set altered the function in sign merely. Examples like the following could then have been given to make the two usages of the term perfectly clear, and to show the exact relation between them. To illustrate the first usage, the expressions

$$ac-bc\,,$$
$$(a-b)(c-d)\,,$$
$$(a-b)(a-c)(b-c)\,,$$

might be taken, where $ac-bc$ is an alternating function with respect to the variables a, b; $(a-b)(c-d)$ an alternating function with respect to a, b, and also with respect to c, d; and $(a-b)(a-c)(b-c)$ an alternating function with respect to a, b, with respect to a, c, and with respect to b, c, or shortly, an alternating function of all its variables. On the other hand, the expressions

$$a^2b-c^2d\,,$$
$$a\,b-c\,d\,,$$

would illustrate the second usage; a^2b-c^2d being an alternating function with respect to the sets of variables ab, cd; and $ab-cd$ an alternating function with respect to the sets ab, cd, and also with respect to the sets ac, bd. In a word, the alteration which produces change of sign is, in the case of the first usage, interchange of two individual elements; in the case of the second usage it is interchange of two ranks or sets of elements.

The entity to which the new name *somme alternée* is given is explained as follows (p. 160):—

"Soit

$$\mathfrak{f}(x,\ y,\ z,\ \ldots\ldots)$$

une fonction quelconque de n variables

$$x,\ y,\ z,\ \ldots\ldots$$

et ajoutons à cette fonction toutes celles qu'on peut en déduire par la transposition des variables, ou, ce qui revient au même, par un ou plusieurs échanges opérés chacun entre deux variables seulement, chaque nouvelle fonction étant prise avec le signe + ou le signe −, suivant qu'elle se déduit de la première à l'aide d'un nombre pair ou impair de semblables échanges. La somme s ainsi obtenue sera la *somme alternée* que nous représentons par la notation

$$\mathrm{S}[\pm \mathrm{f}(x, y, z, \ldots)].$$

On trouvera, par exemple, en supposant $n=2$,

$$s = \mathrm{f}(x, y) - \mathrm{f}(y, x);$$

en supposant $n=3$,

$$s = \mathrm{f}(x,y,z) - \mathrm{f}(x,z,y) + \mathrm{f}(y,z,x) - \mathrm{f}(y,x,z) + \mathrm{f}(z,x,y) - \mathrm{f}(z,y,x),$$

etc."

The only matter now remaining for explanation is the mode of transition from *sommes alternées* to *résultantes*, the difficult point being, as in the memoir of 1812, to include all kinds of the latter as special cases of the former. The two pages which Cauchy devotes to the subject are curious to read, and deserve a little attention. He says (p. 161):—

"Concevons maintenant que la fonction

$$\mathrm{f}(x,y,z, \ldots)$$

se reduise au produit de divers facteurs dont chacun renferme une suite des variables

$$x, y, z, \ldots .$$

en sorte que l'on ait, par exemple,

$$\mathrm{f}(x,y,z, \ldots) = \phi(x)\chi(y)\psi(z) \ldots .$$

alors, pour obtenir la somme alternée

$$s = \mathrm{S}[\pm\phi(x)\chi(y)\psi(z) \ldots]$$

il suffira . . ."

and having shown the mode of formation, and given the examples

$$s = \phi(x)\chi(y) - \phi(y)\chi(x),$$
$$s = \phi(x)\chi(y)\psi(z) - \phi(x)\chi(z)\psi(y) + \ldots$$

he adds

"Les sommes de cette espèce sont celles que M. Laplace a désignées sous le nom de *résultantes.*"

In regard to this the first comment clearly must be that it is not a little misleading. The sums referred to are only a very special class of those functions which Laplace called resultants; they belong, in fact, to that peculiar type for which in later times the name *alternant* was coined. In the second place, Cauchy's virtual renunciation of his own word "determinant" must be noted,—a renunciation all the more curious when we consider that the word had now been adopted by Jacobi, and had thereby become the recognised term in Germany. It may be that Laplace's word "resultant" had proved more acceptable in France, and that Cauchy merely bowed to the fact; but there is little or no evidence to support this.*

In the paragraph following the above Cauchy proceeds, as it were, to rectify matters. He says (p. 162):—

"Les formes des fonctions désignées par

$$\phi(x),\ \chi(x),\ \psi(x),\ \text{etc.}$$

étant arbitraires, aussi bien que les variables

$$x,\ y,\ z,\ \ldots,$$

permettent aux divers termes qui composent le tableau (2) d'acquérir des valeurs quelconques, et représentons ces variables à l'aide de lettres diverses

$$x,\ y,\ z,\ \ldots\ldots,\ t$$

affectés d'indices différents

$$0,\ 1,\ 2,\ \ldots,\ n-1,$$

dans les diverses lignes verticales. Alors, au lieu du tableau (2), on obtiendra le suivant

$$(5)\qquad \left\{\begin{array}{l} x_0,\ x_1,\ x_2,\ \ldots\ldots,\ x_{n-1} \\ y_0,\ y_1,\ y_2,\ \ldots\ldots,\ y_{n-1} \\ z_0,\ z_1,\ z_2,\ \ldots\ldots,\ z_{n-1} \\ \ldots\ldots\ldots\ldots \\ t_0,\ t_1,\ t_2,\ \ldots\ldots,\ t_{n-1} \end{array}\right.$$

* Liouville, in a paper published in the same year as Cauchy's memoirs, uses *resultant*, but adds in a footnote, "Au lieu du mot *résultante*, les géomètres emploient souvent le mot *déterminant*" (*Liouville's Journ.*, vi. p. 348).

et la résultante s des termes dans ce dernier tableau sera

$$s = \mathrm{S}[\pm x_0 y_1 z_2 \ldots t_{n-1}].\text{"}$$

The general determinant is doubtless here reached, but the transition requisite for the attainment of it, viz., from $\phi(x)$, $\chi(x)$, $\psi(x)$, ... to the perfectly independent x_0, x_1, x_2, ... is not made without considerable strain. This is all the more surprising, too, when we consider, that a much less troublesome and less objectionable mode of bringing determinants under alternating aggregates lay ready to Cauchy's hand. Bearing in mind the definition given above, of *fonctions alternées par rapport à diverses suites*, we see that a determinant of the n^{th} order could have been made to appear as an alternating function with respect to n ranks of n variables each. For example, the determinant

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2\,,$$

could have been introduced as a function alternating with respect to any two of the three ranks,

$$\begin{array}{ccc} a_1 & a_2 & a_3\,, \\ b_1 & b_2 & b_3\,, \\ c_1 & c_2 & c_3\,; \end{array}$$

and indeed, as we know, it is alternating also with respect to any two of the ranks

$$\begin{array}{ccc} a_1 & b_1 & c_1\,, \\ a_2 & b_2 & c_2\,, \\ a_3 & b_3 & c_3\,, \end{array}$$

that is to say, according to another phrase of Cauchy's, used above, it is alternating with respect to the indices, 1, 2, 3.

The fourteen pages (pp. 163–176) which follow, are taken up with the properties of determinants as thus defined and with the application of them to the solution of simultaneous linear equations. Most of the matter is already familiar to us, and may be altogether passed over. One of the theorems it is necessary to give verbatim, not because of its importance, but because it serves to make evident the untenable position Cauchy had taken up in so peculiarly bringing determinants under the head of alternating aggregates. The theorem is (p. 164):—

"Si, avec les variables comprises dans le tableau (5), on forme une fonction entière, du degré n, qui offre, dans chaque terme, n facteurs dont un seul appartienne à chacune des suites horizontales de ce tableau, et qui soit alternée par rapport à ces mêmes suites, la fonction entière dont il s'agit devra se réduire, au signe près, à la résultante s."

This not only justifies the definition proposed above to be substituted for Cauchy's, but it also entitles us to say that Cauchy having started by including determinants among alternating functions of one kind, viz., functions alternating with respect to every pair of n variables, soon succeeds in showing that they are alternating functions of an entirely different kind, viz., functions alternating with respect to every pair of n *ranks* of variables.

The only other noteworthy matter is a theorem in regard to the solution of a set of simultaneous equations. Viewing the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= \xi \\ a_2x + b_2y + c_2z &= \eta \\ a_3x + b_3y + c_3z &= \zeta \end{aligned}\right\}$$

as giving each of the three variables ξ, η, ζ, in terms of the other three x, y, z, we see that on solving for x, y, z, we obtain a converse system, that is to say, a system giving each of the three x, y, z, in terms of ξ, η, ζ. The latter system is, as we know,

$$\left.\begin{aligned} x &= \frac{A_1}{\Delta}\xi + \frac{A_2}{\Delta}\eta + \frac{A_3}{\Delta}\zeta, \\ y &= \frac{B_1}{\Delta}\xi + \frac{B_2}{\Delta}\eta + \frac{B_3}{\Delta}\zeta, \\ z &= \frac{C_1}{\Delta}\xi + \frac{C_2}{\Delta}\eta + \frac{C_3}{\Delta}\zeta, \end{aligned}\right\}$$

where Δ is the determinant of the original system and

$$A_1,\ B_1,\ C_1,\ A_2,\ \ldots\ ,$$

are the cofactors in Δ of a_1, b_1, c_1, a_2, ... , respectively. Multiplying the determinants of the two systems, we obtain the determinant of the quantities

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{matrix}$$

Hence (p. 176):—

"Si, n variables

$$x, y, z, \ldots, t,$$

étant liées à n autres variables

$$\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots, \mathrm{t},$$

par n équations linéaires, on suppose les unes exprimées en fonctions linéaires des autres, et réciproquement; les deux résultantes formées avec les coefficients que renfermeront ces fonctions linéaires dans les deux hypothèses, offriront un produit équivalent à l'unité." (XXI. 4)

CHAPTER X.

DETERMINANTS IN GENERAL, FROM 1813 TO 1841: A RETROSPECT.

THE characteristics of this period are best brought out by comparison with those of the preceding period, it being carefully borne in mind, in making the comparison, that the two are markedly unequal in length, the period of pioneering, as we may term it, extending to 120 years, and the next to only about 30.

In the first place, then, the evidence shows that as time went on there was considerable increase of interest in the subject, and a more widely spread knowledge of it; for, whereas to the longer period there belong 21 papers by 16 writers, for the shorter period the corresponding numbers are 38 and 19. Among the 19 writers, too, are represented nationalities which had previously not put in an appearance, viz., English, Italian, and Polish. In both periods the French language greatly predominates in the writings, even although in the second period the number of German contributors is about equal to the number of French. The details on this point are:—

	(1693–1812)	(1813–1841)
French, - - - -	16	17
Latin, - - - -	3	9
German, - - - -	2	6
English, - - - -	—	5
Italian, - - - -	—	1

The Latin papers are mainly those of Germans, Jacobi alone being responsible for 8 in the later period.

In the second place, we have proof that the early period was by far the more fruitful in original results. The pioneers had mapped out most of the prominent features of the new country; their successors had consequently to concern themselves in a considerable degree with filling in the details. During the second period one finds the fundamental propositions of the first period reproduced in new varieties of form; also, there are not awanting new proofs, extensions, and specialisations of old theorems; but of absolutely fresh departures there are comparatively few. An examination of the results numbered XLIV.–LIX. will show the character of these departures. It will be seen that they are due to Desnanot, Scherk, Schweins, Jacobi, Sylvester, and Cauchy. The most notable name of the period is Jacobi's, and next to it perhaps that of Schweins. There is no one name, however, which stands out in this period so conspicuously as Cauchy's does in the first period. Sylvester, unlike the others, it must be remembered, was only beginning his career, and we have yet to see him in the fulness of his power. It is worthy of note, too, that the striking figure of the first period is not by any means dwarfed in the second, his name occurring five times in the chronological list, and his papers at the close of the period showing much of his old insight and vigour.

In the next place, the second period contrasts with the first in that during it important work was done on the subject of *special forms* of determinants. This will become more apparent after consideration of the chapters which follow. It will then be seen that of the five most important forms there dealt with, viz., those subsequently known as Axisymmetric Determinants, Alternants, Jacobians, Skew Determinants, Orthogonants, three had their origin during the second period; and, further, that although the two others originated during the first period the greater bulk of the work done on them belongs to the second. Here, again, the noteworthy names are those of Jacobi, Cauchy, and Schweins.

Lastly, it having been noted in the retrospect of the first period that the subject of determinants was almost entirely a creation of the French intellect, we must not fail to take cognisance now of the fact that in the second period the pre-eminence belongs to Germany, France however taking still a fairly good second place.

To aid in bringing all these facts more clearly home to the reader a table similar to that supplied for the elucidation of the first period (see page 132) is annexed. The cautions formerly given as regards the imperfections of such a table and the care consequently necessary in using it are expected to be again borne in mind.

TABLE—SHOWING THE ADVANCE OF THE

	Gergonne, 1813.	Garnier, 1814.	Wronski, 1815.	Desnanot, 1819.	Cauchy, 1821.	Scherk, 1825.	Schweins,	Jacobi, 1827.	Reiss, 1829.	Cauchy,	Jacobi,
II.											
III.	134-5					156, 158			179		
VI.				145		159	162				
VII.							160		179		
VIII.											
IX.							162		182(a)		
XI.							161				
XII.	135			143			165				
XIII.				148		151					
XIV.							164				
XV.			136				160				
XVII.											
XVIII.											
XX.										188	
XXI.											
XXIII.				140, 142, 145			171		184, 185		
XXIV.											
XXIX.											
XXXVII.							162				
XL.											
1813. Gergonne,											
1814. Garnier,											
1815. Wronski,											
1819. Desnanot, XLIV.				139, 140, 142, 145					185		
XLV.				145			171				
1821. Cauchy,											
1825. Scherk, XLVI.						154					
XLVII.						154					
XLVIII.						158					
Schweins, XLIX.							167, 169, 170(a), 171				
L.							174				
1827. Jacobi, LI.								178			
1829. Reiss,											
Cauchy,											
Jacobi, LII.											189, 192, 193
Minding,											
1831. Drinkwater,											
1832. Mainardi,											
1833. Jacobi,											
1834. Jacobi, LIII.											
1835. Jacobi,											
1836. Grunert,											
1837. Lebesgue,											
1838. Reiss,											
1839. Catalan,											
Sylvester,											
Molins,											
1840. Sylvester, LIV.											
Richelot,											
Cauchy,											
1841. Sylvester,											
Craufurd,											
Cauchy, LV.											
Jacobi, LVI.											
LVII.											
LVIII.											
LIX.											
Cauchy.											

Minding,	Drinkwater, '31.	Mainardi, 1832.	Jacobi, 1833.	Jacobi, 1834.	Jacobi, 1835.	Grunert, 1836.	Lebesgue, 1837.	Reiss, 1838.	Catalan, 1839.	Sylvester,	Molins,	Sylvester, 1840.	Richelot,	Cauchy,	Sylvester, 1841.	Craufurd,	Cauchy,	Jacobi,	Cauchy.
	199					218, 219											250 249(2)	256, 257, 258(2), 260 261	276(2), 277
			212																
					214					228									
							220												
								221 222											
197																			
			212	213		216			225	231	236								
																		261	
										228									
		204																271(2)	
																		270, 271	
197			208 211		214													268	
																			285
										233								264, 265 256	
		201																	
							220(2)											262	
	199 199							223											
			209																
				213															
												237, 238	239	242	244				
																	249, 250(3), 252		
																		262 263 265 272	

CHAPTER XI.

AXISYMMETRIC DETERMINANTS, FROM 1773 TO 1841.

ATTENTION has already been drawn to certain identities of Lagrange's which might possibly be viewed as contributions to the theory of determinants. Among these were the following published in 1773:—

$$\begin{aligned}(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'')^2 \\ = \quad & (x^2+y^2+z^2)(x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2) \\ & + 2(xx'+yy'+zz')(xx''+yy''+zz'')(x'x''+y'y''+z'z'') \\ & - (x^2+y^2+z^2)(x'x''+y'y''+z'z'')^2 \\ & - (x'^2+y'^2+z'^2)(xx''+yy''+zz'')^2 \\ & - (x''^2+y''^2+z''^2)(xx'+yy'+zz')^2 ;\end{aligned}$$

$$\begin{aligned}(y'z''-y''z')^2 + (z'x''-z''x')^2 + (x'y''-x''y')^2 \\ = (x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2) - (x'x''+y'y''+z'z'')^2 ;\end{aligned}$$

and

$$\begin{aligned}(pr-q^2)(\mathrm{M}n-\mathrm{N}m)^2 = (p\mathrm{M}^2+2q\mathrm{M}m+rm^2)(p\mathrm{N}^2+2q\mathrm{N}n+rn^2) \\ - (p\mathrm{MN}+q\mathrm{M}n+q\mathrm{N}m+rmn)^2.\end{aligned}$$

Four of the expressions here occurring would doubtless at a later date have been viewed as axisymmetric determinants, and in Cayley's notation of 1841 would have been written

$$\begin{vmatrix} x^2+y^2+z^2 & xx'+yy'+zz' & xx''+yy''+zz'' \\ xx'+yy'+zz' & x'^2+y'^2+z'^2 & x'x''+y'y''+z'z'' \\ xx''+yy''+zz'' & x'x''+y'y''+z'z'' & x''^2+y''^2+z''^2 \end{vmatrix}, \text{ etc.;}$$

but a reference to the original papers, already described, will make it almost perfectly certain that Lagrange did not view them in this light.

The like is true of Gauss (1801) who discovered the next case of the third of the preceding identities.

ROTHE (1800).

[Ueber Permutationen, in Beziehung auf die Stellen ihrer Elemente. Anwendung der daraus abgeleiteten Sätzen auf das Eliminations-problem. *Sammlung combinatorisch-analytischer Abhandlungen, herausg. v. C. F. Hindenburg*, ii. pp. 263–305.]

The position of Rothe was quite different from that of Lagrange and Gauss, as his paper dealt explicitly with determinants (or, rather, with the functions afterwards known as determinants), and the case of axisymmetry is definitely referred to, although not by name.

His one theorem may be illustrated by the case where the number of given equations is 4, and is then to the effect that if we have

$$\left.\begin{array}{l} ax_1 + bx_2 + cx_3 + dx_4 = s_1 \\ bx_1 + ex_2 + fx_3 + gx_4 = s_2 \\ cx_1 + fx_2 + hx_3 + ix_4 = s_3 \\ dx_1 + gx_2 + ix_3 + jx_4 = s_4 \end{array}\right\},$$

where the array of coefficients on the left is axisymmetric, then the same peculiarity of axisymmetry must make its appearance in the derived set which gives each of the x's in terms of the four s's.

Starting with the more general set of n equations

$$\left.\begin{array}{l} 11.x_1 + 12.x_2 + \ldots + 1n.x_n = s_1 \\ 21.x_1 + 22.x_2 + \ldots + 2n.x_n = s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ n1.x_1 + n2.x_2 + \ldots + nn.x_n = s_n \end{array}\right\},$$

and denoting the determinant formed from the coefficients on the left by N, and the cofactor in N of any coefficient pq by fpq, he proves in Laplace's method that

$$\left.\begin{array}{l} \mathrm{f}11.s_1 + \mathrm{f}21.s_2 + \ldots\ldots + \mathrm{f}n1.s_n = \mathrm{N}.x_1 \\ \mathrm{f}12.s_1 + \mathrm{f}22.s_2 + \ldots\ldots + \mathrm{f}n2.s_n = \mathrm{N}.x_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{f}1n.s_1 + \mathrm{f}2n.s_2 + \ldots\ldots + \mathrm{f}nn.s_n = \mathrm{N}.x_n \end{array}\right\}$$

where, be it observed, the coefficients of s_1 are not the cofactors of the coefficients x_1 in the original set of equations but the cofactors of the coefficients of $x_1, x_2, \ldots, x_n$ in the first equation of that set: in other words, the first column of coefficients in the derived set of equations corresponds to the first row of coefficients in the original set. Then taking another set of n equations having the same coefficients 11, 12, ... differently disposed, viz.,

$$\left.\begin{array}{l} 11.y_1 + 21.y_2 + \ldots\ldots + n1.y_n = v_1 \\ 12.y_1 + 22.y_2 + \ldots\ldots + n2.y_n = v_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 1n.y_1 + 2n.y_2 + \ldots\ldots + nn.y_n = v_n \end{array}\right\},$$

but where of course the determinant of the coefficients is in substance the same as before, and therefore denotable by N, and where consequently the cofactors of the elements of which the determinant is composed are also the same as before, he proves, rather unnecessarily, that

$$\left.\begin{array}{l} \mathrm{f}11.v_1 + \mathrm{f}12.v_2 + \ldots\ldots + \mathrm{f}1n.v_n = \mathrm{N}.y_1 \\ \mathrm{f}21.v_1 + \mathrm{f}22.v_2 + \ldots\ldots + \mathrm{f}2n.v_n = \mathrm{N}.y_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{f}n1.v_1 + \mathrm{f}n2.v_2 + \ldots\ldots + \mathrm{f}nn.v_n = \mathrm{N}.y_n \end{array}\right\}.$$

In this way it is made to appear that the coefficients of the one set of derived equations are the same as the coefficients of the other set of derived equations, the difference in the arrangement of them being exactly the difference observable in regard to the primitive sets.

From this he passes to the case where the array of coefficients of the primitive set of equations possesses the property of axisymmetry, his words being (p. 301)—

"Ist endlich für jedes p und q, $pq = qp$, oder ist bey den gegebenen Gleichungen, für jedes m, die $m$te Horizontalreihe der Coefficienten mit der $m$ten Verticalreihe derselben einerley; die Horizontalreihen nehmlich von oben herab, und die Verticalreihen, von der Linken nach der Rechten zu gerechnet, so ist auch allgemein $\mathfrak{f}pq = \mathfrak{f}qp$, oder die $m$te Horizontalreihe der Coefficienten, mit der $m$ten Verticalreihe derselben, auch bey den Auflösungsgleichungen einerley."

It may be noticed in passing that as the determinant of the coefficients in the derived set of equations is the conjugate of the adjugate of the determinant of the original set, there is involved in Rothe's proposition the well-known proposition of later times, viz., that *the adjugate of an axisymmetric determinant is also axisymmetric.*

BINET (1811).

[Mémoire sur la théorie des axes conjugués .... *Journ. de l'École Polytechnique*, ix. (pp. 41–67), pp. 45, 46.]

[Sur quelques formules d'algèbre, et sur leur application à des expressions qui ont rapport aux axes conjugués des corps. *Nouv. Bull. des Sciences par la Société Philomatique*, ii. pp. 389–392.]

With Binet we have a recurrence to those axisymmetric determinants which appear as equivalents to second powers of determinants or to sums of second powers. His theorems

$$\begin{vmatrix} mx^2+m_1x_1^2+m_2x_2^2+\ldots & mxy+m_1x_1y_1+m_2x_2y_2+\ldots & mxz+m_1x_1z_1+m_2x_2z_2+\ldots \\ mxy+m_1x_1y_1+m_2x_2y_2+\ldots & my^2+m_1y_1^2+m_2y_2^2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots \\ mxz+m_1x_1z_1+m_2x_2z_2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots & mz^2+m_1z_1^2+m_2z_2^2+\ldots \end{vmatrix}$$

$$= mm_1m_2\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix}^2 + mm_1m_3\begin{vmatrix} x & x_1 & x_3 \\ y & y_1 & y_3 \\ z & z_1 & z_3 \end{vmatrix}^2 + \ldots\ldots;$$

$$\begin{array}{c|ccc|} & g & h & i \\ g & mx^2+m_1x_1^2+\ldots & mxy+m_1x_1y_1+\ldots & mxz+m_1x_1z_1+\ldots \\ h & mxy+m_1x_1y_1+\ldots & my^2+m_1y_1^2+\ldots & myz+m_1y_1z_1+\ldots \\ i & mxz+m_1x_1z_1+\ldots & myz+m_1y_1z_1+\ldots & mz^2+m_1z_1^2+\ldots \end{array}$$

$$= mm_1 \begin{vmatrix} g & x & x_1 \\ h & y & y_1 \\ i & z & z_1 \end{vmatrix}^2 + mm_2 \begin{vmatrix} g & x & x_2 \\ h & y & y_2 \\ i & z & z_2 \end{vmatrix}^2 + m_1m_2 \begin{vmatrix} g & x_1 & x_2 \\ h & y_1 & y_2 \\ i & z_1 & z_2 \end{vmatrix}^2 + \ldots ;$$

$$\begin{vmatrix} u^2+u_1^2+\ldots & ux+u_1x_1+\ldots & uy+u_1y_1+\ldots & uz+u_1z_1+\ldots \\ ux+u_1x_1+\ldots & x^2+x_1^2+\ldots & xy+x_1y_1+\ldots & xz+x_1z_1+\ldots \\ uy+u_1y_1+\ldots & xy+x_1y_1+\ldots & y^2+y_1^2+\ldots & yz+y_1z_1+\ldots \\ uz+u_1z_1+\ldots & xz+x_1z_1+\ldots & yz+y_1z_1+\ldots & z^2+z_1^2+\ldots \end{vmatrix}$$

$$= \begin{vmatrix} u & u_1 & u_2 & u_3 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \end{vmatrix}^2 + \begin{vmatrix} u & u_1 & u_2 & u_4 \\ x & x_1 & x_2 & x_4 \\ y & y_1 & y_2 & y_4 \\ z & z_1 & z_2 & z_4 \end{vmatrix}^2 + \ldots .$$

These all indicate most important advances, and, be it noted, the last of them is given by its author as the third of a series the law of which he considered "facile à saisir." In view of this, and the fact that in the following year he published his great memoir containing the multiplication-theorem in all its generality, we are bound to conclude that whatever credit in the latter he must share with Cauchy, the axisymmetric case of it is entirely his own.

Further details need not be given, as this has already been done when dealing with determinants in general.

JACOBI (1827).

[Ueber die Hauptaxen der Flächen der zweiten Ordnung. *Crelle's Journal*, ii. pp. 227–233; or *Werke*, iii. pp. 45–53.]

[De singulari quadam duplicis integralis transformatione. *Crelle's Journal*, ii. pp. 234–242; or *Werke*, iii. pp. 55–66.]

In these two papers, which owe their inspiration to the famous memoir* of Gauss on the "Determinatio Attractionis . . . ," Jacobi concerns himself with two problems of transformation,

* *Commentationes societatis regiæ scientiarum Gottingensis recentiores*, iv. (1818): or Gauss, *Werke*, iii. pp. 331–355. For abstract see *Göttingische gelehrte Anzeigen* (1818, Feb.), pp. 233–237: or *Werke*, iii. pp. 357–360.

the first of which explicitly deals with the transformation of the ternary quadric

$$Ax^2 + By^2 + Cz^2 + 2ayz + 2bzx + 2cxy$$

into the form

$$L\xi^2 + M\eta^2 + N\zeta^2,$$

and the other implicitly with the corresponding change in the case of a quaternary quadric. The papers will be fully discussed when we come to deal with "determinants of an orthogonal substitution." It suffices for the present to note that in the first Jacobi virtually gives as an equivalent for the axisymmetric determinant which we should now write in the form

$$\begin{vmatrix} x-A & x\cos\nu-c & x\cos\mu-b \\ x\cos\nu-c & x-B & x\cos\lambda-a \\ x\cos\mu-b & x\cos\lambda-a & x-C \end{vmatrix}$$

the expansion

$$\begin{aligned}(x-A)(x-B)(x-C) &- (x-A)(x\cos\lambda-a)^2 \\ &- (x-B)(x\cos\mu-b)^2 \\ &- (x-C)(x\cos\nu-c)^2 \ + 2(x\cos\lambda-a)(x\cos\mu-b)(x\cos\nu-c)\,;\end{aligned}$$

and in the second paper for the axisymmetric determinant

$$\begin{vmatrix} a-x & b' & b'' & b''' \\ b' & a'+x & c''' & c'' \\ b'' & c''' & a''+x & c' \\ b''' & c'' & c' & a'''+x \end{vmatrix}$$

the expansion

$$\begin{aligned}(a-x)(a'+x)(a''+x)(a'''+x) &- (a-x)(a'+x)c'^2 - (a''+x)(a'''+x)b'^2 \\ &- (a-x)(a''+x)c''^2 - (a'''+x)(a'+x)b''^2 \\ &- (a-x)(a'''+x)c'''^2 - (a'+x)(a''+x)b'''^2 \\ + 2c'c''c'''(a-x) + 2c'b''b'''(a'+x) &+ 2c''b'''b'(a''+x) + 2c'''b'b''(a'''+x) \\ + b'^2c'^2 + b''^2c''^2 + b'''^2c'''^2 &- 2b'b''c'c'' - 2b''b'''c''c''' - 2b'''b'c'''c',\end{aligned}$$

—that is to say, the expansion arranged according to products of elements of the principal diagonal.

A clause of the paper refers to the writings of Laplace, Vandermonde, Gauss, and Binet.

CAUCHY (1829).

[Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes. *Exercices de Math.*, iv. pp. 140–160; or *Œuvres complètes*, 2ᵉ sér. ix. pp. 172–195.]

The equation which Cauchy refers to in his title is exactly the equation with which we have just seen Jacobi occupied. Cauchy, however, comes upon it from a different direction, and it is no longer with him a cubic or quartic, but an n^{thic}.

The problem he sets out to solve is the finding of the maxima and minima of what we should nowadays call an n-ary quadric, viz.,

$$A_{xx}x^2 + A_{yy}y^2 + A_{zz}z^2 + \ \ldots \ + 2A_{xy}xy + 2A_{xz}xz + \ \ldots .$$

subject to the condition that the sum of the squares of the n variables $x, y, z, \ldots$ equals 1. In a few lines it is ascertained that the equation in s, $S=0$ say, whose roots are the extreme values in question, is obtainable on eliminating $x, y, z, \ldots$ from the set of n equations

$$\left.\begin{array}{rrrl}
(A_{xx}-s)x + & A_{xy}y + & A_{xz}z + & \ldots . = 0 \\
A_{yx}x + & (A_{yy}-s)y + & A_{yz}z + & \ldots . = 0 \\
A_{zx}x + & A_{zy}y + & (A_{zz}-s)z + & \ldots . = 0 \\
\ldots & \ldots & \ldots & \ldots
\end{array}\right\}$$

where $A_{yx}=A_{xy}, \ldots .$ Remembering Cauchy's great paper of 1812, we are quite prepared to find him at this stage proceeding to say:—

"S sera une fonction alternée des quantités comprises dans le Tableau

$$\left\{\begin{array}{llll}
A_{xx}-s & A_{xy} & A_{xz} & \ldots . \\
A_{xy} & A_{yy}-s & A_{yz} & \ldots . \\
A_{xz} & A_{yz} & A_{zz}-s & \ldots . \\
\ldots & \ldots & \ldots & \ldots
\end{array}\right.$$

savoir celle dont les différents termes sont représentées, aux signes près, par les produits qu'on obtient, lorsqu'on multiplie ces quantités, n à n, de toutes les manières possibles, en ayant soin de faire entrer dans chaque produit un facteur pris dans chacune des lignes horizontales du Tableau et un facteur pris dans chacune des lignes verticales."

The lengthy discussion of the character of the roots of $S=0$ which thereupon follows, and in which the properties of "fonctions alternées" are freely used, belongs almost entirely to a different portion of our subject: for the present there concerns us only one theorem subsidiary to the said discussion. In modern phraseology this lemma is—S *being any axisymmetric determinant,* R *the determinant got by deleting the first row and first column of* S, Y *the determinant got by deleting the first row and second column of* S, *and* Q *the determinant got from* R *as* R *from* S, *then if* $R=0$, $SQ=-Y^2$. The mode adopted for testing the truth of this is applicable to any determinant S, whether axisymmetric or not; and when the second condition, viz., the vanishing of R, is also removed, there emerges the simplest case of Jacobi's theorem of 1833 regarding a minor of the adjugate.

JACOBI (1831 Dec.).

[De transformatione integralis duplicis indefiniti

$$\int\frac{\partial\phi\partial\psi}{A+B\cos\phi+C\sin\phi+(A'+B'\cos\phi+C'\sin\phi)\cos\psi+(A''+B''\cos\phi+C''\sin\phi)\sin\psi}$$

in formam simpliciorem $\int\frac{\partial\eta\partial\theta}{G-G'\cos\eta\cos\theta-G''\sin\eta\sin\theta}$.

Crelle's Journal, viii. pp. 253–279, 321–357; or *Werke*, iii. pp. 91–158.]

As the algebraical transformation effected in this paper is an extension of that dealt with in Jacobi's second paper of 1827, it is only what might have been expected to find expressions contained in it which may be viewed as axisymmetric determinants. Such expressions are two forms of the square of

$$A(B'C''-B''C')+B(C'A''-C''A')+C(A'B''-A''B'),\quad\text{or}\quad\Delta,$$

and the non-zero side of the cubic equation therewith connected, upon which the whole investigation depends, viz.,

$$\begin{aligned}&x^3-x^2\{A^2+B^2+C^2+A'^2+B'^2+C'^2+A''^2+B''^2+C''^2\}\\&\quad+x\{(B'C''-B''C')^2+\ \dots\dots\qquad\qquad\qquad\qquad\qquad\}\\&\quad-\ \{A(B'C''-B''C')+B(C'A''-C''A')+C(A'B''-A''B')\}^2.\end{aligned}$$

No hint, however, is given of these expressions being determinants,—a fact which is all the more noteworthy in view of the reference made in the second paper of 1827 to the writings of Laplace, Vandermonde, . . . , and in view of the reference made on p. 350 of his present paper to Cauchy's of 1829, where, as we have just seen, "fonctions alternées" are explicitly used throughout. As a mere aid to the memory it would appear to have been worth while to note that if one of the said squares of Δ be the determinant formed from

$$\begin{matrix} l & n' & m' \\ n' & m & l' \\ m' & l' & n \end{matrix}$$

the non-zero side of the fundamental cubic is the determinant formed from

$$\begin{matrix} x-l & n' & m' \\ n' & x-m & l' \\ m' & l' & x-n, \end{matrix}$$

and that the coefficient of $-x^0$ in the cubic is the square of Δ, the coefficient of x^1 the sum of the squares of what came afterwards to be called the "primary minors" of Δ, and the coefficient of x^2 the sum of the squares of the secondary minors.

JACOBI (1832).

[De transformatione et determinatione integralium duplicium commentatio tertia. *Crelle's Journal*, x. pp. 101–128; or *Werke*, iii. pp. 159–189.]

This last paper of the three dealing with the transformation of integrals contains less regarding our present subject than either of the others. The only thing worth noting is the curious cubic equation

$$\begin{aligned} & x^3 \{ abc - ad^2 - be^2 - cf^2 + 2def \} \\ & - x^2 \left\{ \begin{array}{l} \quad a'(bc-d^2) \ + \ b'(ca-e^2) \ + \ c'(ab-f^2) \\ + \ 2d'(ef-ad) \ + \ 2e'(fd-be) \ + \ 2f'(de-cf) \end{array} \right\} \\ & + x \left\{ \begin{array}{l} \quad a(b'c'-d'^2) \ + \ b(c'a'-e'^2) \ + \ c(a'b'-f'^2) \\ + \ 2d(e'f'-a'd') + 2e(f'd'-b'e') + 2f(d'e'-c'f') \end{array} \right\} \\ & - \quad \{ a'b'c' - a'd'^2 - b'e'^2 - c'f'^2 + 2d'e'f' \} \qquad = 0, \end{aligned}$$

where the first and last coefficients are in modern notation

$$\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix} \quad , \quad \begin{vmatrix} a' & f' & e' \\ f' & b' & d' \\ e' & d' & c' \end{vmatrix};$$

the second coefficient from the beginning is

$$\begin{vmatrix} a' & f' & e' \\ f & b & d \\ e & d & c \end{vmatrix} + \begin{vmatrix} a & f & e \\ f' & b' & d' \\ e & d & c \end{vmatrix} + \begin{vmatrix} a & f & e \\ f & b & d \\ e' & d' & c' \end{vmatrix}$$

or

$$\mathrm{A}a' + \mathrm{B}b' + \mathrm{C}c' + 2\mathrm{D}d' + 2\mathrm{E}e' + 2\mathrm{F}f';$$

and the second from the end

$$\begin{vmatrix} a & f & e \\ f' & b' & d' \\ e' & d' & c' \end{vmatrix} + \begin{vmatrix} a' & f' & e' \\ f & b & d \\ e' & d' & c' \end{vmatrix} + \begin{vmatrix} a' & f' & e' \\ f' & b' & d' \\ e & d & c \end{vmatrix}$$

or

$$\mathrm{A}'a + \mathrm{B}'b + \mathrm{C}'c + 2\mathrm{D}'d + 2\mathrm{E}'e + 2\mathrm{F}'f,$$

or

$$\left.\begin{array}{l} \mathrm{A}'a + \mathrm{F}'f + \mathrm{E}'e \\ + \mathrm{F}'f + \mathrm{B}'b + \mathrm{D}'d \\ + \mathrm{E}'e + \mathrm{D}'d + \mathrm{C}'c \end{array}\right\}.$$

JACOBI (1833).

[De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum ..... *Crelle's Journal*, xii. pp. 1–69; or *Werke*, iii. pp. 191–268.]

As in this great memoir Jacobi sums up and generalises the results of his papers of 1827, 1831, 1832, in which, as we have seen, axisymmetric determinants were implicitly made use of, it is at first somewhat surprising to find very little reference to properties of determinants of this special form. The reason, however, doubtless is that when he came to extend his theorems from the third or fourth order to the nth, he also withdrew the restriction as to axisymmetry and gave the results in quite general form. In support of this the fifth and sixth sections (pp. 8–11) may be referred to,—sections which on account of being concerned with determinants in general have already been dealt with in the proper place. Even when he comes, as before in the particular cases, to his equation for determining the coefficients of the squares of the new variables, that is, the equation

$$\Gamma = 0$$

where Γ is described as the expression got from $\Sigma \pm a_{11} a_{22} \dots a_{nn}$ by changing $a_{11}, a_{22}, \dots$ into $a_{11}-x, a_{22}-x, \dots$ he gives an expansion of Γ according to ascending powers of x, which holds whether $a_{\kappa\lambda} = a_{\lambda\kappa}$ or not. The passage is—

"Quod attinet ipsam ipsius Γ formationem, observo, si signo summatorio S amplectamur expressiones *inter se diversas*, quæ permutatis indicibus 1, 2, 3, . . . , n proveniunt, fieri :

$$\begin{aligned}
\Gamma = \quad & \Sigma \pm a_{11} a_{22} \dots a_{nn} \\
& - x\, S\Sigma \pm a_{11} a_{22} \dots a_{n-1,\, n-1} \\
& + x^2 S\Sigma \pm a_{11} a_{22} \dots a_{n-2,\, n-2} \\
& \dots \dots \dots \dots \\
& \pm x^{n-2} S\Sigma \pm a_{11} a_{22} \\
& \mp x^{n-1} S\Sigma \pm a_{11} \\
& \pm x^n .
\end{aligned}$$

Qua in formula, expressio

$$S\Sigma \pm a_{11} a_{22} \dots a_{mm}$$

designât summam $\dfrac{n(n-1)\dots(n-m+1)}{1.2\dots m}$ expressionum, quæ e

$$\Sigma \pm a_{11} a_{22} \dots a_{mm}$$

proveniunt, si in

$$a_{11} a_{22} \dots a_{mm}$$

loco indicum priorum simul ac posteriorum 1, 2, . . . , m scribimus omnibus modis, quibus fieri potest, m alios e numeris 1, 2, 3, . . . , n."

There are, however, two minor instances in which it is the special determinant that is alone concerned. The first occurs after proving (p. 13, footnote) the theorem (see Rothe's paper of 1800) that if the solution of

$$a_{1\lambda}x_1 + a_{2\lambda}x_2 + \dots + a_{n\lambda}x_n = u_\lambda \qquad (\lambda = 1, 2, \dots, n)$$

be

$$x_\kappa \Sigma \pm a_{11} a_{22} \dots a_{nn} = b_{\kappa 1}u_1 + b_{\kappa 2}u_2 + \dots + b_{\kappa n}u_n \quad (\kappa = 1, 2, \dots, n)$$

then the solution of

$$a_{\lambda 1}y_1 + a_{\lambda 2}y_2 + \ldots + a_{\lambda n}y_n = v_\lambda \qquad (\lambda=1, 2, \ldots, n)$$

must be

$$y_\kappa \Sigma \pm a_{11} a_{22} \ldots a_{nn} = b_{1\kappa}v_1 + b_{2\kappa}v_2 + \ldots + b_{n\kappa}v_n \quad (\kappa=1, 2, \ldots, n)$$

when he adds the corollary that *if* $a_{\kappa\lambda} = a_{\lambda\kappa}$ *then also* $b_{\kappa\lambda} = b_{\lambda\kappa}$. The second occurs quite similarly when, having pointed out (p. 20) that the coefficients $b_{\kappa\lambda}$ in either solution are expressible as differential-quotients of $\Sigma \pm a_{11}a_{22} \ldots a_{nn}$, he adds the sentence, "Quoties $a_{\kappa\lambda} = a_{\lambda\kappa}$ differentialis semisse tantum sumi debet si κ et λ diversi sunt."

JACOBI (1834).

[Dato systemate n æquationum linearium inter n incognitas, valores incognitarum per integralia definita $(n-1)$tuplicia exhibentur. *Crelle's Journal*, xiv. pp. 51–55; or *Werke*, vi. pp. 79–85.]

Jacobi having already pointed out in his long memoir of the preceding year that the cofactor of $a_{\kappa\lambda}$ in

$$\Sigma \pm a_{11} a_{22} \ldots a_{nn}, \quad \text{or N say,}$$

is

$$\frac{\partial \mathrm{N}}{\partial a_{\kappa\lambda}};$$

and having now to deal with the case where $a_{\kappa\lambda} = a_{\lambda\kappa}$, draws attention again to the fact that in solving the equations

$$\left.\begin{array}{l} a_{11} y_1 + a_{12} y_2 + \ldots\ldots + a_{1n} y_n = m_1, \\ a_{21} y_1 + a_{22} y_2 + \ldots\ldots + a_{2n} y_n = m_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} y_1 + a_{n2} y_2 + \ldots\ldots + a_{nn} y_n = m_n, \end{array}\right\}$$

we no longer obtain

$$\mathrm{N}y_1 = \left(\frac{\partial \mathrm{N}}{\partial a_{11}}\right)m_1 + \left(\frac{\partial \mathrm{N}}{\partial a_{21}}\right)m_2 + \ldots + \left(\frac{\partial \mathrm{N}}{\partial a_{n1}}\right)m_n,$$

but

$$\mathrm{N}y_1 = \left(\frac{\partial \mathrm{N}}{\partial a_{11}}\right)m_1 + \tfrac{1}{2}\left(\frac{\partial \mathrm{N}}{\partial a_{21}}\right)m_2 + \ldots + \tfrac{1}{2}\left(\frac{\partial \mathrm{N}}{\partial a_{n1}}\right)m_n,$$

—his explanation being that the differential-quotient of N with respect to $a_{\kappa\lambda}$, where κ and λ are unequal, is obtained by first viewing $a_{\kappa\lambda}$ and $a_{\lambda\kappa}$ as being different, adding together the differential-quotient with respect to $a_{\kappa\lambda}$ and the differential-quotient with respect to $a_{\lambda\kappa}$, and then putting $a_{\kappa\lambda} = a_{\lambda\kappa}$. His own words are—

"Si vero $a_{\kappa\lambda} = a_{\lambda\kappa}$ differentiale partiale secundum $a_{\kappa\lambda}$ sumtum, quoties non $\kappa = \lambda$, obtinetur, si primum $a_{\kappa\lambda}$ et $a_{\lambda\kappa}$ diversæ statuuntur, atque differentialia partialia secundum $a_{\kappa\lambda}$ et secundum $a_{\lambda\kappa}$ sumta iunguntur, ac deinde $a_{\kappa\lambda} = a_{\lambda\kappa}$ statuitur: quo facto cum utraque differentialia æqualia fiant, casu quo $a_{\kappa\lambda} = a_{\lambda\kappa}$ valor duplus emergit eius qui in formulis (3) locum habere debet."

LEBESGUE (1837).

[Thèses de Mécanique et d'Astronomie. Première Partie: Formules pour la transformation des fonctions homogènes du second degré à plusieurs inconnues. *Journ.* (*de Liouville*) *de Math.*, ii. pp. 337–355.]

Lebesgue's subject is exactly that dealt with in the first part of Jacobi's memoir of 1833, viz., the transformation of a general homogeneous function of the second degree into one containing only squares of the variables. Indebtedness to Jacobi, Cauchy, and Sturm is indirectly intimated at the outset, and the paper is modestly offered as being new in manner rather than in matter.

Like Cauchy and Jacobi, the author of course is led to the set of equations from which by elimination there is deduced the equation for the determination of the coefficients of the new variables; and recognising that "le premier membre de cette équation n'est qu'une de ces fonctions nommées déterminants," he devotes his second section of five pages to the properties of these functions. Throughout this section prominence is notably given to determinants having the elements $A_{\alpha\beta}$, $A_{\beta\alpha}$ equal; and such determinants are spoken of as "symétriques,"—a noteworthy fact, since up to this time no separate name had been applied to any specific form. "On peut dire alors," Lebesgue says,

"que le système est symétrique, puisque les nombres qui le forment sont placés symétriquement par rapport aux nombres à indices égaux $A_{11}, A_{22}, \ldots, A_{nn}$ qui forme la diagonale du système."

The first proposition is that in a symmetric determinant $[g, i] = [i, g]$, where $[g, i]$ is used to denote the determinant got from the original determinant D by suppressing the g^{th} row and i^{th} column.

The second is that—

"Pour tout déterminant nul on a

$$[g, g] \cdot [i, i] = [i, g] \cdot [g, i]$$

et par conséquent pour un déterminant à la fois nul et symétrique

$$[g, g] \cdot [i, i] = [i, g]^2 = [g, i]^2."$$

This is proved independently, but, of course, it is nowadays best viewed as a special case of Jacobi's theorem (1833) regarding a minor of the adjugate. The third and fourth propositions combined are to the effect that in every perfectly general determinant

$$\frac{d\mathrm{D}}{d\mathrm{A}_{i,g}} = (-1)^{i+g}[i, g],$$

while in a symmetric determinant

$$\frac{d\mathrm{D}}{d\mathrm{A}_{gg}} = [g, g], \qquad \frac{d\mathrm{D}}{d\mathrm{A}_{i,g}} = (-1)^{i+g}\, 2[i, g].$$

A proof of the last of these is given,* the starting-point being the identity

$$\mathrm{D} = \mathrm{A}_{n,n}[n,n] - \mathrm{A}_{n,n-1}[n,n-1] + \mathrm{A}_{n,n-2}[n,n-2] - \ldots .$$

where D is expressed in terms of the elements of the last row and their cofactors. By differentiating both sides of this with respect to the *particular* non-diagonal element $\mathrm{A}_{n,n-1}$ there is obtained

$$\frac{d\mathrm{D}}{d\mathrm{A}_{n,n-1}} = 0 - \left\{[n,n-1] + \mathrm{A}_{n,n-1}\frac{d[n,n-1]}{d\mathrm{A}_{n,n-1}}\right\} + \mathrm{A}_{n,n-2}\frac{d[n,n-2]}{d\mathrm{A}_{n,n-1}} - \ldots$$

The differentiands on the right of this, viz. $[n,n-1]$, $[n,n-2]$, ... although not involving $\mathrm{A}_{n,n-1}$ do involve $\mathrm{A}_{n-1,n}$ which is the same as $\mathrm{A}_{n,n-1}$: consequently their differential coefficients are other than zero and have to be found,—that is, we have to find

* There are several misprints in the original, and the paging of the volume is hereabouts all wrong.

$$\frac{d[n,i]}{dA_{n-1,n}} \qquad \text{where } i<n.$$

Expanding $[n, i]$ after the manner of D above, but now in terms of the elements of the last *column*, we obtain

$$[n,i] = A_{n-1,n}\begin{bmatrix} n,i \\ n-1,n \end{bmatrix} - A_{n-2,n}\begin{bmatrix} n,i \\ n-2,n \end{bmatrix} + A_{n-3,n}\begin{bmatrix} n,i \\ n-3,n \end{bmatrix} - \cdots,$$

and therefore, since the second factors on the right do not contain $A_{n-1,n}$ or $A_{n,n-1}$ (both the n^{th} row and n^{th} column being gone in all of them), there results

$$\frac{d[n,i]}{dA_{n-1,n}} = \begin{bmatrix} n,i \\ n-1,n \end{bmatrix}.$$

Substituting this above we see that

$$\begin{aligned}\frac{dD}{dA_{n,n-1}} &= -[n,n-1] - A_{n,n-1}\begin{bmatrix} n,n-1 \\ n-1,n \end{bmatrix} + A_{n,n-2}\begin{bmatrix} n,n-2 \\ n-1,n \end{bmatrix} - \cdots, \\ &= -[n,n-1] - [n,n-1], \\ &= -2[n,n-1].\end{aligned}$$

The theorem having thus been proved for the case of the suffixes $(n-1, n)$, the passage to the case of *any* unequal suffixes is made by saying "Par un déplacement de séries horizontales et de séries verticales, on trouvera

$$\frac{dD}{dA_{i,g}} = (-1)^{i+g}\,2[i,g]$$

comme il est dit dans l'énoncé."

Save for a page in which the development of a symmetric determinant for the cases $n=2, 3, 4$ is given, the rest of the paper is taken up with the concluding portion of the solution of the problem of transformation. It may be well to note, however, that on the page referred to (p. 347) the determinant of the system

$$\begin{array}{cccc} A_{11}-u & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22}-u & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn}-u \end{array}$$

is denoted by

$$\text{dét.}\,[A_{11}-u,\quad A_{22}-u,\ \ldots\ ,\quad A_{nn}-u].$$

JACOBI (1841).

[De formatione et proprietatibus Determinantium. *Crelle's Journal*, xxii. pp. 285–318; or *Werke*, iii. pp. 355–392; or Stäckel's translation 'Die Bildung und die Eigenschaften der Determinanten,' 73 pp., Leipzig, 1896.]

As already noted (see above p. 271) Jacobi formally enunciated in this his great memoir Binet's case of the multiplication-theorem when the product-determinant is axisymmetric.

CAUCHY (1841).

[Note sur la formation des fonctions alternées qui servent à résoudre le problème de l'élimination. *Comptes Rendus* ... *Paris*, xii. pp. 414–426; or *Œuvres complètes*, 1re sér. vi. pp. 87–99.]

The early part of this paper, in which the finding of the terms of a general determinant ("fonction alternée") is made dependent on a study of the properties of "groups," or *index-cycles* as they would more appropriately be called, has already been described. The nature of it will be readily recalled from the mode of writing the expansion of the determinant of the 4th order, viz.,

$$\begin{aligned} a_{11}a_{22}a_{33}a_{44} \;-\; \Sigma a_{11}a_{22}a_{34}a_{43} \;+\; \Sigma a_{11}a_{23}a_{34}a_{42} \\ +\; \Sigma a_{12}a_{21}a_{34}a_{43} \;-\; \Sigma a_{12}a_{23}a_{34}a_{41}\,, \end{aligned}$$

where under the last Σ are included all terms (6 in number) whose indices form one quaternary cycle, under the preceding Σ all terms (3 in number) whose indices form two binary cycles, and so on.

On coming to consider a determinant in which $a_{ij}=a_{ji}$, Cauchy points out that because of this peculiarity every term will be found *repeated* unless those whose index-cycles are all lower than ternary: for example, in the case of the determinant of the 4th order, the six terms having a quaternary index-cycle are condensed into three with the coefficient 2 prefixed, and the eight terms having a ternary index-cycle into four with the same coefficient, the whole result being—

$$a_{11}a_{22}a_{33}a_{44} - \Sigma a_{11}a_{22}a_{34}^2 + 2\Sigma a_{11}a_{23}a_{34}a_{24}$$
$$+ \Sigma a_{13}^2 a_{24}^2 - 2\Sigma a_{12}a_{23}a_{34}a_{14}.$$

The definite theorem reached by him on this point may be formulated in later phraseology as follows:—

If $\nu_3, \nu_4, \ldots$ be the number of ternary, quaternary, and higher index-cycles in any term of an axisymmetric determinant, the coefficient of the term when condensation takes place is $2^{\nu_3+\nu_4+\cdots}$

By way of proof it is stated that when we have got a term with index-cycles higher than binary, we may, by reversing the order of the indices in one of the said cycles, obtain another term of the development, and that this will be equal to the former. For example, if a term have the quaternary cycle (1, 2, 3, 4), another term is obtainable by simply changing this into (4, 3, 2, 1), the effect on the original term being to change it from

$$\ldots\ldots a_{12}a_{23}a_{34}a_{41} \ldots\ldots$$

into

$$\ldots\ldots a_{43}a_{32}a_{21}a_{14} \ldots\ldots$$

which, in the circumstances, is equivalent to no substantial change at all.

"Pour fixer les idées" he takes the case of the 6th order, giving the following as the development of what we should nowadays denote by $|a_{11}a_{22}a_{33}a_{44}a_{55}a_{66}|_{rs=sr}$, viz.,

$$a_{11}a_{22}a_{33}a_{44}a_{55}a_{66} - \Sigma a_{11}a_{22}a_{33}a_{44}a_{56}^2 + \Sigma a_{11}a_{22}a_{34}^2a_{56}^2 - \Sigma a_{12}^2a_{34}^2a_{56}^2$$
$$+ 2\Sigma a_{11}a_{22}a_{33}a_{45}a_{56}a_{64} - 2\Sigma a_{11}a_{23}^2a_{45}a_{56}a_{64} + 4\Sigma a_{12}a_{23}a_{31}a_{45}a_{56}a_{64}$$
$$- 2\Sigma a_{11}a_{22}a_{34}a_{45}a_{56}a_{63} + 2\Sigma a_{12}^2a_{34}a_{45}a_{56}a_{63} + 2\Sigma a_{11}a_{23}a_{34}a_{45}a_{56}a_{62}$$
$$- 2\Sigma a_{12}a_{23}a_{34}a_{45}a_{56}a_{61},$$

where it will be seen that the first four types of terms correspond to the following partitions of 6, viz.,

1, 1, 1, 1, 1, 1   1, 1, 1, 1, 2   1, 1, 2, 2   2, 2, 2

and the remaining types to the remaining partitions,

1, 1, 1, 3	1, 2, 3	3, 3
1, 1, 4	2, 4	1, 5
6.		

CHAPTER XII.

ALTERNANTS FROM THE YEAR 1771 TO 1841.

THE first traces of the special functions now known as *alternating functions* are said by Cauchy to be discernible in certain work of Vandermonde's; and if we view the functions as originating in the study of the number of values which a function can assume through permutation of its variables,* such an early date may in a certain sense be justifiable. To all intents and purposes, however, the theory is a creation of Cauchy's, and it is almost absolutely certain that its connection with determinants was never thought of until his time.

PRONY (1795).

[Leçons d'analyse. Considérations sur les principes de la méthode inverse des différences. *Journ. de l'Éc. Polyt.*, i. (pp. 211–273) pp. 264, 265.]

In the course of his investigations Prony comes upon a set of equations

$$\left.\begin{array}{rcl}\mu_1 + \mu_2 + \cdots + \mu_n &=& z_0\\ \rho_1\mu_1 + \rho_2\mu_2 + \cdots + \rho_n\mu_n &=& z_1\\ \rho_1^2\mu_1 + \rho_2^2\mu_2 + \cdots + \rho_n^2\mu_n &=& z_2\\ \cdots\cdots\cdots\cdots\cdots && \\ \rho_1^{n-1}\mu_1 + \rho_2^{n-1}\mu_2 + \cdots + \rho_n^{n-1}\mu_n &=& z_{n-1}\end{array}\right\}$$

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* The history of this subject is referred to in Serret, M. J.-A.: "Sur le nombre de valeurs qui peut prendre une fonction quand on y permute les lettres qu'elle renferme," *Journ. (de Liouville) de Math.*, xv. pp. 1–70 (1849).

where the coefficients of each unknown are the 0th, 1st, 2nd, &c., powers of one and the same quantity, and where, therefore, the determinant of the set is that special form long afterwards known as the simplest form of alternant. The full solution is given for the first four cases, but without any indication of the method employed. Thus for four variables the results appear in the form

$$\mu_1 = \frac{-\rho_2\rho_3\rho_4 z_0 + (\rho_2\rho_3+\rho_2\rho_4+\rho_3\rho_4)z_1 - (\rho_2+\rho_3+\rho_4)z_2 + z_3}{(\rho_1-\rho_2)(\rho_1-\rho_3)(\rho_1-\rho_4)},$$

$$\mu_2 = \frac{-\rho_1\rho_3\rho_4 z_0 + (\rho_1\rho_3+\rho_1\rho_4+\rho_3\rho_4)z_1 - (\rho_1+\rho_3+\rho_4)z_2 + z_3}{(\rho_2-\rho_1)(\rho_2-\rho_3)(\rho_2-\rho_4)},$$

$$\mu_3 = \ldots\ldots$$

$$\mu_4 = \ldots\ldots$$

and the writer then adds :—

"En général, quelque soit le nombre n, pour avoir le numérateur de la fraction qui donne la constante μ_κ, il faut prendre toutes les racines, excepté la racine ρ_κ, et des $n-1$ racines restantes, en trouver le produit total, la somme des produits $n-2$ à $n-2$, $n-3$ à $n-3$, $n-4$ à $n-4$, ... , 2 à 2, 1 à 1, multiplier, respectivement, le produit total et chacune des sommes par z_0, z_1, z_2, ... , z_{n-2}, ajouter z_{n-1}, et donner à tous les termes des signes alternatifs, en commençant par − ou +, selon que n est pair ou impair.

"Pour avoir le dénominateur, on soustraira, successivement, de ρ_κ chacune des autres racines, et on fera un produit de toutes les différences données par ces soustractions."

It is, of course, quite possible that Prony was not acquainted with Vandermonde's memoir of 1771, or Laplace's of 1772, or Bezout's of 1779; and, further, that in seeking for the solution of his equations he was lucky enough to hit upon the set of multipliers which, being used, would, on the performance of addition, eliminate all the unknowns except one; *e.g.*, in the case of four variables the multipliers

$$\begin{array}{l} -\ \rho_2\rho_3\rho_4, \\ +\ (\rho_2\rho_3+\rho_2\rho_4+\rho_3\rho_4), \\ -\ \ \rho\ +\rho_3+\rho_4), \\ \quad 1. \end{array}$$

If, however, he was familiar with the method of any one of these memoirs, and applied it to the set of equations under discussion, it would scarcely be possible for him not to anticipate Cauchy and Schweins in the discovery of the elementary properties of alternants. Thus, to take again the case of four variables, say the equations

$$\left.\begin{array}{r} x + \ \ y + \ \ z + \ \ w = p \\ ax + by + cz + dw = q \\ a^2x + b^2y + c^2z + d^2w = r \\ a^3x + b^3y + c^3z + d^3w = s \end{array}\right\},$$

Laplace's process would have given the value of x in the form

$$\frac{|b^1c^2d^3|\,p - |b^0c^2d^3|\,q + |b^0c^1d^3|\,r - |b^0c^1d^2|\,s}{|b^1c^2d^3|\,a^0 - |b^0c^2d^3|\,a + |b^0c^1d^3|\,a^2 - |b^0c^1d^2|\,a^3},$$

and Prony obtaining it in the form

$$\frac{bcd\,.\,p - (bc+bd+cd)q + (b+c+d)r - s}{bcd\,.\,a^0 - (bc+bd+cd)a + (b+c+d)a^2 - a^3}$$

could not have failed to know in their general forms the theorems

$$|b^1c^2d^3| \div |b^0c^1d^2| = bcd,$$
$$|b^0c^2d^3| \div |b^0c^1d^2| = bc+bd+cd,$$
$$|b^0c^1d^3| \div |b^0c^1d^2| = b+c+d,$$

and

$$|a^0b^1c^2d^3| \div |b^0c^1d^2| = (d-a)(c-a)(b-a),$$

and $\therefore$ $\qquad |a^0b^1c^2d^3| = (d-a)(c-a)(b-a)(c-b)(c-a)(b-a).$

CAUCHY (1812).

[Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment. *Journ. de l'Éc. Polyt.*, x. pp. 29–51, 51–112; or *Œuvres complètes*, 2^e sér. i.]

By reason of the fact that Cauchy viewed determinants as a class of alternating functions, it has already been necessary to give an account of a considerable portion of the first part (pp. 29–51) of this memoir: in fact, only five pages (pp. 45–51) remain to be dealt with if the portion referred to be borne in mind.

From observing the substitutions which result in the vanishing of the function, he derives the following theorem:—

"Soit $S(\pm K)$ une fonction symétrique alternée quelconque. Désignons par α, β, γ, &c., les indices qu'elle renferme, et par

$$\begin{array}{llll} a_\alpha, & a_\beta, & a_\gamma, & \ldots\ldots \\ b_\alpha, & b_\beta, & b_\gamma, & \ldots\ldots \\ c_\alpha, & c_\beta, & c_\gamma, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{array}$$

les quantités qui dans cette fonction se trouvent affectées des indices $\alpha, \beta, \gamma, \ldots.$ Si l'on remplace

$$b_\alpha, c_\alpha, \ldots., \quad b_\beta, c_\beta, \ldots., \quad b_\gamma, c_\gamma \ldots.,$$

par des fonctions semblables des quantités a_α, a_β, a_γ, ...; la fonction symétrique alternée deviendra divisible par chacune des quantités

$$\begin{array}{l} a_\alpha - a_\beta, \\ a_\alpha - a_\gamma^1, \\ \ldots\ldots \\ a_\beta - a_\gamma, \\ \ldots\ldots \end{array}$$"

From this he passes to alternating functions "which contain only one kind of quantities," and deduces the result that

$$S\left(\pm a_1^p a_2^q \ldots. a_n^t\right) \text{ is divisible by}$$

$$(a_2-a_1)(a_3-a_1)\ldots.(a_n-a_1)(a_3-a_2)\ldots.(a_n-a_2)\ldots.(a_n-a_{n-1})$$

The question as to the remaining factor is then dealt with in the three simplest cases:—

(1) In the case of $S\left(\pm a_1^0 a_2^1 \ldots. a_n^{n-1}\right)$ it is found as follows to be 1.

"La somme des exposans des lettres $a_1, a_2, \ldots, a_n$ dans chaque terme de la fonction symétrique alternée

$$S\left(\pm a_1^0 a_2^1 a_3^2 \ldots. a_{n-1}^{n-2} a_n^{n-1}\right)$$

sera

$$0+1+2+\ \ldots.\ +(n-2)+(n-1) = \frac{n(n-1)}{2}.$$

Mais les facteurs du produit A [*i.e.*, $(a_2-a_1)(a_3-a_1)\ldots\ldots(a_n-a_{n-1})$] étant aussi en nombre égal à $\frac{1}{2}n(n-1)$, la somme des exposans des lettres $a_1, a_2, \ldots\ldots, a_n$ dans chaque terme du développement de ce produit sera encore égale à ce nombre; par suite, le quotient qu'on obtiendra, en divisant la fonction symétrique alternée par le produit, sera une quantité constante. Soit c la quantité dont il s'agit, on aura

$$\mathrm{S}^n(\pm a_1^0 a_2^1 a_3^2 \ldots\ldots a_n^{n-1}) = c\mathrm{A}.$$

Pour déterminer c on observera que le terme

$$a_1^0 a_2^1 a_3^2 \ldots\ldots a_n^{n-1}$$

a pour coefficient l'unité dans la fonction donnée et dans le produit A; on doit donc avoir $c=1$."

Before proceeding to the next case he calls to mind the fact that
the product or quotient of two alternating functions of order n *is a symmetric function of the same order*,
and is thus enabled to amplify one of the preceding propositions by affirming that
the result of dividing $\mathrm{S}(\pm a_1^p a_2^q \ldots a_n^t)$ *by* $\mathrm{S}(\pm a_1^0 a_2^1 \ldots a_n^{n-1})$ *is a symmetric function of* $a_1, a_2, \ldots, a_n$.

(2) In the case of $\mathrm{S}(\pm a_1^0 a_2^1 \ldots a_{n-1}^{n-2} a_n^n)$ the quotient is found to be $a_1+a_2+\ldots+a_n$.
For the quotient "sera nécessairement du premier degré par rapport aux quantités $a_1, a_2, \ldots, a_n$: et comme elle doit être symétrique et permanente par rapport à ces quantités, on sera obligé de supposer égale à

$$c(a_1+a_2+\ldots+a_n) = c\mathrm{S}^n(a_1),$$

c étant une constante qui ne peut différer ici de l'unité."

(3) In the case of $\mathrm{S}(\pm a_1^1 a_2^2 \ldots a_n^n)$ the quotient is, of course, found to be $a_1 a_2 \ldots a_n$.

The memoir closes with the conditions for the identity of two alternating functions, these being stated to be (1) that all the terms of the first functions be contained in the second; (2) that the terms have the same numerical coefficients in both; (3) that one of the terms of the first has the same sign as the corresponding term of the second.

SCHWEINS (1825).

[Theorie der Differenzen und Differentiale, u. s. w. Von Ferd. Schweins. vi+666 pp. Heidelberg, 1825. (Pp. 317–431: *Theorie der Producte mit Versetzungen.*)]

It may be remembered that Schweins' large volume contains seven separate treatises, that the third treatise deals with determinants (*Producte mit Versetzungen*), and is divided into four sections (*Abtheilungen*). The first of the four almost entirely concerns *general* determinants, and consequently an account of it has already been given. The second section (pp. 369–398) now falls to be undertaken, its heading being "Determinants in which the upper index denotes a power" (*Producte mit Versetzungen, wenn die oberen Elemente das Potentiiren angeben*).

His first theorem is

$$\overset{h}{\mathrm{A}}_1\overset{h}{\mathrm{A}}_2\overset{h}{\mathrm{A}}_3 \ldots \overset{h}{\mathrm{A}}_n \cdot \Big\| \overset{a_1}{\mathrm{A}_1}\, \overset{a_2}{\mathrm{A}_2}\, \overset{a_3}{\mathrm{A}_3} \ldots . \overset{a_n}{\mathrm{A}_n} \Big) = \Big\| \overset{h+a_1}{\mathrm{A}_1}\, \overset{h+a_2}{\mathrm{A}_2}\, \overset{h+a_3}{\mathrm{A}_3} \ldots . \overset{h+a_n}{\mathrm{A}_n} \Big),$$

which is seen to be an extension of one of Cauchy's; but, besides this, in the first chapter there is practically nothing worth noting. The remaining four chapters, however, are full of interest, and deserve every attention, as until the present day they have been utterly lost sight of and contain a theorem or two which are still quite new.

The second chapter concerns the multiplication of an alternant of the n^{th} order by the sum of the p-ary combinations of the variables in their h^{th} power. In Schweins' notation this product is represented by

$$\Big(\overset{h}{\mathrm{A}}_1, \overset{h}{\mathrm{A}}_2, \ldots, \overset{h}{\mathrm{A}}_n\Big)^p . \Big\| \overset{a_1}{\mathrm{A}_1}\, \overset{a_2}{\mathrm{A}_2} \ldots . \overset{a_n}{\mathrm{A}_n} \Big);$$

in later notation, the case where $n=3$, $p=2$, $h=5$ would be written

$$(a^5b^5+a^5c^5+b^5c^5) . \begin{vmatrix} a^r & a^s & a^t \\ b^r & b^s & b^t \\ c^r & c^s & c^t \end{vmatrix}, \quad \text{or} \quad \Sigma a^5b^5 . |a^r b^s c^t| .$$

The case where $p=1$ is first dealt with, and the proof is written out at length without specialising n; but as this does not add to clearness or conviction, n may here, for convenience in writing, be taken $=4$. Let, then, the alternant be

$$|a^r b^s c^t d^u|$$

so that the multiplier is

$$a^h + b^h + c^h + d^h.$$

Expanding the multiplicand first according to powers of a, we perform the multiplication by a^h; expanding next according to powers of b, we perform the multiplication by b^h; and so on, the sum of the products being naturally arrangeable as a square array of sixteen terms, viz.,

$$\begin{aligned} & a^{r+h}|b^s c^t d^u| - a^{s+h}|b^r c^t d^u| + a^{t+h}|b^r c^s d^u| - a^{u+h}|b^r c^s d^t| \\ & - b^{r+h}|a^s c^t d^u| + b^{s+h}|a^r c^t d^u| - b^{t+h}|a^r c^s d^u| + b^{u+h}|a^r c^s d^t| \\ & + c^{r+h}|a^s b^t d^u| - c^{s+h}|a^r b^t d^u| + c^{t+h}|a^r b^s d^u| - c^{u+h}|a^r b^s d^t| \\ & - d^{r+h}|a^s b^t c^u| + d^{s+h}|a^r b^t c^u| - d^{t+h}|a^r b^s c^u| + d^{u+h}|a^r b^s c^t|. \end{aligned}$$

Recombination of these, however, is possible by taking them in vertical sets of four, and the result of doing this is

$$|a^{r+h} b^s c^t d^u| - |a^{s+h} b^r c^t d^u| + |a^{t+h} b^r c^s d^u| - |a^{u+h} b^r c^s d^t|;$$

so that we have

$$|a^r b^s c^t d^u| \,.\, \Sigma a^h = |a^{r+h} b^s c^t d^u| + |a^r b^{s+h} c^t d^u| + |a^r b^s c^{t+h} d^u| + |a^r b^s c^t d^{u+h}|,$$

and generally

$$\begin{aligned} |a^r b^s c^t d^u e^v \ldots\ldots| \,.\, \Sigma a^h = |a^{r+h} b^s c^t d^u e^v \ldots\ldots| + |a^r b^{s+h} c^t d^u e^v \ldots\ldots| \\ + |a^r b^s c^{t+h} d^u e^v \ldots\ldots| + \ldots\ldots \end{aligned}$$

The special case where $r, s, t, u, \ldots\ldots$ proceed by a common difference, h, is drawn attention to, as then all the alternants on the right vanish except the last: that is to say, we have

$$|a_1^r a_2^{r+h} a_3^{r+2h} \ldots\ldots a_n^{r+(n-1)h}| \,.\, \Sigma a_1^h = |a_1^r a_2^{r+h} a_3^{r+2h} \ldots\ldots a_{n-1}^{r+(n-2)h} a_n^{r+nh}|,$$

a result which may be looked upon as an immediate generalisation of one of Cauchy's.

When $p>1$, the mode of proof is totally different, being an attempt at so-called "mathematical induction." It is not by any means readily convincing, and is much less so than it might have been, as, although there are *two* general integers involved, viz., p and n, Schweins attends only to the second of them. He begins with the case of $n=4$, $p=2$,—that is to say, the multiplication of

$$|a^r b^s c^t d^u| \quad \text{by} \quad \Sigma a^h b^h,$$

the result being

$$\begin{aligned}\left(\overset{h}{A}_1, \overset{h}{A}_2, \overset{h}{A}_3, \overset{h}{A}_4\right)^{(2)} . \left\|\overset{a_1}{A}_1 \overset{a_2}{A}_2 \overset{a_3}{A}_3 \overset{a_4}{A}_4\right) = \; & \left\|\overset{h+a_1}{A}_1 \overset{h+a_2}{A}_2 \overset{a_3}{A}_3 \overset{a_4}{A}_4\right) \\ & + \left\|\overset{h+a_1}{A}_1 \overset{a_2}{A}_2 \overset{h+a_3}{A}_3 \overset{a_4}{A}_4\right) \\ & + \left\|\overset{h+a_1}{A}_1 \overset{a_2}{A}_2 \overset{a_3}{A}_3 \overset{h+a_4}{A}_4\right) \\ & + \left\|\overset{a_1}{A}_1 \overset{h+a_2}{A}_2 \overset{h+a_3}{A}_3 \overset{a_4}{A}_4\right) \\ & + \left\|\overset{a_1}{A}_1 \overset{h+a_2}{A}_2 \overset{a_3}{A}_3 \overset{h+a_4}{A}_4\right) \\ & + \left\|\overset{a_1}{A}_1 \overset{a_2}{A}_2 \overset{h+a_3}{A}_3 \overset{h+a_4}{A}_4\right).\end{aligned}$$

To indicate the mode of formation of the alternants on the right from the given alternant on the left, he says:—

"Hier entstehen alle Vertheilungen von h, h zu zweien in vier Abtheilungen nämlich

$h+a_1$	$h+a_2$	a_3	a_4
$h+a_1$	a_2	$h+a_3$	a_4
$h+a_1$	a_2	a_3	$h+a_4$
a_1	$h+a_2$	$h+a_3$	a_4
a_1	$h+a_2$	a_3	$h+a_4$
a_1	a_2	$h+a_3$	$h+a_4$

."

He next takes the case where $n=5$ and $p=3$: that is to say, the case of

$$|a^r b^s c^t d^u e^v| . \Sigma a^h b^h c^h,$$

and gives as his result

$$\text{“}\left(A_1^h,\ A_2^h,\ A_3^h,\ A_4^h,\ A_5^h\right)^{(3)} . \left\| A_1^{a_1} A_2^{a_2} A_3^{a_3} A_4^{a_4} A_5^{a_5} \right)$$
$$= \left\| A_1^{h+a_1} A_2^{h+a_2} A_3^{h+a_3} A_4^{a_4} A_5^{a_5} \right)$$
$$+ \left\| A_1^{h+a_1} A_2^{h+a_2} A_3^{a_3} A_4^{h+a_4} A_5^{a_5} \right)$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \left\| A_1^{a_1} A_2^{a_2} A_3^{h+a_3} A_4^{h+a_4} A_5^{h+a_5} \right),$$

wo h, h, h in fünf Abtheilungen zu dreien vertheilt werden, nämlich

$h+a_1$	$h+a_2$	$h+a_3$	a_4	a_5
$h+a_1$	$h+a_2$	a_3	$h+a_4$	a_5
$h+a_1$	$h+a_2$	a_3	a_4	$h+a_5$
$h+a_1$	a_2	$h+a_3$	$h+a_4$	a_5
$h+a_1$	a_2	$h+a_3$	a_4	$h+a_5$
$h+a_1$	a_2	a_3	$h+a_4$	$h+a_5$
a_1	$h+a_2$	$h+a_3$	$h+a_4$	a_5
a_1	$h+a_2$	$h+a_3$	a_4	$h+a_5$
a_1	$h+a_2$	a_3	$h+a_4$	$h+a_5$
a_1	a_2	$h+a_3$	$h+a_4$	$h+a_5$

,”

the table being intended to make clear the fact that the five indices of each of the ten alternants on the right of the identity are got from the five

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5$$

of the given alternant on the left by adding h to three of them. The mode of formation, seen to hold in these two cases, being then supposed to hold for

$$\left(A_1^h,\ A_2^h,\ \ldots,\ A_{n-1}^h\right)^{(p)} . \left\| A_1^{a_1} A_2^{a_2} \ldots A_{n-1}^{a_{n-1}} \right),$$

is attempted to be shown to hold for

$$\left(\overset{h}{A}_1, \overset{h}{A}_2, \ldots, \overset{h}{A}_{n-1}, \overset{h}{A}_n\right)^{(p)} . \left\| \overset{a_1}{A}_1 \overset{a_2}{A}_2 \ldots \overset{a_{n-1}}{A}_{n-1} \overset{a_n}{A}_n \right) ;$$

that is to say, the case for n variables, $A_1, \ldots, A_n$, is sought to be made dependent on the case for $n-1$ variables, $A_1, \ldots, A_{n-1}$, p remaining the same in both. The process followed is to change the first factor into

$$\left(\overset{h}{A}_1, \overset{h}{A}_2, \ldots, \overset{h}{A}_{x-1}, \overset{h}{A}_{x+1}, \ldots, \overset{h}{A}_n\right)^{(p)} + \left(\overset{h}{A}_1, \overset{h}{A}_2, \ldots, \overset{h}{A}_{x-1}, \overset{h}{A}_{x+1}, \ldots, \overset{h}{A}_n\right)^{(p-1)} . \overset{h}{A}_x$$

express the second factor—the alternant—in terms of n alternants of the $(n-1)^{\text{th}}$ order, and then perform the required multiplication and condense the result. This being satisfactorily accomplished, it would not of course follow from the two special cases previously dealt with that the theorem had been established in all its generality, but merely that it held for any number of variables $A_1, A_2, \ldots$ so long as p was not greater than 3. The passage from one value of p to the next higher—which is left unattempted by Schweins—is not free from difficulty, as will be seen on trying a particular instance,—say the passage from

$$|a^r b^s c^t d^u| \, . \, (a^h b^h + a^h c^h + a^h d^h + b^h c^h + b^h d^h + c^h d^h)$$

to

$$|a^r b^s c^t d^u| \, . \, (a^h b^h c^h + a^h b^h d^h + a^h c^h d^h + b^h c^h d^h).$$

Several special cases of the general theorem are noted, where a number of the alternants on the right vanish and where consequently a comparatively simple result is attained. The first of these is where the indices of the alternant to be multiplied proceed by a common difference h: the identity then is

$$\left(\overset{h}{A}_1, \overset{h}{A}_2, \ldots, \overset{h}{A}_n\right)^{(p)} . \left\| \overset{a+h}{A}_1 \;\; \overset{a+2h}{A}_2 \;\; \ldots . \;\; \overset{a+nh}{A}_n \right)$$

$$= \left\| \overset{a+h}{A}_1 \;\; \overset{a+2h}{A}_2 \;\; \ldots . \;\; \overset{a+(n-p)h}{A}_{n-p} \;\; \overset{a+(n-p+2)h}{A}_{n-p+1} \;\; \ldots . \;\; \overset{a+(n+1)h}{A}_n \right).$$

The second is where $h = -h$, and the indices proceed by a common difference h, the result then being

$$\left(\overset{-h}{A}_1, \overset{-h}{A}_2, \ldots . , \overset{-h}{A}_n\right)^{(p)} . \left\| \overset{a+h}{A}_1 \;\; \overset{a+2h}{A}_2 \;\; \ldots . \;\; \overset{a+nh}{A}_n \right)$$

$$= \left\| \overset{a}{A}_1 \; \overset{a+h}{A}_2 \;\; \ldots . \;\; \overset{a+(p-1)h}{A}_p \;\; \overset{a+(p+1)h}{A}_{p+1} \;\; \ldots . \;\; \overset{a+nh}{A}_n \right).$$

The third is where the series of indices consists of two progressions proceeding by the common difference h, and where, of course, there are fewer vanishing terms in the product.

In the next chapter the subject matter is quite similar: in fact, the only difference is in the constitution of the multiplier, which is more extensive than before by reason of the fact that in forming the p-ary combinations there is now no restriction as to non-repetition of an element. Thus, instead of the example

$$|a^r b^s c^t| \,.\, (a^h b^h + a^h c^h + b^h c^h)$$

we should now have

$$|a^r b^s c^t| \,.\, (a^h b^h + a^h c^h + b^h c^h + a^{2h} + b^{2h} + c^{2h}).$$

The method followed is exactly the same as before. Three simple cases are carefully worked out, viz.,

$$|a^r b^s| \,.\, (a^{2h} + b^{2h} + a^h b^h)\,,$$
$$|a^r b^s c^t| \,.\, (a^{2h} + b^{2h} + c^{2h} + a^h b^h + a^h c^h + b^h c^h)\,,$$
$$|a^r b^s c^t| \,.\, (a^{3h} + b^{3h} + c^{3h} + a^{2h} b^h + a^{2h} c^h + b^{2h} a^h + b^{2h} c^h + c^{2h} a^h + c^{2h} b^h + a^h b^h c^h),$$

the results in Schweins' notation—where the change to rectangular brackets should be noted—being

$$\left[A_1^h, A_2^h\right]^{(2)} . \left\| A_1^{a_1} A_2^{a_2} \right) = \left\| A_1^{2h+a_1} A_2^{a_2} \right) + \left\| A_1^{h+a_1} A_2^{h+a_2} \right) + \left\| A_1^{a_1} A_2^{2h+a_2} \right)$$

$$\begin{aligned} \left[A_1^h, A_2^h, A_3^h\right]^{(2)} . \left\| A_1^{a_1} A_2^{a_2} A_3^{a_3} \right) = {} & \left\| A_1^{2h+a_1} A_2^{a_2} A_3^{a_3} \right) + \left\| A_1^{a_1} A_2^{2h+a_2} A_3^{a_3} \right) \\ & + \left\| A_1^{a_1} A_2^{a_2} A_3^{2h+a_3} \right) + \left\| A_1^{h+a_1} A_2^{h+a_2} A_3^{a_3} \right) \\ & + \left\| A_1^{h+a_1} A_2^{a_2} A_3^{h+a_3} \right) + \left\| A_1^{a_1} A_2^{h+a_2} A_3^{h+a_3} \right), \end{aligned}$$

$$\begin{aligned} \left[A_1^h, A_2^h, A_3^h\right]^{(3)} . \left\| A_1^{a_1} A_2^{a_2} A_3^{a_3} \right) = {} & \left\| A_1^{3h+a_1} A_2^{a_2} A_3^{a_3} \right) + \left\| A_1^{a_1} A_2^{3h+a_2} A_3^{a_3} \right) \\ & + \left\| A_1^{a_1} A_2^{a_2} A_3^{3h+a_3} \right) + \left\| A_1^{2h+a_1} A_2^{h+a_2} A_3^{a_3} \right) \\ & + \left\| A_1^{2h+a_1} A_2^{a_2} A_3^{h+a_3} \right) + \left\| A_1^{a_1} A_2^{2h+a_2} A_3^{h+a_3} \right) \\ & + \left\| A_1^{h+a_1} A_2^{2h+a_2} A_3^{a_3} \right) + \left\| A_1^{h+a_1} A_2^{a_2} A_3^{2h+a_3} \right) \\ & + \left\| A_1^{a_1} A_2^{h+a_2} A_3^{2h+a_3} \right) + \left\| A_1^{h+a_1} A_2^{h+a_2} A_3^{h+a_3} \right). \end{aligned}$$

Each result is seen, as in the preceding case, to be a sum of alternants differing only in the indices from the alternant which is the subject of multiplication. Further, it is observed that this difference is a difference in excess, the indices of the multiplicand appearing in all the terms of the product, so that the only difficulty is to ascertain what addendum is to be made to each. The next observation is that the addendum is a multiple of h, and that in the three cases the multiples are the following:—

$$\begin{array}{l} 2h,\ 0h \\ 1h,\ 1h \\ 0h,\ 2h \end{array} \qquad \begin{array}{l} 2h,\ 0h,\ 0h \\ 0h,\ 2h,\ 0h \\ 0h,\ 0h,\ 2h \\ \hline 1h,\ 1h,\ 0h \\ 1h,\ 0h,\ 1h \\ 0h,\ 1h,\ 1h \end{array} \qquad \begin{array}{l} 3h,\ 0h,\ 0h \\ 0h,\ 3h,\ 0h \\ 0h,\ 0h,\ 3h \\ \hline 2h,\ 1h,\ 0h \\ 2h,\ 0h,\ 1h \\ 0h,\ 2h,\ 1h \\ \hline 1h,\ 2h,\ 0h \\ 1h,\ 0h,\ 2h \\ 0h,\ 1h,\ 2h \\ \hline 1h,\ 1h,\ 1h\,. \end{array}$$

The law of formation seen by Schweins in these coefficients of h is to be gathered from the sentence, "Hier werden alle mögliche Zerfällungen einer Zahl in mehrere Abtheilungen gebracht," and is nothing more nor less than the solution of the problem of putting p things in every possible way into n compartments. Thus, to take another example, if p were 2 and n were 4, the coefficients would be

$$\begin{array}{l} 2,\ 0,\ 0,\ 0 \\ 0,\ 2,\ 0,\ 0 \\ 0,\ 0,\ 2,\ 0 \\ 0,\ 0,\ 0,\ 2 \\ \\ 1,\ 1,\ 0,\ 0 \\ 1,\ 0,\ 1,\ 0 \\ 1,\ 0,\ 0,\ 1 \\ 0,\ 1,\ 1,\ 0 \\ 0,\ 1,\ 0,\ 1 \\ 0,\ 0,\ 1,\ 1\,. \end{array}$$

Assuming this law to hold in the case of $n-1$ variables $A_1, \ldots,$ A_{n-1}, his mode of writing it being

$$\left[\overset{h}{A_1}, \overset{h}{A_2}, \ldots, \overset{h}{A_{n-1}}\right]^{(p)} . \left\| \overset{a_1}{A_1} \overset{a_2}{A_2} \ldots \overset{a_{n-1}}{A_{n-1}} \right) = \sum\nolimits_{p,\, n-1} \left\| \overset{ph+a_1}{A_1} \overset{a_2}{A_2} \ldots \overset{a_{n-1}}{A_{n-1}} \right),$$

he tries to show that it will hold in the case of one additional variable A_n, the possible variation of p being ignored as before. To do this he changes the first factor

$$\left[\overset{h}{A_1}, \overset{h}{A_2}, \ldots, \overset{h}{A_n}\right]^{(p)}$$

into

$$\left[\overset{h}{A_1}, \overset{h}{A_2}, \ldots, \overset{h}{A_{n-1}}\right]^{(p)} + \left[\overset{h}{A_1}, \overset{h}{A_2}, \ldots, \overset{h}{A_{n-1}}\right]^{(p-1)} . \overset{h}{A_n},$$

and the second factor exactly as it was changed in the preceding chapter, performs the required multiplication, and condenses the result.

The rest of the chapter is occupied with the consideration of special cases, the lines of specialisation being exactly those followed in the case of the previous general theorem. Only the first need be noted: it is

$$\begin{aligned} \left[\overset{h}{A_1}, \overset{h}{A_2}, \ldots, \overset{h}{A_n}\right]^{(p)} . \left\| \overset{a+h}{A_1} \overset{a+2h}{A_2} \ldots \overset{a+nh}{A_n} \right) \\ = \left\| \overset{a+h}{A_1} \overset{a+2h}{A_2} \ldots \overset{a+(n-1)h}{A_{n-1}} \overset{a+(n+p)h}{A_n} \right). \end{aligned}$$

The fourth chapter does not impress one favourably, although the author speaks of its importance in connection with later investigations. It is almost entirely dependent on a very special case of the theorem of the second chapter, viz., the case where all the indices, except the last, of the multiplicand proceed by a common difference h, and where consequently all the alternants in the result vanish except two. In the original notation it is

$$\begin{aligned} \left(\overset{h}{A_1}, \overset{h}{A_2}, \ldots, \overset{h}{A_n}\right)^{(n-p)} . \left\| \overset{a+h}{A_1} \overset{a+2h}{A_2} \ldots \overset{a+(n-1)h}{A_{n-1}} \overset{s}{A_n} \right) \\ = \left\| \overset{a+h}{A_1} \ldots \overset{a+ph}{A_p} \overset{a+(p+2)h}{A_{p+1}} \ldots \overset{a+nh}{A_{n-1}} \overset{s+h}{A_n} \right) \\ + \left\| \overset{a+h}{A_1} \ldots \overset{a+(p-1)h}{A_{p-1}} \overset{a+(p+1)h}{A_p} \ldots \overset{a+nh}{A_{n-1}} \overset{s}{A_n} \right), \end{aligned}$$

but for convenience in what follows it may be shortly written

$$N_{n-p} . A_s = M_{p+1,\, s+h} + M_{p,\, s}.$$

Using it $n-p+1$ times in succession, we have

$$\begin{array}{rcl}
\mathrm{N}_{n-p} \,.\, \mathrm{A}_s & = & \mathrm{M}_{p+1,s+h} + \mathrm{M}_{p,s}, \\
-\mathrm{N}_{n-p-1} \,.\, \mathrm{A}_{s+h} & = & -\mathrm{M}_{p+2,s+2h} - \mathrm{M}_{p+1,s+h}, \\
\mathrm{N}_{n-p-2} \,.\, \mathrm{A}_{s+2h} & = & \mathrm{M}_{p+3,s+3h} + \mathrm{M}_{p+2,s+2h}, \\
-\mathrm{N}_{n-p-3} \,.\, \mathrm{A}_{s+3h} & = & -\mathrm{M}_{p+4,s+4h} - \mathrm{M}_{p+3,s+3h}, \\
\cdots & & \cdots \\
(-)^{n-p}\mathrm{N}_0 \,.\, \mathrm{A}_{s+(n-p)h} & = & 0 + (-)^{n-p}\mathrm{M}_{n,s+(n-p)h},
\end{array}$$

and therefore by addition

$$\mathrm{M}_{p,s} = \mathrm{N}_{n-p} \,.\, \mathrm{A}_s - \mathrm{N}_{n-p-1} \,.\, \mathrm{A}_{s+h} + \mathrm{N}_{n-p-2} \,.\, \mathrm{A}_{s+2h} - \ldots (-)^{n-p}\mathrm{N}_0 \,.\, \mathrm{A}_{s+(n-p)h},$$

or

$$\begin{aligned}
\Big\| \mathrm{A}_1^{a+h}\ \mathrm{A}_2^{a+2h} \ldots . \mathrm{A}_{p-1}^{a+(p-1)h}\ \mathrm{A}_p^{a+(p+1)h} \ldots . \mathrm{A}_{n-1}^{a+nh}\ \mathrm{A}_n^{s}\Big) \\
= \ & \left(\mathrm{A}_1^h, \mathrm{A}_2^h, \ldots . \mathrm{A}_n^h\right)^{(n-p)} . \Big\| \mathrm{A}_1^{a+h} \ldots . \mathrm{A}_{n-1}^{a+(n-1)h}\ \mathrm{A}_n^{s}\Big) \\
- & \left(\mathrm{A}_1^h, \mathrm{A}_2^h, \ldots . \mathrm{A}_n^h\right)^{(n-p-1)} . \Big\| \mathrm{A}_1^{a+h} \ldots . \mathrm{A}_{n-1}^{a+(n-1)h}\ \mathrm{A}_n^{s+h}\Big) \\
+ & \left(\mathrm{A}_1^h, \mathrm{A}_2^h, \ldots . \mathrm{A}_n^h\right)^{(n-p-2)} . \Big\| \mathrm{A}_1^{a+h} \ldots . \mathrm{A}_{n-1}^{a+(n-1)h}\ \mathrm{A}_n^{s+2h}\Big) \\
& \cdots \\
+ & (-1)^{n-p}\left(\mathrm{A}_1^h, \mathrm{A}_2^h, \ldots . \mathrm{A}_n^h\right)^{(0)} . \Big\| \mathrm{A}_1^{a+h} \ldots . \mathrm{A}_{n-1}^{a+(n-1)h}\ \mathrm{A}_n^{s+(n-p)h}\Big),
\end{aligned}$$

a theorem which may be described as giving an expression for an alternant having two breaks in its series of indices in terms of alternants which have only one such break and that at the very last index. On account of the fact, however, that alternants of the latter kind are multiples of the alternant which has no break at all—that is to say on account of the theorem

$$\begin{aligned}
\left[\mathrm{A}_1^h, \mathrm{A}_2^h, \ldots, \mathrm{A}_n^h\right]^{(p)} . \Big\| \mathrm{A}_1^{a+h}\ \mathrm{A}_2^{a+2h} \ldots . \mathrm{A}_n^{a+nh}\Big) \\
= \Big| \mathrm{A}_1^{a+h}\ \mathrm{A}_2^{a+2h} \ldots . \mathrm{A}_{n-1}^{a+(n-1)h}\ \mathrm{A}_n^{a+(n+p)h}\Big)
\end{aligned}$$

above given as an important special case of the general theorem of the third chapter—substitutions may be made which will result in the appearance of the last mentioned simple alternant in every term. Consequently, if we divide by this alternant and put $s=a+(n+m)h$, we have the theorem

$$\frac{\left| A_1^{a+h} \; A_2^{a+2h} \; \ldots\ldots \; A_{p-1}^{a+(p-1)h} \; A_p^{a+(p+1)h} \; \ldots\ldots \; A_{n-1}^{a+nh} \; A_n^{a+(n+m)h} \right)}{\left| A_1^{a+h} \; A_2^{a+2h} \; \ldots\ldots\ldots\ldots\ldots \; A_n^{a+nh} \right)}$$

$$\begin{aligned}
= \quad & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p)} \,.\, \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m)} \\
- & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p-1)} \,.\, \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m+1)} \\
+ & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p-2)} \,.\, \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m+2)} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& (-)^{n-p}\left(A_1^h, A_2^h, \ldots, A_n^h\right)^{0} \,.\, \left[A_1^h, A_2^h, \ldots, A_n^h\right]^{(m+n-p)}
\end{aligned}$$

Again starting from the same initial identity we obtain the analogous series

$$\begin{aligned}
M_{p,s} \quad &+ M_{p-1,s-h} &= \quad & N_{n-p+1} \,.\, A_{s-h}, \\
- M_{p-1,s-h} &- M_{p-2,s-2h} &= - & N_{n-p+2} \,.\, A_{s-2h}, \\
+ M_{p-2,s-2h} &+ M_{p-3,s-3h} &= + & N_{n-p+3} \,.\, A_{s-3h}, \\
& \ldots\ldots\ldots\ldots\ldots\ldots & & \\
(-)^{p-1}M_{1,s-(p-1)h} &+ \quad 0 &= & (-)^{p-1}N_n \,.\, A_{s-ph},
\end{aligned}$$

and therefore by addition have

$$M_{p,s} = N_{n-p+1} \,.\, A_{s-h} - N_{n-p+2} \,.\, A_{s-2h} + \ldots\ldots (-)^{p-1}N_n \,.\, A_{s-ph},$$

or

$$\left| A_1^{a+h} \; A_2^{a+2h} \; \ldots\ldots \; A_{p-1}^{a+(p-1)h} \; A_p^{a+(p+1)h} \; \ldots\ldots \; A_{n-1}^{a+nh} \; A_n^{s} \right)$$

$$\begin{aligned}
= \quad & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p+1)} \,.\, \left| A_1^{a+h} \; A_2^{a+2h} \; \ldots\ldots \; A_{n-1}^{a+(n-1)h} \; A_n^{s-h} \right) \\
- & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p+2)} \,.\, \left| A_1^{a+h} \; A_2^{a+2h} \; \ldots\ldots \; A_{n-1}^{a+(n-1)h} \; A_n^{s-2h} \right) \\
+ & \left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n-p+3)} \,.\, \left| A_1^{a+h} \; A_2^{a+2h} \; \ldots\ldots \; A_{n-1}^{a+(n-1)h} \; A_n^{s-3h} \right) \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& (-)^{p-1}\left(A_1^h, A_2^h, \ldots, A_n^h\right)^{(n)} \,.\, \left| A_1^{a+h} \; A_2^{a+2h} \; \ldots\ldots \; A_{n-1}^{a+(n-1)h} \; A_n^{s-ph} \right);
\end{aligned}$$

so that by substituting as above for each of the alternants on the right and dividing both sides by $\left\| A_1^{a+h}\ A_2^{a+2h} \ldots A_n^{a+nh} \right)$ there results the alternative theorem

$$\begin{aligned}
\frac{\left\| A_1^{a+h}\ A_2^{a+2h} \ldots A_{p-1}^{a+(p-1)h}\ A_p^{a+(p+1)h} \ldots A_{n-1}^{a+nh}\ A_n^{a+(n+m)h} \right)}{\left\| A_1^{a+h}\ A_2^{a+2h} \ldots\ldots A_n^{a+nh} \right)}
&= \left(A_1^h, \ldots, A_n^h\right)^{(n-p+1)} . \left[A_1^h, \ldots, A_n^h\right]^{(m-1)} \\
&\quad - \left(A_1^h, \ldots, A_n^h\right)^{(n-p+2)} . \left[A_1^h, \ldots, A_n^h\right]^{(m-2)} \\
&\quad + \left(A_1^h, \ldots, A_n^h\right)^{(n-p+3)} . \left[A_1^h, \ldots, A_n^h\right]^{(m-3)} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad (-)^{p-1} \left(A_1^h, \ldots, A_n^h\right)^{(n)} . \left[A_1^h, \ldots, A_n^h\right]^{(m-p)} .
\end{aligned}$$

Lastly, attention is drawn to the case where $a=0$, $h=1$, $s=1$, and to a case where the order of the alternants is infinite, viz., to the fraction

$$\frac{\left\| A_1^{b}\ A_2^{a}\ A_3^{a+h}\ A_4^{a+2h} \ldots A_n^{a+(n-2)h}\ A_{n+1}^{a+nh} \ldots A_\infty^{\infty} \right)}{\left\| A_1^{a}\ A_2^{a+h}\ A_3^{a+2h} \ldots\ldots A_\infty^{\infty} \right)} .$$

The fifth and last chapter (pp. 395–398) concerns the simplest form of alternant above met with, viz., that in which the indices proceed throughout by a common difference, the main proposition being in regard to the resolvability of the alternant into binomial factors. The property with which Cauchy and almost all later writers start is thus that with which Schweins ends. The mode of proof is interesting from its farfetchedness and ingenuity, but need not be given in full generality or in the original notation; the case of $|a^0b^1c^2d^3|$ will suffice.

The first step, then, is to select a row, say the last, and express the alternant in terms of the elements of this row and their complementary minors. In this way we obtain

$$|a^0b^1c^2d^3| = d^3|a^0b^1c^2| - d^2|a^0b^1c^3| + d|a^0b^2c^3| - |a^1b^2c^3| .$$

Now each of the alternants on the right is expressible as a multiple of $|a^0b^1c^2|$ by means of the theorem above given regarding alternants with one break in the continuity of the equidifferent progression of their indices. Using this we obtain

$$\begin{aligned}|a^0b^1c^2d^3| &= \left\{d^3-d^2(a,b,c)^1+d(a,b,c)^2-(a,b,c)^3\right\}.|a^0b^1c^2|,\\ &= \left\{d^3-d^2(a+b+c)+d(ab+ac+bc)-abc\right\}.|a^0b^1c^2|,\\ &= (d-a)(d-b)(d-c).|a^0b^1c^2|,\end{aligned}$$

when it only remains to continue the selfsame process upon the alternant of lower order now reached.

It may be remarked in passing that the identity

$$|a^0b^1c^2d^3| = d^3|a^0b^1c^2| - d^2|a^0b^1c^3| + d|a^0b^2c^3| - |a^1b^2c^3|,$$

which expresses the alternant in descending powers of d, when taken along with the identity known to Cauchy

$$|a^0b^1c^2d^3| = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a),$$

the right side of which may likewise be arranged in descending powers of d, viz.,

$$\left\{d^3-d^2(a+b+c)+d(ab+ac+bc)-abc\right\}(c-b)(c-a)(b-a),$$

may have been the means of suggesting to Schweins his theorem regarding alternants like $|a^0b^2c^3|$, $|a^0b^1c^3|$ which have one break in their series of indices. In other words, the order in which he gives his theorems was very probably not the order of discovery.

The remaining portion of the chapter is an investigation of the quotient of two alternants of infinite order, viz.,

$$\frac{\left\| \overset{a}{\mathrm{B}}\ \overset{a+h}{\mathrm{A}_1}\ \overset{a+2h}{\mathrm{A}_2}\ \ldots.\ \overset{a+(n-1)h}{\mathrm{A}_{n-1}}\ \overset{a+nh}{\mathrm{A}_{n+1}}\ \ldots.\ \overset{\infty}{\mathrm{A}_\infty}\right)}{\left\| \overset{a}{\mathrm{A}_1}\overset{a+h}{\mathrm{A}_2}\ \overset{a+2h}{\mathrm{A}_3}\ \ldots\ldots\ldots\ldots\ldots\ \overset{\infty}{\mathrm{A}_\infty}\right)}.$$

SYLVESTER (1839).

[On Derivation of Coexistence: Part I. Being the theory of simultaneous simple homogeneous equations. *Philos. Magazine*, xvi. pp. 37–43; or *Collected Math. Papers*, i. pp. 47–53.]

As has been already shown, Sylvester's first approach to the subject of determinants was similar to Cauchy's, the basis of both being the outward resemblance of the two expressions

$$bc^2 + a^2c + ab^2 - a^2b - ac^2 - b^2c,$$

$$b_1c_2 + a_2c_1 + a_1b_2 - a_2b_1 - a_1c_2 - b_2c_1.$$

As the former is equal to

$$(c-b)(c-a)(b-a) \quad \text{or} \quad \mathrm{PD}(abc),$$

i.e., product of the differences of a, b, c, Sylvester denoted the other, viz., the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix},$$

by $\zeta\mathrm{PD}(abc)$, ζ being the sign for multiplication according to the law $a_r . a_s = a_{r+s}$. Using this notation he rediscovered, as has also already been seen, Schweins' theorem regarding the multiplication of the alternant

$$|a^1b^2c^3d^4 \ldots .|$$

by such symmetric functions as

$$(a+b+c+ \ldots), \quad (ab+ac+ \ldots +bc+ \ldots), \quad \ldots \ldots,$$

his form of statement being

$$\zeta\left\{\mathrm{S}_r(abc \ldots l) . \zeta\mathrm{PD}(0abc \ldots l)\right\} = \zeta_{-r}\mathrm{PD}(0abc \ldots l),$$

where ζ_{-r} implies that after 'zeta-ic' multiplication the subscripts are all to be diminished by r.

His attempted generalisation of this theorem has likewise been spoken of, its validity, however, being left undecided upon. Instead of the multiplier $\mathrm{S}_r(abc \ldots l)$ he proposed to take *any symmetric function whatever* of a, b, c, . . . , l,—or, rather, *any function whatever followed by any symmetric function.* This would have been a most noteworthy extension which Schweins had not foreseen, but unfortunately there are grave doubts as to the truth of it,—indeed, one may go so far as to say that there would be no doubt whatever about the author's inaccuracy, were it not that there are doubts also as to his meaning. By way of test let us take the case where the multiplier of $|a^1b^2c^3d^4|$ is the symmetric function Σa^2bc. From later work * it is known that

$$|a^1b^2c^3d^4| . \Sigma a^2b^1c^1d^0 = |a^1b^3c^4d^6| - 3|a^2b^3c^4d^5|,$$

* See Muir, "Theory of Determinants," p. 176 (1882).

whereas, according to Sylvester, there ought to be on the left only one alternant. Now although we know that Sylvester was in the habit of making guesses, and that these guesses though often brilliant were not always so,* it would be next to impossible to find a generalisation of his which had no individual instances in support of it. There thus remains the curious and interesting question as to what amount of truth there is in the theorem as enunciated, and whether an amendment of the enunciation would not give something not merely unexceptionable but of important value.

In trying to pass from symmetric functions like Σa, Σab, Σabc, . . . which are linear in regard to each of the variables, and to extend the theorem to *any* symmetric function, Sylvester probably thought—at least it would be quite natural for him to do so—of expressing the latter in terms of the former and then applying the theorem already obtained. It is desirable, therefore, to see what such a process may lead to. Taking the case of the multiplier Σa^2bc we have

$$
\begin{aligned}
|a^1b^2c^3d^4|.\Sigma a^2bc &= |a^1b^2c^3d^4|.\Big\{\Sigma a.\Sigma abc - 4\Sigma abcd\Big\},\\
&= \Big\{|a^1b^2c^3d^4|.\Sigma a\Big\}.\Sigma abc - |a^1b^2c^3d^4|.4\Sigma abcd,\\
&= |a^1b^2c^3d^5|.\Sigma abc - 4|a^2b^3c^4d^5|.
\end{aligned}
$$

At this point we encounter a difficulty, for the previous theorem, although it teaches us to multiply $|a^1b^2c^3d^4|$ by Σab, does not help us in the case where the multiplicand is $|a^1b^2c^3d^5|$. Proceeding, however, with other assistance we find the desired product

$$
\begin{aligned}
&= |a^2b^3c^4d^5| + |a^1b^3c^4d^6| - 4|a^2b^3c^4d^5|,\\
&= |a^1b^3c^4d^6| - 3|a^2b^3c^4d^5|,
\end{aligned}
$$

agreeing of course with what has already been found. Now the difficulty referred to would present itself to Sylvester also, but in a slightly different form by reason of the periodicity which he assumed in the elements. Thus, instead of writing

$$
\begin{aligned}
\Big\{|a^1b^2c^3d^4|.\Sigma a\Big\}\Sigma abc &= |a^1b^2c^3d^5|.\Sigma abc,\\
&= |a^2b^3c^4d^5| + |a^1b^3c^4d^6|,
\end{aligned}
$$

* See *Crelle's Journal*, lxxxix. pp. 82–85.

he would write

$$\zeta\Big\{\zeta\mathrm{PD}(0abcd)\,.\,\mathrm{S}_1(abcd)\Big\}\,.\,\mathrm{S}_3(abcd) = \zeta\Big\{\zeta_{-1}\mathrm{PD}(0abcd)\,.\,\mathrm{S}_3(abcd)\Big\}$$

and there pause for a little, not having specifically provided for the 'zeta-ic' multiplication of such an expression as $\zeta_{-1}\mathrm{PD}(0abcd)$ by $\mathrm{S}_3(abcd)$. The result forced upon him, however, would be the single term

$$\zeta_{-4}\mathrm{PD}(0abcd),$$

which in modern notation is

$$|\,a^2b^3c^4d^5\,|.$$

In the course of the work, therefore, the term $|\,a^1b^3c^4d^6\,|$ would be dropped altogether out of sight. The cause of this is undoubtedly the imposition of the condition just mentioned;—indeed, if we take the result of the work as above performed in the modern notation, viz.:—

$$|\,a^1b^3c^4d^6\,| - 3\,|\,a^2b^3c^4d^5\,|,$$

and make the elements periodic, *i.e.*, make

$$a^6,\ b^6,\ c^6,\ d^6 \ = \ a^1,\ b^1,\ c^1,\ d^1,$$

the first alternant will vanish by reason of having two indices alike, and we shall be left with a result agreeing with Sylvester's.

The conclusion, therefore, which we are tempted to draw is that if Sylvester's general theorem be correct it is only when the elements are subjected to periodicity.

JACOBI (1841).

[De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum. *Crelle's Journal*, xxii. pp. 360–371; or *Werke*, iii. pp. 439–452; or Stäckel's translation in 'Ueber die Bildung und die Eigenschaften der Determinanten,' 73 pp., Leipzig (1896).]

After having treated of determinants in general (pp. 285–318), and of the special form which afterwards came to bear his own name (pp. 319–359), Jacobi turned to another special form which he had learned about from his great predecessor Cauchy. As however, he differed from Cauchy in his mode of defining a determinant, Cauchy's definition, which, it will be remembered, made use of the difference-product, now appears as a theorem, and with it Jacobi makes his start; that is to say, he proves that

If in the determinant

$$\Sigma \pm \mathrm{a}_0\mathrm{b}_1\mathrm{c}_2\mathrm{d}_3 \ldots \mathrm{l}_{\mathrm{n}-1}$$

the suffixes be changed into exponents of powers, the result obtained is equal to the product of the $\frac{1}{2}\mathrm{n}(\mathrm{n}-1)$ *differences of* a, b, c, ..., l, *viz., the product*

$$\begin{array}{r} (\mathrm{b}-\mathrm{a})(\mathrm{c}-\mathrm{a})(\mathrm{d}-\mathrm{a}) \ldots\ldots (\mathrm{l}-\mathrm{a}) \\ (\mathrm{c}-\mathrm{b})(\mathrm{d}-\mathrm{b}) \ldots\ldots (\mathrm{l}-\mathrm{b}) \\ (\mathrm{d}-\mathrm{c}) \ldots\ldots (\mathrm{l}-\mathrm{c}) \\ \ldots\ldots\ldots \end{array}$$

With the help of Sylvester's notation, which symbolizes the opposite change, viz., from exponents of powers to suffixes, this may be expressed in the compact form

$$\zeta\mathrm{PD}(abc \ldots l) = \Sigma \pm a_0b_1c_2 \ldots l_{n-1}.$$

In proving it he takes for granted (1) that *the product in question merely changes sign on the interchange of any two of the elements*, and (2) that *in the development of any function of this character there can be no term in which two or more exponents are equal*, for the reason that, if there were one such, there must be another exactly like it but of the opposite sign. Combining with this latter—which includes of course the case where the index 0 is repeated—the fact that, for the particular function under consideration, the indices must all be + and the sum of them equal to $\frac{1}{2}n(n-1)$, he concludes that no term can have any other indices than

$$0,\ 1,\ 2,\ \ldots,\ n-1.$$

Next, as there is only one way of getting an element, k say, in the $(n-1)^{\text{th}}$ power, viz., by multiplying all the $n-1$ binomial factors $k-a$, $k-b$, ... in which k occurs, and after that only one way of getting an element, h say, in the $(n-2)^{\text{th}}$ power, viz., by taking from out the remaining binomial factors all the $n-2$ factors in which h occurs, and so on, it is inferred that no term can have any other coefficient than $+1$ or -1. Summing up rather hurriedly, he consequently finds that the development of the product may be got by permuting in every possible way the indices of the term

$$a^0 b^1 c^2 \ldots l^{n-1}$$

and determining the signs in accordance with the law that the interchange of any pair causes the aggregate of all the terms to pass into the opposite value. This being exactly the mode of formation of the determinant $\Sigma \pm a_0 b_1 c_2 \ldots l_{n-1}$ with the difference that suffixes take the place of exponents of powers, the theorem is held to be established (. . . . "signis insuper ea lege definitis ut binorum indicum commutatione Aggregatum omnium terminorum in valorem oppositum abeat. Quæ ipsa est Determinantis formatio, siquidem exponentes pro indicibus habentur").

In passing, he remarks on the large number of vanishing terms in the development of the product, viz., $2^{\frac{1}{2}n(n^2-1)} - n\,!$, and the consequent desirability of obtaining this development from that of the determinant and not *vice versa.*

The fundamental relation between the determinant

$$\Sigma \pm a_0 b_1 c_2 \ldots l_{n-1}$$

and the product of the differences of $a, b, c, \ldots, l$ having been established, it is then sought to find properties of the latter from the known properties of the former. What properties of the determinant are used Jacobi does not mention, all that is given being a bare enunciation of the results. It may be as well, however, to point out at once that all of them flow from one general theorem, viz., that of Laplace regarding the expansion of a determinant in terms of products of its minors.

The first is indicated by using as examples the case of three elements, a_1, a_2, a_3, and the case of four elements, a_1, a_2, a_3, a_4, viz.,

$$\begin{aligned}(a_2-a_1)(a_3-a_1)(a_3-a_2) = \ & a_2a_3(a_3-a_2)\\ &+a_3a_1(a_1-a_3)\\ &+a_1a_2(a_2-a_1),\end{aligned}$$

$$\begin{aligned}(a_2-a_1)(a_3-a_1)\ldots\ldots(a_4-a_3) = \ & a_2a_3a_4(a_3-a_2)(a_4-a_2)(a_4-a_3)\\ &-a_3a_4a_1(a_4-a_3)(a_1-a_3)(a_1-a_4)\\ &+a_4a_1a_2(a_1-a_4)(a_2-a_4)(a_2-a_1)\\ &-a_1a_2a_3(a_2-a_1)(a_3-a_1)(a_3-a_2)\end{aligned}$$

it being pointed out that any term of the expression is got from the preceding by cyclical permutation of the suffixes, and that the signs are all + when the number of elements is odd, and alternately + and − when the number of elements is even. The case of Laplace's expansion-theorem, which is here used, is easily seen to be that where the orders of the minors are $n-1$ and 1. Thus using later notation, we have

$$\zeta^{\frac{1}{2}}(abcd) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix},$$

$$= |b^1c^2d^3| - |a^1c^2d^3| + |a^1b^2d^3| - |a^1b^2c^3|,$$
$$= bcd|b^0c^1d^2| - acd|a^0c^1d^2| + abd|a^0b^1d^2| - abc|a^0b^1c^2|,$$

which is the desired result.

In connection with this, it is perhaps worth noting that the result being, by the same case of Laplace's theorem, also equal to

$$-\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix},$$

we may view Jacobi's first theorem as being equivalent to one of later date, viz.—

$$\zeta^{\frac{1}{2}}(a_1a_2a_3\ldots a_n) = (-)^{n-1}\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-2} & a_2a_3a_4\ldots a_n \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-2} & a_1a_3a_4\ldots a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-2} & a_1a_2a_3\ldots a_{n-1} \end{vmatrix}.$$

When the determinant is of even order, it is possible to use that case of Laplace's expansion-theorem in which all the minors are of the 2^{nd} order. Thus

$$\zeta^{\frac{1}{2}}(abcd) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix},$$

$$= \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} \cdot \begin{vmatrix} c^2 & c^3 \\ d^2 & d^3 \end{vmatrix} - \begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix} \cdot \begin{vmatrix} b^2 & b^3 \\ d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a \\ 1 & d \end{vmatrix} \cdot \begin{vmatrix} b^2 & b^3 \\ c^2 & c^3 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} \cdot \begin{vmatrix} a^2 & a^3 \\ d^2 & d^3 \end{vmatrix} - \begin{vmatrix} 1 & b \\ 1 & d \end{vmatrix} \cdot \begin{vmatrix} a^2 & a^3 \\ c^2 & c^3 \end{vmatrix} - \begin{vmatrix} 1 & c \\ 1 & d \end{vmatrix} \cdot \begin{vmatrix} a^2 & a^3 \\ b^2 & b^3 \end{vmatrix},$$

$$= (b-a)(d-c)c^2d^2 - (c-a)(d-b)b^2d^2 + (d-a)(c-b)b^2c^2$$

$$+ (c-b)(d-a)a^2d^2 - (d-b)(c-a)a^2c^2 + (d-c)(b-a)a^2b^2,$$

$$= (b-a)(d-c)\{a^2b^2+c^2d^2\}$$

$$+ (c-a)(b-d)\{a^2c^2+d^2b^2\}$$

$$+ (d-a)(c-b)\{a^2d^2+b^2c^2\}.$$

By Jacobi, however, the result here established is given merely as an example of an improved general theorem, which is enunciated in the form of a 'rule,' as follows:—

"Fingatur expressio

$$(a_1-a_0)(a_3-a_2)\ldots(a_n-a_{n-1})\Sigma a_2^2a_3^2a_4^4a_5^4\ldots\ldots a_{n-1}^{n-1}a_n^{n-1}$$

quam quo clarius lex appareat sic scribam

$$(a_1-a_0)(a_3-a_2)\ldots(a_n-a_{n-1})\Sigma(a_0a_1)^0(a_2a_3)^2(a_4a_5)^4\ldots(a_{n-1}a_n)^{n-1},$$

sub signo Σ omnimodis permutatis exponentibus

$$0,\ 2,\ 4, \ldots,\ n-1.$$

In expressione illa cyclum percurrant *primo* elementa tria

$$a_{n-2},\ a_{n-1},\ a_n,$$

secundo elementa quinque

$$a_{n-4},\ a_{n-3},\ a_{n-2},\ a_{n-1},\ a_n,$$

et sic deinceps ita ut *postremo* cyclum percurrant elementa

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n.$$

Omnium expressionum provenientium aggregatum æquabitur ipsi P."

The meaning will be made quite apparent by taking a case other than Jacobi's above referred to, say the case where there are *six* elements, $a_0, a_1, a_2, \ldots, a_5$. According to the rule, what we have got to do at the outset is to form the term

$$(a_1-a_0)(a_3-a_2)(a_5-a_4)\Sigma(a_0a_1)^0(a_2a_3)^2(a_4a_5)^4;$$

then derive from it two others by the cyclical substitution

$$\begin{pmatrix} a_3 & a_4 & a_5 \\ a_4 & a_5 & a_1 \end{pmatrix};$$

and finally, from each of these three derive four others by the cyclical substitution

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{pmatrix}.$$

This being done, the sum of the fifteen terms so obtained can be taken as an expansion of the difference-product of $a_0, a_1, a_2, \ldots, a_5$.

Although, as has been said, the theorem is given without proof, it has to be noted that Jacobi draws attention to the fact that the number of ultimate terms in the expansion of the compound term

$$(a_1-a_0)(a_3-a_2)\ldots(a_n-a_{n-1})\Sigma(a_0a_1)^0(a_2a_3)^2(a_4a_5)^4\ldots(a_{n-1}a_n)^{n-1}$$

is

$$2^{\frac{n+1}{2}}.\left(1.2.3\ldots.\frac{n+1}{2}\right):$$

that the number of ultimate terms obtainable from all the compound terms of this form is

$$2^{\frac{n+1}{2}}.\left(1.2.3\ldots.\frac{n+1}{2}\right).(3.5\ldots.n):$$

and finally that this is equal to

$$1.2.3\ldots.(n+1),$$

a result which agrees with what we know of the difference-product from its determinant form.

From this general theorem regarding the difference-product of an even number of elements, an advance is made to a theorem of still greater generality, the means employed in obtaining it being in all probability the same as before, viz., Laplace's expansion-theorem. The most general form of the latter theorem, it will be remembered, gives an expansion in terms of products of more than two minors. Jacobi was familiar with this, for in his famous fundamental memoir regarding general determinants a whole page (pp. 298, 299) is devoted to an illustration of it. Now, if we take the case where the number of minors is three, and apply it to the determinant which is the equivalent of the difference-product, we obtain a result which is transformable without difficulty into

$$\Pi(a_0, a_1, \ldots, a_n)$$
$$= \sum \pm \left\{ \begin{array}{c} (a_{i+1}a_{i+2} \ldots a_k)^{i+1}(a_{k+1}a_{k+2} \ldots a_n)^{k+1} \\ \times \Pi(a_0, a_1, \ldots, a_i)\Pi(a_{i+1}a_{i+2} \ldots a_k)\Pi(a_{k+1}a_{k+2} \ldots a_n) \end{array} \right\};$$

and this is the theorem "of still greater generality" above referred to.

Jacobi then proceeds to the consideration of alternating functions in general.

The definition which he gives, and to which he attaches Cauchy's name, is somewhat different from Cauchy's, being to the effect that *an alternating function is one which, by permutation of its variables, is either not changed at all, or is changed only in sign.*

In the matter of notation he also introduces a variation, but this time with more success. It will be remembered that, when Cauchy denoted a determinant by prefixing $\mathrm{S}\pm$ to the typical term, he was simply following his practice in regard to alternating functions in general, which he denoted by

$$\mathrm{S} \pm \phi(a,b,c, \ldots, l),$$

the rule for determining the sign of any term of the aggregate being left unexpressed. Instead of this, Jacobi uses

$$\mathrm{P}\sum\left(\frac{\phi(a,b,c, \ldots, l)}{\mathrm{P}}\right),$$

where P stands for the product of the differences of $a, b, c, \ldots, l$; and as the P which is inside the brackets is subject to permutation of its variables, and therefore automatically, as it were, changes sign with every interchange of a pair of variables, while the P which is outside the brackets remains unaltered, it is clear that the rule of signs is here fully expressed. Thus if $\phi(a,b,c,\ldots,l)$ were ab^2c^4, we should have

$$\begin{aligned}\sum\left(\frac{a^1b^2c^4}{\mathrm{P}}\right) = & \frac{a^1b^2c^4}{(b-a)(c-a)(c-b)} + \frac{a^1c^2b^4}{(c-a)(b-a)(b-c)} \\ & + \frac{b^1a^2c^4}{(a-b)(c-b)(c-a)} + \frac{b^1c^2a^4}{(c-b)(a-b)(a-c)} \\ & + \frac{c^1a^2b^4}{(a-c)(b-c)(b-a)} + \frac{c^1b^2a^4}{(b-c)(a-c)(a-b)}, \\ = & \frac{a^1b^2c^4 - a^1c^2b^4 - b^1a^2c^4 + b^1c^2a^4 + c^1a^2b^4 - c^1b^2a^4}{(b-a)(c-a)(c-b)},\end{aligned}$$

and therefore

$$\mathrm{P}\sum\left(\frac{a^1b^2c^4}{\mathrm{P}}\right) = a^1b^2c^4 - a^1c^2b^4 - b^1a^2c^4 + b^1c^2a^4 + c^1a^2b^4 - c^1b^2a^4,$$

which is an alternating function written by Cauchy in the form $\mathrm{S}(\pm a^1b^2c^4)$, and which, being a determinant, was written by Jacobi himself also in the form $\Sigma \pm a^1b^2c^4$.

It is pointed out that any term of ϕ which remains unchanged by the interchange of two of the variables may be left out of account; but the question raised by Cauchy regarding possible and impossible forms of ϕ is not touched upon. As a corollary, it is stated that if

$$\phi(a_0, a_1, \ldots, a_n) = a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n},$$

the indices $\alpha_0, \alpha_1, \ldots, \alpha_n$ must be all different if the alternating function is not to vanish.

He then recalls the known fact that, when the indices $\alpha_0, \alpha_1, \ldots, \alpha_n$ are integral, the alternating function

$$\Sigma \pm a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n} \quad \text{or} \quad \mathrm{P}\sum\frac{a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n}}{\mathrm{P}}$$

is divisible by P, the difference-product of $a_0, a_1, \ldots, a_n$, and puts to himself the problem of finding the generating function of the quotient

$$\sum\frac{a_0^{\alpha_0}a_1^{\alpha_1} \ldots a_n^{\alpha_n}}{\mathrm{P}}.$$

In the course of this quest his first proposition is—

If ϕ be any rational integral function of $\mathrm{m}+1$ *variables,* Π *their difference-product, and* f *be a function of the* $(\mathrm{n}+1)^{\mathrm{th}}$ *degree in one variable and be of the form* $(\mathrm{x}-\mathrm{a}_0)(\mathrm{x}-\mathrm{a}_1)\ldots.(\mathrm{x}-\mathrm{a}_\mathrm{n})$, *then when* $\mathrm{m}>\mathrm{n}$ *no single term of the expansion of*

$$\frac{\Pi(\mathrm{t}_0,\mathrm{t}_1,\ldots.,\mathrm{t}_\mathrm{m})\,.\,\phi(\mathrm{t}_0,\mathrm{t}_1,\ldots.,\mathrm{t}_\mathrm{m})}{\mathrm{f}(\mathrm{t}_0)\mathrm{f}(\mathrm{t}_1)\ldots.\mathrm{f}(\mathrm{t}_\mathrm{m})},$$

according to descending powers of $\mathrm{t}_0, \mathrm{t}_1, \ldots, \mathrm{t}_\mathrm{m}$, *can contain negative powers of all these variables.*

To prove it he of course uses the identity

$$\frac{1}{f(x)}\quad i.e.,\quad \frac{1}{(x-a_0)(x-a_1)\ldots.(x-a_m)}$$

$$=\frac{1}{f'(a_0).(x-a_0)}+\frac{1}{f'(a_1).(x-a_1)}+\ldots\ldots+\frac{1}{f'(a_m).(x-a_m)},$$

and thus changes the expression into the form

$$\begin{aligned}\Pi.\phi.&\left\{\frac{1}{f'(a_0).(t_0-a_0)}+\frac{1}{f'(a_1).(t_0-a_1)}+\ldots.+\frac{1}{f'(a_m).(t_0-a_m)}\right\}\\ \times&\left\{\frac{1}{f'(a_0).(t_1-a_0)}+\frac{1}{f'(a_1).(t_1-a_1)}+\ldots.+\frac{1}{f'(a_n).(t_1-a_m)}\right\}\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \times&\left\{\frac{1}{f'(a_0).(t_m-a_0)}+\frac{1}{f'(a_1).(t_m-a_1)}+\ldots.+\frac{1}{f'(a_n).(t_m-a_m)}\right\}.\end{aligned}$$

He then says that the result of performing the multiplication of these bracketed factors is to produce terms of the form

$$\frac{\Pi.\phi}{f'(a)f'(b)\ldots.f'(p).(t_0-a)(t_1-b)\ldots.(t_m-p)},$$

where each of the $m+1$ quantities $a, b, \ldots, p$ is necessarily one of the $n+1$ quantities $a_0, a_1, \ldots, a_n$, and where, therefore, on account of m being greater than n, the quantities $a, b, \ldots, p$ cannot be all different. But terms of this form can be changed into

$$\frac{\phi}{f'(a)f'(b)\ldots.f'(p)}\cdot\frac{\Pi}{t_1-b-t_0+a}\left\{\frac{1}{t_0-a}-\frac{1}{t_1-b}\right\}\cdot\frac{1}{(t_2-c)(t_3-d)\ldots.(t_m-p)},$$

which shows that in the case of two of the quantities $a, b, \ldots, p$ being alike, say a and b, the second factor would become

$$\frac{\Pi}{t_1 - t_0},$$

and therefore could be simplified by having $t_1 - t_0$ struck out of both numerator and denominator. This means that when $m > n$ the second factor, like the first, can have only positive integral powers of the variables. As for the third and fourth factors, their product is the difference of the two fractions

$$\frac{1}{(t_0-a)(t_2-c)(t_3-d)\ldots(t_m-p)} \quad \text{and} \quad \frac{1}{(t_1-a)(t_2-c)(t_3-d)\ldots(t_m-p)},$$

the former of which yields no negative powers of t_1, and the latter no negative powers of t_0. The proposition is thus established.

To prove the next proposition he utilizes the theorem that

If F *be any rational integral function of a number of variables, the coefficient of* $\mathrm{x}^{-1}\mathrm{y}^{-1}\mathrm{z}^{-1}\ldots$ *in the expansion of*

$$\frac{\mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots)}{(\mathrm{x}-\mathrm{a})(\mathrm{y}-\mathrm{b})(\mathrm{z}-\mathrm{c})\ldots}$$

according to descending powers of x, y, z, ... *is*

$$\mathrm{F}(\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots).$$

This is spoken of as being well-known, and no proof of it is given. It is readily seen, however, that as the expansion referred to is got by performing the multiplications indicated in

$$\begin{aligned} F(x,y,z,\ldots)\,.\,&\{x^{-1}+ax^{-2}+a^2x^{-3}+\ldots\} \\ &\{y^{-1}+by^{-2}+b^2y^{-3}+\ldots\} \\ &\{z^{-1}+cz^{-2}+c^2z^{-3}+\ldots\} \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

any term in F, say the term $\mathrm{A}x^\alpha y^\beta z^\gamma \ldots$, would require to be multiplied by $x^{-\alpha-1}$, $y^{-\beta-1}$, $z^{-\gamma-1}, \ldots$ in order to produce a term in $x^{-1}y^{-1}z^{-1}\ldots$, and that these multipliers being only found associated with the coefficients a^α, b^β, c^γ, ... the term so produced would have for its coefficient $\mathrm{A}a^\alpha b^\beta c^\gamma \ldots$. The full coefficient of $x^{-1}y^{-1}z^{-1}\ldots$ would thus be $\mathrm{F}(a,b,c,\ldots)$.

He also uses an identity regarding difference-products which it may be as well to state separately,* viz., that

$$\Pi(a_0, a_1, \ldots\ldots a_n) \cdot \Pi(a_{n-m}, a_{n-m+1}, \ldots, a_n)$$
$$= (-1)^{\frac{1}{2}m(m+1)}\Pi(a_0, a_1, \ldots, a_{n-m-1}) \cdot f'(a_{n-m})f'(_{n-m+1}) \ldots f'(a_n),$$

where $f'(a_r)$ stands for the product of the n factors got by subtracting from a_r each of the quantities $a_0, a_1, \ldots, a_n$ except a_r. This he holds to be true,* because the product

$$f'(a_{n-m})f'(a_{n-m+1}) \ldots f'(a_n)$$

* The factors of a difference-product may always be, and usually are, arranged in the form of a right-angled isosceles triangle; for example,

$$\begin{array}{rl} \zeta^{\frac{1}{2}}(abcdefg) = & (b-a)(c-a)(d-a)(e-a)(f-a)(g-a) \\ & \quad (c-b)(d-b)(e-b)(f-b)(g-b) \\ & \qquad (d-c)(e-c)(f-c)(g-c) \\ & \qquad\quad (e-d)(f-d)(g-d) \\ & \qquad\qquad (f-e)(g-e) \\ & \qquad\qquad\quad (g-f). \end{array}$$

Consequently there must be an algebraic identity corresponding to the geometrical proposition—*If from a point in the hypotenuse of an isosceles right-angled triangle straight lines be drawn parallel to the other sides, the triangle is thereby divided into two triangles of the same kind and a rectangle.* This identity it is which is at the basis of Jacobi's, for drawing the lines thus—

$$\begin{array}{r|l} (b-a)(c-a)(d-a) & (e-a)(f-a)(g-a) \\ (c-b)(d-b) & (e-b)(f-b)(g-b) \\ (d-c) & (e-c)(f-c)(g-c) \\ & (e-d)(f-d)(g-d) \\ \hline & \quad (f-e)(g-e) \\ & \qquad (g-f), \end{array}$$

we obtain

$$\begin{array}{rl} \zeta^{\frac{1}{2}}(abcdefg) = \zeta^{\frac{1}{2}}(abcd) \cdot \zeta^{\frac{1}{2}}(efg) \cdot & (e-a)(f-a)(g-a) \\ & (e-b)(f-b)(g-b) \\ & (e-c)(f-c)(g-c) \\ & (e-d)(f-d)(g-d). \end{array} \qquad (\omega)$$

But the expression here which corresponds to the rectangle in the geometrical proposition

$$= \left.\begin{array}{l} (e-a)(f-a)(g-a) \\ (e-b)(f-b)(g-b) \\ (e-c)(f-c)(g-c) \\ (e-d)(f-d)(g-d) \\ \quad\cdot\quad (f-e)(g-e) \\ (e-f) \quad\cdot\quad (g-f) \\ (e-g)(f-g) \quad\cdot \end{array}\right\} \div \zeta^{\frac{1}{2}}(efg) \cdot \zeta^{\frac{1}{2}}(gfe)$$

$$= f'(e) \cdot f'(f) \cdot f'(g) \div (-)^3\zeta^{\frac{1}{2}}(efg) \cdot \zeta^{\frac{1}{2}}(efg).$$

contains as factors the differences of all the elements $a_0, a_1, \ldots, a_n$ except those which go to make $\Pi(a_0, a_1, \ldots, a_{n-m-1})$ and contains a second time but with opposite signs the $\frac{1}{2}m(m+1)$ factors which go to make $\Pi(a_{n-m}, a_{n-m+1}, \ldots, a_n)$.

These preliminaries having been given, the second proposition may now be proceeded with. It is—

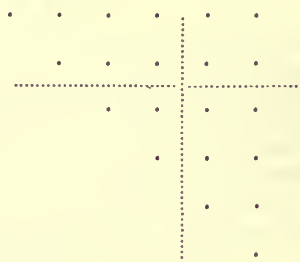
If ϕ be any rational integral function of m + 1 *variables,* Π *their difference-product, and* f *be a function of the* $(\mathrm{n}+1)^{\mathrm{th}}$ *degree in*

Consequently

$$\frac{\zeta^{\frac{1}{2}}(abcdefg)\,.\,\zeta^{\frac{1}{2}}(efg)}{\zeta^{\frac{1}{2}}(abcd)} = (-)^3 f'(e)\,.\,f'(f)\,.\,f'(g). \qquad (\Omega)$$

which is Jacobi's identity.

It is easily seen that there is a corresponding theorem to (ω) obtained when the point through which the parallels are drawn is taken inside the triangle: thus, corresponding to the diagram

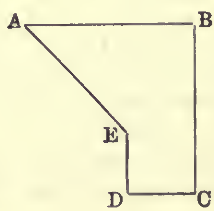


we have the identity

$$\zeta^{\frac{1}{2}}(abcdefg) = \frac{\zeta^{\frac{1}{2}}(abcde)\,.\,\zeta^{\frac{1}{2}}(cdefg)}{\zeta^{\frac{1}{2}}(cde)}\,.\,(f-a)(g-a)(f-b)(g-b). \qquad (\omega')$$

Here, however, it is no longer possible to treat the final group of factors as was done in the case of (ω).

To Jacobi's identity (Ω) the absolutely perfect geometrical analogue is got by taking a rectilineal figure of the form ABCDE, where AB=BC, CD=DE, B=C=D=90°, and then equating the sum of the two parts got by drawing CE to the sum of the two parts got by producing DE to meet AB in F. Further, the exact analogue to his proof would be to say that the rectangle BCDF contains all of the triangle ABC except the triangle AEF, and contains the triangle CDE in addition.



one variable and be of the form $(x-a_0)(x-a_1)\ldots\ldots(x-a_n)$, *then when* $m \not> n$ *the coefficient of* $t_0^{-1}t_1^{-1}\ldots\ldots t_m^{-1}$ *in the expansion of*

$$\frac{\Pi(t_0, t_1, \ldots, t_m)\,.\,\phi(t_0, t_1, \ldots, t_m)}{f(t_0)f(t_1)\ldots\ldots f(t_m)}$$

according to descending powers of $t_0, t_1, \ldots\ldots, t_m$ *is*

$$(-)^{\frac{1}{2}m(m+1)}\sum\frac{a_0^0a_1^1a_2^2\ldots\ldots a_{n-m-1}^{n-m-1}\phi(a_{n-m}, a_{n-m+1}, \ldots\ldots, a_n)}{\Pi(a_0, a_1, \ldots\ldots, a_n)},$$

effect being given to the sign of summation by permuting in every possible way the quantities $a_0, a_1, \ldots\ldots, a_n$.

As has already been seen the expression to be expanded is equal to an aggregate of terms of the form

$$\frac{\phi(t_0, t_1, \ldots\ldots, t_m)\,.\,\Pi(t_0, t_1, \ldots\ldots, t_m)}{f'(a)f'(b)\ldots\ldots f'(p)\,.\,(t_0-a)(t_1-b)\ldots\ldots(t_m-p)},$$

where each of the $m+1$ quantities $a, b, \ldots\ldots, p$ is one of the $n+1$ quantities $a_0, a_1, \ldots\ldots, a_n$. Since, however, we are now in search of the coefficient of $t_0^{-1}t_1^{-1}\ldots\ldots t_m^{-1}$ we may leave out of account all terms of this aggregate which have two or more of the $m+1$ quantities $a, b, \ldots\ldots, p$ alike, for it has been shown that the expansion of such a term cannot contain $t_0^{-1}t_1^{-1}\ldots\ldots t_m^{-1}$. We are thus left with an aggregate which may be represented by

$$S\frac{\phi(t_0, t_1, \ldots\ldots, t_m)\,.\,\Pi(t_0, t_1, \ldots\ldots, t_m)}{f'(a_{n-m})f'(a_{n-m+1})\cdots f'(a_n)\,.\,(t_0-a_{n-m})(t_1-a_{n-m+1})\cdots(t_m-a_n)},$$

it being understood that for $a_{n-m}, a_{n-m+1}, \ldots\ldots, a_n$ is to be taken any permutation of $m+1$ quantities of the group $a_0, a_1, \ldots\ldots, a_n$. But, if the coefficient of $t_0^{-1}t_1^{-1}\ldots\ldots t_m^{-1}$ in this be denoted by H, we have by the first of our auxiliary theorems

$$H = S\frac{\phi(a_{n-m}, a_{n-m+1}, \ldots\ldots, a_n)\,.\,\Pi(a_{n-m}, a_{n-m+1}, \ldots\ldots, a_n)}{f'(a_{n-m})f'(a_{n-m+1})\ldots\ldots f'(a_n)},$$

and using the second to substitute

$$(-1)^{\frac{1}{2}m(m+1)}\Pi(a_0, a_1, \ldots\ldots, a_{n-m-1})\Pi(a_0, a_1, \ldots\ldots, a_n)$$

for $\quad \Pi(a_{n-m}, a_{n-m+1}, \ldots\ldots, a_n)/f'(a_{n-m})f'(a_{n-m+1})\ldots\ldots f'(a_n)$, we have

$$H=(-1)^{\frac{1}{2}m(m+1)}S\frac{\Pi(a_0,a_1,\ldots,a_{n-m-1})\cdot\phi(a_{n-m},a_{n-m+1},\ldots,a_n)}{P},$$

where, be it remembered, the $n+1$ elements $a_0, a_1, \ldots, a_n$ are to be separated in every possible way into two classes containing $n-m$ and $m+1$ elements respectively, and all permutations of the elements of the second class are to be taken. In this expression, however, another substitution can be made by reason of the identity

$$\frac{\Pi(a_0,a_1,\ldots,a_{n-m-1})}{P}=\sum\frac{a_0^0 a_1^1\ldots a_{n-m-1}^{n-m-1}}{P},$$

where under the sign Σ all possible permutations of the indices $0, 1, \ldots, n-m-1$ are to be taken. When this substitution has been made, we shall consequently have to take every possible permutation of *both* classes of elements. But to take every possible separation into two classes and permute the elements of each of the classes in every possible way is the same as to take every possible permutation of *all* the elements. Our result will therefore be

$$H=(-1)^{\frac{1}{2}m(m+1)}\sum\frac{a_0^0 a_1^1\ldots a_{n-m-1}^{n-m-1}\cdot\phi(a_{n-m},a_{n-m+1},\ldots,a_n)}{P},$$

if it be understood that under the sign of summation all possible permutations of $a_0, a_1, \ldots, a_n$ are to be taken: and this is what we set out to prove.

The case where $m=n$ is then considered, because of its special interest. The first expression obtained above for H becomes in this case

$$\sum\frac{P\cdot\phi(a_0,a_1,\ldots,a_n)}{f'(a_0)f'(a_1)\ldots f'(a_n)},$$

where under Σ all permutations of $a_0, a_1, \ldots, a_n$ are to be taken. Making in this the substitution which is possible by reason of the identity

$$f'(a_0)f'(a_1)\ldots f'(a_n) = (-1)^{\frac{1}{2}n(n+1)}P^2,$$

we have

$$H=(-1)^{\frac{1}{2}n(n+1)}S\frac{\phi(a_0,a_1,\ldots,a_n)}{P}.$$

The formal enunciation of the result thus obtained is:—

If ϕ *be any rational integral function of* $n+1$ *variables,* Π *their difference-product, and* f *be a function of the* $(n+1)^{th}$ *degree in one variable and be of the form* $(x-a_0)(x-a_1)\ldots\ldots(x-a_n)$; *then the coefficient of* $t_0^{-1}t_1^{-1}\ldots\ldots t_n^{-1}$ *in the expansion of*

$$(-1)^{\frac{1}{2}n(n+1)}\frac{\Pi(t_0,t_1,\ldots\ldots,t_n)\,.\,\phi(t_0,t_1,\ldots\ldots,t_n)}{f(t_0)f(t_1)\ldots\ldots f(t_n)}$$

is

$$\sum\frac{\phi(a_0,a_1,\ldots\ldots,a_n)}{\Pi(a_0,a_1,\ldots\ldots,a_n)},$$

effect being given to the sign of summation by permuting in every possible way the elements $a_0, a_1, \ldots\ldots, a_n$.

As we have seen above that

$$\sum\frac{\phi(a_0,a_1,\ldots\ldots,a_n)}{\Pi(a_0,a_1,\ldots\ldots,a_n)}$$

is the quotient of any rational integral alternating function by the difference-product of its elements, and that this quotient is often in request, it is important for practical purposes to note that what this last theorem of Jacobi's gives is the generating function of the said quotient.

After giving a line or two to the case where $m=n-1$, Jacobi returns to the general theorem and specializes in another direction, viz., by putting

$$\phi(t_0,t_1,\ldots\ldots,t_m) = t_0^{\gamma}t_1^{\gamma_1}\ldots\ldots t_m^{\gamma_m}.$$

Division of both sides by ϕ is in this case possible, and the resulting theorem is one of considerable importance:—

The expression

$$\sum\frac{a_0^0a_1^1\ldots\ldots a_{n-m-1}^{n-m-1}a_{n-m}^{\gamma}a_{n-m+1}^{\gamma_1}\ldots\ldots a_n^{\gamma_m}}{(a_1-a_0)(a_2-a_0)\ldots\ldots(a_n-a_{n-1})},$$

which is the quotient of an alternating function by the difference-product of its elements, is equal to the coefficient of

$$t_0^{-(\gamma+1)}t_1^{-(\gamma_1+1)}\ldots\ldots t_m^{-(\gamma_m+1)}$$

in the expansion of

$$\frac{(t_0-t_1)(t_0-t_2)\ldots.(t_{n-1}-t_n)}{f(t_0)f(t_1)\ldots.f(t_m)}$$

according to descending powers of $t_0, t_1, \ldots., t_m$, *where*

$$f(x) = (x-a_0)(x-a_1)\ldots.(x-a_n).$$

This is followed up by actually working out the expansion in question, the numerator being of course changed into

$$\Sigma \pm t_0^m t_1^{m-1}\ldots. t_{m-1},$$

and its cofactor

$$\frac{1}{f(t_0)}\cdot\frac{1}{f(t_1)}\ldots.\frac{1}{f(t_m)}$$

into

$$\left(\frac{1}{t_0^{n+1}}+\frac{C_1}{t_0^{n+2}}+\frac{C_2}{t_0^{n+3}}+\ldots.+\frac{C_s}{t_0^{n+1+s}}+\ldots.\right)$$

$$\times\left(\frac{1}{t_1^{n+1}}+\frac{C_1}{t_1^{n+2}}+\frac{C_2}{t_1^{n+3}}+\ldots.+\frac{C_s}{t_1^{n+1+s}}+\ldots.\right)$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\times\left(\frac{1}{t_m^{n+1}}+\frac{C_1}{t_m^{n+2}}+\frac{C_2}{t_m^{n+3}}+\ldots.+\frac{C_s}{t_m^{n+1+s}}+\ldots.\right),$$

where C_s is the sum of all the products of s elements, different or equal, taken from $a_0, a_1, \ldots., a_n$. Multiplication of these $m+1$ partial factors has next to be performed, the general term of the result being seen to be

$$\frac{C_{s_0}C_{s_1}\ldots.C_{s_m}}{t_0^{n+1+s_0}t_1^{n+1+s_1}\ldots.t_m^{n+1+s_m}}.$$

All that remains, then, is the multiplication of this result by the corresponding expression for the original numerator, *i.e.*, by $\Sigma \pm t_0^m t_1^{m-1}\ldots. t_{m-1}$, which, be it noted, consists of $(m+1)^2$ terms, the Σ referring to permutation of the indices, $m, m-1, \ldots. 1, 0$. Without further delay, Jacobi merely adds that the general term will therefore become

$$\sum \pm \frac{C_{s_0}C_{s_1}\ldots.C_{s_m}}{t_0^{n-m+1+s_0}t_1^{n-m+1+s_1}\ldots.t_m^{n-m+1+s_m}},$$

and that consequently the proposition last formulated will "suggest" the identity

$$\sum \frac{a_1^1 a_2^2 \ldots\ldots a_{n-m-1}^{n-m-1} a_{n-m}^{\gamma} a_{n-m+1}^{\gamma_1} \ldots\ldots a_n^{\gamma_m}}{(a_1-a_0)(a_2-a_0)\ldots\ldots(a_n-a_{n-1})}$$
$$= \Sigma \pm \mathrm{C}_{\gamma+m-n}\mathrm{C}_{\gamma_1+m-n-1}\ldots\ldots \mathrm{C}_{\gamma_m-n},$$

where the Σ in the first case refers to permutation of a_0, a_1, ..., a_n, and in the second case to permutation of $\gamma, \gamma_1, \ldots\ldots, \gamma_m$. In a couple of lines it is next pointed out that the putting of $m=0$, $m=1$, ... in this suggested identity gives

$$\sum \frac{a_1^1 a_2^2 \ldots\ldots a_{n-1}^{n-1} a_n^{\gamma}}{\mathrm{P}} = \mathrm{C}_{\gamma-n},$$

$$\sum \frac{a_1^1 a_2^2 \ldots\ldots a_{n-2}^{n-2} a_{n-1}^{\gamma} a_n^{\gamma_1}}{\mathrm{P}} = \mathrm{C}_{\gamma+1-n}\mathrm{C}_{\gamma_1-n} - \mathrm{C}_{\gamma_1+1-n}\mathrm{C}_{\gamma-n},$$

. . . .;

then, rather unexpectedly, there is given a mere restatement of the identity itself, viz.:—

"Generaliter *æquatur quotiens propositus*

$$\sum \frac{a_1^1 a_2^2 \ldots\ldots a_{n-m-1}^{n-m-1} a_{n-m}^{\gamma} a_{n-m+1}^{\gamma_1} \ldots\ldots a_n^{\gamma_m}}{\mathrm{P}}$$

determinanti quod pertinet ad systema quantitatum

$$\begin{array}{llll}
\mathrm{C}_{\gamma+m-n} & \mathrm{C}_{\gamma_1+m-n} & \ldots\ldots & \mathrm{C}_{\gamma_m+m-n} \\
\mathrm{C}_{\gamma+m-n-1} & \mathrm{C}_{\gamma_1+m-n-1} & \ldots\ldots & \mathrm{C}_{\gamma_m+m-n-1} \\
\cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot \\
\mathrm{C}_{\gamma-n} & \mathrm{C}_{\gamma_1-n} & \ldots\ldots & \mathrm{C}_{\gamma_m-n}.\text{''}
\end{array}$$

This is the last result of the memoir, the few additional lines used being merely for the purpose of showing how the determinant just mentioned may be simplified. The simplification consists in leaving out the element a_n in forming the C's of the second row from the end, the elements a_n, a_{n-1} in forming the C's of the third row from the end, and so on. The reason in the first case is that this will have the same effect as subtracting from each element of the row a_n times the corresponding element of the last row, and the reason in other cases is similar. If C′ be used to stand for the same as C, but to concern one element less, viz., a_n, and C″ be used in similar manner, the identities at the bottom of the simplification are—

$$C_{s+1} - a_n C_s = C'_{s+1},$$
$$C_{s+2} - (a_n + a_{n-1})C_{s+1} + a_n a_{n-1} C_s = C''_{s+2},$$
$$\cdots\cdots\cdots\cdots$$

the truth of which is apparent when we remember that C_1, C_2, ... are practically defined by the equation

$$\frac{1}{(x-a_0)(x-a_1)\ldots.(x-a_n)} = \frac{1}{x^{n+1}} + \frac{C_1}{x^{n+2}} + \frac{C_2}{x^{n+3}} + \ldots.$$

It is noted also that in the determinant a C with the suffix 0 is to be taken as 1, and a C with a negative suffix as 0.

CAUCHY (1841).

[Mémoire sur les fonctions alternées et sur les sommes alternées. *Exercices d'analyse et de phys. math.*, ii. pp. 151–159; or *Œuvres complètes*, 2ᵉ sér. xii.]

As has before been pointed out, the preceding paper of Jacobi's was the last of a triad which was followed up by a similar triad from the pen of Cauchy. Cauchy's first paper, which corresponds in subject to Jacobi's third, comes up therefore quite appropriately for discussion now.

What is really new in the first part of it concerns the finding of the symmetric function which is the quotient of an alternating function by the difference-product of the elements; that is to say, in Cauchy's notation, the finding of

$$\frac{S[\pm f(x, y, z, \ldots.)]}{(x-y)(x-z)\ldots.(y-z)\ldots.},$$

or, in Jacobi's notation, the finding of

$$\sum \frac{f(x, y, z, \ldots.)}{\Pi(x, y, z, \ldots.)}$$

It therefore opens with the reminder :—

"Une fraction rationnelle qui a pour dénominateur une fonction symétrique et pour numérateur une fonction alternée des variables $x, y, z, \ldots.$ est évidemment elle-même une fonction alternée de ces variables. Réciproquement, si une fonction alternée de $x, y, z, \ldots.$ se trouve représentée par une fraction rationnelle dont le dénominateur se réduise à une fonction symétrique, le numérateur de la même fraction rationnelle sera nécessairement une autre fonction alternée de $x, y, z. \ldots.$"

This prepares us for the consideration of the alternating aggregate

$$S[\pm f(x, y, z, \ldots.)],$$

where f is fractional and rational, and where, although Cauchy does not explicitly say so, the numerator and denominator are integral. In regard to this he asserts that the various fractions which compose the aggregate may be combined into one fraction U/V, where V is an integral symmetric function divisible by all the denominators, and where, therefore, U will necessarily be an integral alternating function and, as such, be divisible by the difference-product of its variables. We are thus led to the proposition that the given alternating function of $x, y, z, \ldots.$ can be resolved into two factors, one of which is the difference-product (P) of $x, y, z, \ldots.$, and the other of the form W/V, where W and V are integral symmetric functions of the same variables.

As an illustration of this, full consideration is given to the case where

$$f(x, y, z, \ldots.) = \frac{1}{(x-a)(y-b)(z-c)\ldots.},$$

the number of variables being n. The appropriate symmetric function V, which is divisible by all the denominators of the aggregate $\Sigma[\pm f(x, y, z, \ldots.)]$ is evidently in this case

$$(x-a)(x-b)(x-c)\ldots(y-a)(y-b)(y-c)\ldots(z-a)(z-b)(z-c)\ldots$$

or say,

$$\mathrm{F}(x) . \mathrm{F}(y) . \mathrm{F}(z) \ldots.;$$

and the corresponding numerater U, always divisible by the difference-product of $x, y, z, \ldots.$ is in this case, because of the peculiar form* of the denominator of the function f, also

* The form is such that the result of any interchange among $x, y, z, \ldots.$ is attainable by a corresponding interchange among $a, b, c, \ldots.$

divisible by the difference-product of $a, b, c, \ldots.$ It is thus seen that the given alternating aggregate

$$\sum\left[\pm\frac{1}{(x-a)(x-b)(x-c)\ldots.}\right] = k.\frac{\mathrm{PP}'}{\mathrm{V}},$$

where P, P′, V are known, and k has still to be found. An easy step further is made by inquiring as to the *degree* of k, it being noted in this connection that the degree on the one side is $-n$, and that on the other side the degree of $\mathrm{P}=\frac{1}{2}n(n-1)$, the degree of P′ likewise $=\frac{1}{2}n(n-1)$, and the degree of $\mathrm{V}=n^2$. The resultant degree of PP′/V on the right is therefore inferred to be

$$\begin{aligned} &= \tfrac{1}{2}n(n-1) + \tfrac{1}{2}n(n-1) - n^2, \\ &= -n; \end{aligned}$$

and as a consequence the degree of k must be zero. In other words, k must be constant in regard to $x, y, z, \ldots., a, b, c, \ldots.$. so that for its full determination the best thing to do is to select as easy a special case as possible. Cauchy's choice falls on the case where $x=a$, $y=b$, $z=c, \ldots.$; and preparatory for this substitution he transforms the above result,

$$\sum\left[\pm\frac{1}{(x-a)(y-b)(z-c)\ldots.}\right] = k.\frac{\mathrm{PP}'}{\mathrm{V}},$$

into

$$\begin{aligned} k.\mathrm{PP}' &= \mathrm{V}.\sum\left[\pm\frac{1}{(x-a)(y-b)(z-c)\ldots.}\right], \\ &= \sum\left[\pm\frac{\mathrm{V}}{(x-a)(y-b)(z-c)\ldots.}\right]. \end{aligned}$$

As for the right side of this, it has to be noted that, since V contains each of the binomials $x-a$, $y-b$, $z-c, \ldots.$ once and once only, any one of the $1.2.3\ldots.n$ terms under Σ will vanish when the substitution

$$x, y, z, \ldots. = a, b, c, \ldots.$$

is made, unless the denominator of the term also contains *all* the said binomials. But by reason of the interchanges which produce the other denominators, the first term is the only one of this kind: and the value of it after the substitution has been made is

$$(a-b)(a-c)\ldots.(b-a)(b-c)\ldots.(c-a)(c-b)\ldots.$$

an expression which, as we have already seen in the preceding paper of Jacobi's,* is equal to

$$(-1)^{\frac{1}{2}n(n-1)}\mathrm{P}^2.$$

As the left-hand side, $k\mathrm{PP}'$, becomes under the same circumstances

$$k.\mathrm{P}^2,$$

we have as our last desideratum

$$k = (-1)^{\frac{1}{2}n(n-1)},$$

and are thus enabled to formulate the proposition

$$\sum\left[\pm\frac{1}{(x-a)(y-b)(z-c)\ldots}\right]$$

$$= (-1)^{\frac{1}{2}n(n-1)}\frac{\mathrm{P}(x,y,z,\ldots).\mathrm{P}(a,b,c,\ldots)}{(x-a)(x-b)(x-c)\ldots(y-a)(y-b)(y-c)\ldots(z-a)(z-b)(z-c)\ldots},$$

a noteworthy result which in later notation takes the form

$$\begin{vmatrix} (x-a)^{-1} & (x-b)^{-1} & (x-c)^{-1} & \ldots \\ (y-a)^{-1} & (y-b)^{-1} & (y-c)^{-1} & \ldots \\ (z-a)^{-1} & (z-b)^{-1} & (z-c)^{-1} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = (-1)^{\frac{1}{2}n(n-1)}\frac{\zeta^{\frac{1}{2}}(x,y,z,\ldots).\zeta^{\frac{1}{2}}(a,b,c,\ldots)}{\mathrm{F}(x).\mathrm{F}(y).\mathrm{F}(z)\ldots},$$

where n is the number of variables, and

$$\mathrm{F}(x) = (x-a)(x-b)(x-c)\ldots.$$

---

* Since $\mathrm{V}=\mathrm{F}(x).\mathrm{F}(y).\mathrm{F}(z)\ldots.$, the first term of the alternating aggregate may be written

$$\frac{\mathrm{F}(x)}{x-a}\cdot\frac{\mathrm{F}(y)}{y-b}\cdot\frac{\mathrm{F}(z)}{z-c}\cdot\ldots,$$

which, on the substitution being made, becomes

$$\mathrm{F}'(a).\mathrm{F}'(b).\mathrm{F}'(c)\ldots.;$$

and it is this form which in Jacobi is replaced by $(-1)^{\frac{1}{2}n(n-1)}\mathrm{P}^2$.

CHAPTER XIII.

JACOBIANS FROM THE YEAR 1815 TO 1841.

It is not improbable that determinants in which the number of a row is distinguished by differentiation with respect to a definite variable, and in which the number of a column is distinguished by a particular function set for differentiation, may have appeared long before the time of Cauchy and Jacobi, the likelihood probably being the greater the fewer the number of functions and variables involved. There can be little doubt, for example, that expressions like

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

may be found repeatedly in the writings of mathematicians belonging to the eighteenth century. It would appear, however, that the first who got beyond the second order, and clearly associated the expressions with determinants, was Cauchy.

CAUCHY (1815).

[Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie. *Mém. présentés par divers savants à l'Acad. roy. des Sci. de l'Inst. de France* .... i. pp. 1–312 (1827); or *Œuvres complètes*, 1ᵉ sér. i. pp. 5–318.]

Cauchy was a competitor for the prize for mathematical analysis in the 'concours' of 1815, and gained the prize. His work, however, like others belonging to that interesting political period, was not printed until long afterwards. In the form which it takes in the collected works the essay proper extends to only 108 pages, the remaining 210 being occupied with notes: this was probably due to the circumstances under which the paper was first written. In the same way is explained the writer's action in referring in it to himself by name, the object being to preserve his anonymity.

There is only one passage in it which directly concerns the student of determinants, but it is interesting from more than one point of view. The exact wording of the passage (pp. 11, 12) is as follows:—

"Cela posé, concevons que le sommet de la molécule m, auquel appartenaient, dans le premier instant, les trois coordonnées a, b, c, se trouve, au bout de temps t, transporté en un point dont les coordonnées soient x, y, z. Les trois arêtes de la molécule qui aboutissaient au sommet dont il s'agit, et qui, dans l'origine, se trouvaient parallèles aux trois axes des coordonnées, auront alors cessé de l'être, et les projections de ces mêmes arêtes sur les axes dont il s'agit, projections qui dans l'origine étaient respectivement égales,

$$\begin{array}{lccc} \text{pour la première arête à} & da, & 0, & 0, \\ \text{pour la seconde, à} \;\ldots\ldots & 0, & db, & 0, \\ \text{pour la troisième, à} \;\ldots\; & 0, & 0, & dc, \end{array}$$

seront alors devenues

$$\begin{array}{llll} \text{pour la première arête} & \dfrac{dx}{da}da, & \dfrac{dy}{da}da, & \dfrac{dz}{da}da, \\[2ex] \text{pour la seconde} & \dfrac{dx}{db}db, & \dfrac{dy}{db}db, & \dfrac{dz}{db}db, \\[2ex] \text{pour la troisième} & \dfrac{dx}{dc}dc, & \dfrac{dy}{dc}dc, & \dfrac{dz}{dc}dc. \end{array}$$

Il est aisé d'en conclure (voir la Note I.) que le volume de la molécule, qui était primitivement égal à

$$da\,db\,dc,$$

sera devenu, aut bout du temps t,

$$\left(\frac{dx}{da}\frac{dy}{db}\frac{dz}{dc} - \frac{dx}{da}\frac{dy}{dc}\frac{dz}{db} - \frac{dx}{db}\frac{dy}{da}\frac{dz}{dc} + \frac{dx}{db}\frac{dy}{dc}\frac{dz}{da}\right.$$
$$\left. - \frac{dx}{dc}\frac{dy}{db}\frac{dz}{da} + \frac{dx}{dc}\frac{dy}{da}\frac{dz}{db}\right) da\,db\,dc$$

et, comme ces deux volumes doivent être équivalents, on aura, par suite,

$$\frac{dx}{da}\frac{dy}{db}\frac{dz}{dc} - \frac{dx}{da}\frac{dy}{dc}\frac{dz}{db} - \frac{dx}{db}\frac{dy}{da}\frac{dz}{dc} + \frac{dx}{db}\frac{dy}{dc}\frac{dz}{da} - \frac{dx}{dc}\frac{dy}{db}\frac{dz}{da} + \frac{dx}{dc}\frac{dy}{da}\frac{dz}{db} = 1.$$

Si, pour plus de simplicité, on fait usage de la notation adoptée par M. Cauchy dans son *Mémoire sur les fonctions symétriques*,* l'équation prendra la forme suivante :

$$S\left(\pm\frac{dx}{da}\frac{dy}{db}\frac{dz}{dc}\right) = 1,$$

le signe S étant relatif à la permutation des trois lettres a, b, c."

Here we have clearly the Jacobian of x, y, z with respect to a, b, c: and we have it expressed also in the determinant notation then in use.

The second point of interest is centred in the note to which the author directs his reader. This note, which consists merely of the formal statement of a theorem, and extends to only ten lines, is as follows:—

"Si l'on rapporte la position des sommets d'un parallélépipède à trois plans rectangulaires des x, y, et z; que l'on désigne par A, B, C, les longueurs des trois arêtes de ce parallélépipède qui aboutissent à un même sommet, et par

$$\begin{matrix} A_1, & B_1, & C_1, \\ A_2, & B_2, & C_2, \\ A_3, & B_3, & C_3, \end{matrix}$$

les projections respectives des mêmes arêtes sur les axes des x, y, et z, le volume du parallélépipède aura pour mesure

$$A_1B_2C_3 - A_1B_3C_2 + A_2B_3C_1 - A_2B_1C_3 + A_3B_1C_2 - A_3B_2C_1 = S(\pm A_1B_2C_3)."$$

* There is a curious oversight here. In a footnote, Cauchy says "Le Mémoire dont il est ici question a été imprimé en partie dans le xvii^e. Cahier du *Journal de l'École Polytechnique*." Now, as a matter of fact, there is no memoir bearing this title. The well-known memoirs contained in Cahier xvii. are headed "Mémoire sur le nombre des valeurs . . . ." and "Mémoire sur les fonctions qui . . . ." The second part of the latter bears the approximate designation, "Des fonctions symétriques alternées . . . ."; but the notation in question occurs in both parts. It is also not clear what was intended by the words 'imprimé *en partie*' in Cauchy's footnote. Both in the original and in the reprint four signs are twice printed incorrectly, and in the reprint ∂'s have been substituted for d's of the original.

Here we have one of those so-called "applications of determinants to geometry" which are often supposed to belong to a much later date.

CAUCHY (1822).

[Mémoire sur l'intégration des équations linéaires aux différences partielles et à coefficiens constans. *Journ. de l'Éc. Polyt.*, xii. 19e Cahier pp. 511–592; or *Œuvres complètes*, 2e sér. i.]

Here again but under quite different circumstances the same form of determinant presents itself to Cauchy. Having found the value of a certain multiple integral to be

$$\frac{(2\pi)^n}{\sqrt{D^2}},$$

where $D = \Sigma \pm ab'c'' \ldots$, he adds

"Il est essentiel d'observer que si l'on désigne par L le dénominateur commun des fractions qui représente les valeurs de $u, v, w, \ldots$ tirées des équations

$$u\frac{dM}{d\mu} + v\frac{dM}{d\nu} + w\frac{dM}{d\pi} + \ldots = 1,$$

$$u\frac{dN}{d\mu} + v\frac{dN}{d\nu} + w\frac{dN}{d\pi} + \ldots = 1,$$

$$u\frac{dP}{d\mu} + v\frac{dP}{d\nu} + w\frac{dP}{d\pi} + \ldots = 1,$$

$$\cdots\cdots\cdots\cdots$$

et par L_0 ce qui devient L quand on y pose

$$\mu = \mu_0, \quad \nu = \nu_0, \quad \pi = \pi_0, \quad \ldots.$$

on aura identiquement

$$D = L_0."$$

JACOBI (1829).

[Exercitatio algebraica circa discerptionem singularem fractionum, quae plures variabiles involvunt. *Crelle's Journal*, v. pp. 344–364; or *Werke*, iii. pp. 67–90: also abstract in *Nouv. Annales de Math.*, iv. pp. 533–535.]

To the great mathematician whose name was ultimately associated with determinants of this special form, they first appeared in a totally different connection. He was considering a problem of the partition of a fraction with composite denominator into others whose denominators are factors of the original, and the paper to which we have come concerns given fractions of the form

$$(ax+by-t)^{-1}(b'y+a'x-t')^{-1},$$
$$(ax+by+cz-t)^{-1}(b'y+c'z+a'x-t')^{-1}(c''z+a''x+b''y-t'')^{-1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

The expansions of these clearly contain a variety of terms, the reciprocal of each linear expression contributing negative powers of its first term and positive powers of the others; and the 'discerptio singularis' consists in obtaining fractions which produce, each of them, the aggregate of the terms of a particular type found in the expansion. Thus, to take the simplest example, viz., $$(ax+by)^{-1}(b'y+a'x)^{-1},$$

it is seen that the expansion of $(ax+by)^{-1}$ will contain one term with negative power of x and others with a negative power of x and a positive power of y, that the reverse will be the case in the expansion of $(b'y+a'x)^{-1}$, and that the product of the two expansions will therefore contain a term in $x^{-1}y^{-1}$, a series of terms with negative powers of x and positive powers of y, and a series of terms with positive powers of x and negative powers of y. Now Jacobi establishes the identity

$$(ax+by)^{-1}(b'y+a'x)^{-1} = |ab'|^{-1}\left\{\frac{1}{xy} - \frac{1}{x}\cdot\frac{b}{ax+by} - \frac{1}{y}\cdot\frac{a'}{b'y+a'x}\right\},$$

where on the right there are three parts; and as the first is a term in $x^{-1}y^{-1}$, the second equivalent to a series of terms consisting of negative powers of x and positive powers of y, and the third equivalent to a series of terms consisting of negative powers of y and positive powers of x, it is clear that the three portions of the expansion of $(ax+by)^{-1}(b'y+a'x)^{-1}$ have been isolated and summed.

Now it will be noticed that a common factor of the three parts is the reciprocal of the determinant $|ab'|$, or as Jacobi, following Cauchy, writes it (ab'). The corresponding factor in the next case, where there are three linear expressions and three variables, is found to be $(ab'c'')^{-1}$; and Jacobi then makes a generalisation regarding the first of the partial fractions in each case, viz., to the effect that the coefficient of

$$x^{-1}x_1^{-1}x_2^{-1}\ldots\ldots x_{n-1}^{-1}$$

in the expansion of

$$u^{-1}u_1^{-1}u_2^{-1}\ldots\ldots u_{n-1}^{-1},$$

i.e. of

$$(ax+bx_1+cx_2+\ldots\ldots)^{-1}(b'x_1+c'x_2+\ldots\ldots)^{-1}(c''x_2+\ldots\ldots)^{-1}\ldots\ldots,$$

is

$$(ab'c''\ldots\ldots)^{-1},$$

—the result being, so to speak, the discovery of the generating function of the reciprocal of a determinant.

Shortly after this follows the passage which is interesting in the history of Jacobians. It stands as follows:

"At theorematis, de quibus in hac commentatione agimus et quorum modo mentionem injecimus, latissimam conciliare licet extensionem. Ponamus enim, $u-t$, u_1-t', ... iam series esse quaslibet, sive finitas sive infinitas, ad dignitates integras positivas elementorum x, x_1, ... procedentes, quarum serierum t, t', ... sint termini constantes. Sint porro in seriebus illis u, u_1, u_2, ... termini, qui primus ipsorum x, x_1, x_2, ... dignitates continent, respective ax, $b'x_1$, $c''x_2$, ..., ac ponamus, uti in casu lineari, fractiones $(u-t)^{-1}$, $(u_1-t')^{-1}$, $(u_2-t'')^{-1}$, ... evolvi respective ad dignitates descendentes terminorum ax, $b'x_1$, $c''x_2$, ... Vocemus porro Δ determinantem differentialium partialium sequentium:

$$\begin{array}{lllll}
\frac{\partial u}{\partial x}, & \frac{\partial u}{\partial x_1}, & \frac{\partial u}{\partial x_2}, & \ldots\ldots, & \frac{\partial u}{\partial x_{n-1}}, \\[2ex]
\frac{\partial u_1}{\partial x}, & \frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial x_2}, & \ldots\ldots, & \frac{\partial u_1}{\partial x_{n-1}}, \\[2ex]
\frac{\partial u_2}{\partial x}, & \frac{\partial u_2}{\partial x_1}, & \frac{\partial u_2}{\partial x_2}, & \ldots\ldots, & \frac{\partial u_2}{\partial x_{n-1}}, \\[2ex]
\cdot & \cdot & \cdot & \cdot & \cdot \\[2ex]
\frac{\partial u_{n-1}}{\partial x}, & \frac{\partial u_{n-1}}{\partial x_1}, & \frac{\partial u_{n-1}}{\partial x_2}, & \ldots\ldots, & \frac{\partial u_{n-1}}{\partial x_{n-1}}.
\end{array}$$

Erit *e.g.* pro tribus functionibus u, u_1, u_2, tribusque variabilibus x, y, z:

$$\begin{aligned}\Delta = {} & \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial y}\cdot\frac{\partial u_2}{\partial z} - \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial z}\cdot\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\cdot\frac{\partial u_2}{\partial x}\cdot\frac{\partial u}{\partial z} \\ & - \frac{\partial u_2}{\partial z}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial u_1}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial u_1}{\partial z}\cdot\frac{\partial u_2}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial u_1}{\partial x}\cdot\frac{\partial u_2}{\partial y},\end{aligned}$$

quam patet expressionem casu, quo u, u_1, u_2, sunt expressiones lineares, in expressionem ipsius Δ supra exhibitam redire. Quibus positis dico, siquidem $x=p$, $x_1=p_1$, $x_2=p_2$, ..., $x_{n-1}=p_{n-1}$ satisfaciant aequationibus $u=t$, $u_1=t'$, $u_2=t''$, ..., $u_{n-1}=t^{(n-1)}$, producti

$$\frac{\Delta}{(u-t)(u_1-t')(u_2-t'')\ldots.(u_{n-1}-t^{(n-1)})},$$

dictum in modum evoluti, partem eam, quae omnium simul elementorum $x, x_1, \ldots$ dignitates negativas neque ullius positivas continet, ut supra in casu multo simpliciore, fieri

$$\frac{1}{(x-p)(x_1-p_1)(x_2-p_2)\ldots.(x_{n-1}-p_{n-1})}."$$

It will be observed that Jacobi looks temporarily upon the ordinary determinant $(ab'c''\ldots)$ as the particular case of the Jacobian in which the involved functions are linear in all the variables concerned.

JACOBI (1830).

[De resolutione aequationum per series infinitas. *Crelle's Journal*, vi. pp. 257–286; or *Werke*, vi. pp. 26–61.]

Although the general subject here is new, there is a certain link of connection with the preceding paper, in that one of the results of that paper is employed, and also that Jacobi is using once more the method of 'generating functions.'

Passing over the first two cases, let us note how he proceeds with a set of three equations and three variables. As a preliminary he introduces after the manner of the preceding cases the determinant of the partial differential coefficients, the sentence in regard to it being [page 263]—

"Ut similia eruamus de tribus functionibus, tres variabiles x, y, z involventibus $f(x,y,z)$, $\phi(x,y,z)$, $\psi(x,y,z)$ adnotetur aequatio identica:

$$\frac{\partial[\phi'(y)\psi'(z)-\phi'(z)\psi'(y)]}{\partial x}+\frac{\partial[\phi'(z)\psi'(x)-\phi'(x)\psi'(z)]}{\partial y}$$
$$+\frac{\partial[\phi'(x)\psi'(y)-\phi'(y)\psi'(x)]}{\partial z}=0,$$

quam differentiationibus exactis facile probas. E qua, posito brevitatis causa

$$\begin{aligned}\nabla = \quad & f'(x)[\phi'(y)\psi'(z)-\phi'(z)\psi'(y)]\\ &+f'(y)[\phi'(z)\psi'(x)-\phi'(x)\psi'(z)]\\ &+f'(z)[\phi'(x)\psi'(y)-\phi'(y)\psi'(x)],\end{aligned}$$

fluit sequens:

$$\frac{\partial f[\phi'(y)\psi'(z)-\phi'(z)\psi'(y)]}{\partial x}+\frac{\partial f[\phi'(z)\psi'(x)-\phi'(x)\psi'(z)]}{\partial y}$$
$$+\frac{\partial f[\phi'(x)\psi'(y)-\phi'(y)\psi'(x)]}{\partial z}=\nabla."$$

Here the concluding identity has to be noted. He then establishes certain results concerning the coefficient of $x^{-1}y^{-1}z^{-1}$ in the expansion of ∇, or, as he writes this coefficient,

$$\Big[\nabla\Big]_{x^{-1}y^{-1}z^{-1}}.$$

Thus prepared he attacks the given equations (p. 284)—

$$\begin{aligned}\sigma &= ax + by + cz + dx^2 + exy + \ldots,\\ \tau &= a'x + b'y + c'z + d'x^2 + e'xy + \ldots,\\ v &= a''x + b''y + c''z + d''x^2 + e''xy + \ldots;\end{aligned}$$

obtains first the derived set

$$\begin{aligned}s &= \Delta x + \alpha x^2 + \beta xy + \gamma y^2 + \ldots\\ t &= \Delta y + \alpha' x^2 + \beta' xy + \gamma' y^2 + \ldots\\ u &= \Delta z + \alpha'' x^2 + \beta'' xy + \gamma'' y^2 + \ldots\end{aligned}$$

where the values of $s, \Delta, \alpha, \ldots, t, \Delta, \alpha', \ldots, u, \Delta, \alpha'', \ldots$ are sufficiently suggested* by giving one of them, viz.,

$$\Delta = (ab'c'');$$

and then seeks to find any function of the roots, $F(x, y, z)$ say, in the form of a series proceeding according to powers of the constants s, t, u, the result being that the coefficient of $s^p t^q u^r$ in the said series is shown to be in general

$$\left[\frac{F(x, y, z)\,.\,\nabla}{X^{p+1}Y^{q+1}Z^{r+1}}\right]_{x^{-1}y^{-1}z^{-1}},$$

where X, Y, Z are the variable members of the derived set of equations, and ∇ is *the determinant of their partial differential coefficients with respect to* x, y, z.

---

* In later notation the derived equations would of course be written—

$$\begin{vmatrix}\sigma & b & c\\ \tau & b' & c'\\ v & b'' & c''\end{vmatrix} = \begin{vmatrix}a & b & c\\ a' & b' & c'\\ a'' & b'' & c''\end{vmatrix}x + \begin{vmatrix}d & b & c\\ d' & b' & c'\\ d'' & b'' & c''\end{vmatrix}x^2 + \ldots$$

No other case is dealt with, but the paper closes with the sentence—

"Quae autem hactenus de duabus, tribus aequationibus inter duas, tres variabiles propositis protulimus, eadem facilitate ad *numerum quemlibet aequationum et variabilium* extenduntur."

JACOBI (1832, 1833).

[De transformatione et determinatione integralium duplicium commentatio tertia. *Crelle's Journal*, x. pp. 101–128; or *Werke*, iii. pp. 159–189.]

[De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integralium multiplicium. *Crelle's Journal*, xii. pp. 1–69; or *Werke*, iii. pp. 191–268.]

The latter of these memoirs is by far the more important; in fact, it may be fully described as the summation and development of a series of memoirs of which the former is the last. As is natural, therefore, six pages of it at the outset are occupied with an introduction, in which the main points of the said series of papers are recapitulated. Seven-and-twenty pages are then devoted to "Problema I," which may be roughly characterised in later phraseology as the problem of the linear transformation of an n-ary quadric. Then follows "Problema II," the solution of which occupies pages 34–50. Its subject is the transformation of a very general multiple integral, and is closely connected with the subject of the preceding problem by reason of the fact that the integrand involves a power of an n-ary quadric. It is in this portion of the memoir that the special form of determinant which we are now considering makes its appearance.

From two particular results in the previous papers referred to, Jacobi infers the existence of a general theorem which he states, adding that the demonstration is, however, not so easy, and that as a contribution towards it he will enunciate certain general theorems, the proofs of which for the sake of brevity are suppressed. His starting-point is the following:—

If $\xi_1, \xi_2, \ldots, \xi_{n-2}$ *be any given functions of* $v_1, v_2, \ldots, v_{n-2}$, *then*

$$\partial\xi_1\,\partial\xi_2\ldots\partial\xi_{n-2} = \left(\sum\pm\frac{\partial\xi_1}{\partial v_1}\frac{\partial\xi_2}{\partial v_2}\cdots\frac{\partial\xi_{n-2}}{\partial v_{n-2}}\right)\partial v_1\,\partial v_2\ldots\partial v_{n-2}.$$

From this he proceeds to cases where there are additional ξ functions not independent of the others, enunciating and proving the result where there is one connecting equation, contenting himself with the mere enunciation for the case where there are two connecting equations, and leaving these to suggest the general theorem. The two enunciations are—

THEOREMA 1. *Datis* $\xi_1, \xi_2, \ldots, \xi_{n-1}$ *ut functionibus ipsarum* $v_1, v_2, \ldots, v_{n-1}$, *si inter variabiles illas datur aequatio*

$$F(\xi_1, \xi_2, \ldots, \xi_{n-1}) = 0,$$

erit

$$\frac{\partial\xi_1\,\partial\xi_2\ldots\partial\xi_{n-2}}{\dfrac{\partial F}{\partial\xi_{n-1}}} = \left(\sum\pm\frac{\partial\xi_1}{\partial v_1}\frac{\partial\xi_2}{\partial v_2}\cdots\frac{\partial\xi_{n-1}}{\partial v_{n-1}}\right)\cdot\frac{\partial v_1\,\partial v_2\ldots\partial v_{n-2}}{\dfrac{\partial F}{\partial v_{n-1}}}.$$

THEOREMA 2. *Datis* $\xi_1, \xi_2, \ldots, \xi_n$ *ut functionibus ipsarum* $v_1, v_2, \ldots, v_n$, *si inter variabiles illas proponuntur duae aequationes*:

$$F(\xi_1, \xi_2, \ldots, \xi_n) = 0, \quad \Phi(\xi_1, \xi_2, \ldots, \xi_n) = 0,$$

$$\frac{\partial\xi_1\,\partial\xi_2\ldots\partial\xi_{n-2}}{\dfrac{\partial F}{\partial\xi_{n-1}}\dfrac{\partial\Phi}{\partial\xi_n}-\dfrac{\partial F}{\partial\xi_n}\dfrac{\partial\Phi}{\partial\xi_{n-1}}} = \left(\sum\pm\frac{\partial\xi_1}{\partial v_1}\frac{\partial\xi_2}{\partial v_2}\cdots\frac{\partial\xi_n}{\partial v_n}\right)\frac{\partial v_1\,\partial v_2\ldots\partial v_{n-2}}{\dfrac{\partial F}{\partial v_{n-1}}\dfrac{\partial\Phi}{\partial v_n}-\dfrac{\partial F}{\partial v_n}\dfrac{\partial\Phi}{\partial v_{n-1}}}.$$

His third theorem is a special case of his first, and may therefore be passed over. Then we have

THEOREMA 4. *Supponamus* $\xi_1, \xi_2, \ldots, \xi_{n-1}$ *datas esse sub forma fractionum*

$$\xi_1 = \frac{u_1}{u}, \quad \xi_2 = \frac{u_2}{u}, \quad \ldots, \quad \xi_{n-1} = \frac{u_{n-1}}{u},$$

fit

$$\sum\pm\frac{\partial\xi_1}{\partial v_1}\frac{\partial\xi_2}{\partial v_2}\cdots\frac{\partial\xi_{n-1}}{\partial v_{n-1}} = \frac{1}{u^n}\cdot\sum\pm u\frac{\partial u_1}{\partial v_1}\frac{\partial u_2}{\partial v_2}\cdots\frac{\partial u_{n-1}}{\partial v_{n-1}}$$

ubi in altera summa inter indices permutandos etiam referri debet index 0 *seu index deficiens.*

From this is deduced

THEOREMA 5. *Si loco functionum* $u, u_1, u_2, \ldots u_{n-1}$ *ponitur*

$$\frac{u}{t}, \ \frac{u_1}{t}, \ \frac{u_2}{t}, \ \ldots, \ \frac{u_{n-1}}{t},$$

designante t *aliam functionem quamlibet, expressio*

$$\sum \pm \mathrm{u} \frac{\partial \mathrm{u}_1}{\partial v_1} \frac{\partial \mathrm{u}_2}{\partial v_2} \cdots\cdot \frac{\partial \mathrm{u}_{n-1}}{\partial v_{n-1}}$$

abit in

$$\frac{1}{\mathrm{t}^n} \sum \pm \mathrm{u} \frac{\partial \mathrm{u}_1}{\partial v_1} \frac{\partial \mathrm{u}_2}{\partial v_2} \cdots\cdot \frac{\partial \mathrm{u}_{n-1}}{\partial v_{n-1}},$$

sive in differentiationibus instituendis denominatorem communem t *ut constantem considerare licet.*

The last of the series is

THEOREMA 6. *Sint* $\mathrm{u}, \mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_{n-1}$ *expressiones lineares aliarium functionum* $\mathrm{w}, \mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_{n-1}$, *datae per aequationes huius modi*

$$\mathrm{u}_k = a_k \mathrm{w} + a'_k \mathrm{w}_1 + a''_k \mathrm{w}_2 + \ldots + a_k^{(n-1)} \mathrm{w}_{n-1}$$

fit

$$\sum \pm \mathrm{u} \frac{\partial \mathrm{u}_1}{\partial v_1} \frac{\partial \mathrm{u}_2}{\partial v_2} \cdots \frac{\partial \mathrm{u}_{n-1}}{\partial v_{n-1}} = \left(\sum \pm a a'_1 a''_2 \ldots a_{n-1}^{(n-1)}\right)\left(\sum \pm \mathrm{w} \frac{\partial \mathrm{w}_1}{\partial v_1} \cdots \frac{\partial \mathrm{w}_{n-1}}{\partial v_{n-1}}\right),$$

to which is added the remark that if there were one additional independent variable v we should similarly have

$$\sum \pm \frac{\partial u}{\partial v} \frac{\partial u_1}{\partial v_1} \cdots \frac{\partial u_{n-1}}{\partial v_{n-1}} = \left(\sum \pm a a'_1 \ldots a_{n-1}^{(n-1)}\right)\left(\sum \pm \frac{\partial w}{\partial v} \frac{\partial w_1}{\partial v_1} \cdots \frac{\partial w_{n-1}}{\partial v_{n-1}}\right).$$

CATALAN (1839).

[Sur la transformation des variables dans les intégrales multiples. *Mémoires couronnés par l'Académie . . . de Bruxelles*, xiv. 2e partie, 47 pp.]

Having devoted the first part (pp. 7–18) of his memoir to the properties of determinants (see above pp. 224–226) Catalan is ready to use them in the second part in dealing with his main problem, viz., that specified in the title.

The integral as given being

$$\int F(x_1, x_2, \ldots, x_n) dx_1 . dx_2 \ldots . dx_n,$$

and the equations connecting $x_1, x_2, \ldots, x_n$ with the new variables $u_1, u_2, \ldots, u_n$ being

$$\phi_1 = 0, \quad \phi_2 = 0, \quad \ldots, \quad \phi_n = 0,$$

he first seeks to remove the variable x_1. In doing this

$x_2, x_3, \ldots, x_n$ are constants, and the connecting equations involve the $n+1$ variables $x_1, u_1, u_2, \ldots, u_n$, so that he obtain

$$\begin{aligned}
&\frac{d\phi_1}{dx_1}dx_1 + \frac{d\phi_1}{du_1}du_1 + \frac{d\phi_1}{du_2}du_2 + \ldots. + \frac{d\phi_1}{du_n}du_n = 0,\\
&\frac{d\phi_2}{dx_1}dx_1 + \frac{d\phi_2}{du_1}du_1 + \frac{d\phi_2}{du_2}du_2 + \ldots. + \frac{d\phi_2}{du_n}du_n = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{d\phi_n}{dx_1}dx_1 + \frac{d\phi_n}{du_1}du_1 + \frac{d\phi_n}{du_2}du_2 + \ldots. + \frac{d\phi_n}{du_n}du_n = 0,
\end{aligned}$$

and thence

$$dx_1 = -\frac{\mathrm{N}_1}{\mathrm{D}_1}du_1.$$

By dealing similarly with $x_2, x_3, \ldots.$ in succession the result reached is

$$dx_1 \,.\, dx_2 \ldots. dx_n = (-1)^n \frac{\mathrm{N}_1}{\mathrm{D}_1}\cdot\frac{\mathrm{N}_2}{\mathrm{D}_2} \cdots \frac{\mathrm{N}_n}{\mathrm{D}_n}\,.\,du_1\,.\,du_2 \ldots. du_n,$$

and as $\mathrm{N}_2=\mathrm{D}_1$, $\mathrm{N}_3=\mathrm{D}_2, \ldots, \mathrm{N}_n=\mathrm{D}_{n-1}$, the ultimate form is

$$dx_1 \,.\, dx_2 \ldots. dx_n = (-)^n \frac{\mathrm{N}_1}{\mathrm{D}_n}\,.\,du_1\,.\,du_2 \ldots. du_n,$$

where "N_1 est le dénominateur de la valeur des inconnues dans les équations

$$\left.\begin{aligned}
&\frac{d\phi_1}{du_1}z_1 + \frac{d\phi_1}{du_2}z_2 + \ldots. + \frac{d\phi_1}{du_n}z_n = 1\\
&\frac{d\phi_2}{du_1}z_1 + \frac{d\phi_2}{du_2}z_2 + \ldots. + \frac{d\phi_2}{du_n}z_n = 1\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{d\phi_n}{du_1}z_1 + \frac{d\phi_n}{du_2}z_2 + \ldots. + \frac{d\phi_n}{du_n}z_n = 1
\end{aligned}\right\}$$

tandis que D_n est celui qui correspond aux équations

$$\left.\begin{aligned}
&\frac{d\phi_1}{dx_1}y_1 + \frac{d\phi_1}{dx_2}y_2 + \ldots. + \frac{d\phi_1}{dx_n}y_n = 1\\
&\frac{d\phi_2}{dx_1}y_1 + \frac{d\phi_2}{dx_2}y_2 + \ldots. + \frac{d\phi_2}{dx_n}y_n = 1\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{d\phi_n}{dx_1}y_1 + \frac{d\phi_n}{dx_2}y_2 + \ldots. + \frac{d\phi_n}{dx_n}y_n = 1\text{"}
\end{aligned}\right\}$$

—a lengthy and (because of its y's and z's and 1's) an awkward way of saying that

$$\mathrm{N}_1 = \sum\left(\pm\frac{d\phi_1}{du_1}\cdot\frac{d\phi_2}{du_2}\cdots\cdots\frac{d\phi_n}{du_n}\right)$$

and
$$\mathrm{D}_n = \sum\left(\pm\frac{d\phi_1}{dx_1}\cdot\frac{d\phi_2}{dx_2}\cdots\cdots\frac{d\phi_n}{dx_n}\right).$$

Catalan refers to Lagrange and 'others' for the cases $n=2$, $n=3$, but does not mention Jacobi's paper of 1833.*

JACOBI (1841).

[De determinantibus functionalibus. *Crelle's Journal*, xxii. pp. 319–359; or *Werke*, iii. pp. 393–438; or Stäckel's translation 'Ueber die Functionaldeterminanten,' 72 pp. Leipzig (1896).]

Up to this point, as will have been evident, the special determinants which we are considering have turned up merely incidentally in the course of other work. Now, however, we come upon a separate and direct investigation of their properties, the memoir under consideration being the second of the three portions into which Jacobi divided his formal exposition of the theory of determinants. From the mere fact that separate treatment is bestowed by him on only one other special form, it is clear that the subject of the memoir had come to be considered of particular importance. The same is rendered still more strikingly apparent when it is recalled that of the 87 pages occupied by the whole exposition, as many as 41 are devoted to this second portion concerning a subordinate form, while only 34 are assigned to what we are bound to consider the main portion, viz., that dealing with determinants in general.

At the outset the preceding memoir '*De formatione et proprietatibus determinantium*' is referred to, and intimation made that there is now about to be considered the special case where the elements are partial differential-quotients of a set of n functions each of the same n independent variables, and that in this case the special name *functional determinants* may with

* On the question of the authorship of the theorem of transformation see MANSION, P.: Discours sur les travaux mathématiques de M. Eugène-Charles Catalan. *Mém. de la Soc. R. des Sci.* (Liège), 2ᵉ sér. xii. pp. (1–38) 10–12.

convenience be used. Jacobi takes pains, however, to explain that this relation of general to particular may appropriately be taken in reverse order, going, in fact, so far as to say that from the properties of functional determinants the properties of what he calls algebraic determinants may be deduced. He is careful to note also another relationship of the same kind, his statement being that in various questions relating to a system of functions the functional determinant is the analogue of the single differential-quotient in the case of a function of one variable.

The subject of the notation of partial differential-quotients is then entered on at some length (pp. 320–323), and the decision made to use ∂ in the manner which soon afterwards came to be familiar. The insufficiency of this notation is not forgotten, however, although its advantages over the different devices of Euler and Lagrange are recognised, his illustrative example being the case of $\dfrac{\partial z}{\partial x}$ where z is a function of x and u, and u is a function of x and y. He puts the whole matter in a nutshell when he says that it is not enough to specify the function to be operated on and the particular independent variable with respect to which the differentiation is to be performed, but that it is equally necessary to indicate the involved quantities which are to be viewed as constants during the operation.*

* I may state in passing that in 1869 when lecturing on the subject I found it very useful to write

$$\phi\overline{x, y, z}, \quad f\overline{s, t, u, v}, \quad \ldots .$$

in place of

$$\phi(x, y, z), \quad f(s, t, u, v), \quad \ldots .$$

and then indicate the number of times the function had to be differentiated with respect to any one of the variables by writing that number on the opposite side of the vinculum from the said variable; thus

$$\phi\overset{1\ \ 3\ \ 2}{\overline{x, y, z}}$$

meant the result of differentiating once with respect to x, thrice with respect to y, and twice with respect to z.

Using this notation to illustrate Jacobi's example, we see that if it were given that

$$z = \phi\overline{x, u}$$

we should have

$$\frac{\partial z}{\partial x} = \phi\overset{1\ \ \ }{\overline{x, u}};$$

but that if it were given that

$$z = \phi\overline{x, u} \quad \text{and} \quad u = \psi\overline{x, y}$$

The *dependence* or *independence of equations* is the next preliminary subject (pp. 323-325), the starting-point being the definition of an identical equation as one in which every term is destructive of another, and from which, therefore, it is impossible to express one of the involved quantities in terms of the rest. On this the definition of mutually independent equations is made to hang, such equations being defined as those of which no one at the outset is an identical equation nor can be transformed into an identical equation by aid of the others. Then taking $m+1$ equations,

$$u = 0, \quad u_1 = 0, \quad \ldots, \quad u_m = 0$$

involving $n+1$ quantities $x, x_1, \ldots, x_n$ he contemplates the possibility of solving $u=0$ for x in terms of $x_1, x_2, \ldots, x_n$, and the substitution of the expression in place of x in the remaining equations. The latter equations as altered he supposes to be dealt with in the same way, and the process continued until $k+1$ quantities have been *eliminated* and $m-k$ equations left involving $x_{k+1}, x_{k+2}, \ldots, x_n$. Reasoning from this, he concludes that a number of given equations are mutually independent or not according as by their help the same number of involved quantities can or can not be expressed in terms of the remaining quantities. In this connection he does not omit to draw attention to the existence of exceptional cases, such as that in which two of the quantities, x_h, x_k, say, occur indeed in all the equations, but always in the form x_h+x_k; and this leads him, for the sake of greater definiteness, to introduce the qualifying phrase, 'with respect to certain quantities' in using the expression 'mutually independent.' His words are—

"Aequationes $u=0,\ u_1=0,\ \ldots,\ u_m=0$ quibus totidem quantitates $x, x_1, \ldots, x_m$ quas involvunt, determinantur, *harum quantitatum respectu* dico a se independentes."

From the independence of equations he naturally passes (§ 4) to the independence of *functions*, with the remark that exactly

---

then we should not be certain as to the meaning of $\frac{\partial z}{\partial x}$, as it would stand for

$$\phi\overline{x,u}^{\,1} \quad \text{or} \quad \phi\overline{x,u}^{\,1} + \phi\overline{x,u}^{\,1} . \psi\overline{x,y}^{\,1}$$

according as u or y was to be considered constant.

similar propositions are found to hold in regard to the latter,—a statement which it is not hard to believe when we recall that any function, x^2+y^2-4xy say, may be denoted by a functional symbol, f say, the equation $f=x^2+y^2-4xy$ thus resulting; and that any non-identical equation connecting two or more quantities implies that any one of the latter is a function of the others. Functions of several variables are said to be mutually independent when no one of them is constant or can be expressed in terms of the rest. This is extended and made more definite by saying that if functions of $x, x_1, \ldots, x_n$ involve also the quantities $a, a_1, a_2, \ldots$, the functions are said to be mutually independent with respect to the quantities $x, x_1, \ldots, x_n$ if no equation subsists between the functions and the quantities $a, a_1, a_2, \ldots$ These definitions will suffice to indicate the analogy above referred to, and the deduced propositions (pp. 325–327) need not be entered on.

All this introductory matter having been disposed of, Jacobi proceeds (§ 5, p. 327) to deal with the subject proper, his starting-point being the fact that if there be $n+1$ functions $f, f_1, f_2, \ldots, f_n$ of the same number of variables $x, x_1, \ldots, x_n$ there arise in connection with these the $(n+1)^2$ quantities

$$\frac{\partial f_i}{\partial x_k}.$$

The determinant formed therefrom, viz.,

$$\sum \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\cdot\frac{\partial f_n}{\partial x_n},$$

he calls the "determinant pertaining to the functions $f, f_1, \ldots, f_n$ of the variables $x, x_1, \ldots, x_n$," or the "determinant of the functions $f, f_1, \ldots, f_n$, with respect to the variables $x, x_1, \ldots, x_n$." The case where $n=0$ is then referred to in a line, after which cases are taken up where it is the functions that are specialised. The first of these is that in which

$$f_{m+1} = x_{m+1}, \quad f_{m+2} = x_{m+2}, \quad \ldots, \quad f_n = x_n,$$

it being pointed out of course that the order of the determinant is then lowered, being equal to

$$\sum \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\cdot\frac{\partial f_m}{\partial x_m}.$$

Another is that in which the functions $f_{m+1}, f_{m+2}, \ldots, f_n$ do not involve the variables $x, x_1, \ldots, x_m$, the peculiarity then being that the determinant breaks up into two factors similar to itself, being equal to

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_m}{\partial x_m} \cdot \sum \pm \frac{\partial f_{m+1}}{\partial x_{m+1}} \cdot \frac{\partial f_{m+2}}{\partial x_{m+2}} \cdots \frac{\partial f_n}{\partial x_n}.$$

The important proposition regarding a vanishing functional determinant is then dealt with (§ 6), viz., the proposition "functionum a se non independentium evanescere Determinans, functiones quarum Determinans evanescat non esse a se independentes." The proof of the first part of it opens with the assertion that since the functions are not mutually independent, there must exist an equation

$$\Pi(f, f_1, \ldots, f_n) = 0$$

such that on substituting for $f, f_1, \ldots f_n$ their expressions in terms of $x, x_1, \ldots, x_n$ we shall obtain an identity. From this by differentiating separately with respect to $x, x_1, \ldots, x_n$ there is obtained the set of equations

$$\begin{aligned}
0 &= \frac{\partial f}{\partial x} \cdot \frac{\partial \Pi}{\partial f} + \frac{\partial f_1}{\partial x} \cdot \frac{\partial \Pi}{\partial f_1} + \cdots + \frac{\partial f_n}{\partial x} \cdot \frac{\partial \Pi}{\partial f_n}, \\
0 &= \frac{\partial f}{\partial x_1} \cdot \frac{\partial \Pi}{\partial f} + \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial \Pi}{\partial f_1} + \cdots + \frac{\partial f_n}{\partial x_1} \cdot \frac{\partial \Pi}{\partial f_n}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= \frac{\partial f}{\partial x_n} \cdot \frac{\partial \Pi}{\partial f} + \frac{\partial f_1}{\partial x_n} \cdot \frac{\partial \Pi}{\partial f_1} + \cdots + \frac{\partial f_n}{\partial x_n} \cdot \frac{\partial \Pi}{\partial f_n}.
\end{aligned}$$

Then it is recalled that in a set of linear equations

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{aligned}\right\}$$

the determinant of the coefficients must vanish unless all the unknowns vanish. And as the vanishing of

$$\frac{\partial \Pi}{\partial f}, \frac{\partial \Pi}{\partial f_1}, \ldots, \frac{\partial \Pi}{\partial f_n}$$

would imply that the expression $\Pi(f, f_1, \ldots, f_n)$ was free of

$f, f_1, \ldots, f_n$, the conclusion is reached that the determinant of the coefficients of these differential-quotients must vanish, *i.e.*, that the functional determinant

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} = 0.$$

The proof of the converse proposition, Jacobi owns, is 'paullo prolixior.' It is of the kind improperly known as 'inductive,' and the first part (§ 7) of it goes to show that if the proposition holds in the case of $\sum \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$ it will also hold in the case of $\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$. As a preliminary, there is established the lemma that—

If $\mathrm{f}, \mathrm{f}_1, \mathrm{f}_2, \ldots, \mathrm{f}_n$ *are mutually independent, then*

$$\sum \pm \frac{\partial \mathrm{f}}{\partial \mathrm{x}} \cdot \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1} \cdots \frac{\partial \mathrm{f}_n}{\partial \mathrm{x}_n} = \left(\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\right) \sum \pm \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1} \cdot \frac{\partial \mathrm{f}_2}{\partial \mathrm{x}_2} \cdots \frac{\partial \mathrm{f}_n}{\partial \mathrm{x}_n},$$

where the brackets enclosing a differential-quotient are meant to indicate that f *there is to be taken as a function of* $\mathrm{x}, \mathrm{f}_1, \mathrm{f}_2, \ldots, \mathrm{f}_n$.

Denoting the cofactors of

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$$

in the determinant $\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$ by

$$\mathrm{A}, \mathrm{A}_1, \ldots, \mathrm{A}_n$$

we have of course

$$\left.\begin{array}{l}
\frac{\partial f}{\partial x}\mathrm{A} + \frac{\partial f}{\partial x_1}\mathrm{A}_1 + \ldots + \frac{\partial f}{\partial x_n}\mathrm{A}_n = \sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} \\
\frac{\partial f_1}{\partial x}\mathrm{A} + \frac{\partial f_1}{\partial x_1}\mathrm{A}_1 + \ldots + \frac{\partial f_1}{\partial x_n}\mathrm{A}_n = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial f_n}{\partial x}\mathrm{A} + \frac{\partial f_n}{\partial x_1}\mathrm{A}_1 + \ldots + \frac{\partial f_n}{\partial x_n}\mathrm{A}_n = 0
\end{array}\right\}.$$

But since $f_1, f_2, \ldots, f_n$ are mutually independent it is possible by solution to obtain $x_1, x_2, \ldots, x_n$ each in terms of the remaining variable x and $f_1, f_2, \ldots, f_n$; and as a consequence it is possible by substituting for $x_1, x_2, \ldots, x_n$ to obtain f in terms also of $x, f_1, f_2, \ldots, f_n$. Differentiating f with respect to $x, x_1, \ldots, x_n$, and using brackets to indicate that the f within them is to be viewed as a function not of $x, x_1, \ldots,$ but of $x, f_1, \ldots, f_n$, we have

$$\left.\begin{aligned}
\frac{\partial f}{\partial x} &= \left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial f_1}\right)\cdot\frac{\partial f_1}{\partial x} + \left(\frac{\partial f}{\partial f_2}\right)\cdot\frac{\partial f_2}{\partial x} + \ldots + \left(\frac{\partial f}{\partial f_n}\right)\cdot\frac{\partial f_n}{\partial x},\\
\frac{\partial f_1}{\partial x} &= \qquad\quad \left(\frac{\partial f}{\partial f_1}\right)\cdot\frac{\partial f_1}{\partial x_1} + \left(\frac{\partial f}{\partial f_2}\right)\cdot\frac{\partial f_2}{\partial x_1} + \ldots. + \left(\frac{\partial f}{\partial f_n}\right)\cdot\frac{\partial f_n}{\partial x_1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{\partial f_n}{\partial x} &= \qquad\quad \left(\frac{\partial f}{\partial f_1}\right)\cdot\frac{\partial f_1}{\partial x_n} + \left(\frac{\partial f}{\partial f_2}\right)\cdot\frac{\partial f_2}{\partial x_n} + \ldots. + \left(\frac{\partial f}{\partial f_n}\right)\cdot\frac{\partial f_n}{\partial x_n}.
\end{aligned}\right\}$$

From these, on multiplying both sides of the first by A, both sides of the second by A_1, and so on, and then adding in columns, there is obtained, with the help of the $n+1$ equations immediately preceding,

$$\sum\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\ldots.\frac{\partial f_n}{\partial x_n} = \left(\frac{\partial f}{\partial x}\right)A + 0 + 0 + \ldots. + 0,$$

which is what was to be proved.*

Now suppose that in any way it has been established that if $\sum\pm\frac{\partial f_1}{\partial x_1}\ldots.\frac{\partial f_n}{\partial x_n} = 0$, the functions involved are not mutually

* Using this theorem upon itself we have

$$\sum\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\ldots.\frac{\partial f_n}{\partial x_n} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f_1}{\partial x_1}\right)\sum\pm\frac{\partial f_2}{\partial x_2}\ldots.\frac{\partial f_n}{\partial x_n}$$

provided that on the right f is expressed as a function of $x, f_1, f_2, \ldots., f_n$, and f_1 as a function of $x, x_1, f_2, \ldots., f_n$; and ultimately

$$\sum\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\ldots.\frac{\partial f_n}{\partial x_n} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f_1}{\partial x_1}\right)\ldots.\left(\frac{\partial f_n}{\partial x_n}\right),$$

provided that in every instance on the right f_k is expressed as a function of $x, x_1, x_2, \ldots., x_k, f_{k+1}, \ldots., f_n$.

A theorem like this ultimate case Jacobi enunciates and proves quite independently at the end of his memoir (v. § 18). The one, however, is seen to include the other if we note the simple fact that

$$\sum\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\ldots.\frac{\partial f_n}{\partial x_n} = \sum\pm\frac{\partial f_n}{\partial x_n}\cdot\frac{\partial f_{n-1}}{\partial x_{n-1}}\ldots.\frac{\partial f}{\partial x}.$$

independent, and that the next higher case is to be investigated, viz., where $\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} = 0$. Of the $n+1$ functions involved in the latter determinant the last n of them must either be independent or not. If they are not independent, there is nothing more to be proved. And if they be mutually independent, then the lemma gives

$$\left(\frac{\partial f}{\partial x}\right) \mathrm{A} = 0,$$

and therefore

$$\left(\frac{\partial f}{\partial x}\right) = 0,$$

for it is impossible that the functional determinant A could be 0, as the functions involved in it are by hypothesis mutually independent. The vanishing of $\left(\frac{\partial f}{\partial x}\right)$ however implies that x is not involved in f; and this, if we bear in mind the meaning of the brackets, further implies that f is a function of only $f_1, f_2, \ldots, f_n$ —that is to say, that $f, f_1, f_2, \ldots, f_n$ are not mutually independent.

There now only remains to establish the proposition for the case where the determinant is of the second order,—that is to say, where

$$\frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} - \frac{\partial f}{\partial x_1} \cdot \frac{\partial f_1}{\partial x} = 0.$$

Here the function f_1 must involve both variables, or only one of them, x say, or be a constant. If it be not a constant, x_1 is expressible in terms of f_1 or at most in terms of f_1 and x, and thus, by substitution, f is expressible in terms of the same. In this way we have

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial f_1}\right) \cdot \frac{\partial f_1}{\partial x} \\ \frac{\partial f}{\partial x_1} &= \qquad\qquad\; \left(\frac{\partial f}{\partial f_1}\right) \cdot \frac{\partial f_1}{\partial x_1} \end{aligned}\right\}$$

and therefore, by elimination of $\left(\frac{\partial f}{\partial f_1}\right)$,

$$\begin{aligned} \left(\frac{\partial f}{\partial x}\right) \cdot \frac{\partial f_1}{\partial x_1} &= \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} - \frac{\partial f}{\partial x_1} \cdot \frac{\partial f_1}{\partial x}, \\ &= 0. \end{aligned}$$

Now it cannot be that the second factor on the left vanishes, because f_1 is known to involve x_1; consequently

$$\left(\frac{\partial f}{\partial x}\right) = 0.$$

From this it follows that of the two variables f_1 and x, supposed to be involved in f, the second is awanting, and that therefore f is expressible in terms of f_1 alone. Our result thus is that either f_1 is a constant or that f and f_1 are dependent.

The general theorem being thus established, Jacobi refers in a line or two to the corresponding contrapositive proposition, viz., that *if the functional determinant do not vanish, the functions are mutually independent,* and to the contrapositive of the converse, viz., that *if the functions be mutually independent, the functional determinant cannot vanish.* Finally, in recalling the ultimate case where the determinant is of order 1, he notes that, as in that case, so generally the four propositions may be combined in a single enunciation, viz., *According as the functions* f, $\mathrm{f}_1, \ldots, \mathrm{f}_n$ *of* x, $\mathrm{x}_1, \ldots, \mathrm{x}_n$ *are not or are mutually independent, the functional determinant does or does not vanish.*

The next subject taken up (§ 8) is the solution of a set of linear equations

$$\left.\begin{array}{l}
\dfrac{\partial f}{\partial x} r + \dfrac{\partial f}{\partial x_1} r_1 + \ldots + \dfrac{\partial f}{\partial x_n} r_n = s, \\
\dfrac{\partial f_1}{\partial x} r + \dfrac{\partial f_1}{\partial x_1} r_1 + \ldots + \dfrac{\partial f_1}{\partial x_n} r_n = s_1, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\dfrac{\partial f_n}{\partial x} r + \dfrac{\partial f_n}{\partial x_1} r_1 + \ldots + \dfrac{\partial f_n}{\partial x_n} r_n = s_n,
\end{array}\right\}$$

in which the determinant of the coefficients of the unknowns is the functional determinant

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}.$$

Of course a condition of solution is that this determinant does not vanish, and therefore that the functions $f, f_1, \ldots, f_n$ are mutually independent. But this being the case it follows that $x, x_1, \ldots, x_n$ are expressible in terms of $f, f_1, \ldots, f_n$, and, as a consequence of substitution, that any other function ϕ of $x, x_1, \ldots, x_n$ is expressible in terms of the same, thus giving us

$$\frac{\partial\phi}{\partial f_k} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial f_k} + \frac{\partial\phi}{\partial x_1}\cdot\frac{\partial x_1}{\partial f_k} + \ldots + \frac{\partial\phi}{\partial x_n}\cdot\frac{\partial x_n}{\partial f_k}. \qquad (a).$$

Now if we turn to our set of equations, and multiply by

$$\frac{\partial x_k}{\partial f},\quad \frac{\partial x_k}{\partial f_1},\quad \ldots,\quad \frac{\partial x_k}{\partial f_n}$$

respectively, with the result

$$\left.\begin{array}{l}
\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x_k}{\partial f}r + \dfrac{\partial f}{\partial x_1}\cdot\dfrac{\partial x_k}{\partial f}r_1 + \ldots + \dfrac{\partial f}{\partial x_n}\cdot\dfrac{\partial x_k}{\partial f}r_n = \dfrac{\partial x_k}{\partial f}s, \\
\dfrac{\partial f_1}{\partial x}\cdot\dfrac{\partial x_k}{\partial f_1}r + \dfrac{\partial f_1}{\partial x_1}\cdot\dfrac{\partial x_k}{\partial f_1}r_1 + \ldots + \dfrac{\partial f_1}{\partial x_n}\cdot\dfrac{\partial x_k}{\partial f_1}r_n = \dfrac{\partial x_k}{\partial f_1}s_1, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\dfrac{\partial f_n}{\partial x}\cdot\dfrac{\partial x_k}{\partial f_n}r + \dfrac{\partial f_n}{\partial x_1}\cdot\dfrac{\partial x_k}{\partial f_n}r_1 + \ldots + \dfrac{\partial f_n}{\partial x_n}\cdot\dfrac{\partial x_k}{\partial f_n}r_n = \dfrac{\partial x_k}{\partial f_n}s_n,
\end{array}\right\}$$

it is evident from (a) that on adding column-wise we shall find the coefficients of all the unknowns equal to zero, except the coefficient of x_k which is unity, and therefore that

$$r_k = \frac{\partial x_k}{\partial f}s + \frac{\partial x_k}{\partial f_1}s_1 + \ldots + \frac{\partial x_k}{\partial f_n}s_n.$$

Jacobi is thus led to formulate the proposition:—

"Sint variabilium $x, x_1, \ldots, x_n$ functiones $f, f_1, \ldots, f_n$ a se invicem independentes, si proponetur hoc aequationum linearium systema,

$$\left.\begin{array}{l}
\dfrac{\partial f}{\partial x}r + \dfrac{\partial f}{\partial x_1}r_1 + \ldots + \dfrac{\partial f}{\partial x_n}r_n = s, \\
\dfrac{\partial f_1}{\partial x}r + \dfrac{\partial f_1}{\partial x_1}r_1 + \ldots + \dfrac{\partial f_1}{\partial x_n}r_n = s_1, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\dfrac{\partial f_n}{\partial x}r + \dfrac{\partial f_n}{\partial x_1}r_1 + \ldots + \dfrac{\partial f_n}{\partial x_n}r_n = s_n
\end{array}\right\}$$

earum resolutio semper est possibilis et determinata eruntque incognitarum valores:

$$\left.\begin{aligned}
r &= \frac{\partial x}{\partial f}s + \frac{\partial x}{\partial f_1}s_1 + \ . \ . \ . \ + \frac{\partial x}{\partial f_n}s_n, \\
r_1 &= \frac{\partial x_1}{\partial f}s + \frac{\partial x_1}{\partial f_1}s_1 + \ . \ . \ . \ + \frac{\partial x_1}{\partial f_n}s_n, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
r_n &= \frac{\partial x_n}{\partial f}s + \frac{\partial x_n}{\partial f_1}s_1 + \ . \ . \ . \ + \frac{\partial x_n}{\partial f_n}s_n."
\end{aligned}\right\}$$

Put into the form of a 'rule' this amounts to saying that any one of the r's is got by taking its 'reversed' coefficients and multiplying each of them by the corresponding s, and then adding.

Of course, having obtained a solution by using the peculiar eliminating multipliers

$$\frac{\partial x_k}{\partial f}, \quad \frac{\partial x_k}{\partial f_1}, \quad . \ . \ . \ , \quad \frac{\partial x_k}{\partial f_n}$$

and being aware of the existence of a general set of such multipliers, Jacobi had already to hand the means, which he did not fail to use, of finding an important identity. For, if the functional determinant be denoted by R, it had long been known that

$$r_k = \frac{1}{\mathrm{R}}\left\{\frac{\partial \mathrm{R}}{\partial\dfrac{\partial f}{\partial x_k}}s + \frac{\partial \mathrm{R}}{\partial\dfrac{\partial f_1}{\partial x_k}}s_1 + \ \ldots \ + \frac{\partial \mathrm{R}}{\partial\dfrac{\partial f_n}{\partial x_k}}s_n\right\}$$

and a comparison of this with the value of r_k above obtained gives at once

$$\frac{1}{\mathrm{R}}\cdot\frac{\partial \mathrm{R}}{\partial\dfrac{\partial f_i}{\partial x_k}} = \frac{\partial x_k}{\partial f_i},$$

a particular case being

$$\frac{1}{\mathrm{R}}\cdot\sum\pm\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n} = \frac{\partial x}{\partial f}.$$

This result may be viewed as giving any one of the differential-quotients $\dfrac{\partial x_k}{\partial f_i}$ in terms of the differential-quotients $\dfrac{\partial f_i}{\partial x_k}$; or, if we multiply both sides by R, it may be viewed as giving an expression for the *cofactor* of any element of the functional determinant.

Before leaving this part of the subject it may be noted that Jacobi discusses with equal fulness a set of linear equations in which the determinant of the coefficients is the *conjugate* (as it afterwards came to be called) of the functional determinant, with the result that the practical 'rule' above given is shown to hold here also.

The expression obtained for the cofactor of any element of the functional determinant is utilised (§ 9) to find an equally interesting expression for the differential-quotient of the functional determinant with respect to a quantity a which may be x, x_1, ... or any other involved in the functions. As R can be considered a function of its $(n+1)^2$ elements $\frac{\partial f_i}{\partial x_k}$, it is clear that $\frac{\partial \mathrm{R}}{\partial a}$ is expressible as the sum of $(n+1)^2$ terms of the form

$$\frac{\partial \mathrm{R}}{\partial \frac{\partial f_i}{\partial x_k}} \cdot \frac{\partial \frac{\partial f_i}{\partial x_k}}{\partial a}.$$

This, however, by substitution of the expression just referred to, becomes

$$\mathrm{R}\frac{\partial x_k}{\partial f_i} \cdot \frac{\partial \frac{\partial f_i}{\partial x_k}}{\partial a},$$

or

$$\mathrm{R}\frac{\partial x_k}{\partial f_i} \cdot \frac{\partial \frac{\partial f_i}{\partial a}}{\partial x_k};$$

so that we have

$$\begin{aligned}\frac{\partial \mathrm{R}}{\partial a} &= \mathrm{R}\sum_{i=0}^{i=n} \sum_{k=0}^{k=n} \frac{\partial \frac{\partial f_i}{\partial a}}{\partial x_k} \cdot \frac{\partial x_k}{\partial f_i}, \\ &= \mathrm{R}\sum_{i=0}^{i=n} \frac{\partial \frac{\partial f_i}{\partial a}}{\partial f_i},\end{aligned}$$

or, as Jacobi puts it,

$$\frac{\partial \log R}{\partial a} = \frac{\partial \frac{\partial f}{\partial a}}{\partial f} + \frac{\partial \frac{\partial f_1}{\partial a}}{\partial f_1} + \ldots + \frac{\partial \frac{\partial f_n}{\partial a}}{\partial f_n}.$$

With this theorem is associated another having no connection with it save the fact that the proof of it is dependent on the use of the same expression for the cofactor of an element of the functional determinant. Recalling the general theorem proved in his memoir *De formatione* ... viz.,

$$\Sigma \pm AA_1' \ldots A_m^{(m)} = (\Sigma \pm aa_1' \ldots a_n^{(n)})^m . \Sigma \pm a_{m+1}^{(m+1)} \ldots a_n^{(n)},$$

he applies it to the functional determinant, viz., to the case where

$$a_k^{(i)} = \frac{\partial f_i}{\partial x_k},$$

and where the cofactor

$$A_k^{(i)} = R\frac{\partial x_k}{\partial f_i}.$$

The immediate result, on dividing both sides by R^m, is

$$\sum \pm \frac{\partial f}{\partial x} . \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} . \sum \pm \frac{\partial x}{\partial f} . \frac{\partial x_1}{\partial f_1} \cdots \frac{\partial x_m}{\partial f_m} = \sum \pm \frac{\partial f_{m+1}}{\partial x_{m+1}} \cdots \frac{\partial f_n}{\partial x_n}.$$

The theorem

$$R . \frac{\partial x}{\partial f} = \sum \pm \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n},$$

already obtained, is the special case of this where $m = 0$. Another special case is at the other extreme, viz., where $m = n$, when we have

$$\sum \pm \frac{\partial x}{\partial f} . \frac{\partial x_1}{\partial f_1} \cdots \frac{\partial x_n}{\partial f_n} = \frac{1}{\sum \pm \frac{\partial f}{\partial x} . \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}},$$

the generalisation—or the analogue, as Jacobi would seem to prefer to view it—of the theorem

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

The next part (§ 10) of the subject relates to the case where the functions, whose functional determinant is sought, are not given explicitly in terms of the variables—where, in fact, we have $n+1$ functions of $x, x_1, \ldots, x_n, f, f_1, \ldots, f_n$, viz.,

$$F = 0, \quad F_1 = 0, \quad \ldots\ldots, \quad F_n = 0,$$

and where we are asked to find

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_n}{\partial x_n}.$$

Differentiating any one of the F's with respect to any one of the x's we have

$$0 = \frac{\partial F_i}{\partial x_k} + \frac{\partial F_i}{\partial f} \cdot \frac{\partial f}{\partial x_k} + \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_k} + \cdots\cdots + \frac{\partial F_i}{\partial f_n} \cdot \frac{\partial f_n}{\partial x_k},$$

which may be viewed as giving an expression for $-\frac{\partial F_i}{\partial x_k}$. Using this expression $(n+1)^2$ times we obtain for

$$(-1)^{n+1} \sum \pm \frac{\partial F}{\partial x} \cdot \frac{\partial F_1}{\partial x_1} \cdots\cdots \frac{\partial F_n}{\partial x_n}$$

an equivalent determinant each of whose elements is the sum of $n+1$ products, and which from the multiplication-theorem we know to be equal to

$$\sum \pm \frac{\partial F}{\partial f} \cdot \frac{\partial F_1}{\partial f_1} \cdots\cdots \frac{\partial F_n}{\partial f_n} \cdot \sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_n}{\partial x_n}.$$

It thus follows that the result sought is

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_n}{\partial x_n} = (-1)^{n+1} \frac{\sum \pm \frac{\partial F}{\partial x} \cdot \frac{\partial F_1}{\partial x_1} \cdots\cdots \frac{\partial F_n}{\partial x_n}}{\sum \pm \frac{\partial F}{\partial f} \cdot \frac{\partial F_1}{\partial f_1} \cdots\cdots \frac{\partial F_n}{\partial f_n}},$$

a theorem which Jacobi again takes pains to have noted as the analogue of the theorem which holds when $F(f, x) = 0$, viz.,

$$\frac{df}{dx} = -\frac{\partial F}{\partial f} \div \frac{\partial F}{\partial x}.$$

By way of corollary it is remarked that as the equations

$$F = 0, \quad F_1 = 0, \quad \ldots\ldots, \quad F_n = 0$$

cannot be more appropriately viewed as giving the f's in terms of the x's than as giving the x's in terms of the f's, we therefore have the twin result

$$\sum \pm \frac{\partial x}{\partial f} \cdot \frac{\partial x_1}{\partial f_1} \cdots\cdots \frac{\partial x_n}{\partial f_n} = (-1)^{n+1} \frac{\sum \pm \frac{\partial F}{\partial f} \cdot \frac{\partial F_1}{\partial f_1} \cdots\cdots \frac{\partial F_n}{\partial f_n}}{\sum \pm \frac{\partial F}{\partial x} \cdot \frac{\partial F_1}{\partial x_1} \cdots\cdots \frac{\partial F_n}{\partial x_n}},$$

and consequently from the two a theorem already obtained by a different method, viz.,

$$\sum \pm \frac{\partial x}{\partial f} \cdot \frac{\partial x_1}{\partial f_1} \cdots\cdots \frac{\partial x_n}{\partial f_n} = \frac{1}{\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_n}{\partial x_n}}.$$

Steadily pursuing his analogy, Jacobi next takes up (§ 11) the case where the f's are not given immediately in terms of the x's, but are given in terms of functions $\phi, \phi_1, \ldots, \phi_p$ of the x's. Here, of course, we have

$$\frac{\partial f_i}{\partial x_k} = \frac{\partial f_i}{\partial \phi} \cdot \frac{\partial \phi}{\partial x_k} + \frac{\partial f_i}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial x_k} + \cdots\cdots + \frac{\partial f_i}{\partial \phi_p} \cdot \frac{\partial \phi_p}{\partial x_k}$$

and therefore by $(n+1)^2$ substitutions there is obtained for

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_n}{\partial x_n}$$

an equivalent determinant, each of whose elements is the sum of $p+1$ products. This latter determinant, however, we know from Binet's multiplication-theorem is equal to 0 when $p<n$; is equal to the product of two determinants

$$\sum \pm \frac{\partial f}{\partial \phi} \cdot \frac{\partial f_1}{\partial \phi_1} \cdots\cdots \frac{\partial f_n}{\partial \phi_n} \cdot \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdots\cdots \frac{\partial \phi_n}{\partial x_n}$$

when $p=n$; and when $p>n$, is equal to a sum of such products, viz.,

$$\mathrm{S}\left\{\sum \pm \frac{\partial f}{\partial \phi} \cdot \frac{\partial f_1}{\partial \phi_1} \cdots\cdots \frac{\partial f_n}{\partial \phi_n} \cdot \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdots\cdots \frac{\partial \phi_n}{\partial x_n}\right\}$$

where the different terms included under the S are got by taking all the different sets of $n+1$ ϕ's from the $p+1$ available. This tripartite result Jacobi carefully enunciates at length in the form of three propositions. He notes, too, that the first is practically a result already obtained, because the functions in that case are not independent; that the second has for its analogue or ultimate case the theorem

$$\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx};$$

and similarly that the third is the extension of a result actually used in the proof, viz.,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial x} + \dots + \frac{\partial f}{\partial \phi_p} \cdot \frac{\partial \phi_p}{\partial x}.$$

He even enunciates formally a variant of the second proposition, calling the variant "Proposition iv," viz., *If* $\mathrm{f}, \mathrm{f}_1, \dots, \mathrm{f}_n$ *be functions of* $\mathrm{y}, \mathrm{y}_1, \dots, \mathrm{y}_n$, *and it be possible to express both the* f's *and the* y's *in terms of* $n+1$ *other quantities* $\mathrm{x}, \mathrm{x}_1, \dots, \mathrm{x}_n$, *then*

$$\sum \pm \frac{\partial \mathrm{f}}{\partial \mathrm{y}} \cdot \frac{\partial \mathrm{f}_1}{\partial \mathrm{y}_1} \dots \frac{\partial \mathrm{f}_n}{\partial \mathrm{y}_n} = \frac{\sum \pm \frac{\partial \mathrm{f}}{\partial \mathrm{x}} \cdot \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1} \dots \frac{\partial \mathrm{f}_n}{\partial \mathrm{x}_n}}{\sum \pm \frac{\partial \mathrm{y}}{\partial \mathrm{x}} \cdot \frac{\partial \mathrm{y}_1}{\partial \mathrm{x}_1} \dots \frac{\partial \mathrm{y}_n}{\partial \mathrm{x}_n}}.$$

The analogue also is again referred to in the form

$$\frac{df}{dy} = \frac{\frac{df}{dx}}{\frac{dy}{dx}},$$

and the special case, already twice obtained, where $f=x$, $f_1=x_1$, $\dots$, $f_n=x_n$.

Still further importance is given to the second proposition by assigning the next section (§ 12, pp. 341–343) to the consideration of certain deductions therefrom. First there is taken the special case where

$$\phi = x, \quad \phi_1 = x_1, \quad \dots, \quad \phi_m = x_m,$$

and where therefore

$$\sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \dots \frac{\partial \phi_n}{\partial x_n} = \sum \pm \frac{\partial \phi_{m+1}}{\partial x_{m+1}} \cdot \frac{\partial \phi_{m+2}}{\partial x_{m+2}} \dots \frac{\partial \phi_n}{\partial x_n},$$

the result clearly being that *If* $\mathrm{f}, \mathrm{f}_1, \dots, \mathrm{f}_n$ *be functions of* $\mathrm{x}, \mathrm{x}_1, \dots, \mathrm{x}_m, \phi_{m+1}, \dots, \phi_n$ *and* $\phi_{m+1}, \dots, \phi_n$ *be functions* *of* x, $x_1, \ldots, x_n$, *then the functional determinant of* f, $f_1, \ldots, f_n$ *with respect to* x, $x_1, \ldots, x_n$

$$= \sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_m}{\partial x_m} \cdot \frac{\partial f_{m+1}}{\partial \phi_{m+1}} \cdots\cdots \frac{\partial f_n}{\partial \phi_n} \cdot \sum \pm \frac{\partial \phi_{m+1}}{\partial x_{m+1}} \cdots\cdots \frac{\partial \phi_n}{\partial x_n}.$$

Here the first factor would reduce to

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_m}{\partial x_m}$$

if it were possible to put

$$\phi_{m+1} = f_{m+1}, \quad \phi_{m+2} = f_{m+2}, \quad \ldots, \quad \phi_n = f_n;$$

hence there follows the further proposition, which Jacobi speaks of as "prae ceteris memorabilis," that *the functional determinant of* f, $f_1, \ldots, f_n$ *with respect to* x, $x_1, \ldots, x_n$ *is equal to*

$$\sum \pm \left(\frac{\partial f}{\partial x}\right) \cdot \left(\frac{\partial f_1}{\partial x_1}\right) \cdots\cdots \left(\frac{\partial f_m}{\partial x_m}\right) \cdot \sum \pm \frac{\partial f_{m+1}}{\partial x_{m+1}} \cdot \frac{\partial f_{m+2}}{\partial x_{m+2}} \cdots\cdots \frac{\partial f_n}{\partial x_n},$$

if the brackets in the first factor be taken to mean that the functions therein occurring, viz., f, $f_1, \ldots, f_m$, *are considered to be expressed in terms of* x, $x_1, \ldots, x_m, f_{m-1}, f_{m+2}, \ldots, f_n$. An extreme case of this, viz., where the first determinant factor is of the order 1, has already been given.

In order that we may be able to substitute

$$\frac{1}{\sum \pm \frac{\partial x_{m+1}}{\partial \phi_{m+1}} \cdot \frac{\partial x_{m+2}}{\partial \phi_{m+2}} \cdots\cdots \frac{\partial x_n}{\partial \phi_n}} \quad \text{for} \quad \sum \pm \frac{\partial \phi_{m+1}}{\partial x_{m+1}} \cdot \frac{\partial \phi_{m+2}}{\partial x_{m+2}} \cdots\cdots \frac{\partial \phi_n}{\partial x_n},$$

in the former of these two propositions it is necessary that from the equations which give $\phi_{m+1}, \phi_{m+2}, \ldots, \phi_n$ in terms of x, $x_1, \ldots, x_n$ we obtain $x_{m+1}, x_{m+2}, \ldots, x_n$ in terms of the other x's and $\phi_{m+1}, \phi_{m+2}, \ldots, \phi_n$. Consequently, we have the proposition, *If* f, $f_1, \ldots, f_n$ *and* $x_{m+1}, x_{m+2}, \ldots, x_n$ *be expressed in terms of* x, $x_1, \ldots, x_m, \phi_{m+1}, \phi_{m+2}, \ldots, \phi_n$ *the functional determinant of* f, $f_1, \ldots, f_n$ *with respect to* x, $x_1, \ldots, x_n$ *is equal to*

$$\frac{\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_m}{\partial x_m} \cdot \frac{\partial f_{m+1}}{\partial \phi_{m+1}} \cdot \frac{\partial f_{m+2}}{\partial \phi_{m+2}} \cdots \frac{\partial f_n}{\partial \phi_n}}{\sum \pm \frac{\partial x_{m+1}}{\partial \phi_{m+1}} \cdot \frac{\partial x_{m+2}}{\partial \phi_{m+2}} \cdots \frac{\partial x_n}{\partial \phi_n}}.$$

Similarly, in order to be able to substitute

$$\frac{1}{\sum \pm \frac{\partial x_{m+1}}{\partial f_{m+1}} \cdot \frac{\partial x_{m+2}}{\partial f_{m+2}} \cdots \frac{\partial x_n}{\partial f_n}} \quad \text{for} \quad \sum \pm \frac{\partial f_{m+1}}{\partial x_{m+1}} \cdot \frac{\partial f_{m+2}}{\partial x_{m+2}} \cdots \frac{\partial f_n}{\partial x_n}$$

in the other proposition it is necessary that from the equations which give $f_{m+1}, f_{m+2}, \ldots, f_n$ in terms of $x, x_1, \ldots, x_n$ we obtain $x_{m+1}, x_{m+2}, \ldots, x_n$ in terms of the other x's and $f_{m+1}, f_{m+2}, \ldots, f_n$. We therefore have the result—*If*

$$f, \; f_1, \; \ldots, \; f_m, \; x_{m+1}, \; x_{m+2}, \; \ldots, \; x_n$$

be expressed in terms of

$$x, \; x_1, \; \ldots, \; x_m, \; f_{m+1}, \; f_{m+2}, \; \ldots, \; f_n,$$

the functional determinant of $f, f_1, \ldots, f_n$ *with respect to* $x, x_1, \ldots, x_n$, *is equal to*

$$\frac{\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_m}{\partial x_m}}{\sum \pm \frac{\partial x_{m+1}}{\partial f_{m+1}} \cdot \frac{\partial x_{m+2}}{\partial f_{m+2}} \cdots \frac{\partial x_n}{\partial f_n}}.$$

Leaving this, Jacobi harks back (§ 13) to an earlier proposition with a view to a generalisation now possible, viz., the proposition where the functions are given implicitly in terms of the independent variables by means of $n+1$ equations

$$F = 0, \quad F_1 = 0, \quad \ldots, \quad F_n = 0.$$

The extension arises from the number of equations now given being $n+m$, viz.,

$$F = 0, \quad F_1 = 0, \quad \ldots, \quad F_{n+m} = 0,$$

and each of the F's being a function not only of $x, x_1, \ldots, x_n$, $f, f_1, \ldots, f_n$ but also of

$$f_{n+1}, \; f_{n+2}, \; \ldots, \; f_{n+m}.$$

If the last m of the equations were solved for the last m of the f's, and these f's thus eliminated from the other $n+1$ equations, the functional determinant desired would, by the proposition sought to be generalised, be equal to

$$(-1)^{n+1}\frac{\sum\pm\left(\frac{\partial \mathrm{F}}{\partial x}\right)\left(\frac{\partial \mathrm{F}_1}{\partial x_1}\right)\cdots\cdot\left(\frac{\partial \mathrm{F}_n}{\partial x_n}\right)}{\sum\pm\left(\frac{\partial \mathrm{F}}{\partial f}\right)\left(\frac{\partial \mathrm{F}_1}{\partial f_1}\right)\cdots\cdot\left(\frac{\partial \mathrm{F}_n}{\partial f_n}\right)},$$

where the brackets are used to indicate that the enclosed F's are in the altered forms resulting from the substitution referred to. Multiplying numerator and denominator by

$$\sum\pm\frac{\partial \mathrm{F}_{n+1}}{\partial f_{n+1}}\cdot\frac{\partial \mathrm{F}_{n+2}}{\partial f_{n+2}}\cdots\cdot\frac{\partial \mathrm{F}_{n+m}}{\partial f_{n+m}}$$

we obtain a new numerator which by a later proposition is equal to

$$\sum\pm\frac{\partial \mathrm{F}}{\partial x}\cdot\frac{\partial \mathrm{F}_1}{\partial x_1}\cdots\cdot\frac{\partial \mathrm{F}_n}{\partial x_n}\cdot\frac{\partial \mathrm{F}_{n+1}}{\partial f_{n+1}}\cdot\frac{\partial \mathrm{F}_{n+2}}{\partial f_{n+2}}\cdots\cdot\frac{\partial \mathrm{F}_{n+m}}{\partial f_{n+m}},$$

and a new denominator which for a similar reason is equal to

$$\sum\pm\frac{\partial \mathrm{F}}{\partial f}\cdot\frac{\partial \mathrm{F}_1}{\partial f_1}\cdots\cdot\frac{\partial \mathrm{F}_{n+m}}{\partial f_{n+m}}.$$

The functional determinant desired is thus found to be equal to

$$(-1)^{n+1}\frac{\sum\frac{\partial \mathrm{F}}{\partial x}\cdot\frac{\partial \mathrm{F}_1}{\partial x_1}\cdots\cdot\frac{\partial \mathrm{F}_n}{\partial x_n}\cdot\frac{\partial \mathrm{F}_{n+1}}{\partial f_{n+1}}\cdot\frac{\partial \mathrm{F}_{n+2}}{\partial f_{n+2}}\cdots\cdot\frac{\partial \mathrm{F}_{n+m}}{\partial f_{n+m}}}{\sum\pm\frac{\partial \mathrm{F}}{\partial f}\cdot\frac{\partial \mathrm{F}_1}{\partial f_1}\cdots\cdot\frac{\partial \mathrm{F}_{n+m}}{\partial f_{n+m}}},$$

where, it will be observed, all the F's are in their original form.

As usual, the extreme case is noted, viz., the case where a single function f of one variable x is given by means of $m+1$ equations connecting $x, f, f_1, \ldots, f_n$ the result then being

$$\frac{dx}{df}=-\frac{\sum\pm\frac{\partial \mathrm{F}}{\partial x}\cdot\frac{\partial \mathrm{F}_1}{\partial f_1}\cdots\cdot\frac{\partial \mathrm{F}_m}{\partial f_m}}{\sum\pm\frac{\partial \mathrm{F}}{\partial f}\cdot\frac{\partial \mathrm{F}_1}{\partial f_1}\cdots\cdot\frac{\partial \mathrm{F}_m}{\partial f_m}};$$

and also, as usual, the occasion is utilised to draw attention to the analogy between the differential-quotient and the functional-determinant. "Quam formulam si cum generali comparas, et hic vides perfectam locum habere analogiam inter differentiale primum functionis unius variabilis atque Determinans systematis functionum plurium variabilium."

The theorem next brought forward (§ 14) is said to be useful in connection with the preceding general theorem for finding the functional determinant when the functions 'quocumque modo implicito dantur,' and is also spoken of as being in itself 'prae ceteris memorabilis.' It is—*If* $\mathrm{f}, \mathrm{f}_1, \ldots, \mathrm{f}_\mathrm{n}$ *be functions of* $\mathrm{x}, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}$ *and there be given the equations*

$$\mathrm{f} = a, \quad \mathrm{f}_1 = a_1, \quad \ldots, \quad \mathrm{f}_\mathrm{n} = a_\mathrm{n}$$

in which $a, a_1, \ldots, a_n$ *are constants, then the functional determinant*

$$\sum \pm \frac{\partial \mathrm{f}}{\partial \mathrm{x}} \cdot \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1} \cdots \cdot \frac{\partial \mathrm{f}_\mathrm{n}}{\partial \mathrm{x}_\mathrm{n}}$$

will not be altered by any transformation made upon the f's *by the use of the given equations,* it being understood, of course, that *the equation* $\mathrm{f}_\mathrm{i} = a_\mathrm{i}$ *is not used in the transformation of* f_i.

Taking first the case where only one of the functions, say f, is transformed, this becoming ϕ by the use of the equations

$$f_1 = a_1, \quad f_2 = a_2, \quad \ldots, \quad f_n = a_n,$$

we see that ϕ in addition to $x, x_1, \ldots, x_n$ may involve $a_1, a_2, \ldots, a_n$, and that therefore we have

$$\frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial a_1} \cdot \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial a_2} \cdot \frac{\partial f_2}{\partial x} + \ldots + \frac{\partial \phi}{\partial a_n} \cdot \frac{\partial f_n}{\partial x},$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial a_1} \cdot \frac{\partial f_1}{\partial x_1} + \frac{\partial \phi}{\partial a_2} \cdot \frac{\partial f_2}{\partial x_1} + \ldots + \frac{\partial \phi}{\partial a_n} \cdot \frac{\partial f_n}{\partial x_1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial f}{\partial x_n} = \frac{\partial \phi}{\partial x_n} + \frac{\partial \phi}{\partial a_1} \cdot \frac{\partial f_1}{\partial x_n} + \frac{\partial \phi}{\partial a_2} \cdot \frac{\partial f_2}{\partial x_n} + \ldots + \frac{\partial \phi}{\partial a_n} \cdot \frac{\partial f_n}{\partial x_n}.$$

By substituting in the functional determinant

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \cdot \frac{\partial f_n}{\partial x_n},$$

the equivalents here given for

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial x_1}, \quad \ldots, \quad \frac{\partial f}{\partial x_n},$$

there is obtained a determinant which is expressible as the sum of $n+1$ determinants, all of which vanish except the first. We thus arrive at

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdot \frac{\partial f_n}{\partial x_n},$$

as was to be proved.

Passing to the case where *two* of the functions are changed, he says, first, that if, by the use of the equations

$$\phi = a, \quad f_2 = a_2, \quad f_3 = a_3, \quad \ldots, \quad f_n = a_n$$

the function f_1 is changed into ϕ_1, then exactly as before it can be shown that

$$\sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots\cdot \frac{\partial f_n}{\partial x_n}.$$

More questionable is the logic of his second step, which is to the effect that from this and the previous result it follows that

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots\cdot \frac{\partial f_n}{\partial x_n}.$$

His third step is simply the assertion that by proceeding in this way we may prove generally that if by use of the equations

$$f = a, \quad f_1 = a_1, \quad \ldots, \quad f_{\iota-1} = a_{\iota-1}, \quad f_{\iota+1} = a_{\iota+1}, \quad \ldots, \quad f_n = a_n$$

f_ι becomes changed into ϕ_ι, then

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdot \frac{\partial f_n}{\partial x_n} = \sum \pm \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi_1}{\partial x_1} \cdots\cdot \frac{\partial \phi_n}{\partial x_n};$$

and the matter is concluded with the further assertion that if in the elements of the second determinant there be substituted for $a, a_1, \ldots, a_n$ the functions which they represent, that determinant will be identically equal to the other.

Considerable space is next given (§ 15) to the discussion of the case where the number of variables $x, x_1, \ldots, x_{n+m}$ which the functions involve is m greater than the number of functions. First it is noted that if the functions be not mutually independent, they are not independent with respect to any $n+1$ of the variables, and therefore each functional determinant formed with respect to $n+1$ of the $n+m+1$ variables must vanish. Then the converse proposition is taken up, viz., that if all these determinants vanish, the functions are not independent. The method of proof is that known as mathematical induction, that is to say, the assumption being made that the proposition holds for n functions $f, f_1, \ldots, f_{n-1}$ it is shown to hold for $n+1$. Clearly we may start by viewing $f, f_1, \ldots, f_{n-1}$ as being independent, for if they be not, there is nothing to prove; and this being the case, the various determinants of these functions with respect to n of the variables cannot vanish. Denoting the first of the said determinants, viz.,

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \text{ by B,}$$

and choosing from the given vanishing determinants of the $(n+1)^{\text{th}}$ order those having n of their variables the same as those of B, viz., the $m+1$ determinants,

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial f_n}{\partial x_n},$$

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial f_n}{\partial x_{n+1}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial f_n}{\partial x_{n+m}},$$

we see that from a previous proposition these are respectively equal to

$$\mathrm{B}\left(\frac{\partial f_n}{\partial x_n}\right), \quad \mathrm{B}\left(\frac{\partial f_n}{\partial x_{n+1}}\right), \quad \mathrm{B}\left(\frac{\partial f_n}{\partial x_{n+2}}\right), \quad \ldots, \quad \mathrm{B}\left(\frac{\partial f_n}{\partial x_{n+m}}\right),$$

where the operation indicated within brackets is meant to be performed on f_n as expressed in terms of

$$f, \; f_1, \; \ldots, \; f_{n-1}, \; x_n, \; x_{n+1}, \; \ldots, \; x_{n+m}.$$

As B does not vanish, it follows from this that

$$\left(\frac{\partial f_n}{\partial x_n}\right) = 0, \quad \left(\frac{\partial f_n}{\partial x_{n+1}}\right) = 0, \quad \ldots, \quad \left(\frac{\partial f_n}{\partial x_{n+m}}\right) = 0,$$

and consequently that f_n involves only $f, f_1, \ldots, f_{n-1}$; that is to say, that $f, f_1, \ldots, f_n$ are not independent. This result being obtained, it only needs to be noted that the proposition being manifestly true in the case where the number of functions is *one*, must be true generally.

As an addendum, it is pointed out that since the vanishing of the $m+1$ determinants

$$\begin{gathered}
\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial f_n}{\partial x_n}, \\
\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial f_n}{\partial x_{n+1}}, \\
\cdots\cdots\cdots\cdots\cdots \\
\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial f_n}{\partial x_{n+m}},
\end{gathered}$$

when the determinant, B, and the n^2 elements common to all these do not vanish, implies that f_n is a function of $f, f_1, \ldots, f_{n-1}$: and since this mutual dependence of $f, f_1, \ldots, f_{n-1}, f_n$ implies the vanishing of all the functional determinants formed with respect to any $n+1$ of the $n+m+1$ independent variables, we are led to the conclusion that, provided B does not vanish, the vanishing of all these functional determinants of which the number is

$$\frac{(n+m+1)(n+m)\ldots(m+1)}{1.2.3\ldots(n+1)} \quad \text{or} \quad \frac{(n+m+1)(n+m)\ldots(n+1)}{1.2.3\ldots(m+1)}$$

is a consequence of the vanishing of a certain $m+1$ of them.

In order that the connection between the members of this group of functional determinants formed from the differential-quotients of $f, f_1, \ldots, f_n$ with respect to any $n+1$ of the variables $x, x_1, \ldots, x_{n+m}$ may be better looked into, several identities regarding square arrays of functional determinants are next given (§ 16).

Taking in addition to $f, f_1, \ldots, f_n$ the m arbitrary functions

$$f_{n+1}, \ f_{n+2}, \ \ldots, \ f_{n+m}$$

of the same $n+m+1$ variables, and denoting the determinant

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial f_{n+i}}{\partial x_{n+k}} \quad \text{by} \quad b_k^{(i)}$$

where i, k may each have the values 0, 1, 2, . . . , m, we see that from a previous result we have

$$b_k^{(i)} = \mathrm{B} \cdot \left(\frac{\partial f_{n+i}}{\partial x_{n+k}} \right),$$

if f_{n+i} within the brackets involves $f, f_1, \ldots, f_{n-1}$ in place of $x, x_1, \ldots, x_{n-1}$. From this it immediately follows that

$$\sum \pm b b_1' \cdots\cdots b_m^{(m)} = \mathrm{B}^{m+1} \cdot \sum \pm \left(\frac{\partial f_n}{\partial x_n} \right) \cdot \left(\frac{\partial f_{n+1}}{\partial x_{n+1}} \right) \cdots\cdots \left(\frac{\partial f_{n+m}}{\partial x_{n+m}} \right).$$

But the theorem already obtained regarding the factorisation of any functional determinant gives

$$\begin{aligned} &\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n+m}}{\partial x_{n+m}} \\ &= \sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \cdot \sum \pm \left(\frac{\partial f_n}{\partial x_n} \right) \left(\frac{\partial f_{n+1}}{\partial x_{n+1}} \right) \cdots\cdots \left(\frac{\partial f_{n+m}}{\partial x_{n+m}} \right), \\ &= \mathrm{B} \cdot \sum \pm \left(\frac{\partial f_n}{\partial x_n} \right) \cdot \left(\frac{\partial f_{n+1}}{\partial x_{n+1}} \right) \cdots\cdots \left(\frac{\partial f_{n+m}}{\partial x_{n+m}} \right). \end{aligned}$$

Consequently, by substitution we have finally

$$\sum \pm b b_1' \cdots\cdots b_m^{(m)} = \mathrm{B}^m \cdot \sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{n+m}}{\partial x_{n+m}},$$

a result which Jacobi says is of frequent use in dealing with questions regarding determinants.*

* A curious interest attaches to this result. On the right-hand side are two determinants whose elements are differential-quotients; but the first, B, being a minor of the second, the total number of different elements is simply the number in the second determinant, viz., $(n+m+1)^2$. On the left-hand side is a compound determinant of the $(m+1)^{\text{th}}$ order, each of whose elements is a determinant of the $(n+1)^{\text{th}}$ order; nevertheless the number of different elements is again $(n+m+1)^2$ and not $(m+1)^2(n+1)^2$, because all the $(m+1)^2$ elements of the compound determinant have the n^2 elements of B in common, and of the $2n+1$ which border these n^2 elements, only one, viz., the cofactor of B, is different throughout, each of the $2n$ others being repeated $m+1$ times, so

Next, $-\beta_k^{(i)}$ being used to stand for

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots\cdots \frac{\partial f_{i-1}}{\partial x_{i-1}} \cdot \frac{\partial f_i}{\partial x_{n+k}} \cdot \frac{\partial f_{i+1}}{\partial x_{i+1}} \cdots\cdots \frac{\partial f_{n-1}}{\partial x_{n-1}},$$

it is sought to find an equivalent for $\Sigma \pm \beta \beta_1^{(i)} \ldots\ldots \beta_m^{(m)}$. Clearly $-\beta_k^{(i)}$ is the determinant formed from B by using x_{n+k} as an independent variable instead of x_i; but for our purpose it is of more importance to note that it is the cofactor of $\dfrac{\partial f_{n+i}}{\partial x_i}$ in $b_k^{(i)}$, for then the determinant under consideration, viz.,

$$\Sigma \pm \beta \beta_1^{(1)} \ldots\ldots \beta_m^{(m)}$$

being thus the cofactor of

$$\frac{\partial f_n}{\partial x} \cdot \frac{\partial f_{n+1}}{\partial x_1} \cdots\cdots \frac{\partial f_{n+m}}{\partial x_m}$$

in the left-hand member of the identity just found, it only remains to seek the cofactor of the same expression in the

that the total number of different elements is $n^2+(m+1)^2+2n(m+1^2)(m+1)$. Further, on both sides the degree in these $(n+m+1)^2$ differential-quotients is the same, being clearly $(n+1)(m+1)$ on the left, and $mn+(n+m+1)$ on the right. It is thus at once suggested to us that the identity is not necessarily an identity connecting differential-quotients only, but is true of any $(n+m+1)^2$ elements whatever; and the suggestion is readily verified when the ubiquitous presence of B as a coaxial minor raises the suspicion that the identity must be an 'extensional.' The case where $n=2$ and $m=3$ is given on p. 215 of my text-book (*Treatise on the Theory of Determinants*) in the form—

$$\begin{vmatrix} |a_1\ e_5\ f_6| & |a_2\ e_5\ f_6| & |a_3\ e_5\ f_6| & |a_4\ e_5\ f_6| \\ |b_1\ e_5\ f_6| & |b_2\ e_5\ f_6| & |b_3\ e_5\ f_6| & |b_4\ e_5\ f_6| \\ |c_1\ e_5\ f_6| & |c_2\ e_5\ f_6| & |c_3\ e_5\ f_6| & |c_4\ e_5\ f_6| \\ |d_1\ e_5\ f_6| & |d_2\ e_5\ f_6| & |d_3\ e_5\ f_6| & |d_4\ e_5\ f_6| \end{vmatrix} = |\,a_1b_2c_3d_4e_5f_6\,| \cdot |\,e_5f_6\,|^3$$

where it is viewed as an extensional of the manifest identity

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = |a_1b_2c_3d_4|.$$

The theorem in its general form may be enunciated as follows:—*If from the determinant* $|\,a_{1,n+m+1}\,|$ *there be formed all minors of the* $(n+1)^{th}$ *order which have* $|\,a_{1n}\,|$ *for the cofactor of their final element, and these be orderly arranged in square array, the determinant of this square array of the* $(m+1)^{th}$ *order is equal to*

$$|\,a_{1n}\,|^m \cdot |\,a_{1,n+m+1}\,|.$$

right-hand member. Now, in the first factor, B_m, of this right-hand member the expression does not occur at all, and in the other factor, which is transformable into

$$(-1)^{n(m+1)}\sum\pm\frac{\partial f}{\partial x_{m+1}}\cdot\frac{\partial f_1}{\partial x_{m+2}}\cdots\cdot\frac{\partial f_{n-1}}{\partial x_{m+n}}\cdot\frac{\partial f_n}{\partial x}\cdots\cdot\frac{\partial f_{n+m}}{\partial x_m},$$

it occurs multiplied by

$$(-1)^{n(m+1)}\sum\pm\frac{\partial f}{\partial x_{m+1}}\,\frac{\partial f_1}{\partial x_{m+2}}\cdots\cdot\frac{\partial f_{n-1}}{\partial x_{m+n}}.$$

The desired result thus is

$$\sum\pm\beta\beta_1^{(1)}\cdots\cdot\beta_m^{(m)}=(-1)^{n(m+1)}\mathrm{B}^m\cdot\sum\pm\frac{\partial f}{\partial x_{m+1}}\cdot\frac{\partial f_1}{\partial x_{m+2}}\cdots\cdot\frac{\partial f_{n-1}}{\partial x_{m+n}},$$

which Jacobi formally enunciates as follows:

"E Determinante

$$\mathrm{B}=\sum\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\cdot\frac{\partial f_{n-1}}{\partial x_{n-1}}$$

deducantur $(m+1)^2$ alia Determinantia, uni cuilibet differentialium ipsarum $x, x_1, \ldots, x_m$ respectu sumtorum substituendo successive differentialia ipsarum $x_n, x_{n+1}, \ldots, x_{n+m}$ respectu sumta: illarum $(m+1)^2$ quantitatum Determinans aequatur expressioni

$$(-1)^{(m+1)(n+1)}\mathrm{B}^m\cdot\sum\pm\frac{\partial f}{\partial x_{m+1}}\cdot\frac{\partial f_1}{\partial x_{m+2}}\cdots\cdot\frac{\partial f_{n-1}}{\partial x_{m+n}}.\text{"}$$

From equating cofactors of

$$\frac{\partial f_n}{\partial x}\cdot\frac{\partial f_{n+1}}{\partial x_1}\cdots\cdot\frac{\partial f_{m+n}}{\partial x_m},$$

Jacobi proceeds to equate cofactors of

$$\frac{\partial f_{n+1}}{\partial x}\cdots\cdot\frac{\partial f_{m+n}}{\partial x_{m-1}}$$

in the same fundamental identity, the resulting theorem now being

$$\sum\pm b\beta_1\beta_2'\cdots\cdot\beta_m^{(m-1)}=(-1)^{m(n+1)}\mathrm{B}^m\cdot\sum\pm\frac{\partial f}{\partial x_m}\cdot\frac{\partial f_1}{\partial x_{m+1}}\cdots\cdot\frac{\partial f_n}{\partial x_{m+n}};$$

and then he adds,

"Eodem modo obtinetur generaliter

$$\sum\pm bb_1'\cdots\cdot b_{i-1}^{(i-1)}\beta_i\beta_{i+1}'\cdots\cdot\beta_m^{(m-1)}$$
$$=\pm\mathrm{B}^m\cdot\sum\pm\frac{\partial f}{\partial x_{m-i+1}}\cdot\frac{\partial f_1}{\partial x_{m-i+2}}\cdots\cdot\frac{\partial f_{n+i-1}}{\partial x_{m+n}}$$

qua in formula signo $\pm$ substituendum est aut $(-1)^{n(m+1)}$ aut $(-1)^{m(n+1)}$ prout i par aut impar est."*

The second derived identity,—that is to say, the case of the final general identity where $i=1$,—Jacobi proceeds to utilise for the purpose of proving his proposition regarding the effect of the vanishing of certain $m+1$ functional determinants. The path which he follows to reach his result is not a little surprising. Instead of saying that the vanishing of $b, b_1, b_2, \ldots, b_m$,—for these are the $m+1$ determinants in question,—entails the vanishing of the left-hand side of the identity, viz.,

$$\Sigma \pm b\beta_1\beta_2^{(1)} \ldots \beta_m^{(m-1)},$$

* The fact that these identities can be derived in the way here indicated from another which the preceding footnote has shown to be true, not merely of functional determinants but of determinants in general, is convincing proof that they also (*i.e.*, the derived identities) are not restricted to any special form of determinant. Using the fundamental identity as enunciated in the footnote, and taking the special case of it where $n=4$ and $m=2$, and where therefore the given determinant may be written $|a_1b_2c_3d_4e_5f_6g_7|$, we have

$$\begin{vmatrix} |a_1 b_2 c_3 d_4 e_5| & |a_1 b_2 c_3 d_4 e_6| & |a_1 b_2 c_3 d_4 e_7| \\ |a_1 b_2 c_3 d_4 f_5| & |a_1 b_2 c_3 d_4 f_6| & |a_1 b_2 c_3 d_4 f_7| \\ |a_1 b_2 c_3 d_4 g_5| & |a_1 b_2 c_3 d_4 g_6| & |a_1 b_2 c_3 d_4 g_7| \end{vmatrix} = |a_1 b_2 c_3 d_4|^2 \cdot |a_1 b_2 c_3 d_4 e_5 f_6 g_7|.$$

Now in each of the determinants forming the first row on the left here, e_1 occurs as an element, in the second row f_2 similarly occurs, and in the third row g_3, while on the right these only occur in $|a_1b_2c_3d_4e_5f_6g_7|$. Consequently, equating cofactors of $e_1f_2g_3$ we have

$$\begin{vmatrix} |a_2 b_3 c_4 d_5| & |a_2 b_3 c_4 d_6| & |a_2 b_3 c_4 d_7| \\ -|a_1 b_3 c_4 d_5| & -|a_1 b_3 c_4 d_6| & -|a_1 b_3 c_4 d_7| \\ |a_1 b_2 c_4 d_5| & |a_1 b_2 c_4 d_6| & |a_1 b_2 c_4 d_7| \end{vmatrix} = |a_1 b_2 c_3 d_4|^2 \cdot |a_4 b_5 c_6 d_7|$$

which when put in the form

$$\begin{vmatrix} |a_5 b_2 c_3 d_4| & |a_6 b_2 c_3 d_4| & |a_7 b_2 c_3 d_4| \\ |a_1 b_5 c_3 d_4| & |a_1 b_6 c_3 d_4| & |a_1 b_7 c_3 d_4| \\ |a_1 b_2 c_5 d_4| & |a_1 b_2 c_6 d_4| & |a_1 b_2 c_7 d_4| \end{vmatrix} = -|a_1 b_2 c_3 d_4|^2 \cdot |a_4 b_5 c_6 d_7|$$

is a case of the first derived theorem.

The original theorem, it should be noted, is true for all values of n and m; the derived holds only when $m<n$,—in fact, if we do not, in seeking to obtain the latter, take $m<n$ in the former, we shall fail in our aim. Thus, taking $n=2=m$ in the former, the given determinant being $|a_1b_2c_3d_4e_5|$, we have quite correctly

$$\begin{vmatrix} |a_1 b_2 c_3| & |a_1 b_2 c_4| & |a_1 b_2 c_5| \\ |a_1 b_2 d_3| & |a_1 b_2 d_4| & |a_1 b_2 d_5| \\ |a_1 b_2 e_3| & |a_1 b_2 e_4| & |a_1 b_2 e_5| \end{vmatrix} = |a_1 b_2|^2 \cdot |a_1 b_2 c_3 d_4 e_5|;$$

and that therefore, if the factor B on the right-hand side do not vanish, the other factor

$$\sum \pm \frac{\partial f}{\partial x_m} \cdot \frac{\partial f_1}{\partial x_{m+1}} \cdots \cdot \frac{\partial f_n}{\partial x_{m+n}}$$

must vanish, he takes some pains to obtain a new identity and then applies this very reasoning to it. Denoting the cofactors of b, $b_1, \ldots, b_m$ in $\Sigma \pm b\beta_1\beta_2^{(1)} \cdots \beta_m^{(m-1)}$ by λ, $\lambda_1, \ldots, \lambda_m$, he points out that as none of the β's involves f_n the same is true of the λ's. On the other hand, the b's do involve f_n, but only one of them, viz., b_k, involves the differential-quotients of f_n with respect to x_{n+k}, this differential-quotient being in fact the last element of all and having B for its cofactor. In this way it appears that on the left-hand side of the identity

$$\text{the cofactor of } \frac{\partial f_n}{\partial x_{n+k}} \text{ is } \lambda_\kappa \mathrm{B}.$$

If in the same manner we take $\sum \pm \frac{\partial f}{\partial x_m} \cdot \frac{\partial f_1}{\partial x_{m+1}} \cdots \cdot \frac{\partial f_n}{\partial x_{m+n}}$ and denote* the cofactor of $\frac{\partial f_n}{\partial x_{n+k}}$ in it by μ_κ, it must follow that on

---

but while c_1 occurs in each element of the first row on the left, and d_2 similarly in the second row, e_3 does not so occur in the third, and consequently the cofactor of $c_1d_2e_3$ on the left takes a different form from that given by Jacobi.

The first derived theorem in its general form may be enunciated as follows:—*If there be two determinants* D *and* Δ *of the* n^{th} *order such that the last* n − m *columns of* D *are the same as the first* n − m *columns of* Δ, *and if there be formed a square array of new determinants by supplanting each of the first* m *columns of* D *by each of the last* m *columns of* Δ, *the determinant of this square array of the* m^{th} *order is equal to*

$$(-1)^{m(n+1)}\, \mathrm{D}^{m-1} \Delta.$$

To illustrate the second derived theorem we may equate cofactors of f_1g_2 where we formerly equated cofactors of $e_1f_2g_3$, the result clearly being

$$\begin{Vmatrix} |a_1b_2c_3d_4e_5| & |a_1b_2c_3d_4e_6| & |a_1b_2c_3d_4e_7| \\ |a_2b_3c_4d_5| & |a_2b_3c_4d_6| & |a_2b_3c_4d_7| \\ |a_1b_3c_4d_5| & |a_1b_3c_4d_6| & |a_1b_3c_4d_7| \end{Vmatrix} = -|a_1b_2c_3d_4|^2 . |a_3b_4c_5d_6e_7|.$$

The next of the series would be got by equating cofactors of g_1.

* This is not the same as putting, with Jacobi,

$$\sum \pm \frac{\partial f}{\partial x_m} \cdot \frac{\partial f_1}{\partial x_{m+1}} \cdots \cdot \frac{\partial f_n}{\partial x_{m+n}} = \mu \frac{\partial f_n}{\partial x_n} + \mu_1 \frac{\partial f_n}{\partial x_{n+1}} + \ldots + \mu_m \frac{\partial f_n}{\partial x_{m+n}},$$

for the determinant on the left being of the $(n+1)^{\text{th}}$ order there should be $n+1$ terms on the right instead of $m+1$.

the right-hand side

$$\text{the cofactor of } \frac{\partial f_n}{\partial x_{n+k}} \text{ is } (-1)^{m(n+1)} \mathrm{B}\mu_k .$$

The connection between the λ's and the μ's is thus

$$\lambda_k = (-1)^{m(n+1)} \mathrm{B}^{m-1} \mu_k ,$$

so that

$$\lambda b + \lambda_1 b_1 + \ldots + \lambda_m b_m = (-1)^{m(n+1)} \mathrm{B}^{m-1} (\mu b + \mu_1 b_1 + \ldots + \mu_m b_m).$$

The left-hand member here, however, being equal to the left-hand member of the identity with which we started, it follows that the two right-hand members must also be equal, and therefore that

$$\mu b + \mu_1 b_1 + \ldots + \mu_m b_m = \mathrm{B} . \sum \pm \frac{\partial f}{\partial x_m} \cdot \frac{\partial f_1}{\partial x_{m+1}} \cdots \frac{\partial f_n}{\partial x_{m+n}} .$$

Of course this shows, exactly as the original identity did, that if

$$b = b_1 = \ldots = b_n = 0 \quad \text{and} \quad \mathrm{B} \gtrless 0$$

then

$$\sum \pm \frac{\partial f}{\partial x_m} \cdot \frac{\partial f_1}{\partial x_{m+1}} \cdots \frac{\partial f_n}{\partial x_{m+n}} = 0$$

—that is to say, the functional determinant of $f, f_1, \ldots, f_n$ with respect to another set of $n+1$ variables vanishes also. Jacobi, however, does not at once say this, but drawing his reader's attention to the fact that the new set of variables contains $n-m$ taken from $x, x_1, \ldots, x_{n-1}$ and $m+1$ others, viz., $x_n, x_{n+1}, \ldots, x_{n+m}$, he affirms that the identity reached shows how the functional determinant of $f, f_1, \ldots, f_n$ with respect to *any* set of $n+1$ variables is expressible in terms of the $m+1$ functional determinants whose variables are

$$\begin{array}{l} x,\ x_1,\ \ldots\ldots,\ x_{n-1},\ x_n, \\ x,\ x_1,\ \ldots\ldots,\ x_{n-1},\ x_{n+1}, \\ \ldots\ldots\ldots\ldots\ldots \\ x,\ x_1,\ \ldots\ldots,\ x_{n-1},\ x_{n+m}. \end{array}$$

His words are—

"Unde formula docet quomodo e functionum $f, f_1, \ldots, f_n$ Determinantibus b_k per idoneos factores multiplicatis et additis proveniat earundem functionum Determinans *quarumcunque* variabilium respectu formatum atque per ipsum B multiplicatum. Hinc bene patet, quod § pr. demonstravi, quomodo omnibus b_k evanescentibus neque ipso B evanescente, simul cuncta illa Determinantia evanescant."*

Continuing the work of deduction, Jacobi lastly equates the cofactors of $\frac{\partial f_n}{\partial x}$ in the two members of the identity

$$\mu b + \mu_1 b_1 + \ . \ . \ . \ + \mu_m b_m = \mathrm{B} \, . \sum \pm \frac{\partial f}{\partial x_m} \cdot \frac{\partial f_1}{\partial x_{m+1}} \cdots \frac{\partial f_n}{\partial x_{m+n}}$$

noting that this differential-quotient does not occur at all on the right-hand side, nor in the μ's on the left-hand side, but in b_k occurs with the cofactor

$$-\sum \pm \frac{\partial f}{\partial x_{n+k}} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_{n-1}}{\partial x_{n-1}}.$$

The result is the proposition †

"Sit μ_k functionum $f, f_1, \ldots, f_{n-1}$ Determinans quod in Determinante

$$\sum \pm \frac{\partial f}{\partial x_m} \cdot \frac{\partial f_1}{\partial x_{m+1}} \cdots \frac{\partial f_n}{\partial x_{m+n}}$$

per $\frac{\partial f_n}{\partial x_{n+k}}$ multiplicatur, ubi $m \leqq n$, erit

$$\begin{aligned} &\mu \sum \pm \frac{\partial f}{\partial x_n} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \\ + \ &\mu_1 \sum \pm \frac{\partial f}{\partial x_{n+1}} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \\ &\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ + \ &\mu_m \sum \pm \frac{\partial f}{\partial x_{n+m}} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} = 0." \end{aligned}$$

The case where $m = n$ is specially noted.

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* Of course this theorem also is not limited to determinants having differential-quotients for their elements. The general enunciation may be put as follows :— *If* m *determinants of the* nth *order all have the same* n − 1 *columns in common, and vanish independently, then every determinant of the* nth *order whose* n *columns are chosen from the* m + n − 1 *different columns must vanish likewise.* (See *Proc. Roy. Soc. Edin.*, xviii. pp. 73–82.)

† This proposition, and that from which it is derived, are again propositions which hold regarding determinants in general, the class to which they belong being that which concerns *aggregates of products of pairs of determinants*,—a class, the first instances of which have been seen to occur in Bezout (1779). In connection with Jacobi's remark regarding the case where $m = n$, it is worth while to note Sylvester's enunciation in *Philos. Magazine* (1839), xvi. p. 42.

Leaving now these general theorems which involve two suffixes m and n, and which concern *groups* of functional determinants, Jacobi returns (§ 17) to the consideration of the properties of a single functional determinant, the specialisation being made, not by giving a particular value to m the second suffix introduced, but by leaving it unrestricted, and putting the original $n=1$. In the theorem

$$\sum \pm b b_1^{(1)} \dots . b_m^{(m)} = \mathrm{B}^m . \sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \dots . \frac{\partial f_{m+1}}{\partial x_{m+1}}$$

B then stands for $\frac{\partial f}{\partial x}$ and $b_k^{(i)}$ for $\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_{i+1}}{\partial x_{k+1}}$. Making in this the further specialisation, $f=0$, so that x has to be considered as a function of $x_1, x_2, \dots, x_{m-1}$ we have

$$\frac{\partial f}{\partial x_{k+1}} + \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x_{k+1}} = 0,$$

and consequently

$$\begin{aligned} b_k^{(i)} \quad i.e. \quad & \frac{\partial f}{\partial x} \cdot \frac{\partial f_{i+1}}{\partial x_{k+1}} - \frac{\partial f}{\partial x_{k+1}} \cdot \frac{\partial f_{i+1}}{\partial x} \\ &= \frac{\partial f}{\partial x} \left\{ \frac{\partial f_{i+1}}{\partial x_{k+1}} + \frac{\partial f_{i+1}}{\partial x} \cdot \frac{\partial x}{\partial x_{k+1}} \right\}, \\ &= \frac{\partial f}{\partial x} \left(\frac{\partial f_{i+1}}{\partial x_{k+1}} \right), \end{aligned}$$

if the brackets in the last line be taken to indicate that the f_{i+1} enclosed by them does not involve x but its equivalent in terms of $x_1, x_2, \dots, x_m$. By use of this substitute for $b_k^{(i)}$ there results

$$\frac{\partial f}{\partial x} \cdot \sum \pm \left(\frac{\partial f_1}{\partial x_1} \right) \left(\frac{\partial f_2}{\partial x_2} \right) \dots . \left(\frac{\partial f_{m+1}}{\partial x_{m+1}} \right) = \sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \dots . \frac{\partial f_{m+1}}{\partial x_{m+1}},$$

and if we denote the cofactor of $\frac{\partial f}{\partial x_k}$ in the determinant on the right by A_k we have of course the said determinant

$$= \mathrm{A} \frac{\partial f}{\partial x} + \mathrm{A}_1 \frac{\partial f}{\partial x_1} + \dots . + \mathrm{A}_{m+1} \frac{\partial f}{\partial x_{m+1}};$$

and $\therefore$

$$= \mathrm{A} \frac{\partial f}{\partial x} - \mathrm{A}_1 \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x_1} - \dots . - \mathrm{A}_{m+1} \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x_{m+1}};$$

by reason of the deduction from $f=0$. Division by $\frac{\partial f}{\partial x}$ thus gives us the 'formula memorabilis'

$$\sum\pm\left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right)\cdots\left(\frac{\partial f_{m+1}}{\partial x_{m+1}}\right) = \mathrm{A}-\mathrm{A}_1\frac{\partial x}{\partial x_1}-\mathrm{A}_2\frac{\partial x}{\partial x_2}-\ldots-\mathrm{A}_{m+1}\frac{\partial x}{\partial x_{m+1}},$$

where
$$\mathrm{A} = \sum\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\cdot\frac{\partial f_{m+1}}{\partial x_{m+1}}$$

and A_k is derivable from A by putting differentiation with respect to x in place of differentiation with respect to x_k, and prefixing the proper sign.

Jacobi adds "Formula inter egregia inventa illustrissimi Lagrange censetur," and asserts that for the purposes of proof it is not necessary to put $f=0$.*

This is immediately followed by the theorem—*If* $\mathrm{u}, \mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ *be functions of* $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$, *then*

$$\sum\pm\frac{\partial \mathrm{u}_1\mathrm{u}^{-1}}{\partial \mathrm{x}_1}\cdot\frac{\partial \mathrm{u}_2\mathrm{u}^{-1}}{\partial \mathrm{x}_2}\cdots\cdot\frac{\partial \mathrm{u}_n\mathrm{u}^{-1}}{\partial \mathrm{x}_n} = \frac{1}{\mathrm{u}^{n+1}}\sum\pm\mathrm{u}\cdot\frac{\partial \mathrm{u}_1}{\partial \mathrm{x}_1}\cdot\frac{\partial \mathrm{u}_2}{\partial \mathrm{x}_2}\cdots\cdot\frac{\partial \mathrm{u}_n}{\partial \mathrm{x}_n},$$

the connection being somewhat distant, and probably lying in the fact that in both the b's are of the 2nd order. The mode of proof is curious. In the first place, by putting

$$\frac{\partial f_i}{\partial x_k} = a_k^{(i)}$$

* No indication is given of where Lagrange published the theorem attributed to him. As for the particular way in which x is given as a function of $x_1, x_2, \ldots, x_{m+1}$, whether by the equation $f=0$ or not, it is clear that this cannot affect the truth of the result, because the latter contains no f at all, the requisites for validity being (1) that $f_1, f_2, \ldots, f_{m+1}$ are functions of $x, x_1, x_2, \ldots, x_{m+1}$; (2) that x is a function of $x_1, x_2, \ldots, x_{m+1}$; (3) that on the left the f's have by substitution been freed of x before differentiation; (4) that on the right this has not been done, but they are there differentiated as if x were a constant. The subsidiary result

$$\sum\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_{i+1}}{\partial x_{k+1}} = \frac{\partial f}{\partial x}\left(\frac{\partial f_{i+1}}{\partial x_{k+1}}\right)$$

is certainly true whether $f=0$ or not, all that is requisite (see p. 374 above) being that f_{i+1} on the right shall be what f_{i+1} becomes by substituting for x its value in terms of $f, x_1, x_2, \ldots, x_{m+1}$, and that in the differentiation of it f shall be viewed as constant.

and therefore

$$b_k^{(i)} = aa_{k+1}^{(i+1)} - a^{(i+1)}a_{k+1},$$

there is obtained the identity

$$\Sigma \pm bb_1^{(1)} \ldots . \, b_m^{(m)} = a^m \Sigma \pm aa_1^{(1)} \ldots . \, a_{m+1}^{(m+1)}.$$

Then since the a's here may denote any quantities whatever ("qua in formula cum ipsa $a_k^{(i)}$ quantitates quascunque designare possint") a further substitution * bringing us back to differential-quotients is made, viz.,

$$a^{(i+1)} = u_{i+1}, \quad a_{k+1}^{(i+1)} = \frac{\partial u_{i+1}}{\partial x_{k+1}},$$

where u, u_1, u_2, ..., u_{m+1} are functions of x_1, x_2, ..., x_{m+1}. In this way

$$b_k^{(i)} = u \, . \frac{\partial u_{i+1}}{\partial x_{k+1}} - u_{i+1} \, . \frac{\partial u}{\partial x_{k+1}} = uu \, . \frac{\partial \frac{u_{i+1}}{u}}{\partial x_{k+1}},$$

and the aforesaid identity gives us

$$u^{2m+2} \sum \pm \frac{\partial \frac{u_1}{u}}{\partial x_1} \cdot \frac{\partial \frac{u_2}{u}}{\partial x_2} \cdots \frac{\partial \frac{u_{m+1}}{u}}{\partial x_{m+1}} = u^m \, . \sum \pm u \, . \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdots \frac{\partial u_{m+1}}{\partial x_{m+1}},$$

as was to be proved.

As a corollary to this it follows that if in the determinant

$$\sum \pm u \, . \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdots \frac{\partial u_n}{\partial x_n},$$

tu, tu_1, ..., be put instead of u, u_1, ..., t being any function whatever, the effect is the same as if the determinant were simply multiplied by t^{n+1}. "Quod iam olim alia occasione adnotavi," the reference being to the 5th theorem of his paper of the year 1833, already dealt with.

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* A combination of the successive substitutions is impossible, by reason of the fact that in the second case the equation

$$a_{k+1}^{(i+1)} = \frac{\partial u_{i+1}}{\partial x_{k+1}}$$

is not meant, as in the first case, to include the definition of $a_{(i+1)}$, which has thus to be defined by a supplementary equation.

The result of the first substitution is very noteworthy, in view of previous footnotes.

The next section, the 18th and penultimate, shows how by modifications made upon the functions, the functional determinant may reduce to one term. The first function f being expressed in terms of $x, x_1, x_2, \ldots, x_n$, it follows that theoretically x is expressible in terms of $f, x_1, x_2, \ldots, x_n$, and that therefore by substitution so also are $f_1, \ldots, f_n$. Similarly f_1, being now a function of $f, x_1, x_2, \ldots, x_n$, we conclude that theoretically x_1 is expressible in terms of $f, f_1, x_2, \ldots, x_n$, and that therefore by substitution so also are $f_2, f_3, \ldots, f_n$. Supposing this process to be completed, Jacobi denotes the new forms of $f_1, f_2, \ldots, f_n$ by

$$\begin{gathered} f_1(f, x_1, x_2, \ldots, x_n), \\ f_2(f, f_1, x_2, \ldots, x_n), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_n(f, f_1, \ldots\, f_{n-1}, x_n), \end{gathered}$$

and the difference between any one of the old forms and the corresponding new by F with the appropriate suffix. He thus has $n+1$ equations

$$\begin{aligned} 0 &= \mathrm{F} = f - f\,(x, x_1, x_2, \ldots, x_n), \\ 0 &= \mathrm{F}_1 = f_1 - f_1\,(f, x_1, x_2, \ldots, x_n), \\ 0 &= \mathrm{F}_2 = f_2 - f_2\,(f, f_1, x_2, \ldots, x_n), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ 0 &= \mathrm{F}_n = f_n - f_n(f, f_1, \ldots, f_{n-1}, x_n), \end{aligned}$$

connecting $2n+2$ variables

$$x,\ x_1,\ \ldots,\ x_n,\ f,\ f_1,\ \ldots,\ f_n$$

now viewed as independent. By a previous theorem there is thus obtained

$$\sum \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\cdot\frac{\partial f_n}{\partial x_n} = (-1)^{n+1}\frac{\sum \pm \dfrac{\partial \mathrm{F}}{\partial x}\cdot\dfrac{\partial \mathrm{F}_1}{\partial x_1}\cdots\cdot\dfrac{\partial \mathrm{F}_n}{\partial x_n}}{\sum \pm \dfrac{\partial \mathrm{F}}{\partial f}\cdot\dfrac{\partial \mathrm{F}_1}{\partial f_1}\cdots\cdot\dfrac{\partial \mathrm{F}_n}{\partial f_n}}.$$

Now the numerator here reduces to one term, viz.,

$$\frac{\partial \mathrm{F}}{\partial x}\cdot\frac{\partial \mathrm{F}_1}{\partial x_1}\cdots\cdot\frac{\partial \mathrm{F}_n}{\partial x_n},$$

and the denominator in similar fashion to

$$\frac{\partial F}{\partial f} \cdot \frac{\partial F_1}{\partial f_1} \cdots \frac{\partial F_n}{\partial f_n},$$

as Jacobi might briefly have justified by a reference to Prop. III. of § 5 of his "De formatione et proprietatibus Determinantium"—this proposition being that which concerns a determinant whose elements on one side of the 'diagonal,' as it afterwards came to be called, all vanish. Further, the factors of the reduced numerator are equal to

$$-\left(\frac{\partial f}{\partial x}\right), \quad -\left(\frac{\partial f_1}{\partial x_1}\right), \quad -\left(\frac{\partial f_2}{\partial x_2}\right), \quad \cdots\cdot$$

and those of the reduced denominator to

$$1, \quad 1, \quad 1, \quad \cdots\cdot$$

Our final result thus is

$$\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} = \left(\frac{\partial f}{\partial x}\right) \cdot \left(\frac{\partial f_1}{\partial x_1}\right) \cdots \left(\frac{\partial f_n}{\partial x_n}\right),$$

where the brackets on the right are meant as a reminder that f_i is there expressed in terms of $f, f_1, \ldots, f_{i-1}, x_i, x_{i+1}, \ldots, x_n$.

The last section (§ 19) is occupied with a theorem of the Integral Calculus, already twice enunciated (see pp. 357–358), namely,

$$\int U \partial f . \partial f_1 \ldots . \partial f_n = \int U . \left(\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}\right) . \partial x . \partial x_1 \ldots . \partial x_n,$$

it being noted that the cases where the number of variables to be changed are 2 and 3 had been already dealt with by Euler and Lagrange.* Catalan's memoir of 1839 is not referred to. Then come the final words, "Et haec formula egregie analogiam differentialis et Determinantis functionalis declarat,"—a not inappropriate ending in view of the author's attitude throughout the memoir.

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* The memoirs referred to here and by Catalan seem to be—

Euler, L.—De formulis integralibus duplicatis. . . . *Nov. Comm. Acad. Petrop.* (1769), xiv. i. pp. 72–103.

Lagrange, J. L.—Sur l'attraction des sphéroïdes elliptiques. *Nouv. Mém. Acad. . . . Berlin* (1773), pp. 121–148; or *Œuvres*, iii. pp. 619–658.

CAUCHY (1841).

[Mémoire sur les fonctions différentielles alternées. *Exercices d'analyse et de phys. math.*, ii. pp. 177–187; or *Œuvres complètes*, 2e sér. xii.]

This was evidently suggested by Jacobi's memoir just dealt with, but it belongs to a quite different class, being merely a simply-written exposition containing two of the fundamental properties of the functions and a few illustrations.

In the first part ('Considérations générales') he explains the meaning of

$$S[\pm D_x\mathrm{x} \,.\, D_y\mathrm{y} \,.\, D_z\mathrm{z} \,\ldots.\, D_t\mathrm{t}]$$

on the understanding that the n variables

$$\mathrm{x},\ \mathrm{y},\ \mathrm{z},\ \ldots,\ \mathrm{t}$$

are connected with the n others

$$x,\ y,\ z,\ \ldots,\ t$$

by n equations: and then establishes the theorems

$$S[\pm D_x\mathrm{x} \,.\, D_y\mathrm{y} \,\ldots.\, D_t\mathrm{t}] \,.\, S[\pm D_{\mathrm{x}}x \,.\, D_{\mathrm{y}}y \,\ldots.\, D_{\mathrm{t}}t] = 1, \qquad (\alpha)$$

$$S[\pm D_x\mathrm{x} \,.\, D_y\mathrm{y} \,\ldots.\, D_t\mathrm{t}] = (-1)^n \frac{S[\pm D_x\phi \,.\, D_y\chi \,\ldots.\, D_t\omega]}{S[\pm D_{\mathrm{x}}\phi \,.\, D_{\mathrm{y}}\chi \,\ldots.\, D_{\mathrm{t}}\omega]} \qquad (\beta)$$

making reference in both cases to Jacobi's memoir, and pointing out in connection with the latter theorem that it leads to "la formule donnée par M. Catalan pour la transformation d'une intégrale multiple."

In the second part ('Exemples') he instances first the case where $\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots.\, \mathrm{t}$ are *linear* functions of $x, y, z, \ldots.\, t$, and shows how a result obtained in a previous memoir (see above, p. 285) affords a verification of the theorem (α). The next example is of greater interest, being the case where

$$\begin{aligned}
\mathrm{x} &= A(x-a)(y-a)(z-a) \ldots. (t-a),\\
\mathrm{y} &= B(x-b)(y-b)(z-b) \ldots. (t-b),\\
\mathrm{z} &= C(x-c)(y-c)(z-c) \ldots. (t-c),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\mathrm{t} &= H(x-h)(y-h)(z-h) \ldots. (t-h).
\end{aligned}$$

As is readily seen we have here

$$\begin{aligned}&\mathrm{S}[\pm \mathrm{D}_x\mathrm{x}\,.\,\mathrm{D}_y\mathrm{y}\,.\,\mathrm{D}_z\mathrm{z}\,\ldots .\,\mathrm{D}_t\mathrm{t}]\\ &\qquad = \mathrm{xyz}\ldots\mathrm{t}\,.\,\mathrm{S}\left[\pm\frac{1}{x-a}\cdot\frac{1}{y-b}\cdot\frac{1}{z-c}\cdots\frac{1}{t-h}\right],\end{aligned}$$

which, according to a previously obtained result (see above, p. 345),

$$\begin{aligned}&= (-1)^{\frac{1}{2}n(n-1)}\,.\,\mathrm{xyz}\ldots\mathrm{t}\\ &\quad\cdot\frac{\mathrm{S}[\pm a^0b^1c^2\ldots h^{n-1}]\,.\,\mathrm{S}[\pm x^0y^1z^2\ldots t^{n-1}]}{(x-a)(x-b)\ldots(y-a)(y-b)\ldots(z-a)(z-b)\ldots(t-a)(t-b)\ldots}:\end{aligned}$$

so that if we substitute the given expressions for x, y, z, ... t there is obtained finally

$$\begin{aligned}&\mathrm{S}[\pm \mathrm{D}_x\mathrm{x}\,.\,\mathrm{D}_y\mathrm{y}\,.\,\mathrm{D}_z\mathrm{z}\,\ldots .\,\mathrm{D}_t\mathrm{t}]\\ &\quad = (-1)^{\frac{1}{2}n(n-1)}\mathrm{ABC}\ldots\mathrm{H}\,\mathrm{S}[\pm a^0b^1c^2\ldots h^{n-1}]\,.\,\mathrm{S}[\pm x^0y^1z^2\ldots t^{n-1}].\end{aligned}$$

The third example is equally worth attention, the connecting equations being

$$\begin{aligned}\mathrm{x} &= \mathrm{A}\frac{(x-a)(y-a)(z-a)\ldots .(t-a)}{(x-k)(y-k)(z-k)\ldots .(t-k)},\\ \mathrm{y} &= \mathrm{B}\frac{(x-b)(y-b)(z-b)\ldots .(t-b)}{(x-k)(y-k)(z-k)\ldots .(t-k)},\\ \mathrm{z} &= \mathrm{C}\frac{(x-c)(y-c)(z-c)\ldots .(t-c)}{(x-k)(y-k)(z-k)\ldots .(t-k)},\\ &\ldots\ldots\ldots\ldots\\ \mathrm{t} &= \mathrm{H}\frac{(x-h)(y-h)(z-h)\ldots .(t-h)}{(x-k)(y-k)(z-k)\ldots .(t-k)};\end{aligned}$$

so that at the outset it is found that

$$\begin{aligned}&\mathrm{S}[\pm \mathrm{D}_x\mathrm{x}\,.\,\mathrm{D}_y\mathrm{y}\,.\,\mathrm{D}_z\mathrm{z}\,\ldots .\,\mathrm{D}_t\mathrm{t}]\\ &\quad = \mathrm{xyz}\ldots .\,\mathrm{t}\,.\,\frac{(a-k)\ldots .(h-k)}{(x-k)\ldots .(t-k)}\cdot\mathrm{S}\left[\pm\frac{1}{x-a}\cdot\frac{1}{y-b}\cdot\frac{1}{z-c}\cdots\frac{1}{t-h}\right],\end{aligned}$$

and ultimately after substituting as before

$$\begin{aligned}&\mathrm{S}[\pm \mathrm{D}_x\mathrm{x}\,.\,\mathrm{D}_y\mathrm{y}\,.\,\mathrm{D}_z\mathrm{z}\,\ldots .\,\mathrm{D}_t\mathrm{t}]\\ &\qquad = (-1)^{\frac{1}{2}n(n-1)}\,.\,\mathrm{ABC}\ldots .\,\mathrm{H}\,.\,\frac{(a-k)\ldots(h-k)}{\{(x-k)\ldots(t-k)\}^{n+1}}\\ &\qquad\qquad\cdot\mathrm{S}[\pm a^0b^1c^2\ldots h^{n-1}]\,.\,\mathrm{S}[\pm x^0y^1z^2\ldots t^{n-1}].\end{aligned}$$

CHAPTER XIV.

SKEW DETERMINANTS FROM 1827 TO 1845.

SETS of equations of the form

$$\left.\begin{array}{lllllll} & a_{12}x_2 + & a_{13}x_3 + & a_{14}x_4 + & \ldots & + a_{1n}x_n & = \xi_1 \\ -a_{12}x_1 & & + a_{23}x_3 + & a_{24}x_4 + & \ldots & + a_{2n}x_n & = \xi_2 \\ -a_{13}x_1 - & a_{23}x_2 & & + a_{34}x_4 + & \ldots & + a_{3n}x_n & = \xi_3 \\ -a_{14}x_1 - & a_{24}x_2 - & a_{34}x_3 & & + \ldots & + a_{4n}x_n & = \xi_4 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_{1n}x_1 - & a_{2n}x_2 - & a_{3n}x_3 - & a_{4n}x_4 - & \ldots & & = \xi_n \end{array}\right\},$$

where the coefficient of x_r in the s^{th} equation differs only in sign from the coefficient of x_s in the r^{th} equation, had often made their appearance in analytical investigations before the determinant of such a set came to be considered. An instance is to be found in a memoir of Poisson's, read before the Institute in October 1809, and printed in the *Journal de l'École Polytechnique*, viii. pp. 266–344;* and similar instances of an earlier date in writings of Lagrange and Laplace are therein referred to. A curious example occurs in one of Monge's papers already dealt with (see above, pp. 67, 68), there being additional interest attaching to it by reason of the fact that in it the a's and x's are themselves determinants. It is to be found in an earlier portion (pp. 107–109) of the same volume as Poisson's. Denoting by

$$\alpha, \beta, \gamma, \delta; \quad M, N, P, Q, R, S$$

the six-termed expressions which at a later date would have been written

$$|b_1c_2d_3|, \ |a_1c_2d_3|, \ |a_1b_2d_3|, \ |a_1b_2c_3|;$$
$$|a_1b_2e_3|, \ |b_1c_2e_3|, \ |c_1d_2e_3|, \ -|a_1d_2e_3|, \ |a_1c_2e_3|, \ |b_1d_2e_3|,$$

* See especially p. 288.

Monge established the set of equations

$$\left.\begin{array}{llllll} Q\alpha + & S\beta - & P\gamma & & = 0 \\ R\alpha - & N\beta & & + P\delta & = 0 \\ M\alpha & & - N\gamma & + S\delta & = 0 \\ & - M\beta + & R\gamma & + Q\delta & = 0 \end{array}\right\}$$

from which he eliminated α, β, γ, δ, and obtained

$$RS + QN - PM = 0.$$

On altering the signs of the last two equations the result of the elimination would a generation afterwards have been put at once in the form

$$\begin{vmatrix} Q & S & -P & . \\ R & -N & . & P \\ -M & . & N & -S \\ . & M & -R & -Q \end{vmatrix} = 0,$$

and the left-hand side would have been recognised as a 'skew' determinant and altered into

$$(RS+QN-PM)^2 = 0.$$

No prophetic glimpse of this, however, occurred to Monge. The mathematician who first referred definitely to the determinant appears to have been Jacobi.

PFAFF (1815).

[Methodus generalis, aequationes differentiarum partialium, nec non aequationes differentiales vulgares, utrasque primi ordinis, inter quotcunque variabiles, completi integrandi. *Abhandl. .... Akad. der Wiss. (math. Klasse)*, Berlin, 1814–1815, pp. 76–136; or Kowalewski's German Translation, 84 pp., Leipzig, 1902.]

After seven pages of historical introduction and preliminary explanation Pfaff arranges the subject of his memoir in the form of a series of fourteen problems with their solutions. Problem i. is to integrate completely a partial differential equation in three variables; problem ii. is to transform any differential equation of the first order in four variables into an equation in three variables, and to integrate the latter by means of a system of two equations; and problem iii. is to integrate an ordinary differential equation of the first order in five variables by means of a system of three equations. Problems iv., v., vi. correspond respectively to i., ii., iii., the number of variables being one more in any problem of the second triad than in the corresponding problem of the first triad. A similar step onward is taken in problems vii., viii., ix. which form a third triad, and again in problems x., xi., which are the first two members of a fourth triad. At this stage the 'methodus generalis' is supposed to be sufficiently foreshadowed, and in the remaining three problems the restriction to a definite number of variables is withdrawn.

Of these three it is the thirteenth (xiii.) which concerns us here, namely, the reduction of an ordinary equation of the first order between $2m$ variables to a similar equation between $2m-1$ variables, and the performance of the integration by means of a system of m equations. It (xiii.) is the generalization of problems ii., v., viii., xi., these being the cases of it where $m=2, 3, 4, 5$.

The solution consists in expressing $2m-1$ of the given variables as functions of the $2m^{\text{th}}$ and $2m-1$ new quantities, and introducing the latter in place of the former. By considering the new quantities as constants $2m-1$ auxiliary differential-equations arise, the integration of which supplies the desired functions; and for the formation of the auxiliary equations $2m-1$ quantities are needed whose values are determinable from the same number of conditional equations. It is in the solution of this set of conditional equations that our interest centres. As Cramer and Bezout had dealt with a more general set, Pfaff naturally made trial of their methods; but they were found, he says, of little service. He therefore sought for and discovered two laws of formation which sufficed for his wants. His words are (p. 119)—

"Haec determinatio, si consueta eliminandi methodo tractetur, calculos nimium complicatos et operosos postulat: ipsaque precepta generalia, quae Bezout et Cramer de eliminatione tradiderunt, in casu substrato parum commodi afferre videntur. Accuratius vero considerando praedictas aequationes conditionales et formulas ex earum solutione actu evolutas, ad duas leges satis simplices easque generales perveni, quas hic breviter exponere sufficiat."

In exposition of the first law he begins by repeating the results for the cases $m=2, 3, 4$, using for brevity's sake the symbol *

$$(\mathrm{ACE})$$

to stand for

$$\frac{\mathrm{A}d\mathrm{C}-\mathrm{C}d\mathrm{A}}{de}+\frac{\mathrm{C}d\mathrm{E}-\mathrm{E}d\mathrm{C}}{da}+\frac{\mathrm{E}d\mathrm{A}-\mathrm{A}d\mathrm{E}}{dc}.$$

That is to say, he recalls (1) that when the given equation is

$$\mathrm{A}da + \mathrm{B}db + \mathrm{C}dc + \mathrm{E}de = 0$$

the auxiliary equations are

$$0 = \frac{da}{(\mathrm{BCE})} + \frac{db}{(\mathrm{ACE})} = \ldots\ldots;$$

(2) that when the given equation is

$$\mathrm{A}da + \mathrm{B}db + \mathrm{C}dc + \mathrm{E}de + \mathrm{F}df + \mathrm{G}dg = 0,$$

the auxiliary equations are

$$\begin{aligned} 0 &= \frac{da}{(\mathrm{CBE})(\mathrm{CFG}) - (\mathrm{CBF})(\mathrm{CEG}) + (\mathrm{CBG})(\mathrm{CEF})} \\ &\quad + \frac{db}{(\mathrm{CAE})(\mathrm{CFG}) - (\mathrm{CAF})(\mathrm{CEG}) + (\mathrm{CAG})(\mathrm{CEF})}, \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}$$

and (3) that when the given equation is

$$\mathrm{A}da + \mathrm{B}db + \mathrm{C}dc + \mathrm{E}de + \mathrm{F}df + \mathrm{G}dg + \mathrm{H}dh + \mathrm{I}di = 0,$$

the auxiliary equations are

$$0 = \frac{da}{\mathfrak{A}} + \frac{dc}{\mathfrak{C}} = \ldots\ldots,$$

* This may be nothing more than a coincidence; but if so, it is a curious one, the expression replaced by (ACE) being the determinant

$$\begin{vmatrix} \mathrm{A} & d\mathrm{A} & \frac{1}{da} \\ \mathrm{C} & d\mathrm{C} & \frac{1}{dc} \\ \mathrm{E} & d\mathrm{E} & \frac{1}{de} \end{vmatrix},$$

and Pfaff even drawing attention to the fact that

$$(\mathrm{AEC}) = -(\mathrm{ACE}).$$

where

$$\begin{aligned}\mathfrak{A} = \;& (BCE)(BFG)(BHI) - (BCE)(BFH)(BGI) + (BCE)(BFI)(BGH)\\ & - (BCF)(BEG)(BHI) + (BCF)(BEH)(BGI) - (BCF)(BEI)(BGH)\\ & + (BCG)(BEF)(BHI) - (BCG)(BEH)(BFI) + (BCG)(BEI)(BFH)\\ & - (BCH)(BEF)(BGI) + (BCH)(BEG)(BFI) - (BCH)(BEI)(BFG)\\ & + (BCI)(BEF)(BGH) - (BCI)(BEG)(BFH) + (BCI)(BEH)(BFG).\end{aligned}$$

Now the denominators here are functions of the kind afterwards known as Pfaffians, and such as would now be written

$$(BCE), \qquad \left| \begin{matrix} (CBE) & (CBF) & (CBG) \\ & (CEF) & (CEG) \\ & & (CFG) \end{matrix} \right|,$$

$$\left| \begin{matrix} (BCE) & (BCF) & (BCG) & (BCH) & (BCI) \\ & (BEF) & (BEG) & (BEH) & (BEI) \\ & & (BFG) & (BFH) & (BFI) \\ & & & (BGH) & (BGI) \\ & & & & (BHI) \end{matrix} \right|.$$

Their law of formation as given by Pfaff is therefore of the greatest interest. His words as regards $\mathfrak{A}$ are (p. 124):

"Separando litteram B, termini hujus expressionis complectuntur permutationes litterarum reliquarum C, E, F, G, H, I (exclusa prima A), quae sub hac restrictione fieri possunt, ut litterae in quavis complexione (ex. gr. C, G, E, H, F, I in termino octavo τοῦ $\mathfrak{A}$) prima, tertia, quinta (ex. gr. C, E, F) in genere imparem locum obtinentes inter se rite sint ordinatae, et litterarum quaevis pari loco constituta (G, H, I) sit ordine alphabetico posterior littera in loco impari proxime praecedente (C, E, F). His formis rite inter se ordinatis i.e. secundum ordinem lexicographicum (e.g. C, G, E, H, F, I ante C, G, E, I, F, H) terminorum signa alternant. Haec lex restrictiva permutationum etiam sic enuntiari potest, ut singulas complexiones dispertiendo in dyades, sive classes binorum elementorum, ipsae dyades tam quoad sua elementa, quam inter se invicem rite debeant esse ordinatae."

This practically means that the terms of $\mathfrak{A}$ are got (1) by taking every permutation of C, E, F, G, H, I which is such that each odd-placed letter in it and the letter immediately following are not an inverted-pair, and that the full group of odd-placed letters is also free of inversions; (2) by placing a B before each odd-placed letter and marking off with brackets the triads so formed; and (3) by making the signs alternately + and − when the terms have been arranged in dictionary order.

Not content, however, with this rule Pfaff immediately gives another of equal interest, the passage being (p. 125):

"Processus autem combinatorius, quo permutationes praedictae exhibentur, satis commodus hic est: Sint litterae, quarum permutationes sub restrictione supra commemorata quaeruntur a, b, c, e, ..., k, l, m, n: supponamus inventas esse permutationes litterarum c, e, ..., m, n, exclusas duabus a, b: tum 1) singulis his permutationibus vel complexionibus praeponatur binio ab; 2) ex hac prima serie complexiones totidem aliae formentur, permutando b et c; 3) ex his porro aliae, permutando c et d, sicque progrediendo ex quavis serie complexionum nova formetur, litteram aliquam cum proxime sequente permutando, donec postremo m et n invicem permutentur. Qua ratione obtinentur omnes permutationes litterarum a, b, c, ..., m, n quas restrictio praedicta admittit."

Here the case for $2m$ letters is made dependent on the case for $2m-2$ letters, so that if six letters a, b, c, d, e, f were given, we should begin by forming the permutation for the last two letters, namely

$$ef;$$

to this we should prefix cd, and by performing the interchanges $d \rightleftharpoons e$, $e \rightleftharpoons f$ obtain the permutations for the four letters c, d, e, f, namely

$$cd\,.\,ef - ce\,.\,df + cf\,.\,de;$$

and lastly we should prefix ab to each of these and perform the interchanges $b \rightleftharpoons c$, $c \rightleftharpoons d$, $d \rightleftharpoons e$, $e \rightleftharpoons f$, the result reached being

$$\begin{aligned} &ab\,.\,cd\,.\,ef - ac\,.\,bd\,.\,ef + ad\,.\,bc\,.\,ef - ae\,.\,bc\,.\,df + af\,.\,bc\,.\,de \\ -\,&ab\,.\,ce\,.\,df + ac\,.\,be\,.\,df - ad\,.\,be\,.\,cf + ae\,.\,bd\,.\,cf - af\,.\,bd\,.\,ce \\ +\,&ab\,.\,cf\,.\,de - ac\,.\,bf\,.\,de + ad\,.\,bf\,.\,ce - ae\,.\,bf\,.\,cd + af\,.\,be\,.\,cd. \end{aligned}$$

As a manifest deduction from this rule* Pfaff states that for $2m$ letters the number of such restricted permutations is

$$1\,.\,3\,.\,5\,\ldots\,(2m-1).$$

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* The other rule, however, would have been equally useful towards this end. For if we remove the restriction each one of the N restricted permutations would give rise to $2^m\,.\,(1\,.\,2\,.\,3\ldots m)$ unrestricted permutations; so that we should have

$$2^m\,.\,(1\,.\,2\,.\,3\ldots m)\,.\,N = 1\,.\,2\,.\,3\ldots(2m);$$

and therefore

$$N = \frac{1\,.\,2\,.\,3\ldots(2m)}{2^m\,.\,(1\,.\,2\,.\,3\ldots m)} = 1\,.\,3\,.\,5\ldots(2m-1).$$

Later he points out that since each factor of each of the fifteen terms of $\mathfrak{A}$ is itself six-termed, the total number of terms in the final expansion of $\mathfrak{A}$ ought to be

$$6^3 . 15, \quad i.e. \quad 3240;$$

but that 2400 of them cancel each other. To obtain the 840 really needed is the reason for his propounding a second law, which he does in § 18 (pp. 126–129).

JACOBI (1827).

[Ueber die Pfaffsche Methode, eine gewöhnliche lineare Differential-gleichung zwischen $2n$ Variabeln durch ein System von n Gleichungen zu integriren. *Crelle's Journal*, ii. pp. 347–357; or *Werke*, iv. pp. 17–29.]

An essential part of Pfaff's method is the solution of a set of equations which Jacobi writes in the form

$$\left.\begin{array}{l} \mathrm{NX}\partial x = \quad * \quad + (0,1)\partial x_1 + (0,2)\partial x_2 + \ldots\ldots + (0,p)\partial x_p \\ \mathrm{NX}_1\partial x = (1,0)\partial x + \quad * \quad + (1,2)\partial x_2 + \ldots\ldots + (1,p)\partial x_p \\ \mathrm{NX}_2\partial x = (2,0)\partial x + (2,1)\partial x_1 + \quad * \quad + \ldots\ldots + (2,p)\partial x_p \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{NX}_p\partial x = (p,0)\partial x + (p,1)\partial x_1 + (p,2)\partial x_2 + \ldots\ldots + \quad * \end{array}\right\},$$

where $(0,0) = -(1,0)$ and generally $(\alpha,\beta)+(\beta,\alpha)=0$. This form of his own he frankly characterises as "elegant and completely symmetrical"; but the same description would apply equally appropriately to the solution which he gives. Unfortunately, the method by which the latter was obtained is not indicated, and we can only hazard a guess in regard to it. The balance of probability would seem to be in favour of the method of devising a set of multipliers which, when applied to the given equations, would after the performance of addition bring about the elimination of all the unknowns except one. In the case of four equations this would not be at all difficult. For example, if we wish to eliminate x_2, x_3, x_4 from the equations

$$\left.\begin{array}{rrrrl} \cdot & ax_2 & + bx_3 & + cx_4 & = \xi_1 \\ -ax_1 & \cdot & + dx_3 & + ex_4 & = \xi_2 \\ -bx_1 & - dx_2 & \cdot & + fx_4 & = \xi_3 \\ -cx_1 & - ex_2 & - fx_3 & \cdot & = \xi_4 \end{array}\right\},$$

the multipliers are readily seen to be

$$0, \quad f, \quad -e, \quad d,$$

so that after multiplication and addition there results

$$(-af+be-cd)x_1 = \quad f\xi_2-e\xi_3+d\xi_4.$$

Similarly by using the multipliers $-f$, 0, c, $-b$, we find

$$(-af+be-cd)x_2 = -f\xi_1+c\xi_3-b\xi_4;$$

and the other two are

$$(-af+be-cd)x_3 = \quad e\xi_1-c\xi_2+a\xi_4,$$
$$(-af+be-cd)x_4 = -d\xi_1+b\xi_2-a\xi_3.$$

Jacobi's corresponding result is to the effect that the numerators of the values of the four unknowns are

$$\begin{array}{l} \mathrm{N}\partial x\Big\{ \quad * \quad + (2,3)\mathrm{X}_1 + (3,1)\mathrm{X}_2 + (1,2)\mathrm{X}_3\Big\}, \\ \mathrm{N}\partial x\Big\{(3,2)\mathrm{X} + \quad * \quad + (0,3)\mathrm{X}_2 + (2,0)\mathrm{X}_3\Big\}, \\ \mathrm{N}\partial x\Big\{(1,3)\mathrm{X} + (3,0)\mathrm{X}_1 + \quad * \quad + (0,1)\mathrm{X}_3\Big\}, \\ \mathrm{N}\partial x\Big\{(2,1)\mathrm{X} + (0,2)\mathrm{X}_1 + (1,0)\mathrm{X}_2 + \quad * \quad \Big\}, \end{array}$$

and the common denominator

$$(0,1)(3,2) + (0,3)(2,1) + (0,2)(1,3),$$

or, as he thereafter writes it

$$(0,1,3,2).$$

When the similar set of six equations had to be dealt with, the devising of the multipliers requisite for elimination would necessarily be harder; but keeping in view the analogous mode of arriving at the solution of

$$\left.\begin{aligned} a_1x_1 + a_2x_2 &= \xi_1 \\ b_1x_1 + b_2x_2 &= \xi_2 \end{aligned}\right\}$$

and then proceeding to the solution of

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= \xi_1 \\ b_1x_1 + b_2x_2 + b_3x_3 &= \xi_2 \\ c_1x_1 + c_2x_2 + c_3x_3 &= \xi_3 \end{aligned}\right\},$$

where, it will be remembered, the multipliers requisite for elimination are of the same form as the common denominator of the values of the unknowns in the preceding case, Jacobi would have little real difficulty in finding that corresponding to the four multipliers requisite for eliminating ∂x_1, ∂x_2, ∂x_3 in his first case, viz.,—

$$0, \quad (2,3), \quad (3,1), \quad (1,2)$$

he would now require to have the six multipliers

$$0, \quad (2345), \quad (3451), \quad (4512), \quad (5123), \quad (1234).$$

As a matter of fact, he gives for the numerator of the first unknown

$$\mathrm{N}\partial x\Big\{\ {*}\ + (2345)\mathrm{X}_1 + (3451)\mathrm{X}_2 + (4512)\mathrm{X}_3 + (5123)\mathrm{X}_4 + (1234)\mathrm{X}_5\Big\},$$

the others being

$$\mathrm{N}\partial x\Big\{(3245)\mathrm{X} + \ {*}\ + (4350)\mathrm{X}_2 + (5402)\mathrm{X}_3 + (0523)\mathrm{X}_4 + (2034)\mathrm{X}_5\Big\}$$

. . . .

The common denominator is not mentioned; we should have expected him to say that it was

$$(10)(2345) + (20)(3451) + (30)(4512) + (40)(5123) + (50)(1234)$$

or $$-(012345).$$

It is then pointed out that when the first coefficient has been got in one of the numerators, the others are arrived at by circular permutation, the elements permuted being 12345 in the case of the first numerator, 02345 in the case of the second, 01345 in the case of the third, and so on; also that the first coefficient in one line is got from the last in the preceding line by changing 012345 into 123450 and then transposing the first two elements; and that these laws hold generally.

A general mode of finding the ordinary expression for the new functions here introduced and symbolized by

$$(1234), (123456), \ldots .$$

is next explained. It is first stated that the number of terms represented by

$$(2, 3, 4, \ldots, p)$$

where p is necessarily an odd integer is

$$1.3.5. \ldots . (p-2),$$

and that one of them is

$$(23).(45).(67) \ldots . (p-1, p).$$

We are then told to permute cyclically the last $p-2$ elements $3, 4, 5 \ldots, p$, and we shall obtain from this $p-2$ terms in all; thereafter to permute cyclically the last $p-4$ elements $5, 6, 7, \ldots p$ in each of the $p-2$ terms just obtained, and so on. For example,

$$\begin{aligned}(234567) = \quad & (23)(45)(67) + (23)(46)(75) + (23)(47)(56) \\ & + (24)(56)(73) + (24)(57)(36) + (24)(53)(67) \\ & + (25)(67)(34) + (25)(63)(47) + (25)(64)(73) \\ & + (26)(73)(45) + (26)(74)(53) + (26)(75)(34) \\ & + (27)(34)(56) + (27)(35)(64) + (27)(36)(45).\end{aligned}$$

It is important to note in conclusion, that the case of an *odd* number of equations is not neglected by Jacobi, a proof being given by him that in that case the determinant of the system vanishes. In his own words—which are interesting in view of what has been said elsewhere regarding the evidence which the paper affords of the progress made by him in the study of determinants—

"Nun bleibt nach dem bekannten Algorithmus, nach welchem die Determinante gebildet wird, diese unverändert, wenn man die Horizontalreihen und Verticalreihen der Coefficienten mit einander vertauscht. Für unsern besondern Fall nun wird, wenn wir die Determinante mit Δ bezeichnen, hieraus folgen: $\Delta=(-1)^{p+1}\Delta$, da jedes Glied der Determinante ein Product aus $p+1$ Coefficienten ist, von denen jeder durch Vertauschung der Horizontal- und Verticalreihen sich in sein Negatives verwandelt. Diese Gleichung $\Delta=(-1)^{p+1}\Delta$ aber kann nur bestehen, wenn $p+1$ eine gerade Zahl ist, wofern nicht $\Delta=0$ sein soll."

Thus, though only Pfaff's expositor as regards the functions which came long afterwards to be known and are still known as 'Pfaffians,' Jacobi was the first to discover and prove the now familiar-worded theorem "*A zero-axial skew determinant of odd order vanishes.*"

JACOBI (1845).

[Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi. *Crelle's Journal*, xxvii. pp. 199–268, xxix. pp. 213–279, 333–376; or *Math. Werke* (1846), i. pp. 47–226; or *Gesammelte Werke*,* iv. pp. 317–509.]

As is well known, this long and exhaustive memoir of Jacobi's is broken up into three chapters,—the first giving the definition and various properties of the new muitiplier, the second explaining the application of it to the integration of differential equations, and the third illustrating this application by means of particular differential equations of historical interest. One of the latter is the equation associated then, and still more since, with the name of Pfaff, the discussion of it occupying §§ 20, 21 on pp. 236–253 of vol. xxix. We are thus prepared to find the function, defined by Jacobi eighteen years before, again referred to.

The old definition of the function, which he here denotes by R, is practically repeated, the initial and originating term being

* In all preceding references of this kind *Werke* has been used for *Gesammelte Werke*: here the longer but more correct name is necessary for distinction's sake.

now of the form $a_{12}a_{34}\ldots a_{2m-1,2m}$: and then he makes the pregnant general remark that the properties of R are analogous to those of determinants. Prominence is given to the theorem regarding the effect of interchanging two indices. This is followed by the twin pair of identities

$$\mathrm{R} = a_{1,s}\frac{\partial \mathrm{R}}{\partial a_{1,s}} + a_{2,s}\frac{\partial \mathrm{R}}{\partial a_{2,s}} + \ \ldots\ldots\ + a_{2m,s}\frac{\partial \mathrm{R}}{\partial a_{2m,s}},$$

$$0 = a_{1,s}\frac{\partial \mathrm{R}}{\partial a_{1,r}} + a_{2,s}\frac{\partial \mathrm{R}}{\partial a_{2,r}} + \ \ldots\ldots\ + a_{2m,s}\frac{\partial \mathrm{R}}{\partial a_{2m,r}},$$

in the latter of which s differs from r, and the term $a_{r,s}\dfrac{\partial \mathrm{R}}{\partial a_{r,r}}$ is awanting; and finally, it is pointed out that the differential-quotients of R with respect to one or more elements are functions of the same kind as the original, and, probably as a consequence, that certain second differential-quotients are identical. No proofs are given; indeed, the statements themselves are in the most concise form possible, the whole passage being as follows:—

"Designantibus i, i', i'', etc., indices inter se diversos, si sumuntur differentialia partialia

$$\frac{\partial \mathrm{R}}{\partial a_{i,i'}},\quad \frac{\partial^2 \mathrm{R}}{\partial a_{i,i'}\,\partial a_{i'',i'''}},\quad \ldots\ldots$$

ea erunt aggregata ad instar aggregati R formata, respective reiectis Coëfficientium binis, quatuor, . . . seriebus cum horizontalibus tum verticalibus, eritque

$$\frac{\partial^2 \mathrm{R}}{\partial a_{ii'}\,\partial a_{i''i'''}} = \frac{\partial^2 \mathrm{R}}{\partial a_{i,i''}\,\partial a_{i''',i'}} = \frac{\partial^2 \mathrm{R}}{\partial a_{i,i'''}\,\partial a_{i',i''}}."$$

It should be carefully noted that both in this paper and in the preceding, Jacobi views the new functions as separate from and independent of determinants, and not at all in the light in which, at a later time, they came to be looked upon—viz., as a subsidiary function arising out of the study of a particular kind of determinant with which it had a definite quantitative relation.

CHAPTER XV.

ORTHOGONANTS FROM THE YEAR 1827 TO 1841.

THE special form of determinant to which we have now come is connected with a problem in coordinate geometry—the problem of transformation from one set of rectangular axes to another set having the same origin. The actual appearance of determinants in any of the attempts to solve the geometrical problem did not take place until comparatively late in its history, probably because the connection between the two subjects was less patent than in other cases, the problem when transformed into algebraical language being not a mere matter of elimination of unknowns from a set of linear equations. The earlier portion of the history of orthogonal substitution, although of considerable interest, is thus not sufficiently germane to our subject to warrant detailed treatment of it. For those interested in this earlier portion it will suffice to give the following chronologically arranged list of papers from 1770 to 1840:—

1748. EULER. Introductio in Analysin Infinitorum. Tomi duo. Lausannae et Genevae (v. ii. Appendix de Superficiebus *).

1770. EULER. Problema algebraicum ob affectiones prorsus singulares memorabile. *Novi Commentarii Acad. Petrop.*, xv. pp. 75–106; or *Commentationes Arith. Collectae*, i. pp. 427–443.

1772. LAPLACE. Recherches sur le calcul intégral et sur le système du monde. *Hist. de l'acad. roy. des sciences* (Paris), 2^{e} partie, pp. 267–376.

* Or in Labey's French Translation, ii. pp. 370–378.

1773. LAGRANGE. Nouvelle solution du problème du mouvement de rotation d'un corps de figure quelconque qui n'est animé par aucune force accélératrice. *Nouv. mém. de l'acad. roy.* . . . (Berlin), pp. 85–120.

1775. EULER. Formulae generales pro translatione quacunque corporum rigidorum. *Novi Commentarii Acad. Petrop.*, xx. pp. 189–207.

1776. EULER. Nova methodus motum corporum rigidorum determinandi. *Novi Commentarii Acad. Petrop.*, xx. pp. 208–238.

1776. LEXELL. Theoremata nonnulla generalia de translatione corporum rigidorum. *Novi Commentarii Acad. Petrop.*, xx. pp. 239–270.

1784. MONGE. Sur l'expression analytique de la génération des surfaces courbes. *Mém. de l'acad. roy. des sciences* (Paris) [pp. 85–117], p. 114.

1802. HACHETTE et POISSON. Addition au mémoire précédent. *Journ. de l'éc. polyt.*, cahier xi. pp. 170–172.

1806. CARNOT, L. N. M. Sur la relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace, suivi d'un . . . Paris, 1806.

1810. LACROIX, S. F. Traité du calcul différentiel et du calcul intégral. 2ᵉ édition, i. p. 533 . . . .

1811. LAGRANGE. Mécanique analytique. 2ᵉ édit., i. p. 267.

1818. GAUSS. Determinatio attractionis. . . . *Commentationes Soc.* . . . *Gottingensis*, (*Classis math.*) iv. pp. 21–48; or *Werke*, iii. pp. 331–355.

1827. JACOBI. Euleri formulae de transformatione coordinatarum. *Crelle's Journal*, ii. pp. 188–189; or *Gesammelte Werke*, vii. pp. 3–5.

1827. JACOBI. Ueber die Hauptaxen der Flächen der zweiten Ordnung. *Crelle's Journal*, ii. pp. 227–233; or *Gesammelte Werke*, iii. pp. 45–53.

1827. JACOBI. De singulari quadam duplicis integralis transformatione. *Crelle's Journal*, ii. pp. 234–242; or *Gesammelte Werke*, iii. pp. 55–66.

1828. CAUCHY. Sur les centres, les plans principaux et les axes principaux des surfaces du second degré. *Exercices de Math.*, iii. pp. 1–22; or *Œuvres complètes*, 2ᵉ sér. viii. pp. 8–35.

1828. CAUCHY. Discussion des lignes et des surfaces du second degré. *Exercices de Math.*, iii. pp. 65–120; or *Œuvres complètes*, 2e sér. viii. pp. 83–149.

1829. CHASLES. Sur les propriétés des diamètres conjugués des hyperboloïdes. *Corresp. math. et phys.*, v. pp. [137-157] 139–141.

1829. CLAUSEN. Ueber die Bestimmung der Lage des Haupt-Umdrehungs-Axen eines Körpers *Crelle's Journal*, v. pp. 383-385; or *Nouv. Annales de Math.*, v. pp. 81–83.

1829. CAUCHY. Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes. *Exercices de Math.*, iv. pp. 140-160; or *Œuvres complètes*, 2e sér. ix. pp. 172–195.

1831. JACOBI. De transformatione integralis duplicis indefiniti .... in formam simpliciorem ...... *Crelle's Journal*, viii. pp. 253–279, 321–357; or *Gesammelte Werke*, iii. pp. 91–158.

1832. GRUNERT. Ueber die Verwandlung der Coordinaten im Raume. *Crelle's Journal*, viii. pp. 153–159; or *Nouv. Annales de Math.*, v. pp. 414-419.

1832. ENCKE. Ableitung der Formeln von Monge für die Transformation der Coordinaten in Raume. *Berliner Astron. Jahrbuch* (1832), pp. 305-310; or *Corresp. math. et phys.*, vii. pp. 273-277.

1832. JACOBI. De transformatione et determinatione integralium duplicium commentatio tertia. *Crelle's Journal*, x. pp. 101–128; or *Gesammelte Werke*, iii. pp. 159–189.

1833. JACOBI. De finis quibuslibet functionibus homogeneis secundi ordinis. . . . *Crelle's Journal*, xii. pp. 1–69; or *Gesammelte Werke*, iii. pp. 191–268.

1833. GRUNERT. Supplemente zu Klügel's Wörterbuch: Art. "Coordinaten."

1835. JACOBI. Observationes geometricae. *Crelle's Journal*, xv. pp. 309–312; or *Gesammelte Werke*, vii. pp. 20–23.

1839. CATALAN. Sur la transformation des variables dans les intégrales multiples. *Mém. couronnés par l'Acad. de Bruxelles*, xiv. ii. pp. 1–47.

1839. REISS. Sur les neuf angles que forment réciproquement deux systèmes d'axes rectangulaires. *Correspond. math. et phys.*, xi. pp. 119–173.

1840. RODRIGUES. Des lois géometriques qui régissent les déplacements d'un système solide dans l'espace, . . . *Journ. (de Liouville) de Math.*, v. pp. [380–440] 404–405.

Of these only seven need be taken account of because of their connection with determinants, viz., two by Jacobi in 1827, one by Cauchy in 1829, three by Jacobi in 1831–3, and one by Catalan in 1839.

JACOBI (1827).

[Ueber die Hauptaxen der Flächen der zweiten Ordnung. *Crelle's Journal*, ii. pp. 227–233; or *Gesammelte Werke*, iii. pp. 45–53.]

Without unnecessary preliminaries Jacobi enunciates the problem which he wishes to solve, viz., the transformation of an expression of the form

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2ayz + 2bzx + 2cxy,$$

where x, y, z are the coordinates of a point referred to an oblique coordinate-system, into an expression of the form

$$\mathrm{L}\xi^2 + \mathrm{M}\eta^2 + \mathrm{N}\zeta^2,$$

where ξ, η, ζ are the coordinates of the same point referred to a rectangular system having the same origin. This implies that the things directly sought are the nine coefficients which give each of the original coordinates in terms of the new.

Jacobi, however, prefers to begin with a related set of unknowns, taking the equations which give the new coordinates in terms of the old. These being assumed to be

$$\left.\begin{aligned} \xi &= \alpha x + \beta y + \gamma z \\ \eta &= \alpha' x + \beta' y + \gamma' z \\ \zeta &= \alpha'' x + \beta'' y + \gamma'' z \end{aligned}\right\},$$

the equivalent set giving the old in terms of the new is of course

$$\left.\begin{aligned} \Delta.x &= (\beta'\gamma'' - \beta''\gamma')\xi + (\beta''\gamma - \beta\gamma'')\eta + (\beta\gamma' - \beta'\gamma)\zeta \\ \Delta.y &= (\gamma'\alpha'' - \gamma''\alpha')\xi + (\gamma''\alpha - \gamma\alpha'')\eta + (\gamma\alpha' - \gamma'\alpha)\zeta \\ \Delta.z &= (\alpha'\beta'' - \alpha''\beta')\xi + (\alpha''\beta - \alpha\beta'')\eta + (\alpha\beta' - \alpha'\beta)\zeta \end{aligned}\right\},$$

where

$$\Delta = \alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \alpha\beta''\gamma' - \beta\gamma''\alpha' - \gamma\alpha''\beta'.$$

Denoting the known angles between the original axes by λ, μ, ν, there is obtained at once the set of six equations

$$\begin{aligned}
\alpha^2 + \alpha'^2 + \alpha''^2 &= 1, \\
\beta^2 + \beta'^2 + \beta''^2 &= 1, \\
\gamma^2 + \gamma'^2 + \gamma''^2 &= 1, \\
\beta\gamma + \beta'\gamma' + \beta''\gamma'' &= \cos\lambda, \\
\gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' &= \cos\mu, \\
\alpha\beta + \alpha'\beta' + \alpha''\beta'' &= \cos\nu;
\end{aligned}$$

and, since the expression

$$\mathrm{L}(\alpha x+\beta y+\gamma z)^2 + \mathrm{M}(\alpha' x+\beta' y+\gamma' z)^2 + \mathrm{N}(\alpha'' x+\beta'' y+\gamma'' z)^2$$

has to be identical with

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2ayz + 2bzx + 2cxy,$$

we have thus by implication another set of six equations, viz.:

$$\begin{aligned}
\mathrm{L}\alpha^2 + \mathrm{M}\alpha'^2 + \mathrm{N}\alpha''^2 &= \mathrm{A}, \\
\mathrm{L}\beta^2 + \mathrm{M}\beta'^2 + \mathrm{N}\beta''^2 &= \mathrm{B}, \\
\mathrm{L}\gamma^2 + \mathrm{M}\gamma'^2 + \mathrm{N}\gamma''^2 &= \mathrm{C}, \\
\mathrm{L}\beta\gamma + \mathrm{M}\beta'\gamma' + \mathrm{N}\beta''\gamma'' &= a, \\
\mathrm{L}\gamma\alpha + \mathrm{M}\gamma'\alpha' + \mathrm{N}\gamma''\alpha'' &= b, \\
\mathrm{L}\alpha\beta + \mathrm{M}\alpha'\beta' + \mathrm{N}\alpha''\beta'' &= c.
\end{aligned}$$

What, therefore, remains to be done is the solution of these twelve equations in the twelve unknowns

$$\alpha, \beta, \gamma : \alpha', \beta', \gamma' : \alpha'', \beta'', \gamma'' : \mathrm{L}, \mathrm{M}, \mathrm{N}.$$

Jacobi's mode of accomplishing this is very interesting. He notes first that Δ may be looked upon as known, by reason of the fact that it is expressible in terms of λ, μ, ν, the connection in modern notation being

$$\begin{aligned}
\Delta^2 &= \begin{vmatrix} \alpha^2+\alpha'^2+\alpha''^2 & \alpha\beta+\alpha'\beta'+\alpha''\beta'' & \alpha\gamma+\alpha'\gamma'+\alpha''\gamma'' \\ \alpha\beta+\alpha'\beta'+\alpha''\beta'' & \beta^2+\beta'^2+\beta''^2 & \gamma\beta+\gamma'\beta'+\gamma''\beta'' \\ \alpha\gamma+\alpha'\gamma'+\alpha''\gamma'' & \beta\gamma+\beta'\gamma'+\beta''\gamma'' & \gamma^2+\gamma'^2+\gamma''^2 \end{vmatrix}, \\
&= \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}.
\end{aligned}$$

In the next place he draws attention to the resemblance between the two sets of six equations, and points out that as a consequence any equation legitimately obtainable from the second set is matched by an equation which might in like manner be obtained from the first set, but which is much more readily got by using the substitution

$$\left.\begin{array}{l} \mathrm{L} = \mathrm{M} = \mathrm{N} = \mathrm{A} = \mathrm{B} = \mathrm{C} = 1, \\ a,\ b,\ c = \cos\lambda,\ \cos\mu,\ \cos\nu \end{array}\right\}.$$

He then from the second set of six equations forms three groups—

$$\left.\begin{array}{l} \mathrm{L}\alpha\,.\,\alpha + \mathrm{M}\alpha'\,.\,\alpha' + \mathrm{N}\alpha''\,.\,\alpha'' = \mathrm{A} \\ \mathrm{L}\alpha\,.\,\beta + \mathrm{M}\alpha'\,.\,\beta' + \mathrm{N}\alpha''\,.\,\beta'' = c \\ \mathrm{L}\alpha\,.\,\gamma + \mathrm{M}\alpha'\,.\,\gamma' + \mathrm{N}\alpha''\,.\,\gamma'' = b \end{array}\right\}$$

$$\left.\begin{array}{l} \mathrm{L}\beta\,.\,\alpha + \mathrm{M}\beta'\,.\,\alpha' + \mathrm{N}\beta''\,.\,\alpha'' = c \\ \mathrm{L}\beta\,.\,\beta + \mathrm{M}\beta'\,.\,\beta' + \mathrm{N}\beta''\,.\,\beta'' = \mathrm{B} \\ \mathrm{L}\beta\,.\,\gamma + \mathrm{M}\beta'\,.\,\gamma' + \mathrm{N}\beta''\,.\,\gamma'' = a \end{array}\right\}$$

$$\left.\begin{array}{l} \mathrm{L}\gamma\,.\,\alpha + \mathrm{M}\gamma'\,.\,\alpha' + \mathrm{N}\gamma''\,.\,\alpha'' = b \\ \mathrm{L}\gamma\,.\,\beta + \mathrm{M}\gamma'\,.\,\beta' + \mathrm{N}\gamma''\,.\,\beta'' = a \\ \mathrm{L}\gamma\,.\,\gamma + \mathrm{M}\gamma'\,.\,\gamma' + \mathrm{N}\gamma''\,.\,\gamma'' = \mathrm{C} \end{array}\right\},$$

and solves the first group for $\mathrm{L}\alpha$, $\mathrm{M}\alpha'$, $\mathrm{N}\alpha''$; the second for $\mathrm{L}\beta$, $\mathrm{M}\beta'$, $\mathrm{N}\beta''$; and the third for $\mathrm{L}\gamma$, $\mathrm{M}\gamma'$, $\mathrm{N}\gamma''$; the results being

$$\left.\begin{array}{l} \Delta\,.\,\mathrm{L}\alpha = (\beta'\gamma''-\beta''\gamma')\mathrm{A} + (\gamma'\alpha''-\gamma''\alpha')c + (\alpha'\beta''-\alpha''\beta')b \\ \Delta\,.\,\mathrm{M}\alpha' = (\beta''\gamma-\beta\gamma'')\mathrm{A} + (\gamma''\alpha-\gamma\alpha'')c + (\alpha''\beta-\alpha\beta'')b \\ \Delta\,.\,\mathrm{N}\alpha'' = (\beta\gamma'-\beta'\gamma)\mathrm{A} + (\gamma\alpha'-\gamma'\alpha)c + (\alpha\beta'-\alpha'\beta)b \end{array}\right\}$$

$$\left.\begin{array}{l} \Delta\,.\,\mathrm{L}\beta = (\beta'\gamma''-\beta''\gamma')c + (\gamma'\alpha''-\gamma''\alpha')\mathrm{B} + (\alpha'\beta''-\alpha''\beta')a \\ \Delta\,.\,\mathrm{M}\beta' = (\beta''\gamma-\beta\gamma'')c + (\gamma''\alpha-\gamma\alpha'')\mathrm{B} + (\alpha''\beta-\alpha\beta'')a \\ \Delta\,.\,\mathrm{N}\beta'' = (\beta\gamma'-\beta'\gamma)c + (\gamma\alpha'-\gamma'\alpha)\mathrm{B} + (\alpha\beta'-\alpha'\beta)a \end{array}\right\}$$

$$\left.\begin{array}{l} \Delta\,.\,\mathrm{L}\gamma = (\beta'\gamma''-\beta''\gamma')b + (\gamma'\alpha''-\gamma''\alpha')a + (\alpha'\beta''-\alpha''\beta')\mathrm{C} \\ \Delta\,.\,\mathrm{M}\gamma' = (\beta''\gamma-\beta\gamma'')b + (\gamma''\alpha-\gamma\alpha'')a + (\alpha''\beta-\alpha\beta'')\mathrm{C} \\ \Delta\,.\,\mathrm{N}\gamma'' = (\beta\gamma'-\beta'\gamma)b + (\gamma\alpha'-\gamma'\alpha)a + (\alpha\beta'-\alpha'\beta)\mathrm{C} \end{array}\right\}.$$

Making the substitution above referred to he derives the corresponding results which are obtainable from the first set of six, viz.:

$$\left.\begin{aligned}\Delta . \alpha &= (\beta'\gamma''-\beta''\gamma') + (\gamma'\alpha''-\gamma''\alpha')\cos\nu + (\alpha'\beta''-\alpha''\beta')\cos\mu\\ \Delta . \alpha' &= (\beta''\gamma-\beta\gamma'') + (\gamma''\alpha-\gamma\alpha'')\cos\nu + (\alpha''\beta-\alpha\beta'')\cos\mu\\ \Delta . \alpha'' &= (\beta\gamma'-\beta'\gamma) + (\gamma\alpha'-\gamma'\alpha)\cos\nu + (\alpha\beta'-\alpha'\beta)\cos\mu\end{aligned}\right\}$$

$$\left.\begin{aligned}\Delta . \beta &= (\beta'\gamma''-\beta''\gamma')\cos\nu + (\gamma'\alpha''-\gamma''\alpha') + (\alpha'\beta''-\alpha''\beta')\cos\lambda\\ \Delta . \beta' &= (\beta''\gamma-\beta\gamma'')\cos\nu + (\gamma''\alpha-\gamma\alpha'') + (\alpha''\beta-\alpha\beta'')\cos\lambda\\ \Delta . \beta'' &= (\beta\gamma'-\beta'\gamma)\cos\nu + (\gamma\alpha'-\gamma'\alpha) + (\alpha\beta'-\alpha'\beta)\cos\lambda\end{aligned}\right\}$$

$$\left.\begin{aligned}\Delta . \gamma &= (\beta'\gamma''-\beta''\gamma')\cos\mu + (\gamma'\alpha''-\gamma''\alpha')\cos\lambda + (\alpha'\beta''-\alpha''\beta')\\ \Delta . \gamma' &= (\beta''\gamma-\beta\gamma'')\cos\mu + (\gamma''\alpha-\gamma\alpha'')\cos\lambda + (\alpha''\beta-\alpha\beta'')\\ \Delta . \gamma'' &= (\beta\gamma'-\beta'\gamma)\cos\mu + (\gamma\alpha'-\gamma'\alpha)\cos\lambda + (\alpha\beta'-\alpha'\beta)\end{aligned}\right\}.$$

He then takes each of these nine equations along with the one of which it is a special case, and by subtraction obtains nine new equations, which he groups as follows:—

$$\begin{aligned}0 &= (\mathrm{L}-\mathrm{A})\cdot(\beta'\gamma''-\beta''\gamma') + (\mathrm{L}\cos\nu-c)\cdot(\gamma'\alpha''-\gamma''\alpha') + (\mathrm{L}\cos\mu-b)\cdot(\alpha'\beta''-\alpha''\beta')\\ 0 &= (\mathrm{L}\cos\nu-c)\cdot(\beta'\gamma''-\beta''\gamma') + (\mathrm{L}-\mathrm{B})\cdot(\gamma'\alpha''-\gamma''\alpha') + (\mathrm{L}\cos\lambda-a)\cdot(\alpha'\beta''-\alpha''\beta')\\ 0 &= (\mathrm{L}\cos\mu-b)\cdot(\beta'\gamma''-\beta''\gamma') + (\mathrm{L}\cos\lambda-a)\cdot(\gamma'\alpha''-\gamma''\alpha') + (\mathrm{L}-\mathrm{C})\cdot(\alpha'\beta''-\alpha''\beta')\end{aligned}$$

$$\begin{aligned}0 &= (\mathrm{M}-\mathrm{A})\cdot(\beta''\gamma-\beta\gamma'') + (\mathrm{M}\cos\nu-c)\cdot(\gamma''\alpha-\gamma\alpha'') + (\mathrm{L}\cos\mu-b)\cdot(\alpha''\beta-\alpha\beta'')\\ 0 &= (\mathrm{M}\cos\nu-c)\cdot(\beta''\gamma-\beta\gamma'') + (\mathrm{M}-\mathrm{B})\cdot(\gamma''\alpha-\gamma\alpha'') + (\mathrm{M}\cos\lambda-a)\cdot(\alpha''\beta-\alpha\beta'')\\ 0 &= (\mathrm{M}\cos\mu-b)\cdot(\beta''\gamma-\beta\gamma'') + (\mathrm{M}\cos\lambda-a)\cdot(\gamma''\alpha-\gamma\alpha'') + (\mathrm{M}-\mathrm{C})\cdot(\alpha''\beta-\alpha\beta'')\end{aligned}$$

$$\begin{aligned}0 &= (\mathrm{N}-\mathrm{A})\cdot(\beta\gamma'-\beta'\gamma) + (\mathrm{N}\cos\nu-c)\cdot(\gamma\alpha'-\gamma'\alpha) + (\mathrm{N}\cos\mu-b)\cdot(\alpha\beta'-\alpha'\beta)\\ 0 &= (\mathrm{N}\cos\nu-c)\cdot(\beta\gamma'-\beta'\gamma) + (\mathrm{N}-\mathrm{B})\cdot(\gamma\alpha'-\gamma'\alpha) + (\mathrm{N}\cos\lambda-a)\cdot(\alpha\beta'-\alpha'\beta)\\ 0 &= (\mathrm{N}\cos\mu-b)\cdot(\beta\gamma'-\beta'\gamma) + (\mathrm{N}\cos\lambda-a)\cdot(\gamma\alpha'-\gamma'\alpha) + (\mathrm{N}-\mathrm{C})\cdot(\alpha\beta'-\alpha'\beta).\end{aligned}$$

Now from the first of these groups of three it is possible to eliminate $\beta'\gamma''-\beta''\gamma'$, $\gamma'\alpha''-\gamma''\alpha'$, $\alpha'\beta''-\alpha''\beta'$; from the second, $\beta''\gamma-\beta\gamma''$, $\gamma''\alpha-\gamma\alpha''$, $\alpha''\beta-\alpha\beta''$; and from the third, $\beta\gamma'-\beta'\gamma$, $\gamma\alpha'-\gamma'\alpha$, $\alpha\beta'-\alpha'\beta$; and this being done there is obtained the set of three equations

$$\begin{aligned}0 = {} & (\mathrm{L}-\mathrm{A})(\mathrm{L}-\mathrm{B})(\mathrm{L}-\mathrm{C}) + 2(\mathrm{L}\cos\lambda-a)(\mathrm{L}\cos\mu-b)(\mathrm{L}\cos\nu-c)\\ & -(\mathrm{L}-\mathrm{A})(\mathrm{L}\cos\lambda-a)^2-(\mathrm{L}-\mathrm{B})(\mathrm{L}\cos\mu-b)^2-(\mathrm{L}-\mathrm{C})(\mathrm{L}\cos\nu-c)^2,\end{aligned}$$

$$\begin{aligned}0 = {} & (\mathrm{M}-\mathrm{A})(\mathrm{M}-\mathrm{B})(\mathrm{M}-\mathrm{C}) + 2(\mathrm{M}\cos\lambda-a)(\mathrm{M}\cos\mu-b)(\mathrm{M}\cos\nu-c)\\ & -(\mathrm{M}-\mathrm{A})(\mathrm{M}\cos\lambda-a)^2-(\mathrm{M}-\mathrm{B})(\mathrm{M}\cos\mu-b)^2-(\mathrm{M}-\mathrm{C})(\mathrm{M}\cos\nu-c)^2,\end{aligned}$$

$$\begin{aligned}0 = {} & (\mathrm{N}-\mathrm{A})(\mathrm{N}-\mathrm{B})(\mathrm{N}-\mathrm{C}) + 2(\mathrm{N}\cos\lambda-a)(\mathrm{N}\cos\mu-b)(\mathrm{N}\cos\nu-c)\\ & -(\mathrm{N}-\mathrm{A})(\mathrm{N}\cos\lambda-a)^2-(\mathrm{N}-\mathrm{B})(\mathrm{N}\cos\mu-b)^2-(\mathrm{N}-\mathrm{C})(\mathrm{N}\cos\nu-c)^2;\end{aligned}$$

from which it is clear that the unknowns L, M, N are the three roots of the equation in x,

$$0 = (x-A)(x-B)(x-C) + 2(x\cos\lambda - a)(x\cos\mu - b)(x\cos\nu - c) \\ -(x-A)(x\cos\lambda - a)^2 - (x-B)(x\cos\mu - b)^2 - (x-C)(x\cos\nu - c)^2,$$

and therefore may be considered as expressible in terms of the nine knowns, A, B, C, a, b, c, λ, μ, ν.

To obtain the remaining unknowns—which, be it noted, are not

$$\begin{array}{l} \alpha\ ,\ \beta\ ,\ \gamma \\ \alpha'\ ,\ \beta'\ ,\ \gamma' \\ \alpha'',\ \beta'',\ \gamma'' \end{array}$$

but

$$\begin{array}{lll} \beta'\gamma''-\beta''\gamma', & \beta''\gamma-\beta\gamma'', & \beta\gamma'-\beta'\gamma, \\ \gamma'\alpha''-\gamma''\alpha', & \gamma''\alpha-\gamma\alpha'', & \gamma\alpha'-\gamma'\alpha, \\ \alpha'\beta''-\alpha''\beta', & \alpha''\beta-\alpha\beta'', & \alpha\beta'-\alpha'\beta, \end{array}$$

—recourse is had to the two original sets of six equations. In the first equation of each set α^2 occurs, in the second β^2, and in the sixth $\alpha\beta$. Eliminating these in succession we have

$$\begin{aligned} (L-M)\alpha'^2 \ + (L-N)\alpha''^2 &= L-A, \\ (L-M)\beta'^2 \ + (L-N)\beta''^2 &= L-B, \\ (L-M)\alpha'\beta' + (L-N)\alpha''\beta'' &= L\cos\nu - c; \end{aligned}$$

and thence

$$(L-M)(L-N)(\alpha'\beta''-\alpha''\beta')^2 = (L-A)(L-B) - (L\cos\nu - c)^2;$$

so that one of the nine unknowns

$$= \pm\sqrt{\frac{(L-A)(L-B) - (L\cos\nu - c)^2}{(L-M)(L-N)}},$$

the others being like it, and indeed derivable from it, although Jacobi does not say so, by cyclical permutation of triads of letters.

The solution thus reached we may formulate as follows:—

The Cartesian equation

$$Ax^2 + By^2 + Cz^2 + 2ayz + 2bzx + 2cxy = 0,$$

where the axes are inclined to one another at angles λ, μ, ν, *may be transformed into*

$$L\xi^2 + M\eta^2 + N\zeta^2 = 0,$$

where the axes are rectangular, by means of the substitution

$$x = \left\{\frac{(L-B)(L-C) - (L\cos\lambda - a)^2}{\Delta^2(L-M)(L-N)}\right\}^{\frac{1}{2}}\xi + \left\{\frac{(M-B)(M-C) - (M\cos\lambda - a)^2}{\Delta^2(M-N)(M-L)}\right\}^{\frac{1}{2}}\eta$$
$$+ \left\{\frac{(N-B)(N-C) - (N\cos\lambda - a)^2}{\Delta^2(N-L)(N-M)}\right\}^{\frac{1}{2}}\zeta$$
$$y = \left\{\frac{(L-C)(L-A) - (L\cos\mu - b)^2}{\Delta^2(L-M)(L-N)}\right\}^{\frac{1}{2}}\xi + \ldots\ldots\ldots\ldots$$
$$z = \left\{\frac{(L-A)(L-B) - (L\cos\nu - c)^2}{\Delta^2(L-M)(L-N)}\right\}^{\frac{1}{2}}\xi + \ldots\ldots\ldots\ldots$$

where L, M, N *are the roots of the equation*

$$\begin{vmatrix} x - A & x\cos\nu - c & x\cos\mu - b \\ x\cos\nu - c & x - B & x\cos\lambda - a \\ x\cos\mu - b & x\cos\lambda - a & x - C \end{vmatrix} = 0,$$

and

$$\Delta^2 = \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}.$$

The paper closes with a reference to the case where $\cos\lambda = \cos\mu = \cos\nu = 0$, and to the case where $a = b = c = 0$; the equation for the determination of L, M, N being in the former case

$$\mathrm{x}^3 - (A+B+C)\mathrm{x}^2 + (AB+BC+CA-a^2-b^2-c^2)\mathrm{x}$$
$$- ABC + Aa^2 + Bb^2 + Cc^2 - 2abc = 0,$$

and in the latter case

$$\Delta^2\mathrm{x}^3 - (A\sin^2\lambda + B\sin^2\mu + C\sin^2\nu)x^2$$
$$+ (AB+BC+CA)x - ABC = 0,$$

"welche beide Gleichungen schon sonst gegeben sind."

JACOBI (1827).

[De singulari quadam duplicis integralis transformatione. *Crelle's Journal*, ii. pp. 234–242; or *Gesammelte Werke*, iii. pp. 55–66.]

Although the title of this paper is quite unlike that of the preceding, it will be seen that the two are in essence most closely related.

The double integral referred to is

$$\iint \frac{\sin\psi . \partial\psi . \partial\phi}{\rho}$$

where

$$\begin{aligned}\rho = a + a'\cos^2\psi + a''\sin^2\psi\cos^2\phi + a'''\sin^2\psi\sin^2\phi \\ + 2b'\cos\psi + 2b''\sin\psi\cos\phi + 2b'''\sin\psi\sin\phi \\ + 2c'\sin^2\psi\cos\phi\sin\phi + 2c''\cos\psi\sin\psi\sin\phi + 2c'''\cos\psi\sin\psi\cos\phi,\end{aligned}$$

—that is to say, where ρ is a quadratic function of $\cos\psi$, $\sin\psi\cos\phi$, $\sin\psi\sin\phi$; and the purpose of the paper is to show that the integral can be transformed into

$$\iint \frac{\sin P . \partial P . \partial\theta}{G + G'\cos^2 P + G''\sin^2 P\cos^2\theta + G'''\sin^2 P\sin^2\theta},$$

where the denominator is a quadratic function of $\cos P$, $\sin P\cos\theta$, $\sin P\sin\theta$, but contains only the squares of these quantities. The transformation is avowedly suggested by Gauss' solution of a simpler problem of the same kind, viz., the transformation of

$$\int \frac{\partial E}{\sqrt{[(A - a\cos E)^2 + (B - b\sin E)^2 + C^2]}}$$

into the form

$$\int \frac{\partial P}{\sqrt{(G + G'\cos^2 P + G''\sin^2 P)}}.$$

As in the preceding paper, Jacobi does not begin with the substitution which is really sought, but with the reverse substitution,—that is to say, the substitution necessary for the transformation of

$$\iint \frac{\sin P . \partial P . \partial\theta}{G + G'\cos^2 P + G''\sin^2 P\cos^2\theta + G'''\sin^2 P\sin^2\theta} \quad \text{into} \quad \iint \frac{\sin\psi . \partial\psi . \partial\phi}{\rho}$$

—knowing that from the latter substitution, when found, the former will be obtainable. This substitution he takes in the form

$$\cos P = \frac{\alpha + \alpha'\cos\psi + \alpha''\sin\psi\cos\phi + \alpha'''\sin\psi\sin\phi}{\delta + \delta'\cos\psi + \delta''\sin\psi\cos\phi + \delta'''\sin\psi\sin\phi},$$

$$\sin P\cos\theta = \frac{\beta + \beta'\cos\psi + \beta''\sin\psi\cos\phi + \beta'''\sin\psi\sin\phi}{\delta + \delta'\cos\psi + \delta''\sin\psi\cos\phi + \delta'''\sin\psi\sin\phi},$$

$$\sin P\sin\theta = \frac{\gamma + \gamma'\cos\psi + \gamma''\sin\psi\cos\phi + \gamma'''\sin\psi\sin\phi}{\delta + \delta'\cos\psi + \delta''\sin\psi\cos\phi + \delta'''\sin\psi\sin\phi},$$

the three new facients, $\cos P$, $\sin P \cos\theta$, $\sin P \sin\theta$ being expressible as fractions whose numerators and common denominator are linear functions of the original facients. It rests with him therefore to prove that the sixteen quantities

$$\begin{matrix} \alpha, & \alpha', & \alpha'', & \alpha''' \\ \beta, & \beta', & \beta'', & \beta''' \\ \gamma, & \gamma', & \gamma'', & \gamma''' \\ \delta, & \delta', & \delta'', & \delta''' \end{matrix}$$

and the four

$$G, \quad G', \quad G'', \quad G'''$$

are so determinable that the performance of the substitution may bring back the original integral.

By reason of the fact that

$$\cos^2 P + \sin^2 P \cos^2\theta + \sin^2 P \sin^2\theta = 1$$

for all values of P and θ, it follows that the expression

$$\begin{aligned} &(\alpha+\alpha'\cos\psi+\alpha''\sin\psi\cos\phi+\alpha'''\sin\psi\sin\phi)^2 \\ +\ &(\beta+\beta'\cos\psi+\beta''\sin\psi\cos\phi+\beta'''\sin\psi\sin\phi)^2 \\ +\ &(\gamma+\gamma'\cos\psi+\gamma''\sin\psi\cos\phi+\gamma'''\sin\psi\sin\phi)^2 \\ -\ &(\delta+\delta'\cos\psi+\delta''\sin\psi\cos\phi+\delta'''\sin\psi\sin\phi)^2 \end{aligned}$$

must vanish for all values of ψ and ϕ, and that therefore a number of relations must exist between products of pairs of the coefficients. These relations Jacobi might have obtained by giving special values to ψ and ϕ: for example, by putting $\psi=0$ and $\psi=\pi$ he might have obtained

$$(\alpha^2+\beta^2+\gamma^2-\delta^2)+2(\alpha\alpha'+\beta\beta'+\gamma\gamma'-\delta\delta')+(\alpha'^2+\beta'^2+\gamma'^2-\delta'^2) = 0$$

and

$$(\alpha^2+\beta^2+\gamma^2-\delta^2)-2(\alpha\alpha'+\beta\beta'+\gamma\gamma'-\delta\delta')+(\alpha'^2+\beta'^2+\gamma'^2-\delta'^2) = 0$$

and thence

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'-\delta\delta' = 0$$

and $$\alpha^2+\ \beta^2+\ \gamma^2-\ \delta^2 = -(\alpha'^2+\beta'^2+\gamma'^2-\delta'^2).$$

As a matter of fact, however, taking a hint from Gauss, he concludes that since

$$\cos^2\psi+\sin^2\psi\cos^2\phi+\sin^2\psi\sin^2\phi = 1,$$

the expression must be of the form

$$k(\cos^2\psi+\sin^2\psi\cos^2\phi+\sin^2\psi\sin^2\phi-1)$$

and that therefore by equalisation of coefficients

$$\begin{array}{lllll}
a^2 & +\beta^2 & +\gamma^2 & -\delta^2 & = -k, \\
a'^2 & +\beta'^2 & +\gamma'^2 & -\delta'^2 & = \quad k, \\
a''^2 & +\beta''^2 & +\gamma''^2 & -\delta''^2 & = \quad k, \\
a'''^2 & +\beta'''^2 & +\gamma'''^2 & -\delta'''^2 & = \quad k, \\
aa' & +\beta\beta' & +\gamma\gamma' & -\delta\delta' & = \quad 0, \\
aa'' & +\beta\beta'' & +\gamma\gamma'' & -\delta\delta'' & = \quad 0, \\
aa''' & +\beta\beta''' & +\gamma\gamma''' & -\delta\delta''' & = \quad 0, \\
a''a''' & +\beta''\beta''' & +\gamma''\gamma''' & -\delta''\delta''' & = \quad 0, \\
a'''a' & +\beta'''\beta' & +\gamma'''\gamma' & -\delta'''\delta' & = \quad 0, \\
a'a'' & +\beta'\beta'' & +\gamma'\gamma'' & -\delta'\delta'' & = \quad 0,
\end{array}$$

where k is arbitrary.* Again, since by making the substitution in the denominator

$$\mathrm{G}+\mathrm{G}'\cos^2\mathrm{P}+\mathrm{G}''\sin^2\mathrm{P}\cos^2\theta+\mathrm{G}'''\sin^2\mathrm{P}\sin^2\theta$$

a multiple of the original denominator ρ must be obtained, it follows that the expression

$$\begin{array}{l}
\mathrm{G}'\,(a+a'\cos\psi+a''\sin\psi\cos\phi+a'''\sin\psi\sin\phi)^2 \\
+\,\mathrm{G}''\,(\beta+\beta'\cos\psi+\beta''\sin\psi\cos\phi+\beta'''\sin\psi\sin\phi)^2 \\
+\,\mathrm{G}'''(\gamma+\gamma'\cos\psi+\gamma''\sin\psi\cos\phi+\gamma'''\sin\psi\sin\phi)^2 \\
+\,\mathrm{G}\,(\delta+\delta'\cos\psi+\delta''\sin\psi\cos\phi+\delta'''\sin\psi\sin\phi)^2
\end{array}$$

must also be a multiple of ρ. Putting it equal to $k\rho$, and equalising the coefficients, we obtain another set of ten equations

$$\begin{array}{lllll}
\mathrm{G}'a^2 & +\mathrm{G}''\beta^2 & +\mathrm{G}'''\gamma^2 & +\mathrm{G}\delta^2 & = ak, \\
\mathrm{G}'a'^2 & +\mathrm{G}''\beta'^2 & +\mathrm{G}'''\gamma'^2 & +\mathrm{G}\delta'^2 & = a'k, \\
\mathrm{G}'a''^2 & +\mathrm{G}''\beta''^2 & +\mathrm{G}'''\gamma''^2 & +\mathrm{G}\delta''^2 & = a''k, \\
\mathrm{G}'a'''^2 & +\mathrm{G}''\beta'''^2 & +\mathrm{G}'''\gamma'''^2 & +\mathrm{G}\delta'''^2 & = a'''k, \\
\mathrm{G}'aa' & +\mathrm{G}''\beta\beta' & +\mathrm{G}'''\gamma\gamma' & +\mathrm{G}\delta\delta' & = b'k, \\
\mathrm{G}'aa'' & +\mathrm{G}''\beta\beta'' & +\mathrm{G}'''\gamma\gamma'' & +\mathrm{G}\delta\delta'' & = b''k, \\
\mathrm{G}'aa''' & +\mathrm{G}''\beta\beta''' & +\mathrm{G}'''\gamma\gamma''' & +\mathrm{G}\delta\delta''' & = b'''k, \\
\mathrm{G}'a''a''' & +\mathrm{G}''\beta''\beta''' & +\mathrm{G}'''\gamma''\gamma''' & +\mathrm{G}\delta''\delta''' & = c'k, \\
\mathrm{G}'a'''a' & +\mathrm{G}''\beta'''\beta' & +\mathrm{G}'''\gamma'''\gamma' & +\mathrm{G}\delta'''\delta' & = c''k, \\
\mathrm{G}'a'a'' & +\mathrm{G}''\beta'\beta'' & +\mathrm{G}'''\gamma'\gamma'' & +\mathrm{G}\delta'\delta'' & = c'''k.
\end{array}$$

* The fact that these equations imply $|\,a\beta'\gamma''\delta'''\,| = \pm k^2$ is not alluded to.

We have thus a score of equations from which to determine the score of unknowns, $\alpha, \beta, \gamma, \delta, \alpha', \ldots, \mathrm{G}, \mathrm{G}', \mathrm{G}'', \mathrm{G}'''$.

From this point onward the procedure closely follows that of the preceding paper.

Noting that the specialising substitution

$$\left.\begin{array}{rlll} -\mathrm{G} = \mathrm{G}' & = \mathrm{G}'' & = \mathrm{G}''' & = 1 \\ -a \;\;= a' & = a'' & = a''' & = 1 \\ b' & = b'' & = b''' & = 0 \\ c' & = c'' & = c''' & = 0 \end{array}\right\}$$

changes the second set of ten equations into the first, he confines himself at the outset to the second set. From this four sets of four equations are selected, *e.g.*, the set

$$\left.\begin{array}{l} a.\mathrm{G}'\alpha + \beta.\mathrm{G}''\beta + \gamma.\mathrm{G}'''\gamma + \delta.\mathrm{G}\delta = ak \\ a'.\mathrm{G}'\alpha + \beta'.\mathrm{G}''\beta + \gamma'.\mathrm{G}'''\gamma + \delta'.\mathrm{G}\delta = b'k \\ a''.\mathrm{G}'\alpha + \beta''.\mathrm{G}''\beta + \gamma''.\mathrm{G}'''\gamma + \delta''.\mathrm{G}\delta = b''k \\ a'''.\mathrm{G}'\alpha + \beta'''.\mathrm{G}''\beta + \gamma'''.\mathrm{G}'''\gamma + \delta'''.\mathrm{G}\delta = b'''k \end{array}\right\},$$

and solved as sets of linear equations, the results being put in the form

$$\begin{array}{l} k(\mathrm{A}a + \mathrm{A}'b' + \mathrm{A}''b'' + \mathrm{A}'''b''') = \mathrm{G}'\alpha, \\ k(\mathrm{B}a + \mathrm{B}'b' + \mathrm{B}''b'' + \mathrm{B}'''b''') = \mathrm{G}''\beta, \\ k(\mathrm{C}a + \mathrm{C}'b' + \mathrm{C}''b'' + \mathrm{C}'''b''') = \mathrm{G}'''\gamma, \\ k(\mathrm{D}a + \mathrm{D}'b' + \mathrm{D}''b'' + \mathrm{D}'''b''') = \mathrm{G}\delta; \end{array}$$

$$\begin{array}{l} k(\mathrm{A}b' + \mathrm{A}'a' + \mathrm{A}''c''' + \mathrm{A}'''c'') = \mathrm{G}'\alpha', \\ k(\mathrm{B}b' + \mathrm{B}'a' + \mathrm{B}''c''' + \mathrm{B}'''c'') = \mathrm{G}''\beta', \\ k(\mathrm{C}b' + \mathrm{C}'a' + \mathrm{C}''c''' + \mathrm{C}'''c'') = \mathrm{G}'''\gamma', \\ k(\mathrm{D}b' + \mathrm{D}'a' + \mathrm{D}''c''' + \mathrm{D}'''c'') = \mathrm{G}\delta'; \end{array}$$

$$\begin{array}{l} k(\mathrm{A}b'' + \mathrm{A}'c''' + \mathrm{A}''a'' + \mathrm{A}'''c') = \mathrm{G}'\alpha'', \\ k(\mathrm{B}b'' + \mathrm{B}'c''' + \mathrm{B}''a'' + \mathrm{B}'''c') = \mathrm{G}''\beta'', \\ k(\mathrm{C}b'' + \mathrm{C}'c''' + \mathrm{C}''a'' + \mathrm{C}'''c') = \mathrm{G}'''\gamma'', \\ k(\mathrm{D}b'' + \mathrm{D}'c''' + \mathrm{D}''a'' + \mathrm{D}'''c') = \mathrm{G}\delta''; \end{array}$$

$$\begin{array}{l} k(\mathrm{A}b''' + \mathrm{A}'c'' + \mathrm{A}''c' + \mathrm{A}'''a''') = \mathrm{G}'\alpha''', \\ k(\mathrm{B}b''' + \mathrm{B}'c'' + \mathrm{B}''c' + \mathrm{B}'''a''') = \mathrm{G}''\beta''', \\ k(\mathrm{C}b''' + \mathrm{C}'c'' + \mathrm{C}''c' + \mathrm{C}'''a''') = \mathrm{G}'''\gamma''', \\ k(\mathrm{D}b''' + \mathrm{D}'c'' + \mathrm{D}''c' + \mathrm{D}'''a''') = \mathrm{G}\delta'''; \end{array}$$

where it is readily seen what is denoted by A, B, C, D, A′, B′, . . .* The corresponding results from the other set of ten equations are

$$\begin{array}{llll}
\alpha = -k\mathrm{A} & \beta = -k\mathrm{B} & \gamma = -k\mathrm{C} & \delta = k\mathrm{D}, \\
\alpha' = k\mathrm{A}' & \beta' = k\mathrm{B}' & \gamma' = k\mathrm{C}' & \delta' = -k\mathrm{D}', \\
\alpha'' = k\mathrm{A}'' & \beta'' = k\mathrm{B}'' & \gamma'' = k\mathrm{C}'' & \delta'' = -k\mathrm{D}'', \\
\alpha''' = k\mathrm{A}''' & \beta''' = k\mathrm{B}''' & \gamma''' = k\mathrm{C}''' & \delta''' = -k\mathrm{D}''',
\end{array}$$

these being most quickly obtainable by means of the specialising substitution just referred to. By taking each result of the former collection along with the corresponding one of the latter, four new sets of four are deduced, which on rearrangement stand thus:—

$$\begin{array}{lllll}
\mathrm{A}(a+\mathrm{G}') & +\mathrm{A}'b' & +\mathrm{A}''b'' & +\mathrm{A}'''b''' & =0, \\
\mathrm{A}b' & +\mathrm{A}'(a'-\mathrm{G}') & +\mathrm{A}''c''' & +\mathrm{A}'''c'' & =0, \\
\mathrm{A}b'' & +\mathrm{A}'c''' & +\mathrm{A}''(a''-\mathrm{G}') & +\mathrm{A}'''c' & =0, \\
\mathrm{A}b''' & +\mathrm{A}'c'' & +\mathrm{A}''c' & +\mathrm{A}'''(a'''-\mathrm{G}') & =0; \\
\\
\mathrm{B}(a+\mathrm{G}'') & +\mathrm{B}'b' & +\mathrm{B}''b'' & +\mathrm{B}'''b''' & =0, \\
\mathrm{B}b' & +\mathrm{B}'(a'-\mathrm{G}'') & +\mathrm{B}''c''' & +\mathrm{B}'''c'' & =0, \\
\mathrm{B}b'' & +\mathrm{B}'c''' & +\mathrm{B}''(a''-\mathrm{G}'') & +\mathrm{B}'''c' & =0, \\
\mathrm{B}b''' & +\mathrm{B}'c'' & +\mathrm{B}''c' & +\mathrm{B}'''(a'''-\mathrm{G}'') & =0; \\
\\
\mathrm{C}(a+\mathrm{G}''') & +\mathrm{C}'b' & +\mathrm{C}''b'' & +\mathrm{C}'''b''' & =0, \\
\mathrm{C}b' & +\mathrm{C}'(a'-\mathrm{G}''') & +\mathrm{C}''c''' & +\mathrm{C}'''c'' & =0, \\
\mathrm{C}b'' & +\mathrm{C}'c''' & +\mathrm{C}''(a''-\mathrm{G}''') & +\mathrm{C}'''c' & =0, \\
\mathrm{C}b''' & +\mathrm{C}'c'' & +\mathrm{C}''c' & +\mathrm{C}'''(a'''-\mathrm{G}''') & =0; \\
\\
\mathrm{D}(a-\mathrm{G}) & +\mathrm{D}'b' & +\mathrm{D}''b'' & +\mathrm{D}'''b''' & =0, \\
\mathrm{D}b' & +\mathrm{D}'(a'+\mathrm{G}) & +\mathrm{D}''c''' & +\mathrm{D}'''c'' & =0, \\
\mathrm{D}b'' & +\mathrm{D}'c''' & +\mathrm{D}''(a''+\mathrm{G}) & +\mathrm{D}'''c' & =0, \\
\mathrm{D}b''' & +\mathrm{D}'c'' & +\mathrm{D}''c' & +\mathrm{D}'''(a'''+\mathrm{G}) & =0.
\end{array}$$

The elimination of A, A′, A″, A‴ from the first set of four; B, B′, B″, B‴ from the second set of four; and so on; gives

* Observe A is not the cofactor of α, viz., $|\beta'\gamma''\delta'''|$, but

$$|\beta'\gamma''\delta'''| \div |\alpha\beta'\gamma''\delta'''|.$$

Attention has been drawn elsewhere to the fact that at this point a passage occurs which contains Jacobi's first printed reference to determinants. The words are "Valores sedecim quantitatum A, B, . . . supprimimus eorum prolixitatis causa; in libris algebraicis passim traduntur, et algorithmus, cuius ope formantur, hodie abunde notus est."

rise to four equations, the first of which is a quartic in G′, and the second, third, and fourth differ from the first merely in having G″, G‴, −G in place of G′. This, of course, is the same as saying that G′, G″, G‴, −G are the roots of a certain quartic in x, which would nowadays be written

$$\begin{vmatrix} a-x & b' & b'' & b''' \\ b' & a'+x & c''' & c'' \\ b'' & c''' & a''+x & c' \\ b''' & c'' & c' & a'''+x \end{vmatrix} = 0,$$

but which Jacobi writes in the form

$$\begin{aligned}&(a-x)(a'+x)(a''+x)(a'''+x)\\ &\left.\begin{aligned}&-(a-x)(a'+x)c'^2-(a-x)(a''+x)c''^2-(a-x)(a'''+x)c'''^2\\ &-(a''+x)(a'''+x)b'^2-(a'''+x)(a'+x)b''^2-(a'+x)(a''+x)b'''^2\end{aligned}\right\}\\ &+2c'c''c'''(a-x)+2c'b''b'''(a'+x)+2c''b'''b'(a''+x)+2c'''b'b''(a'''+x)\\ &+b'^2c'^2+b''^2c''^2+b'''^2c'''^2-2b'b''c'c''-2b''b'''c''c'''-2b'''b'c'''c',\end{aligned}$$

just as if he had expanded the determinant according to products of the elements of the principal diagonal.

Interrupting the process of solution for a moment Jacobi draws attention to the fact that elegant relations between the sixteen quantities $\alpha, \alpha', \alpha'', \alpha''', \ldots$ and the sixteen A, A′, A″, A‴, ... have been handed down by Laplace, Vandermonde, Gauss, and Binet,—an interesting remark as showing what writings on determinants were then known to him. Upon the subject of these relations, however, he does not enter, contenting himself with giving two sets of equations derivable from them with the help of the sixteen results

$$\alpha = -k\mathrm{A}, \quad \beta = -k\mathrm{B}, \quad \ldots .$$

The first set resembles the half-score of equations obtained near the outset, being

$$\begin{aligned} -\alpha^2 + \alpha'^2 + \alpha''^2 + \alpha'''^2 &= k,\\ \ldots\ldots\ldots\ldots\ldots\ldots&\\ -\alpha\beta + \alpha'\beta' + \alpha''\beta'' + \alpha'''\beta''' &= 0,\\ \ldots\ldots\ldots\ldots\ldots\ldots&\\ -\gamma\delta + \gamma'\delta' + \gamma''\delta'' + \gamma'''\delta''' &= 0.\end{aligned}$$

The other set consists of sixteen of the type

$$\alpha\beta' - \alpha'\beta = -(\gamma''\delta''' - \gamma'''\delta'')\epsilon,$$

where $\epsilon = \pm 1$, and is in effect a prolix way of stating the fact, nowadays familiar, that any two-line minor of $|\alpha\beta'\gamma''\delta'''|$ differs from its complementary minor only in sign, if it differ at all.

Further he inserts at this stage the reverse substitution of that with which he started, viz.,

$$\cos\psi = \frac{-\delta' + \alpha'\cos P + \beta'\sin P\cos\theta + \gamma'\sin P\sin\theta}{\delta - \alpha\cos P - \beta\sin P\cos\theta - \gamma\sin P\sin\theta},$$

$$\sin\psi\cos\phi = \frac{-\delta'' + \alpha''\cos P + \beta''\sin P\cos\theta + \gamma''\sin P\sin\theta}{\delta - \alpha\cos P - \beta\sin P\cos\theta - \gamma\sin P\sin\theta},$$

$$\sin\psi\sin\phi = \frac{-\delta''' + \alpha'''\cos P + \beta'''\sin P\cos\theta + \gamma'''\sin P\sin\theta}{\delta - \alpha\cos P - \beta\sin P\cos\theta - \gamma\sin P\sin\theta},$$

to which is added the fact that the common denominator here is the quotient of k by the common denominator in the original substitution. These results, he states, are easily proved,—doubtless by solving the three equations of the original substitution for $\cos\psi$, $\sin\psi\cos\phi$, $\sin\psi\sin\phi$, or by taking the results as already found, and verifying them by substituting the values of $\cos P$, $\sin P\cos\theta$, $\sin P\sin\theta$.

On returning to the main line of investigation, viz., the solution of the set of twenty equations, Jacobi unfortunately does not proceed with the same fulness of explanation as before the interruption. In fact, the values of the remaining sixteen unknowns are merely put on record without any indication of the mode in which they have been obtained, "brevitati ut consulatur," the first four of the sixteen being

$$\frac{\alpha^2}{k} = \frac{(a'-G')(a''-G')(a'''-G') - c'^2(a'-G') - c''^2(a''-G') - c'''^2(a'''-G') + 2c'c''c'''}{(G'+G)(G'-G'')(G'-G''')},$$

$$\frac{\alpha'^2}{k} = \frac{(a''-G')(a'''-G')(a+G') - b'''^2(a''-G') - b''^2(a'''-G') - c'^2(a+G') + 2b''b'''c'}{(G'+G)(G'-G'')(G'-G''')},$$

$$\frac{\alpha''^2}{k} = \frac{(a'''-G')(a+G')(a'-G') - b'^2(a'''-G') - c''^2(a+G') - b'''^2(a'-G') + 2b'''b'c''}{(G'+G)(G'-G'')(G'-G''')},$$

$$\frac{\alpha'''^2}{k} = \frac{(a+G')(a'-G')(a''-G') - c'''^2(a+G') - b''^2(a'-G') - b'^2(a''-G') + 2b'b''c'''}{(G'+G)(G'-G'')(G'-G''')}:$$

and the others obtainable therefrom by the change of

$$a^2, \quad a'^2, \quad a''^2, \quad a'''^2, \quad G, \quad G', \quad G'', \quad G''',$$

into

$$\begin{array}{cccccccc} \beta^2, & \beta'^2, & \beta''^2, & \beta'''^2, & G, & G'', & G', & G''', \\ \gamma^2, & \gamma'^2, & \gamma''^2, & \gamma'''^2, & G, & G''', & G'', & G', \\ -\delta^2, & -\delta'^2, & -\delta''^2, & -\delta'''^2, & -G', & -G, & G'', & G'''. \end{array}$$

The difficulty of the double sign which appears in every case is got over by merely fixing at will the sign of a, β, γ, δ,—the reason being that there are rational expressions for

$$aa', \ aa'', \ aa''', \ \beta\beta', \ \ldots, \ \gamma\gamma', \ \ldots, \ \delta\delta', \ \ldots,$$

and indeed also for

$$a'a'', \ a'a''', \ a''a''', \ \ldots,$$

similar to those just given for $a^2, \ldots$. For example

$$\frac{aa'}{k} = \frac{b'(a''-G')(a'''-G') - c''b'''(a''-G') - c'''b''(a'''-G') - b'c'^2 + b''c'c'' + b'''c'c'''}{(G'+G)(G'-G'')(G'-G''')}.$$

There is nothing to suggest that the numerators of all these expressions are determinants, and still less that in the case of

$$\begin{array}{cccc} \dfrac{a^2}{k}, & \dfrac{aa'}{k}, & \dfrac{aa''}{k}, & \dfrac{aa'''}{k} \\ & \dfrac{a'^2}{k}, & \dfrac{a'a''}{k}, & \dfrac{a'a'''}{k} \\ & & \dfrac{a''^2}{k}, & \dfrac{a''a'''}{k} \\ & & & \dfrac{a'''^2}{k} \end{array}$$

the numerators are* the ten principal minors of

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* For the modern reader the following substitute for the missing demonstration will suffice :—

If the cofactors of the elements in the four-line determinant given at the top of next page be denoted by [11], [12], . . . , then from the equations

$$\left.\begin{array}{l} -(a+G')a + \quad b'a' + \quad b''a'' + \quad b'''a''' = 0 \\ -b'a + (a'-G')a' + \quad c'''a'' + \quad c''a''' = 0 \\ -b''a + \quad c'''a' + (a''-G')a'' + \quad c'a''' = 0 \\ -b'''a + \quad c''a' + \quad c'a'' + (a'''-G')a''' = 0 \end{array}\right\}$$

$$\begin{vmatrix} a+G' & b' & b'' & b''' \\ b' & a'-G' & c''' & c'' \\ b'' & c''' & a''-G' & c' \\ b''' & c'' & c' & a'''-G' \end{vmatrix}.$$

The next and concluding paragraph of the paper is of course occupied in showing that by making the substitution whose coefficients have just been obtained, the given integral can be transformed as desired.

It is worth noting here that although this paper and the previous one are contiguous in the original volume of publication, and the problem solved in the second is in essence quite similar to that solved in the first, there is not a word to indicate that the author viewed them in this common light.

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we have

$$-\frac{a}{[11]} = \frac{a'}{[12]} = \frac{a''}{[13]} = \frac{a'''}{[14]},$$

$$-\frac{a}{[21]} = \frac{a'}{[22]} = \quad . \quad . \quad . \quad . \quad ,$$

$$-\frac{a}{[31]} = . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ,$$

$$-\frac{a}{[41]} = . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Multiplying in these lines by a, a', a'', a''' respectively we see that

$$\frac{a^2}{[11]} = \frac{a'^2}{[22]} = \frac{a''^2}{[33]} = \frac{a'''^2}{[44]}$$
$$= . \quad . \quad . \quad . \quad . \quad . \quad .$$

and therefore that each of them is equal to

$$\frac{a^2 - a'^2 - a''^2 - a'''^2}{[11]-[22]-[33]-[44]},$$

and thus equal to

$$\frac{-k}{[11]-[22]-[33]-[44]}.$$

But by the rule for differentiating a determinant the denominator here is the differential-quotient of the determinant with respect to G′; and this because of the theorem

$$\left[\frac{d}{dx}\{(x-r_1)(x-r_2)(x-r_3) \;.\;.\;.\; \}\right]_{x=r_1} = (r_1-r_2)(r_1-r_3)\;.\;.\;.$$

is equal to $-(G'-G'')(G'-G''')(G'+G)$: consequently

$$\frac{k}{(G'-G'')(G'-G''')(G'+G)} = \frac{a^2}{[11]} = \;.\;.\;.\;.$$

CAUCHY (1829).

[Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes. *Exercices de Math.*, iv. pp. 140–160; or *Œuvres complètes*, 2e sér. ix. pp. 172–195.]

The equation as it arises with Cauchy would be more fitly described as the equation whose roots are the maxima and minima of a homogeneous function of the second degree with real coefficients, and with variables subject to the condition that the sum of the squares equals unity.

Denoting the function by

$$A_{xx}x^2 + A_{yy}y^2 + A_{zz}z^2 + \ \ldots \ + 2A_{xy}xy + 2A_{xz}xz + \ \ldots\ldots ,$$

or for shortness' sake by s, he of course begins with the known equations for determining the extreme values in question, viz., the equations

$$\frac{\dfrac{\partial s}{\partial x}}{x} = \frac{\dfrac{\partial s}{\partial y}}{y} = \frac{\dfrac{\partial s}{\partial z}}{z} = \ \ldots .$$

An elementary algebraical theorem gives each of these ratios

$$= \frac{x\dfrac{\partial s}{\partial x} + y\dfrac{\partial s}{\partial y} + z\dfrac{\partial s}{\partial z} + \ \ldots .}{x^2 + y^2 + z^2 + \ \ldots},$$

and, therefore, by the fundamental theorem regarding the differentiation of homogeneous functions and by the above-mentioned condition,

$$= 2s.$$

He thus obtains the set of equations

$$\tfrac{1}{2}\frac{\partial s}{\partial x} = sx, \quad \tfrac{1}{2}\frac{\partial s}{\partial y} = sy, \quad \tfrac{1}{2}\frac{\partial s}{\partial z} = sz, \quad \ldots\ldots$$

or

$$\left.\begin{array}{rrrl} (A_{xx}-s)x + & A_{xy}y + & A_{xz}z + \ \ldots & = 0 \\ A_{yx}x + & (A_{yy}-s)y + & A_{yz}z + \ \ldots & = 0 \\ A_{zx}x + & A_{zy}y + & (A_{zz}-s)z + \ \ldots & = 0 \\ \ldots & \ldots & \ldots & \ldots \end{array}\right\}$$

and therefore concludes that, on eliminating $x, y, z, \ldots$ from the set, the resulting equations in s,

$$S = 0$$

say, has for its roots the maxima and minima values of s. The third chapter of the *Cours d'Analyse* is then referred to and taken as warrant that

"S sera une fonction alternée des quantités comprises dans le Tableau

$$\begin{array}{llll} A_{xx}-s & A_{xy} & A_{xz} & \cdots \\ A_{xy} & A_{yy}-s & A_{yz} & \cdots \\ A_{xz} & A_{yz} & A_{zz}-s & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$ "

and the developments of the function are given for the cases $n=2$, $n=3$, $n=4$ exactly in the form adopted by Jacobi.

The question of the particular values of the variables $x, y, z, \ldots$ which correspond and give rise to each of the n extreme values of s is next taken up, the equations for the determination of them being clearly the set from which the equation $S=0$ was obtained (a set, be it remarked, which of itself can only give the ratio of any two) and the additional equation

$$x^2+y^2+z^2+\ldots\ldots=1.$$

A series of identities connecting these n^2 values is however first obtained. Denoting by $x_r, y_r, z_r, \ldots$ the values of $x, y, z, \ldots$ which corresponds to the extreme value s_r of s, he has, by a double use of each equation of the set, the n pairs of equations

$$\left.\begin{array}{l}(A_{xx}-s_1)x_1+A_{xy}y_1+A_{xz}z_1+\ldots=0\\(A_{xx}-s_2)x_2+A_{xy}y_2+A_{xz}z_2+\ldots=0\end{array}\right\}$$
$$\left.\begin{array}{l}A_{xy}x_1+(A_{yy}-s_1)y_1+A_{yz}z_1+\ldots=0\\A_{xy}x_2+(A_{yy}-s_2)y_2+A_{yz}z_2+\ldots=0\end{array}\right\}$$
$$\left.\begin{array}{l}A_{xz}x_1+A_{yz}y_1+(A_{zz}-s_1)z_1+\ldots=0\\A_{xz}x_2+A_{yz}y_2+(A_{zz}-s_2)z_2+\ldots=0\end{array}\right\}$$
$$\cdots\cdots\cdots\cdots$$

From the first pair A_{xx} can be eliminated, from the second pair A_{yy}, and so on. Consequently there is in this way obtained the n equations

$$\begin{array}{l}(s_2-s_1)x_1x_2+A_{xy}(x_2y_1-x_1y_2)+A_{xz}(x_2z_1-x_1z_2)+\ldots=0\\A_{xy}(y_2x_1-y_1x_2)+(s_2-s_1)y_1y_2+A_{yz}(y_2z_1-y_1z_2)+\ldots=0\\A_{xz}(z_2x_1-z_1x_2)+A_{zy}(z_2y_1-z_1y_2)+(s_2-s_1)z_1z_2+\ldots=0\\\cdots\cdots\cdots\cdots\end{array}$$

and from these by addition

$$(x_1x_2+y_1y_2+z_1z_2+\ \ldots\ )(s_2-s_1) = 0,$$

the conclusion being

"Donc, toutes les fois que les racines s_1, s_2, seront inégales entre elles, on aura

$$x_1x_2 + y_1y_1 + z_1z_2 + \ \ldots\ = 0\,;$$

et, si l'équation $S=0$ n'offre pas de racines égales, les valeurs de x, y, z, . . . correspondantes à ces racines vérifieront toutes les formules comprises dans le Tableau suivant:

$$\begin{array}{llll}
x_1^2+y_1^2 \quad + \ldots = 1, & x_1x_2+y_1y_2+\ldots = 0, & \ldots\ldots, & x_1x_n+y_1y_n+\ldots = 0 \\
x_2x_1+y_2y_1+\ldots = 0, & x_2^2+\ y_2^2+\ldots = 1, & \ldots\ldots, & x_2x_n+y_2y_n+\ldots = 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_nx_1+y_ny_1+\ldots = 0, & x_nx_2+y_ny_2+\ldots = 0, & \ldots\ldots, & x_n^2+\ y_n^2+\ldots = 0."
\end{array}$$

This interlude over, the fundamental set of equations is returned to, and, the first of them being deleted, there is got from the remainder

$$\frac{x}{P_{xx}} = -\frac{y}{P_{xy}} = -\frac{z}{P_{xz}} = - \ldots\ldots$$

where the denominators are seen to be what we now call certain 'principal minors' of S; or, as Cauchy says, where P_{uv} is

"ce que devient S, lorsqu'on supprime dans le Tableau les termes qui appartiennent à la même colonne horizontale que le binôme $A_{uu}-s$, avec ceux qui appartiennent à la même colonne verticale que $A_{vv}-s$, ou bien encore les termes compris dans la même colonne verticale que $A_{uu}-s$, et ceux qui sont renfermés dans la même colonne horizontale que $A_{vv}-s$."

The ratios $x:y:z:\ \ldots\ .$ having thus been got, there only remains, for the determination of x, y, z, ... , to use the equation

$$x^2+y^2+z^2+\ \ldots\ . = 1\,.$$

But before doing so it is temporarily convenient to introduce an alternative notation, viz., denoting the signed minors

$$P_{xx},\quad -P_{yy},\quad -P_{zz},\ \ldots\ .$$

by

$$X\ ,\quad Y\ ,\quad Z\ ,\ \ldots\ .$$

so that the values of these corresponding to x_r, y_r, z_r, ... , and therefore to s_r, may be denoted by X_r, Y_r, Z_r, . . . We thus have from the additional equation

$$\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z} = \ldots\ldots = \pm \frac{1}{\sqrt{X^2+Y^2+Z^2+\ldots}}$$

and therefore

$$\frac{x_1}{X_1} = \frac{y_1}{Y_1} = \frac{z_1}{Z_1} = \ldots\ldots = \pm \frac{1}{\sqrt{X_1{}^2+Y_1{}^2+Z_1{}^2+\ldots}}$$
$$\frac{x_2}{X_2} = \frac{y_2}{Y_2} = \frac{z_2}{Z_2} = \ldots\ldots = \pm \frac{1}{\sqrt{X_2{}^2+Y_2{}^2+Z_2{}^2+\ldots}}$$
$$\frac{x_n}{X_n} = \frac{y_n}{Y_n} = \frac{z_n}{Z_n} = \ldots\ldots = \pm \frac{1}{\sqrt{X_n{}^2+Y_n{}^2+Z_n{}^2+\ldots}}.$$

Of course this supposes that the special values of X_1, Y_1, Z_1, . . . occurring in the denominators do not vanish; and Cauchy's conclusion therefore is

"les expressions

$$\begin{array}{l} x_1,\ y_1,\ z_1,\ \ldots\ldots \\ x_2,\ y_2,\ z_2,\ \ldots\ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

seront, aux signes près, complètement déterminées . . . . , à moins que des racines de l'équation $S=0$ ne vérifient en même temps la formule

$$P_{xx} = 0."$$

The next step is to prove that the roots of the equation $S=0$ are all real so long as the coefficients of the quadratic s are real. If the contrary be supposed, viz., that one of the roots s_p is of the form $\lambda+\mu\sqrt{-1}$, this will of course entail the existence of another s_q of the form $\lambda-\mu\sqrt{-1}$. Also, X_p being the same function of s_p, that X_q is of s_q, it will follow that X_p and X_q will be of the form

$$M+N\sqrt{-1}\ ,\ M-N\sqrt{-1}$$

and therefore that

$$X_pX_q = M^2+N^2.$$

This means that X_pX_q will be positive or zero, and similar reasoning would prove the same regarding Y_pY_q, Z_pZ_q, . . . None of them, however, can be positive; for since

$$x_px_q + y_py_q + \ldots = 0,$$

it follows from the values obtained for $x_p, x_q, \ldots$, that

$$X_p X_q + Y_p Y_q + \ldots = 0.$$

And since they are all zero, and each the sum of two squares, we are forced to the conclusion that

$$\begin{array}{ccccc} X_p & = & X_q & = & 0, \\ Y_p & = & Y_q & = & 0, \\ Z_p & = & Z_q & = & 0, \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

which is the same as to say that the roots x_p, x_q satisfy the equations

$$0 = X = Y = Z = \ldots.$$

i.e.,

$$0 = P_{xx} = P_{xy} = P_{xz} = \ldots.$$

The supposition therefore that the equation of the n^{th} degree $S=0$ can have a pair of imaginary roots leads us to assert that a perfectly similar equation, $P_{xx}=0$, of the $(n-1)^{\text{th}}$ degree, will have the same pair of roots. It is thus seen that the supposition and reasoning, if persevered in, will ultimately land us in an absurdity, when we reach, as we are bound to do, one of the equations of the first degree

$$A_{xx} - s = 0, \quad A_{yy} - s = 0, \ldots.$$

"Donc l'équation $S=0$ n'a pas de racines imaginaires."

The next object being to fix the limits between which the roots of the equation $S=0$ are comprised, a theorem necessary for the accomplishment of this is first attended to. Formally enunciated in modern phraseology it is:—

S *being any axisymmetric determinant*, R *the determinant got by deleting the first row and first column of* S, Y *the determinant got by deleting the first row and second column of* S, *and* Q *the determinant got from* R *as* R *from* S, then *if* $R=0$,

$$SQ = -Y^2.$$

As the mode of proof employed by Cauchy applies equally well when S is not axisymmetric, let us take $|a_1 b_2 c_3 d_4|$ for the given determinant, and write the proof as it would nowadays be given. To begin with, if $A_1, A_2, \ldots$ be the complementary minors of the elements $a_1, a_2, \ldots$ in $|a_1 b_2 c_3 d_4|$ we have

$$\left.\begin{array}{l} a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4 = |a_1b_2c_3d_4| \\ b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4 = 0 \\ c_1A_1 - c_2A_2 + c_3A_3 - c_4A_4 = 0 \\ d_1A_1 - d_2A_2 + d_3A_3 - d_4A_4 = 0 \end{array}\right\}.$$

Putting $A_1 = 0$, and leaving out one of the last three equations, we obtain

$$\left.\begin{array}{l} -a_2A_2 + a_3A_3 - a_4A_4 = |a_1b_2c_3d_4| \\ -c_2A_2 + c_3A_3 - c_4A_4 = 0 \\ -d_2A_2 + d_3A_3 - d_4A_4 = 0 \end{array}\right\},$$

from which by solving for A_2 there results

$$A_2 = \frac{-|a_1b_2c_3d_4| \cdot |c_3d_4|}{|a_2c_3d_4|},$$

that is,

$$|a_2c_3d_4| \cdot |b_1c_3d_4| = -|a_1b_2c_3d_4| \cdot |c_3d_4|,\text{*}$$

and this, when the original determinant is axisymmetric, becomes

$$|b_1c_3d_4|^2 = -|a_1b_2c_3d_4| \cdot |c_3d_4|,$$

or, as Cauchy writes it,

$$-Y^2 = SQ.$$

The first four cases of $S = 0$ are then considered, viz., the series of equations $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$, ... or, as at a later date they would have been written,

$$A_{uu} - s = 0,$$

$$\begin{vmatrix} A_{zz} - s & A_{zu} \\ A_{zu} & A_{uu} - s \end{vmatrix} = 0,$$

$$\begin{vmatrix} A_{yy} - s & A_{yz} & A_{yu} \\ A_{yz} & A_{zz} - s & A_{zu} \\ A_{yu} & A_{zu} & A_{uu} - s \end{vmatrix} = 0,$$

$$\begin{vmatrix} A_{xx} - s & A_{xy} & A_{xz} & A_{xu} \\ A_{xy} & A_{yy} - s & A_{yz} & A_{yu} \\ A_{xz} & A_{yz} & A_{zz} - s & A_{zu} \\ A_{xu} & A_{yu} & A_{zu} & A_{uu} - s \end{vmatrix} = 0,$$

* We know from a later theorem (Jacobi, 1833) that when A_1 is not 0 the identity is

$$|A_1B_2| = |a_1b_2c_3d_4| \cdot |c_3d_4|.$$

where each determinant is the complementary minor of the element in the place (1, 1) of the next determinant. The root in the first case is evidently A_{uu}. In the second case the solution is

$$s = \tfrac{1}{2}\{A_{zz}+A_{uu}\mp\sqrt{(A_{zz}-A_{uu})^2+(2A_{zu})^2}\},$$

where the reality of the roots s_1, s_2 is manifest; and as their sum is $A_{zz}+A_{uu}$, it follows that s_2-A_{uu} may be substituted for $A_{zz}-s_1$ in

$$\begin{vmatrix} A_{zz}-s_1 & A_{zu} \\ A_{zu} & A_{uu}-s_1 \end{vmatrix} = 0,$$

with the result that we have

$$(A_{uu}-s_2)(A_{uu}-s_1) = -A_{zu}{}^2$$

and are able to conclude that the roots s_1, s_2 of the equation $S_2=0$ lie on opposite sides of the root A_{uu} of the equation $S_1=0$.

Coming now to the case of $S_3=0$ we proceed differently, the three roots being localised by observing the changes of sign in S_3 as we pass from one value of the variable s to another. Four values of s which suffice for the purpose are $-\infty$, s_1, s_2, $+\infty$. No reasoning is necessary to show that, when s is $=-\infty$, S_3 is positive, and when $s=+\infty$, S_3 is negative. When $s=s_1$ we have $S_2=0$, and therefore know from our auxiliary theorem that S_1 and S_3 must have different signs,—a fact from which we deduce that S_3 is then negative. Similarly, when $s=s_2$, it is seen that S_3 is positive. We thus have the set of values

$$s = -\infty,\quad s_1,\quad s_2,\quad +\infty,$$

and $$S_3 = +,\quad -,\quad +,\quad -,$$

which shows that one value of s which makes $S_3=0$ lies between $-\infty$ and s_1, another between s_1 and s_2, and the third between s_2 and $+\infty$. In other words, the roots s', s'', s''' of $S_3=0$ are such that between each consecutive two of them there lies a root of $S_2=0$.

The case of $S_4=0$ is treated similarly, the five values given to s in S_4 being

$$-\infty,\quad s',\quad s'',\quad s''',\quad +\infty.$$

As before, there is no difficulty about the first and last of these, the value of S_4 being seen to be positive for both. When s is put $= s'$ we know that S_3 vanishes, and that therefore S_2 and S_4 must have different signs. The sign of S_2 is settled from recalling that s' lies between $-\infty$ and s_1, and that for these values of the variable S_2 is equal $+\infty$ and 0 respectively: consequently the putting of $s=s'$ makes S_4 negative. Similar reasoning enables us to complete the set

$$\left.\begin{array}{rccccc} s = & -\infty, & s', & s'', & s''', & +\infty \\ S_4 = & +, & -, & +, & -, & + \end{array}\right\},$$

from which we learn that one value of s which makes $S_4=0$ lies between $-\infty$ and s', a second between s' and s'', a third between s'' and s''', and the fourth between s''' and $-\infty$.

Having reached this point Cauchy adds—

"Les mêmes raisonnements, successivement étendus au cas où la fonction s renfermerait cinq, six, . . . , variables, fourniront évidemment la proposition suivante:

THÉORÈME I.—*Quel que soit le nombre* n *de variables* x, y, z, . . . *l'équation*

$$S = 0$$

et les équations de même forme

$$R = 0, \quad Q = 0, \quad . \; . \; . \; .$$

auront toutes leurs racines réelles. De plus, si l'on nomme

$$s', \quad s'', \quad s''', \quad . \; . \; . \; , \quad s^{(n-1)}$$

les racines de l'équation

$$R = 0$$

rangées par ordre de grandeur, les racines réelles de l'équation $S=0$ *seront respectivement comprises entre les limites*

$$-\infty, \quad s' \quad s'', \quad s''', \quad . \; . \; . \; , \quad s^{(n-1)}, \quad \infty."$$

Considerable space (pp. 188–192) is next given to extending this theorem to the case where several values of s satisfy at the same time two consecutive equations of the series $S=0$, $R=0$, $P=0$, ...

Then follows a series of noteworthy deductions, which bring us round to the solution of a general problem of a quite different character, viz., the problem of transformation which we have seen Jacobi attacking in detail. Denoting, as before, the extreme values, all different, of the quadratic function

$$A_{xx}x^2 + A_{yy}y^2 + \ldots + 2A_{xy}xy + \ldots$$

by $s_1, s_2, \ldots, s_n$, and by $x_r, y_r, z_r, \ldots$, the values of the independent variables which give rise to s_r, we know that we have

$$\left.\begin{array}{rrrcl} (A_{xx}-s_1)x_1 + & A_{xy}y_1 + & A_{xz}z_1 + & \ldots & = 0 \\ A_{xy}x_1 + & (A_{yy}-s_1)y_1 + & A_{yz}z_1 + & \ldots & = 0 \\ A_{xz}x_1 + & A_{yz}y_1 + & (A_{zz}-s_1)z_1 + & \ldots & = 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^2 + & y_1^2 + & z_1^2 + & \ldots & = 1 \end{array}\right\}$$

$$\left.\begin{array}{rrrcl} (A_{xx}-s_2)x_2 + & A_{xy}y_2 + & A_{xz}z_2 + & \ldots & = 0 \\ A_{xy}x_2 + & (A_{yy}-s_2)y_2 + & A_{yz}z_2 + & \ldots & = 0 \\ A_{xz}x_2 + & A_{yz}y_2 + & (A_{zz}-s_2)z_2 + & \ldots & = 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_2^2 + & y_2^2 + & z_2^2 + & & = 1 \end{array}\right\}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\left.\begin{array}{rrrcl} (A_{xx}-s_n)x_n + & A_{xy}y_n + & A_{xz}z_n + & \ldots & = 0 \\ A_{xy}x_n + & (A_{yy}-s_n)y_n + & A_{yz}z_n + & \ldots & = 0 \\ A_{xz}x_n + & A_{yz}y_n + & (A_{zz}-s_n)z_n + & \ldots & = 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_n^2 + & y_n^2 + & z_n^2 + & \ldots & = 1 \end{array}\right\}$$

and that, further, when r and s are unequal

$$x_rx_s + y_ry_s + z_rz_s + \ldots = 0.$$

Recalling this, Cauchy says that if a new set of n variables be taken

$$\xi, \eta, \zeta, \ldots .$$

related to the old by the equations

$$\left.\begin{array}{l} x = x_1\xi + x_2\eta + x_3\zeta + \ldots \\ y = y_1\xi + y_2\eta + y_3\zeta + \ldots \\ z = z_1\xi + z_2\eta + z_3\zeta + \ldots \\ \ldots \ldots \ldots \ldots \end{array}\right\}$$

it is at once verifiable that

$$x^2 + y^2 + z^2 + \ldots = \xi^2 + \eta^2 + \zeta^2 + \ldots$$

In the second place, if we take any one, say the first, of the set of equations connecting $s_1, x_1, y_1, z_1, \ldots$, the corresponding equation of the set connecting $s_2, x_2, y_2, z_2, \ldots$, and so forth, writing them in the form

$$\begin{array}{l} A_{xx}x_1 + A_{xy}y_1 + A_{xz}z_1 + \ldots = s_1x_1, \\ A_{xx}x_2 + A_{xy}y_2 + A_{xz}z_2 + \ldots = s_2x_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

multiplication by $\xi, \eta, \zeta, \ldots$ respectively, followed by addition, gives

$$\left.\begin{array}{l} A_{xx}x + A_{xy}y + A_{xz}z + \ldots = s_1x_1\xi + s_2x_2\eta + s_3x_3\zeta + \ldots \\ A_{xy}x + A_{yy}y + A_{yz}z + \ldots = s_1y_1\xi + s_2y_2\eta + s_3y_3\zeta + \ldots \\ A_{xz}x + A_{yz}y + A_{zz}z + \ldots = s_1z_1\xi + s_2z_2\eta + s_3z_3\zeta + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}.$$

In the third place, if the equations giving $x, y, z, \ldots$ in terms of $\xi, \eta, \zeta, \ldots$ be taken, multiplication by $x_r, y_r, z_r, \ldots$ respectively, followed by addition, gives

$$\left.\begin{array}{l} \xi = x_1x + y_1y + z_1z + \ldots. \\ \eta = x_2x + y_2y + z_2z + \ldots. \\ \zeta = x_3x + y_3y + z_3z + \ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}.$$

In the fourth place, if we take the second of these derived sets of equations, multiplication by $x, y, z, \ldots$ respectively, followed by addition, gives

$$\begin{aligned} A_{xx}x^2 + A_{yy}y^2 + \ldots + 2A_{xy}xy + \ldots. \\ = s_1\xi^2 + s_2\eta^2 + s_3\zeta^2 + \ldots. \end{aligned}$$

With these results before him Cauchy is led to formulate the following proposition previously given "dans le dernier volume des *Mémoires de l'Academie des Sciences*" :—

"THÉORÈME II. Etant donnée une fonction homogène et du second degré de plusieurs variables $x, y, z, \ldots$, on peut toujours leur substituer d'autres variables $\xi, \eta, \zeta, \ldots$ liées à $x, y, z, \ldots$ par des équations linéaires tellement choisies que la somme des carrés de $x, y, z, \ldots$ soit équivalente à la somme des carrés de $\xi, \eta, \zeta, \ldots$, et que la fonction donnée de $x, y, z, \ldots$ se transforme en une fonction $\xi, \eta, \zeta, \ldots$ homogène et du second degré, mais qui renferme seulement les carrés de $\xi, \eta, \zeta, \ldots$"

The validity of this rests on the supposition that the equation $R=0$ has all its roots unequal; but Cauchy is careful to point out that even if this were not the case, the requisite inequality could be brought about by giving an infinitely small increment ϵ to one of the coefficients A_{xx}, A_{xy} . . . ; and as ϵ could be made to approach indefinitely near to zero without the theorem ceasing to be valid, the validity would remain even at the limit.

After a reference to the special case of three variables, the paper closes with the announcement that Sturm had arrived independently at the theorems marked I. and II., and had offered his paper on the subject to the Academy on the same day as Cauchy's.*

JACOBI (Decr. 1831).

[De transformatione integralis duplicis indefiniti

$$\int\frac{\partial\phi\,\partial\psi}{A+B\cos\phi+C\sin\phi+(A'+B'\cos\phi+C'\sin\phi)\cos\psi+(A''+B''\cos\phi+C''\sin\phi)\sin\psi}$$

in formam simpliciorem $\displaystyle\int\frac{\partial\eta\,\partial\theta}{G-G'\cos\eta\cos\theta-G''\sin\eta\sin\theta}$.

Crelle's Journal, viii. pp. 253–279, 321–357; or *Gesammelte Werke*, iii. pp. 91–158.]

In his previous paper with a similar title to this Jacobi confined himself strictly to the consideration of his double integral, without saying a word as to the purely algebraical problem of transformation which lay at the root of it. Had he acted otherwise he would have been forced to note that this

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* A short account of Cauchy's memoir is given in the *Bulletin des Sciences Math.*, xii. (1829), pp. 301–303, by C. S(turm), who says, "M. Cauchy a bien voulu observer, en terminant son article, que j'étais parvenu, de mon côté, à des théorèmes semblables aux siens, sans avoir connaissance de ses recherches. Le Mémoire de M. Cauchy, et le mien, dont je donne plus loin un extrait, ont été offerts le même jour à l'Académie des Sciences." A few pages further on in the same volume we come to an article entitled "Extrait d'un Mémoire sur l'intégration d'un système d'équations différentielles linéaires, présenté à l'Académie des Sciences le 27 Juillet 1829, par M. Sturm." The abstract occupies nine pages (pp. 313–322), and though it does not contain explicit statement of the two theorems referred to by Cauchy, the theorems themselves are evidently implied. There can be little doubt, therefore, that the memoir here condensed is that which was presented on the same day as Cauchy's.

algebraical problem differed from that dealt with in the earlier paper of the same year merely in having four independent variables instead of three. Using modern phraseology, we may say that the one paper dealt explicitly with the transformation of a ternary quadric into the form $L\xi^2+M\eta^2+N\zeta^2$, and the other implicitly with the transformation of a quaternary quadric into the form $G\xi_0{}^2+G'\xi_1{}^2+G''\xi_2{}^2+G'''\xi_3{}^2$; and such being the case, it is a matter for some surprise that the consideration of the corresponding problem for an n-ary quadric was left to Cauchy.

In the lengthy paper we have now come to, the algebraical problem is no longer kept in the background, but forms one of the three parts into which the subject-matter naturally divides itself. The first is the "Introductio," occupying §§ 1–9, pp. 253–264, and containing a brief account of previous related work, followed by an indication of the new results reached. The second is headed "Problema I." and occupies §§ 10–15, pp. 264–279, its subject being an algebraical transformation pure and simple. The third and longest is headed "Problema II." and concerns the closely related, not to say dependent, problem of the transformation of a double integral. With this clear-cut subdivision there is no need for any process of sifting: we turn at once to Problema I.

It is stated by Jacobi as follows:—*Proponitur, per substitutiones lineares*

$$\begin{aligned} x &= \alpha s + \alpha' s' + \alpha'' s'' \qquad & w &= at + bu + cv \\ y &= \beta s + \beta' s' + \beta'' s'' & w' &= a't + b'u + c'v \\ z &= \gamma s + \gamma' s' + \gamma'' s'' & w'' &= a''t + b''u + c''v \end{aligned}$$

quae identice efficiant

$$\begin{gathered} x^2 + y^2 + z^2 = s^2 + s'^2 + s''^2, \\ w^2 + w'^2 + w''^2 = t^2 + u^2 + v^2, \end{gathered}$$

transformare expressionem

$$(Ax+By+Cz)w + (A'x+B'y+C'z)w' + (A''x+B''y+C''z)w''$$

in hanc simpliciorem

$$Gst + G's'u + G''s''v.$$

Among the problems of the previous papers its closest relative is the first of all, the relation being that of general to particular. In modern symbolism the expression now given for transformation is

$$\begin{array}{ccc|c} x & y & z & \\ \hline \mathrm{A} & \mathrm{B} & \mathrm{C} & w \\ \mathrm{A}' & \mathrm{B}' & \mathrm{C}' & w \\ \mathrm{A}'' & \mathrm{B}'' & \mathrm{C}'' & w'', \end{array}$$

whereas in the first paper of 1827 it is

$$\begin{array}{ccc|c} x & y & z & \\ \hline \mathrm{A} & \mathrm{B} & \mathrm{C} & x \\ \mathrm{B} & \mathrm{B}' & \mathrm{C}' & y \\ \mathrm{C} & \mathrm{C}' & \mathrm{C}'' & z. \end{array}$$

Cauchy's extension was from one set of three or four variables to one set of n variables; Jacobi's from one set of three variables to *two* sets of three variables.

The preparation for solution begins with the reminder that the condition

$$x^2 + y^2 + z^2 = s^2 + s'^2 + s''^2$$

associated with the substitution

$$\left.\begin{aligned} x &= \alpha s + \alpha' s' + \alpha'' s'' \\ y &= \beta s + \beta' s' + \beta'' s'' \\ z &= \gamma s + \gamma' s' + \gamma'' s'' \end{aligned}\right\}$$

entails the six relations

$$\begin{aligned} &\alpha^2 + \beta^2 + \gamma^2 = 1, \qquad &&\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' = 0, \\ &\alpha'^2 + \beta'^2 + \gamma'^2 = 1, &&\alpha''\alpha + \beta''\beta + \gamma''\gamma = 0, \\ &\alpha''^2 + \beta''^2 + \gamma''^2 = 1, &&\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0; \end{aligned}$$

that from these and the given substitution we obtain the reverse substitution

$$\left.\begin{aligned} s &= \alpha x + \beta y + \gamma z \\ s' &= \alpha' x + \beta' y + \gamma' z \\ s'' &= \alpha'' x + \beta'' y + \gamma'' z \end{aligned}\right\};$$

and that this latter substitution when taken along with the original condition gives the second set of six relations

$$\begin{array}{ll} \alpha^2 + \alpha'^2 + \alpha''^2 = 1, & \beta\gamma + \beta'\gamma' + \beta''\gamma'' = 0, \\ \beta^2 + \beta'^2 + \beta''^2 = 1, & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' = 0, \\ \gamma^2 + \gamma'^2 + \gamma''^2 = 1, & \alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0. \end{array}$$

Further, it is pointed out that if we put

$$\epsilon \quad \text{for} \quad \alpha(\beta'\gamma''-\beta''\gamma') + \beta(\gamma'\alpha''-\gamma''\alpha') + \gamma(\alpha'\beta''-\alpha''\beta')$$

the ordinary solution of the given substitution results in

$$\left.\begin{array}{l} \epsilon s \ = x(\beta'\gamma'' - \beta''\gamma') + y(\gamma'\alpha'' - \gamma''\alpha') + z(\alpha'\beta'' - \alpha''\beta') \\ \epsilon s' \ = x(\beta''\gamma - \beta\gamma'') + y(\gamma''\alpha - \gamma\alpha'') + z(\alpha''\beta - \alpha\beta'') \\ \epsilon s'' = x(\beta\gamma' - \beta'\gamma) + y(\gamma\alpha' - \gamma'\alpha) + z(\alpha\beta' - \alpha'\beta) \end{array}\right\},$$

and that a comparison of this with the reverse substitution as already obtained produces

$$\begin{array}{lll} \epsilon\alpha = \beta'\gamma'' - \beta''\gamma', & \epsilon\alpha' = \beta''\gamma - \beta\gamma'', & \epsilon\alpha'' = \beta\gamma' - \beta'\gamma, \\ \epsilon\beta = \gamma'\alpha'' - \gamma''\alpha', & \epsilon\beta' = \gamma''\alpha - \gamma\alpha'', & \epsilon\beta'' = \gamma\alpha' - \gamma'\alpha, \\ \epsilon\gamma = \alpha'\beta'' - \alpha''\beta', & \epsilon\gamma' = \alpha''\beta - \alpha\beta'', & \epsilon\gamma'' = \alpha\beta' - \alpha'\beta. \end{array}$$

In the next place it is noted that with the help of these the left side of the identity

$$(\gamma''\alpha-\gamma\alpha'')(\alpha\beta'-\alpha'\beta) - (\gamma\alpha-\gamma'\alpha)(\alpha''\beta-\alpha\beta'') = \alpha\epsilon$$

becomes first

$$\epsilon^2(\beta'\gamma'' - \beta''\gamma')$$

and then

$$\epsilon^2 . \epsilon\alpha ;$$

and that consequently

$$\epsilon^2 = 1.$$

Lastly, attention is very pointedly drawn to the fact that if the nine quantities α, α', α'', β, β', β'', γ, γ', γ'' be such as the foregoing results imply, and any three quantities X, Y, Z be connected with other three P, Q, R by the equations

$$\left.\begin{array}{l} \mathrm{X} = \alpha\mathrm{P} + \alpha'\mathrm{Q} + \alpha''\mathrm{R} \\ \mathrm{Y} = \beta\mathrm{P} + \beta'\mathrm{Q} + \beta''\mathrm{R} \\ \mathrm{Z} = \gamma\mathrm{P} + \gamma'\mathrm{Q} + \gamma''\mathrm{R} \end{array}\right\}$$

then it follows that

$$\left.\begin{aligned} P &= \alpha X + \beta Y + \gamma Z \\ Q &= \alpha' X + \beta' Y + \gamma' Z \\ R &= \alpha'' X + \beta'' Y + \gamma'' Z \end{aligned}\right\}$$

and

$$X^2 + Y^2 + Z^2 = P^2 + Q^2 + R^2.* \qquad (0)$$

The next preliminary step is to formulate the equations which result from the identity of $(Ax+By+Cz)w+ \ldots$ with $Gst+G's'u+G''s''v$. These are†

$$\begin{array}{lll} A = G\alpha a + G'\alpha' b + G''\alpha'' c & B = G\beta a + G'\beta' b + G''\beta'' c & C = G\gamma a + G'\gamma' b + G''\gamma'' c \\ A' = G\alpha a' + G'\alpha' b' + G''\alpha'' c' & B' = G\beta a' + G'\beta' b' + G''\beta'' c' & C' = G\gamma a' + G'\gamma' b' + G''\gamma'' c' \\ A'' = G\alpha a'' + G'\alpha' b'' + G''\alpha'' c'' & B'' = G\beta a'' + G'\beta' b'' + G''\beta'' c'' & C'' = G\gamma a'' + G'\gamma' b'' + G''\gamma'' c''. \end{array}$$

Along with the twelve relations previously obtained, they give in all twenty-one equations for the determination of the three G's and the eighteen coefficients of the substitutions.

The actual process of solution consists in a long series of deductions from the last-obtained set of nine equations, the repeated use of the twelve other equations being disguised by employing the theorem above called (0). Thus from the first column of equations this theorem gives

$$\left.\begin{aligned} G\alpha &= aA + a'A' + a''A'' \\ G'\alpha' &= bA + b'A' + b''A'' \\ G''\alpha'' &= cA + c'A' + c''A'' \end{aligned}\right\};$$

the second column gives similar expressions for $G\beta$, $G'\beta'$, $G''\beta''$; and the third column for $G\gamma$, $G'\gamma'$, $G''\gamma''$. The whole set is in later notation

$$\left(\begin{vmatrix} G\alpha & G'\alpha' & G''\alpha'' \\ G\beta & G'\beta' & G''\beta'' \\ G\gamma & G'\gamma' & G''\gamma'' \end{vmatrix}\right) = \left(\begin{vmatrix} \dfrac{a\ a'\ a''}{A\ A'\ A''} & \dfrac{b\ b'\ b''}{A\ A'\ A''} & \dfrac{c\ c'\ c''}{A\ A'\ A''} \\ \dfrac{a\ a'\ a''}{B\ B'\ B''} & \dfrac{b\ b'\ b''}{B\ B'\ B''} & \dfrac{c\ c'\ c''}{B\ B'\ B''} \\ \dfrac{a\ a'\ a''}{C\ C'\ C''} & \dfrac{b\ b'\ b''}{C\ C'\ C''} & \dfrac{c\ c'\ c''}{C\ C'\ C''} \end{vmatrix}\right),$$

* In leaving these preliminary deductions, it may be worth remarking that the like results which flow from the second given substitution and its associated condition are not taken entirely for granted by Jacobi, but are given with equal fulness, the two series indeed appearing in parallel columns. † v. next page.

where $\dfrac{g\ \ h\ \ k}{\rho\ \ \sigma\ \ \tau}$ is used to denote $g\rho+h\sigma+k\tau$. Similarly by taking the same set of nine equations in *rows* there is obtained

$$\begin{vmatrix} Ga & G'b & G''c \\ Ga' & G'b' & G''c' \\ Ga'' & G'b'' & G''c'' \end{vmatrix} = \begin{vmatrix} \dfrac{\alpha\ \ \beta\ \ \gamma}{A\ \ B\ \ C} & \dfrac{\alpha'\ \ \beta'\ \ \gamma'}{A\ \ B\ \ C} & \dfrac{\alpha''\ \ \beta''\ \ \gamma''}{A\ \ B\ \ C} \\ \dfrac{\alpha\ \ \beta\ \ \gamma}{A'\ \ B'\ \ C'} & \dfrac{\alpha'\ \ \beta'\ \ \gamma'}{A'\ \ B'\ \ C'} & \dfrac{\alpha''\ \ \beta''\ \ \gamma''}{A'\ \ B'\ \ C'} \\ \dfrac{\alpha\ \ \beta\ \ \gamma}{A''\ \ B''\ \ C''} & \dfrac{\alpha'\ \ \beta'\ \ \gamma'}{A''\ \ B''\ \ C''} & \dfrac{\alpha''\ \ \beta''\ \ \gamma''}{A''\ \ B''\ \ C''} \end{vmatrix}.$$

From these two sets of equations it is clear how the coefficients of one of the substitutions may be obtained when the G's and the coefficients of the other substitution have become known.

Separating the latter of these new sets of nine in a similar fashion into column-sets of three, but solving this time in the ordinary way, Jacobi obtains a further set, which, if only to save space, we may write in the form

$$\begin{vmatrix} \dfrac{\Delta\alpha}{G} & \dfrac{\Delta\alpha'}{G'} & \dfrac{\Delta\alpha''}{G''} \\ \dfrac{\Delta\beta}{G} & \dfrac{\Delta\beta'}{G'} & \dfrac{\Delta\beta''}{G''} \\ \dfrac{\Delta\gamma}{G} & \dfrac{\Delta\gamma'}{G'} & \dfrac{\Delta\gamma''}{G''} \end{vmatrix} = \begin{vmatrix} |aB'C''| & |bB'C''| & |cB'C''| \\ |aC'A''| & |bC'A''| & |cC'A''| \\ |aA'B''| & |bA'B''| & |cA'B''| \end{vmatrix},$$

where $\Delta = |AB'C''|$, or, as Jacobi of course writes it,

$$\Delta = A(B'C''-B''C') + B(C'A''-C''A') + C(A'B''-A''B').$$

From a set giving the Italic coefficients in terms of the Greek coefficients we have thus got a reverse set. The other reverse set obtainable in the same way need not be given; but it is easily seen that the two have the same practical value as the two from which they are derived.

†Jacobi writes the nine equations in one column: they are better arranged in three, however. Cayley at a later date would have preferred to write more luminously

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = \begin{vmatrix} (G, G', G''\between\alpha, \alpha', \alpha''\between a, b, c) & (G, G', G''\between\beta, \beta', \beta''\between a, b, c) & \cdots \\ (G, G', G''\between\alpha, \alpha', \alpha''\between a', b', c') & (G, G', G''\between\beta, \beta', \beta''\between a', b', c') & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

To make another advance, either of our latest sets of nine is taken and separated into *row*-sets of three, and theorem (0) applied. The result which Jacobi gives in nine separate equations of the type

$$\frac{B'C''-B''C'}{\Delta} = \frac{\alpha a}{G} + \frac{\alpha' b}{G'} + \frac{\alpha'' c}{G''}$$

may be written more compactly and more instructively in the form

$$\left(\begin{vmatrix} \frac{|B'C''|}{\Delta} & \frac{|C'A''|}{\Delta} & \frac{|A'B''|}{\Delta} \\ \frac{|B''C|}{\Delta} & \frac{|C''A|}{\Delta} & \frac{|A''B|}{\Delta} \\ \frac{|BC'|}{\Delta} & \frac{|CA'|}{\Delta} & \frac{|A''B'|}{\Delta} \end{vmatrix}\right) = \left(\begin{vmatrix} \frac{\alpha a}{G}+\frac{\alpha' b}{G'}+\frac{\alpha'' c}{G''} & \frac{\beta a}{G}+\frac{\beta' b}{G'}+\frac{\beta'' c}{G''} & \frac{\gamma a}{G}+\frac{\gamma' b}{G'}+\frac{\gamma'' c}{G''} \\ \frac{\alpha a'}{G}+\frac{\alpha' b'}{G'}+\frac{\alpha'' c'}{G''} & \frac{\beta a'}{G}+\frac{\beta' b'}{G'}+\frac{\beta'' c'}{G''} & \frac{\gamma a'}{G}+\frac{\gamma' b'}{G'}+\frac{\gamma'' c'}{G''} \\ \frac{\alpha a''}{G}+\frac{\alpha' b''}{G'}+\frac{\alpha'' c''}{G''} & \frac{\beta a''}{G}+\frac{\beta' b''}{G'}+\frac{\beta'' c''}{G''} & \frac{\gamma a''}{G}+\frac{\gamma' b''}{G'}+\frac{\gamma'' c''}{G''} \end{vmatrix}\right)$$

Any one of the nine here, however, may be matched by one deduced directly from the set of nine which we obtained at the very outset. Thus*

$$\begin{aligned} B'C'' - B''C' = {} & (G\beta a' + G'\beta' b' + G''\beta'' c')(G\gamma a'' + G'\gamma' b'' + G''\gamma'' c'') \\ & - (G\beta a'' + G'\beta' b'' + G''\beta'' c'')(G\gamma a' + G'\gamma' b' + G''\gamma'' c'), \\ = {} & G'G''(\beta'\gamma'' - \beta''\gamma')(b'c'' - b''c') \\ & + G''G(\beta''\gamma - \beta\gamma'')(c'a'' - c''a') \\ & + GG'(\beta\gamma' - \beta'\gamma)(a'b'' - a''b'), \\ = {} & G'G''\alpha a + G''G\alpha' b + GG'\alpha'' c. \end{aligned}$$

With this we have to compare

$$\frac{\Delta}{G}\alpha a + \frac{\Delta}{G'}\alpha' b + \frac{\Delta}{G''}\alpha'' c,$$

the result being that we obtain

$$GG'G'' = \Delta,$$

and thus reach the first resting-stage on our journey.

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* Nowadays we should rather put

$$|B'C''| = \begin{vmatrix} G\beta a' + G'\beta' b' + G''\beta'' c' & G\beta a'' + G'\beta' b'' + G''\beta'' c'' \\ G\gamma a' + G'\gamma' b' + G''\gamma'' c' & G\gamma a'' + G'\gamma' b'' + G''\gamma'' c'' \end{vmatrix},$$

$$= \begin{vmatrix} G\beta & G'\beta' & G''\beta'' \\ G\gamma & G'\gamma' & G''\gamma'' \end{vmatrix} \cdot \begin{vmatrix} a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix},$$

$$= |GG'.|\beta\gamma'|.|a'b''| + GG''.|\beta\gamma''|.|a'c''| + G'G''.|\beta'\gamma''|.|b'c''|.$$

At the outset of the next stage it is found desirable, for brevity's sake, to introduce six additional letters to denote certain functions of the known quantities A, A′, A″, ... viz.

$$\begin{array}{ll} p \text{ for } A^2 + B^2 + C^2, & q \text{ for } A'A'' + B'B'' + C'C'', \\ p' \text{ for } A'^2 + B'^2 + C'^2, & q' \text{ for } A''A + B''B + C''C, \\ p'' \text{ for } A''^2 + B''^2 + C''^2, & q'' \text{ for } AA' + BB' + CC'. \end{array}$$

These are said to entail the six identities

$$\begin{aligned}
p'p'' - q^2 &= (B'C'' - B''C')^2 + (C'A'' - C''A')^2 + (A'B'' - A''B')^2, \\
p''p - q'^2 &= (B''C - BC'')^2 + (C''A - CA'')^2 + (A''B - AB'')^2, \\
pp' - q''^2 &= (BC' - B'C)^2 + (CA' - C'A)^2 + (AB' - A'B)^2, \\
qq'' - pq &= (B''C - BC'')(BC' - B'C) + (C''A - CA'')(CA' - C'A) \\
&\qquad + (A''B - AB'')(AB' - A'B), \\
q''q - p'q' &= (BC' - B'C)(B'C'' - B''C') + (CA' - C'A)(C'A'' - C''A') \\
&\qquad + (AB' - A'B)(A'B'' - A''B'), \\
qq' - p''q'' &= (B'C'' - B''C')(B''C - BC'') + (C'A'' - C''A')(C''A - CA'') \\
&\qquad + (A'B'' - A''B')(A''B - AB''),
\end{aligned}$$

and

$$\Delta^2 = pp'p'' - pq^2 - p'q'^2 - p''q''^2 + 2qq'q''.$$

The original set of nine equations, giving A, A′, A″, ... in terms of the three G's and the coefficients of the substitutions, is then returned to, and the following equations derived,—

$$\begin{aligned}
p &= G^2a^2 + G'^2b^2 + G''^2c^2, \\
p' &= G^2a'^2 + G'^2b'^2 + G''^2c'^2, \\
p'' &= G^2a''^2 + G'^2b''^2 + G''^2c''^2, \\
q &= G^2a'a'' + G'^2b'b'' + G''^2c'c'', \\
q' &= G^2a''a + G'^2b''b + G''^2c''c, \\
q'' &= G^2aa' + G'^2bb' + G''^2cc';
\end{aligned}$$

the first three being got by use of the second part of theorem (0), but all of them readily verifiable by merely substituting the said values of A, A′, A″, ... In exactly the same way from another set of nine equations, viz., those beginning

$$\frac{\Delta a}{G} = (B'C'' - B''C')a + (B''C - BC'')a' + (BC' - B'C)a'',$$

. . . .

there is obtained

$$\frac{p'p'' - q^2}{\Delta^2} = \frac{a^2}{G^2} + \frac{b^2}{G'^2} + \frac{c^2}{G''^2},$$

$$\frac{p''p - q'^2}{\Delta^2} = \frac{a'^2}{G^2} + \frac{b'^2}{G'^2} + \frac{c'^2}{G''^2},$$

$$\frac{pp' - q''^2}{\Delta^2} = \frac{a''^2}{G^2} + \frac{b''}{G'^2} + \frac{c''^2}{G''^2},$$

$$\frac{q'q'' - pq}{\Delta^2} = \frac{a'a''}{G^2} + \frac{b'b''}{G'^2} + \frac{c'c''}{G''^2},$$

$$\frac{q''q - p'q'}{\Delta^2} = \frac{a''a}{G^2} + \frac{b''b}{G'^2} + \frac{c''c}{G''^2},$$

$$\frac{qq' - p''q''}{\Delta^2} = \frac{aa'}{G^2} + \frac{bb'}{G'^2} + \frac{cc'}{G''^2}.$$

Then, by mere addition, half of the first derived set gives

$$G^2 + G'^2 + G''^2 = p + p' + p'';$$

and the corresponding half of the second set

$$\frac{1}{G^2} + \frac{1}{G'^2} + \frac{1}{G''^2} = \frac{p'p' + p''p + pp' - q^2 - q'^2 - q''^2}{\Delta^2}$$

which on putting $GG'G''$ for Δ becomes

$$G'^2G''^2 + G''^2G^2 + G^2G'^2 = p'p'' + p''p + pp' - q^2 - q'^2 - q''^2.$$

Lastly, by taking all of the first derived set and using the first part of theorem (0), there is obtained a reverse set of nine,—

$$\begin{aligned}
G^2a &= pa + q''a' + q'a'',\\
G^2a' &= q''a + p'a' + qa'',\\
G^2a'' &= q'a + qa' + p''a'',\\
G'^2b &= pb + q''b' + q'b'',\\
G'^2b' &= q''b + p'b' + qb'',\\
G'^2b'' &= q'b + qb' + p''b'',\\
G''^2c &= pc + q''c' + q'c'',\\
G''^2c' &= q''c + p'c' + qc'',\\
G''^2c'' &= q'c + qc' + p''c'',
\end{aligned}$$

and by the second part of the same theorem

$$\begin{aligned}
p^2 + q''^2 + q'^2 &= G^4a^2 + G'^4b^2 + G''^4c^2, \\
p'^2 + q^2 + q''^2 &= G^4a'^2 + G'^4b'^2 + G''^4c'^2, \\
p''^2 + q'^2 + q^2 &= G^4a''^2 + G'^4b''^2 + G''^4c''^2.
\end{aligned}$$

The existence of similar results obtainable from the second derived set is pointed out, but separate investigation of the two sets is shown to be clearly unnecessary in view of the following theorem:—

"*E qualibet formularum propositarum derivari posse alteram, si in locum quantitatum*

$$\begin{matrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{matrix} \qquad G, \quad G', \quad G''$$

substituantur respective sequentes:

$$\begin{matrix}
\dfrac{B'C''-B''C'}{\Delta}, & \dfrac{C'A''-C''A'}{\Delta}, & \dfrac{A'B''-A''B'}{\Delta}, & \\[2ex]
\dfrac{B''C-BC''}{\Delta}, & \dfrac{C''A-CA''}{\Delta}, & \dfrac{A''B-AB''}{\Delta}, & \dfrac{1}{G}, \quad \dfrac{1}{G'}, \quad \dfrac{1}{G''}; \\[2ex]
\dfrac{BC'-B'C}{\Delta}, & \dfrac{CA'-C'A}{\Delta}, & \dfrac{AB'-A'B}{\Delta}, &
\end{matrix}$$

unde, e.g., *etiam pro* Δ *ponendum* $\dfrac{1}{\Delta}$. *Quod patet reciprocum esse, id est, ubi illa in haec abeant, simul etiam haec in illa mutari.*"

The reason for this dualism is at once perceived on noting that the original set of nine equations is matched by a derived set, perfectly similar in form, but having $(B'C''-B''C')/\Delta, \ldots$ in place of $A, \ldots$ Of course, as Jacobi notes, the dualism extends to the transformation which is the object of the whole memoir; that is to say, the equation

$$(Ax+By+Cz)w + (A'x+B'y+C'z)w' + (A''x+B''y+C''z)w'' \\ = Gst + G's'u + G''s''v$$

is necessarily accompanied by

$$\begin{aligned}
&[(B'C''-B''C')x + (C'A''-C''A')y + (A'B''-A''B')z]w \\
&+ [(B''C - BC'')x + (C''A - CA'')y + (A''B - AB'')z]w' \\
&+ [(BC' - B'C)x + (CA' - C'A)y + (AB' - A'B)z]w'' \\
&\qquad = G'G''st + G''Gs'u + GG's''v.
\end{aligned}$$

This means, in modern phraseology and nomenclature, that *the linear orthogonal substitutions which change*

$$\begin{array}{ccc|l} x & y & z & \\ \hline A & B & C & w \\ A' & B' & C' & w' \\ A'' & B'' & C'' & w'' \end{array} \quad \textit{into} \quad Gst+G's'u+G''s''v$$

will at the same time change

$$\begin{array}{ccc|l} x & y & z & \\ \hline |B'C''| & |C'A''| & |A'B''| & w \\ |B''C| & |C''A| & |A''B| & w' \\ |BC'| & |CA'| & |AB'| & w'' \end{array} \quad \textit{into} \quad G'G''st+G''Gs'u+GG's''v.$$

In parallel columns with these results regarding the p's and q's Jacobi places a series of others perfectly similar to them, the twin series originating in the fact that in squaring $|AB'C''|$, as we should nowadays put it, the multiplication may be performed either row-wise or column-wise. The chief points in the second series we may state rapidly in modern compact form as follows By way of defining the new letters introduced we start with

$$\left.\begin{matrix} & n' & m' \\ \imath' & m & l' \\ \imath' & l' & n \end{matrix}\right| \equiv \left|\begin{matrix} A^2+A'^2+A''^2 & AB+A'B'+A''B'' & AC+A'C'+A''C'' \\ BA+B'A'+B''A'' & B^2+B'^2+B''^2 & BC+B'C'+B''C'' \\ CA+C'A'+C''A'' & CB+C'B'+C''B'' & C^2+C'^2+C''^2 \end{matrix}\right|,$$

whence it follows that the determinant of either matrix is equal to $|AB'C''|^2$, and the secondary minors equal to

$$\left|\begin{matrix} |B'C''|^2+|B''C|^2+|BC'|^2 & |B'C''||C'A''|+|B''C||C''A|+|BC'||CA'| & |B'C''||A'B''|+\dots \\ & |C'A''|^2+|C''A|^2+|CA'|^2 & |C'A''||A'B''|+\dots \\ & & |A'B''|^2+\dots \end{matrix}\right|$$

Then from the original set of nine equations we have

$$\left.\begin{matrix} l & n' & m' \\ n' & m & l' \\ m' & l' & n \end{matrix}\right| = \left|\begin{matrix} G^2\alpha^2+G'^2\alpha'^2+G''^2\alpha''^2 & G^2\alpha\beta+G'^2\alpha'\beta'+G''^2\alpha''\beta'' & G^2\alpha\gamma+G'^2\alpha'\gamma'+G''^2\alpha''\gamma'' \\ & G^2\beta^2+G'^2\beta'^2+G''^2\beta''^2 & G^2\beta\gamma+G'^2\beta'\gamma'+G''^2\beta''\gamma'' \\ & & G^2\gamma^2+G'^2\gamma'^2+G''^2\gamma''^2 \end{matrix}\right|,$$

and from this, in passing, by the addition of diagonal elements,

$$l+m+n = G^2+G'^2+G''^2.$$

Next, as the matrix on the right

$$= \begin{vmatrix} G^2\alpha & G'^2\alpha' & G''^2\alpha'' \\ G^2\beta & G'^2\beta' & G''^2\beta'' \\ G^2\gamma & G'^2\gamma' & G''^2\gamma'' \end{vmatrix} \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix},$$

there follows

$$\begin{vmatrix} G^2\alpha & G'^2\alpha' & G''^2\alpha'' \\ G^2\beta & G'^2\beta' & G''^2\beta'' \\ G^2\gamma & G'^2\gamma' & G''^2\gamma'' \end{vmatrix} = \begin{vmatrix} l & n' & m' \\ n' & m & l' \\ m' & l' & n \end{vmatrix} \begin{vmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{vmatrix},$$

$$= \begin{vmatrix} l\alpha + n'\beta + m'\gamma & l\alpha' + n'\beta' + m'\gamma' & l\alpha'' + n'\beta'' + m'\gamma'' \\ n'\alpha + m\beta + l'\gamma & n'\alpha' + m\beta' + l'\gamma' & n'\alpha'' + m\beta'' + l'\gamma'' \\ m'\alpha + l'\beta + n\gamma & m'\alpha' + l'\beta' + n\gamma' & m'\alpha'' + l'\beta'' + n\gamma'' \end{vmatrix},$$

whence, by summing the squares of the elements of each row separately, we have

$$\left.\begin{aligned} G''\alpha^2 + G'^4\alpha'^2 + G''^4\alpha''^2 &= l^2 + n'^2 + m'^2, \\ G''\beta^2 + G'^4\beta'^2 + G''^4\beta''^2 &= m^2 + l'^2 + n'^2, \\ G''\gamma^2 + G'^4\gamma'^2 + G''^4\gamma''^2 &= n^2 + m'^2 + l'^2. \end{aligned}\right\}$$

Among the results obtained up to this point, there are sufficient to determine the twenty-one unknowns, and to this Jacobi now definitely devotes a section (§ 14). First the G's are dealt with. There having been obtained

$$\begin{aligned} G^2 + G'^2 + G''^2 &= l + m + n = p + p' + p'', \\ G'^2G''^2 + G''^2G^2 + G^2G'^2 &= (mn - l'^2) + \ldots\ldots = (p'p'' - q^2) + \ldots \\ G^2G'^2G''^2 &= \Delta^2, \end{aligned}$$

it is perceived at once that G^2, G'^2, G''^2 are the roots of the equation

$$x^3 - x^2(l+m+n) + x(mn+nl+lm-l'^2-m'^2-n'^2) \\ - (lmn+2l'm'n'-ll'^2-mm'^2-nn'^2) = 0,$$

or

$$x^3 - x^2(p+p'+p'') + x(p'p''+p''p+pp'-q^2-q'^2-q''^2) \\ - (pp'p''+2qq'q''-pq^2-p'q'^2-p''q''^2) = 0;$$

which respectively are the same as

$$(x-l)(x-m)(x-n)-l'^2(x-l)-m'^2(x-m)-n'^2(x-n)-2l'm'n' \;=0,$$
$$(x-p)(x-p')(x-p'')-q^2(x-p)-q'^2(x-p')-q''^2(x-p'')-2qq'q'' \;=0\,;$$

and either of which is

$$\begin{aligned}&x^3-x^2(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2+\mathrm{A}'^2+\mathrm{B}'^2+\mathrm{C}'^2+\mathrm{A}''^2+\mathrm{B}''^2+\mathrm{C}''^2)\\ &+x\left\{\begin{aligned}&(\mathrm{B'C''}-\mathrm{B''C'})^2+(\mathrm{C'A''}-\mathrm{C''A'})^2+(\mathrm{A'B''}-\mathrm{A''B'})^2\\ &+(\mathrm{B''C}-\mathrm{BC''})^2+(\mathrm{C''A}-\mathrm{CA''})^2+(\mathrm{A''B}-\mathrm{AB''})^2\\ &+(\mathrm{BC'}-\mathrm{B'C})^2+(\mathrm{CA'}-\mathrm{C'A})^2+(\mathrm{AB'}-\mathrm{A'B})^2\end{aligned}\right\}\\ &-\left\{\mathrm{A}(\mathrm{B'C''}-\mathrm{B''C'})+\mathrm{B}(\mathrm{C'A''}-\mathrm{C''A'})+\mathrm{C}(\mathrm{A'B''}-\mathrm{A''B'})\right\}^2=0.\end{aligned}$$

As an alternative to this, however, it is pointed out that we might, by putting the equations

$$\left.\begin{aligned}\mathrm{G}^2\alpha &= l\alpha+n'\beta+m'\gamma\\ \mathrm{G}^2\beta &= n'\alpha+m\beta+l'\gamma\\ \mathrm{G}^2\gamma &= m'\alpha+l'\beta+n\gamma\end{aligned}\right\}\text{ in the form }\left\{\begin{aligned}0 &= (l-\mathrm{G}^2)\alpha+n'\beta+m'\gamma\\ 0 &= n'\alpha+(m-\mathrm{G}^2)\beta+l'\gamma\\ 0 &= m'\alpha+l'\beta+(n-\mathrm{G}^2)\gamma,\end{aligned}\right.$$

eliminate α, β, γ and obtain a cubic in G^2; then by similar action obtain the same cubic in G'^2 and the same cubic in G''^2. In this way the left-hand side of the equation, whose roots are G^2, G'^2, G''^2, would naturally recall determinants, although Jacobi does not say so; and after Cayley (1841) it might have been written

$$\begin{vmatrix} l-x & n' & m' \\ n' & m-x & l' \\ m' & l' & n-x \end{vmatrix} \quad\text{or}\quad \begin{vmatrix} p-x & q'' & q' \\ q'' & p'-x & q \\ q' & q & p''-x \end{vmatrix}.$$

In the next place, four equations having been found in α^2, α'^2, α''^2, viz.,

$$\alpha^2+\alpha'^2+\alpha''^2=1,$$
$$\mathrm{G}^2\alpha^2+\mathrm{G}'^2\alpha'^2+\mathrm{G}''^2\alpha''^2=l,$$
$$\frac{1}{\mathrm{G}^2}\alpha^2+\frac{1}{\mathrm{G}'^2}\alpha'^2+\frac{1}{\mathrm{G}''^2}\alpha''^2=\frac{mn-l'^2}{\Delta^2},$$
$$\mathrm{G}^4\alpha^2+\mathrm{G}'^4\alpha'^2+\mathrm{G}''^4\alpha''^2=l^2+m'^2+n'^2,$$

if the first three be taken there is obtained for α^2 the value

$$\frac{(\mathrm{G}^2-m)(\mathrm{G}^2-n)-l'^2}{(\mathrm{G}^2-\mathrm{G}'^2)(\mathrm{G}^2-\mathrm{G}''^2)};$$

and, if the 1st, 2nd and 4th, the alternative form

$$\frac{(l-\mathrm{G}'^2)(l-\mathrm{G}''^2)+m'^2+n'^2}{(\mathrm{G}^2-\mathrm{G}'^2)(\mathrm{G}^2-\mathrm{G}''^2)},$$

where the identity of the two numerators is readily verifiable. In the same way the expressions for the squares of the six other coefficients of the first substitution may be obtained. The difficulty of the double sign resulting from the extraction of the square root is readily got over, because rational expressions similar to those for α^2, α'^2, ... are given for the nine binary products $\alpha\beta$, $\alpha'\beta'$, $\alpha''\beta''$, $\alpha\gamma$, ..., from which, when the sign of one of the coefficients is fixed, the signs of the others at once follow. It is not noticed, however, that the numerators of these eighteen values are the principal minors of the three eliminants,

$$\begin{vmatrix} l-\mathrm{G}^2 & n' & m' \\ n' & m-\mathrm{G}^2 & l' \\ m' & l' & n-\mathrm{G}^2 \end{vmatrix}, \quad \begin{vmatrix} l-\mathrm{G}'^2 & n' & m' \\ n' & m-\mathrm{G}'^2 & l' \\ m' & l' & n-\mathrm{G}'^2 \end{vmatrix}, \quad \begin{vmatrix} l-\mathrm{G}''^2 & n' & m' \\ n' & m-\mathrm{G}''^2 & l' \\ m' & l' & n-\mathrm{G}''^2 \end{vmatrix}$$

above referred to, the corresponding unknowns being

$$\left(\begin{vmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ & \beta^2 & \beta\gamma \\ & & \gamma^2 \end{vmatrix}\right), \quad \left(\begin{vmatrix} \alpha'^2 & \alpha'\beta' & \alpha'\gamma' \\ & \beta'^2 & \beta'\gamma' \\ & & \gamma'^2 \end{vmatrix}\right), \quad \left(\begin{vmatrix} \alpha''^2 & \alpha''\beta'' & \alpha''\gamma'' \\ & \beta''^2 & \beta''\gamma'' \\ & & \gamma''^2 \end{vmatrix}\right),$$

and the corresponding denominators,

$$(\mathrm{G}^2-\mathrm{G}'^2)(\mathrm{G}^2-\mathrm{G}''^2),\ (\mathrm{G}'^2-\mathrm{G}''^2)(\mathrm{G}'^2-\mathrm{G}^2),\ (\mathrm{G}''^2-\mathrm{G}^2)(\mathrm{G}''^2-\mathrm{G}'^2).$$

As an alternative to this process for finding α^2, α'^2, ... there is given another, which in some respects is the more interesting of the two. Beginning with a different set of equations, viz., the set

$$\left.\begin{array}{r} (l-\mathrm{G}^2)\alpha + \qquad n'\beta + \qquad m'\gamma = 0 \\ n'\alpha + (m-\mathrm{G}^2)\beta + \qquad l'\gamma = 0 \\ m'\alpha + \qquad l'\beta + (n-\mathrm{G}^2)\gamma = 0 \end{array}\right\},$$

Jacobi drops out the first and finds $\alpha:\beta:\gamma$, drops out the second and finds $\beta:\gamma:\alpha$, drops out the third and finds $\gamma:\alpha:\beta$. Then since these three sets of ratios are the same as the three sets $\alpha^2:\alpha\beta:\alpha\gamma$, $\beta^2:\beta\gamma:\beta\alpha$, $\gamma^2:\gamma\alpha:\gamma\beta$; and as the expressions found proportional to $\alpha\beta$, $\alpha\gamma$ in the first set are respectively equal to the expressions found proportional to the same unknowns in the other sets; it follows that

$$\begin{matrix} \alpha^2, & \alpha\beta, & \alpha\gamma \\ & \beta^2, & \beta\gamma \\ & & \gamma^2 \end{matrix}$$

are proportional to

$$\begin{matrix} (m-\mathrm{G}^2)(n-\mathrm{G}^2)-l'^2, & l'm'-n'(n-\mathrm{G}^2), & n'l'-m'(m-\mathrm{G}^2), \\ & (n-\mathrm{G}^2)(l-\mathrm{G}^2)-m'^2, & m'n'-l'(l-\mathrm{G}^2), \\ & & (l-\mathrm{G}^2)(m-\mathrm{G}^2)-n'^2; \end{matrix}$$

and therefore that

$$\frac{\alpha^2}{(m-\mathrm{G}^2)(n-\mathrm{G}^2)-l'^2}, \quad \text{or} \quad \frac{\alpha\beta}{l'm'-n'(n-\mathrm{G}^2)}, \quad \text{or} \quad \ldots\ldots$$

$$=\frac{\alpha^2+\beta^2+\gamma^2}{(m-\mathrm{G}^2)(n-\mathrm{G}^2)+(n-\mathrm{G}^2)(l-\mathrm{G}^2)+(l-\mathrm{G}^2)(m-\mathrm{G}^2)-l'^2-m'^2-n'^2}.$$

Here, however, the numerator is equal to 1: and the denominator, being obtainable by differentiating

$$(x-l)(x-m)(x-n)-l'^2(x-l)-m'^2(x-m)-n'^2(x-n)-2l'm'n'$$

with respect to x, and substituting G^2 for x in the result, must be what is obtainable in the same way from

$$(x-\mathrm{G}^2)(x-\mathrm{G}'^2)(x-\mathrm{G}''^2)$$

and therefore must be equal to

$$(\mathrm{G}^2-\mathrm{G}'^2)(\mathrm{G}^2-\mathrm{G}''^2).$$

There thus result the same values for α^2, $\alpha\beta, \ldots$, as before.

The values of a^2, aa', ... are throughout given side by side with those for α^2, $\alpha\beta$, ... ; thus—

$$\alpha^2 = \frac{(\mathrm{G}^2-m)(\mathrm{G}^2-n)-l'^2}{(\mathrm{G}^2-\mathrm{G}'^2)(\mathrm{G}^2-\mathrm{G}''^2)}, \qquad a^2 = \frac{(\mathrm{G}^2-p')(\mathrm{G}^2-p'')-q^2}{(\mathrm{G}^2-\mathrm{G}'^2)(\mathrm{G}^2-\mathrm{G}''^2)},$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

At this point "Problema I." stands fully solved: one or two interesting addenda,* however, are given in a concluding section (§ 15). From the equations

$$\begin{array}{lll} \mathrm{G}a = \mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{C}\gamma, & & s = \alpha x + \beta y + \gamma z, \\ \mathrm{G}'b = \mathrm{A}\alpha' + \mathrm{B}\beta' + \mathrm{C}\gamma', & \text{and} & s' = \alpha' x + \beta' y + \gamma' z, \\ \mathrm{G}''c = \mathrm{A}\alpha'' + \mathrm{B}\beta'' + \mathrm{C}\gamma'', & & s'' = \alpha'' x + \beta'' y + \gamma'' z, \\ \mathrm{G}a' = \ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots & & \end{array}$$

by multiplication and addition* there are obtained

$$\left.\begin{array}{l} \mathrm{A}x + \mathrm{B}y + \mathrm{C}z = \mathrm{G}as + \mathrm{G}'bs' + \mathrm{G}''cs'', \\ \mathrm{A}'x + \mathrm{B}'y + \mathrm{C}'z = \mathrm{G}a's + \mathrm{G}'b's' + \mathrm{G}''c's'', \\ \mathrm{A}''x + \mathrm{B}''y + \mathrm{C}''z = \mathrm{G}a''s + \mathrm{G}'b''s' + \mathrm{G}''c''s''; \end{array}\right\}$$

and then from these by the second part of theorem (0)

$$(\mathrm{A}x+\mathrm{B}y+\mathrm{C}z)^2 + (\mathrm{A}'x+\mathrm{B}'y+\mathrm{C}'z)^2 + (\mathrm{A}''x+\mathrm{B}''y+\mathrm{C}''z)^2 \\ = \mathrm{G}^2s^2 + \mathrm{G}'^2s'^2 + \mathrm{G}''^2s''^2,$$

which may also be written in the form

$$lx^2 + my^2 + nz^2 + 2l'yz + 2m'zx + 2n'xy = \mathrm{G}^2s^2 + \mathrm{G}'^2s'^2 + \mathrm{G}''^2s''^2.$$

To this of course may be appended the derivative from it by the substitution of $\dfrac{\mathrm{B}'\mathrm{C}''-\mathrm{B}''\mathrm{C}'}{\Delta}$, ... for A, ... viz.,

$$\left\{(\mathrm{B}'\mathrm{C}''-\mathrm{B}''\mathrm{C}')x + (\mathrm{C}'\mathrm{A}''-\mathrm{C}''\mathrm{A}')y + (\mathrm{A}\mathrm{B}''-\mathrm{A}''\mathrm{B})z\right\}^2 \\ + \left\{(\mathrm{B}''\mathrm{C}-\mathrm{B}\mathrm{C}'')x + (\mathrm{C}''\mathrm{A}-\mathrm{C}\mathrm{A}'')y + (\mathrm{A}''\mathrm{B}-\mathrm{A}\mathrm{B}'')z\right\}^2 \\ + \left\{(\mathrm{B}\mathrm{C}'-\mathrm{B}'\mathrm{C})x + (\mathrm{C}\mathrm{A}'-\mathrm{C}'\mathrm{A})y + (\mathrm{A}\mathrm{B}'-\mathrm{A}'\mathrm{B})z\right\}^2 \\ = \mathrm{G}'^2\mathrm{G}''^2s^2 + \mathrm{G}''^2\mathrm{G}^2s'^2 + \mathrm{G}^2\mathrm{G}'^2s''^2.$$

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* We may formulate for use here the following theorem in modern dress:—
If $|\alpha\beta'\gamma''|$ *be an orthogonant, then*

$$\frac{\mathrm{A},\ \mathrm{B},\ \mathrm{C}}{\alpha,\ \beta,\ \gamma}\cdot\frac{\alpha,\ \beta,\ \gamma}{\mathrm{x},\ \mathrm{y},\ \mathrm{z}} + \frac{\mathrm{A},\ \mathrm{B},\ \mathrm{C}}{\alpha',\ \beta',\ \gamma'}\cdot\frac{\alpha',\ \beta',\ \gamma'}{\mathrm{x},\ \mathrm{y},\ \mathrm{z}} + \frac{\mathrm{A},\ \mathrm{B},\ \mathrm{C}}{\alpha'',\ \beta'',\ \gamma''}\cdot\frac{\alpha'',\ \beta''\ \gamma''}{\mathrm{x},\ \mathrm{y},\ \mathrm{z}} = \frac{\mathrm{A},\ \mathrm{B},\ \mathrm{C}}{\mathrm{x},\ \mathrm{y},\ \mathrm{z}}.$$

Further, it will be observed that only one substitution is here involved, and that consequently in connection with the other substitution there must be analogous results, beginning with

$$pw^2+p'w'^2+p''w''^2+2qw'w''+2q'w''w+2q''ww'=G^2t^2+G'^2u^2+G''^2v^2.$$

All of them, manifestly, may be described as *transformations simultaneous with the main transformation*, and, like one which appeared earlier in the paper, may be usefully enunciated in modern form as follows:—

The linear orthogonal substitutions which change

$$\begin{array}{ccc|c} x & y & z & \\ \hline \mathrm{A} & \mathrm{B} & \mathrm{C} & w \\ \mathrm{A}' & \mathrm{B}' & \mathrm{C}' & w' \\ \mathrm{A}'' & \mathrm{B}'' & \mathrm{C}'' & w'' \end{array} \quad \textit{into} \quad \mathrm{G}st + \mathrm{G}'s'u + \mathrm{G}''s''v$$

will at the same time change

$$\begin{array}{ccc|c} x & y & z & \\ \hline l & n' & m' & x \\ n' & m & l' & y \\ m' & l' & n & z \end{array} \quad \textit{into} \quad \mathrm{G}^2s^2 + \mathrm{G}'^2s'^2 + \mathrm{G}''^2s''^2,$$

$$\begin{array}{ccc|c} x & y & z & \\ \hline mn-l'^2 & l'm'-nn' & n'l'-mm' & x \\ l'm'-nn' & nl-m'^2 & m'n'-ll' & y \\ n'l'-mm' & m'n'-ll' & lm-n'^2 & z \end{array} \quad \textit{into} \quad \mathrm{G}'^2\mathrm{G}''^2s^2+\mathrm{G}''^2\mathrm{G}^2s'^2+\mathrm{G}^2\mathrm{G}'^2s''^2,$$

$$\begin{array}{ccc|c} w & w' & w'' & \\ \hline p & q'' & q' & w \\ q'' & p' & q & w' \\ q' & q & p'' & w'' \end{array} \quad \textit{into} \quad \mathrm{G}^2t^2 + \mathrm{G}'^2u^2 + \mathrm{G}''^2v^2,$$

. . . .

The second result, however, is seen to follow from the first, and a fourth from the third by the previously enunciated theorem of this kind.

JACOBI (1832).

[De transformatione et determinatione integralium duplicium commentatio tertia. *Crelle's Journal*, x. pp. 101–128; or *Gesammelte Werke*, iii. pp. 159–189.]

This memoir, although classed by its author with the two others of which we have given an account, is of much less interest on the purely algebraical side. In fact it consists almost entirely of the transformation of integrals like

$$\iint\sqrt{\mathrm{R}}\sin\phi\,d\phi\,d\psi,\qquad \iint\frac{\sin\phi}{\sqrt{\mathrm{R}}}\,d\phi\,d\psi,\qquad \ldots\ldots$$

by means of substitutions like

$$\cos\eta=\frac{m\cos\phi}{\sqrt{\mathrm{R}}},\quad \sin\eta\cos\theta=\frac{n\sin\phi\cos\psi}{\sqrt{\mathrm{R}}},\quad \sin\eta\sin\theta=\frac{p\sin\phi\sin\psi}{\sqrt{\mathrm{R}}},$$

where

$$\mathrm{R} = m^2\cos^2\phi + n^2\sin^2\phi\cos^2\psi + p^2\sin^2\phi\sin^2\psi.$$

When, however, an advance is made from R to U, *i.e.* to

$$a^2\cos^2\phi + b^2\sin^2\phi\cos^2\psi + c^2\sin^2\phi\sin^2\psi + 2d\sin^2\phi\cos\psi\sin\psi$$
$$+ 2e\cos\phi\sin\phi\sin\psi + 2f\cos\phi\sin\phi\cos\psi,$$

the underlying algebraical problem becomes of more importance; for example, such a problem (p. 122) as the finding of the coefficients of the substitution

$$\left.\begin{aligned} u &= gx + hy + iz,\\ v &= g'x + h'y + i'z,\\ w &= g''x + h''y + i''z,\end{aligned}\right\}$$

which transforms

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy,$$
$$a'x^2 + b'y^2 + c'z^2 + 2d'yz + 2e'zx + 2f'xy,$$

into

$$u^2 + v^2 + w^2,$$
$$\frac{u^2}{m^2} + \frac{v^2}{n^2} + \frac{w^2}{p^2},$$

respectively. Still there is nothing calling for more than this passing mention.

JACOBI (1833).

[De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant: una cum .... *Crelle's Journal*, xii. pp. 1–69; or *Gesammelte Werke*, iii. pp. 191–268.]

This memoir, the general plan of which has already been indicated (see above, p. 354), naturally divides into two main portions in accordance with the title, these being prefaced by an introduction referring to both. The first portion, now to be dealt with, is the natural outcome of a thorough re-examination of the author's own previous work viewed in the strong light of Cauchy's memoir of 1829.

In the 'Introduction' (pp. 1–7) the general problem is at the outset concisely stated and shown to be determinate. The opening words are (p. 1):

"Propositis inter variabiles

$$x_1,\ x_2,\ \ldots,\ x_n \quad \text{et} \quad y_1,\ y_2,\ \ldots,\ y_n$$

n aequationibus linearibus huiusmodi

$$y_m = \alpha_1^{(m)}x_1 + \alpha_2^{(m)}x_2 + \ldots + \alpha_n^{(m)}x_n,$$

facile patet, coëfficientes $\alpha_\kappa^{(m)}$, quorum est numerus nn, ita determinari posse, ut data functio quaelibet homogenea secundi ordinis variabilium $x_1, x_2, \ldots, x_n$ transformetur in aliam variabilium $y_1, y_2, \ldots, y_n$ quae solis earum quadratis constet, simulque summa quadratorum variabilium non mutet valorem, sive fiat

$$x_1x_1 + x_2x_2 + \ldots + x_nx_n = y_1y_1 + y_2y_2 + \ldots + y_ny_n.$$

Nam haec altera conditio sibi poscit aequationes conditionales numero $\frac{n(n+1)}{2}$, porro cum de functione transformata supponatur abiisse producta e binis variabilibus conflata, accedunt aequationes $\frac{n(n-1)}{2}$: ita ut habeas aequationes conditionales numero nn, qui est numerus coëfficientium substitutionis adhibitae. Unde problema determinatum est."

Referring shortly to Cauchy he next intimates the chief of his own new results, and illustrates it in recounting the contents of his previous papers.

Problem ii. is then attacked, the direct consequences of the condition

$$x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2$$

being first noted, namely

$$\left.\begin{array}{l} a'_\kappa a'_\lambda + a''_\kappa a''_\lambda + \dots + a^{(n)}_\kappa a^{(n)}_\lambda = 0 \\ a'_\kappa a'_\kappa + a''_\kappa a''_\kappa + \dots + a^{(n)}_\kappa a^{(n)}_\kappa = 1 \end{array}\right\}$$

from which comes

$$x_\kappa = a'_\kappa y_1 + a''_\kappa y_2 + \dots + a^{(n)}_\kappa y_n$$

and thence

$$\left.\begin{array}{l} a^{(\kappa)}_1 a^{(\lambda)}_1 + a^{(\kappa)}_2 a^{(\lambda)}_2 + \dots + a^{(\kappa)}_n a^{(\lambda)}_n = 0 \\ a^{(\kappa)}_1 a^{(\kappa)}_1 + a^{(\kappa)}_2 a^{(\kappa)}_2 + \dots + a^{(\kappa)}_n a^{(\kappa)}_n = 1 \end{array}\right\}.$$

In the second place it is recalled that if the determinant of the coefficients of the substitution, $\Sigma \pm a'_1 a''_2 \dots a^{(n)}_n$, be denoted by A, the cofactor of $a^{(s)}_r$ in A by $\beta^{(s)}_r$, and the determinant $\Sigma \pm \beta'_1 \beta''_2 \dots \beta^{(n)}_n$ by B, there are at our disposal three results independent of the conditioning equation, namely,

$$\mathrm{A}x_\kappa = \beta'_\kappa y_1 + \beta''_\kappa y_2 + \dots + \beta^{(n)}_\kappa y_n,$$

$$\beta = \mathrm{A}^{n-1},$$

$$\Sigma \pm \beta'_1 \beta''_2 \dots \beta^{(m)}_m = \mathrm{A}^{m-1} \,.\, \Sigma \pm a^{(m+1)}_{m+1} a^{(m+2)}_{m+2} \dots a^{(n)}_n ;$$

and it is then pointed out that a comparison of the first of these with one already obtained gives in our special case

$$\beta^{(m)}_\kappa = \mathrm{A} a^{(m)}_\kappa ,$$

from which follows by substitution

$$\mathrm{B} \quad \textit{i.e.} \quad \Sigma \pm \beta'_1 \beta''_2 \dots \beta^{(n)}_n = \mathrm{A}^n \,.\, \Sigma \pm a'_1 a''_2 \dots a^{(n)}_n = \mathrm{A}^{n+1} ;$$

that a comparison of this with the second general result gives

$$\mathrm{A}^2 = 1 ;$$

and that a like substitution changes the third general result into

$$\mathrm{A} \,.\, \Sigma \pm a'_1 a''_2 \dots a^{(m)}_m = \Sigma \pm a^{(m+1)}_{m+1} a^{(m+2)}_{m+2} \dots a^{(n)}_n .$$

In later language these are the propositions: *The square of an orthogonant is unity* and *The product of an orthogonant* *and one of its coaxial minors is equal to the complementary minor.*

The other conditioning equation is next utilized, namely, that the homogeneous function of the second order

$$\sum_{\kappa,\lambda} a_{\kappa\lambda} x_\kappa x_\lambda \quad \text{or} \quad \mathrm{V},$$

where $a_{\kappa\lambda} = a_{\lambda\kappa}$, shall be transformable into and from

$$\mathrm{G}_1 y_1^2 + \mathrm{G}_2 y_2^2 + \ldots\ldots + \mathrm{G}_n y_n^2.$$

The latter transformation at once gives

$$a_{\kappa\lambda} = \mathrm{G}_1 a'_\kappa a'_\lambda + \mathrm{G}_2 a''_\kappa a''_\lambda + \ldots\ldots + \mathrm{G}_n a^{(n)}_\kappa a^{(n)}_\lambda.$$

Taking from this set of n^2 equations the sub-set in which $\kappa = 1, 2, \ldots\ldots, n$, namely,

$$\left.\begin{aligned} a_{1\lambda} &= \mathrm{G}_1 a'_\lambda . a'_1 + \mathrm{G}_2 a''_\lambda . a''_1 + \ldots\ldots + \mathrm{G}_n a^{(n)}_\lambda . a^{(n)}_1 \\ a_{2\lambda} &= \mathrm{G}_1 a'_\lambda . a'_2 + \mathrm{G}_2 a''_\lambda . a''_2 + \ldots\ldots + \mathrm{G}_n a^{(n)}_\lambda . a^{(n)}_2 \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n\lambda} &= \mathrm{G}_1 a'_\lambda . a'_n + \mathrm{G}_2 a''_\lambda . a''_n + \ldots\ldots + \mathrm{G}_\kappa a^{(n)}_\lambda . a^{(n)}_n \end{aligned}\right\}$$

and using the result (0), so strongly insisted on in his paper of the year 1831 (see above, pp. 438–439) and here again spoken of as something "quod maxime tenendum est," he deduces the n (or n^2) equations

$$\mathrm{G}_m a^{(m)}_\lambda = a^{(m)}_1 a_{1\lambda} + a^{(m)}_2 a_{2\lambda} + \ldots\ldots + a^{(m)}_n a_{n\lambda}$$

and the equation

$$a^2_{1\lambda} + a^2_{2\lambda} + \ldots\ldots + a^2_{n\lambda} = (\mathrm{G}_1 a'_\lambda)^2 + (\mathrm{G}_2 a''_\lambda)^2 + \ldots\ldots + (\mathrm{G}_n a^{(n)}_\lambda)^2,$$

pointing out however that from the former a more general result than the latter is obtainable,* namely,

$$\begin{aligned} & a_{1\kappa} a_{1\lambda} + a_{2\kappa} a_{2\lambda} + \ldots\ldots + a_{n\kappa} a_{n\lambda} \\ &\quad = (\mathrm{G}_1 a'_\kappa)(\mathrm{G}_1 a'_\lambda) + (\mathrm{G}_2 a''_\kappa)(\mathrm{G}_2 a'_\lambda) + \ldots\ldots + (\mathrm{G}_n a^{(n)}_\kappa)(\mathrm{G}_n a^{(n)}_\lambda). \end{aligned}$$

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* The mode of deduction is not given, but evidently

$$(\mathrm{G}_1 a'_\kappa)(\mathrm{G}_1 a'_\lambda) + (\mathrm{G}_2 a''_\kappa)(\mathrm{G}_2 a''_\lambda) + \ldots\ldots + (\mathrm{G}_n a^{(n)}_\kappa)(\mathrm{G}_n a^{(n)}_\lambda)$$

From the same source and in the same way he derives

$$\begin{aligned} G_1 a'_\lambda \cdot y_1 + G_2 a''_\lambda \cdot y_2 + \ldots + G_n a^{(n)}_\lambda \cdot y_n \\ &= a_{1\lambda}x_1 + a_{2\lambda}x_2 + \ldots + a_{n\lambda}x_n, \\ &= w_\lambda \text{ say,} \end{aligned}$$

exactly as Cauchy did (see above, p. 434): but taking $\lambda = 1, 2, \ldots, n$ and using the second part of the theorem (0) he steps ahead of Cauchy with the result

$$(G_1y_1)^2 + (G_2y_2)^2 + \ldots + (G_ny_n)^2 = \sum_\lambda [a_{1\lambda}x_1 + a_{2\lambda}x_2 + \ldots + a_{n\lambda}x_n]^2$$

and, what is more important, he notes that the n equations on which (0) has just been used, viewed as connecting

$$G_1y_1,\ G_2y_2,\ \ldots,\ G_ny_n \quad \text{and} \quad w_1,\ w_2,\ \ldots,\ w_n$$

are exactly the equations originally connecting

$$y_1,\ y_2,\ \ldots,\ y_n \quad \text{and} \quad x_1,\ x_2,\ \ldots,\ x_n;$$

and thus draws the important conclusion that *any relation between the* y's *and the* x's *will still hold when* y_m *is changed into* $G_m y_m$ *and* x_r *into* $a_{1r}x_1 + a_{2r}x_2 + \ldots + a_{nr}x_n$. For example, corresponding to and deduced from the relation

$$y_1^2 + y_2^2 + \ldots + y_n^2 = x_1^2 + x_2^2 + \ldots + x_n^2,$$

we have the result just obtained by means of the theorem (0). Further, any new relation derived in this way may be treated in

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by $2n$ substitutions becomes

$$\frac{a'_1,\ a'_2,\ \ldots,\ a'_n}{a_{1\kappa},\ a_{2\kappa},\ \ldots,\ a_{n\kappa}} \cdot \frac{a'_1,\ a'_2,\ \ldots,\ a'_n}{a_{1\lambda},\ a_{2\lambda},\ \ldots,\ a_{n\lambda}}$$
$$+ \frac{a''_1,\ a''_2,\ \ldots,\ a''_n}{a_{1\kappa},\ a_{2\kappa},\ \ldots,\ a_{n\kappa}} \cdot \frac{a''_1,\ a''_2,\ \ldots,\ a''_n}{a_{1\lambda},\ a_{2\lambda},\ \ldots,\ a_{n\lambda}}$$
$$+ \ldots \ldots \ldots \ldots \ldots$$
$$+ \frac{a^{(n)}_1,\ a^{(n)}_2,\ \ldots,\ a^{(n)}_n}{a_{1\kappa},\ a_{2\kappa},\ \ldots,\ a_{n\kappa}} \cdot \frac{a^{(n)}_1,\ a^{(n)}_2,\ \ldots,\ a^{(n)}_n}{a_{1\lambda},\ a_{2\lambda},\ \ldots,\ a_{n\lambda}}$$

which by the proposition already formulated by me (see above, p. 450)

$$= \frac{a_{1\kappa},\ a_{2\kappa},\ \ldots,\ a_{n\kappa}}{a_{1\lambda},\ a_{2\lambda},\ \ldots,\ a_{n\lambda}}.$$

In the case of the next deduction n of the $2n$ substitutions would be for y's.

the same fashion as that from which it was obtained; consequently it is seen that for any positive integer p

$$G_1^p y_1^2 + G_2^p y_2^2 + \ldots + G_n^p y_n^2$$

may be expressed in terms of the x's.

Returning now to the set of n equations

$$a_{1\lambda}x_1 + a_{2\lambda}x_2 + \ldots + a_{n\lambda}x_n = w_\lambda$$

where the w's are known as linear functions of the y's, and solving for the x's we have, on putting b_{rs} for the cofactor of a_{rs} in $\Sigma \pm a_{11}a_{22} \ldots a_{nn}$,

$$x_\kappa \,.\, \Sigma \pm a_{11}a_{22} \ldots a_{nn} = b_{\kappa 1}w_1 + b_{\kappa 2}w_2 + \ldots + b_{\kappa n}w_n.$$

Should we now substitute for the w's in this the appropriate expressions in terms of the y's we should have a set of n equations corresponding and in a sense equivalent to the original set

$$x_\kappa = \alpha'_\kappa y_1 + \alpha''_\kappa y_2 + \ldots + \alpha^{(n)}_\kappa y_n.$$

By doing this and comparing the two sets there is obtained

$$\alpha_\kappa^{(m)} \,.\, \Sigma \pm a_{11}a_{22} \ldots a_{nn} = G_m\left[b_{\kappa 1}\alpha_1^{(m)} + b_{\kappa 2}\alpha_2^{(m)} + \ldots + b_{\kappa n}\alpha_n^{(m)}\right]$$

or

$$\frac{\alpha_\kappa^{(m)}}{G_m} = \frac{b_{\kappa 1}}{\Delta}\alpha_1^{(m)} + \frac{b_{\kappa 2}}{\Delta}\alpha_2^{(m)} + \ldots\ldots + \frac{b_{\kappa n}}{\Delta}\alpha_n^{(m)},$$

a result distinguished as a "formula memorabilis" because of the fact that on comparing it with the previously obtained equation

$$G_m \,.\, \alpha_\kappa^{(m)} = a_{1\kappa} \,.\, \alpha_1^{(m)} + a_{2\kappa} \,.\, \alpha_2^{(m)} + \ldots\ldots + a_{n\kappa} \,.\, \alpha_n^{(m)}$$

we are led to the result that *all equations involving the* a's, *the* G's *and the* α's *will still hold if* $a_{\kappa\lambda}$ *be replaced by* $b_{\kappa\lambda} \div \Delta$, G_m *be replaced by* $1 \div G_m$, *and the* α's *be left unchanged.* For example, having already found that

$$a_{\kappa\lambda} = G_1\alpha'_\kappa\alpha'_\lambda + G_2\alpha''_\kappa\alpha''_\lambda + \ldots\ldots + G_n\alpha^{(n)}_\kappa\alpha^{(n)}_\lambda$$

and

$$G_1\alpha'_\lambda y_1 + G_2\alpha''_\lambda y_2 + \ldots + G_n\alpha^{(n)}_\lambda y_n = a_{1\lambda}x_1 + a_{2\lambda}x_2 + \ldots + a_{n\lambda}x_n,$$

we conclude at once that

$$\frac{b_{\kappa\lambda}}{\Delta} = \frac{a'_\kappa a'_\lambda}{G_1} + \frac{a''_\kappa a''_\lambda}{G_2} + \ldots\ldots + \frac{a^{(n)}_\kappa a^{(n)}_\lambda}{G_n}$$

and

$$\frac{a'_\lambda y_1}{G_1} + \frac{a''_\lambda y_2}{G_2} + \ldots\ldots + \frac{a^{(n)}_\lambda y_n}{G_n} = \frac{b_{1\lambda}x_1 + b_{2\lambda}x_2 + \ldots\ldots + b_{n\lambda}x_n}{\Delta};$$

and similarly, from another previous result, that

$$\frac{y_1^2}{G_1^p} + \frac{y_2^2}{G_2^p} + \ldots\ldots + \frac{y_n^2}{G_n^p}$$

can be expressed in terms of the x's. From the n equations embraced in the second of these we obtain by multiplying by $x_1, x_2, \ldots\ldots x_n$ respectively and by adding

$$\frac{y_1^2}{G_1} + \frac{y_2^2}{G_2} + \ldots\ldots + \frac{y_n^2}{G_n} = \sum_{\kappa\lambda} b_{\kappa\lambda}x_\kappa x_\lambda \div \Delta,$$

a result which may also be viewed as a fourth example of the efficacy of the general theorem. Lastly it is noted that the same second example teaches that *all relations between the* x's *and* y's *will still hold if for* y_m *we substitute* $y_m \div G_m$ *and for* x_λ *put*

$$\frac{b_{1\lambda}x_1 + b_{2\lambda}x_2 + \ldots\ldots + b_{n\lambda}x_n}{\Delta}.$$

The next section (§ 8) concerns the equation $\Gamma = 0$ for determining the G's, and need not detain us because the set of n equations from which the said equation is derived by elimination of $a_1^{(m)}, a_2^{(m)}, \ldots, a_n^{(m)}$ has already been more than once referred to, namely, the set

$$G_m \,.\, a_\kappa^{(m)} = a_{1\kappa} \,.\, a_1^{(m)} + a_{2\kappa} \,.\, a_2^{(m)} + \ldots + a_{n\kappa} a_n^{(m)},$$

or, as it may also be written

$$\left.\begin{array}{rllll} 0 = & (a_{11} - G_m)a_1^{(m)} + & a_{21}a_2^{(m)} + & \ldots\ldots + & a_{n1}a_n^{(m)} \\ 0 = & a_{12}a_1^{(m)} + & (a_{22} - G_m)a_2^{(m)} + & \ldots\ldots + & a_{n2}a_n^{(m)} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 = & a_{1n}a_1^{(m)} + & a_{2n}a_2^{(m)} + & \ldots\ldots + & (a_{nn} - G_m)a_n^{(m)} \end{array}\right\}.$$

The only addition to be made to what has been said above (see p. 229) is that in Γ the term involving the highest power of x evidently comes from the determinant-term

$$(a_{11}-x)(a_{22}-x)\ \ldots.\ (a_{nn}-x),$$

and therefore is $(-1)^n x^n$. Consequently we must have identically

$$\Gamma = (G_1-x)(G_2-x)\ \ldots.\ (G_n-x),$$

which on putting $x=0$ gives

$$\Sigma \pm a_{11}a_{22}\ \ldots.\ a_{nn} \quad \text{or} \quad \Delta = G_1G_2\ \ldots.\ G_n.$$

The ninth section (§ 9, pp. 15–19) deals after the manner of Cauchy with the finding of the values of the coefficients $\alpha_{\kappa\lambda}$, and the character of the roots of the equation $\Gamma=0$. Denoting by $B_{\kappa\lambda}^{(m)}$ what $b_{\kappa\lambda}$ becomes when $a_{11}-G_m$, $a_{22}-G_m$, ... are substituted for a_{11}, a_{22}, ..., a consequence of which is that

$$B_{\kappa\lambda}^{(m)} = B_{\lambda\kappa}^{(m)},$$

Jacobi leaves out the λ^{th} equation from the set used to determine the G's, and derives from the rest

$$\alpha_1^{(m)} : \alpha_2^{(m)} : \ldots : \alpha_n^{(m)} = B_{1\lambda}^{(m)} : B_{2\lambda}^{(m)} : \ldots : B_{n\lambda}^{(m)},$$

and thence

$$\alpha_\kappa^{(m)} = \frac{B_{\kappa\lambda}^{(m)}}{\sqrt{\{(B_{1\lambda}^{(m)})^2+(B_{2\lambda}^{(m)})^2+\ \ldots\ +(B_{n\lambda}^{(m)})^2\}}}$$

exactly as Cauchy did. He also however supplies an alternative procedure and result. Writing the above chain of equal ratios in the form

$$\alpha_1^{(m)}\alpha_\lambda^{(m)} : \alpha_2^{(m)}\alpha_\lambda^{(m)} : \ldots : \alpha_n^{(m)}\alpha_\lambda^{(m)} = B_{1\lambda}^{(m)} : B_{2\lambda}^{(m)} : \ldots : B_{n\lambda}^{(m)},$$

whence it is evident that $B_{\kappa\lambda} \div \alpha_\kappa^{(m)}\alpha_\lambda^{(m)}$ is independent of κ, he puts

$$P_\lambda^{(m)} \,.\, \alpha_\kappa^{(m)}\alpha_\lambda^{(m)} = B_{\kappa\lambda}^{(m)},$$

thence derives of course

$$P_\kappa^{(m)} \,.\, \alpha_\lambda^{(m)}\alpha_\kappa^{(m)} = B_{\lambda\kappa}^{(m)},$$

and consequently $$P_\kappa^{(m)} = P_\lambda^{(m)}.$$

In other words, he proves that the ratio is independent of λ also, and may therefore be denoted by $P^{(m)}$. Knowing this, it only remains to use the equation

$$(a_1^{(m)})^2 + (a_2^{(m)})^2 + \ldots + (a_n^{(m)})^2 = 1$$

as before, and we obtain

$$P^{(m)} = B_{11}^{(m)} + B_{22}^{(m)} + \ldots + B_{nn}^{(m)},$$

whence immediately we have

$$a_\kappa^{(m)} a_\lambda^{(m)} = \frac{B_{\kappa\lambda}^{(m)}}{B_{11}^{(m)} + B_{22}^{(m)} + \ldots + B_{nn}^{(m)}},$$

and finally

$$a_\kappa^{(m)} = \frac{\sqrt{B_{\kappa\kappa}^{(m)}}}{\sqrt{(B_{11}^{(m)} + B_{22}^{(m)} + \ldots + B_{nn}^{(m)})}}.$$

Although the solution is thus complete, Jacobi takes the opportunity to add that from the value P_m found for the ratio it follows that

$$B_{\kappa\lambda}^{(m)} B_{\kappa'\lambda'}^{(m)} = B_{\kappa\lambda'}^{(m)} . B_{\kappa'\lambda}^{(m)},$$

and therefore also that

$$B_{\kappa\kappa}^{(m)} B_{\kappa'\kappa'}^{(m)} = (B_{\kappa\kappa'}^{(m)})^2.$$

Not only so, but he gives another mode of investigating the value of $P^{(m)}$ itself. This consists in noting that if n of the coefficients of the original function V, namely, the coefficients $a_{11}, a_{22}, \ldots, a_{nn}$, all receive the same increment ξ, the corresponding increment of V will be

$$\xi(x_1^2 + x_2^2 + \ldots + x_n^2);$$

and that consequently when V by substitution alters its form into

$$G_1 y_1^2 + G_2 y_2^2 + \ldots + G_n y_n^2,$$

this increment may, by reason of the relation

$$x_1^2 + x_2^2 + \ldots + x_n^2 = y_1^2 + y_2^2 + \ldots + y_n^2,$$

be written

$$\xi(y_1^2 + y_2^2 + \ldots + y_n^2),$$

—a result which shows that $G_1, G_2, \ldots, G_n$ all receive simultaneously the same increment ξ. Now knowing this, and applying it with the increment $-G_m$ to the previously established result

$$b_{\kappa\lambda} = G_1 G_2 \ldots G_n \left\{ \frac{a'_\kappa a'_\lambda}{G_1} + \frac{a''_\kappa a''_\lambda}{G_2} + \ldots + \frac{a^{(n)}_\kappa a^{(n)}_\lambda}{G_n} \right\},$$

we see that the left-hand member becomes $B^{(m)}_{\kappa\lambda}$, and that the terms of the right-hand member all vanish except one, namely, the term

$$G_1 G_2 \ldots G_n . \frac{a^{(m)}_\kappa a^{(m)}_\lambda}{G_m},$$

which becomes

$$(G_1 - G_m)(G_2 - G_m) \ldots (G_{m-1} - G_m)(G_{m+1} - G_m) \ldots (G_n - G_m) . a^{(m)}_\kappa a^{(m)}_\lambda .$$

The new value obtained for the ratio $B^{(m)}_{\kappa\lambda} \div a^{(m)}_\kappa a^{(m)}_\lambda$ is thus the product of the differences got by subtracting G_m from all the other G's in succession. Nor is this all, for since

$$\Gamma = (G_1 - x)(G_2 - x) \ldots (G_m - x),$$

it follows that if we differentiate Γ with respect to x and subsequently put $x = G_m$, we shall obtain this very new value changed only in sign; so that the equation with which Jacobi legitimately closes § 9 is

$$a^{(m)}_\kappa a^{(m)}_\lambda = - \frac{B^{(m)}_{\kappa\lambda}}{\Gamma'_m}.$$

We now come to the section (§ 10, pp. 19–21) containing the notable result to which in his introductory pages Jacobi, as we have indicated, specially directs attention. Using the fact that $\sum_{\kappa\lambda} a_{\kappa\lambda} x_\kappa x_\lambda$ is transformed into

$$G_1 y_1^2 + G_2 y_2^2 + \ldots + G_n y_n^2$$

by the substitution

$$x_\kappa = a'_\kappa y_1 + a''_\kappa y_2 + \ldots + a^{(n)}_\kappa y_n,$$

he deduces the results

$$\sum_{\kappa\lambda} a_{\kappa\lambda} a^{(m)}_\kappa a^{(m')}_\lambda = 0,$$

$$\sum_{\kappa\lambda} a_{\kappa\lambda} a^{(m)}_\kappa a^{(m)}_\lambda = G_m;$$

the latter of which is more clearly comprehended and easily remembered by writing the original function V in the modern notation

$$\begin{array}{cccc|c}
x_1 & x_2 & x_3 & \dots & \\
\hline
a_{11} & a_{12} & a_{13} & \dots & x_1 \\
a_{21} & a_{22} & a_{23} & \dots & x_2 \\
a_{31} & a_{32} & a_{33} & \dots & x_3 \\
\cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \vdots
\end{array}$$

and noting that G_m is what this becomes when $a_1^{(m)}, a_2^{(m)}, \ldots, a_n^{(m)}$ are substituted for the x's. Then, preparatory to differentiating the second of the two results, he points out that the variation of the a's on the left-hand side of this result may be neglected, because, if it be not, the sum of all the terms involving differentials of the a's will be

$$\sum_{\kappa\lambda} a_{\kappa\lambda} \cdot \partial(a_\kappa^{(m)} a_\lambda^{(m)}),$$

and that this

$$\begin{aligned}
&= 2\sum_{\kappa\lambda} a_{\kappa\lambda} a_\kappa^{(m)} \partial a_\lambda^{(m)}, \\
&= 2\sum_\lambda \left[\partial a_\lambda^{(m)} \cdot \sum_\kappa a_{\kappa\lambda} a_\kappa^{(m)}\right]^* = 2\sum_\lambda \left[\partial a_\lambda^{(m)} \cdot G_m a_\lambda^{(m)}\right], \\
&= 2G_m \sum_\lambda a_\lambda^{(m)} \partial a_\lambda^{(m)}, \\
&= G_m \cdot \partial\left\{(a_1^{(m)})^2 + (a_2^{(m)})^2 + \ldots + (a_n^{(m)})^2\right\}, \\
&= 0.
\end{aligned}$$

Thus prepared and differentiating with respect to $a_{\kappa\lambda}$, we obtain at once

$$2a_\kappa^{(m)} a_\lambda^{(m)} = \frac{\partial G_m}{\partial a_{\kappa\lambda}}, \quad (\kappa \neq \lambda),$$

and $$a_\kappa^{(m)} a_\kappa^{(m)} = \frac{\partial G_m}{\partial a_{\kappa\kappa}},$$

—results which Jacobi deservedly styles "formulae perelegantes." As, however, we have another expression for $\dfrac{\partial G_m}{\partial a_{\kappa\lambda}}$, namely,

$$-\frac{\dfrac{\partial \Gamma_m}{\partial a_{\kappa\lambda}}}{\Gamma'_m}$$

* In the original λ is incorrectly placed below the second Σ.

where the numerator is an abbreviation for

$$\left(\frac{\partial \Gamma}{\partial a_{\kappa\lambda}}\right)_{x=\mathrm{G}_m}$$

it follows that

$$2a_\kappa^{(m)}a_\lambda^{(m)} = -\frac{\dfrac{\partial \Gamma_m}{\partial a_{\kappa\lambda}}}{\Gamma'_m} \quad \text{and} \quad a_\kappa^{(m)}a_\kappa^{(m)} = -\frac{\dfrac{\partial \Gamma_m}{\partial a_{\kappa\kappa}}}{\Gamma'_m};$$

and thence with the help of the last result of § 9,

$$2\mathrm{B}_{\kappa\lambda}^{(m)} = \frac{\partial \Gamma_m}{\partial a_{\kappa\lambda}} \quad \text{and} \quad \mathrm{B}_{\kappa\kappa}^{(m)} = \frac{\partial \Gamma_m}{\partial a_{\kappa\kappa}}$$

—a verification of a case of the general theorem regarding the differentiation of a determinant (see above, p. 212). The section closes with an extension of the result of § 7 regarding

$$\mathrm{G}_1^p y_1^2 + \mathrm{G}_2^p y_2^2 + \ldots. + \mathrm{G}_n^p y_n^2.$$

In § 11, the last which concerns our present subject, Jacobi brings himself into touch with Cauchy's starting-point, namely, the problem of finding the extreme values of $\Sigma a_{\kappa\lambda}x_\kappa x_\lambda$. The other sections (§§ 12–16) deal with special forms of V.

Passing over these and the 17 pages devoted to Problem ii. concerning the related subject of the transformation of multiple integrals, we find a return made to the original purely-algebraical subject, the new problem (iii.) being more general than the first in that for the condition

$$x_1^2 + x_2^2 + \ldots. + x_n^2 = y_1^2 + y_2^2 + \ldots. + y_n^2$$

there is substituted

$$\sum_{\kappa\lambda} c_{\kappa\lambda}x_\kappa x_\lambda = \mathrm{H}_1 y_1^2 + \mathrm{H}_2 y_2^2 + \ldots. + \mathrm{H}_n y_n^2.$$

The investigation (pp. 51–57) does not, however, so far as determinants are concerned, contain any new departure.

LEBESGUE (1837).

[Thèses de Mécanique et d'Astronomie. *Journ. (de Liouville) de Math.* ii. pp. 337–355.]

Of the two parts into which Lebesgue's memoir is divided it is the first which concerns us, the sub-title being 'Formules pour la transformation des fonctions homogènes du second degré à plusieurs inconnues.' The authorities cited are Cauchy, Sturm and Jacobi, and little credit is taken for novelty. All the same the exposition is singularly clear and elegant.

The given function $\Sigma A_{rs}x_r x_s$ is written in the form

$$\begin{aligned} & x_1(A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n) \\ + \; & x_2(A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n) \\ + \; & \ldots\ldots\ldots\ldots \\ + \; & x_n(A_{n1}x_1 + A_{n2}x_2 + \ldots + A_{nn}x_n), \end{aligned}$$

where $A_{\alpha\beta} = A_{\beta\alpha}$; and the substitution

$$x_r = a_{r1}y_1 + a_{r2}y_2 + \ldots + a_{rn}y_n \qquad (r = 1, 2, \ldots, n)$$

being made, the result is necessarily taken to be of the form

$$\begin{aligned} & y_1(B_{11}y_1 + B_{12}y_2 + \ldots + B_{1n}y_n) \\ + \; & y_2(B_{21}y_1 + B_{22}y_2 + \ldots + B_{2n}y_n) \\ + \; & \ldots\ldots\ldots\ldots \\ + \; & y_n(B_{n1}y_1 + B_{n2}y_2 + \ldots + B_{nn}y_n). \end{aligned}$$

As for the values of the B's, if for shortness' sake there be put

$$C_{i\alpha} \quad \text{for} \quad A_{i1}a_{1\alpha} + A_{i2}a_{2\alpha} + \ldots + A_{in}a_{n\alpha},$$

it is found that

$$\begin{aligned} B_{\alpha\alpha} &= a_{1\alpha}C_{1\alpha} + a_{2\alpha}C_{2\alpha} + \ldots + a_{n\alpha}C_{n\alpha}, \\ B_{\beta\alpha} &= a_{1\beta}C_{1\alpha} + a_{2\beta}C_{2\alpha} + \ldots + a_{n\beta}C_{n\alpha}, \\ B_{\alpha\beta} &= a_{1\alpha}C_{1\beta} + a_{2\alpha}C_{2\beta} + \ldots + a_{n\alpha}C_{n\beta}, \end{aligned}$$

and, that, because of $A_{\alpha\beta}$ and $A_{\beta\alpha}$ being identical,

$$B_{\alpha\beta} = B_{\beta\alpha}.$$

Should it be desired to have the result of transformation in the form

$$U_1y_1^2 + U_2y_2^2 + \ldots + U_ny_n^2$$

it is necessary to put

$$\begin{array}{lllll} B_{11} = U_1, & B_{12} = 0, & B_{13} = 0, & \ldots, & B_{1n} = 0, \\ B_{21} = 0, & B_{22} = U_2, & B_{23} = 0, & \ldots, & B_{2n} = 0, \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ B_{n1} = 0, & B_{n2} = 0, & B_{n3} = 0, & \ldots, & B_{nn} = U_n, \end{array}$$

a set of equations of which only $\frac{1}{2}(n^2+n)$ are distinct, but in which are involved double that number of unknowns, namely the n^2 a's and the n U's. "On doit donc," says Lebesgue, "encore se donner $\frac{1}{2}(n^2+n)$ équations entre les inconnues, afin d'ôter au problème son indétermination. Il est bien de manières d'obtenir ces nouvelles relations." Taking, first, for this purpose the condition

$$x_1^2 + x_2^2 + \ldots + x_n^2 = y_1^2 + y_2^2 + \ldots + y_n^2,$$

and deducing the said relations, he then shows how by partitioning the thus completed set of n^2+n equations into n sets of $n+1$ equations each, it is possible to find the values of the unknowns $n+1$ at a time. The passage is (p. 341):—

"Par exemple, si l'on veut obtenir le système qui donnera la valeur des $n+1$ inconnues

$$\mathrm{U}_\alpha,\ a_{1\alpha},\ a_{2\alpha},\ \ldots,\ a_{n\alpha}$$

on prendra les équations

$$\mathrm{B}_{1\alpha} = 0,\quad \mathrm{B}_{2\alpha} = 0,\ \ldots,\ \mathrm{B}_{\alpha\alpha} = \mathrm{U}_\alpha,\ \ldots,\ \mathrm{B}_{n\alpha} = 0,$$

auxquelles on joindra l'équation

$$a_{1\alpha}^2 + a_{2\alpha}^2 + \ldots + a_{n\alpha}^2 = 1.$$

Les n premières équations reviennent à

$$\begin{array}{l} a_{11}\mathrm{C}_{1\alpha} + a_{21}\mathrm{C}_{2\alpha} + \ldots + a_{n1}\mathrm{C}_{n\alpha} = 0, \\ a_{12}\mathrm{C}_{1\alpha} + a_{22}\mathrm{C}_{2\alpha} + \ldots + a_{n2}\mathrm{C}_{n\alpha} = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{1\alpha}\mathrm{C}_{1\alpha} + a_{2\alpha}\mathrm{C}_{2\alpha} + \ldots + a_{n\alpha}\mathrm{C}_{n\alpha} = \mathrm{U}_\alpha, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{1n}\mathrm{C}_{1\alpha} + a_{2n}\mathrm{C}_{2\alpha} + \ldots + a_{nn}\mathrm{C}_{n\alpha} = 0, \end{array}$$

d'où l'on tire très facilement

$$\mathrm{C}_{1\alpha} = a_{1\alpha}\mathrm{U}_\alpha,\quad \mathrm{C}_{2\alpha} = a_{2\alpha}\mathrm{U}_\alpha,\ \ldots,\quad \mathrm{C}_{n\alpha} = a_{n\alpha}\mathrm{U}_\alpha;$$

la première, $\mathrm{C}_{1\alpha} = a_{1\alpha}\mathrm{U}_\alpha$, s'obtient en multipliant les équations précédentes par a_{11}, a_{12}, ..., a_{1n} (coefficients de $\mathrm{C}_{1\alpha}$) respectivement, et en faisant la somme des résultats. Les autres s'obtiennent d'une manière toute semblable. Remplaçant $C_{1\alpha}$, $C_{2\alpha}$, ..., $C_{n\alpha}$ par leurs valeurs, on aura donc définitivement le système

$$\begin{array}{rcrcl}
(A_{11}-U_\alpha)a_{1\alpha} + & A_{12}a_{2\alpha} + \ldots + & A_{1n}a_{n\alpha} & = 0, \\
A_{21}a_{1\alpha} + & (A_{22}-U_\alpha)a_{2\alpha} + \ldots + & A_{2n}a_{n\alpha} & = 0, \\
A_{31}a_{1\alpha} + & A_{32}a_{2\alpha} + \ldots + & A_{3n}a_{n\alpha} & = 0, \\
\cdots & \cdots & \cdots & \\
A_{n1}a_{1\alpha} + & A_{n2}a_{2\alpha} + \ldots + & (A_{nn}-U_\alpha)a_{n\alpha} & = 0,
\end{array}$$

$$a_{1\alpha}^2 + a_{2\alpha}^2 + \ldots + a_{n\alpha}^2 = 1,$$

dont la solution n'offre d'autre difficulté que la simplification des résultats auxquels conduit l'élimination."

Here a digression (§ ii.) is naturally made into the subject of determinants (see above, pp. 301–303), after which (in § iii.) the reality of the roots of the equation in U_α or u (namely, $U=0$) is considered, this being done in three steps: (1) when $n=2$; (2) when all the A's vanish except A_{11}, A_{22}, ..., A_{nn}, A_{1n}, A_{2n}, ..., $A_{n-1,n}$; (3) when $n=m$, the previous case $n=m-1$ having been already established.*

The differential expressions for the coefficients of the substitution are then found, the starting point being the equation

$$\frac{a_{\kappa\alpha}}{a_{n\alpha}} = (-1)^{n+k}\frac{[n,\kappa]}{[n,n]}$$

from § ii. By putting $[n,n].[i,i]$ for $[n,i]^2$ and using the equation

$$a_{1\alpha}^2 + a_{2\alpha}^2 + \ldots + a_{n\alpha}^2 = 1$$

there is obtained

$$a_{n\alpha}^2 = \frac{[n,n]}{[1,1]+[2,2]+\ldots+[n,n]},$$

and generally

$$a_{\kappa\alpha}^2 = \frac{[\kappa,\kappa]}{[1,1]+[2,2]+\ldots+[n,n]}.$$

Since, however, § ii. gives us $[n,\kappa]$ as a differential-quotient, and since by reason of the special form of U we know that

$$-\frac{dU}{du} = \frac{dU}{dA_{11}} + \frac{dU}{dA_{22}} + \ldots + \frac{dU}{dA_{nn}},$$

* Lebesgue says this proof is essentially the same as Poisson's for the case $n=3$, reference being made to the latter's memoir of the year 1834 in *Mém. de l'acad. roy. des. sci. ...* (Paris), xiv. pp. 275–432.

it follows that

$$\frac{a_{\kappa\alpha}}{a_{n\alpha}} = \frac{1}{2}\frac{dU}{dA_{\kappa n}} \div \frac{dU}{dA_{nn}}$$

and

$$a_{\kappa\alpha}^2 = -\frac{dU}{dA_{\kappa\kappa}} \div \frac{dU}{du},$$

in the latter of which u is to be changed into U_α after differentiation.

The remaining section (§ iv.) concerns a second mode of obtaining $\frac{1}{2}(n^2+n)$ equations to make the original problem determinate, namely, from laying down the condition

$$x_1^2 + x_2^2 + \ldots + x_{n-1}^2 - x_n^2 = y_1^2 + y_2^2 + \ldots + y_{n-1}^2 - y_n^2.$$

It has no present interest.

CATALAN (1839).

[Sur la transformation des variables dans les intégrales multiples. *Mémoires couronnés par l'Acad. .... de Bruxelles,* xiv. 2e partie, 47 pp.]

After his introduction on the solution of a set of linear equations (see above, pp. 224–226) which he writes in the form

$$\left.\begin{array}{l} a_1x_1 + b_1x_2 + c_1x_3 + \ \ldots\ + h_1x_{n-1} + l_1x_n = \alpha_1 \\ a_2x_1 + b_2x_2 + c_2x_3 + \ \ldots\ + h_2x_{n-1} + l_2x_n = \alpha_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_nx_1 + b_nx_2 + c_nx_3 + \ \ldots\ + h_nx_{n-1} + l_nx_n = \alpha_n \end{array}\right\}$$

he sets himself to consider the special case where the n^2 coefficients are connected by the $\frac{1}{2}n(n-1)$ relations

$$\begin{array}{l} a_1b_1 + a_2b_2 + a_3b_3 + \ \ldots\ + a_nb_n = 0, \\ a_1c_1 + a_2c_2 + a_3c_3 + \ \ldots\ + a_nc_n = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_1l_1 + a_2l_2 + a_3l_3 + \ \ldots\ + a_nl_n = 0, \\ b_1c_1 + b_2c_2 + b_3c_3 + \ \ldots\ + b_nc_n = 0, \\ b_1d_1 + b_2d_2 + b_3d_3 + \ \ldots\ + b_nd_n = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_1l_1 + b_2l_2 + b_3l_3 + \ \ldots\ + b_nl_n = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ k_1l_1 + k_2l_2 + k_3l_3 + \ \ldots\ + k_nl_n = 0, \end{array}$$

with the object "de trouver d'autres relations entre ces coefficients."

In the first place, following, as he says, Poisson and Lacroix, he squares both sides of each of the given equations, and adds, thus obtaining

$$\sum_1^n a_i^2 = \mathrm{A}x_1^2 + \mathrm{B}x_2^2 + \ldots + \mathrm{L}x_n^2,$$

if for shortness' sake there be put

$$\mathrm{A} = \sum_1^n a_i^2, \quad \mathrm{B} = \sum_1^n b_i^2, \quad \ldots, \quad \mathrm{L} = \sum_1^n l_i^2.$$

In the second place, multiplying both sides of each equation by the coefficient of x_r in it, and adding, he obtains in succession the equations

$$\begin{aligned} \mathrm{A}x_1 &= a_1 a_1 + a_2 a_2 + \ldots + a_n a_n\,, \\ \mathrm{B}x_2 &= b_1 a_1 + b_2 a_2 + \ldots + b_n a_n\,, \\ &\cdots\cdots\cdots\cdots \\ \mathrm{L}x_n &= l_1 a_1 + l_2 a_2 + \ldots + l_n a_n\,, \end{aligned}$$

which constitute of course the solution of the original set.

In the third place he treats the derived set as the original set was first treated, save that he divides by A, B, ..., L respectively before performing the addition. This enables him to put his result in the form

$$\begin{aligned} \mathrm{A}x_1^2 + \mathrm{B}x_2^2 + \ldots + \mathrm{L}x_n^2 = &\sum_1^n \left(\frac{a_i^2}{\mathrm{A}} + \frac{b_i^2}{\mathrm{B}} + \ldots + \frac{l_i^2}{\mathrm{L}}\right) a_i^2 \\ &+ \sum_1^n \left(\frac{a_i a_j}{\mathrm{A}} + \frac{b_i b_j}{\mathrm{B}} + \ldots + \frac{l_i l_j}{\mathrm{L}}\right) a_i a_j. \end{aligned}$$

In the fourth place, taking advantage of the fact that the first and third results have a member in common, he equates the other members, and thence concludes that

$$\left.\begin{aligned} \frac{a_1^2}{\mathrm{A}} + \frac{b_1^2}{\mathrm{B}} + \frac{c_1^2}{\mathrm{C}} + \ldots + \frac{l_1^2}{\mathrm{L}} &= 1 \\ \frac{a_2^2}{\mathrm{A}} + \frac{b_2^2}{\mathrm{B}} + \frac{c_2^2}{\mathrm{C}} + \ldots + \frac{l_2^2}{\mathrm{L}} &= 1 \\ \cdots\cdots\cdots\cdots & \\ \frac{a_n^2}{\mathrm{A}} + \frac{b_n^2}{\mathrm{B}} + \frac{c_n^2}{\mathrm{C}} + \ldots + \frac{l_n^2}{\mathrm{L}} &= 1 \end{aligned}\right\}$$

and

$$\left.\begin{array}{l}
\dfrac{a_1a_2}{\mathrm{A}} + \dfrac{b_1b_2}{\mathrm{B}} + \ldots + \dfrac{l_1l_2}{\mathrm{L}} = 0 \\
\dfrac{a_1a_3}{\mathrm{A}} + \dfrac{b_1b_3}{\mathrm{B}} + \ldots + \dfrac{l_1l_3}{\mathrm{L}} = 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\dfrac{a_1a_n}{\mathrm{A}} + \dfrac{b_1b_n}{\mathrm{B}} + \ldots + \dfrac{l_1l_n}{\mathrm{L}} = 0 \\
\dfrac{a_2a_3}{\mathrm{A}} + \dfrac{b_2b_3}{\mathrm{B}} + \ldots + \dfrac{l_2l_3}{\mathrm{L}} = 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\dfrac{a_2a_n}{\mathrm{A}} + \dfrac{b_2b_n}{\mathrm{B}} + \ldots + \dfrac{l_2l_n}{\mathrm{L}} = 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\dfrac{a_{n-1}a_n}{\mathrm{A}} + \dfrac{b_{n-1}b_n}{\mathrm{B}} + \ldots + \dfrac{l_{n-1}l_n}{\mathrm{L}} = 0
\end{array}\right\}.$$

These are additional relations of the kind sought, the number of them being $n+\frac{1}{2}n(n-1)$, that is, $\frac{1}{2}n(n+1)$. At this point opportunity is taken to effect contact with the work of previous writers by means of the sentence "Ordinairement, dans les problèmes de mécanique, on suppose les quantités A, B, ..., L égales á l'unité: et alors les formules ci-dessus se simplifient considérablement."

In the fifth place he takes the ordinary solution of the original set of equations, that is to say, denoting the determinant of the coefficients by Δ and the cofactor of a_r in Δ by D_r, he obtains

$$x_1 = \frac{a_1\mathrm{D}_1 + a_2\mathrm{D}_2 + \ldots + a_n\mathrm{D}_n}{\Delta}.$$

This he compares with the first line of his second result, and deduces

$$\frac{\mathrm{D}_1}{a_1} = \frac{\mathrm{D}_2}{a_2} = \frac{\mathrm{D}_3}{a_3} = \ldots = \frac{\mathrm{D}_n}{a_n} = \frac{\Delta}{\mathrm{A}},$$

and thence

$$\begin{aligned}\mathrm{D}_1^2 + \mathrm{D}_2^2 + \ldots + \mathrm{D}_n^2 &= \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\cdot\frac{\Delta^2}{\mathrm{A}^2}, \\ &= \frac{\Delta^2}{\mathrm{A}}.\end{aligned}$$

Lastly he devotes four pages to establishing in an unconvincing manner* an immediate result of Binet's multiplication-theorem. Thus, according to Binet but in a later notation,

$$\begin{vmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \\ f_1 & f_2 & f_3 & f_4 & f_5 \end{vmatrix}^2 \quad \text{or} \quad \begin{vmatrix} \Sigma d_1^2 & \Sigma d_1e_1 & \Sigma d_1f_1 \\ \Sigma d_1e_1 & \Sigma e_1^2 & \Sigma e_1f_1 \\ \Sigma d_1f_1 & \Sigma e_1f_1 & \Sigma f_1^2 \end{vmatrix}$$

$$= |d_1e_2f_3|^2 + |d_1e_2f_4|^2 + \ldots + |d_3e_4f_5|^2.$$

Should it be given that

$$\begin{aligned} \Sigma d_1e_1 \quad i.e. \quad d_1e_1 + d_2e_2 + \ldots + d_5e_5 &= 0, \\ d_1f_1 + d_2f_2 + \ldots + d_5f_5 &= 0, \\ e_1f_1 + e_2f_2 + \ldots + e_5f_5 &= 0, \end{aligned}$$

the product-determinant reduces to its diagonal term, and we have an instance of Catalan's result, namely, the sum of the squares of ten ($C_{5,3}$) three-line determinants equal to the product of three sums of five squared elements. A special case of the result is of course

$$\Delta^2 = A \,.\, B \,.\, C \ldots L$$

which permits a previous theorem to be changed into

$$D_1^2 + D_2^2 + \ldots + D_n^2 = B \,.\, C \ldots L.$$

POSTSCRIPT.

Le 'dernier volume' referred to on p. 434 turns out to be the *ninth*; and the title of the memoir, which does not extend to three pages (pp. 111–113), is "L'équation qui a pour racines les moments d'inertie principaux d'un corps solide, et sur diverses équations du même genre." As it was read to the Academy on 20th November 1826, and therefore preceded all Jacobi's papers

* The so-called property of general determinants which Catalan uses as his foundation is, when accurately stated, a truism. The 'demonstration' is vitiated by an oversight, in regard to signs, similar to that made in connection with a 'permutation tournante' in the opening portion of the memoir, namely, he takes

$$d_4|e_1f_2g_3| + e_4|f_1g_2d_3| + f_4|g_1d_2e_3| + g_4|d_1e_2f_3|$$

as the equivalent of

$$|d_4e_1f_2g_3|.$$

on the subject, it deserves very special attention. The second theorem enunciated in it is:

"Si l'on nomme s la somme des carrés de n variables indépendantes $x, y, z, u, \ldots$ et r une fonction homogène du second degré, composée avec ces mêmes variables, et si l'on cherche les valeurs *maximum* ou *minimum* du rapport $\frac{r}{s}$, la détermination de ces valeurs dépendra d'une équation du n^{me} degré dont toutes les racines sont réelles."

To this Cauchy adds the remark that the method followed in proving it had led him to other propositions, and he quotes one, namely, that given above on p. 434, adding:

"Le dernier théorème entraine évidemment plusieurs relations entre les coefficients des équations linéaires par lesquelles les variables $\xi, \eta, \zeta, \ldots$ sont liées aux variables $x, y, z, \ldots$. Ces relations sont semblables à celles qui existent entre les cosinus des angles que forment trois axes rectangulaires donnés avec les axes des coordonnées supposés eux-mêmes rectangulaires."

CHAPTER XVI.

MISCELLANEOUS SPECIAL FORMS FROM 1811 TO 1841.

THERE now only remains to deal with those special forms which prior to 1841 had not excited much interest, and whose properties had consequently been little investigated. The most fertile originator of such forms was Wronski: unfortunately he had only one follower, and still more unfortunately the work of this follower, Schweins, was almost immediately lost sight of and remained of none effect until 1884 (see above, p. 175). Others, whose similar contributions fall to be noted, are Scherk, Jacobi and Sylvester. The writings will not be grouped according to subjects, but will be taken in order of date.

WRONSKI (1812).

[Réfutation de la Théorie des Fonctions Analytiques de Lagrange. Dediée à l'Institut impérial de France. 136 pp. Paris.]

As has already been pointed out (see above, pp. 78–79) Wronski's first mention of 'sommes combinatoires' was in connection with a special form of them. The form was not the product of fancy: it made its appearance, like so many others, when a set of linear equations called for solution in the course of an attack on a seemingly unconnected problem. This is important to have noted, and it is made quite clear by a note appended to the highly controversial 'Réfutation' and bearing the title "Sur la démonstration de la loi générale des séries, servant de principe à cet ouvrage." The law itself is stated as follows (p. 15):

"Or, si Fx est la fonction qu'il s'agit de développer en série, ϕx la fonction arbitraire qu'on prend, dans la série, pour la mesure algorithmique de la fonction proposée Fx, c'est-à-dire, pour la fonction génératrice* du développement ; et si, de plus, on considère les séries dans leur plus grande généralité, d'après la forme donnée plus haut, savoir,

$$\mathrm{F}x = \mathrm{A}_0 + \mathrm{A}_1 \,.\, \phi x + \mathrm{A}_2 \,.\, \phi x^{2/\xi} + \mathrm{A}_3 \,.\, \phi x^{3/\xi} + \ldots .$$

qui procède suivant les facultés progressives ϕx, $\phi x^{2/\xi}$, $\phi x^{3/\xi}$, etc., de la fonction génératrice, l'accroissement ξ étant arbitraire, on aura, pour la détermination des coefficients A_1, A_2, A_3, etc., exprimés généralement par A_μ, μ étant un indice quelconque depuis *un* jusqu'à l'*infini*, la loi

$$\mathrm{A}_\mu = \frac{\text{ש}[\Delta^a\phi x \,.\, \Delta^b\phi x^{2/\xi} \,.\, \Delta^c\phi x^{3/\xi} \ldots . \Delta^l\phi x^{(m-1)/\xi} \,.\, \Delta^m \mathrm{F}x]}{\Delta^a\phi x \,.\, \Delta^b\phi x^{2/\xi} \,.\, \Delta^c\phi x^{3/\xi} \ldots . \Delta^l\phi x^{(m-1)/\xi} \,.\, \Delta^m\phi x^{\mu/\xi}},$$

en observant de faire égal à ξ l'accroissement dont dépendent les différences, les valeurs

$$a = 1, \quad b = 2, \quad c = 3, \quad d = 4, \quad \ldots ., \quad l = \mu - 1, \quad m = \mu$$

et à la variable x, une valeur telle que $\phi x = 0$. Quant à la quantité A_0, il faut savoir que, suivant la loi de continuité de ces fonctions, on a $\mathrm{A}_0 = \mathrm{F}x$, en donnant toujours à la variable x la valeur qui résulte de la relation $\phi x = 0$."

No demonstration of the law was given in the original communication to the Institute. That supplied in the 'Réfutation' (pp. 131–133) proceeds as follows:

"Prenant donc, des deux membres ... les différences des ordres successifs 1, 2, 3, 4, etc., et donnant ensuite à x la valeur qui réduit à zéro le facteur ϕx, nous aurons, en vertu de l'expression (91) [*i.e.* $\Delta^\mu\phi x^{\omega/\xi} = 0$], la suite indéfinie d'équations

$$\begin{aligned} \Delta \mathrm{F}x &= \mathrm{A}_1 \,.\, \Delta\phi x, \\ \Delta^2 \mathrm{F}x &= \mathrm{A}_1 \,.\, \Delta^2\phi x + \mathrm{A}_2 \,.\, \Delta^2\phi x^{2/\xi}, \\ \Delta^3 \mathrm{F}x &= \mathrm{A}_1 \,.\, \Delta^3\phi x + \mathrm{A}_2 \,.\, \Delta^3\phi x^{2/\xi} + \mathrm{A}_3 \,.\, \Delta^3\phi x^{3/\xi}, \\ & \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

La première de ces équations donne immédiatement

$$\mathrm{A}_1 = \frac{\Delta \mathrm{F}x}{\Delta\phi x}.$$

En second lieu, puisqu'en vertu de l'expression (91) on a $\Delta\phi x^{2/\xi} = 0$, les deux premières des équations précédentes sont identiques avec celles-ci

$$\left.\begin{aligned} \Delta \mathrm{F}x &= \mathrm{A}_1 \,.\, \Delta\phi x + \mathrm{A}_2 \,.\, \Delta\phi x^{2/\xi} \\ \Delta^2 \mathrm{F}x &= \mathrm{A}_1 \,.\, \Delta^2\phi x + \mathrm{A}_2 \,.\, \Delta^2\phi x^{2/\xi} \end{aligned}\right\},$$

* In Wronski's own use of the word.

équations qui donnent de même immédiatement

$$A_2 = \frac{\text{ש}[\Delta^1\phi x \,.\, \Delta^2 Fx]}{\text{ש}[\Delta^1\phi x \,.\, \Delta^2\phi x^{2/\xi}]}.\text{"}$$

In similar manner A_3 is found from the equations:

$$\left.\begin{array}{l} \Delta\, Fx = A_1.\Delta\, \phi x + A_2.\Delta\, \phi x^{2/\xi} + A_3.\Delta\, \phi x^{3/\xi} \\ \Delta^2 Fx = A_1.\Delta^2\phi x + A_2.\Delta^2\phi x^{2/\xi} + A_3.\Delta^2\phi x^{3/\xi} \\ \Delta^3 Fx = A_1.\Delta^3\phi x + A_2.\Delta^3\phi x^{2/\xi} + A_3.\Delta^3\phi x^{3/\xi}, \end{array}\right\}$$

and the proof is considered complete when it is pointed out that "la somme combinatoire formant le dénominateur se réduit à son premier terme."

On leaving the matter it may be as well to note that this first special determinant-form of Wronski's is not only specialized in having differences for its elements, but in having zeros in all the places included between the diagonal and the last column. Also, attention may be called to the fact that on page 33 he has an instance in which 'differentials' take the place of 'differences' as elements.

WRONSKI (1815).

[Philosophie de la Technie Algorithmique. Première Section, contenant la loi suprême et universelle de mathématiques. xii + 286 pp. Paris.]

Instead of the "loi générale des séries" we have now the much more extensive "loi suprême," which Wronski writes in the form

$$Fx = A_0.\Omega_0 + A_1.\Omega_1 + A_2.\Omega_2 + \ldots,$$

and which appears inscribed on the pedestal of the androsphinx adopted later by him as an authenticating stamp for his works. Notwithstanding the increased generality, the Ω's being now any functions of x whatever,* the law of formation of the A's is expressed by means of the same kind of 'schin' functions as

*The interestingly guarded report of Lagrange and Lacroix on Wronski's first memoir to the Institute (see *Gazette Nationale* for 15th June, 1810) shows that the statement

$$F(x) = A_0\Omega_0 + A_1\Omega_1 + A_2\Omega_2 + \ldots$$

however vague and undefined occurred in that memoir.

before. In the short exposition (pp. 175–182) given of the latter functions preparatory to the treatment of the "loi suprême" there is therefore little new to be expected; and as a matter of fact it is only the last sentence that is worth transcribing, the reason being that it brings out a conspicuous width of view in connection with the functions. The words are:

"Il faut enfin observer que si la caractéristique Δ, au lieu de désigner les différences prises sur les fonctions $X_1, X_2, X_3, \ldots$ dénotait tout autre système de fonctions algorithmiques prises sur les mêmes fonctions $X_1, X_2, X_3, \ldots$ tout ce que nous venons de dire concernant les fonctions schins, se trouverait également vrai."

It may be added that in a later part of the work specimens of such 'schin' functions actually appear, *e.g.*

$$\text{ש}[\Omega_1 a_1 . \Omega_2 a_2 . \Omega_3 a_3 . \Omega_\mu a_\mu],$$

$$\text{ש}[\Phi(\mu+\beta_1)_{\mu+1} . \Phi(\mu+\beta_2)_{\mu+2} . \Phi(\mu+\beta_3)_{\mu+3} \ldots .].$$

The demonstration of the law occupies fifty pages (pp. 188–238), and is preceded by the intimation that it depends on a higher theory of the 'schin' functions and that consequently this theory, reduced to a lemma and a theorem and three corollaries, must first be dealt with. This is done at considerable length, thirty-four of the fifty pages being so occupied. The most important portion to be reproduced is the enunciation of the said theorem, which stands as follows (p. 193):

"Soient $Y_0, Y_1, Y_0, \ldots, Y_\omega$ des fonctions d'une variable x, et soit pour cette fois, Δ la caractéristique des différences prises à volonté, suivant la voie progressive ou la voie régressive, par rapport à un accroissement quelconque de la variable x. Si, avec ces fonctions, on construit, d'une part, les quantités

$$\begin{aligned} X_1 &= \text{ש}[\Delta^{\delta_0} Y_0 . \Delta^{\delta_1} Y_1], \\ X_2 &= \text{ש}[\Delta^{\delta_0} Y_0 . \Delta^{\delta_1} Y_2], \\ X_3 &= \text{ש}[\Delta^{\delta_0} Y_0 . \Delta^{\delta_1} Y_3], \\ &\cdots\cdots\cdots \\ X_\omega &= \text{ש}[\Delta^{\delta_0} Y_0 . \Delta^{\delta_1} Y_\omega]; \end{aligned}$$

et, d'une autre part, les quantités

$$\begin{aligned} T_1 &= \Delta^{\delta_0} Y_0 + i . \Delta^{\delta_1} Y_0, \\ T_2 &= \Delta^{\delta_0} Y_0 + 2i . \Delta^{\delta_1} Y_0 + i^2 . \Delta^{\delta_2} Y_0, \\ T_3 &= \Delta^{\delta_0} Y_0 + 3i . \Delta^{\delta_1} Y_0 + 3i^2 . \Delta^{\delta_2} Y_0 + i^3 . \Delta^{\delta_3} Y_0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

et généralement, pour un indice quelconque ρ,

$$T_\rho = \Delta^{\delta_0}Y_0 + \frac{\rho}{1} i \,.\, \Delta^{\delta_1}Y_0 + \frac{\rho(\rho-1)}{1\,.\,2} i^2 \,.\, \Delta^{\delta_2}Y_0$$
$$+ \frac{\rho(\rho-1)(\rho-2)}{1\,.\,2\,.\,3} i^3 \,.\, \Delta^{\delta_3}Y_0 + \ldots,$$

en faisant $i = +1$ lorsque les différences Δ sont prises suivant la voie progressive, et $i = -1$ lorsque ces différences sont prises suivant la voie régressive; on aura la relation d'égalité

$$\mathfrak{W}[\Delta^{\beta_1}X_1 \,.\, \Delta^{\beta_2}X_2 \,.\, \Delta^{\beta_3}X_3 \ldots \Delta^{\beta_\omega}X_0]$$
$$= (T_1 \,.\, T_2 \,.\, T_3 \ldots T_{\omega-1}) \,.\, \mathfrak{W}[\Delta^{\delta_0}Y_0 \,.\, \Delta^{\delta_1}Y_1 \,.\, \Delta^{\delta_2}Y_2 \ldots \Delta^{\delta_\omega}Y_\omega];$$

en donnant aux exposans $\beta_1, \beta_2, \beta_3, \ldots, \beta_\omega$ et $\delta_0, \delta_1, \delta_2, \ldots, \delta_\omega$ de la permutation desquels dépendent ici les fonctions schins, les valeurs suivantes

$$\beta_1 = 0, \quad \beta_2 = 1 + \beta_1, \quad \beta_3 = 1 + \beta_2, \quad \ldots, \quad \beta_\nu = 1 + \beta_{\nu-1} = \nu - 1,$$
$$\delta_0 = \delta, \quad \delta_1 = 1 + \delta_0, \quad \delta_2 = 1 + \delta_1, \quad \ldots, \quad \delta_\nu = 1 + \delta_{\nu-1} = \delta + 1;$$

δ étant un nombre entier quelconque, et ν un indice arbitraire."

WRONSKI (1816-1817).

[Philosophie de la Technie Algorithmique. Seconde Section, contenant les lois des séries comme préparation à la réforme des mathématiques. xx+646 pp. Paris.]

The contents of this larger volume are more or less methodically concerned with specializations from the "loi suprême," the word 'series' being used by Wronski in a way of his own which enables him to view all series as being derivable from

$$A_0\Omega_0 + A_1\Omega_1 + A_2\Omega_2 + \ldots.$$

by taking

$$\Omega_1 = \phi(x, a, b, c, \ldots)^{1/\xi,\, \alpha,\, \beta,\, \gamma, \ldots} = \phi(x, a, b, c, \ldots),$$

$$\Omega_2 = \phi(x, a, b, c, \ldots)^{2/\xi,\, \alpha,\, \beta,\, \gamma, \ldots} = \phi(x, a, b, c, \ldots)$$
$$.\, \phi(x+\xi, a+\alpha, b+\beta, c+\gamma, \ldots),$$

$$\Omega_3 = \phi(x, a, b, c, \ldots)^{3/\xi,\, \alpha,\, \beta,\, \gamma, \ldots} = \phi(x, a, b, c, \ldots)$$
$$.\, \phi(x+\xi, a+\alpha, b+\beta, c+\gamma, \ldots)$$
$$.\, \phi(x+2\xi, a+2\alpha, b+2\beta, c+2\gamma \ldots)$$

. . . .

This being the case it is natural that the ground covered by his earlier writings, namely, by his first paper to the Institute and by the 'Réfutation' should be partly retraversed, and that specialized forms of the 'schin' functions which form the numerators and denominators of the A's should have to be considered. Thus we at once come again upon the case where

$$\Omega_1 = \phi x, \quad \Omega_2 = \phi x^{2/\xi}, \quad \Omega_3 = \phi x^{3/\xi}, \quad \ldots .$$

and the sub-case where ξ is indefinitely small, and where therefore $\Omega_r = (\phi x)^r$. In the latter case it is shown that

$$A_\mu = \frac{\text{ש}[d^0\phi x^0 \,.\, d^1\phi x^1 \,.\, d^2\phi x^2 \ldots . \, d^{\mu-1}\phi x^{\mu-1} \,.\, d^\mu \mathrm{F}x]}{(1^{0/1} \,.\, 1^{1/1} \,.\, 1^{2/1} \ldots . \, 1^{\mu/1}) \,.\, (d\phi x)^{\frac{1}{2}\mu(\mu+)1}}$$

and further that the A's are then so related that

$$A_1 = \frac{1}{1 \,.\, d\phi x} \,.\, d\mathrm{F}x,$$

$$A_2 = \frac{1}{1^{2/1} \,.\, (d\phi x)^2} \,.\, \{d^2\mathrm{F}x - A_1 \,.\, d^2\phi x\},$$

$$A_3 = \frac{1}{1^{3/1} \,.\, (d\phi x)^3} \,.\, \{d^3 \,.\, \mathrm{F}x - A_1 \,.\, d^3\phi x - A_2 \,.\, d^3\phi x^2\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

in all of which, be it noted, x is ultimately to be given the value a which makes $\phi a = 0$.*

A little further on (p. 60) but in the same connection, and while considering Burmann's series, the result

$$\frac{(\mu-1)^{(\mu-1)/1}}{1^{(\mu-1)/1}} \cdot \frac{d^{\mu-1}\left(\dfrac{x-a}{\phi x}\right)^\mu}{dx^{\mu-1}} = (-1)^{\mu-1} \frac{\text{ש}[d^2\phi x \,.\, d^3\phi x^2 \ldots . \, d^\mu\phi x^{\mu-1}] \,.\, dx}{1^{(\mu-1)/1} \,.\, 1^{(\mu-2)/1} \ldots . \, 1^{1/1} \,.\, (d\phi x)^{\frac{1}{2}\mu(\mu+1)}}$$

is reached.

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* In a foot-note (p. 13) it is curious to find the identity

$$\begin{aligned} & \text{ש}[\Omega_0^{(0)}\Omega_1^{(1)} \ldots . \, \Omega_{\rho-2}^{(\rho-2)}\Omega_{\rho-1}^{(\rho-1)}\mathrm{F}^{(\rho)}] \,.\, \text{ש}[\Omega_0^{(0)}\Omega_1^{(1)} \ldots . \, \Omega_{\rho-2}^{(\rho-2)}\Omega_\rho^{(\rho-1)}] \\ = \; & \text{ש}[\Omega_0^{(0)}\Omega_1^{(1)} \ldots . \, \Omega_{\rho-2}^{(\rho-2)}\Omega_{\rho-1}^{(\rho-1)}\Omega_\rho^{(\rho)}] \,.\, \text{ש}[\Omega_0^{(0)}\Omega_1^{(1)} \ldots . \, \Omega_{\rho-2}^{(\rho-2)}\mathrm{F}^{(\rho-1)}] \\ & + \text{ש}[\Omega_0^{(0)}\Omega_1^{(1)} \ldots . \, \Omega_{\rho-2}^{(\rho-2)}\Omega_\rho^{(\rho-1)}\mathrm{F}^{(\rho)}] \,.\, \text{ש}[\Omega_0^{(0)}\Omega_1^{(1)} \ldots . \, \Omega_{\rho-2}^{(\rho-2)}\Omega_{\rho-1}^{(\rho-1)}], \end{aligned}$$

namely, the 'extensional' of

$$a_1|b_1c_2| - b_1|a_1c_2| + c_1|a_1b_2| = 0$$

reached two years later by Desnanot. This should have been noted in its proper place.

Still later (p. 399), when another form of ϕ is being dealt with, a quite different form of determinant makes its appearance, namely (p. 421) the determinant

$$\text{ש}\,[(1,0)(2,1)(3,2)\ldots\ldots(\omega,\,\omega-1)]$$

where $(\rho,\sigma)=0$ when $\sigma>\rho$. But although Wronski notes that it

$$= (1,0)\,.\,\text{ש}[(2,1)(3,2)\ldots(\omega,\omega-1)] - (1,1)\,\text{ש}[(2,0)(3,2)(4,3)\ldots(\omega,\omega-1)]$$

and evaluates it in a number of particular instances, he does not appear to have gone further.

WRONSKI (1819).

[Critique de la Théorie de Fonctions Génératrices de M. Laplace. iv + 136 pp. Paris.]

In this his last work on pure mathematics Wronski comes across (p. 67) still another special form of 'schin' function, viz.,

$$\text{ש}\,[n_1^{v_1}n_2^{v_2}n_3^{v_3}\ldots\ldots n_\mu^{v_\mu}]$$

where $v_1, v_2, v_3, \ldots\ldots$ are indices of powers. Again, however, in working with them he does not get beyond the use of the identities

$$|a_1b_2c_3d_4\ldots| = a_1|b_2c_3d_4\ldots| - a_2|b_1c_3d_4\ldots| + a_3|b_1c_2d_4\ldots| - a_4|b_1c_2d_3\ldots|$$
$$0 = b_1|b_2c_3d_4\ldots| - b_2|b_1c_3d_4\ldots| + b_3|b_1c_2d_4\ldots| - b_4|b_1c_2d_3\ldots|.$$

SCHERK (1825).

[Mathematische Abhandlungen. iv + 116 pp. Berlin. (pp. 31–66).]

The second portion of the appendix (§ 8) to the second of Scherk's memoirs (see above, pp. 150–159) concerns the solution of the set of equations

$$\begin{aligned}
\underset{1}{s} &= \overset{1}{\underset{11}{ax}}, \\
\underset{2}{s} &= \overset{1}{\underset{21}{ax}} + \overset{2}{\underset{22}{ax}}, \\
\underset{3}{s} &= \overset{1}{\underset{31}{ax}} + \overset{2}{\underset{32}{ax}} + \overset{3}{\underset{33}{ax}}, \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\underset{n}{s} &= \overset{1}{\underset{n1}{ax}} + \overset{2}{\underset{n2}{ax}} + \overset{3}{\underset{n3}{ax}} + \ \ldots\ldots\ + \overset{n}{\underset{nn}{ax}}.
\end{aligned}$$

This, it will be observed, is exactly the kind of set which Wronski obtained for the determination of his coefficients in the Third Note of the 'Réfutation.' As a consequence Scherk is led to seek for the final expansion of a special determinant, which we should nowadays write in the form

$$\begin{vmatrix} \overset{1}{\underset{1}{a}} & \cdot & \cdot \; \cdots & \underset{1}{s} \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & \cdot \; \cdots & \underset{2}{s} \\ \overset{1}{\underset{3}{a}} & \overset{2}{\underset{3}{a}} & \overset{3}{\underset{3}{a}} \; \cdots & \underset{3}{s} \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ \overset{1}{\underset{k}{a}} & \overset{2}{\underset{k}{a}} & \overset{3}{\underset{k}{a}} \; \cdots & \underset{k}{s} \end{vmatrix}.$$

His result is (p. 60)—

"Folglich ist der Zähler von x_k das Aggregat aller der Glieder, die entstehen, wenn man alle Permutationsformen aus den Grössen 1, 2, ..., $h-1$, $h+1$, ..., k, die so gebildet werden, dass nur 1 in der ersten, 2 nur in den beiden ersten, 3 nur in den 3 ersten, ... Stellen steht, bildet, diese nach einander unter $a^1\ a^2 \dots a^{k-1}$ setzt, mit s^1_h multiplicirt, und h nach einander die Werthe k, $k-1$, ..., 2, 1 giebt."

To provide a check on the calculation, he draws attention to the fact that the number of terms in the development is 2^{k-1}; and to establish this, he considers (p. 61) in succession the cases where $h=k$, $k-1$, $k-2$, $k-3$, $k-4$.

As an illustration both of the law of formation of the expression for x_k and of the check upon it, he takes $k=4$, giving

$$x_4 = \frac{a_1a_2a_3s_4 - a_1a_2a_4s_3 + a_1a_3a_4s_2 - a_1a_4a_3s_2 - a_2a_3a_4s_1 + a_2a_4a_3s_1 + a_3a_2a_4s_1 - a_4a_2a_3s_1}{a_1^1a_2^2a_3^3a_4^4}$$

and then specializes by finding the 4th Bernoulli number and the 4th coefficient of the secant-series.

SCHWEINS (1825).

[Theorie der Differenzen und Differentiale, ... vi+666 pp. Heidelberg.]

There can be little doubt that Schweins was one of the few men who were not daunted by the egotistic and exhaustingly wearisome style of Wronski's works. This appears not only from the fact that Wronski is repeatedly referred to by Schweins, but from a striking coincidence which occurs in connection with their study of determinants. As we have seen there are four special forms of those functions which are to be found in Wronski's writings, three of them having appeared there and nowhere else previously: and these four are exactly those which receive attention at the hands of Schweins. Therefore, while refusing to accept the estimate of the 'loi suprême' which its author in season and out of season insisted upon, let us not forget that some of the concepts which sprang from the hot brain of the poor Polish enthusiast provided material for exercising the industry of an exemplary German professor.

Taking the forms in the order in which Schweins deals with them, we have, then, first of all, Wronski's

$$\varpi[(1,0)(2,1)(3,2)\ldots.(\omega,\omega-1)], \quad (\rho,\sigma)_{\sigma>\rho}=0$$

or, in modern notation,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \ldots. \\ b_1 & b_2 & b_3 & b_4 & \ldots. \\ . & c_2 & c_3 & c_4 & \ldots. \\ . & . & d_3 & d_4 & \ldots. \\ . & . & . & e_4 & \ldots. \\ . & . & . & . & . \;\;.\;\;.\;\;. \end{vmatrix}.$$

This Schweins treats of in a portion of his treatise which we have called the fourth 'chapter' of the first Abtheilung (see above, p. 173), a chapter headed "Auflösung der Producte mit Versetzungen, in welchen einige Factoren verschwinden, in Producte bestehend in gedoppelten Verbindungen." His solution of the problem set for himself, namely, the finding of the final expansion of the determinant, is complete though clumsy and unpleasing in form, being stated as follows:—

$$\begin{aligned} \frac{\left\| \overset{(n)}{\mathrm{A}}_{n+1} \overset{(n+1)}{\mathrm{A}}_{n+2} \ldots. \overset{(n+m)}{\mathrm{A}}_{n+m+1} \right)}{\overset{(n)}{\mathrm{A}}_n \overset{(n+1)}{\mathrm{A}}_{n+1} \overset{(n+m)}{\mathrm{A}}_{n+m}} &= \mathrm{D}['n, (n+1, n+2, \ldots., n+m)^{(m)}, n+m+1] \\ &- \mathrm{D}['n, (n+1, n+2, \ldots., n+m)^{(m-1)}, n+m+1] \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &(-)^m \mathrm{D}['n, (n+1, n+2, \ldots., n+m)^{0}, n+m+1], \end{aligned}$$

where $\overset{(n)}{\mathrm{D}}_{n+s} = \dfrac{\overset{(n)}{\mathrm{A}}_{n+s}}{\overset{(n)}{\mathrm{A}}_{n}}$. The notation on the right may be understood from the example

$$\mathrm{D}['0, (1, 2, 3, 4, 5)^3, 6],$$

which stands for the ten-termed sum

$$\overset{0}{\mathrm{D}}_1\overset{1}{\mathrm{D}}_2\overset{2}{\mathrm{D}}_3\overset{3}{\mathrm{D}}_6 + \overset{0}{\mathrm{D}}_1\overset{1}{\mathrm{D}}_2\overset{2}{\mathrm{D}}_4\overset{4}{\mathrm{D}}_6 + \overset{0}{\mathrm{D}}_1\overset{1}{\mathrm{D}}_2\overset{2}{\mathrm{D}}_5\overset{5}{\mathrm{D}}_6 + \ldots. + \overset{0}{\mathrm{D}}_3\overset{3}{\mathrm{D}}_4\overset{4}{\mathrm{D}}_5\overset{5}{\mathrm{D}}_6,$$

the first part of the symbol, '0, indicating that the upper index of the first D is always to be 0, the second part, $(1, 2, 3, 4, 5)^3$, that the last three upper indices and the first three lower indices are to be those of a set chosen from 1, 2, 3, 4, 5, and the third part that 6 is to be always the lower index of the last D. Thus, taking the determinant of the 4th order, we have

$$\begin{vmatrix} a_{01} & a_{02} & a_{03} & a_{04} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ \cdot & a_{22} & a_{23} & a_{24} \\ \cdot & \cdot & a_{33} & a_{34} \end{vmatrix} \div a_{00}a_{11}a_{22}a_{33}$$

$$\begin{aligned} &= \overset{0}{d}_1\overset{1}{d}_2\overset{2}{d}_3\overset{3}{d}_4 - \left(\overset{0}{d}_1\overset{1}{d}_2\overset{2}{d}_4 + \overset{0}{d}_1\overset{1}{d}_3\overset{3}{d}_4 + \overset{0}{d}_2\overset{2}{d}_3\overset{3}{d}_4\right) \\ &\qquad + \left(\overset{0}{d}_1\overset{1}{d}_4 + \overset{0}{d}_2\overset{2}{d}_4 + \overset{0}{d}_3\overset{3}{d}_4\right) \\ &\qquad - \overset{0}{d}_4, \\ &= \frac{a_{01}a_{12}a_{23}a_{34}}{a_{00}a_{11}a_{22}a_{33}} - \left(\frac{a_{02}a_{23}a_{34}}{a_{00}a_{22}a_{33}} + \frac{a_{01}a_{13}a_{34}}{a_{00}a_{11}a_{33}} + \frac{a_{01}a_{12}a_{24}}{a_{00}a_{11}a_{22}}\right) \\ &\qquad + \left(\frac{a_{03}a_{34}}{a_{00}a_{33}} + \frac{a_{02}a_{24}}{a_{00}a_{22}} + \frac{a_{01}a_{14}}{a_{00}a_{11}}\right) \\ &\qquad - \frac{a_{04}}{a_{00}}; \end{aligned}$$

and therefore the determinant

$$= a_{01}a_{12}a_{23}a_{34} - a_{11} \cdot a_{02}a_{23}a_{34} - \ldots..$$

The forced introduction of a_{00} should be noted, and the object gained in doing so. The form is really the same as Scherk's.

The next special form taken up by Schweins is that to which the name *alternant* has since been assigned, and his contributions to the theory of it, occupying all the five chapters of the second Abtheilung, have been already recounted (see above, pp. 311–322).

The third form occupies similarly the whole of the third Abtheilung, but much less space is given to it, there being only one chapter of four pages. The title at once recalls the 'loi suprême'; it is "Producte mit Versetzungen, wenn die oberen Elemente höhere Unterschiede angeben." Beginning with the expression for the Δ of a product in terms of the Δ's of the factors, he deduces from it the Δ of the special determinant form in question, his contracted mode of writing the result being

$$\begin{aligned}
\Delta \| \Delta^a A_1 . \Delta^b A_2 . \Delta^c A_3 . \Delta^d A_4) \\
= \quad & \| \Delta^a A_1 . \Delta^b A_2 . \Delta^c A_3 . \Delta^d A_4) \\
& \begin{array}{cccc} a & b & c+1 & d \\ a & b+1 & c & d \\ a+1 & b & c & d \end{array} \\
+ & \| \Delta^a A_1 . \Delta^b A_2 . \Delta^{c+1} A_3 . \Delta^{d+1} A_4) \\
& \begin{array}{cccc} a & b+1 & c & d+1 \\ a & b+1 & c+1 & d \\ a+1 & b & c & d+1 \\ a+1 & b & c+1 & d \\ a+1 & b+1 & c & d \end{array} \\
+ & \| \Delta^a A_1 . \Delta^{b+1} A_2 . \Delta^{c+1} A_3 . \Delta^{d+1} A_4) \\
& \begin{array}{cccc} a+1 & b & c+1 & d+1 \\ a+1 & b+1 & c & d+1 \\ a+1 & b+1 & c+1 & d \end{array} \\
+ & \| \Delta^{a+1} A_1 . \Delta^{b+1} A_2 . \Delta^{c+1} A_3 . \Delta^{d+1} A_4),
\end{aligned}$$

where there has of course to be noted the large number of determinants which vanish when $b=a+1$, $c=b+1$, ... The only other matter is an investigation of the Δ of the Wronskian quotient

$$\frac{\| \Delta^a A_1 . \Delta^{a+1} A_2 \ldots . \Delta^{a+n-2} A_{n-1} . \Delta^{a+n-1} A_{n+1})}{\| \Delta^a A_1 . \Delta^{a+1} A_2 \ldots\ldots\ldots\ldots \Delta^{a+n-1} A_n \quad)},$$

The mode of procedure is perfectly straightforward, the results used being

$$\Delta\left(\frac{P}{Q}\right) = \frac{Q\Delta P - P\Delta Q}{Q(Q+\Delta Q)},$$

the expression just obtained for the Δ of a determinant, and a theorem giving a product of two determinants as a sum of like products (see above, p. 171, result XLV. 2). Putting A_{n+2} in place of A_{n+1} there is obtained by division the corollary

$$\frac{\| \Delta^a A_1 . \Delta^{a+1} A_2 \ldots . \Delta^{a+n-1} A_n . \Delta^{a+n} A_{n+2})}{\| \Delta^a A_1 . \Delta^{a+1} A_2 \ldots . \Delta^{a+n-1} A_n . \Delta^{a+n} A_{n+1})}$$

$$= \frac{\Delta \left\{ \dfrac{\Delta^a A_1 . \Delta^{a+1} A_2 \ldots . \Delta^{a+n-2} A_{n-1} . \Delta^{a+n-1} A_{n+2}}{\Delta^a A_1 . \Delta^{a+1} A_2 \ldots\ldots\ldots\ldots \Delta^{a+n-1} A_n} \right)}{\Delta \left\{ \dfrac{\Delta^a A_1 . \Delta^{a+1} A_2 \ldots . \Delta^{a+n-2} A_{n-1} . \Delta^{a+n-1} A_{n+1}}{\Delta^a A_1 . \Delta^{a+1} A_2 \ldots\ldots\ldots\ldots \Delta^{a+n-1} A_n} \right)}.$$

To the fourth and last special form is devoted the whole of the fourth and last Abtheilung, which consists of four chapters and occupies pp. 404–431. The title under which it appears is "Producte mit Versetzungen, wenn die Elemente das Differentiiren mit abwechselndem Vervielfachen angeben"; that is to say, the subject is the determinant derivable from that of the 'loi suprême' by changing Δ into Zd, where by Zd is meant the double operation of differentiating and subsequently multiplying by Z. The first chapter, which closely corresponds to the first and only chapter of the third Abtheilung, opens with the expression for the Zd of a product in terms of the Zd's of the factors: and thence there is obtained the Zd of the special determinant-form under consideration, namely,

$$\begin{aligned}
Zd \| (Zd)^{a_1} A_1 . (Zd)^{a_2} A_2 \ldots . (Zd)^{a_n} A_n) \\
= \quad & \| (Zd)^{a_1+1} A_1 . (Zd)^{a_2} A_2 . (Zd)^{a_3} A_3 \ldots . (Zd)^{a_n} A_n) \\
+ & \| (Zd)^{a_1} A_1 . (Zd)^{a_2+1} A_2 . (Zd)^{a_3} A_3 \ldots . (Zd)^{a_n} A_n) \\
+ & \| (Zd)^{a_1} A_1 . (Zd)^{a_2} A_2 . (Zd)^{a_3+1} A_3 \ldots . (Zd)^{a_n} A_n) \\
+ & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\
+ & \| (Zd)^{a_1} A_1 . (Zd)^{a_2} A_2 \ldots . (Zd)^{a_{n-1}+1} A_{n-1} . (Zd)^{a_n} A_n),
\end{aligned}$$

the r^{th} determinant on the right being derivable from the original by increasing the r^{th} upper index by 1. When $a_r = a + r$ all the determinants on the right except the last evidently vanish, and we have

$$Zd \left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-1} A_n \right)$$
$$= \left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n} A_n \right).$$

The Zd of a generalized Wronskian quotient is next investigated, with the result

$$Zd\left\{\frac{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-1} A_n \right)}{\left| (Zd)^{a+1} A_1 \,.\, (Zd)^{a+2} A_2 \ldots (Zd)^{a+n-1} A_{n-1} \right)}\right\}$$

$$= \frac{\left| (Zd)^a A_1 \ldots (Zd)^{a+n-2} A_{n-1} \right)}{\left| (Zd)^{a+1} A_1 \ldots (Zd)^{a+n-1} A_{n-1} \right)} \cdot \frac{\left| (Zd)^{a+1} A_1 \ldots (Zd)^{a+n} A_n \right)}{\left| (Zd)^{a+1} A_1 \ldots (Zd)^{a+n-1} A_{n-1} \right)}.$$

Similarly there is obtained

$$Zd\left\{\frac{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n-1} A_{n+1} \right)}{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n-1} A_n \right)}\right\}$$

$$= \frac{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-2} A_{n-1} \right)}{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-1} A_n \right)} \cdot \frac{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n} A_{n+1} \right)}{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots (Zd)^{a+n-1} A_n \right)}$$

by a double use of which and by division it follows that

$$\frac{Zd\left\{\dfrac{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n-1} A_{n+2} \right)}{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n-1} A_n \right)}\right\}}{Zd\left\{\dfrac{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n-1} A_{n+1} \right)}{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n-1} A_n \right)}\right\}}$$
$$= \frac{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-1} A_n \,.\, (Zd)^{a+n} A_{n+2} \right)}{\left| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-1} A_n \,.\, (Zd)^{a+n} A_{n+1} \right)}.$$

The second chapter, which is much longer, is devoted to the consideration of the quotient

$$\frac{\left| (Zd)^1 A^{a_1} \,.\, (Zd)^2 A^{a_2} \ldots\ldots (Zd)^{n-1} A^{a_{n-1}} \,.\, (Zd)^n B \right)}{\left| (Zd)^1 A^{a_1} \,.\, (Zd)^2 A^{a_2} \ldots\ldots\ldots\ldots (Zd)^n A^{a_n} \right)}, \text{ or } Q_n \text{ say,}$$

in other words, to those special cases in which the functions to be operated on are all of them powers of one function A. Several interesting results are obtained, such as

$$Q_n = \frac{a_{n-1}}{a_n} \cdot \frac{a_{n-1}-a_1}{a_n-a_1} \cdot \frac{a_{n-1}-a_2}{a_n-a_2} \cdots \frac{a_{n-1}-a_{n-2}}{a_n-a_{n-2}} \cdot \frac{1}{d(A^{a_n-a_{n-1}})} \cdot Q_{n-1},$$

the last of them being specialized down until

$$\Big\| (Zd)^0A^0 . (Zd)^1A^1 \dots (Zd)^nA^n \Big) = 1^{1/1} . 1^{2/1} \dots 1^{n/1} . (ZdA)^{1+2+\dots+n};$$

and to this is appended the note "Setzen wir noch in dieser Gleichung $Z=1$, so erhalten wir endlich jene particuläre Gleichung, welche Wronski Seite 110 findet." The third chapter of three pages begins with the consideration of

$$\Big\| d\,(fx)^a . d^2\,(fx)^{a+1} \dots d^{n-1}(fx)^{a+n-2} . d^n\mathrm{F}x \Big),$$

and ends with the result

$$\Big\| d^{n-q+1}(fx)^{n-q} . d^{n-q+2}(fx)^{n-q+1} \dots d^n(fx)^{n-1} \Big)$$

$$= (-)^q \frac{1^{n/1} . 1^{n-1/1} \dots 1^{n-q/1}}{n . 1^{q/1} . 1^{n-q-1/1}} . (dfx)^{n+(n-1)+(n-2)+\dots+(n-q)} . \frac{d^q}{dx^n}\left(\frac{x}{fx}\right)^n_{fx=0}$$

and the sentence "Diesen ganz speciellen Fall findet zuerst Wronski Phil. d. l. T Seite 60 auf einem ganz verschiedenen Wege." The fourth chapter of like extent specializes in a different direction, but ends in quite the same manner.

JACOBI (1835).

[De eliminatione variabilis e duabus aequationibus algebraicis. *Crelle's Journal*, xv. pp. 101–124; or *Nouv. Annales de Math.* vii. pp. 158–171, 287–294; or *Werke*, iii. pp. 295–320.]

This memoir which we have already referred to (see above, p. 214) contains an investigation of questions arising out of Bezout's method of eliminating the unknown from two equations of the n^{th} degree in x. If the given equations be

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_0$$
$$\phi(x) = b_nx^n + b_{n-1}x^{n-1} + \dots + b_0,$$

Bezout, as is well known, reached the desired end by deriving from them a set of n equations of the $(n-1)^{\text{th}}$ degree, and eliminating $x, x^1, x^2, \dots, x^{n-1}$ from the said set. This process

Jacobi explains at the outset, and, writing for shortness' sake the set in the form

$$\left.\begin{array}{l} 0 = a_{00}x^0 \quad + a_{10}x^1 \quad + \ldots + a_{n-1,0}x^{n-1} \\ 0 = a_{01}x^0 \quad + a_{11}x^1 \quad + \ldots + a_{n-1,1}x^{n-1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 = a_{0,n-1}x^0 + a_{1,n-1}x^1 + \ldots + a_{n-1,n-1}x^{n-1} \end{array}\right\},$$

that the determinant of the coefficients is axisymmetric.* Then he first shows that $a_{rs} = a_{sr}$, or, as would have been said later, denoting the cofactor of a_{rs} in this determinant by A_{rs} he next proves not only that $A_{rs} = A_{sr}$ but that $A_{r,s} = A_{r',s'}$ in every case where $r+s = r'+s'$. $A_{r,s}$ thus depending only on the sum of the suffixes he suggests that in what follows it would be better to write

$$A_{r,s} = A_{r+s},$$

and he thereupon formulates his result as follows (p. 105):

"Quo adhibito notationis modo, videmus, *eam* esse naturam coefficientium a_{rs} quae aequationes lineares afficiunt, e quibus eliminatione incognitarum facta aequatio finalis quaesita petitur, ut posito:

$$\left.\begin{array}{l} a_{00}x_0 \quad + a_{01}x_1 \quad + \ \ldots\ldots + a_{0,n-1}x_{n-1} \quad = m_0 \\ a_{10}x_0 \quad + a_{11}x_1 \quad + \ \ldots\ldots + a_{1,n-1}x_{n-1} \quad = m_1 \\ a_{20}x_0 \quad + a_{21}x_1 \quad + \ \ldots\ldots + a_{2,n-1}x_{n-1} \quad = m_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n-1,0}x_0 + a_{n-1,1}x_1 + \ \ldots\ldots + a_{n-1,n-1}x_{n-1} = m_{n-1} \end{array}\right\}$$

---

* Taking for shortness' sake the case where $n=3$, it is immediately evident that if $f(x)=0$ and $\phi(x)=0$, we must also have

$$\begin{vmatrix} a_0 & a_1 + a_2x + a_3x^2 \\ b_1 & b_1 + b_2x + b_3x^2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_0 + a_1x & a_2 + a_3x \\ b_0 + b_1x & b_2 + b_3x \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_0 + a_1x + a_2x^2 & a_3 \\ b_0 + b_1x + b_2x^2 & b_3 \end{vmatrix} = 0,$$

because the first column increased by a multiple of the second column gives in each case a column whose elements are $f(x)$, $\phi(x)$. These are the n derived equations in question. Further, by transforming them into

$$\left.\begin{array}{l} |a_0b_1| + \ |a_0b_2|x \qquad\qquad\qquad + |a_0b_3|x^2 = 0 \\ |a_0b_2| + \{|a_0b_3| + |a_1b_2|\}x + |a_1b_3|x^2 = 0 \\ |a_0b_3| + \ |a_1b_3|x \qquad\qquad\qquad + |a_2b_3|x^2 = 0 \end{array}\right\}$$

the axisymmetry not only comes into evidence, but the reason for it is apparent.

aequationes inversae, quibus quantitates x_r per quantitates m_r exhibentur, formam sequentem induant

$$\left.\begin{array}{lllllll} L.x_0 & = A_0m_0 & + A_1m_1 & + A_2m_2 & + & \dots & + A_{n-1}m_{n-1} \\ L.x_1 & = A_1m_0 & + A_2m_1 & + A_3m_2 & + & \dots & + A_nm_{n-1} \\ L.x_2 & = A_2m_0 & + A_3m_1 & + A_4m_2 & + & \dots & + A_{n+1}m_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L.x_{n-1} & = A_{n-1}m_0 & + A_nm_1 & + A_{n+1}m_2 & + & \dots & + A_{2n-2}m_{n-1} \end{array}\right\}.$$

The determinant of the coefficients of the latter set of equations here is of the type which Sylvester afterwards distinguished by the name 'persymmetric.' Of course L in the same set stands for $\Sigma \pm a_{00}a_{11} \dots a_{n-1, n-1}$.

SYLVESTER (1840).

[A method of determining by mere inspection the derivatives from two equations of any degree. *Philos. Magazine*, xvi. pp. 132–135; or *Collected Math. Papers*, i. pp 54–57.]

As we have already seen (see above, pp. 236-238) the eliminant of

$$\left.\begin{array}{l} a_0 + a_1x + a_2x^2 + \dots + a_mx^m = 0 \\ b_0 + b_1x + b_2x^2 + \dots + a_nx^n = 0 \end{array}\right\}$$

arising from Sylvester's dialytic process is a determinant of the $(m+n)^{\text{th}}$ order, in which the coefficients of the first equation appear as elements of each of n successive rows, and those of the second equation as elements of each of the remaining rows, each coefficient appearing in any row one place in advance of its position in the immediately preceding row. Determinants of this bi-gradient form, *e.g.*

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & \cdot & \cdot \\ \cdot & b_0 & b_1 & b_2 & \cdot \\ \cdot & \cdot & b_0 & b_1 & b_2 \end{vmatrix},$$

were not long in attracting attention. Up to the year with which we close, however, no property of them had been noted, although any one able to compare the new process of elimination with Bezout's process had a result ready to his hand.

CHAPTER XVII.

RETROSPECT ON SPECIAL FORMS FROM 1772 TO 1841.

A GLANCE over the preceding six chapters shows the extent to which the study of special forms of determinants had been carried prior to 1841, an extent probably hitherto unsuspected. In all, ten different forms had made their appearance, and about half of them had more or less engaged the attention of several investigators and had had a number of their properties brought to light. Of the dozen writers to whom one or more special forms had become familiar, by far the most conspicuous was Jacobi, to whom six forms were known and by whom five of them at least were carefully studied. After him came Cauchy, Wronski and Schweins. The only form to which a distinguishing name had been assigned was that called *symmetric* by Lebesgue in 1837. The fruitful era of nomenclature, but not of that alone, was ushered in by Sylvester shortly after the date with which we close.

INDEX TO THE NUMBERED RESULTS IN PART I.

LIST OF AUTHORS WHOSE WRITINGS ARE REPORTED ON.

THE THEORY OF DETERMINANTS in the HISTORICAL ORDER OF DEVELOPMENT

VOLUME TWO

THE PERIOD 1841 TO 1860.

PREFACE.

THE story of the production of this volume is quite similar to that of the first. Its bibliographical basis was the three "Lists of Writings on Determinants" formerly referred to (see *History,* I., Introduction). These were followed at irregular intervals by a long series of papers in the *Proceedings of the Royal Society of Edinburgh,* containing historically arranged accounts of the writings in question up to 1860. A certain amount of outside attention having thus been attracted to the subject, neglected writings known to other mathematicians came gradually to be brought to my notice, and of course these were steadily supplemented by assiduous research on my own part. As a consequence, the "Fourth List of Writings," published in 1906,* was able to show a score or so of titles belonging to the period prior to 1860, and the "Fifth List," published about a fortnight ago,† a still greater number of similarly belated items. Considerable intercalations have thus been made in the Edinburgh series of papers, and much care has been bestowed on a complete revision of the whole. It is confidently hoped that now very little matter of real moment has escaped attention.

T. M.

CAPETOWN, SOUTH AFRICA,
19*th July*, 1911.

* *Quart. Journ. of Math.*, xxxvii. pp. 237–264.
† *Quart. Journ. of Math.*, xlii. pp. 343–378.

CONTENTS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER IX.

CHAPTER X.

CHAPTER XI.

CHAPTER XII.

CHAPTER XIII.

CHAPTER I.

DETERMINANTS IN GENERAL, PRIOR TO 1841.

SINCE the publication of the first volume of this History, two other writings have been discovered which concern the general subject and belong to the period therein dealt with. The authors, Bianchi and Chelini, being very imperfectly acquainted with the relevant literature of the previous part of their own century, could scarcely be expected to make any advance: and, as the following notices will show, no advance was really made.

BIANCHI, G. (1839, January).

[Sopra l'analisi lineare per la risoluzione dei problemi di primo grado. *Mem. della Soc. ital. delle Sci.*, xxii. pp. 184–227.]

Bianchi's knowledge of previous work on simultaneous linear equations must have been slight—confined, probably, to an acquaintance with Cramer's rule and with Cauchy's so-called "symbolical" solution as given in the *Cours d'Analyse* of 1821: unless this were so, he would scarcely have referred to the methods given in such text-books as Ruffini's *Elementary Algebra* and Euler's *Elements.* One is thus prepared to find little new in his conscientiously laboured monograph, consisting of an introduction of five pages, a section of twenty-seven pages on the solution of a set of n equations with n unknowns, and a section of twelve pages on n equations with fewer unknowns. The main interest lies in the first fifteen pages (pp. 189–204) of the earlier section, these being devoted to establishing the validity of Cramer's rule. The procedure consists in eliminating one and the same unknown between the first equation and each of the other equations of the set, then in treating in the same way the set of $n-1$ equations thus derived, and so on until a single equation $\mathrm{N}x_n = \mathrm{D}$ results. As negligible factors are not struck out in the course of the work, the discovery of the law of formation of the coefficients in the successive sets of equations is made unnecessarily difficult, and N and D are obtained in unwieldy forms. Thus, in the case of the six equations

$$\left.\begin{array}{l} a_1x_1 + b_1x_2 + \ldots\ldots + f_1x_6 = s_1 \\ a_2x_1 + b_2x_2 + \ldots\ldots + f_2x_6 = s_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}$$

the expression found for the last coefficient of x_6 is, in later notation,

$$|a_1b_2c_3d_4e_5f_6| \cdot |a_1b_2c_3d_4| \cdot |a_1b_2c_3|^2 \cdot |a_1b_2|^4 \cdot a_1^8,$$

and for the term independent of x_6

$$|a_1b_2c_3d_4e_5s_6| \cdot |a_1b_2c_3d_4| \cdot |a_1b_2c_3|^2 \cdot |a_1b_2|^4 \cdot a_1^8,$$

with the result, of course, that

$$x_6 = \frac{|a_1b_2c_3d_4e_5s_6|}{|a_1b_2c_3d_4e_5f_6|}.$$

It will readily be agreed that this procedure, though fresh, is not an improvement on others previously known.

CHELINI, D. (1840).

[Formazione e dimostrazione della formula che dà i valori delle incognite nelle equazioni di primo grado. *Giornale Arcadico di Sci.* . . . , lxxxv. pp. 3–12.]

The writings known to Chelini were Terquem's *Manuel d'Algèbre*, Bianchi's paper of 1839, and Molins' of the same year. The paper, however, which his short and clearly written exposition most readily calls to mind is Gergonne's of the year 1813. The "formazione" is essentially Bezout's, and the "dimostrazione" essentially Laplace's.

Attention must also be drawn to a certain neglect in regard to the theorem about the effect of increasing each element of a row (or column) of a determinant by a constant multiple of the corresponding element of another row (or column). It is unquestionably curious that this very elementary property should not have been formulated at a comparatively early date. The writers whose work came nearest to it were Scherk (1825) and Drinkwater (1831): indeed it is little short of marvellous that the latter author should have written his eighth and ninth propositions in the form

$$f(v+w, x, y, z, \dots) = f(v, x, y, z, \dots) + f(w, x, y, z, \dots),$$
$$f(mx, y, z, t, \dots) = m . f(x, y, z, t, \dots),$$

(see *History*, i. p. 199), and should not have added

$$f(v+mx, x, y, z, \dots) = f(v, x, y, z, \dots).$$

Though not formulated, the property in question, however, may actually have been used: and there is certainly evidence of such use in the last year of the period, as may be seen on turning to the closing lines of Jacobi's *De functionibus alternantibus* (see *History*, i. pp. 341–342).

CHAPTER II.

DETERMINANTS IN GENERAL, FROM 1841 TO 1860.

THE number of writings to be considered under this heading is sixty-six, and the number of writers thirty-eight, both numbers being slightly in excess of the corresponding numbers for the period of the whole previous history of the subject. For the first time a fair share of the advance is contributed by English mathematicians: indeed, from being practically of no account, England suddenly leaps to a position of predominating influence. As, therefore, we have already spoken of a *French* period (1693–1812) and a *German* period (1813–1841), it would not be inappropriate to style this the *English* period, or the period of Cayley-and-Sylvester.* Italy also became notably active, emerging from a still more unproductive condition: while Germany and France steadily continued their former labours. The period is also marked by the first appearance of text-books specially devoted to the subject.

CAYLEY, A. (1841).

[On a theorem in the geometry of position. *Cambridge Math. Journ.*, ii. pp. 267–271; or *Collected Math. Papers*, i. pp. 1–4.]

Of the two English mathematicians whose names are inseparably associated with the development of what has been called

* It must be remembered, of course, that Sylvester's work began in the latter part of the previous period (see *History*, i. pp. 227, etc.).

Modern Higher Algebra, Sylvester, as we have seen, was the first to direct public attention to the functions then partially known as determinants, but called by him in the heat of supposed discovery "zetaic products of differences." Cayley it was, however, who gave the great impetus to the study of them—an impetus due to two different causes, the choice of an exceedingly apt notation and the masterly manner in which he put the functions to use. How he obtained his knowledge we know not. It may be that Sylvester's two early papers had directed his attention to the matter, and that he had then read some of the authors who preceded Cauchy; but, whether this be true or not, it is certain that by his own independent research he had attained in 1841 a powerful and comprehensive grasp of the subject. The little paper to which we have now come is ample evidence of this. A peculiar interest attaches to it also, as being the first fruits of Cayley's genius, the earliest of that long and varied series of papers which has done so much to extend the bounds of pure mathematics.*

With characteristic directness and concision he opens as follows:—

"We propose to apply the following (new?) theorem to the solution of two problems in Analytical Geometry.

"Let the symbols

$$|\alpha|, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha' & \beta' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}, \text{ \&c.}$$

denote the quantities

$$\alpha, \quad \alpha\beta' - \alpha'\beta, \quad \alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma, \quad \text{\&c.}$$

the law of whose formation is tolerably well known, but may be thus expressed,

$$|\alpha| = \alpha, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix} = \alpha|\beta'| - \alpha'|\beta|,$$

* In a strictly chronological arrangement Cayley's paper would not follow, but precede the papers of Craufurd, Cauchy, and Jacobi of the same year. It was published in February: Cauchy's note was presented to the Academy on 8th March, and Jacobi's memoir bears the date 17th March, though not published for more than two months afterwards. As Cayley's first appearance, however, marks the beginning of a new epoch, and as the other papers referred to belong by their character to the preceding epoch, a slight deviation from the chronological order seems warranted.

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \alpha \begin{vmatrix} \beta', & \gamma' \\ \beta'', & \gamma'' \end{vmatrix} + \alpha' \begin{vmatrix} \beta'', & \gamma'' \\ \beta, & \gamma \end{vmatrix} + \alpha'' \begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \text{ \&c.}$$

the signs + being used when the number of terms in the side of the square is odd, and + and − alternately when it is even. Then the theorem in question is

$$\begin{vmatrix} \rho\alpha + \sigma\beta + \tau\gamma \ldots, & \rho\alpha' + \sigma\beta' + \tau\gamma' \ldots, & \rho\alpha'' + \sigma\beta'' + \tau\gamma'' \ldots \\ \rho'\alpha + \sigma'\beta + \tau'\gamma \ldots, & \rho'\alpha' + \sigma'\beta' + \tau'\gamma' \ldots, & \rho'\alpha'' + \sigma'\beta'' + \tau'\gamma'' \ldots \\ \rho''\alpha + \sigma''\beta + \tau''\gamma \ldots, & \rho''\alpha' + \sigma''\beta' + \tau''\gamma' \ldots, & \rho''\alpha'' + \sigma''\beta'' + \tau''\gamma'' \ldots \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \end{vmatrix}$$

$$= \begin{vmatrix} \rho, & \sigma, & \tau \ldots \\ \rho', & \sigma', & \tau' \ldots \\ \rho'', & \sigma'', & \tau'' \ldots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \ldots \\ \alpha', & \beta', & \gamma' \ldots \\ \alpha'', & \beta'', & \gamma'' \ldots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}.$$

"This theorem admits of a generalisation which we shall not have occasion to make use of, and which therefore we may notice at another opportunity."

Here, then, we have for the first time in the notation of determinants the pair of upright lines so familiar in all the later work. The introduction of them marks an epoch in the history, so important to the mathematician is this apparently trivial matter of notation. By means of them every determinant became representable, no matter how heterogeneous or complicated its elements might be; and the most disguised member of the family could be exhibited in its true lineaments. While the common characteristic of previous notations is their ability to represent the determinant of such a system as

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \qquad \text{or} \qquad \begin{matrix} a_{1.1} & a_{1.2} & a_{1.3} \\ a_{2.1} & a_{2.2} & a_{2.3} \\ a_{3.1} & a_{3.2} & a_{3.3} \end{matrix}$$

and failure to represent in the case of systems like

$$\begin{matrix} a & b & c \\ c & a & b \\ b & c & a, \end{matrix} \qquad \begin{matrix} a & b & c \\ 1 & a & b \\ 0 & 1 & a, \end{matrix} \qquad \begin{matrix} 4 & 5 & 6 \\ 3 & 2 & 7 \\ 8 & 1 & 0: \end{matrix}$$

Cayley's notation is equally suitable for all. To illustrate by analogy,—the infinitesimal calculus supplied with Lagrange's notation for the differential coefficient of $\phi(x)$, but unable to symbolise the differential coefficients of such a special function as ax^3+bx^2, or $\log(1-x)/(1+x)$ would be in the exact predicament of the theory of determinants prior to Cayley.

Of less importance is the fact, which the quotation indicates, that Cayley had discovered for himself the multiplication-theorem, but characteristically hesitated to proclaim it *new*: also, that, probably following Vandermonde, he took the recurrent law of formation for his definition, making the signs all $+$ in one case and $+$ and $-$ alternately in the next, exactly as Vandermonde did.

He then proceeds to the seemingly geometrical problem :—

"To find the relation that exists between the distances of five points in space."

"We have, in general, whatever x_1, y_1, z_1, w_1, &c., denote,

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2+w_1^2, & -2x_1, & -2y_1, & -2z_1, & -2w_1, & 1 \\ x_2^2+y_2^2+z_2^2+w_2^2, & -2x_2, & -2y_2, & -2z_2, & -2w_2, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_5^2+y_5^2+z_5^2+w_5^2, & -2x_5, & -2y_5, & -2z_5, & -2w_5, & 1 \\ 1, & 0, & 0, & 0, & 0, & 0 \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} 1, & x_1, & y_1, & z_1, & w_1, & x_1^2+y_1^2+z_1^2+w_1^2 \\ 1, & x_2, & y_2, & z_2, & w_2, & x_2^2+y_2^2+z_2^2+w_2^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & x_5, & y_5, & z_5, & w_5, & x_5^2+y_5^2+z_5^2+w_5^2 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{vmatrix}$$

$$- \begin{vmatrix} \overline{x_1-x_1}^2+\overline{y_1-y_1}^2+\overline{z_1-z_1}^2+\overline{w_1-w_1}^2, & \overline{x_1-x_2}^2+\ldots, & \overline{x_1-x_3}^2+\ldots, & \overline{x_1-x_4}^2+\ldots, & \overline{x_1-x_5}^2+\ldots, & 1 \\ \overline{x_2-x_1}^2+\ldots\ldots\ldots\ldots, & \overline{x_2-x_2}^2+\ldots, & \overline{x_2-x_3}^2+\ldots, & \overline{x_2-x_4}^2+\ldots, & \overline{x_2-x_5}^2+\ldots, & 1 \\ \cdot\ , & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot\ , & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{x_5-x_1}^2+\ldots\ldots\ldots\ldots, & \overline{x_5-x_2}^2+\ldots, & \overline{x_5-x_3}^2+\ldots, & \overline{x_5-x_4}^2+\ldots, & \overline{x_5-x_5}^2+\ldots, & 1 \\ 1 & 1\ , & 1 & 1\ , & 1\ , & 0 \end{vmatrix}.$$

Putting the w's equal to 0, each factor of the first side of the equation vanishes, and therefore in this case the second side of the equation becomes equal to zero. Hence x_1, y_1, z_1, x_2, y_2, z_2, &c., being the coordinates of the points 1, 2, &c., situated arbitrarily in space, and $\overline{12}^2$, $\overline{13}^2$, &c., denoting the squares of the distances between these points, we have immediately the required relation

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2, & 1 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is easily expanded, though from the mere number of terms the process is somewhat long."

Than this no better example could have been chosen to illustrate what has just been said above regarding the great advantages of Cayley's notation. As is well known, the result arrived at had been given in forms, lengthy and forbidding, many years before by Lagrange and Carnot. What Cayley did was to rob it of all disguise, by expressing it as the vanishing of an elegantly formed determinant; and secondly, to show that the said determinant vanished because it was eight times the square* of another determinant whose zero character could not be overlooked. As has been implied, the result is purely algebraical, its geometrical character only appearing when x, y, z are taken to denote the coordinates of a point.

The corresponding identities for the cases of four points in a plane and three points in a straight line are given; and the latter of the two is most interestingly shown to be deducible also from the general theory of elimination. This is done as follows:—

"Let $\qquad x_{\prime\prime} - x_{\prime\prime\prime} = \alpha, \quad x_{\prime\prime\prime} - x_{\prime} = \beta, \quad x_{\prime} - x_{\prime\prime} = \gamma;$

then $\qquad \overline{12}^2 = \gamma^2, \quad \overline{23}^2 = \alpha^2, \quad \overline{31}^2 = \beta^2, \quad$ and $\quad \alpha + \beta + \gamma = 0;$

from which α, β, γ are to be eliminated. Multiplying the last equation by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, and reducing by the three first,

* The first factor being 16 times the second, and the w's unnecessary.

$$\begin{aligned}
0.\alpha + \overline{12}^2.\beta + \overline{31}^2.\gamma + \quad \alpha\beta\gamma &= 0,\\
\overline{12}^2.\alpha + \quad 0.\beta + \overline{23}^2.\gamma + \quad \alpha\beta\gamma &= 0,\\
\overline{31}^2.\alpha + \overline{32}^2.\beta + \quad 0.\gamma + \quad \alpha\beta\gamma &= 0,\\
\alpha + \quad \beta + \quad \gamma + 0.\alpha\beta\gamma &= 0;
\end{aligned}$$

from which, eliminating α, β, γ, $\alpha\beta\gamma$ by the general theory of simple equations

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0."$$

The conviction that the identity ought to come out as a result of elimination, and the ingenious fulfilment of it by using the identity $\alpha+\beta+\gamma=0$ after the manner of Sylvester's paper of 1840 are very noteworthy.

It is finally noticed that "the additional equation that exists between the distances of five points on a sphere" can be similarly obtained, and the process is given.

GRUNERT, J. A. (1842).

[Ueber die Theorie der Elimination. *Archiv der Math. u. Phys.*, ii. pp. 76–105, 345–377.]

This paper, extending to more than sixty pages, is little else than an amplified reproduction of work by Cauchy. Nine pages at the beginning concern simultaneous linear equations; the rest is entirely taken up with the various modes of eliminating x between two algebraical equations, $\phi(x)=0$, $\psi(x)=0$.

In the former part, which seems based on the third chapter of the *Cours d'Analyse*, the only fresh matter is a lengthy proof of the proposition that *the difference-product of any number of quantities changes sign when two of the quantities are transposed.* It will suffice to note in regard to it that the so-called inductive method is followed, and that two cases have to be considered, viz. (1) when the new quantity is not one of the two which are interchanged, (2) when it is.

The second part follows closely Cauchy's memoir of 1840.

TERQUEM, O. (1842).

[Notice sur l'élimination. Formules de Cramer. *Nouv. Annales de Math.*, i. pp. 125–131.*]

This is merely a simply written exposition of Cramer's rule, and of Bezout's rule of 1779, and contains nothing noteworthy. It is curious, however, to observe the reason given for directing attention to Cramer's rule,—"Comme ce procédé ne se trouve décrit, que je sache, que dans un seul ouvrage élémentaire français, peu répandu (*Manuel d'Algèbre*, p. 80, 2[e] édition, 1836)." This indicates a sad contrast to the state of matters attested to by Gergonne,† showing that there is a fashion which changeth even in things mathematical. The new favourite, it also appears, was Bezout's rule of 1764; for in passing this over, in order to give an account of the same author's rule of later date, Terquem says in regard to it, "Comme ce procédé est décrit dans tous les ouvrages à l'usage des classes, nous ne nous y arrêterons pas."

CAYLEY, A. (1843).

[Demonstration of Pascal's Theorem. *Cambridge Math. Journ.*, iv. pp. 18–20; or *Collected Math. Papers*, i. pp. 43–45.]

At the outset of this paper two lemmas are given, the second of which stands as follows:—

"Lemma 2. Representing the determinants

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}, \text{ \&c.}$$

by the abbreviated notation $\overline{123}$, &c.; the following equation is identically true:

$$\overline{345}\,.\,\overline{126} - \overline{346}\,.\,\overline{125} + \overline{356}\,.\,\overline{124} - \overline{456}\,.\,\overline{123} = 0.$$

* The continuation intimated at the close (p. 131) was never made.

† The passage in question, which we quoted under Cramer, is to be found in the *Annales de Math.*, xx. p. 45.

This is an immediate consequence of the equations

$$\begin{vmatrix} \cdot & \cdot & x_3, & x_4, & x_5, & x_6 \\ \cdot & \cdot & y_3, & y_4, & y_5, & y_6 \\ \cdot & \cdot & z_3, & z_4, & z_5, & z_6 \\ x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \\ y_1, & y_2, & y_3, & y_4, & y_5, & y_6 \\ z_1, & z_2, & z_3, & z_4, & z_5, & z_6 \end{vmatrix} = \begin{vmatrix} \cdot & \cdot & x_3, & x_4, & x_5, & x_6 \\ \cdot & \cdot & y_3, & y_4, & y_5, & y_6 \\ \cdot & \cdot & z_3, & z_4, & z_5, & z_6 \\ x_1, & x_2, & \cdot & \cdot & \cdot & \cdot \\ y_1, & y_2, & \cdot & \cdot & \cdot & \cdot \\ z_1, & z_2, & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0."$$

The identity is readily recognisable as Bezout's (1779). The mode of arriving at it, however, is fresh, and worthy of every attention. The determinant of the sixth order on the left is shown to be equal to zero; and it is implied that the identity is got by transforming the said vanishing determinant into an aggregate of products of pairs of determinants by means of Laplace's expansion-theorem. The method is far-reaching in its application, and manifestly Cayley could have used it to produce a host of identities of similar kind.

The equatement of the two determinants of the sixth order deserves also to be noted, and may be taken as evidence that Cayley was familiar with the theorem that a determinant is not altered if each element of one row be diminished by the corresponding element of another row.

Lastly, it may be pointed out that we have here the first instance of a practice which afterwards became very general, viz., putting a dot instead of a zero element when writing a determinant.

The other lemma and the main body of the paper are geometrical; but as an important determinant identity is implicitly established in the course of the investigation, and as it is of the greatest historical importance to make evident the wonderful command which Cayley with his new notation had suddenly obtained over determinants, we shall give the full text of these portions also, at least up to a certain point.

"Lemma 1. Let $U = Ax + By + Cz = 0$ be the equation of a plane passing through a given point taken for the origin, and consider the planes

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = 0, \quad U_4 = 0, \quad U_5 = 0, \quad U_6 = 0;$$

the condition which expresses that the intersections of the planes (1) and (2), (3) and (4), (5) and (6), lie in the same plane, may be written down under the form.*

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 & . & . \\ B_1 & B_2 & B_3 & B_4 & . & . \\ C_1 & C_2 & C_3 & C_4 & . & . \\ . & . & A_3 & A_4 & A_5 & A_6 \\ . & . & B_3 & B_4 & B_5 & B_6 \\ . & . & C_3 & C_4 & C_5 & C_6 \end{vmatrix} = 0.$$

"Consider now the points 1, 2, 3, 4, 5, 6, the coordinates of these being respectively $x_1, y_1, z_1, \ldots\ldots, x_6, y_6, z_6$. I represent, for shortness, the equation to the plane passing through the origin, and the points 1, 2, which may be called the plane $\overline{12}$, in the form

$$x\,\overline{12}_x + y\,\overline{12}_y + z\,\overline{12}_z = 0;$$

consequently the symbols $\overline{12}_x$, $\overline{12}_y$, $\overline{12}_z$ denote respectively $y_1z_2 - y_2z_1$, $z_1x_2 - z_2x_1$, $x_1y_2 - x_2y_1$, and similarly for the planes $\overline{13}$, &c. If now the intersections of $\overline{12}$ and $\overline{45}$, $\overline{23}$ and $\overline{56}$, $\overline{34}$ and $\overline{61}$ lie in the same plane, we must have by lemma (1) the equation

$$\begin{vmatrix} 12_x & 45_x & 23_x & 56_x & . & . \\ 12_y & 45_y & 23_y & 56_y & . & . \\ 12_z & 45_z & 23_z & 56_z & . & . \\ . & . & 23_x & 56_x & 34_x & 61_x \\ . & . & 23_y & 56_y & 34_y & 61_y \\ . & . & 23_z & 56_z & 34_z & 61_z \end{vmatrix} = 0.$$

Multiplying the two sides of this equation by the two sides respectively of the equation

$$\begin{vmatrix} x_6 & x_1 & x_2 & . & . & . \\ y_6 & y_1 & y_2 & . & . & . \\ z_6 & z_1 & z_2 & . & . & . \\ . & . & . & x_3 & x_4 & x_5 \\ . & . & . & y_3 & y_4 & y_5 \\ . & . & . & z_3 & z_4 & z_5 \end{vmatrix} = \overline{612}\,.\overline{345},$$

and observing the equations

$$x_6\overline{12}_x + y_6\overline{12}_y + z_6\overline{12}_z = \overline{612}, \quad \overline{112} = 0, \quad \&c.$$

* The commas which Cayley prints after the elements in a determinant we omit here and henceforth.

this becomes

$$\begin{vmatrix} \overline{612} & . & . & . & . & . \\ \overline{645} & \overline{145} & \overline{245} & . & . & . \\ \overline{623} & \overline{123} & . & . & \overline{423} & \overline{523} \\ . & \overline{156} & \overline{256} & \overline{356} & \overline{456} & . \\ . & . & . & . & . & \overline{534} \\ . & . & . & \overline{361} & \overline{461} & \overline{561} \end{vmatrix} = 0,$$

reducible to

$$\overline{612}\,.\,\overline{534} \begin{vmatrix} \overline{145} & \overline{245} & . & . \\ \overline{123} & . & . & \overline{423} \\ \overline{156} & \overline{256} & \overline{356} & \overline{456} \\ . & . & \overline{361} & \overline{461} \end{vmatrix} = 0:$$

or, omitting the factor $\overline{612}\,.\,\overline{534}$, and expanding

$$\overline{145}\,.\,\overline{256}\,.\,\overline{423}\,.\,\overline{361} + \overline{245}\,.\,\overline{123}\,.\,\overline{456}\,.\,\overline{361}$$
$$- \overline{245}\,.\,\overline{123}\,.\,\overline{356}\,.\,\overline{461} - \overline{245}\,.\,\overline{156}\,.\,\overline{423}\,.\,\overline{361} = 0.\text{"}$$

The purely algebraical identity involved in this is in later notation

$$\begin{vmatrix} |y_1z_2| & |y_4z_5| & |y_2z_3| & |y_5z_6| & . & . \\ |z_1x_2| & |z_4x_5| & |z_2x_3| & |z_5x_6| & . & . \\ |x_1y_2| & |x_4y_5| & |x_2y_3| & |x_5y_6| & . & . \\ . & . & |y_2z_3| & |y_5z_6| & |y_3z_4| & |y_6z_1| \\ . & . & |z_2x_3| & |z_5x_6| & |z_3x_4| & |z_6x_1| \\ . & . & |x_2y_3| & |x_5y_6| & |x_3y_4| & |x_6y_1| \end{vmatrix} = \begin{vmatrix} |x_1y_2z_3| & . & . & |x_4y_2z_3| \\ |x_1y_4z_5| & |x_2y_4z_5| & . & . \\ |x_1y_5z_6| & |x_2y_5z_6| & |x_3y_5z_6| & |x_4y_5z_6| \\ . & . & |x_3y_6z_1| & |x_4y_6z_1| \end{vmatrix}.$$

BOOLE, G. (1843).

[On the transformation of multiple integrals. *Cambridge Math. Journ.*, iv. pp. 20–28.]

Boole had to use in his paper the resultant of a system of n linear homogeneous equations, and he therefore thought proper, by way of introduction, to state a mode of forming the resultant, and to prove that the result was correct. As the mode is that in which the rule of signs is dependent on the number of interchanges,* or, as Boole calls them, "binary permutations," any

* See Rothe's paper of the year 1800.

interest attaching to the little exposition is connected with the "proof." The first essential paragraph is:—

"The result of the elimination of the variables from the equations

$$\begin{array}{c} a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0, \\ b_1x_1 + b_2x_2 + \ldots + b_nx_n = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r_1x_1 + r_2x_2 + \ldots + r_nx_n = 0, \end{array}$$

is an equation of which the second member is 0, and of which the first member is formed from the coefficient of $x_1x_2\ldots x_n$ in the product of the given equations, by assuming a particular term, as $a_1b_2\ldots r_n$, positive, and applying to every other term a change of sign for every binary permutation which it may exhibit, when compared with the proposed term $a_1b_2\ldots r_n$."

The curious point worth noting here is that we are directed first to form the terms of the expression afterwards denoted by $\overset{+}{|} a_1b_2\ldots r_n \overset{+}{|}$ and called a "permanent," and then to alter the signs of certain terms of it. Boole then proceeds:—

"The truth of the above theorem is shown by the following considerations. The elimination of x_1 from the first and second equation of the system introduces terms of the form $a_1b_2 - a_2b_1$, $a_1b_3 - a_3b_1$, etc., in which the law of binary permutation is apparent, and as we may begin the process of elimination with any variable and with any pair of equations, the law is universal. From the same instance it is evident that no proposed suffix can occur twice in a given term, which condition is also characteristic of the coefficient of $x_1x_2\ldots x_n$ in the product of the equations of the system, whence the theorem is manifest."

It will be observed that neither the word "determinant" nor the word "resultant" occurs: indeed, throughout the paper, instead of resultant he uses "final derivative," a term which probably may be traced to Sylvester.*

CAYLEY, A. (1843).

[Chapters in the analytical geometry of n dimensions. *Cambridge Math. Journ.*, iv. pp. 119–127; or *Collected Math. Papers*, i. pp. 55–62.]

Of the four short chapters which compose this paper, the only one which concerns us is the first, although in the others deter-

* See Sylvester's paper of 1840.

minants are constantly made use of. At the outset an important notation is introduced which afterwards came to be generally adopted. The passage in regard to it is:—

"Consider the series of terms—

$$\begin{array}{ccccc} x_1 & x_2 & \ldots\ldots & x_n \\ A_1 & A_2 & \ldots\ldots & A_n \\ \cdot & \cdot & \cdots\cdots & \cdot \\ K_1 & K_2 & \ldots\ldots & K_n, \end{array}$$

the number of quantities A, . . . , K being equal to $q(q<n)$. Suppose $q+1$ vertical rows selected, and the quantities contained in them formed into a determinant, this may be done in

$$\frac{n(n-1)\ \ldots\ (q+2)}{1\,.\,2\ \ldots\ (n-q-1)}$$

different ways. The system of determinants so obtained will be represented by the notation

$$\left\| \begin{array}{ccccc} x_1 & x_2 & \ldots\ldots & x_n \\ A_1 & A_2 & \ldots\ldots & A_n \\ \cdot & \cdot & \cdots\cdots & \cdot \\ K_1 & K_2 & \ldots\ldots & K_n \end{array} \right\|;$$

and the system of equations, obtained by equating each of these determinants to zero, by the notation

$$(3) \qquad \left\| \begin{array}{ccccc} x_1 & x_2 & \ldots\ldots & x_n \\ A_1 & A_2 & \ldots\ldots & A_n \\ \cdot & \cdot & \cdots\cdots & \cdot \\ K_1 & K_2 & \ldots\ldots & K_n \end{array} \right\| = 0."$$

A theorem is next enunciated in regard to the expression of any one of the determinants in terms of $n-q$ of them.

"The $\dfrac{n(n-1)\ \ldots\ (q+2)}{1\,.\,2\ \ldots\ (n-q-1)}$ equations represented by this formula reduce themselves to $n-q$ independent equations. Imagine these expressed by

$$(1)=0, \quad (2)=0, \quad \ldots\ldots, \quad (n-q)=0,$$

any one of the determinants is reducible to the form

$$\Theta_1(1) + \Theta_2(2) + \ldots + \Theta_{n-q}(n-q)$$

where $\Theta_1, \Theta_2, \ldots, \Theta_{n-q}$ are coefficients independent of $x_1, x_2, \ldots, x_n$."

No proof is given.

The introduction of the notation is fully justified by two theorems which follow. The first is virtually to the effect that we may multiply both sides of (3) by the determinant

$$(5)\qquad \begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \cdot & \cdot & \cdots & \cdot \\ \tau_1 & \tau_2 & \cdots & \tau_n \end{vmatrix}$$

just as if (3) were a single equation instead of $C_{n,q+1}$ equations, and as if the left-hand side were a determinant; and the result, written in the form

$$(6)\quad \left\|\begin{matrix} \lambda_1 x_1 + \ldots + \lambda_n x_n & \mu_1 x_1 + \ldots + \mu_n x_n & \ldots & \tau_1 x_1 + \ldots + \tau_n x_n \\ \lambda_1 A_1 + \ldots + \lambda_n A_n & \mu_1 A_1 + \ldots + \mu_n A_n & \ldots & \tau_1 A_1 + \ldots + \tau_n A_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_1 K_1 + \ldots + \lambda_n K_n & \mu_1 K_1 + \ldots + \mu_n K_n & \ldots & \tau_1 K_1 + \ldots + \tau_n K_n \end{matrix}\right\| = 0$$

will be true; that is to say, we shall have a new set of $C_{n,q+1}$ equations, which follows logically from the original set. Further. and conversely, if the set (6) hold, we can deduce the set (3) provided that the determinant (5) be not zero. The other theorem is quite similar, being to the effect that the equations (3) may be replaced by the set

$$(8)\quad \left\|\begin{matrix} x_1 & x_2 & \ldots & x_n \\ \lambda_1 A_1 + \ldots + \omega_1 K_1 & \lambda_1 A_2 + \ldots + \omega_1 K_2 & \ldots & \lambda_1 A_n + \ldots + \omega_1 K_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdots & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_q A_1 + \ldots + \omega_q K_1 & \lambda_q A_2 + \ldots + \omega_q K_2 & \ldots & \lambda_q A_n + \ldots + \omega_q K_n \end{matrix}\right\| = 0,$$

and that conversely from the set (8) the set (3) is deducible provided the determinant

$$\begin{vmatrix} \lambda_1 & \mu_1 & \cdots & \omega_1 \\ \lambda_2 & \mu_2 & \cdots & \omega_2 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_q & \mu_q & \cdots & \omega_q \end{vmatrix}$$

be not zero.

As the "derivation of coexistence" came prominently before us in examining Sylvester's early work, it may be noted here in passing that Cayley's second chapter, extending to about a page, consists of the enunciation of a theorem on this subject.

HESSE, O. (1843).

[Ueber die Bildung der Endgleichung, welche durch Elimination einer Variabeln aus zwei algebraischen Gleichungen hervorgeht, und die Bestimmung ihres Grades. *Crelle's Journ.*, xxvii. pp. 1–5; or *Werke*, pp. 83–88.]

Hesse, at this time, must have been unaware of Richelot's paper (dated from the same University), and Grunert's paper, not to speak of writings published outside Germany, for the method which he gives of finding the final equation is nothing more nor less than Sylvester's dialytic method. His exposition, to say the least, is not preferable to Grunert's, and the determinant of the $(m+n)^{\text{th}}$ order which he prints is misleading in points of detail.

CAYLEY, A. (1843).

[On the theory of determinants. *Transac. Cambridge Philos. Soc.*, viii. pp. 1–16; or *Collected Math. Papers*, i. pp. 63–79.]

Up to this point Cayley had dealt with determinants, only, as it were, incidentally. Now, however, he devotes a memoir of sixteen quarto pages to the study of them.

The introductory page shows a pretty wide acquaintance with previous writings on the subject, the authors mentioned being Cramer, Bezout (1764), Laplace, Vandermonde, Lagrange,* Bezout (1779), Gauss, Binet, Cauchy (1812), Lebesgue, Jacobi (1841), and Cauchy (1841).

The first section of the paper is said to deal with "the properties of determinants considered as *derivational functions*."

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* As the memoir of Lagrange which Cayley refers to is not one of those brought into notice in the early part of our history, but is one bearing the title "Sur le problème de la détermination des orbites des comètes d'après trois observations," it may be well to mention that the substance of the only sentence in it which concerns us had already appeared in the memoir of 1773. The sentence is

"De là il s'ensuit aussi qu'on aura

$$(t''u'-t'u'')^2 = (x''z'-x'z'')^2 + (y''z'-y'z'')^2 + (x''y'-x'y'')^2,$$
$$= (x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2) - (x'x'+y'y''+z'z'')^2."$$

—*Nouv. Mém. de l'Acad. Roy.* . . . (Berlin), ann. 1778, p. 160.

As a matter of fact, however, a close examination shows that the functions whose properties are investigated are not strictly determinants, but belong to a class afterwards named *bipartites* by Cayley himself. It is true that it is the determinant notation which is employed in specifying the functions, but this is due to the fact that the bipartite under discussion is of a very special type, and so happens to be expressible as a determinant.

The function U from which he considers his three determinants to be "derived" is

$$\begin{array}{l} \quad x(\alpha\xi+\beta\eta+\ \ldots\ldots) \\ +\ y(\alpha'\xi+\beta'\eta+\ \ldots\ldots) \\ +\ \ldots\ldots\ldots\ldots\ldots \end{array}$$

there being n lines and n terms in each line. This at a somewhat later date (1855) he would have denoted by

$$\left(\begin{array}{|l|} \alpha\quad\beta\ \ldots\ldots \\ \alpha'\quad\beta'\ \ldots\ldots \\ \ldots\ldots\ldots\ldots \end{array}\between\ \xi,\eta,\ \ldots\ \between\ x, y,\ \ldots\right)$$

and called a *bipartite*. A still later notation is

$$\begin{array}{l|l} \xi\quad\eta\ \ldots\ldots & \\ \hline \alpha\quad\beta\ \ldots\ldots & x \\ \alpha'\quad\beta'\ \ldots\ldots & y \\ \ldots\ldots\ldots\ldots & . \end{array}$$

from which each term of the final expansion is very readily obtained by multiplying an element, β' say, of the square array by the two elements (y, η) which lie in the same row and column with it but outside the array. The three determinants which are viewed as "derivational functions" of this function U are

$$\left|\begin{array}{l} \alpha\quad\beta\ \ldots\ldots \\ \alpha'\quad\beta'\ \ldots\ldots \\ \ldots\ldots\ldots\ldots \end{array}\right|,$$

$$-\left|\begin{array}{cccc} & \mathrm{A}x+\mathrm{A}'y+\ldots & \mathrm{B}x+\mathrm{B}'y+\ldots & \ldots\ldots \\ \mathrm{R}\xi+\mathrm{S}\eta+\ldots & \alpha & \beta & \ldots\ldots \\ \mathrm{R}'\xi+\mathrm{S}'\eta+\ldots & \alpha' & \beta' & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}\right|,$$

and

$$-\left|\begin{array}{cccc} & \mathrm{R}x+\mathrm{R}'y+\dots & \mathrm{S}x+\mathrm{S}'y+\dots & \dots\dots \\ \mathrm{A}\xi+\mathrm{B}\eta+\dots & \alpha & \beta & \dots\dots \\ \mathrm{A}'\xi+\mathrm{B}'\eta+\dots & \alpha' & \beta' & \dots\dots \\ \dots & \dots & \dots & \dots \end{array}\right|.$$

These are denoted by KU, FU, ℲU; and the closing sentence of the introduction is, "The symbols K, F, Ⅎ possess properties which it is the object of this section to investigate."

KU, it will be observed, is what afterwards came to be called the *discriminant* of U; and FU, ℲU are the results of making certain linear substitutions for the elements of the first row and of the first column of the determinant

$$\left|\begin{array}{ccccc} & x & y & z & \dots\dots \\ \xi & \alpha & \beta & \gamma & \dots\dots \\ \eta & \alpha' & \beta' & \gamma' & \dots\dots \\ \zeta & \alpha'' & \beta'' & \gamma'' & \dots\dots \\ \dots & \dots & \dots & \dots & \dots \end{array}\right|.$$

It is this determinant, therefore, which is under investigation and under comparison with U. That it is a bipartite function of $x, y, z, \dots$ and $\xi, \eta, \zeta, \dots$ is manifest when we think of expanding it according to binary products of the elements of the first row and of the first column, the expression for it in the notation of bipartites being thus seen to be

$$\begin{array}{cccc|c} x & y & z & \dots\dots & \\ \hline -|\beta'\gamma''\dots| & |\alpha'\gamma''\dots| & -|\alpha'\beta''\dots| & \dots\dots & \xi \\ |\beta\gamma''\dots| & -|\alpha\gamma''\dots| & |\alpha\beta''\dots| & \dots\dots & \eta \\ -|\beta\gamma'\dots| & |\alpha\gamma'\dots| & -|\alpha\beta'\dots| & \dots\dots & \zeta \\ \dots & \dots & \dots & \dots & . \end{array}$$

Now the properties of this which are investigated by Cayley are properties possessed by the more general bipartite

$$\begin{array}{cccc|c} x & y & z & \dots\dots & \\ \hline a_1 & a_2 & a_3 & \dots\dots & \xi \\ b_1 & b_2 & b_3 & \dots\dots & \eta \\ c_1 & c_2 & c_3 & \dots\dots & \zeta \\ \dots & \dots & \dots & \dots & . \end{array}$$

which is not expressible in the form of a determinant. So far, therefore, as this section of the memoir is concerned, it is evident that the title is somewhat misleading, and it is unnecessary to enter into detail regarding the properties in question.

In the course of the section, however, having occasion to use Jacobi's theorem regarding a coaxial minor of the adjugate, Cayley gives at the outset a formal proof which it is most important to note, as it is the natural generalisation of Cauchy's proof for the ultimate case, and consequently has since become the standard proof given in text-books. The passage is

"Let A, B, ..., A′, B′, ... be given by the equations

$$\mathrm{A} = \begin{vmatrix} \beta' & \gamma' & \ldots \\ \beta'' & \gamma'' & \ldots \\ \cdot & \cdot & \cdot \end{vmatrix}, \qquad \mathrm{B} = \pm \begin{vmatrix} \gamma' & \delta' & \ldots \\ \gamma'' & \delta'' & \ldots \\ \cdot & \cdot & \cdot \end{vmatrix}, \quad \ldots$$

$$\mathrm{A}' = \pm \begin{vmatrix} \beta'' & \gamma'' & \ldots \\ \beta''' & \gamma''' & \ldots \\ \cdot & \cdot & \cdot \end{vmatrix}, \qquad \mathrm{B}' = \begin{vmatrix} \gamma'' & \delta'' & \ldots \\ \gamma''' & \delta''' & \ldots \\ \cdot & \cdot & \cdot \end{vmatrix}, \quad \ldots$$

the upper or lower signs being taken according as n is odd or even.

"These quantities satisfy the double series of equations

$$\left.\begin{array}{l} \mathrm{A}\alpha + \mathrm{B}\beta + \ldots\ldots = \kappa \\ \mathrm{A}\alpha' + \mathrm{B}\beta' + \ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots \\ \mathrm{A}'\alpha + \mathrm{B}'\beta + \ldots\ldots = 0 \\ \mathrm{A}'\alpha' + \mathrm{B}'\beta' + \ldots\ldots = \kappa \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad (6)$$

$$\left.\begin{array}{l} \mathrm{A}\alpha + \mathrm{A}'\alpha' + \ldots\ldots = \kappa \\ \mathrm{A}\beta + \mathrm{A}'\beta' + \ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots \\ \mathrm{B}\alpha + \mathrm{B}'\alpha' + \ldots\ldots = 0 \\ \mathrm{B}\beta + \mathrm{B}'\beta' + \ldots\ldots = \kappa \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

the second side of each equation being 0, except for the r^{th} equation of the r^{th} set of equations in the systems.

"Let λ, μ, ... represent the r^{th}, $(r+1)^{\text{th}}$, ... terms of the series $\alpha, \beta, \ldots$; L, M, ... the corresponding terms of the series A, B, ..., where r is any number less than n, and consider the determinant

$$\begin{vmatrix} A & \dots\dots & L \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ A^{(r-1)} & \dots\dots & L^{(r-1)} \end{vmatrix}$$

which may be expressed as a determinant of the n^{th} order, in the form

$$\begin{vmatrix} A & \dots\dots & L & 0 & 0 & \dots \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot \\ A^{(r-1)} & \dots\dots & L^{(r-1)} & 0 & 0 & \dots \\ 0 & \dots\dots & 0 & 1 & 0 & \dots \\ 0 & \dots\dots & 0 & 0 & 1 & \dots \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot \end{vmatrix}.$$

Multiplying this by the two sides of the equation

$$\kappa = \begin{vmatrix} \alpha & \beta & \dots \\ \alpha' & \beta' & \dots \\ \cdot & \cdot & \cdot\;\cdot\;\cdot \end{vmatrix}$$

and reducing the result by the equation ($\odot$) [*i.e.* the multiplication-theorem] and the equations (6), the second side becomes

$$\begin{vmatrix} \kappa & 0 & \dots & & & & \\ 0 & \kappa & \dots & & & & \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & & & & \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \kappa & 0 & 0 & \dots \\ & & & 0 & \mu^{(r)} & \nu^{(r)} & \dots \\ & & & 0 & \mu^{(r+1)} & \nu^{(r+1)} & \dots \\ & & & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot \end{vmatrix}$$

which is equivalent to

$$\kappa^r \begin{vmatrix} \mu^{(r)} & \nu^{(r)} & \dots\dots \\ \mu^{(r+1)} & \nu^{(r+1)} & \dots\dots \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot \end{vmatrix},$$

or we have the equation

$$\begin{vmatrix} A & \dots\dots & L \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ A^{(r-1)} & \dots\dots & L^{(r-1)} \end{vmatrix} = \kappa^{r-1} \begin{vmatrix} \mu^{(r)} & \nu^{(r)} & \dots \\ \mu^{(r+1)} & \nu^{(r+1)} & \dots \\ \cdot & \cdot & \cdot\;\cdot\;\cdot \end{vmatrix}.$$

which in the particular case of $r = n$ becomes

$$\begin{vmatrix} A & B & \dots \\ A' & B' & \dots \\ \cdot & \cdot & \cdot\;\cdot\;\cdot \end{vmatrix} = \kappa^{n-1}."$$

The Second Section is said to concern "the notation and properties of certain functions resolvable into a series of determinants," and it is at once seen that the functions in question are obtainable from the use of m sets of n indices in the way in which a determinant is obtainable from only two sets. Sylvester spoke of them later (1851) as *commutants*.*

CAUCHY, A. L. (1844).

[Mémoire sur les arrangements que l'on peut former avec des lettres données, et sur les permutations ou substitutions à l'aide desquelles on passe d'un arrangement à un autre. *Exercices d'Analyse et de Phys. Math.*, iii. pp. 151–252; or *Œuvres complètes*, (2) xiii.]

The nature of the connection of this with the theory of determinants is evident from the title. Some of the elementary portions of the memoir had in fact already appeared in Cauchy's determinant papers of the years 1812, 1840, 1841, and have been noted in our accounts of the latter. In these papers, as was natural, only such isolated properties were given as might be of immediate application to the main subject: here we have a methodically arranged and lucidly written *treatise*. As, however, in dealing with permutations the question of signature is not taken up, there is no explicit reference to determinants: and all that is therefore necessary is to direct attention to a storehouse of information regarding a subject closely connected with them.

CAUCHY, A. L. (1844).

[Mémoire sur quelques propriétés des résultantes à deux termes. *Exercices d'Analyse et de Phys. Math.*, iii. pp. 274–304; or *Œuvres complètes*, (2) xiii.]

By "résultantes à deux termes" are meant *determinants of the second order*. The expression recalls "résultantes à deux lettres," used by Binet in his memoir of November 1812; and as

* See Postscript to Cayley's paper "On the Theory of Permutants," *Camb. and Dub. Math. Journ.*, vii. pp. 40–51; or *Collected Math. Papers*, ii. pp. 16–26.

the said memoir is here referred to by Cauchy and contains the foundation of the latter's results, it is not improbable that the one expression suggested the other.

Five theorems with attendant corollaries are carefully formulated and proved, extreme simplicity and fulness of exposition being in evidence throughout. The first three theorems are mere variants of the first case of Binet's multiplication-theorem for two non-quadrate matrices, namely, in later notation—

$$\begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 + \ldots & a_1y_1 + a_2y_2 + a_3y_3 + \ldots \\ \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \ldots & \beta_1y_1 + \beta_2y_2 + \beta_3y_3 + \ldots \end{vmatrix}$$

$$\equiv \begin{vmatrix} a_1 & a_2 & a_3 & \ldots. \\ \beta_1 & \beta_2 & \beta_3 & \ldots. \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 & x_3 & \ldots. \\ y_1 & y_2 & y_3 & \ldots. \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ \beta_1 & \beta_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \ldots. + \begin{vmatrix} \iota_2 & a_3 \\ \beta_2 & \beta_3 \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \ldots.$$

The real interest arises when the a's and β's of this are so taken that the first determinant of every pair on the right is of the same form as the determinant on the left, and can therefore be expanded in exactly the same way as the latter. The outcome is

"4^e *Théorème.* Soient

$$P_x, \quad P_y, \quad P_z, \quad \ldots.$$

n fonctions homogènes et linéaires de n variables

$$x, \quad y, \quad z, \quad \ldots.$$

[viz., $P_x = xP_{x,x} + yP_{x,y} + zP_{x,z} + \ldots, P_y = \ldots$] et nommons

$$\mathrm{P}_{\mathrm{x}}, \quad \mathrm{P}_{\mathrm{y}}, \quad \mathrm{P}_{\mathrm{z}}, \quad \ldots.$$

ce que deviennent les fonctions P_x, P_y, P_z, ... quand on remplace les n variables x, y, z, ... par n autres variables

$$\mathrm{x}, \quad \mathrm{y}, \quad \mathrm{z}, \quad \ldots.$$

[viz., $\mathrm{P}_{\mathrm{x}} = \mathrm{x}P_{x,x} + \mathrm{y}P_{x,y} + \mathrm{z}P_{x,z} + \ldots, \mathrm{P}_{\mathrm{y}} = \ldots$]. Concevons d'ailleurs, que l'on ajoute entre eux les termes de la suite

$$P_x, \quad P_y, \quad P_z, \quad \ldots.,$$

ou de la suite

$$P_{\mathrm{x}}, \quad P_{\mathrm{y}}, \quad P_{\mathrm{z}}, \quad \ldots.,$$

respectivement multipliés par les variables

$$x, \quad y, \quad z, \quad \ldots\ldots,$$

ou par les variables

$$\mathrm{x}, \quad \mathrm{y}, \quad \mathrm{z}, \quad \ldots\ldots;$$

et nommons

$$\begin{matrix} P & Q \\ \mathrm{Q} & \mathrm{P} \end{matrix}$$

les quatre sommes ainsi obtenus, P étant celle qui renferme les seules variables $x, y, z, \ldots$, et P celle qui renferme les seules variables $\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots$, en sorte qu'on ait

$$P = xP_x + yP_y + zP_z + \ldots, \quad Q = \mathrm{x}P_x + \mathrm{y}P_y + \mathrm{z}P_z + \ldots$$
$$\mathrm{Q} = x\mathrm{P_x} + y\mathrm{P_y} + z\mathrm{P_z} + \ldots, \quad \mathrm{P} = \mathrm{x}\mathrm{P_x} + \mathrm{y}\mathrm{P_y} + \mathrm{z}\mathrm{P_z} + \ldots$$

La résultante

$$P\mathrm{P} - Q\mathrm{Q},$$

formée avec ces quatres sommes, dépendra uniquement des binômes qui représentent les divers termes de la série

$$x\mathrm{y} - \mathrm{x}y, \quad x\mathrm{z} - \mathrm{x}z, \quad \ldots\ldots, \quad y\mathrm{z} - \mathrm{y}z, \quad \ldots\ldots$$

et sera une fonction de ces binômes, non-seulement entière, mais encore homogène et du second degré."

The full meaning of the theorem and the mode of establishing it will be readily understood from working out the case where the number of terms in each element of the initial determinant is *three*. The process is—

$$\begin{vmatrix} P & Q \\ \mathrm{Q} & \mathrm{P} \end{vmatrix} \equiv \begin{vmatrix} xP_x+yP_y+zP_z & \mathrm{x}P_x+\mathrm{y}P_y+\mathrm{z}P_z \\ x\mathrm{P_x}+y\mathrm{P_y}+z\mathrm{P_z} & \mathrm{x}\mathrm{P_x}+\mathrm{y}\mathrm{P_y}+\mathrm{z}\mathrm{P_z} \end{vmatrix},$$

$$\equiv \begin{vmatrix} P_x & P_y & P_z \\ \mathrm{P_x} & \mathrm{P_y} & \mathrm{P_z} \end{vmatrix} \cdot \begin{vmatrix} x & y & z \\ \mathrm{x} & \mathrm{y} & \mathrm{z} \end{vmatrix}$$

$$= \begin{vmatrix} P_x & P_y \\ \mathrm{P_x} & \mathrm{P_y} \end{vmatrix} \cdot \begin{vmatrix} x & y \\ \mathrm{x} & \mathrm{y} \end{vmatrix} + \begin{vmatrix} P_x & P_z \\ \mathrm{P_x} & \mathrm{P_z} \end{vmatrix} \cdot \begin{vmatrix} x & z \\ \mathrm{x} & \mathrm{z} \end{vmatrix} + \begin{vmatrix} P_y & P_z \\ \mathrm{P_y} & \mathrm{P_z} \end{vmatrix} \cdot \begin{vmatrix} y & z \\ \mathrm{y} & \mathrm{z} \end{vmatrix},$$

$$= \begin{vmatrix} xP_{x,x}+yP_{x,y}+zP_{x,z} & xP_{y,x}+yP_{y,y}+zP_{y,z} \\ \mathrm{x}P_{x,x}+\mathrm{y}P_{x,y}+\mathrm{z}P_{x,z} & \mathrm{x}P_{y,x}+\mathrm{y}P_{y,y}+\mathrm{z}P_{y,z} \end{vmatrix} \cdot \begin{vmatrix} x & y \\ \mathrm{x} & \mathrm{y} \end{vmatrix}$$
$$+ \begin{vmatrix} xP_{x,x}+yP_{x,y}+zP_{x,z} & xP_{z,x}+yP_{z,y}+zP_{z,z} \\ \mathrm{x}P_{x,x}+\mathrm{y}P_{x,y}+\mathrm{z}P_{x,z} & \mathrm{x}P_{z,x}+\mathrm{y}P_{z,y}+\mathrm{z}P_{z,z} \end{vmatrix} \cdot \begin{vmatrix} x & z \\ \mathrm{x} & \mathrm{z} \end{vmatrix}$$
$$+ \begin{vmatrix} xP_{y,x}+yP_{y,y}+zP_{y,z} & xP_{z,x}+yP_{z,y}+zP_{z,z} \\ \mathrm{x}P_{y,x}+\mathrm{y}P_{y,y}+\mathrm{z}P_{y,z} & \mathrm{x}P_{z,x}+\mathrm{y}P_{z,y}+\mathrm{z}P_{z,z} \end{vmatrix} \cdot \begin{vmatrix} y & z \\ \mathrm{y} & \mathrm{z} \end{vmatrix},$$

$$= \begin{vmatrix} x & y & z \\ \mathrm{x} & \mathrm{y} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{x,x} & P_{x,y} & P_{x,z} \\ P_{y,x} & P_{y,y} & P_{y,z} \end{vmatrix} \cdot \begin{vmatrix} x & y \\ \mathrm{x} & \mathrm{y} \end{vmatrix} + \begin{vmatrix} x & y & z \\ \mathrm{x} & \mathrm{y} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{x,x} & P_{x,y} & P_{x,z} \\ P_{z,x} & P_{z,y} & P_{z,z} \end{vmatrix} \cdot \begin{vmatrix} x & z \\ \mathrm{x} & \mathrm{z} \end{vmatrix}$$
$$+ \begin{vmatrix} x & y & z \\ \mathrm{x} & \mathrm{y} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{y,x} & P_{y,y} & P_{y,z} \\ P_{z,x} & P_{z,y} & P_{z,z} \end{vmatrix} \cdot \begin{vmatrix} y & z \\ \mathrm{y} & \mathrm{z} \end{vmatrix},$$

$$= \left\{ \begin{vmatrix} x & y \\ \mathrm{x} & \mathrm{y} \end{vmatrix} \cdot \begin{vmatrix} P_{x,x} & P_{x,y} \\ P_{y,x} & P_{y,y} \end{vmatrix} + \begin{vmatrix} x & z \\ \mathrm{x} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{x,x} & P_{x,z} \\ P_{y,x} & P_{y,z} \end{vmatrix} + \begin{vmatrix} y & z \\ \mathrm{y} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{x,y} & P_{x,z} \\ P_{y,y} & P_{y,z} \end{vmatrix} \right\} \cdot \begin{vmatrix} x & y \\ \mathrm{x} & \mathrm{y} \end{vmatrix}$$
$$+ \left\{ \begin{vmatrix} x & y \\ \mathrm{x} & \mathrm{y} \end{vmatrix} \cdot \begin{vmatrix} P_{x,x} & P_{x,y} \\ P_{z,x} & P_{z,y} \end{vmatrix} + \begin{vmatrix} x & z \\ \mathrm{x} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{x,x} & P_{x,z} \\ P_{z,x} & P_{z,z} \end{vmatrix} + \begin{vmatrix} y & z \\ \mathrm{y} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{x,y} & P_{x,z} \\ P_{z,y} & P_{z,z} \end{vmatrix} \right\} \cdot \begin{vmatrix} x & z \\ \mathrm{x} & \mathrm{z} \end{vmatrix}$$
$$+ \left\{ \begin{vmatrix} x & y \\ \mathrm{x} & \mathrm{y} \end{vmatrix} \cdot \begin{vmatrix} P_{y,x} & P_{y,y} \\ P_{z,x} & P_{z,y} \end{vmatrix} + \begin{vmatrix} x & z \\ \mathrm{x} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{y,x} & P_{y,z} \\ P_{z,x} & P_{z,z} \end{vmatrix} + \begin{vmatrix} y & z \\ \mathrm{y} & \mathrm{z} \end{vmatrix} \cdot \begin{vmatrix} P_{y,y} & P_{y,z} \\ P_{z,y} & P_{z,z} \end{vmatrix} \right\} \cdot \begin{vmatrix} y & z \\ \mathrm{y} & \mathrm{z} \end{vmatrix},$$

or, in still later notation,

$$\begin{array}{ccc|c} |x\,\mathrm{y}| & |x\,\mathrm{z}| & |y\,\mathrm{z}| & \\ \hline |P_{x,x}\,P_{y,y}| & |P_{x,x}\,P_{y,z}| & |P_{x,y}\,P_{y,z}| & |x\,\mathrm{y}| \\ |P_{x,x}\,P_{z,y}| & |P_{x,x}\,P_{z,z}| & |P_{x,y}\,P_{z,z}| & |x\,\mathrm{z}| \\ |P_{y,x}\,P_{z,y}| & |P_{y,x}\,P_{z,z}| & |P_{y,y}\,P_{z,z}| & |y\,\mathrm{z}| \end{array},$$

—a result which loses half its interest if we do not note that each element of the initial determinant is presentable in the same form, viz.,

$$P = \begin{array}{ccc|c} x & y & z & \\ \hline P_{x,x} & P_{x,y} & P_{x,z} & x \\ P_{y,x} & P_{y,y} & P_{y,z} & y \\ P_{z,x} & P_{z,y} & P_{z,z} & z \end{array}, \qquad Q = \begin{array}{ccc|c} \mathrm{x} & \mathrm{y} & \mathrm{z} & \\ \hline & & & x \\ & & & y \\ & & & z \end{array},$$

$$\mathrm{Q} = \begin{array}{ccc|c} x & y & z & \\ \hline & & & \mathrm{x} \\ & & & \mathrm{y} \\ & & & \mathrm{z} \end{array}, \qquad \mathrm{P} = \begin{array}{ccc|c} \mathrm{x} & \mathrm{y} & \mathrm{z} & \\ \hline & & & \mathrm{x} \\ & & & \mathrm{y} \\ & & & \mathrm{z} \end{array}.$$

The fifth theorem, which is obtained from the fourth by further specialisation, viz., by putting in every instance $P_{r,s} = P_{s,r}$, is enunciated at equal length; and then, evidently for the sake of historical connection, it is illustrated by the two simplest cases, that is to say, the case where the number of variables is two and where the number is three. In the former case "on obtiendra l'équation identique

$$(ax^2+by^2+2cxy)(a\mathrm{x}^2+b\mathrm{y}^2+2c\mathrm{xy}) - \{ax\mathrm{x}+by\mathrm{y}+c(x\mathrm{y}+\mathrm{x}y)\}^2$$
$$=(ab-c^2)(x\mathrm{y}-\mathrm{x}y)^2,$$

qui a été donnée par Lagrange dans les *Mémoires de Berlin* de 1773";

in the latter case, it being explained that

$$\mathrm{A}=bc-d^2,\quad \mathrm{B}=ca-e^2,\quad \mathrm{C}=ab-f^2,$$
$$\mathrm{D}=ef-ad,\quad \mathrm{E}=fd-be,\quad \mathrm{F}=de-cf,$$

and

$$\mathrm{X}=y\mathrm{z}-\mathrm{y}z,\quad \mathrm{Y}=z\mathrm{x}-\mathrm{z}x,\quad \mathrm{Z}=x\mathrm{y}-\mathrm{x}y,$$

"on obtiendra l'équation identique

$$(ax^2+by^2+cz^2+2dyz+2ezx+2fxy)(a\mathrm{x}^2+b\mathrm{y}^2+c\mathrm{z}^2+2d\mathrm{yz}+2e\mathrm{zx}+2f\mathrm{xy})$$
$$-\Big\{ax\mathrm{x}+by\mathrm{y}+cz\mathrm{z}+d(y\mathrm{z}+\mathrm{y}z)+e(z\mathrm{x}+\mathrm{z}x)+f(x\mathrm{y}+\mathrm{x}y)\Big\}^2$$
$$=\mathrm{AX}^2+\mathrm{BY}^2+\mathrm{CZ}^2+2\mathrm{DYZ}+2\mathrm{EZX}+2\mathrm{FXY},$$

que l'on pourrait déduire de l'une des formules données par M. Binet dans le xvi^e cahier du *Journal de l'École Polytechnique.*"

This latter, for the sake of future reference, it is well to restate in the form

$$\begin{vmatrix} \begin{array}{ccc|c} x & y & z & \\ \hline a & f & e & x \\ f & b & d & y \\ e & d & c & z \end{array} & \begin{array}{ccc|c} \mathrm{x} & \mathrm{y} & \mathrm{z} & \\ \hline a & f & e & x \\ f & b & d & y \\ e & d & c & z \end{array} \\ \\ \begin{array}{ccc|c} x & y & z & \\ \hline a & f & e & \mathrm{x} \\ f & b & d & \mathrm{y} \\ e & d & c & \mathrm{z} \end{array} & \begin{array}{ccc|c} \mathrm{x} & \mathrm{y} & \mathrm{z} & \\ \hline a & f & e & \mathrm{x} \\ f & b & d & \mathrm{y} \\ e & d & c & \mathrm{z} \end{array} \end{vmatrix} = \begin{array}{ccc|c} \mathrm{X} & \mathrm{Y} & \mathrm{Z} & \\ \hline \mathrm{A} & \mathrm{F} & \mathrm{E} & \mathrm{X} \\ \mathrm{F} & \mathrm{B} & \mathrm{D} & \mathrm{Y} \\ \mathrm{E} & \mathrm{D} & \mathrm{C} & \mathrm{Z}. \end{array}$$

A concluding paragraph is devoted to noting that $P_{x,x}, P_{x,y}, \ldots$ in the fourth theorem are expressible as halved differential-quotients of P, viz.,

$$P_{x,x}=\tfrac{1}{2}\mathrm{D}_x^2P,\qquad P_{y,y}=\tfrac{1}{2}\mathrm{D}_y^2P,\qquad P_{z,z}=\tfrac{1}{2}\mathrm{D}_z^2P,$$
$$P_{x,y}=\tfrac{1}{2}\mathrm{D}_x\mathrm{D}_yP,\quad P_{x,z}=\tfrac{1}{2}\mathrm{D}_x\mathrm{D}_zP,\quad P_{y,z}=\tfrac{1}{2}\mathrm{D}_y\mathrm{D}_zP;$$

that therefore

$$2P = x^2 D_x^2 P + y^2 D_y^2 P + z^2 D_z^2 P + \ldots$$
$$+ 2xy D_x D_y P + 2xz D_x D_z P + \ldots .$$

and

$$P_x = \tfrac{1}{2} D_x P, \quad P_y = \tfrac{1}{2} D_y P, \quad P_z = \tfrac{1}{2} D_z P, \quad \ldots .$$

After this the second part of the memoir, consisting of geometrical applications, is entered upon.

GRASSMANN, H. (June 1844).

[DIE WISSENSCHAFT DER EXTENSIVEN GRÖSSE, oder die Ausdehnungslehre, eine neue mathematische Disciplin dargestellt und durch Anwendungen erläutert. Erster Theil, die lineale Ausdehnungslehre enthaltend. xxxii+279 pp. Leipzig, 1844. Abstract in *Archiv d. Math. u. Phys.* vi. (1845), pp. 337–369.]

A quite peculiar form of the law of formation of a determinant had its origin with Grassmann. Grassmann, it will be remembered, was one of the most distinguished of the mathematicians who occupied themselves with the search for an *Algebra of directed quantities*, or with the allied problem of the geometrical interpretation of the so-called imaginary expressions of ordinary algebra. By the beginning of the third decade of the century, the way had been gradually, though intermittingly, prepared for important discoveries on the subject by the writings of Wallis (1685), Buée (1805), Argand (1806), Servois (1813), Mourey (1828), Warren (1828), and Gauss (1831).* With Hamilton and Grassmann important discoveries came. Hamilton, whose writings of 1833 and 1835 show that even then he had meditated to some purpose on the matter, announced in 1843 his great invention of Quaternions. In 1844 Grassmann followed with the first part of the Ausdehnungslehre.

In his preface Grassmann explains the steps by which he had been led to his theory. First, there was the question of the addition of directed straight lines (Strecken), or *vectors*, to use

* See art. "Quaternions," by Professor Tait, in *Encyclopædia Britannica*; or Hamilton's *Lectures on Quaternions*.

Hamilton's widely accepted term. This it was unnecessary to linger over, as his predecessors had already dealt satisfactorily with it. Then came the question of multiplication of vectors. Seeing that when a and b represent two lines in magnitude only, in other words, are scalars and not vectors, the product ab represents the rectangle of which a and b are adjacent sides, Grassmann ventured to denote by the product ab, when a and b are vectors, a parallelogram having the vectors for adjacent sides. This definition of multiplication manifestly entailed the result.

$$a^2 = 0;$$

and along with the definition of addition required further that

$$a(b+c) = ab+ac.$$

These two again involved a third, viz.,

$$ab = -ba;$$

for from the two we have

$$\begin{aligned} 0 &= (a+b)^2, \\ &= (a+b)a+(a+b)b, \\ &= a^2+ba+ab+b^2, \\ &= ba+ab. \end{aligned}$$

The remaining steps of the building up of the theory need not be told, as these laws of *outer* multiplication ("*äussere Multiplication*") suffice for the purpose we have in view.

The exposition of the theory itself is broken up into an introduction and nine chapters, all of them marked by ability and much originality. It is the second chapter which deals specially with outer multiplication, and at the end of it (pp. 70–73) occurs the application which concerns determinants. The matter is introduced by a sentence or two pointing out that it is scarcely to be expected that outer multiplication can be so directly applied to ordinary algebra as to geometry and dynamics, because in ordinary algebra the quantities are essentially alike (*gleichartige*, in the sense of the Ausdehnungslehre), and outer multiplication presupposes the idea of unlikeness. In certain circumstances, however, we are told that we may impose distinctions upon the quantities, and then outer multiplication may be applied with notable results.

"Um hiervon eine Idee zu geben, will ich n Gleichungen ersten Grades mit n Unbekannten setzen, von der Form

$$\begin{array}{l} a_1x_1 + a_2x_2 + \dots + a_nx_n = a_0, \\ b_1x_1 + b_2x_2 + \dots + b_nx_n = b_0, \\ \cdots\cdots\cdots\cdots \\ s_1x_1 + s_2x_2 + \dots + s_nx_n = s_0, \end{array}$$

wo $x_1, \dots, x_n$ die Unbekannten seien. Hier können wir die Zahlencoefficienten, welche verschiedenen Gleichungen angehören, sofern wir diese Verschiedenheit an ihrem Begriff noch festhalten, als verschiedenartig ansehen, und zwar alle als an sich verschiedenartig, d. h. als unabhängig in dem Sinne unserer Wissenschaft, die einer und derselben Gleichung als unter sich in derselben Beziehung gleichartig. Addiren wir nun in diesem Sinne alle n Gleichungen und bezeichnen die Summe des Verschiedenartigen in dem Sinne unserer Wissenschaft mit dem Verknüpfungszeichen $\dot{+}$, indem die gleichen Stellen in den so gebildeten Summenausdrücken immer dem Gleichartigen zukommen sollen, so erhalten wir

$$(a_1 \dot{+} b_1 \dot{+} \dots \dot{+} s_1)x_1 \dot{+} (a_2 \dot{+} b_2 \dot{+} \dots \dot{+} s_2)x_2$$
$$\dots\dots \dot{+} (a_n \dot{+} b_n \dot{+} \dots \dot{+} s_n)x_n = a_0 \dot{+} b_0 \dot{+} \dots \dot{+} s_0,$$

oder bezeichnen wir $(a_1 \dot{+} b_1 \dot{+} \dots \dot{+} s_1)$ mit p_1, und entsprechend die übrigen Summen, so haben wir

$$p_1x_1 \dot{+} p_2x_2 \dot{+} \dots \dot{+} p_nx_n = p_0.$$

Aus dieser Gleichung, welche die Stelle jener n Gleichungen vertritt, lässt sich nun auf der Stelle jede der Unbekannten, z. B. x_1 finden, wenn wir die beiden Seiten mit dem äusseren Produkte aus den Coefficienten der übrigen Unbekannten äusserlich multipliciren, also hier mit $p_2p_3 \dots p_n$. Da nämlich, wenn man die Glieder der linken Seite einzeln multiplicirt, nach dem Begriff des äusseren Produktes, alle Produkte wegfallen, welche zwei gleiche Factoren enthalten, so erhält man

$$p_1p_2p_3 \cdots p_nx_1 = p_0p_2p_3 \cdots p_n.$$

Also da beide Produkte, als demselben System n-ter Stufe angehörig einander gleichartig sind, so hat man

$$x_1 = \frac{p_0p_2p_3 \cdots p_n}{p_1p_2p_3 \cdots p_n}."$$

The method is thus seen to consist in the deduction of a new equation by addition, and in the elimination of all the unknowns, except one, from the equation, by multiplying both sides by the product of the coefficients of the other unknowns,—the multiplication in question being "outer," and for the purposes of the multiplication, any two coefficients of one and the same equation being considered as "like," and any two belonging to different equations as "unlike." For example, in the case of $n=3$, we have

$$\begin{aligned} x_1 &= \frac{(a_0+b_0+c_0).(a_2+b_2+c_2).(a_3+b_3+c_3)}{(a_1+b_1+c_1).(a_2+b_2+c_2).(a_3+b_3+c_3)}, \\ &= \frac{(a_0a_2+a_0b_2+a_0c_2+b_0a_2+b_0b_2+\ \ldots).(a_3+b_3+c_3)}{(a_1a_2+a_1b_2+a_1c_2+b_1a_2+b_1b_2+\ \ldots).(a_3+b_3+c_3)}, \\ &= \frac{(a_0b_2+a_0c_2+b_0a_2+b_0c_2+c_0a_2+c_0b_2).(a_3+b_3+c_3)}{(a_1b_2+a_1c_2+b_1a_2+b_1c_2+c_1a_2+c_1b_2).(a_3+b_3+c_3)}, \end{aligned}$$

since $a_0a_2 = b_0b_2 = \ldots = c_1c_2 = 0$; and finally

$$x_1 = \frac{a_0b_2c_3-a_0b_3c_2+a_2b_3c_0-a_2b_0c_3+a_3b_0c_2-a_3b_2c_0}{a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1},$$

"worin wir, da alles entsprechend geordnet ist, wieder die gewöhnliche Multiplicationsbezeichnung einführen konnten."

All this semblance of demonstration—for it is nothing else—is of little moment compared with the fact sought to be demonstrated, viz., that a determinant is expressible as the outer product of the sums of the elements of its columns. Grassman, however, makes no reference to determinants.

In a paragraph of a subsequent chapter (p. 129), he takes up the problem of elimination between two equations of the m^{th} and n^{th} degrees. What it contains is a reproduction of Sylvester's dialytic method, without any reference to the author of the method.

CAUCHY, A. L. (1845 Augt.).

[Mémoire sur divers théorèmes d'analyse et de calcul intégral. *Comptes rendus .... Acad. des Sci.* (Paris), xxi. pp. 407–415; or *Œuvres complètes* (1), ix. pp. 266–275.]

Cauchy's results are arrived at by taking two different ways of eliminating α, β, γ from the equations

$$\left.\begin{aligned} u_1\alpha + u_2\beta + u_3\gamma &= \alpha s \\ v_1\alpha + v_2\beta + v_3\gamma &= \beta s \\ w_1\alpha + w_2\beta + w_3\gamma &= \gamma s \end{aligned}\right\}$$

where the u's, v's, w's are definable in Cayley's notation by

$$\left(\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}\right) \equiv \left(\begin{vmatrix} a_1x_1+b_1y_1+c_1z_1 & a_2x_1+b_2y_1+c_2z_1 & a_3x_1+b_3y_1+c_3z_1 \\ a_1x_2+b_1y_2+c_1z_2 & a_2x_2+b_2y_2+c_2z_2 & a_3x_2+b_3y_2+c_3z_2 \\ a_1x_3+b_1y_3+c_1z_3 & a_2x_3+b_2y_3+c_2z_3 & a_3x_3+b_3y_3+c_3z_3 \end{vmatrix}\right).$$

The most direct mode of elimination gives of course

$$\begin{vmatrix} u_1-s & u_2 & u_3 \\ v_1 & v_2-s & v_3 \\ w_1 & w_2 & w_3-s \end{vmatrix} = 0. \tag{1}$$

But by writing X, Y, Z for

$$\alpha x_1+\beta x_2+\gamma x_3, \quad \alpha y_1+\beta y_2+\gamma y_3, \quad \alpha z_1+\beta y_2+\gamma z_3,$$

the given set of equations is changed into

$$\left.\begin{aligned} a_1\mathrm{X} + b_1\mathrm{Y} + c_1\mathrm{Z} &= \alpha s \\ a_2\mathrm{X} + b_2\mathrm{Y} + c_2\mathrm{Z} &= \beta s \\ a_3\mathrm{X} + b_3\mathrm{Y} + c_3\mathrm{Z} &= \gamma s \end{aligned}\right\},$$

whence by solution we obtain

$$\left.\begin{aligned} \mathrm{X}\Delta &= (\alpha\mathrm{A}_1+\beta\mathrm{A}_2+\gamma\mathrm{A}_3)s \\ \mathrm{Y}\Delta &= (\alpha\mathrm{B}_1+\beta\mathrm{B}_2+\gamma\mathrm{B}_3)s \\ \mathrm{Z}\Delta &= (\alpha\mathrm{C}_1+\beta\mathrm{C}_2+\gamma\mathrm{C}_3)s \end{aligned}\right\},$$

where Δ is put for $|a_1b_2c_3|$ and $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{C}_3$ are respectively the cofactors of $a_1, a_2, \ldots, c_3$ in Δ. This, however, if we dispense with the use of X, Y, Z is again a set of linear homogeneous equations in α, β, γ, and elimination gives

$$\begin{vmatrix} x_1\Delta - s\mathrm{A}_1 & x_2\Delta - s\mathrm{A}_2 & x_3\Delta - s\mathrm{A}_3 \\ y_1\Delta - s\mathrm{B}_1 & y_2\Delta - s\mathrm{B}_2 & y_3\Delta - s\mathrm{B}_3 \\ z_1\Delta - s\mathrm{C}_1 & z_2\Delta - s\mathrm{C}_2 & z_3\Delta - s\mathrm{C}_3 \end{vmatrix} = 0. \tag{2}$$

Now when (1) and (2) have their left-hand members arranged according to ascending powers of s, they become, in later notation,

$$|u_1v_2w_3| - s\Big\{|v_2w_3| + |u_1w_3| + |u_1v_2|\Big\} + s^2(u_1+v_2+w_3) - s^3 = 0,$$

and

$$\Delta^3|x_1y_2z_3| - \Delta^2 s\left\{\begin{vmatrix} x_1 & x_2 & \mathrm{A}_3 \\ y_1 & y_2 & \mathrm{B}_3 \\ z_1 & z_2 & \mathrm{C}_3 \end{vmatrix} + \begin{vmatrix} x_1 & \mathrm{A}_2 & x_3 \\ y_1 & \mathrm{B}_2 & y_3 \\ z_1 & \mathrm{C}_2 & z_3 \end{vmatrix} + \begin{vmatrix} \mathrm{A}_1 & x_2 & x_3 \\ \mathrm{B}_1 & y_2 & y_3 \\ \mathrm{C}_1 & z_2 & z_3 \end{vmatrix}\right\}$$

$$+ \Delta s^2 \left\{ \begin{vmatrix} x_1 & A_2 & A_3 \\ y_1 & B_2 & B_3 \\ z_1 & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & x_2 & A_3 \\ B_1 & y_2 & B_3 \\ C_1 & z_2 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & A_2 & x_3 \\ B_1 & B_2 & y_3 \\ C_1 & C_2 & z_3 \end{vmatrix} \right\}$$
$$- s^3 |A_1B_2C_3| = 0;$$

and, by equating like powers of s in the two, three results are reached, namely,

$$|u_1v_2w_3| = \frac{\Delta^3}{|A_1B_2C_3|} |x_1y_2z_3|,$$

$$|v_2w_3| + |u_1w_3| + |u_1v_2| = \frac{\Delta^2}{|A_1B_2C_3|} \left\{ \begin{vmatrix} x_1 & x_2 & A_3 \\ y_1 & y_2 & B_3 \\ z_1 & z_2 & C_3 \end{vmatrix} + \begin{vmatrix} x_1 & A_2 & x_3 \\ y_1 & B_2 & y_3 \\ z_1 & C_2 & z_3 \end{vmatrix} + \begin{vmatrix} A_1 & x_2 & x_3 \\ B_1 & y_2 & y_3 \\ C_1 & z_2 & z_3 \end{vmatrix} \right\},$$

$$u_1 + v_2 + w_3 = \frac{\Delta}{|A_1B_2C_3|} \left\{ \begin{vmatrix} x_1 & A_2 & A_3 \\ y_1 & B_2 & B_3 \\ z_1 & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & x_2 & A_3 \\ B_1 & y_2 & B_3 \\ C_1 & z_2 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & A_2 & x_3 \\ B_1 & B_2 & y_3 \\ C_1 & C_2 & z_3 \end{vmatrix} \right\}.$$

In the first of these the cofactor of $|x_1y_2z_3|$ is independent of the x's, y's, z's, so that it does not alter on putting

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix};$$

and as on doing so we have

$$\begin{vmatrix} a_1x_1 & a_2x_1 & a_3x_1 \\ b_1y_2 & b_2y_2 & b_3y_2 \\ c_1z_3 & c_2z_3 & c_3z_3 \end{vmatrix} = \frac{\Delta^3}{|A_1B_2C_3|} \cdot x_1y_2z_3,$$

it is seen that the said cofactor is equal to Δ. This is Cauchy's old theorem regarding the adjugate determinant, and substituting Δ for the cofactor, we have another equally old, namely, the multiplication-theorem.

The reader should note, however, that it is better not to view these old results as the goal of the memoir; but, viewing them as already known, to note the theorem then reached regarding the sum of the m-line coaxial minors of the product-determinant.

CAYLEY, A. (1845).

[On the theory of linear transformations. *Camb. Math. Journ.*, iv. pp. 193–209; or *Collected Math. Papers*, i. pp. 80–94.]

[Mémoire sur les hyperdéterminants. *Crelle's Journ.*, xxx. pp. 1–37.]*

[On linear transformations. *Camb. and Dub. Math. Journ.*, i. pp. 104–122; or *Collected Math. Papers*, i. pp. 95–112.]

These memoirs, afterwards so famous in the history of what is now known as the algebra of quantics, contain exceedingly little on determinants. It is important, however, to direct attention to them, because the basis of them is a generalisation of determinants. Using language which came into vogue two or three years later, we may say that just as the idea and notation of determinants provided the means of expressing *one* of the invariants (viz., the discriminant) of a function, the idea and notation of hyperdeterminants were brought forward for the purpose of expressing *all* the invariants.† The generalisation is of great width, hyperdeterminants including as a very special case the generalisation previously made, viz., *commutants.*

The first memoir gives incidentally a more general mode of using what we may call the *notation of multiple determinants* than that specified in his paper of 1843. The first usage, it will be remembered, is exemplified by

$$\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}\right\| = 0$$

which is meant to signify that

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} = 0.$$

A corresponding example of the new usage is

$$\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}\right\| = \left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix}\right\|,$$

*This is stated to be a translation of the preceding paper, with certain additions by the author; and as such it is not reprinted in *Collected Math. Papers*. It also contains the substance of the paper which follows, the latter having been delayed in publication.

†And indeed the covariants also.

where six equations are again intended to be specified, viz.,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \quad \ldots\ldots$$

each determinant of the one group of six being meant to be equal to the corresponding determinant of the other group.

The example actually employed by Cayley is a result of the multiplication-theorem, and fully justifies the usage. It is

$$\left\|\begin{matrix} \lambda\alpha+\lambda'\alpha'+\ldots, & \lambda\beta+\lambda'\beta'+\ldots, & \ldots \\ \mu\alpha+\mu'\alpha'+\ldots, & \mu\beta+\mu'\beta'+\ldots, & \ldots \\ \cdot & \cdot & \cdot \end{matrix}\right\| = \begin{vmatrix} \lambda & \mu & \ldots \\ \lambda' & \mu' & \ldots \\ \cdot & \cdot & \cdot \end{vmatrix} \cdot \left\|\begin{matrix} \alpha & \beta & \ldots \\ \alpha' & \beta' & \ldots \\ \cdot & \cdot & \cdot \end{matrix}\right\|$$

where, of course, the number of columns in the multiplier must be greater than the number in the determinant which is its cofactor.

It may be worth adding that the *Mémoire sur les hyperdéterminants* affords the first instance of the occurrence of Cayley's vertical-line notation in *Crelle's Journal*.*

FÉRUSSAC, DE (1845).

[Sur la résolution d'un système général de m équations du premier degré entre m inconnues. *Nouv. Annales de Math.*, iv. pp. 28–32.]

This is a belated contribution, having no connection with any of those immediately preceding it. The author in all probability knew nothing of the subject, with the exception of Cramer's rule, which by this time was almost a century old.

The theorem which he seeks to establish is:—

"Connaissant les valeurs des inconnues d'un système de n équations à n inconnues, pour avoir le dénominateur commun des valeurs d'un système de $n+1$ équations à $n+1$ inconnues, on multiplie le dénominateur du valeur du premier système, par le coefficient de la nouvelle inconnue dans la nouvelle équation. Puis on en retranche les produits respectifs des numérateurs des n inconnues du premier système par leurs coefficients dans la dernière du nouveau système. Quant au numérateur il se forme toujours du dénominateur en remplaçant le coefficient de l'inconnue que l'on considère par le terme tout connu."

* In *Liouville's Journal* brackets, [ ] or { }, were used in Cayley's own papers of the year 1845. See vol. x.

The method of proof is that known as "mathematical induction." The details of it need not be given, as they correspond closely with what are to be found in Scherk's paper of the year 1825, the main differences being that Férussac uses no special determinant notation, and, while clear and simple, is not nearly so lengthy nor so laboriously logical.

TERQUEM, O. (1846).

[Notice sur l'élimination. *Nouv. Annales de Math.*, v. pp. 153–162.]

This is a continuation of Terquem's paper of the year 1842. Just as the previous portion dealt with Cramer and Bezout, this deals with Fontaine (des Bertins), Vandermonde, and Laplace, explaining concisely and clearly their main contributions to the subject.

The only portion of it calling for notice is that in which attention is drawn to the curious fact that Laplace makes no reference to Vandermonde's paper read to the Academy in the preceding year. In regard to this Terquem's remark is—

"Il est extrêmement probable que Laplace n'a pas pris connaissance du mémoire de son confrère : on sait, d'ailleurs, que les analystes français lisent peu les ouvrages les uns des autres. Ceci nous explique également comment la résolution de l'équation du onzième degré à deux termes, la plus importante découverte de Vandermonde, soit restée ignorée jusqu'à ce qu'elle ait attiré l'attention de Lagrange, après la découverte similaire de M. Gauss."

Not only, however, does this explanation not carry us far, but the question arises whether the point sought to be explained is really the point which stands most in need of explanation. Vandermonde's paper was read at the very beginning of 1771 and Laplace's in 1772: yet in the History of the Academy for the latter year Laplace's occupies pp. 267–376 and Vandermonde's pp. 516-532, and neither refers to the other's work.

It may be noted here that, notwithstanding Terquem's knowledge of the early history of determinants and his manifest desire to induce his readers to take up the subject, he does not himself hold the new weapon with a very firm grasp. For example, in giving in this volume an account of a paper of Grunert's in *Crelle's Journal*, viii. pp. 153–159, in which the author says—

"Entwickeln wir nemlich x', y', z', durch Elimination aus den Gleichungen:

$$\begin{aligned} x &= Ax' + By' + Cz', \\ y &= A'x' + B'y' + C'z', \\ z &= A''x' + B''y' + C''z', \end{aligned}$$

so erhalten wir:

$$\begin{aligned} x' &= \frac{(B'C'' - B''C')x + (B''C - BC'')y + (BC' - B'C)z}{L}, \\ y' &= \ldots\ldots \\ z' &= \ldots\ldots \end{aligned}$$

wenn wir

$$L = AB'C'' - A'BC'' + A''BC' - AB''C' + A'B''C - A''B'C$$

setzen"—

he paraphrases the passage as follows:

"Les équations donnent

$$\begin{aligned} x' &= \frac{x[B'C''] + y[B''C] + z[BC']}{L}, \\ y' &= \ldots\ldots \\ z' &= \ldots\ldots \end{aligned}$$

où les crochets représentent des *binômes alternés*;

$$[B'C''] = B'C'' - B''C',$$

et ainsi des autres: L est la *résultante*, dénominateur commun."

The simultaneous use of *binôme alterné* and *résultante* is far from happy.*

---

*Two years later we find him, in referring to a paper of Cayley's where the determinant

$$\begin{vmatrix} L & T & S & \xi \\ T & M & R & \eta \\ S & R & N & \zeta \\ \xi & \eta & \zeta & \end{vmatrix}$$

occurs, calling it, as he did in 1845, a "fonction cramérienne," and writing it

$$\left\{\begin{matrix} L & T & S & \xi \\ T & M & R & \eta \\ S & R & N & \zeta \\ \xi & \eta & \zeta & \end{matrix}\right\}.$$

See *Nouv. Annales de Math.*, iv. (1845), p. 535; vii. (1848), p. 420.

CATALAN, E. (1846).

[Recherches sur les déterminants. *Bull. de l'Acad. roy. ... de Belgique*, xiii. pp. 534–555.]

As is known, Catalan had already dealt with determinants in the year 1839 in a memoir regarding the change of variables in a multiple integral. In the paper which we have now come to he leads up to examples of the same kind of transformation; but the greater part of it—seventeen out of the total twenty-two pages—is occupied with determinants pure and simple. Half of this amount consists of an elementary exposition of known properties, and calls for no remark save that what Cauchy called "produit principal" or "terme indicatif" is here called "terme caractéristique," and that he makes constant use of the symbolism

$$\text{dét.}(A, B, C, \dots)$$

to stand for the determinant whose first row consists of a's, second row of b's, and so on: for example,

$$\begin{aligned} \text{dét.}(B, A, C, \dots) &= -\,\text{dét.}(A, B, C, \dots), \\ \text{dét.}(A, A, C, \dots) &= 0, \\ \text{dét.}(A + M, B) &= \text{dét.}(A, B) + \text{dét.}(M, B), \\ &\dots\dots\dots\dots \end{aligned}$$

When we come to § 13, however, we find fresh ground struck. The exact words are:—

"Supposons maintenant qu'étant donné la système—

$$\left.\begin{array}{c} A_1, \\ A_2, \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{array}\right\} \quad \dots . \quad (A)$$

dont le déterminant est Δ, on ait combiné par voie d'addition et de soustraction les équations dont les premiers membres sont représentés par $A_1, A_2, \dots, A_n$; et, par exemple, qu'on ait déduit du système (A) le système suivant

$$\left.\begin{array}{c} A_1 + A_2 + \dots + A_n, \\ A_1 - A_2, \\ A_2 - A_3, \\ \dots \\ A_{n-1} - A_n \end{array}\right\} \quad \dots . \quad (B)$$

dont la considération nous sera utile plus loin. Soit Δ' le déterminant de ce nouveau système : d'après les n^{os} (3) et (4), nous aurons

$$\begin{aligned}\Delta' = \; & \text{dét.}(A_1, -A_2, -A_3, \ldots, -A_n) \\ & + \text{dét.}(A_1, A_2, -A_3, -A_4, \ldots, -A_n) \\ & + \text{dét.}(A_1, A_2, A_3, -A_4, \ldots, -A_n) \\ & + \ldots\ldots\ldots\ldots \\ & + \text{dét.}(A_n, A_1, A_2, \ldots, A_{n-1}).\end{aligned}$$

On sait que si l'on change les signes des termes d'une colonne horizontale, le déterminant change de signe ; donc

$$\begin{aligned}\Delta' = & (-1)^{n-1}\text{dét.}(A_1, A_2, \ldots, A_n) + (-1)^{n-2}\text{dét.}(A_2, A_1, A_3, \ldots, A_n) \\ & + (-1)^{n-3}\text{dét.}(A_3, A_1, A_2, A_4, \ldots, A_n) + \ldots\ldots\ldots \\ & + (-1)\text{dét.}(A_{n-1}, A_1, A_2, \ldots, A_{n-2}, A_n) + \text{dét.}(A_n, A_1, A_2, \ldots A_{n-1}).\end{aligned}$$

Dans la première parenthèse, il n'y a pas d'inversion ; dans la seconde, il y a une inversion, etc. ; donc

$$\Delta' = (-1)^{n-1} n \Delta."$$

The theorem thus reached may be enunciated as follows:—*If from a determined Δ of the* nth *order, we form another Δ' such that the first row of Δ' is the sum of all the rows of Δ and every other row of Δ' is got by subtracting the corresponding row of Δ from the row preceding it in Δ, then*

$$\Delta' = (-1)^{n-1} n \Delta.$$

In Catalan's notation it is

$$\begin{aligned}\text{dét.}(A_1+A_2+\ldots+A_n,\ A_1-A_2,\ A_2-A_3, \ldots, A_{n-1}-A_n) \\ = (-1)^{n-1} n \,.\, \text{dét.}(A_1, A_2, \ldots, A_n),\end{aligned}$$

although, strange to say, it is never so formulated by him.

A generalisation of it is next given by saying :—

"Si la première ligne du système (B) avait renfermé seulement p des quantites $A_1, A_2, \ldots, A_n$, nous aurions trouvé, pour la déterminant de ce système,

$$\Delta' = (-1)^{n-1} p \Delta,"$$

and then there follow a number of applications to the evaluation of certain special determinants. Thus, to take the simplest example, having

$$\Delta = \begin{vmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{vmatrix} = 1$$

the theorem gives

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & . & . \\ . & 1 & -1 & . \\ . & . & -1 & -1 \end{vmatrix} = (-1)^3 4\Delta = -4.$$

The other illustrations all concern determinants of the special form afterwards known as "circulants"; for example, $C(-1, 1, 1, \ldots, 1)$, $C(-1, -1, 1, 1, \ldots, 1)$, etc., $C(1, 1, \ldots, 1, 0)$, $C(1, 1, \ldots, 1, 0, 0)$, etc. They therefore fall to be dealt with in a different place.

SARRUS, P. F. (1846).

[Finck, P. J. E. Éléments d'Algèbre. Seconde édition. iv + 544 pages. Strasbourg.]

In the course of his discussion of the solution of a set of linear equations with three unknowns, the author interjects the following paragraph (No. 52, p. 95):—

"Pour calculer, dans un exemple donné, les valeurs de x, y et z, M. Sarrus a imaginé la méthode pratique suivante, qui est fort ingénieuse. D'abord on peut calculer le dénominateur, et à cet effet on écrit les coefficients des inconnues ainsi

$$\begin{array}{rccc} & a & b & c \\ & a' & b' & c' \\ & a'' & b'' & c'' \\ \text{On répète les trois premiers} & a & b & c \\ \text{et les trois suivants} & a' & b' & c'. \end{array}$$

Actuellement partant de a, on prend diagonalement du haut en bas, en descendant à la fois d'un rang, et reculant d'autant à droite, $ab'c''$: on part de a' de même, et on a $a'b''c$; de a'', et on trouve $a''bc'$; on a ainsi les trois termes positifs (c'est-à-dire à prendre avec leur signes) du dénominateur. On commence ensuite par c et descendant de même vers la gauche on a $cb'a''$, $c'b''a$, $c''ba'$, ou les trois termes négatifs (ou plutôt les termes qu'il faut changer de signe)."

This "méthode pratique" or mnemonic is the original form of the so-called "règle de Sarrus" which came later to have unnecessary prominence given to it by writers on determinants when dealing with those of the third order.*

* The date 1833 has been assigned to this "rule" in a recent German text-book on determinants (Weichold's), but without adducing any evidence.

HANSEN, P. A. (1846, Aug.).

[Ueber eine allgemeine Auflösung eines beliebigen Systems von linearischen Gleichungen. *Berichte* . . . *k. sächs. Ges. d. Wiss.* (Leipzig), pp. 333–339.]

The solution of a set of linear equations is here looked at from the point of view of an astronomical computer, and though determinants are not used—are, in fact, eschewed—it is interesting to note that the hidden identities on which the new process of solution depends are

$$\frac{|a_3b_2|}{|a_1b_2|} = \frac{a_3}{a_1} + \frac{|a_3b_1|}{|a_1b_2|}\cdot\frac{a_2}{a_1},$$

$$\frac{|a_4b_2c_3|}{|a_1b_2c_3|} = \frac{a_4}{a_1} + \frac{|a_4b_1|}{|a_1b_2|}\cdot\frac{a_2}{a_1} + \frac{|a_4b_1c_2|}{|a_1b_2c_3|}\cdot\frac{|a_2b_3|}{|a_1b_2|},$$

$$\frac{|a_5b_2c_3d_4|}{|a_1b_2c_3d_4|} = \frac{a_5}{a_1} + \frac{|a_5b_1|}{|a_1b_2|}\cdot\frac{a_2}{a_1} + \frac{|a_5b_1c_2|}{|a_1b_2c_3|}\cdot\frac{|a_2b_3|}{|a_1b_2|} + \frac{|a_5b_1c_2d_3|}{|a_1b_2c_3d_4|}\cdot\frac{|a_2b_3c_4|}{|a_1b_2c_3|},$$

. . . .

and that these are to be found explicitly stated in Schweins' memoir of the year 1825.

TERQUEM, O. (1846).

[Note sur les équations du premier degré en nombre plus grand que celui des inconnues. . . . *Nouv. Annales de Math.*, v. pp. 551–556.]

Knowing from the four equations (Terquem uses n)

$$\left.\begin{array}{l} a_1x + a_2y + a_3z + a_4w = a_5 \\ b_1x + b_2y + b_3z + b_4w = b_5 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}$$

the usual expressions for $w, z, \ldots$ Terquem affirms that if w is to be equal to 0 we must have

$$|a_1\,b_2\,c_3\,d_5| = 0,$$

and that therefore this last equation is the equation of condition for the simultaneous existence of four equations between three unknowns. Continuing, he says that if we are to have $z = w = 0$, we must have

$$| a_1 b_2 c_3 d_5 | = | a_1 b_2 c_4 d_5 | = 0,^* \qquad (\beta)$$

and that therefore these two equations are "les deux équations de condition pour que 4 équations entre 2 inconnues puissent être satisfaites par les mêmes valeurs." The words "et ainsi de suite" are added to draw attention to the general theorem.

On this we can only remark that the giving of the equations of condition in the form (β) in the second case, even although the real equations of condition

$$\left\| \begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_5 & b_5 & c_5 & d_5 \end{matrix} \right\| = 0$$

are thence deducible, seems quite inexcusable, especially in an exposition meant to be elementary.

CAYLEY, A. (1847).

[Sur les déterminants gauches. *Crelle's Journ.*, xxxviii. pp. 93–96; or *Collected Math. Papers*, i. pp. 410–413.]

As the title implies, the subject of this paper is not *general* determinants. Part of the purpose, however, which the author had in view necessitated reflection on the definition of such determinants, and the outcome was a suggestion which it would be a serious mistake to pass over. In explaining the character of the functions known afterwards as Pfaffians, and which he was about to show were closely connected with skew determinants, it was natural that he should be struck with certain points of resemblance between them and general determinants, and that in consequence he should seek a general definition which would include both. The new definition given is

"\. . . . en exprimant par (1 2 . . . n) une fonction quelconque dans laquelle entrent les nombres symboliques 1, 2, . . . , n, et par $\pm$ le signe

* The second determinant is incorrectly printed in the original.

correspondant à une permutation quelconque de ces nombres, la fonction

$$\Sigma \pm (1\ 2\ \ldots\ n)$$

(où Σ désigne la somme de tous les termes qu'on obtient en permutant ces nombres d'une manière quelconque) est ce qu'on nomme *Déterminant.*"

It is readily seen that this is much more general than any definition in use up to that time, and that it agrees with the ordinary definition only when the function $(1\ 2\ \ldots\ n)$ takes the particular form $\lambda_{\alpha 1}\lambda_{\beta 2}\ldots\lambda_{\kappa n}$ or $\lambda_{1\alpha}\lambda_{2\beta}\ldots\lambda_{n\kappa}$. Further, it is not the same generalisation as we are familiar with from Cauchy's great memoir of 1812, where determinants are viewed as a special class of *alternating symmetric functions.* This is shown quite clearly by the only other case brought forward by Cayley, viz., the case where the function $(1\ 2\ \ldots\ n)$ is given the form $\lambda_{12}\lambda_{34}\ldots\lambda_{n-1,n}$. Other examples, not given by Cayley, are—

$$\sum \pm a_{123} \quad i.e. \quad a_{123}-a_{132}-a_{213}+a_{231}+a_{312}-a_{321},$$

$$\sum \pm \frac{a_{12}}{a_{23}} \quad i.e. \quad \frac{a_{12}}{a_{23}}-\frac{a_{13}}{a_{32}}-\frac{a_{21}}{a_{13}}+\frac{a_{23}}{a_{31}}+\frac{a_{31}}{a_{12}}-\frac{a_{32}}{a_{21}},$$

. . . .

There is also in the same paper a more direct contribution to the theory of general determinants, viz., the theorem afterwards associated with Cayley's name, and which,—to use later phraseology,—gives the expression for a determinant in terms of its own devertebrated coaxial minors and its primary diagonal elements. In the actual wording of the description of the theorem it is, not unnaturally, applied to a skew determinant only; but there is clearly nothing in the nature of the case to confine it to this special form. The description is—

"En effet, soit Ω le déterminant gauche dont il s'agit, cette fonction peut être présentée sous la forme

$$\Omega = \Omega_0 + \Omega_1\lambda_{11} + \Omega_2\lambda_{22} + \ldots + \Omega_{12}\lambda_{11}\lambda_{22} + \ldots$$

où Ω_0 est ce que devient Ω si λ_{11}, λ_{22}, . . . sont réduits à zéro. Ω_1 est ce que devient le coëfficient de λ_{11} sous la même condition, et ainsi de suite; c'est-à-dire: Ω_0 est le déterminant formé par les quantités λ_{rs} en supposant que ces quantités satisfassent aux condition (2), et en donnant à r les valeurs 1, 2, . . , n. Ω_1 est le déterminant formé pareillement en donnant à r, s les valeurs 2, 3, . . . , n; Ω_2 s'obtient en donnant à r, s les valeurs 1, 3, . . . , n, et ainsi de suite: cela est aisé de voir si l'on range les quantités λ_{rs} en forme de carré."

CAUCHY, A. L. (1847).

[Mémoire sur les clefs algébriques. *Exercices d'Analyse et de Phys. Math.*, iv. pp. 356–400, § 11; or *Œuvres complètes* (2), xiv.]

In this longish memoir it is the second section (§ ii.) that is of interest to us, its title being "Décomposition des sommes alternées, connues sous le nom de résultantes, en facteurs symboliques." The only previous writing with which it is clearly connected in subject is Grassmann's of the year 1844. To ensure the possibility of proper comparison between the two, it is necessary to do now as was done in the previous case, viz., to give the opening paragraph verbatim. No general explanation of 'algebraic keys' need be offered, all that is requisite for our present purpose being obtainable from the paragraph itself. It runs as follows:—

"En effet, considérons d'abord la somme alternée s, formée avec les quatre termes du tableau

$$(1) \qquad \begin{cases} a_1 & b_1 \\ a_2 & b_2 \end{cases}$$

et fournie par l'équation

$$(2) \qquad s = S(\pm a_1 b_2) = a_1 b_2 - a_2 b_1.$$

Si l'on donne pour coefficients à deux clefs algébriques α, β dans deux fonctions linéaires λ, μ les termes qui renferment la première et la seconde ligne horizontale du tableau (1) on aura non-seulement

$$(3) \qquad \begin{aligned} \lambda &= a_1\alpha + b_1\beta, \\ \mu &= a_2\alpha + b_2\beta, \end{aligned}$$

mais encore

$$(4) \qquad |\lambda\mu| = a_1 a_2 |\alpha^2| + a_1 b_2 |\alpha\beta| + b_1 a_2 |\beta\alpha| + b_1 b_2 |\beta^2|;$$

et, pour que le produit symbolique $|\lambda\mu|$ se réduise à la résultante s, il suffira évidemment de poser

$$(5) \qquad |\alpha^2| = 0, \quad |\alpha\beta| = 1, \quad |\beta\alpha| = -1, \quad |\beta^2| = 0.$$

Sous cette condition, l'on aura

$$(6) \qquad s = |\lambda\mu|;$$

et λ, μ seront les *facteurs symboliques* de la résultante s.

Si aux formules (5) on substituait les suivantes:

$$(7) \qquad |\alpha^2| = 0, \quad |\beta\alpha| = -|\alpha\beta|, \quad |\beta^2| = 0,$$

alors, à la place de l'équation (6) on obtiendrait la formule

$$|\lambda\mu| = s\,|\alpha\beta|,$$

ou

(8) $$s = \frac{|\lambda\mu|}{|\alpha\beta|},$$

dans laquelle il suffirait de poser $|\alpha\beta| = 1$ pour retrouver l'équation (6). Remarquons d'ailleurs que la seconde des formules (7) peut s'écrire comme il suit:

(9) $$|\alpha\beta| + |\beta\alpha| = 0,$$

et que la formule (9) donne $|\alpha^2| = 0$ ou $|\beta^2| = 0$, quand on y suppose $\beta = \alpha$. Donc, et définitive, les transmutations (7) sont toutes trois comprises dans la formule (9). Donc il suffit de recourir à cette formule et à celles qui s'en déduisent, pour obtenir l'équation (8), et, par suite, pour décomposer en facteurs symboliques la somme alternée s, c'est-à-dire la rèsultante algébrique formée avec les quatre termes du tableau (1)."

From the case of the second order he proceeds at once to the case of the n^{th} order, inquiring as before under what conditions the symbolic product

$$\begin{aligned}
&(a_1\alpha + b_1\beta + c_1\gamma + \ \ldots\ + h_1\eta)\\
.\,&(a_2\alpha + b_2\beta + c_2\gamma + \ \ldots\ + h_2\eta)\\
.\,&(a_3\alpha + b_3\beta + c_3\gamma + \ \ldots\ + h_3\eta)\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
.\,&(a_n\alpha + b_n\beta + c_n\gamma + \ \ldots\ + h_n\eta),
\end{aligned}$$

where $\alpha, \beta, \gamma, \ldots, \eta$ are n distinct 'algebraic keys,' reduces to

$$S(\pm a_1 b_2 c_3 \ldots h_n).$$

Denoting by $|\kappa|$ the symbolic portion of any term of the final product, he finds that

"on devra poser

(14) $$|\kappa| = 0$$

quand l'une quelconque des lettres

$$\alpha,\ \beta,\ \gamma,\ \ldots,\ \eta$$

entrera deux ou plusieurs fois comme facteur dans le produit κ; et poser, au contraire,

(15) $$|\kappa| = 1,$$

ou

(16) $$|\kappa| = -1$$

quand le produit κ renfermera une seule fois chacune des lettres

$$\alpha,\ \beta,\ \gamma,\ \ldots,\ \eta,$$

la formule (15) étant relative au cas où l'on sera obligé d'opérer entre ces lettres prises deux à deux un nombre pair d'échanges, pour passer du produit $|\,\alpha\beta\gamma\,\ldots\,\eta\,|$ au produit $|\,\kappa\,|$."

The obtaining of the results

$$\left.\begin{array}{rl} |\,\alpha\beta\gamma\,| = \quad |\,\beta\gamma\alpha\,| = \quad |\,\gamma\alpha\beta\,| \\ = -\,|\,\alpha\gamma\beta\,| = -\,|\,\beta\alpha\gamma\,| = -\,|\,\gamma\beta\alpha\,| \end{array}\right\}$$

from transformations of the form

$$|\,\alpha\beta\,| = -\,|\,\beta\alpha\,|$$

is then shortly considered; and this is followed by a concluding paragraph, in which the statement occurs that "cette décomposition (des sommes alternées en facteurs symboliques) une fois operée on peut s'en servir avec avantage pour découvrir ou pour démontrer les principales propriétés des sommes alternées."

Cauchy's position is thus seen to be very different from Grassmann's. Grassmann was not concerned with determinants: his problem was to solve the set of equations

$$\left.\begin{array}{l} a_1x + a_2y + a_3z = a_0 \\ b_1x + b_2y + b_3z = b_0 \\ c_1x + c_2y + c_3z = c_0 \end{array}\right\}$$

and he satisfied himself by a curious process of reasoning that

$$x = \frac{(a_0+b_0+c_0)(a_2+b_2+c_2)(a_3+b_3+c_3)}{(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)},$$

provided that the multiplications indicated be performed according to the laws of "outer multiplication." Cauchy, on the other hand, starts with the determinant formed from

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3, \end{array}$$

his problem being to find under what conditions, as regards the arbitrarily introduced symbols α, β, γ, the product

$$(a_1\alpha+a_2\beta+a_3\gamma)(b_1\alpha+b_2\beta+b_3\gamma)(c_1\alpha+c_2\beta+c_3\gamma)$$

will be identical with the determinant; and he is led to the need for imposing laws essentially the same as those of "outer multiplication." Grassmann's factors are each the sum of the elements of a *column* of the determinant, and, according to his quasi-demonstration, could not be anything else: Cauchy's factors are each formed from the elements of a *row*; but had they been formed from the elements of a column, the result would not have been different. The root idea, viz., *the expression of a determinant as a product of factors subject to multiplication of a special kind,* was certainly first reached by Grassmann: Cauchy attained the same result, adding somewhat to its width, and presenting it in a fresh and more reasonable form.

HERMITE, C. (1849, January).

[Sur une question relative à la théorie des nombres. *Journ.* (*de Liouville*) *de Math.*, xiv. pp. 21–30; or *Œuvres*, i. pp. 265–273.]

As a lemma in the process of attaining the main purpose of his paper Hermite gives an identity which for the 4th order we should nowadays write in the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = -\frac{1}{b_1c_1}\begin{vmatrix} |a_2b_1| & |a_3b_1| & |a_4b_1| \\ |b_2c_1| & |b_3c_1| & |b_4c_1| \\ |c_2d_1| & |c_3d_1| & |c_4d_1| \end{vmatrix}.$$

This he establishes rather circuitously by taking four quantities $\xi_1, \xi_2, \xi_3, \xi_4$ which satisfy the equations

$$\left.\begin{aligned} a_1\xi_1 + b_1\xi_2 + c_1\xi_3 + d_1\xi_4 &= 1 \\ a_2\xi_1 + b_2\xi_2 + c_2\xi_3 + d_2\xi_4 &= 0 \\ a_3\xi_1 + b_3\xi_2 + c_3\xi_3 + d_3\xi_4 &= 0 \\ a_4\xi_1 + b_4\xi_2 + c_4\xi_3 + d_4\xi_4 &= 0 \end{aligned}\right\},$$

and then multiplying the original determinant columnwise by $(-1)^3b_1c_1$ in the form

$$\begin{vmatrix} \xi_1 & b_1 & \cdot & \cdot \\ \xi_2 & -a_1 & c_1 & \cdot \\ \xi_3 & \cdot & -b_1 & d_1 \\ \xi_4 & \cdot & \cdot & -c_1 \end{vmatrix}.$$

JOACHIMSTHAL, F. (1849, Nov.).

[Sur quelques applications des déterminants à la géométrie. *Crelle's Journ.*, xi. pp. 21–47.]

Joachimsthal's interesting series of 'applications' being mainly connected with the multiplication of determinants, he introduces them by enunciating the multiplication-theorem, and indicating a mode of proof "pour éviter aux lecteurs la peine de la chercher ailleurs."

The enunciation is

$$\text{dét.}\left\{\begin{matrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix}\right\} \times \text{dét.}\left\{\begin{matrix} \xi & \eta & \zeta \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \end{matrix}\right\}$$

$$= \text{dét.}\left\{\begin{matrix} x\xi + y\eta + z\zeta & x\xi_1 + y\eta_1 + z\zeta_1 & x\xi_2 + y\eta_2 + z\zeta_2 \\ x_1\xi + y_1\eta + z_1\zeta & x_1\xi_1 + y_1\eta_1 + z_1\zeta_1 & x_1\xi_2 + y_1\eta_2 + z_1\zeta_2 \\ x_2\xi + y_2\eta + z_2\zeta & x_2\xi_1 + y_2\eta_1 + z_2\zeta_1 & x_2\xi_2 + y_2\eta_2 + z_2\zeta_2 \end{matrix}\right\}$$

and the proof, it is stated, consists in taking the equations

$$\begin{aligned} h &= x\mathrm{U} + y\mathrm{V} + z\mathrm{W}, & \mathrm{U} &= \xi u + \xi_1 v + \xi_2 w, \\ h_1 &= x_1\mathrm{U} + y_1\mathrm{V} + z_1\mathrm{W}, & \mathrm{V} &= \eta u + \eta_1 v + \eta_2 w, \\ h_2 &= x_2\mathrm{U} + y_2\mathrm{V} + z_2\mathrm{W}, & \mathrm{W} &= \zeta u + \zeta_1 v + \zeta_2 w, \end{aligned}$$

deducing from them two different triads of equations independent of U, V, W, solving these triads for u, v, w, and then comparing the results.*

In connection with this two points have to be noted. The first is the use of the notation

$$\text{dét.}\left\{\qquad\qquad\right\},$$

which may be compared with Catalan's of the year 1846, and

* The details of the proof not being given, one cannot guess how it was that a second theorem was not obtained, viz., the theorem

$$\begin{vmatrix} h & x\xi_1 + y\eta_1 + z\zeta_1 & x\xi_2 + y\eta_2 + z\zeta_2 \\ h_1 & x_1\xi_1 + y_1\eta_1 + z_1\zeta_1 & x_1\xi_2 + y_1\eta_2 + z_1\zeta_2 \\ h_2 & x_2\xi_1 + y_2\eta_1 + z_2\zeta_1 & x_2\xi_2 + y_2\eta_2 + z_2\zeta_2 \end{vmatrix} = \begin{Vmatrix} |hy_1z_2| & \xi_1 & \xi_2 \\ |xh_1z_2| & \eta_1 & \eta_2 \\ |xy_1h_2| & \zeta_1 & \zeta_2 \end{Vmatrix}.$$

with the modification of the vertical-line notation which the printers of Crelle's and Liouville's journals employed for two or three years in setting up Cayley's papers. That Joachimsthal was familiar with Catalan's paper of 1846 is made more probable by the occurrence of a footnote (p. 28) giving

$$\text{dét.}\left\{\begin{matrix} x+l & y & z \\ x'+l' & y' & z' \\ x''+l'' & y'' & z'' \end{matrix}\right\} = \text{dét.}\left\{\begin{matrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{matrix}\right\} + \text{dét.}\left\{\begin{matrix} l & y & z \\ l' & y' & z' \\ l'' & y'' & z'' \end{matrix}\right\};$$

the identity which Catalan was the first to formulate in a similar way.

The second point is the unnaturalness, in view of the mode of proof, of not writing the second determinant of the theorem in the form

$$\text{dét.}\left\{\begin{matrix} \xi & \xi_1 & \xi_2 \\ \eta & \eta_1 & \eta_2 \\ \zeta & \zeta_1 & \zeta_2 \end{matrix}\right\}.$$

Towards the close of the paper (p. 44), his geometrical work having led him to use the identity

$$\begin{aligned}(\alpha'\beta'+\alpha''\beta''+\alpha'''\beta''')(\gamma'\delta'+\gamma''\delta''+\gamma'''\delta''') &- (\alpha'\delta'+\alpha''\delta''+\alpha'''\delta''')(\beta'\gamma'+\beta''\gamma''+\beta'''\gamma''') \\ = (\alpha''\gamma'''-\alpha'''\gamma'')(\beta''\delta'''-\beta'''\delta'') &+ (\alpha'\gamma'''-\alpha'''\gamma')(\beta'\delta'''-\beta'''\delta') \\ &+ (\alpha'\gamma''-\alpha''\gamma')(\beta'\delta''-\beta''\delta'),\end{aligned}$$

he devotes the last two pages to proving a generalisation of it,—a generalisation not so wide as that of Binet and Cauchy, but interesting because of the way in which it is arrived at. Putting

$$\text{D} \equiv \text{dét.}\left\{\begin{matrix} x_0 & y_0 & z_0 & \dots & t_0 \\ x_1 & y_1 & z_1 & \dots & t_1 \\ x_2 & y_2 & z_2 & \dots & t_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n & y_n & z_n & \dots & t_n \end{matrix}\right\}, \quad \Delta \equiv \text{dét.}\left\{\begin{matrix} \xi_0 & \eta_0 & \zeta_0 & \dots & \tau_0 \\ \xi_1 & \eta_1 & \zeta_1 & \dots & \tau_1 \\ \xi_2 & \eta_2 & \zeta_2 & \dots & \tau_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_n & \eta_n & \zeta_n & \dots & \tau_n \end{matrix}\right\}$$

$$\text{and} \quad \text{N} \equiv \text{dét.}\left\{\begin{matrix} l_{0,0} & l_{0,1} & l_{0,2} & \dots & l_{0,n} \\ l_{1,0} & l_{1,1} & l_{1,2} & \dots & l_{1,n} \\ l_{2,0} & l_{2,1} & l_{2,2} & \dots & l_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l_{n,0} & l_{n,1} & l_{n,} & \dots & l_{n,n} \end{matrix}\right\}$$

where

$$\begin{array}{l} l_{0,0} = x_0\xi_0 + y_0\eta_0 + z_0\zeta_0 + \ldots + t_0\tau_0, \\ l_{0,1} = x_0\xi_1 + y_0\eta_1 + z_0\zeta_1 + \ldots + t_0\tau_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ l_{n,0} = x_n\xi_0 + y_n\eta_0 + z_n\zeta_0 + \ldots + t_n\tau_0, \\ l_{n,1} = x_n\xi_1 + y_n\eta_1 + z_n\zeta_1 + \ldots + t_n\tau_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ l_{n,n} = x_n\xi_n + y_n\eta_n + z_n\zeta_n + \ldots + t_n\tau_n; \end{array}$$

and where therefore

$$D\Delta = N,$$

he differentiates both members of this identity with respect to $x_n, y_n, z_n, \ldots t_n$, obtaining

$$\begin{array}{l} \Delta\dfrac{\partial D}{\partial x_n} = \xi_0\dfrac{\partial N}{\partial l_{n,0}} + \xi_1\dfrac{\partial N}{\partial l_{n,1}} + \ldots + \xi_n\dfrac{\partial N}{\partial l_{n,n}}, \\ \Delta\dfrac{\partial D}{\partial y_n} = \eta_0\dfrac{\partial N}{\partial l_{n,0}} + \eta_1\dfrac{\partial N}{\partial l_{n,1}} + \ldots + \eta_n\dfrac{\partial N}{\partial l_{n,n}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta\dfrac{\partial D}{\partial t_n} = \tau_0\dfrac{\partial N}{\partial l_{n,0}} + \tau_1\dfrac{\partial N}{\partial l_{n,1}} + \ldots + \tau_n\dfrac{\partial N}{\partial l_{n,n}}. \end{array}$$

From these last equations, on multiplying by

$$\frac{\partial \Delta}{\partial \xi_n}, \frac{\partial \Delta}{\partial \eta_n}, \ldots, \frac{\partial \Delta}{\partial \tau_n}$$

respectively, and adding, there is obtained with the help of the known identities

$$\begin{array}{l} \xi_0\dfrac{\partial \Delta}{\partial \xi_n} + \eta_0\dfrac{\partial \Delta}{\partial \eta_n} + \ldots + \tau_0\dfrac{\partial \Delta}{\partial \tau_n} = 0, \\ \xi_1\dfrac{\partial \Delta}{\partial \xi_n} + \eta_1\dfrac{\partial \Delta}{\partial \eta_n} + \ldots + \tau_1\dfrac{\partial \Delta}{\partial \tau_n} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_n\dfrac{\partial \Delta}{\partial \xi_n} + \eta_n\dfrac{\partial \Delta}{\partial \eta_n} + \ldots + \tau_n\dfrac{\partial \Delta}{\partial \tau_n} = \Delta, \end{array}$$

the result

$$\frac{\partial D}{\partial x_n}\cdot\frac{\partial \Delta}{\partial \xi_n} + \frac{\partial D}{\partial y_n}\cdot\frac{\partial \Delta}{\partial \eta_n} + \ldots + \frac{\partial D}{\partial t_n}\cdot\frac{\partial \Delta}{\partial \tau_n} = \frac{\partial N}{\partial l_{n,n}},$$

i.e.

$$\text{dét.}\begin{Bmatrix} y_0 & z_0 & \cdots & t_0 \\ y_1 & z_1 & \cdots & t_1 \\ \cdot & \cdot & \cdot & \cdot \\ y_{n-1} & z_{n-1} & \cdots & t_{n-1} \end{Bmatrix} . \text{dét.}\begin{Bmatrix} \eta_0 & \zeta_0 & \cdots & \tau_0 \\ \eta_1 & \zeta_1 & \cdots & \tau_1 \\ \cdot & \cdot & \cdot & \cdot \\ \eta_{n-1} & \zeta_{n-1} & \cdots & \tau_{n-1} \end{Bmatrix}$$

$$+ \text{dét.}\begin{Bmatrix} x_0 & z_0 & \cdots & t_0 \\ x_1 & z_1 & \cdots & t_1 \\ \cdot & \cdot & \cdot & \cdot \\ x_{n-1} & z_{n-1} & \cdots & t_{n-1} \end{Bmatrix} . \text{dét.}\begin{Bmatrix} \xi_0 & \zeta_0 & \cdots & \tau_0 \\ \xi_1 & \zeta_1 & \cdots & \tau_1 \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{n-1} & \zeta_{n-1} & \cdots & \tau_{n-1} \end{Bmatrix} + \cdots$$

$$= \text{dét.}\begin{Bmatrix} l_{0,0} & l_{0,1} & \cdots & l_{0,n-1} \\ l_{1,0} & l_{1,1} & \cdots & l_{1,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ l_{n-1,0} & l_{n-1,1} & \cdots & l_{n-1,n-1} \end{Bmatrix}.$$

Using later phraseology we may say in describing this that from the theorem regarding the product of two square matrices of order $n+1$ there is obtained the theorem regarding the product of two rectangular but non-quadrate matrices, the latter product appearing as a principal coaxial minor of the former. Cauchy's generalisation concerned *any* minor of the former product, but even this further extension was not beyond Joachimsthal's reach, for he ends with the remark "En différentiant de nouveau par rapport à x_{n-1}, y_{n-1}, . . . on obtiendra d'autres formules ; et ainsi de suite."

SYLVESTER, J. J. (1850, Sep.).

[Additions to the articles in the September number of this journal "On a new class of theorems . . . ." and "On Pascal's theorem." *Philos. Magazine* (3), xxxvii. pp. 363–370 ; *Collected Math. Papers*, i. pp. 145–151.]

Of the three additions referred to in the title it is the last which concerns us, viz., that in which Sylvester introduces and explains his use of the term *minor* as applied to a determinant. Starting with the 'square array' of a given determinant, and leaving out one 'line' and one 'column,' he calls the determinant of the minor array which remains a 'First Minor Determinant';

similarly 'Second Minor Determinant' is explained; and then he adds,

"and so in general we can form a system of r^{th} minor determinants by the exclusion of r lines and r columns, and such system in general will contain

$$\left\{\frac{n(n-1)\ .\ .\ .\ (n-r+1)}{1\cdot 2\ .\ .\ .\ r}\right\}^2$$

distinct determinants."

It is thus seen that 'minor determinant is used as 'partial determinant' had already been used by Lebesgue (1837), and as 'determinant of a derived system' had been used by Cauchy (1812), but that, whereas Cauchy added a distinguishing epithet to specify the order of the determinant, Sylvester did so to indicate how many lines or columns fewer it had than the 'principal' or 'complete' determinant originally started with.

The following proposition or 'law' is next given, viz.: *The whole of a system of* r^{th} *minors being zero implies only* $(\mathrm{r}+1)^2$ *equations, that is, by making* $(\mathrm{r}+1)^2$ *of these minors zero, all will become zero: and this is true, no matter what may be the dimensions or form of the complete determinant.* Then, after some geometrical applications concerned with *first* minors of a symmetrical determinant, there follows the explanation—

"The law which I have stated for assigning the number of independent or, to speak more accurately, non-coevanescent determinants belonging to a given system of minors, I call the Homaloidal law, because it is a corollary to a proposition which represents analytically the indefinite extension of a property, common to lines and surfaces, to all loci (whether in ordinary or transcendental space) of the first order, all of which loci may, by an abstraction derived from the idea of levelness common to straight lines and planes, be called Homaloids."

A further advance is made just before the close of his paper. Leaving the *square* array and taking m lines and n columns, he says

"This will not represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p and selecting at will p lines and p columns, the squares corresponding to which may be termed determinants of the p^{th} order."

Here there is to be noticed the first use of the word *matrix* in connection with determinants, as well as the change back to Cauchy's mode of particularising the minors. The corresponding more general 'law' is said to be—*The number of uncoevanescent determinants constituting a system of the* pth *order derived from a given matrix,* n *terms broad and* m *terms deep, may equal but can never exceed in number*

$$(n-p+1)(m-p+1).$$

No proof of this is given, nor does he refer to Cayley's related theorem of 1843. (See p. 15 above.)

SCHLÄFLI, L. (1851, Jan.).

[Ueber die Resultante eines Systems mehrerer algebraischen Gleichungen: ein Beitrag zur Theorie der Elimination. *Denkschr. d. k. Akad. d. Wiss.* (Wien): *math.-naturw. Cl.*, iv. (2), pp. 1–54.]

The last section (§ 29, pp. 52–54) of this long memoir bears the heading "Ein zur Theorie der Determinanten gehörender Satz," being introduced as an aid to the reading of a geometrical memoir (pp. 54–74) on the ternary cubic.

The 'Satz' is not formally enunciated, but we may express it for ourselves as follows: *If a set of* n *homogeneous equations of the first degree in* n *unknowns be given, the determinant of the set being* Δ, *and there be formed another set consisting of all equations of the* rth *degree derivable from the equations of the given set by multiplication among themselves, the determinant of the latter set is equal to*

$$\Delta^{C_{n+r-1,\,n}}.$$

For example, the initial set of equations being

$$\left.\begin{aligned} a_1x + a_2y &= 0 \\ b_1x + b_2y &= 0 \end{aligned}\right\}$$

and $r = 3$, the derived set of equations is

$$\left.\begin{aligned} (a_1x+a_2y)^3 &= 0 \\ (a_1x+a_2y)^2(b_1x+b_2y) &= 0 \\ (a_1x+a_2y)\,(b_1x+b_2y)^2 &= 0 \\ (b_1x+b_2y)^3 &= 0 \end{aligned}\right\}$$

and it has to be shown that

$$\begin{vmatrix} a_1^3 & 3a_1^2a_2 & 3a_1a_2^2 & a_2^3 \\ a_1^2b_1 & 2a_1a_2b_1 + a_1^2b_2 & 2a_1a_2b_2 + a_2^2b_1 & a_2^2b_2 \\ a_1b_1^2 & 2b_1b_2a_1 + b_1^2a_2 & 2b_1b_2a_2 + b_2^2a_1 & a_2b_2^2 \\ b_1^3 & 3b_1^2b_2 & 3b_1b_2^2 & b_2^3 \end{vmatrix} = |a_1b_2|^6.$$

To do this Schläfli takes the conjugate of the adjugate of $|a_1b_2|$, namely,

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix};$$

forms with its elements the like determinant

$$\begin{vmatrix} A_1^3 & 3A_1^2B_1 & 3A_1B_1^2 & B_1^3 \\ A_1^2A_2 & 2A_1B_1A_2 + A_1^2B_2 & 2A_1B_1B_2 + B_1^2A_2 & B_1^2B_2 \\ A_1A_2^2 & 2A_2B_2A_1 + A_2^2B_1 & 2A_2B_2B_1 + B_2^2A_1 & B_1B_2^2 \\ A_2^3 & 3A_2^2B_2 & 3A_2B_2^2 & B_2^3 \end{vmatrix};$$

and then in row-by-column fashion multiplies the former four-line determinant by the latter, obtaining for the product

$$\begin{vmatrix} |a_1b_2|^3 & . & . & . \\ . & |a_1b_2|^3 & . & . \\ . & . & |a_1b_2|^3 & . \\ . & . & . & |a_1b_2|^3 \end{vmatrix}$$

whence the desired result follows.

It is the penultimate step of the demonstration, namely, the obtaining of the final form of the elements of the product, that is attended with difficulty: and the difficulty is not lessened to Schläfli by his taking n and r in all their generality. As soon, however, as the determinant under investigation is known in this or any other way to be a power of Δ, the index of the power is readily found to be

$$\frac{r}{n} \cdot C_{n+r-1,\,r}$$

by noting that the determinant is of the order $C_{n+r-1,\,r}$ and each of its elements of the r^{th} degree.

SPOTTISWOODE, W. (1851).

[ELEMENTARY THEOREMS RELATING TO DETERMINANTS. viii + 63 pp. London.]

This is noteworthy as being the first separately-published elementary work on the subject, the author explaining that he had been led to write it because determinants had come to be in frequent use, and there was no accessible text-book to which students could be referred. It consists of a preface and ten short chapters or sections, the mode of partitioning and arranging the matter being such as has often subsequently been followed.

The preface contains a short sketch of the history of the theory, the first of the kind that had appeared. In the first section (§ 1) the reader is introduced to determinants of the second and third orders as they actually occur in the solution of geometrical problems, and certain of the simpler properties are incidentally pointed out; the next seven sections (§§ 2-8) deal with determinants in general; and the rest of the book is occupied with determinants of special form, viz., § 9 with skew determinants and § 10 with functional determinants. The concluding portions of most of the sections consist of worked examples illustrative of applications of the theory to co-ordinate geometry.

The definition employed is that of Vandermonde * as expressed in Cayley's vertical-line notation, viz.—

$$|(1,1)| = (1,1),$$

$$\begin{vmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{vmatrix} = (1,1)\,|(2,2)| - (1,2)\,|(2,1)|,$$

* It may be worth noting that while both Vandermonde and Schweins used the recurrent law of formation as a definition, they did not write it in exactly the same form. Schweins followed closely the form used by the original discoverer, Bezout, putting for example

$$|a_1b_2c_3d_4| = d_4\,|a_1b_2c_3| - d_3\,|a_1b_2c_4| + d_2\,|a_1b_3c_4| - d_1\,|a_2b_3c_4|,$$

the connecting signs being in all cases alternately positive and negative; whereas Vandermonde wrote

$$|a_1b_2c_3d_4| = a_1\,|b_2c_3d_4| - a_2\,|b_3c_4d_1| + a_3\,|b_4c_1d_2| - a_4\,|b_1c_2d_3|,$$

where the cyclical change of suffixes causes the connecting signs to be alternately positive and negative when the order of the determinant is even, and to be uniformly positive when the order is odd.

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{vmatrix} = (1,1)\begin{vmatrix} (2,2) & (2,3) \\ (3,2) & (3,3) \end{vmatrix} + (1,2)\begin{vmatrix} (2,3) & (2,1) \\ (3,3) & (3,1) \end{vmatrix} + (1,3)\begin{vmatrix} (2,1) & (2,2) \\ (3,1) & (3,2) \end{vmatrix},$$

. . . .

The quantities $(1,1)$, $(1,2)$, . . . which Cauchy called 'terms' and Jacobi 'elements,' are named '*constituents*'; and the determinant of the n^{th} order having these constituents is denoted shortly by

$$\sum \pm (1,1)(2,2)\ldots(n,n)$$

The first result, deduced in somewhat loose fashion from the definition, is "Cramer's rule"; but the first that is actually formulated and numbered is one of much later date than Cramer, viz.—

"Theorem I.—*If the whole of a vertical or horizontal row be multiplied by the same quantity, the determinant is multiplied by that quantity.*"

In this form, as is well known, it afterwards became almost stereotyped. The second result, which is of about the same age, is that regarding a determinant whose vertical row consists of p-termed expressions, second vertical row of q-termed expressions third vertical row of r-termed expressions, and so on, being to the effect that such a determinant is expressible as a sum of pqr . . . determinants with monomial constituents. The next seven results are, like the first, new only in form, the wording being, as in Theorem I., more topographical in character than formerly, on account of the determinant being now more consciously viewed as connected with a square holding $n \cdot n$ quantities situate in n vertical rows and at the same time in n horizontal rows. The tenth result, which is a converse of the ninth, is new but unimportant, viz.—

"Theorem X.—*If a determinant of the* n^{th} *order vanishes, a system of* n *homogeneous linear equations, the coefficients of which are the constituents of the given determinant, may always be established.*"

The ninth he establishes by the method of so-called 'mathematical induction,' deducing from it the solution of

$$(r1)x_1 + (r2)x_2 + \ldots + (rn)x_n = u_r \Big\}_{r=1}^{r=n}$$

and thence the ratios of $x_0, x_1, \ldots, x_n$ in the case of

$$(r0)x_0 + (r1)x_1 + (r2)x_2 + \ldots + (rn)x_n = 0 \quad \Big\}{\scriptstyle r=n \atop r=1}$$

as originally noted by Jacobi.*

The eleventh result is the multiplication-theorem; and here anything that is noteworthy is not in the enunciation but in the proof. Beginning with two sets of equations, exactly after the manner of Joachimsthal, viz.,

$$\left.\begin{array}{l} a_1x + a_2y + a_3z = u_1 \\ b_1x + b_2y + b_3z = u_2 \\ c_1x + c_2y + c_3z = u_3 \end{array}\right\}, \quad \left.\begin{array}{l} l_1u_1 + l_2u_2 + l_3u_3 = v_1 \\ m_1u_1 + m_2u_2 + m_3u_3 = v_2 \\ n_1u_1 + n_2u_2 + n_3u_3 = v_3 \end{array}\right\},$$

he views them as six linear equations in x, y, z, u_1, u_2, u_3, and seeks to find the value of one of the first three unknowns, say x, in two different ways. Firstly, by using the first set to eliminate u_1, u_2, u_3 from the second set, he obtains

$$\left.\begin{array}{l} (l_1a_1 + l_2b_1 + l_3c_1)x + (l_1a_2 + l_2b_2 + l_3c_2)y + (l_1a_3 + l_2b_3 + l_3c_3)z = v_1 \\ (m_1a_1 + m_2b_1 + m_3c_1)x + (m_1a_2 + m_2b_2 + m_3c_2)y + (m_1a_3 + m_2b_3 + m_3c_3)z = v_2 \\ (n_1a_1 + n_2b_1 + n_3c_1)x + (n_1a_2 + n_2b_2 + n_3c_2)y + (n_1a_3 + n_2b_3 + n_3c_3)z = v_3 \end{array}\right\},$$

and thence

$$x = \frac{\begin{vmatrix} v_1 & l_1a_2 + l_2b_2 + l_3c_2 & l_1a_3 + l_2b_3 + l_3c_3 \\ v_2 & m_1a_2 + m_2b_2 + m_3c_2 & m_1a_3 + m_2b_3 + m_3c_3 \\ v_3 & n_1a_2 + n_2b_2 + n_3c_2 & n_1a_3 + n_2b_3 + n_3c_3 \end{vmatrix}}{\begin{vmatrix} l_1a_1 + l_2b_1 + l_3c_1 & l_1a_2 + l_2b_2 + l_3c_2 & l_1a_3 + l_2b_3 + l_3c_3 \\ m_1a_1 + m_2b_1 + m_3c_1 & m_1a_2 + m_2b_2 + m_3c_2 & m_1a_3 + m_2b_3 + m_3c_3 \\ n_1a_1 + n_2b_1 + n_3c_1 & n_1a_2 + n_2b_2 + n_3c_2 & n_1a_3 + n_2b_3 + n_3c_3 \end{vmatrix}}.$$

Secondly,—and here he differs from Joachimsthal,—by writing the six equations as one set in the form

$$\left.\begin{array}{llllll} a_1x + a_2y + a_3z & - u_1 & & & = 0 \\ b_1x + b_2y + b_3z & & - u_2 & & = 0 \\ c_1x + c_2y + c_3z & & & - u_3 & = 0 \\ & l_1u_1 & + l_2u_2 & + l_3u_3 & = v_1 \\ & m_1u_1 & + m_2u_2 & + m_3u_3 & = v_2 \\ & n_1u_1 & + n_2u_2 & + n_3u_3 & = v_3 \end{array}\right\}$$

* *Crelle's Journ.*, xv. (1835), p. 104, and xxii. (1841), p. 296.

he obtains directly for x the value

$$\begin{vmatrix} . & a_2 & a_3 & -1 & . & . \\ . & b_2 & b_3 & . & -1 & . \\ . & c_2 & c_3 & . & . & -1 \\ v_1 & . & . & l_1 & l_2 & l_3 \\ v_2 & . & . & m_1 & m_2 & m_3 \\ v_3 & . & . & n_1 & n_2 & n_3 \end{vmatrix} \div \begin{vmatrix} a_1 & a_2 & a_3 & -1 & . & . \\ b_1 & b_2 & b_3 & . & -1 & . \\ c_1 & c_2 & c_3 & . & . & -1 \\ . & . & . & l_1 & l_2 & l_3 \\ . & . & . & m_1 & m_2 & m_3 \\ . & . & . & n_1 & n_2 & n_3 \end{vmatrix}$$

A comparison of the denominators in the two values of x then gives the desired result.*

The theorems of § 6, though, for some unexplained reason not formulated and numbered like the others, are of the highest importance, the subject-matter being the determinant of what is called the "inverse" system, that is to say, the determinant

$$\begin{vmatrix} [1,1] & [1,2] & \dots & [1,n] \\ [2,1] & [2,2] & \dots & [2,n] \\ . & . & . & . \\ [n,1] & [n,2] & \dots & [n,n] \end{vmatrix},$$

where $[r,s]$ is the cofactor of $(1,1)$ in $|1,2,\dots,n|$. Cauchy's theorem regarding the whole determinant is first proved, and then, instead of Jacobi's more general theorem, there is established a theorem said to include Jacobi's, viz.—

$$\begin{vmatrix} [i+1,\, i+1] & [i+1,\, i+2] & \dots & [i+1,\, n] \\ [i+2,\, i+1] & [i+2,\, i+2] & \dots & [i+2,\, n] \\ . & . & . & . \\ [n,\, i+1] & [n,\, i+2] & \dots & [n,\, n] \end{vmatrix}$$

$$= \begin{vmatrix} [i+j+1,\, i+j+1] & [i+j+1, i+j+2] & \dots & [i+j+1,\, n] \\ [i+j+2,\, i+j+1] & [i+j+2, i+j+2] & \dots & [i+j+2,\, n] \\ . & . & . & . \\ [n,\, i+j+1] & [n,\, i+j+2] & \dots & [n,\, n] \end{vmatrix} \cdot |1,2,\dots,n|^j \cdot \frac{|1,2,\dots,i|}{|1,2,\dots,i+j|}.$$

*Spottiswoode, like Joachimsthal, it will be observed, deduces nothing from a comparison of the *numerators*. Thus, by equating the two cofactors of v_1, he might have obtained

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & a_3 & -1 & . & . \\ b_2 & b_3 & . & -1 & . \\ c_2 & c_3 & . & . & -1 \\ . & . & m_1 & m_2 & m_3 \\ . & . & n_1 & n_2 & n_3 \end{vmatrix}.$$

That it does include Jacobi's is at once seen on putting $i+j+1=n$, when we have

$$\begin{vmatrix} [i+1, i+1] & [i+1, i+2] & \dots & [i+1, n] \\ [i+2\ i+1] & [i+2, i+2] & \dots & [i+2, n] \\ \cdot & \cdot & \cdot & \cdot \\ [n, i+1] & [n, i+2] & \dots & [n, n] \end{vmatrix} = [n, n]\cdot|1, 2, \dots, n|^{n-i-1}\cdot\frac{|1, 2, \dots, i|}{|1, 2, \dots, n-1|}.$$
$$= |1, 2, \dots, n|^{n-i-1}\cdot|1, 2, \dots, i|.$$

It is equally true, however, that by a double use of Jacobi's theorem Spottiswoode's follows immediately.

The next section (§ 7) deals with the *differentiation* of a determinant, and with an application of the same by Malmsten to find the n^{th} particular integral of a certain differential equation when $n-1$ particular integrals are already known.

The eighth section concerns the solution of what is called a "redundant system" of linear equations, that is to say, a system of m equations in n unknowns where $m > n$. Theorem XII. is the result obtained therefrom, and this being applied to the method of least squares, the last formulated result, Theorem XIII., is reached.

SYLVESTER, J. J. (1851, March).

[On the relation between the minor determinants of linearly equivalent quadratic functions. *Philos. Magazine* (4), i. pp. 295–305, 415; *Collected Math. Papers*, i. pp. 241–250, 251.]

In order to formulate the relation referred to in the title, that is to say, the relation between any minor of the determinant (later, discriminant) of a quadratic form and the 'pari-ordinal' minors of the determinant of the new form obtained from the old by means of a linear substitution, Sylvester found it necessary to introduce "a most powerful, because natural, method of notation" for determinants. He says—

"My method consists in expressing the same quantities biliterally as below:

$$\begin{matrix} a_1\alpha_1 & a_1\alpha_2 & \dots & a_1\alpha_n \\ a_2\alpha_1 & a_2\alpha_2 & \dots & a_2\alpha_n \\ \cdot & \cdot & \cdot & \cdot \\ a_n\alpha_1 & a_n\alpha_2 & \dots & a_n\alpha_n \end{matrix}$$

where, of course, whenever desirable, instead of $a_1, a_2, \ldots, a_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$, we may write simply $a, b, \ldots, l$ and $\alpha, \beta, \ldots, \lambda$ respectively. Each quantity is now represented by two letters; the letters themselves, taken separately, being symbols neither of quantity nor of operation, but mere umbræ or ideal elements of quantitative symbols. We have now a means of representing the determinant above given in a compact form: for this purpose we need but to write one set of umbræ over the other as follows: $\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}$. If we now wish to obtain the algebraic value of this determinant, it is only necessary to take $\alpha_1, \alpha_2, \ldots, \alpha_n$ in all its $1\cdot 2\cdot 3 \ldots n$ different positions, and we shall have

$$\begin{Bmatrix} a_1 & a_2 & \ldots & a_n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{Bmatrix} = \sum \pm \{a_1\alpha_{\theta_1} \times a_2\alpha_{\theta_2} \times \ldots \times a_n\alpha_{\theta_n}\},$$

in which expression $\theta_1, \theta_2, \ldots, \theta_n$ represents some order of the numbers $1, 2, \ldots, n$, and the positive or negative sign is to be taken according to the well-known dichotomous law."

An obvious extension of the notation is also indicated, whereby what he calls "compound" determinants may be appropriately represented. Since, in accordance with the foregoing,

$$\begin{Bmatrix} a & b \\ \alpha & \beta \end{Bmatrix}$$

is used to denote

$$a\alpha \,.\, b\beta - a\beta \,.\, b\alpha,$$

he considers that

$$\begin{Bmatrix} \overline{ab} & \overline{cd} \\ \alpha\beta & \gamma\delta \end{Bmatrix}$$

"will naturally denote

$$\begin{matrix} ab \\ \alpha\beta \end{matrix} \times \begin{matrix} cd \\ \gamma\delta \end{matrix} - \begin{matrix} ab \\ \gamma\delta \end{matrix} \times \begin{matrix} cd \\ \alpha\beta, \end{matrix}$$

that is

$$\begin{Bmatrix} (a\alpha \times b\beta) \\ -(a\beta \times b\alpha) \end{Bmatrix} \times \begin{Bmatrix} (c\gamma \times d\delta) \\ -(c\delta \times d\gamma) \end{Bmatrix} - \begin{Bmatrix} (a\gamma \times b\delta) \\ -(a\delta \times b\gamma) \end{Bmatrix} \times \begin{Bmatrix} (c\alpha \times d\beta) \\ -(c\beta \times d\alpha) \end{Bmatrix}.$$

And in general the compound determinant

$$\begin{Bmatrix} \overline{a_1 \; b_1 \; \ldots \; l_1} & \overline{a_2 \; b_2 \; \ldots \; l_2} & \ldots & \overline{a_r \; b_r \; \ldots \; l_r} \\ \alpha_1 \; \beta_1 \; \ldots \; \lambda_1 & \alpha_2 \; \beta_2 \; \ldots \; \lambda_2 & \ldots & \alpha_r \; \beta_r \; \ldots \; \lambda_r \end{Bmatrix}$$

will denote

$$\sum \pm \begin{Bmatrix} a_1 & b_1 & \ldots & l_1 \\ \alpha_{\theta_1} & \beta_{\theta_1} & \ldots & \lambda_{\theta_1} \end{Bmatrix} \times \begin{Bmatrix} a_2 & b_2 & \ldots & l_2 \\ \alpha_{\theta_2} & \beta_{\theta_2} & \ldots & \lambda_{\theta_2} \end{Bmatrix} \times \ldots \times \begin{Bmatrix} a_r & b_r & \ldots & l_r \\ \alpha_{\theta_r} & \beta_{\theta_r} & \ldots & \lambda_{\theta_r} \end{Bmatrix},$$

where, as before, we have the disjunctive equation

$$\theta_1, \theta_2, \ldots, \theta_r = 1, 2, \ldots, r."$$

As an example of the power of this notation he gives the theorem

$$\left\{\begin{matrix} \overline{a_1 \ a_2 \ \ldots \ a_r \ a_{r+1}} & \overline{a_1 \ a_2 \ \ldots \ a_r \ a_{r+2}} & \ldots\ldots & \overline{a_1 \ a_2 \ \ldots \ a_r \ a_{r+s}} \\ a_1 \ a_2 \ \ldots \ a_r \ a_{r+1} & a_1 \ a_2 \ \ldots \ a_r \ a_{r+2} & \ldots\ldots & a_1 \ a_2 \ \ldots \ a_r \ a_{r+s} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} a_1 \ a_2 \ \ldots \ a_r \\ a_1 \ a_2 \ \ldots \ a_r \end{matrix}\right\}^{s-1} \times \left\{\begin{matrix} a_1 \ a_2 \ \ldots \ a_r \ a_{r+1} \ a_{r+2} \ \ldots \ a_{r+s} \\ a_1 \ a_2 \ \ldots \ a_r \ a_{r+1} \ a_{r+2} \ \ldots \ a_{r+s} \end{matrix}\right\},$$

adding, in his enthusiasm, that without the aid of his "system of umbral or biliteral notation, this important theorem could not be made the subject of statement without an enormous periphrasis, and could never have been made the object of distinct contemplation or proof."

The main object of his paper is then taken up, but the subject of the notation is twice recurred to,—once to say that it is "very similiar to that of Vandermonde" as he had learned from "Mr. Spottiswoode's valuable treatise," and the second time to point out that on second thoughts it is better to unite two umbral elements in the form

$$\begin{matrix} a \\ a \end{matrix}$$

rather than in the form

$$aa,$$

because then the analogy upon which the extension of the notation from simple to compound determinants is grounded will be better apprehended.

The theorem used above to illustrate the power of the 'umbral' notation should not be lightly passed over, being far more deserving of the epithet 'new' than the notation employed in formulating it. Although at a later date it came to be of less moment because of its inclusion as merely one of a *class* of theorems, viz., the class known as 'Extensionals,' there can be no doubt that at the time of its discovery it was, as Sylvester himself styled it, "a remarkable theorem." Taking, for the sake of illustration, the case of it where $r=3$ and $s=4$, viz.,

$$\begin{Vmatrix} |a_1b_2c_3d_4| & |a_1b_2c_3d_5| & |a_1b_2c_3d_6| & |a_1b_2c_3d_7| \\ |a_1b_2c_3e_4| & |a_1b_2c_3e_5| & |a_1b_2c_3e_6| & |a_1b_2c_3e_7| \\ |a_1b_2c_3f_4| & |a_1b_2c_3f_5| & |a_1b_2c_3f_6| & |a_1b_2c_3f_7| \\ |a_1b_2c_3g_4| & |a_1b_2c_3g_5| & |a_1b_2c_3g_6| & |a_1b_2c_3g_7| \end{Vmatrix}$$
$$= |a_1b_2c_3|^3 \cdot |a_1b_2c_3d_4e_5f_6g_7|,$$

we see that if we delete $a_1b_2c_3$ everywhere on both sides we are left with

$$\begin{vmatrix} d_4 & d_5 & d_6 & d_7 \\ e_4 & e_5 & e_6 & e_7 \\ f_4 & f_5 & f_6 & f_7 \\ g_4 & g_5 & g_6 & g_7 \end{vmatrix} = |d_4e_5f_6g_7|;$$

so that the theorem is seen to be the Extensional of a manifest identity.*

SYLVESTER, J. J. (1851, Aug.).

[On a certain fundamental theorem of determinants. *Philos. Magazine* (4), ii. pp. 142–145; *Collected Math. Papers*, i. pp. 252–255.]

After a characteristic introductory paragraph about the importance of the new theorem and his reasons for publishing it, Sylvester proceeds—

"The theorem is as follows:—Suppose that there are two determinants of the ordinary kind, each expressed by a square array of terms made up of n lines and n columns, so that in each square there are n^2 terms. Now let n be broken up in any given manner into two parts p and q, so that $p+q=n$. Let, firstly, one of the two given squares be divided in a given *definite* manner into two parts, one containing p of the n given lines, and the other part q of the same; and secondly, let the other of the two given squares be divided *in every possible way* into two parts, consisting of q and p lines respectively, so that on tacking on the part containing q lines of the second square to the part containing p lines of the first square, and the part containing p lines of the second square to the part containing q of the first, we get back a new couple of squares, each denoting a determinant different from the two given determinants: the number of such new couples will evidently be

$$\frac{n(n-1)\ \ldots\ (n-p+1)}{1\cdot 2\ \ldots\ p};$$

* See *Trans. R. Soc. Edinburgh*, xxx. p. 4.

and my theorem is that *the product of the given couple of determinants is equal to the sum of the products (affected with the proper algebraical sign) of each of the new couples formed as above described.*"

The same is then stated in symbols, namely,

$$\begin{Bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{Bmatrix} \times \begin{Bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_1 & \beta_2 & \dots & \beta_n \end{Bmatrix}$$

$$= \sum \pm \begin{Bmatrix} a_1\ a_2 \dots\dots\dots\dots\dots\dots\dots a_n \\ b_1\ b_2 \dots b_p \beta_{\theta_{p+1}} \beta_{\theta_{p+2}} \dots \beta_{\theta_n} \end{Bmatrix} \begin{Bmatrix} \alpha_1\ \alpha_2 \dots\dots\dots\dots\dots\dots \alpha_n \\ \beta_{\theta_1} \beta_{\theta_2} \dots \beta_{\theta_p} b_{p+1} b_{p+2} \dots b_n \end{Bmatrix}$$

where $\theta_1, \theta_2, \dots, \theta_p$ are any p different integers taken from 1, 2, ..., n, or where, as Sylvester puts it, $\theta_1, \theta_2, \dots, \theta_n$ are 'disjunctively' equal to 1, 2, ..., n.

A proof of a verificatory character is given at some length, the aim being to show that the terms arising from the expansion of the products occurring on the right-hand side are classifiable under two heads: (1) those which appear twice, namely, once with a positive sign and once with a negative sign; (2) those which appear only once, and which in their aggregate agree with those arising from the expansion of the single product on the left. An improvement of it by Faà di Bruno will be given later.

By removing to the right the solitary product at present on the left, the theorem is seen to belong to the class of *vanishing aggregates of products of pairs of determinants*, and therefore to be not entirely new, the first instances of it, viz.,

$$|ab'|\cdot|cd'| - |ac'|\cdot|bd'| + |ad'|\cdot|bc'| = 0,$$

$$|ab'c''|\cdot|de'f''| - |ab'd''|\cdot|ce'f''| + |ac'd''|\cdot|be'f''| - |bc'd''|\cdot|ae'f''| = 0,$$

that is to say, the cases where $n=2, p=1$, and $n=3, p=1$, being found in Bezout (1779). Nothing, however, done by Bezout, Cauchy, or Schweins ought to dissociate Sylvester's name from the theorem. His claim, too, is all the stronger from the fact that he it was who in 1839 first formulated the case $n=n$, $p=1$,—a fact which seems to have dropped entirely out of remembrance, writers giving the date 1851 when referring to the case enunciated a dozen years earlier.

SYLVESTER, J. J. (1851, Oct.).

[On a remarkable discovery in the theory of canonical forms and of hyperdeterminants. *Philos. Magazine* (4), ii. pp. 391–410; *Collected Math. Papers*, i. pp. 265–283.]

Here, amid matter of great algebraical importance, there is incidentally suggested the discarding of the use of the term 'determinant' as connected with a single function,—that is to say, Gauss' original use of the term,—and the substitution of the term 'discriminant' in its place. The introduction of a new word, it is explained, is for the purpose of avoiding the obscurity and confusion which arises from employing the same word in two different senses, and "'discriminant' because it affords the *discrimen* or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally of the existence or otherwise of multiple points in the locus represented or characterised by any algebraical function." *

CAYLEY, A. (1851, end).

[On the theory of permutants. *Cambridge and Dubl. Math. Journ.*, vii. pp. 40–51; *Collected Math. Papers*, ii. pp. 16–26.]

The second part of his paper of 1843, as we have seen, Cayley devoted to the consideration of a class of functions obtainable from the use of m sets of n indices in the way in which a determinant is obtainable from only two sets. The general symbol used for such a function was

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* Apropos of this happy coinage, Sylvester adds in a footnote the general remark:—"Progress in these researches is impossible without the aid of clear expression; and the first condition of a good nomenclature is that different things shall be called by different names. The innovations in mathematical language here and elsewhere (not without high sanction) introduced by the author, have been never adopted except under actual experience of the embarrassment arising from the want of them, and will require no vindication to those who have reached that point where the necessity of some such additions becomes felt." The truth of the remark is not appreciably diminished by the occurrence of the word 'meso-catalecticism' in another footnote two pages further on. The year of the paper (1851) was for Cayley and Sylvester a year teeming with fresh ideas as well as with fresh words.

$$\begin{Bmatrix} \mathrm{A}_{\rho_1} & \sigma_1 & \tau_1 & \cdots \\ \rho_2 & \sigma_2 & \tau_2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \rho_n & \sigma_n & \tau_n & \cdots \end{Bmatrix},$$

this standing for the sum of all the different terms of the form

$$\pm_r \cdot \pm_s \cdot \pm_t \ldots \mathrm{A}_{\rho_{r_1}\ \sigma_{s_1}\ \tau_{t_1} \ldots} \times \ldots . \times \mathrm{A}_{\rho_{r_n}\ \sigma_{s_n}\ \tau_{t_n} \ldots}$$

where

$$r_1, r_2, \ldots, r_n; \quad s_1, s_2, \ldots, s_n; \quad t_1, t_2, \ldots, t_n; \quad \ldots\ldots$$

denote any permutation whatever, the same or different, of the series of integers

$$1, 2, \ldots, n,$$

and where $\pm_r$ denotes the sign $+$ or $-$ according as the number of inversions in $r_1, r_2, \ldots, r_n$ is even or odd. Using a † over the column of ρ's to indicate that these are unpermutable, he shows that

$$\begin{Bmatrix} \mathrm{A}_{\rho_1} & \sigma_1 & \cdots \\ \cdot & \cdot & \cdots \\ \rho_n & \sigma_n & \cdots \end{Bmatrix} = 1 \cdot 2 \ldots . m \begin{Bmatrix} \overset{\dagger}{\mathrm{A}}_{\rho_1} & \sigma_1 & \cdots \\ \cdot & \cdot & \cdots \\ \rho_n & \sigma_n & \cdots \end{Bmatrix}$$

when the number of columns, m, is even; and vanishes when m is odd. In the former case it is clear that the placing of the † over any other column would have had the same result, and therefore that it is better to mark this indifference by placing it over the A. The other theorems, including a multiplication theorem, need not be given, our object being simply to show the position occupied by determinants among the new functions; and this we can now do by quoting one sentence, viz., "an ordinary determinant is represented by

$$\begin{Bmatrix} \overset{\dagger}{\mathrm{A}}_{\alpha_1} & \beta_1 \\ \cdot\ \cdot & \cdot\ \cdot \\ \alpha_n & \beta_n \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \overset{\dagger}{\mathrm{A}}_1 & 1 \\ \cdot\ \cdot & \cdot \\ n & n \end{Bmatrix},$$

the latter form being obviously equally general with the former one."

In his paper of 1845 a further generalisation was made, the functions then reached being called 'hyperdeterminants,' and a hyperdeterminant defined as an expression representable as a homogeneous p^{th}-degree function, H_p, of certain of the determinants of a rectangular array, each of whose elements is denoted by n umbræ, and each umbra one of the integers 1, 2, ..., m. Thus when $n=3$ and $m=2$, the array is

$$\begin{matrix} 111 & 112 & 121 & 122 \\ 211 & 212 & 221 & 222, \end{matrix}$$

or

$$\begin{matrix} 111 & 112 & 211 & 212 \\ 121 & 122 & 221 & 222, \end{matrix}$$

or

$$\begin{matrix} 111 & 121 & 211 & 221 \\ 112 & 122 & 212 & 222, \end{matrix}$$

according as the first, second, or third umbra is made invariable throughout the two rows; and if $p=1$ we have the 'incomplete' hyperdeterminants

$$\begin{vmatrix} 111 & 122 \\ 211 & 222 \end{vmatrix} + \begin{vmatrix} 112 & 121 \\ 212 & 221 \end{vmatrix},$$

$$\begin{vmatrix} 111 & 212 \\ 121 & 222 \end{vmatrix} + \begin{vmatrix} 112 & 211 \\ 122 & 221 \end{vmatrix},$$

$$\begin{vmatrix} 111 & 221 \\ 112 & 222 \end{vmatrix} + \begin{vmatrix} 121 & 211 \\ 122 & 212 \end{vmatrix};$$

whereas if $p=2$ we have the hyperdeterminants

$$\left\{\begin{vmatrix} 111 & 122 \\ 211 & 222 \end{vmatrix} + \begin{vmatrix} 112 & 121 \\ 212 & 221 \end{vmatrix}\right\}^2 - 4\begin{vmatrix} 111 & 121 \\ 211 & 221 \end{vmatrix} \cdot \begin{vmatrix} 112 & 122 \\ 212 & 222 \end{vmatrix},$$

$$\left\{\begin{vmatrix} 111 & 212 \\ 121 & 222 \end{vmatrix} + \begin{vmatrix} 112 & 211 \\ 122 & 221 \end{vmatrix}\right\}^2 - 4\begin{vmatrix} 111 & 211 \\ 121 & 221 \end{vmatrix} \cdot \begin{vmatrix} 112 & 212 \\ 122 & 222 \end{vmatrix},$$

$$\left\{\begin{vmatrix} 111 & 221 \\ 112 & 222 \end{vmatrix} + \begin{vmatrix} 121 & 211 \\ 122 & 212 \end{vmatrix}\right\}^2 - 4\begin{vmatrix} 111 & 211 \\ 112 & 212 \end{vmatrix} \cdot \begin{vmatrix} 121 & 221 \\ 122 & 222 \end{vmatrix},$$

which being really identical, have their common expression designated a 'complete' hyperdeterminant. The general form of H_p is not specified. Further details would here be out of place:

the one important point to be noted is the relation between hyperdeterminants and the functions of Cayley's paper of the year 1843, and this is shortly indicated by saying that the latter functions are hyperdeterminants in which $p=1$ and n is even.

In his paper of the year 1847 an altogether different generalisation was formulated, the corresponding symbol being

$$\sum \pm (1, 2, \ldots, n),$$

and one of the objects aimed at being to extend the definition of a determinant so as to include within it the Pfaffian. (See *Collected Math. Papers*, i. p. 589.)

Having thus attempted to make clear the stage which the process of generalisation had reached with Cayley prior to 1851, we are prepared to appreciate the notable advance made by him in his paper of that year. The widely embracing conception therein formulated was that of the functions called on the suggestion of Sylvester '*permutants*.' For the sake of easy exposition we shall follow him in his special usage of the words 'form,' 'blank,' 'characters,' 'symbol.' "A *form*," he says, "may be considered as composed of *blanks* which are to be filled up by inserting in them specialising *characters*, and a form the blanks of which are so filled up becomes a *symbol*." If the 'characters' (previously called by him 'nombres symboliques') be 1, 2, 3, 4, ... the 'symbol' may always be represented in the first instance, and without reference to the nature or constitution of the 'form,' by $V_{1234\ldots}$; for example $V_{1234\ldots}$ may stand for

$$P_{12}Q_3R_4\ldots, \quad \text{or} \quad P_{12}P_{34}\ldots, \quad \text{or} \quad \ldots.$$

Now, let the characters 1, 2, 3, 4, ... in such a symbol be permuted in every possible way, and the resulting symbols have the sign + or − prefixed to them in accordance with Cramer's rule, then the aggregate of all these symbols is a '*simple permutant*.' The originating symbol being $V_{1234\ldots}$, the corresponding permutant might have been denoted by $\Sigma \pm V_{1234\ldots}$ as in his paper of the year 1847, but as a matter of fact Cayley now makes use of

$$(V_{1234\ldots\ldots}).$$

It is thus seen that, taking for shortness' sake only three characters, we have

$$\begin{aligned}&(V_{123}) = V_{123} + V_{231} + V_{312} - V_{132} - V_{213} - V_{321},\\&(V_{123}) = (V_{231}) = -(V_{132}) = \ldots .\\&(V_{113}) = 0.\end{aligned}$$

These preliminaries having been grasped, "it is easy," in Cayley's own words, "to pass to the general definition of a permutant. We have only to consider the blanks as forming, not as heretofore a single set, but any number of distinct sets, and to consider the characters in each set of blanks as permutable *inter se*, and not otherwise, giving to the symbol the sign compounded of the signs corresponding to the arrangements of the characters in the different sets of blanks." Thus, if the first and second blanks form a set, and the third and fourth blanks form a set, the permutant whose originating symbol is V_{1234} is

$$V_{1234} - V_{2134} - V_{1243} + V_{2143}.$$

The idea is hereupon suggested of arranging the blanks of a compound permutant so as to show in what manner they are grouped into sets. For example, instead of doing as we have just done, viz., using V_{1234} accompanied by a verbal explanation as to its sets, we might write

$$V_{\substack{12\\34}}$$

and so obtain

$$\left(V_{\substack{12\\34}}\right) = V_{\substack{12\\34}} - V_{\substack{21\\34}} - V_{\substack{12\\43}} + V_{\substack{21\\43}}.$$

From this it is a simple step to the idea of grouping the blanks in lines and columns, that is to say, to such a symbol as

$$V_{\begin{array}{llll}\alpha & \beta & \gamma & \ldots\\ \alpha' & \beta' & \gamma' & \ldots\\ \cdot & \cdot & \cdot & \cdot\end{array}}$$

One case of this is that in which it is viewed as a function of the symbols $V_{\alpha\beta\gamma\ldots}$, $V_{\alpha'\beta'\gamma'\ldots}$, etc., and a less general case that in which it is viewed as the *product*. Cayley then proceeds:—

"Upon this assumption it becomes important to distinguish the different ways in which the blanks of a set are distributed in the different lines and columns. The cases to be considered are: (A) The blanks of a single set or of single sets are situated in more than one column, (B) The blanks of each single set are situated in the same column, (C) The blanks of each single set form a separate column. The case B (which includes the case C) and the case C merit particular consideration. In fact, the case B is that of the functions which I have, in my memoir of Linear Transformations in the *Journal*, called hyperdeterminants, and the case C is the particular class of hyperdeterminants previously treated by me in the *Cambridge Philosophical Transactions*, and also particularly noticed in the memoir on Linear Transformations. The functions of the case B, I now propose to call 'Intermutants,' and those in the case C, 'Commutants.' Commutants include as a particular case 'Determinants,' which term will be used in its ordinary signification."

To arrive at the position of determinants, therefore, in the great theory of permutants, we have first to seek out the particular permutants whose originating symbol is of the form

$$\mathrm{V}\begin{array}{llll} \alpha & \beta & \gamma & \ldots. \\ \alpha' & \beta' & \gamma' & \ldots. \\ . & . & . & \ldots.; \end{array}$$

then in this restricted field to look for those in which the symbol just given is viewed as a product of symbols $\mathrm{V}_{\alpha\beta\gamma\ldots.}$, $\mathrm{V}_{\alpha'\beta'\gamma'\ldots.}$, ...; next to confine ourselves in this smaller domain to those in which the 'blanks' of each single 'set' form a separate column; and lastly to isolate those in which the number of such columns is 2.

In his paper Cayley goes on to expound the theory first of *commutants*, and then of *intermutants*. Neither exposition, however, needs attention here, because the one has already been dealt with under the year 1843, and the other is outside our subject.

SYLVESTER, J. J. (1852, Jan.).

[On the principles of the calculus of forms. *Cambridge and Dublin Math. Journ.*, vii. pp. 52–97; *Collected Math. Papers*, i. pp. 284–327.]

A postscript added by Cayley to his paper of the year 1851 makes evident that Sylvester had a share in the latest generalisation, and, as was natural, did not wish that share to be lost sight of. It made clear that the two workers had during the year been deeply engrossed in what Sylvester then called the 'calculus of forms,' that they had been in close communication with one another, and that Sylvester's discovery that the function

$$ace + 2bcd - ad^2 - b^2e - c^3$$

could be expressed as a commutant, namely,

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$$

by considering $00 = a$, $01 = 10 = b$, $02 = 11 = 20 = c$, $12 = 21 = d$, $22 = e$ had led Cayley to the conception of intermutants.

The famous paper which we have now reached, and which was doubtless completed very shortly after Cayley's, contains the results—numerous and suggestive—of Sylvester's labours. The only section, however, which directly concerns the theory of determinants is the third, bearing the heading "On Commutants." It opens with a page regarding the simplest species, "the well-known common determinant," and then proceeds:—

"If, instead of two lines of umbræ, three or more be taken, the same principle of solution will continue to be applicable. Thus, if there be a matrix of any even number r of lines each of n umbræ

$$\begin{matrix} a_1 & b_1 & \dots & l_1 \\ a_2 & b_2 & \dots & l_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_r & b_r & \dots & l_r, \end{matrix}$$

the first may be supposed to remain stationary, and the remaining $r-1$ lines each be taken in $1 \cdot 2 \cdot \dots n$ different orders: every order in each line will be accompanied by its appropriate sign $+$ or $-$; and each different grouping in each line will give rise to a particular grouping of the letters read off in columns. The value of the commutant expressed by the above matrix will therefore consist of the sum of $(1 \cdot 2 \cdot \dots n)^{r-1}$ terms, each term being the product of n quantities respectively symbolised by a group of r letters and affected with the sign $+$ or $-$ according as the number of negative signs in the total of the arrangements of the lines (from the columnar reading off of which each such term is derived) is even or odd.

For example, the value of

$$\begin{matrix} a & b \\ c & d \\ e & f \\ g & h \end{matrix}$$

will be found by taking the $(1 \cdot 2)^3$ arrangements, as below,

$$\begin{matrix} ab & ab & ab & ab & ab & ab & ab & ab \\ cd & dc & cd & dc & cd & dc & cd & dc \\ ef & ef & fe & fe & ef & ef & fe & fe \\ gh & gh & gh & gh & hg & hg & hg & hg, \end{matrix}$$

the signs of cd, ef, gh being supposed $+$, those of dc, fe, hg will be each $-$. Consequently the sum of the terms will be expressed by

$$aceg \cdot bdfh - adeg \cdot bcfh - acfg \cdot bdeh + adfg \cdot bceh$$
$$- aceh \cdot bdfg + adeh \cdot bcfg + acfh \cdot bdeg - adfh \cdot bceg."$$

It will be observed that the example here given is the quadratic function which Cayley would have denoted by

$$\left(\overset{\dagger}{\mathrm{V}} \begin{matrix} \\ 0\,0\,0\,0 \\ 1\,1\,1\,1 \end{matrix} \right),$$

and which, on the supposition that generally $\mathrm{V}_{\alpha,\beta,\gamma,\delta} = \mathrm{V}_{\alpha+\beta+\gamma+\delta}$ and in particular that $\mathrm{V}_0 = a$, $\mathrm{V}_1 = b$, $\mathrm{V}_2 = c$, ... would represent

$$ae - bd - bd + c^2 - bd + c^2 + c^2 - db,$$

i.e. $$ae - 4bd + 3c^2.$$

In his applications of the theory of commutants to that of 'forms,' Sylvester uses for the first time differential operators as umbræ, speaking, for example, of the commutant

$$\begin{matrix} \dfrac{\partial}{\partial x_1}, & \dfrac{\partial}{\partial y_1}, & \dfrac{\partial}{\partial z_1}, & \cdots\cdot \\ \dfrac{\partial}{\partial x_2}, & \dfrac{\partial}{\partial y_2}, & \dfrac{\partial}{\partial z_2}, & \cdots\cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ , \end{matrix}$$

this being possible from the fact that the coefficients of a 'form' u are, save for a constant factor, representable as differential-quotients of u. This we may illustrate for ourselves by the result

$$\left[\begin{array}{cc} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{array}\right](a,\ b,\ c,\ d,\ e \between x,\ y)^4 = \left(\frac{1}{4\,!}\right)^2 (ae - 4bd + 3c^2\,.$$

He also uses effectively such umbræ as

$$\alpha^n, \quad \alpha^{n-1}\beta, \quad \alpha^{n-2}\beta^2, \quad \ldots,$$

—a usage which is most suitably illustrated by taking a commutant having two lines of three quadratic umbræ each, that is to say, the determinant of the third order

$$\begin{array}{ccc} \alpha^2 & \alpha\beta & \beta^2 \\ \alpha^2 & \alpha\beta & \beta^2. \end{array}$$

This, by a similar convention, is taken to represent

$$\alpha^4 \cdot \alpha^2\beta^2 \cdot \beta^4 - \alpha^4 \cdot \alpha\beta^3 \cdot \alpha\beta^3 - \alpha^3\beta \cdot \alpha^3\beta \cdot \beta^4 + \alpha^3\beta \cdot \alpha\beta^3 \cdot \alpha^2\beta^2$$
$$+ \alpha^2\beta^2 \cdot \alpha^3\beta \cdot \alpha\beta^3 - \alpha^2\beta^2 \cdot \alpha^2\beta^2 \cdot \alpha^2\beta^2,$$

and consequently if

$$(\alpha x + \beta y)^4 \equiv ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

it is an expression for

$$ace + 2bcd - ad^2 - b^2e - c^3,$$

that is to say, for the above-mentioned (p. 69) *special* determinant

$$\left|\begin{array}{ccc} a & b & c \\ b & c & d \\ c & d & e \end{array}\right|.$$

When he comes to speak of 'partial' commutants, which are identical with intermutants, he devotes a page (pp. 88–89) to the subject of his relations with Cayley. As it would appear that he was not quite satisfied with the wording of the postscript above referred to, Cayley published a modified form of it as a note, headed "Correction of the Postscript to the Paper on Permutants," and there the matter between the two friends happily rested.

BETTI, E. (1852, Feb.).

[Sulla risoluzione delle equazioni algebriche. *Annali di Sci. mat. e fis.*, iii. pp. 49–115; or *Opere mat.*]

In this important memoir dealing with the theory of substitutions, and with the application of the same towards finding the conditions of solvability of algebraic equations, the author following Sylvester* defines on page 80, in the manner afterwards so familiar, the expression 'determinant of a substitution': and on the following page there occurs the passage—

"Quindi dal noto teorema della moltiplicazione delle determinanti è facile dedurre che, se si chiama Δ la determinante del prodotto delle due sostituzioni (h) e (h') [whose determinants are D and D′], avremo

$$\Delta = \mathrm{D}\mathrm{D}';$$

cioè *la determinante del prodotto di due sostituzioni è eguale al prodotto delle loro determinanti.*"

BRUNO, F. FAÀ DI (1852, May).

[Démonstration d'un théorème de M. Sylvester relatif à la décomposition d'un produit de deux déterminants. *Journ. (de Liouville) de Math.* (1), xvii. pp. 190–192.]

The theorem is that which appeared in the *Philosophical Magazine* for August 1851, and which is there formulated in the umbral notation as follows:—

$$\begin{Bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{Bmatrix} \times \begin{Bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_1 & \beta_2 & \dots & \beta_n \end{Bmatrix}$$

$$= \sum\left[\pm \begin{Bmatrix} a_1 & a_2 & \dots & & & a_n \\ b_1 & b_2 \dots b_p & \beta_{\theta_{p+1}} & \beta_{\theta_{p+2}} & \dots & \beta_{\theta_n} \end{Bmatrix} \begin{Bmatrix} \alpha_1 & \alpha_2 & \dots & & & \alpha_n \\ \beta_{\theta_1} & \beta_{\theta_2} \dots \beta_{\theta_p} & b_{p+1} & b_{p+2} & \dots & b_n \end{Bmatrix}\right]$$

where $\theta_1, \theta_2, \dots, \theta_n$ are 'disjunctively' equal to $1, 2, \dots n$. Faà di Bruno prefers to write it in the form

* See footnote to p. 52 of his paper just described.

$$\sum(\pm a_1^{\phi_1} a_2^{\phi_2} \ldots a_p^{\phi_p} \ldots a_n^{\phi_n}) \cdot \sum(\pm b_1^{\psi_1} b_2^{\psi_2} \ldots b_p^{\psi_p} \ldots b_n^{\psi_n})$$

$$= \sum\left[\sum\left(\pm a_1^{\phi_1} a_2^{\phi_2} \ldots a_p^{\phi_p} b_{\theta_{p+1}}^{\phi_{p+1}} b_{\theta_{p+2}}^{\phi_{p+2}} \ldots b_{\theta_n}^{\phi_n}\right) \cdot \sum\left(\pm b_{\theta_1}^{\psi_1} b_{\theta_2}^{\psi_2} \ldots b_{\theta_p}^{\psi_p} a_{p+1}^{\psi_{p+1}} a_{p+2}^{\psi_{p+2}} \ldots a_n^{\psi_n}\right)\right]$$

using two other sets of letters like $\theta_1, \theta_2, \ldots, \theta_n$. This change in notation being allowed for, the new proof is in general character exactly the same as the old; it is, however, more concise and more clearly set forth. It starts with the fact that any term arising from the expansion of the typical product on the right-hand side may be written

$$a_1^{\phi_1} a_2^{\phi_2} \ldots a_p^{\phi_p} a_{p+1}^{\psi_{p+1}} \ldots a_n^{\psi_n} \cdot b_{\theta_1}^{\psi_1} b_{\theta_2}^{\psi_2} \ldots b_{\theta_p}^{\psi_p} b_{\theta_{p+1}}^{\phi_{p+1}} b_{\theta_{p+2}}^{\phi_{p+2}} \ldots b_{\theta_n}^{\phi_n}.$$

Then observing the 'indices supérieures' attached to the b's in this, we are asked to consider two possible cases. In the first place, we have to note that if no one of the ψ's be identical with any one of the ϕ's, the term is a term of the expansion of the product on the left-hand side, and that the number of such terms in the expansion of each product on the right-hand side being

$$(1.2.3 \ldots n) \cdot (1.2.3 \ldots p) \cdot (1.2.3 \ldots \overline{n-p})$$

and the number of products

$$\frac{n(n-1)(n-2) \ldots\ldots (n-p+1)}{1.2.3 \ldots p}$$

the total number of such terms is

$$(1.2.3 \ldots n)^2,$$

which is exactly the total number on the left-hand side. In the second place, if one of the ψ's be identical with one of the ϕ's, say

$$\psi_i = \phi_{p+h},$$

it is pointed out that there must exist another term in which, in place of having

$$\ldots\ldots b_{\theta_i}^{\psi_i} \ldots\ldots b_{\theta_{p+h}}^{\psi_{p+h}} \ldots\ldots$$

we shall have

$$\ldots\ldots b_{\theta_{p+h}}^{\psi_i} \ldots\ldots b_{\theta_i}^{\psi_{p+h}} \ldots\ldots$$

and that these two terms having necessarily different signs, must cancel each other.

SALMON, G. (1852).

[A Treatise on the Higher Plane Curves: . . . By the Rev. George Salmon, M.A. . . . xii+316 pp. Dublin, 1852.]

For the convenience of his readers Salmon appends a fifteen-page note on the subject of *Elimination*, and, as was natural, the note opens with a sketch (pp. 285–292) of the theory of determinants. Short and simple as this is, it contains one paragraph (§ 11) worthy of note, namely, in regard to the multiplication-theorem.

The determinant

$$\begin{vmatrix} A_1a_1+B_1b_1+C_1c_1 & A_2a_1+B_2b_1+C_2c_1 & A_3a_1+B_3b_1+C_3c_1 \\ A_1a_2+B_1b_2+C_1c_2 & A_2a_2+B_2b_2+C_2c_2 & A_3a_2+B_3b_2+C_3c_2 \\ A_1a_3+B_1b_3+C_1c_3 & A_2a_3+B_2b_3+C_2c_3 & A_3a_3+B_3b_3+C_3c_3 \end{vmatrix},$$

he says, is evidently the result of eliminating x, y, z from the equations

$$\left.\begin{aligned} a_1S_1+b_1S_2+c_1S_3 &= 0 \\ a_2S_1+b_2S_2+c_2S_3 &= 0 \\ a_3S_1+b_3S_2+c_3S_3 &= 0 \end{aligned}\right\}$$

when

$$\left.\begin{aligned} S_1 &= A_1x+A_2y+A_3z \\ S_2 &= B_1x+B_2y+B_3z \\ S_3 &= C_1x+C_2y+C_3z \end{aligned}\right\}.$$

But this elimination may be effected at once by eliminating S_1, S_2, S_3: consequently $|a_1\ b_2\ c_3|$ must be a factor of the resultant. In the second place, since a set of values of x, y, z can be found to satisfy simultaneously the given equations if a set can be found to satisfy simultaneously the equations $S_1=0$, $S_2=0$, $S_3=0$: and since the condition that the latter shall be possible is $|\ A_1\ B_2\ C_3\ | = 0$, it follows that $|\ A_1\ B_2\ C_3\ |$ must also be a factor of the result. The remaining factor being manifestly 1, the desired end, in Salmon's opinion, is attained. We only remark in passing that a little careful scrutiny of the reasoning would have suggested the need for additional support.

Salmon also proposes a fresh enunciation of the same theorem, namely, *If any set of linear equations*

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+\ldots\ldots = 0 \\ a_2x+b_2y+c_2z+\ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

be transformed by any linear substitution

$$\left.\begin{array}{l} x = \mathrm{A}_1\xi+\mathrm{B}_1\eta+\mathrm{C}_1\zeta+\ldots\ldots \\ y = \mathrm{A}_2\xi+\mathrm{B}_2\eta+\mathrm{C}_2\zeta+\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

then the determinant of the new set will be equal to the determinant of the original set multiplied by the determinant of transformation. This new wording will be recognised as a sign of the advent of the "algebra of linear transformation."

SYLVESTER, J. J. (1852, Oct.).

[On Staudt's theorems concerning the contents of polygons and polyhedrons, with a note on a new and resembling class of theorems. *Philos. Magazine* (4), iv. pp. 335–345; *Collected Math. Papers*, i. pp. 382–391.]

After a page of introduction, written in a light semi-historical, semi-critical style, Sylvester prepares the way for considering his main subject by giving as a basis two algebraical lemmas. The first he formulates as follows :—

"If the determinants represented by two square matrices are to be multiplied together, any number of columns may be cut off from the one matrix, and a corresponding number of columns from the other. Each of the lines in either one of the matrices so reduced in width as aforesaid being then multiplied by each line of the other, and the results of the multiplication arranged as a square matrix and bordered with the two respective sets of columns cut off, arranged symmetrically (the one set parallel to the new columns, the other set parallel to the new lines), the complete determinant represented by the new matrix so bordered (abstraction made of the algebraical sign) will be the product of the two original determinants."

In illustration he gives three forms for the product of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$$

viz.—

$$\begin{vmatrix} a\alpha+b\beta & a\gamma+b\delta \\ c\alpha+d\beta & c\gamma+d\delta \end{vmatrix}, \quad \begin{vmatrix} a\alpha & a\gamma & b \\ c\alpha & c\gamma & d \\ \beta & \delta & . \end{vmatrix}, \quad \begin{vmatrix} 2 & 2 & a & b \\ 2 & 2 & c & d \\ \alpha & \beta & . & . \\ \gamma & \delta & . & . \end{vmatrix}.$$

In regard to the 2's which occur in the last form his remark is:—

"Any quantities might be substituted instead of 2 . . . . , as such terms do not influence the result: this figure is probably, however, the proper quantity arising from the application of the rule, because . . . . the value of the determinant represented by a matrix of *no* places is not zero but unity."

In the case where the two given determinants are of the third order, say

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix},$$

he gives only the second and third of the four $(n+1)$ possible forms, namely—

$$-\begin{vmatrix} a\alpha+b\beta & a\alpha'+b\beta' & a\alpha''+b\beta'' & c \\ a'\alpha+b'\beta & a'\alpha'+b'\beta' & a'\alpha''+b'\beta'' & c' \\ a''\alpha+b''\beta & a''\alpha'+b''\beta' & a''\alpha''+b''\beta'' & c'' \\ \gamma & \gamma' & \gamma'' & . \end{vmatrix}$$

and

$$\begin{vmatrix} a\alpha & a\alpha' & a\alpha'' & b & c \\ a'\alpha & a'\alpha' & a'\alpha'' & b' & c' \\ a''\alpha & a''\alpha' & a''\alpha'' & b'' & c'' \\ \beta & \beta' & \beta'' & . & . \\ \gamma & \gamma' & \gamma'' & . & . \end{vmatrix},$$

pointing out by way of demonstration that the former of these is arrived at by transforming the given determinants into

$$\begin{vmatrix} a & b & c & . \\ a' & b' & c' & . \\ a'' & b'' & c'' & . \\ . & . & . & 1 \end{vmatrix} \quad \text{and} \quad -\begin{vmatrix} \alpha & \beta & . & \gamma \\ \alpha' & \beta' & . & \gamma' \\ \alpha'' & \beta'' & . & \gamma'' \\ . & . & 1 & . \end{vmatrix}$$

and applying the ordinary rule of multiplication, and similarly that the latter is got by multiplying

$$\begin{vmatrix} a & b & c & . & . \\ a' & b' & c' & . & . \\ a'' & b'' & c'' & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} \alpha & . & . & \beta & \gamma \\ \alpha' & . & . & \beta' & \gamma' \\ \alpha'' & . & . & \beta'' & \gamma'' \\ . & 1 & . & . & . \\ . & . & 1 & . & . \end{vmatrix}.$$

He thereupon leaves the subject with the remark :—

"This rule is interesting as exhibiting .... a complete scale whereby we may descend from the ordinary mode of representing the product of two determinants to the form .... where the two original determinants are made to occupy opposite quadrants of a square whose places in one of the remaining quadrants are left vacant, and shows us that under one aspect at least this latter form may be regarded as a matrix *bordered* by the two given matrices."

The second lemma is the identity—

$$\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & 1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & 1 \\ . & . & . \quad . & . & . \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} & 1 \\ 1 & 1 & \ldots\ldots & 1 & . \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & \ldots\ldots & A_{1n} & 1 \\ A_{21} & A_{22} & \ldots\ldots & A_{2n} & 1 \\ . & . & . \quad . & . & . \\ A_{n1} & A_{n2} & \ldots\ldots & A_{nn} & 1 \\ 1 & 1 & \ldots\ldots & 1 & . \end{vmatrix}$$

where $A_{rs} = a_{rs} + h_r + k_s$, and the h's and k's are perfectly arbitrary quantities, the transformation being of course effected by adding multiples of the last column to the other columns, and thereafter multiples of the last row to the other rows.

The geometrical applications which follow, it may be interesting to note, are connected with the subject of Cayley's paper of 1841,—his well-known first paper on determinants.

CAUCHY, A. L. (1853, January).

[Sur les clefs algébriques. *Comptes rendus ... Acad. des Sci.* (Paris), xxxvi. pp. 70–75, 129–136; or *Œuvres complètes* (1), xi. pp. 439–445, xii. pp. 12–20.]

[Sur les avantages que présente, dans un grand nombre de questions, l'emploi des clefs algébriques. *Comptes rendus ... Acad. des Sci.* (Paris), xxxvi. pp. 161–169; or *Œuvres complètes* (1), xii. pp. 21–30.]

These papers add nothing of algebraic importance to the contents of Cauchy's memoir of the year 1847: in fact, they may be looked on as short and simply-worded abstracts of parts of that memoir. It is worthy of note, however, that even where problems of elimination are being dealt with "sommes alternées" are not now explicitly referred to.

SAINT VENANT, B. DE (1853, March).

[De l'interprétation géométrique des *clefs algébriques* et des *déterminants. Comptes rendus ... Acad. des Sci.* (Paris), xxxvi. pp. 582–585.]

De Saint Venant's suggestion is that Cauchy's "algebraic keys" $\alpha, \beta, \gamma, \ldots$ may be viewed as *directed magnitudes*, and this leads up to the so-called geometric interpretation of determinants. "Un déterminant du $n$ième ordre," he says, "me paraît être le produit géométrique de n sommes algébriques de n lignes ayant, chacune à chacune, les mêmes directions dans les diverses sommes: en sorte que l'on a pour celui du troisième ordre, par exemple,

$xy'z'' - xy''z' + \ldots =$ le produit $(\bar{x}+\bar{y}+\bar{z})(\bar{x}'+\bar{y}'+\bar{z}')(\bar{x}''+\bar{y}''+\bar{z}'')$

où x, x', x'' ont un même direction (c'est-à-dire sont parallèles), y, y', y'' une autre direction qui est la même pour toutes trois, et z, z', z'' aussi une même troisième direction."

HESSE, O. (1853, April).

[Ueber Determinanten und ihre Anwendung in der Geometrie, ... *Crelle's Journal*, xlix. pp. 243–264; or *Werke*, pp. 319–343.]

The product of two determinants A and B being C, Hesse's professed object is to show "wie die partiellen Differentialquotienten der Determinante C nach ihren Elementen c genommen durch die partiellen Differentialquotienten der Factoren A und B nach ihren Elementen genommen sich ausdrücken lassen." We are prepared, therefore, to find his ground already pretty well covered by Joachimsthal's paper of November 1849. The latter established the result

$$\frac{\partial C}{\partial c_{\kappa\lambda}} = \frac{\partial A}{\partial a_{0\kappa}} \cdot \frac{\partial B}{\partial b_{0\lambda}} + \frac{\partial A}{\partial a_{1\kappa}} \cdot \frac{\partial B}{\partial b_{1\lambda}} + \ldots + \frac{\partial A}{\partial a_{n\kappa}} \cdot \frac{\partial B}{\partial b_{n\lambda}},$$

and said that others could be found: Hesse established one of these others, namely,

$$\frac{\partial^2 C}{\partial c_{\kappa\lambda}\partial c_{\mu\nu}} = \frac{1}{1\cdot 2}\sum \frac{\partial^2 A}{\partial a_{p\kappa}\partial a_{q\mu}} \cdot \frac{\partial^2 B}{\partial b_{p\lambda}\partial b_{q\nu}},$$

and said that the next would be

$$\frac{\partial^3 C}{\partial c_{\kappa\lambda}\partial c_{\mu\nu}\partial c_{\rho\sigma}} = \frac{1}{1\cdot 2\cdot 3}\sum \frac{\partial^3 A}{\partial a_{p\kappa}\partial a_{q\mu}\partial a_{r\rho}} \cdot \frac{\partial^3 B}{\partial b_{p\lambda}\partial b_{q\nu}\partial b_{r\sigma}},$$

where $p, q, \ldots$ have the values $0, 1, 2, \ldots, n$.

We can only remark that the second and third results are not so simple as they ought to have been: for Hesse does not point out that (1) when p and q are identical the term vanishes; (2) putting $p, q = \alpha, \beta$ gives the same term as putting $p, q = \beta, \alpha$; and (3) therefore the second result should be

$$\frac{\partial^2 C}{\partial c_{\kappa\lambda}\partial c_{\mu\nu}} = \sum \frac{\partial^2 A}{\partial a_{p\kappa}\partial a_{q\mu}} \cdot \frac{\partial^2 B}{\partial b_{p\lambda}\partial b_{q\nu}},$$

where p has any of the values $0, 1, 2, \ldots, n-1$, and q any of the values $1, 2, \ldots, n$, subject to the condition that $p < q$. It would then agree with the extended multiplication-theorem of Binet and Cauchy, and especially with the latter's form of it.

CHIO, F. (1853, June).

[Mémoire sur les fonctions connues sous le nom de résultantes ou de déterminants. 32 pp. Turin.]

The title here is not sufficiently descriptive, almost the whole of the thirty-two pages being occupied with the consideration of determinants *whose elements are binomial.* Beginning with the "tableau"

$$\begin{matrix} a_0+m_0 & a_1+m_1 & \ldots & a_{i-1}+m_{i-1} \\ b_0+n_0 & b_1+n_1 & \ldots & b_{i-1}+n_{i-1} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & & \cdot\;\cdot\;\cdot \\ l_0+t_0 & l_1+t_1 & \ldots & l_{i-1}+t_{i-1} \end{matrix}$$

Chio seeks, of course, to express its determinant as a sum of determinants with monomial elements, and thereafter applies his result to particular cases.

The first matter of real interest is reached on p. 11, where the following theorem is given: '*Soient* s *la résultante de l'ordre* i *formée avec les termes du tableau*

$$\begin{array}{cccc} a_0 & a_1 & \dots & a_{i-1} \\ b_0 & b_1 & \dots & b_{i-1} \\ \cdot & \cdot & \cdot & \cdot \\ l_0 & l_1 & \dots & l_{i-1}, \end{array}$$

et s″ *résultante de l'ordre* i − 1 *formée avec les termes compris dans le tableau*

$$\begin{array}{cccc} S(\pm a_0b_1) & S(\pm a_0b_2) & \dots & S(\pm a_0b_{i-1}) \\ S(\pm a_0c_1) & S(\pm a_0c_2) & \dots & S(\pm a_0c_{i-1}) \\ \cdot & \cdot & \cdot & \cdot \\ S(\pm a_0l_1) & S(\pm a_0l_2) & \dots & S(\pm a_0l_{i-1}). \end{array}$$

La résultante s″ *sera égale à* s, *au facteur près* a_0^{i-2}, *en sorte qu'on aura* $s'' = a_0^{i-2}\,s$."

This is one form of the theorem afterwards well known as effecting the transformation of any determinant into one of the next lower order. It may be viewed as a case of Hermite's result of the year 1849.

On p. 17 particular cases cease to be considered, and the multiplication of an array of i rows and $2i$ columns by a similar array is taken up, with a result in accordance with that arrived at by Binet and Cauchy in 1812. From this result, by specialisation, the ordinary multiplication-theorem is then deduced, and with it (Chio's "théorème ix.") the first part of the memoir closes.

The second part, which begins on p. 23, concerns the solving of a set of $2n$ equations of a type which will be sufficiently specified by giving the set where $n=3$, namely,

$$\left.\begin{array}{l} x\ \ +y\ \ +z\ \ = d_1 \\ x\xi +y\eta +z\zeta = d_2 \\ x\xi^2+y\eta^2+z\zeta^2 = d_3 \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ x\xi^5+y\eta^5+z\zeta^5 = d_6 \end{array}\right\}.$$

The connection of this with what precedes consists in the fact, arrived at by Sylvester in his solution of the problem of the canonisation of the quintic, that ξ, η, ζ are then the roots of the equation in ω

$$\begin{vmatrix} d_2-\omega d_1 & d_3-\omega d_2 & d_4-\omega d_3 \\ d_3-\omega d_2 & d_4-\omega d_3 & d_5-\omega d_4 \\ d_4-\omega d_3 & d_5-\omega d_4 & d_6-\omega d_5 \end{vmatrix} = 0.$$

SPOTTISWOODE, W. (1853).

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journal*, li. pp. 209–271, 328–381.]

A more correct description of Spottiswoode's second edition would be *rearranged, partly rewritten, and much enlarged*, the majority of the titles of the old sections or chapters occurring again but in a different order, the majority of the sections being enlarged, and two or three new sections being inserted. Although the total increase of matter is from 71 pages to 117, there is comparatively little to be noted concerning general determinants.

In § 2, which bears the title "Addition and Subtraction of Determinants," the following appears (p. 232) for the first time:—THEOREM ix. *The sum of two determinants in which* i *rows (on a certain level) are respectively equal, is equal to the determinant whose* ith *minors on the aforesaid level are identical with the corresponding* ith *minors of each of the two given determinants, and whose* $(n-i)^{th}$ *complementary minors are respectively the sum of the complementary minors of the given determinants.* No instance is given where the two determinants have more than one row different.

In § 4, which deals with the multiplication of determinants, much space (pp. 238–248) is given to Sylvester's theorem of 1852 (October). Spottiswoode's own mode of treating the subject is to begin apparently with the two factors and arrive at the product, whereas in reality the opposite is the case. For example, his proof that

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \cdot \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \begin{vmatrix} a\alpha & a\alpha' & a\alpha'' & b & c \\ a'\alpha & a'\alpha' & a'\alpha'' & b' & c' \\ a''\alpha & a''\alpha' & a''\alpha'' & b'' & c'' \\ \beta & \beta' & \beta'' & . & . \\ \gamma & \gamma' & \gamma'' & . & . \end{vmatrix}$$

essentially consists in expanding the right-hand determinant in terms of minors formed from the first three rows and minors formed from the last two rows. His other fresh proof is dependent on the connection between determinants and simultaneous linear equations. Taking the two sets of equations

$$\left.\begin{array}{l} ax+a'y+a''z = u_1 \\ \beta x+\beta'y+\beta''z = u_2 \\ \gamma x+\gamma'y+\gamma''z = u_3 \end{array}\right\} \qquad \left.\begin{array}{l} au_1+bu_2+cu_3 = v_1 \\ a'u_1+b'u_2+c'u_3 = v_2 \\ a''u_1+b''u_2+c''u_3 = v_3 \end{array}\right\}$$

and substituting for u_1, u_2, u_3, in the second set and solving, there is obtained for x an expression whose denominator is known to be

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \begin{vmatrix} a & a' & a'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{vmatrix}.$$

In the second place, by substituting for u_1 only there is obtained

$$\left.\begin{array}{rl} aax+aa'y+aa''z+bu_2+cu_3 & = v_1 \\ a'ax+a'a'y+a'a''z+b'u_2+c'u_3 & = v_2 \\ a''ax+a''a'y+a''a''z+b''u_2+c''u_3 & = v_3 \\ \beta x+\ \beta'y+\ \beta''z-\ u_2 \qquad & = 0 \\ \gamma x+\ \gamma'y+\ \gamma''z \qquad -\ u_3 & = 0 \end{array}\right\},$$

whence comes for x an expression whose denominator is

$$\begin{vmatrix} aa & aa' & aa'' & b & c \\ a'a & a'a' & a'a'' & b' & c' \\ a''a & a''a' & a''a'' & b'' & c'' \\ \beta & \beta' & \beta'' & -1 & . \\ \gamma & \gamma' & \gamma'' & . & -1 \end{vmatrix}.$$

A comparison of the two denominators is supposed to establish the desired result; but, although the dropping of the two negative units in the five-line determinant is quite justifiable, no allusion is made to it. It may be added that Sylvester's umbral notation is used throughout in dealing with the subjects just referred to,

$$\begin{Bmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{Bmatrix} \quad \text{or} \quad \begin{vmatrix} (11) & (12) & \dots & (1n) \\ (21) & (22) & \dots & (2n) \\ \cdot & \cdot & \cdot & \cdot \\ (n1) & (n2) & \dots & (nn) \end{vmatrix}$$

being used for one of the two determinants, and

$$\begin{Bmatrix} 1' & 2' & \dots & n' \\ 1' & 2' & \dots & n' \end{Bmatrix} \quad \text{or} \quad \begin{vmatrix} (11)' & (12)' & \dots & (1n)' \\ (21)' & (22)' & \dots & (2n)' \\ \cdot & \cdot & \cdot & \cdot \\ (n1)' & (n2)' & \dots & (nn)' \end{vmatrix}$$

for the other. The reading is thus rendered tiresome, and inaccurate printing exaggerates the trouble.

In § 8 (pp. 335–337) Sylvester's proposition of the year 1850 regarding the vanishing of the minors of a non-quadrate matrix is attempted to be proved. The matrix being, for example,

$$\begin{matrix} 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 \end{matrix} \quad \text{or} \quad \begin{Bmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,3\,4 \qquad \end{Bmatrix}$$

and $\{rstu\}$ being the determinant whose columns are identical with the r^{th}, s^{th}, t^{th}, u^{th} columns of the matrix, it is required to show that if the minors $\{1234\}$, $\{1235\}$, $\{1236\}$, $\{1237\}$ vanish, the thirty-one other four-line minors of the matrix must vanish also. In support of this Spottiswoode says truly enough that if

$$\text{A, B, C, D}$$

stand for

$$-\begin{Bmatrix}123\\234\end{Bmatrix}, \quad \begin{Bmatrix}123\\134\end{Bmatrix}, \quad -\begin{Bmatrix}123\\124\end{Bmatrix}, \quad \begin{Bmatrix}123\\123\end{Bmatrix},$$

we know that

$$\begin{aligned} \{1234\} &= 14\cdot\text{A} + 24\cdot\text{B} + 34\cdot\text{C} + 44\cdot\text{D}, \\ \{1235\} &= 15\cdot\text{A} + 25\cdot\text{B} + 35\cdot\text{C} + 45\cdot\text{D}, \\ \{1236\} &= 16\cdot\text{A} + 26\cdot\text{B} + 36\cdot\text{C} + 46\cdot\text{D}, \\ \{1237\} &= 17\cdot\text{A} + 27\cdot\text{B} + 37\cdot\text{C} + 47\cdot\text{D}; \end{aligned}$$

but then to this he merely adds, "and, if these vanish, it is obvious that by direct elimination all the others may be at once deduced." He notes in addition that in the case of the matrix

$$\begin{Bmatrix} 1\,2\,3\,4 \dots n \\ 1\,2 \qquad\qquad \end{Bmatrix} \quad \text{or} \quad \begin{matrix} 11 & 12 & 13 & 14 & \dots & 1n \\ 21 & 22 & 23 & 24 & \dots & 2n \end{matrix}$$

the vanishing of

$$\begin{Bmatrix}12\\12\end{Bmatrix}, \quad \begin{Bmatrix}13\\12\end{Bmatrix}, \quad \begin{Bmatrix}14\\12\end{Bmatrix}, \quad \dots, \quad \begin{Bmatrix}1n\\12\end{Bmatrix}$$

if brought about by the vanishing of 11 and 21 will not ensure the vanishing of the other 2-line minors of the matrix; but he does not see that the vanishing of {1234}, {1235}, {1236}, {1237} in the previous example, if brought about by the vanishing of A, B, C, D, will be equally ineffectual.

GRASSMANN [H.] (1854, February, April).

[Sur les différents genres de multiplication. *Crelle's Journ.*, xlix. pp. 123–141.]

[Extrait d'un mémoire de M. Grassmann. *Comptes rendus . . . Acad. des Sci.* (Paris), xxxviii. pp. 743–744.]

Grassmann, having become aware of Cauchy's three communications to the French Academy in January of 1853, claims that the principles there established and the results deduced are absolutely the same as those published by himself in 1844. He says (p. 127), "Les clefs algébriques de M. Cauchy ne sont au fond que les unités relatives; et ses facteurs symboliques conviennent, du moins dans un certain rapport, aux quantités extensives telles que je les ai définies. La différence ne consiste qu'en ce que M. Cauchy regarde les clefs algébriques seulement comme un moyen pour résoudre divers problèmes de l'analyse et de la mécanique et qui, les problèmes étant résolus, disparaissent, tandis que d'après les principes établis par moi, on est en état, à chaque pas du procédé, d'attribuer une signification indépendante aux unités relatives et aux quantités qui en sont composées, qu'elle que soit d'ailleurs la marche que l'on suive."

MAJO, L. DE (1854, March).

[Metodi e formole generali per l' eliminazione nelle equazioni di primo grado. *Memorie . . . Accad. delle Sci.* (Napoli), i. pp. 101–116.]

This is a carefully written but curiously belated exposition, the author apparently being quite out of touch with the writers of his own time, and possibly not familiar with any of the older writers save Cramer, Bezout, and Hindenburg. In the first six pages he defines "il polinomio $P_m(a_1b_2c_3 \ldots s_m)$" after the fashion of Bezout (1764), and gives one or two very elementary properties of it. The remaining ten pages are occupied with simultaneous linear equations, and are notable as containing (§§ 15–19) a clear exposition of Bezout's peculiar rule-of-thumb process of 1779. Herein lies the value of the paper, Majo being not only the first since Hindenburg to recall attention to a neglected process of real practical value, but also the first to give (§ 16) a reason for its validity.

CAYLEY, A. (1854, May).

[Remarques sur la notation des fonctions algébriques. *Crelle's Journal*, l. pp. 282–285; or *Collected Math. Papers*, ii. pp. 185–188.]

The notation referred to is that of *matrices*, and is exemplified by

$$\begin{array}{|ccc|} (\;\alpha_1 & \alpha_2 & \alpha_3\;) \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array},$$

a matrix being defined as a system of quantities arranged in the form of a square, but otherwise quite independent. With its help the set of equations

$$\left.\begin{aligned} \xi &= a_1x+a_2y+a_3z \\ \eta &= b_1x+b_2y+b_3z \\ \zeta &= c_1x+c_2y+c_3z \end{aligned}\right\}$$

may, he says, be written in the form

$$\xi,\ \eta,\ \zeta = \begin{array}{|ccc|} (\;a_1 & a_2 & a_3 \between x,\ y,\ z), \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}$$

and consequently the solution of the set in the form

$$x,\ y,\ z = \begin{array}{|ccc|} (\;\frac{A_1}{\Delta} & \frac{B_1}{\Delta} & \frac{C_1}{\Delta} \between \xi,\ \eta,\ \zeta). \\ \frac{A_2}{\Delta} & \frac{B_2}{\Delta} & \frac{C_2}{\Delta} \\ \frac{A_3}{\Delta} & \frac{B_3}{\Delta} & \frac{C_3}{\Delta} \end{array}$$

The latter matrix he calls the *inverse* of the former, and is naturally led to propose that it be denoted by

$$\begin{array}{l} (\; a_1 \quad a_2 \quad a_3 \;)^{-1} \\ \left| \begin{array}{ccc} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right| . \end{array}$$

Next, supposing that along with the original set there exists the set

$$x,\ y,\ z = \begin{array}{l} (\; \alpha_1 \quad \alpha_2 \quad \alpha_3 \between \mathrm{X},\ \mathrm{Y},\ \mathrm{Z}), \\ \left| \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array} \right| \end{array}$$

so that by substitution ξ, η, ζ are expressible in terms of X, Y, Z, Cayley is led by comparison of the old and the new notations to the conception of the *product* of two matrices, and to the use of

$$\begin{array}{l} (\; a_1 \quad a_2 \quad a_3 \between \alpha_1 \quad \alpha_2 \quad \alpha_3 \;) \\ \left| \begin{array}{ccc} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right| \left| \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array} \right| \end{array}$$

for

$$\begin{array}{l} (\; (a_1a_2a_3 \between \alpha_1\beta_1\gamma_1) \quad (a_1a_2a_3 \between \alpha_2\beta_2\gamma_2) \quad (a_1a_2a_3 \between \alpha_3\beta_3\gamma_3) \;) \\ \left| \begin{array}{ccc} (b_1b_2b_3 \between \alpha_1\beta_1\gamma_1) & (b_1b_2b_3 \between \alpha_2\beta_2\gamma_2) & (b_1b_2b_3 \between \alpha_3\beta_3\gamma_3) \\ (c_1c_2c_3 \between \alpha_1\beta_1\gamma_1) & (c_1c_2c_3 \between \alpha_2\beta_2\gamma_2) & (c_1c_2c_3 \between \alpha_3\beta_3\gamma_3) \end{array} \right| . \end{array}$$

Lastly, he explains his related notations for lineo-linear functions and quantics.* These we need only exemplify by saying that

$$\begin{array}{l} (\; a_1 \quad a_2 \quad a_3 \between \xi,\ \eta,\ \zeta \between x,\ y,\ z\,), \\ \left| \begin{array}{ccc} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right| \end{array} \qquad \begin{array}{l} (\; a \quad h \quad g \between x,\ y,\ z\,)^2, \\ \left| \begin{array}{ccc} h & b & f \\ g & f & c \end{array} \right| \end{array}$$

are made to stand for

$$\begin{array}{l} (a_1\xi + a_2\eta + a_3\zeta)x \\ +(b_1\xi + b_2\eta + b_3\zeta)y \\ +(c_1\xi + c_2\eta + c_3\zeta)z \end{array} \quad \text{and} \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

respectively, and that the latter is also denoted by

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2,$$

* Cayley's first memoir on *quantics* was presented to the Royal Society of London on 20th April, and this paper on the notation of *matrices* is the first of five which appeared together in *Crelle's Journal* with the date 24th May affixed by the author.

and the binary cubics

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3, \quad ax^3 + bx^2y + cxy^2 + dy^3$$

by $$(a,\ b,\ c,\ d \between x,\ y)^3, \qquad (a,\ b,\ c,\ d \between x,\ y)^3$$

respectively.

We may suggest for consideration in passing the following order of ideas, as leading up to Cayley's contracted mode of writing a set of linear equations. First, a *row* of separate quantities, *e.g.* $(a, b, c, \ldots)$; second, the statement of the identity of two rows, *e.g.* $(a, b, c, \ldots) = (x, y, z, \ldots)$, or simply $a, b, c, \ldots = x, y, z, \ldots$; third, the so-called product of two rows, *e.g.* $(a, b, c, \ldots \between x, y, z, \ldots)$; fourth, a *square* of separate quantities, *i.e.* a matrix; fifth, the result of multiplying a matrix and a row being a row. It is unfortunate that, from the point of view of notation merely, this does not at once suggest, in the sixth place, the result of multiplying two matrices, where, as Cayley is careful to point out, the multiplication is row-by-column and not row-by-row.

BRIOSCHI, F. (1854).

[LA TEORICA DEI DETERMINANTI, E LE SUE PRINCIPALI APPLICAZIONI; del Dr. Francesco Brioschi; viii + 116 pp.; Pavia. Translation into French, by Combescure; ix + 216 pp.; Paris, 1856. Translation into German, by Schellbach; vii + 102 pp.; Berlin, 1856.]

This, the second separately-published text-book on determinants, is mainly on the same lines as the first, but is marked by greater attention to verbal and logical accuracy. It consists of an historical preface and eleven short chapters or sections, seven of the latter being devoted to determinants in general, and the remaining four to special forms.

Sylvester's umbral notation is given in the form

$$|\, a_1\ a_2\ \ldots\ a_n \,|$$

but is not afterwards employed. The same author's term "minor" (*minore*) is adopted, this being represented in the French translation by "mineur," and in the German by "Unterdeterminante." "Complete" is used as opposed to "minor," and "principal minor" for what nowadays we call "coaxial."

Conspicuously frequent use is made of differentiation in the specification of minors; and it is well to note that, though the work in this way becomes cumbrous, there is a certain effectiveness attained by the usage. Thus, Δ standing for $\Sigma(\pm a_{11}a_{22}\ldots a_{nn})$. Brioschi, like Jacobi, obtains

$$\left.\begin{array}{l}
\dfrac{\partial\Delta}{\partial a_{r1}} = 0 + a_{s2}\dfrac{\partial^2\Delta}{\partial a_{r1}\,\partial a_{s2}} + \ldots + a_{sn}\dfrac{\partial^2\Delta}{\partial a_{r1}\,\partial a_{sn}} \\
\dfrac{\partial\Delta}{\partial a_{r2}} = a_{s1}\dfrac{\partial^2\Delta}{\partial a_{r2}\,\partial a_{s1}} + 0 + \ldots + a_{sn}\dfrac{\partial^2\Delta}{\partial a_{r2}\,\partial a_{sn}} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dfrac{\partial\Delta}{\partial a_{rn}} = a_{s1}\dfrac{\partial^2\Delta}{\partial a_{rn}\,\partial a_{s1}} + a_{s2}\dfrac{\partial^2\Delta}{\partial a_{rn}\,\partial a_{s2}} + \ldots + 0
\end{array}\right\},$$

and then, by using the multipliers $a_{r1}, a_{r2}, \ldots, a_{rn}$ and adding, finds

$$\Delta = \begin{vmatrix} a_{r1} & a_{r2} \\ a_{s1} & a_{s2} \end{vmatrix} \frac{\partial^2\Delta}{\partial a_{r1}\partial a_{s2}} + \begin{vmatrix} a_{r1} & a_{r3} \\ a_{s1} & a_{s3} \end{vmatrix} \frac{\partial^2\Delta}{\partial a_{r1}\partial a_{s3}} + \ldots + \begin{vmatrix} a_{r,n-1} & a_{rn} \\ a_{s,n-1} & a_{sn} \end{vmatrix} \frac{\partial^2\Delta}{\partial a_{r1}\partial a_{sn}},$$

which is Laplace's expansion-theorem for the case where the minors of one set are of the second order. The remaining cases, he says, can be established in the same way.

Again, having proved the multiplication-theorem (row-by-row)

$$PQ = R,$$

where

$$P = \Sigma(\pm a_{11}a_{22}\ldots a_{nn}), \quad Q = \Sigma(\pm b_{11}b_{22}\ldots b_{nn}),$$
$$R = \Sigma(\pm c_{11}c_{22}\ldots c_{nn}),$$

he obtains by differentiation with respect to elements of P,

$$\frac{\partial P}{\partial a_{rs}}Q = \frac{\partial R}{\partial c_{r1}}b_{1s} + \frac{\partial R}{\partial c_{r2}}b_{2s} + \ldots + \frac{\partial R}{\partial c_{rn}}b_{ns}, \tag{1}$$

$$\frac{\partial^2 P}{\partial a_{rs}\partial a_{\rho\sigma}}Q = \sum_x\sum_y \begin{vmatrix} b_{x\sigma} & b_{xs} \\ b_{y\sigma} & b_{ys} \end{vmatrix} \frac{\partial^2 R}{\partial c_{ry}\partial c_{\rho x}}, \tag{2}$$

and by twice differentiating with respect to an element of P and an element of Q,

$$\frac{\partial P}{\partial a_{rs}}\frac{\partial Q}{\partial b_{\rho\sigma}} - \frac{\partial P}{\partial a_{r\sigma}}\frac{\partial Q}{\partial b_{\rho s}} = \sum_x \sum_y \begin{vmatrix} a_{x\sigma} & a_{xs} \\ b_{y\sigma} & b_{ys} \end{vmatrix} \frac{\partial^2 R}{\partial c_{ry}\partial c_{\rho x}}. \qquad (3)$$

Taking $n=4$ and $r, s, \rho, \sigma = 1, 2, 3, 4$, we can best illustrate these by writing them thus:—

$$-\,|a_{21}\,a_{33}\,a_{44}| \cdot |b_{11}\,b_{22}\,b_{33}\,b_{44}| = \begin{vmatrix} b_{12} & b_{22} & b_{32} & b_{42} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix}$$

$$= b_{12}\,|c_{22}\,c_{33}\,c_{44}| - \ldots\ldots, \qquad (1')$$

$$|a_{21}\,a_{43}| \cdot |b_{11}\,b_{22}\,b_{33}\,b_{44}| = \begin{vmatrix} b_{12} & b_{22} & b_{32} & b_{42} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ b_{14} & b_{24} & b_{34} & b_{44} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix}$$

$$= -\,|b_{12}\,b_{24}| \cdot |c_{23}\,c_{44}| + \ldots\ldots, \qquad (2')$$

$$\begin{Vmatrix} |a_{21}\,a_{33}\,a_{44}| & |a_{21}\,a_{32}\,a_{43}| \\ |b_{11}\,b_{23}\,b_{44}| & |b_{11}\,b_{22}\,b_{43}| \end{Vmatrix} = \begin{vmatrix} b_{12} & b_{22} & . & b_{42} \\ c_{21} & c_{22} & a_{24} & c_{24} \\ c_{31} & c_{32} & a_{34} & c_{34} \\ c_{41} & c_{42} & a_{44} & c_{44} \end{vmatrix} - \begin{vmatrix} b_{14} & b_{24} & . & b_{44} \\ c_{21} & c_{22} & a_{22} & c_{24} \\ c_{31} & c_{32} & a_{32} & c_{34} \\ c_{41} & c_{42} & a_{42} & c_{44} \end{vmatrix}$$

$$= -\,|a_{24}\,b_{12}| \cdot |c_{32}\,c_{44}| + \ldots\ldots \qquad (3')$$

The right-hand member of (1) is equivalent to a direction to substitute for the r^{th} row of R the s^{th} column of Q: similarly, the right-hand member of (2) is a direction, though not so evident, to substitute for the r^{th} and ρ^{th} rows of R the s^{th} and σ^{th} columns of Q; and it is clear that (1) and (2) are but the first two identities of many. On the other hand, (3) is quite diverse in character, being got by the combination of two results analogous to (2). This is best brought out by noting that in the examples the right-hand members of (1′) and (2′) are got by multiplying

$$\begin{vmatrix} . & 1 & . & . \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} . & 1 & . & . \\ a_{21} & a_{22} & a_{23} & a_{24} \\ . & . & . & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

respectively by $|b_{11}\, b_{22}\, b_{33}\, b_{44}|$; and that similarly the two four-line determinants on the right of (3) are got by multiplying

$$\begin{vmatrix} \cdot & 1 & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \cdot & \cdot & \cdot & 1 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix}$$

and

$$\begin{vmatrix} \cdot & \cdot & \cdot & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \cdot & 1 & \cdot & \cdot \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix}.$$

It should be carefully noted also that, while in (2) the number of terms in the development is $\frac{1}{2}n(n-1)$, in (3) the number is $(n-1)^2$.

Lastly, putting

$$\left.\begin{array}{l} b_{11}\dfrac{\partial P}{\partial a_{r1}} + b_{12}\dfrac{\partial P}{\partial a_{r2}} + \ldots\ldots + b_{1n}\dfrac{\partial P}{\partial a_{rn}} = H_{r1} \\ b_{21}\dfrac{\partial P}{\partial a_{r1}} + b_{22}\dfrac{\partial P}{\partial a_{r2}} + \ldots\ldots + b_{2n}\dfrac{\partial P}{\partial a_{rn}} = H_{r2} \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ b_{n1}\dfrac{\partial P}{\partial a_{r1}} + b_{n2}\dfrac{\partial P}{\partial a_{r2}} + \ldots\ldots + b_{nn}\dfrac{\partial P}{\partial a_{rn}} = H_{rn} \end{array}\right\}$$

so that H_{rs} stands for what P becomes when its r^{th} row is replaced by the s^{th} row of Q, and using the multipliers $\partial Q/\partial b_{11}$, $\partial Q/\partial b_{21}, \ldots\ldots, \partial Q/\partial b_{n1}$ prior to addition, Brioschi obtains

$$Q\frac{\partial P}{\partial a_{r1}} = H_{r1}\frac{\partial Q}{\partial b_{11}} + H_{r2}\frac{\partial Q}{\partial b_{21}} + \ldots\ldots + H_{rn}\frac{\partial Q}{\partial b_{n1}},$$

and similarly

$$Q\frac{\partial P}{\partial a_{r2}} = H_{r1}\frac{\partial Q}{\partial b_{12}} + H_{r2}\frac{\partial Q}{\partial b_{22}} + \ldots\ldots + H_{rn}\frac{\partial Q}{\partial b_{n2}},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$Q\frac{\partial P}{\partial a_{rn}} = H_{r1}\frac{\partial Q}{\partial b_{1n}} + H_{r2}\frac{\partial Q}{\partial b_{2n}} + \ldots\ldots + H_{rn}\frac{\partial Q}{\partial b_{nn}}.$$

With this derived set of equations the multipliers $a_{r1}, a_{r2}, \ldots, a_{rn}$ are then used, and addition performed, the result being Sylvester's theorem of 1839, namely,

$$QP = H_{r1}K_{r1} + H_{r2}K_{r2} + \dots + H_{rn}K_{rn},$$

where K_{rs} stands for what Q becomes when its s^{th} row is replaced by the r^{th} row of P.*

In his treatment of the minors of the adjugate determinant Brioschi (pp. 36–39) closely follows Spottiswoode; that is to say, from a set of linear equations he derives one result, then from the adjugate set another result, and finally draws a deduction from a comparison of the two. His thus obtained extension of Spottiswoode's theorem is open to the same criticism as Spottiswoode's extension of Jacobi's.

The section (§ 7) on "determinanti di determinanti" is founded on Cauchy, and contains known extensions of two or three theorems above given in the notation of differentiation.

CANTOR [M. B.] (1855, March).

[Théorème sur les déterminants Cramériens. *Nouv. Annales de Math.* (1), xiv. pp. 113–114.]

The theorem in question may be formulated thus—*If the permutations of* 1, 2, 3, . . . n *be arranged in order of magnitude as if they were integral numbers, the sign of the* k[th] *permutation is independent of* n. Reference is appropriately made to Reiss' paper of 1825, but the theorem is virtually contained in Hinderburg's rule of the year 1784.

Another author who dealt with the 'rule of signs' in this year was Mainardi; his paper is referred to along with a kindred one by Zehfuss of the year 1858.

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* Brioschi does not note the independent importance of his second set of equations, which may be condensed into

$$Q\frac{\partial P}{\partial a_{rs}} = H_{r1}\frac{\partial Q}{\partial b_{1s}} + H_{r2}\frac{\partial Q}{\partial b_{2s}} + \dots + H_{rn}\frac{\partial Q}{\partial b_{ns}},$$

and which, when $r, s = 1, 1$ and $n = 3$, is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \beta_2 & \beta_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} - \begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_2 & a_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} + \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_2 & a_3 \\ \beta_2 & \beta_3 \end{vmatrix}.$$

This, however, may be viewed also as a case of Sylvester's theorem, namely, where the first row of P is 1, 0, 0.

CAUCHY, A. L. (1856, Feb.).

[Sur une formule très-simple et très-général .... *Comptes rendus* .... *Acad. des Sci.* (Paris), xlii. pp. 366–374; or *Œuvres complètes* (1), xii. pp. 302–311.]

The main theorem concerns the set of equations

$$\left.a_{r1}x_1 + a_{r2}x_2 + \ .\ .\ .\ . + a_{rm}x_m = u_r\right\}_{r=1}^{r=n}$$

where the number m of unknowns is greater than the number n of equations, and the other theorem deals with the case of this where the u's all vanish. The weapons employed in the discussion are the 'produit symbolique' of the u's and 'clefs algébriques,' or 'clefs anastrophiques' as they are now called. The paper may be compared with Sylvester's of the year 1839 on 'the derivation of coexistence.'

HERMITE, C. (1856).

[Sur la théorie des polynomes homogènes du second degré. Note vi. (pp. 154–190) of *Programme .... d'Arithmétique, d'Algèbre et de Géométrie Analytique,* .... par MM. Gerono et Roguet: 4e éd. 215 pp. Paris.]

This note of thirty-seven pages, which is said to be 'd'après M. Hermite,' consists of five sections, the first of which (pp. 154–157) deals expressly and the others incidentally with déterminants. The latter sections concern the invariance of the discriminant (called '*the* invariant'), orthogonal transformation, etc., and are simply but suggestively written.

HEGER, I. (1856, July).

[Ueber die Auflösung eines Systemes von mehreren unbestimmten Gleichungen des ersten Grades in ganzen Zahlen. *Denkschr. d. k. Akad. d. Wiss.* (Wien): *math.-naturw. Cl.*, xiv. (2), pp. 1–122.]

Although in this lengthy paper the vanishing of the determinants of a 2-by-n array is repeatedly under consideration (*e.g.* § 24, p. 87), nothing new on the subject presents itself.

SCHLÖMILCH, O. (1856).

[Brioschi's Theorie der Determinante und ihre hauptsächlichsten Anwendungen. *Zeitschrift f. Math. u. Phys.*, I. *Literaturzeitung*, pp. 80–87.]

After a faithful account of Schellbach's translation of Brioschi's text-book, Schlömilch inveighs against the adoption of "die miserable englische Terminologie," instancing *Unterdeterminante, Determinante mit reciproken Elementen*, and *Hessian*, for the last of which he proposes to substitute "Inflexionsdeterminante."

RUBINI, R. (1857, May).

[Applicazione della teorica dei determinanti. *Annali de Sci. mat. e fis.*, viii. pp. 179–200.]

This resembles Chio's paper of 1853, having the same fundamental theorem, but different illustrative examples. In the mere enunciation of the theorem Rubini is the more successful. Taking the n-line determinant whose element in the place r,s is $a_{rs}+b_{rs}$, and denoting by A the determinant of the a's, and by A_r a determinant obtainable from A on substituting for r columns of a's the corresponding r columns of b's, he writes the expansion in the form

$$A+\Sigma A_1+\Sigma A_2+\ \ldots\ .+\Sigma A_{n-1}+A_n.$$

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinante. *Memorie . . . Istituto Veneto*, . . . vii. pp. 67–144.]

Notwithstanding its place of publication, this writing of Bellavitis' is exactly what its title implies; and as a text-book it could scarcely have failed to be useful, so simple and clear is it in style. It consists of two chapters, one on determinants in general (pp. 3–30), and one on special forms (pp. 30–72): a note of six pages on permutations appears as an appendix.

To Bellavitis we owe the modification of Laplace's notation which is now in common use. The passage introducing it is:

"Quando gli elementi sieno indicati in modo che chiaramente apparisca la loro formazione, noi porremo tra le due | i soli elementi della *diagonale* (intendendo sempre per *diagonale* quella da sinistra verso destra descendendo). Cosi

$$|a_1 b_2 c_3 \ldots .| \text{ equivalerà a } \begin{vmatrix} a_1 & b_1 & c_1 & \ldots . \\ a_2 & b_2 & c_2 & \ldots . \\ a_3 & b_3 & c_3 & \ldots . \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$|a_q^{(h)} a_s^{(r)}| \text{ equivalerà a } \begin{vmatrix} a_q^{(h)} & a_q^{(r)} \\ a_s^{(h)} & a_s^{(r)} \end{vmatrix}, \text{ ec.''}$$

Throughout the exposition this notation is employed. "Riga" he uses either for a "fila orizzontale" or a "fila verticale," and "colonna" for a "fila perpendicolare a quella che s'intese per riga."

Two well-known developments he specifies thus:—

$$|a_1 b_2 c_3 \ldots| = \left(a_1 \frac{\partial}{\partial a_1} + b_1 \frac{\partial}{\partial b_1} + c_1 \frac{\partial}{\partial c_1} + \ldots \right) |a_1 b_2 c_3 \ldots|,$$

$$\begin{aligned} |a_1 b_2 c_3 \ldots h_n| = \Big(a_1 \frac{\partial}{\partial a_1} &+ a_2 b_1 \frac{\partial^2}{\partial a_2 \partial b_1} + \ldots . + a_2 h_1 \frac{\partial^2}{\partial a_2 \partial h_1} \\ &+ a_3 b_1 \frac{\partial^2}{\partial a_3 \partial b_1} + \ldots . + a_3 h_1 \frac{\partial^2}{\partial a_3 \partial h_1} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ a_n b_1 \frac{\partial^2}{\partial a_n \partial b_1} + \ldots . + a_n h_1 \frac{\partial^2}{\partial a_n \partial h_1}\Big) |a_1 b_2 c_3 \ldots h_n|. \end{aligned}$$

In reference to determinants with binomial elements (§ 13) he says: "Compiendo questo sviluppo si ottiene la formula

$$\begin{aligned} |a_1 + \alpha_1 \; b_2 + \beta_2 \; c_3 + \gamma_3| = \; & |a_1 \, b_2 \, c_3| + |a_1 \, b_2 \, \gamma_3| + |a_1 \, \beta_2 \, c_3| + |a_1 \, \beta_2 \, \gamma_3| \\ & + |\alpha_1 \, b_2 \, c_3| + |\alpha_1 \, b_2 \, \gamma_3| + |\alpha_1 \, \beta_2 \, c_3| + |\alpha_1 \, \beta_2 \, \gamma_3| \end{aligned}$$

che è facile da tenersi a memoria per la sua perfetta analogia collo sviluppo del prodotto di tre binomii."

After giving a sufficient condition for the vanishing of a determinant, he enunciates (§ 15) the converse, namely, *When a determinant vanishes, one of the rows is equal to a sum of multiples of the other rows,* basing its validity on the fact that the multipliers referred to can actually be found by solving a set of simultaneous linear equations.

The multiplication-theorem for determinants Δ_1, Δ_2 of the third order he seeks to establish (§ 31) by partitioning the product-determinant into twenty-seven determinants, and showing that the sum of the six which do not vanish is $\Delta_1\Delta_2$.

Chio's theorem of 1853 is introduced (§ 38) by noting that the resultant of

$$a_r x + b_r y + c_r = 0 \qquad (r=1,2,3)$$

may be viewed as the resultant of

$$\left.\begin{matrix} |a_1\, b_2|\, y + |a_1\, c_2| = 0 \\ |a_1\, b_3|\, y + |a_1\, c_3| = 0 \end{matrix}\right\},$$

and that therefore

$$\left\| \begin{matrix} |a_1\, b_2| & |a_1\, c_2| \\ |a_1\, b_3| & |a_1\, c_3| \end{matrix} \right\| \text{ must be a multiple of } |a_1\, b_2\, c_3|.$$

That it is so he proves by diminishing the 2nd and 3rd columns of $|\, a_1\, b_2\, c_3\,|$ by b_1/a_1 times the 1st column and c_1/a_1 times the 1st column respectively. Further, he points out (§§ 39, 40) a practical application, namely, in evaluating a determinant whose elements are given in figures.

The adjugate determinant (unfortunately renamed *associato*) is dealt with (§§ 55–58) in connection with the solution of a set of simultaneous linear equations, the special cases being considered where the determinant of the set is 1 and 0. In the former special case he notes the theorem, *The adjugate of the product of two unit determinants is identical in all its elements with the product of the adjugates of the said determinants*; and in the latter the theorem all but reached by Jacobi in 1835 and 1841, *In a zero determinant the cofactors of the elements of a row are proportional to the cofactors of the elements of any other row.*

Cauchy's "clefs algébriques" (*chiavi algebriche*) are expounded at some length (§§ 81–88).

In the last three paragraphs he draws attention to the existence of expressions which may be viewed as "determinanti simbolici," his first kind being those in which symbols of differentiation take the place of elements; *e.g.* the negative of the expression

$$\mathrm{P}\left(\frac{\partial \mathrm{Q}}{\partial z}-\frac{\partial \mathrm{R}}{\partial y}\right)+\mathrm{Q}\left(\frac{\partial \mathrm{R}}{\partial x}-\frac{\partial \mathrm{P}}{\partial z}\right)+\mathrm{R}\left(\frac{\partial \mathrm{P}}{\partial y}-\frac{\partial \mathrm{Q}}{\partial x}\right),$$

whose vanishing is the condition for the derivability of the equation

$$\mathrm{P}\partial x + \mathrm{Q}\partial y + \mathrm{R}\partial z = 0$$

from a single primitive, is denoted by

$$\left| \mathrm{P} \ \frac{\partial}{\partial y} \ \mathrm{R} \right|$$

—a notation which is even less satisfactory than that for which it is a contraction, namely,

$$\begin{vmatrix} \mathrm{P} & \frac{\partial}{\partial x} & \mathrm{P} \\ \mathrm{Q} & \frac{\partial}{\partial y} & \mathrm{Q} \\ \mathrm{R} & \frac{\partial}{\partial z} & \mathrm{R} \end{vmatrix}.$$

The other kind of expressions originated with Binet, who in 1812 gave the identities

$$\begin{aligned} \Sigma ab' &= \Sigma a \Sigma b - \Sigma ab, \\ \Sigma ab'c'' &= \Sigma a \Sigma b \Sigma c + 2\Sigma abc - \Sigma a \Sigma bc - \Sigma b \Sigma ca - \Sigma c \Sigma ab; \\ & \cdots\cdots\cdots\cdots \end{aligned}$$

but in this case, though the close resemblance of the right-hand expressions to the developments of axisymmetric determinants is pointed out, no notation founded on the fact is suggested.

As an appendix there is a note on permutations, explaining circular substitutions, interchanges (*alternazioni*), inversions of order (*rovesciamenti d'ordine*), and their relations to one another. Cauchy's sign-rule depending on the number of circular substitutions is replaced by a simpler rule, which requires the counting of only the *even* circular substitutions. Thus the permutation 3265417 being got from the standard permutation 1234567 by means of the circular substitutions

$$(316), \quad (2), \quad (54), \quad (7),$$

and only one of these being even, the sign of 3265417 is $(-)^1$. Bellavitis' enunciation is: "*Il numero delle alternazioni, con* *cui una disposizione può mutarsi in un' altra è pari o dispari insieme col numero di tutte le sostituzioni binomie, quadrinomie, sestinomie, ecc. che occorrono per passare da una disposizione all' altra."*

BALTZER, R. (1857).

[THEORIE UND ANWENDUNG DER DETERMINANTEN, mit Beziehung auf die Originalquellen, dargestellt von Dr. Richard Baltzer . . .; vi+129 pp.; Leipzig. French translation by J. Houel, xii+235 pp.; Paris, 1861.]

The good qualities spoken of above as belonging to Brioschi's text-book are still more conspicuous in the German text-book of three years later, and the historical footnotes in Baltzer's give it a special additional value. The *theory* is dealt with in eight little chapters or sections, and the so-called *applications* in ten; several of the latter, however, might quite well have been classed with the former, as they are merely concerned with determinants of special form.

The first section corresponds closely in subject with Bellavitis' appendix: and in connection therewith may be noted Baltzer's remark (§ 2, 3) that *any term got from the diagonal term by substituting* $k_1, k_2, \ldots, k_n$ *for the second suffixes* $1, 2, \ldots, n$ *may also be got by substituting* $1, 2, \ldots, n$ *for* $k_1, k_2, \ldots, k_n$ *in the set of first suffixes.*

Brioschi's mode of proving Sylvester's theorem of 1839 is improved upon (§ 3, 11) by taking Q one order lower than P, and using the multipliers $\partial Q/\partial b_{11}, \partial Q/\partial b_{21}, \ldots, \partial Q/\partial b_{n-1,1}$ on the identities

$$\left.\begin{array}{lllll} a_{11}\dfrac{\partial P}{\partial a_{n1}} & + \; a_{12}\dfrac{\partial P}{\partial a_{n2}} & + \cdots + & a_{1n}\dfrac{\partial P}{\partial a_{nn}} & = 0 \\ a_{21}\dfrac{\partial P}{\partial a_{n1}} & + \; a_{22}\dfrac{\partial P}{\partial a_{n2}} & + \cdots + & a_{2n}\dfrac{\partial P}{\partial a_{nn}} & = 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1}\dfrac{\partial P}{\partial a_{n1}} & + \; a_{n-1,2}\dfrac{\partial P}{\partial a_{n2}} & + \cdots + & a_{n-1,n}\dfrac{\partial P}{\partial a_{nn}} & = 0 \end{array}\right\},$$

the result of addition then being

$$\begin{aligned}&\left(a_{11}\frac{\partial Q}{\partial b_{11}}+a_{21}\frac{\partial Q}{\partial b_{21}}+\dots\dots+a_{n-1,1}\frac{\partial Q}{\partial b_{n-1,1}}\right)\frac{\partial P}{\partial a_{n1}}\\+&\left(a_{12}\frac{\partial Q}{\partial b_{11}}+a_{22}\frac{\partial Q}{\partial b_{21}}+\dots\dots+a_{n-1,2}\frac{\partial Q}{\partial b_{n-1,1}}\right)\frac{\partial P}{\partial a_{n2}}\\+&\quad\cdots\cdots\cdots\cdots\cdots\cdots\\+&\left(a_{1n}\frac{\partial Q}{\partial b_{11}}+a_{2n}\frac{\partial Q}{\partial b_{21}}+\dots\dots+a_{n-1,n}\frac{\partial Q}{\partial b_{n-1,1}}\right)\frac{\partial P}{\partial a_{nn}}=0,\end{aligned}$$

which, if we bear in mind what single determinants the expressions in brackets stand for, is seen to be Sylvester's theorem in its alternative form as the assertion of the vanishing of an aggregate of products of pairs of determinants.

Of Jacobi's theorem regarding any coaxial minor of the adjugate an obvious extension is made (§ 7, 2), namely, *Any* m^{th} *order minor of the adjugate of any determinant* Δ *is equal to the product obtained by multiplying the cofactor of the corresponding minor in* Δ *by* Δ^{m-1}. The mode of proof followed in Cayley's of 1843.

The theorem formulated by Bellavitis regarding a zero determinant is appropriately based (§ 7, 5) on the vanishing of the two-line minors of the adjugate determinant—a course suggested by what Lebesgue did in 1837.

Cayley's development of 1847 is well stated (§ 8, 6) in the form

$$D+\Sigma a_{ii}D_i+\Sigma a_{ii}a_{kk}D_{ik}+\Sigma a_{ii}a_{kk}a_{ll}D_{ikl}+\dots+a_{11}a_{22}\dots a_{nn},$$

where D is what the given determinant becomes when all its diagonal elements are made 0, and $D_{ik\dots}$ is the minor of D got by deleting the i^{th}, k^{th}, . . . rows and the i^{th}, k^{th}, . . . columns; and the proof consists in showing that no term is thus neglected or repeated.

NEWMAN, F. (1857).

[On determinants, better called eliminants. *Proceedings Roy. Soc. London,* viii. pp. 426–431; or *Philos. Magazine* (4), xiv. p. 392.]

The author's object was merely to recommend the introduction of the subject into elementary text-books.

DEL GROSSO, R. (1857).

[Sulla regola secondo la quale debbono procedere i segni nello sviluppo d'un determinante in prodotti di determinanti minori. *Rendic. . . . Accad. Pontaniana* (Napoli), Ann. v. pp. 196–198. See also pp. 198–206.]

When a determinant is expressed in accordance with Laplace's theorem as an aggregate of products of complementary minors, Del Grosso directs that the sign of any product is to be $(-1)^\sigma$, where σ is the sum of the odd row-numbers and odd column-numbers of one of the factors. The rule is not stated with sufficient care, and the author in reaching it concludes too hastily that the simplest case is all that need be established.

JANNI, G. (1858).

[SAGGIO DI UNA TEORICA ELEMENTARE DE' DETERMINANTI, del Sacerdote Giuseppe Janni. . . . 40 pp. Napoli.]

Janni's professed object was to make determinants more readily accessible, previous text-books having, he says, either totally neglected demonstrations or used those of great difficulty. He speaks of the work as the first of a series, and its contents certainly look like the first five chapters of a text-book planned on a fairly large scale. The theorems, twenty-three in number, are carefully enunciated and are printed in italics; but, although the proofs receive every attention, it is very doubtful whether the object aimed at was to any extent accomplished. There is at any rate nothing sufficiently fresh in the treatment to warrant attention here.

ZEHFUSS, G. (1858).

[Ueber die Auflösung der linearen endlichen Differenzengleichungen mit variabeln Coefficienten. *Zeitschrift f. Math. u. Phys.*, iii. pp. 175–177.]

His solution suggests to Zehfuss the remark (p. 177) that every determinant can be expressed as a multiple integral. It will suffice to give the result in the case of a determinant of the 4th order. Denoting $\cos 2\pi\theta + \sqrt{-1}\sin 2\pi\theta$ by 1^θ, and putting P for

$$1^\alpha 1^\beta 1^\gamma 1^\delta (1^\delta - 1^\gamma)(1^\delta - 1^\beta)(1^\delta - 1^\alpha)(1^\gamma - 1^\beta)(1^\gamma - 1^\alpha)(1^\beta - 1^\alpha)$$

and Q for

$$\begin{aligned} &(a_1 1^{-\alpha} + b_1 1^{-\beta} + c_1 1^{-\gamma} + d_1 1^{-\delta}) \\ \times &(a_2 1^{-2\alpha} + b_2 1^{-2\beta} + c_2 1^{-2\gamma} + d_2 1^{-2\delta}) \\ \times &(a_3 1^{-3\alpha} + b_3 1^{-3\beta} + c_3 1^{-3\gamma} + d_3 1^{-3\delta}) \\ \times &(a_4 1^{-4\alpha} + b_4 1^{-4\beta} + c_4 1^{-4\gamma} + d_4 1^{-4\delta}), \end{aligned}$$

Zehfuss says that

$$\sum \pm a_1 b_2 c_3 d_4 = \overline{\underline{\iiiint}}_0^1 \mathrm{PQ}\, d\alpha\, d\beta\, d\gamma\, d\delta.$$

He does not, however, note in passing that

$$\mathrm{P} = \begin{vmatrix} 1^\alpha & 1^\beta & 1^\gamma & 1^\delta \\ 1^{2\alpha} & 1^{2\beta} & 1^{2\gamma} & 1^{2\delta} \\ 1^{3\alpha} & 1^{3\beta} & 1^{3\gamma} & 1^{3\delta} \\ 1^{4\alpha} & 1^{4\beta} & 1^{4\gamma} & 1^{4\delta} \end{vmatrix}.$$

BELLAVITIS, G. (1858, June).

[Studii sulle Memorie pubblicate dal Prof. Mainardi negli Atti del' i. r. Istituto Lombardo, vol. i., 1855, p. 90. *Atti . . . Istituto Veneto* . . . An. 1857–58, pp. 623–629.]

One note (p. 627), closely following Sylvester's paper of 1840, points out that if the equations

$$\left.\begin{aligned} a + bx + cx^2 &= 0 \\ a + \beta x + \gamma x^2 + \delta x^3 &= 0 \end{aligned}\right\}$$

have a common root, it is given by the equation

$$\begin{vmatrix} a + bx & c & . \\ a + \beta x & \gamma & \delta \\ ax & b & c \end{vmatrix} = 0\,;$$

and if the equations

$$\left.\begin{aligned} a + bx + cx^2 + dx^3 &= 0 \\ a + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4 &= 0 \end{aligned}\right\}$$

have two common roots, these roots are given by the equation*

$$\begin{vmatrix} a + bx + cx^2 & d & . \\ \alpha + \beta x + \gamma x^2 & \delta & \epsilon \\ ax + bx^2 & c & d \end{vmatrix} = 0.$$

Another note (pp. 627–628), in view of Mainardi's 'regola' above referred to, very properly draws attention to the rule given in the appendix to Bellavitis' *Sposizione* of 1857.

ZEHFUSS, G. (1858): MAINARDI, G. (1855).

[Ueber die Zeichen der einzelnen Glieder einer Determinante. *Zeitschrift f. Math. u. Phys.*, iii. pp. 249–250.]

[Una regola per attribuire il segno proprio ad ogni parte di un determinante numerico. *Atti ... Istituto Lombardo* (Milano), i. pp. 105–106.]

Neither of these communications is of importance. Zehfuss, using the recurrent law of formation and giving "dérangement" the very opposite of its original meaning, so that the principal term of an n-line determinant has $\frac{1}{2}n(n-1)$ derangements, seeks to show that the sign of any other term having μ derangements is $(-1)^{\frac{1}{2}n(n-1)-\mu}$.

Mainardi, employing Cauchy's "clefs algébriques," finds himself also face to face with derangements, and seriously advises that in counting them we should say, not 1, 2, 3, 4, 5, . . . , but 1, 2, 1, 2, 1, . . . , the sign being − or + according as we end with 1 or 2.

GALLENKAMP, W. (1858).

[Die einfachsten Eigenschaften und Anwendungen der Determinanten. 12 pp. Sch. Progr. Duisburg.]

A workmanlike twelve-page exposition.

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*No reference is made by either Sylvester or Bellavitis to the two other similarly derivable quadratics

$$\begin{vmatrix} b + cx + dx^2 & a & . \\ a + bx + cx^2 & . & d \\ \beta + \gamma x + \delta x^2 & a & \epsilon \end{vmatrix} = 0, \quad \begin{vmatrix} c + dx & a & b \\ b + cx + dx^2 & . & a \\ \gamma + \delta x + \epsilon x^2 & a & \beta \end{vmatrix} = 0.$$

SPERLING, I. (1858).

[Teorija opredělitelej i eja važnějsija priloženija. Č. 1. St. Petersburg.]

This dissertation I have failed to see. In English the title is, *The Theory of Determinants and its most important applications.* The letters used here in transliterating the Russian title have German values.

ZEHFUSS, G. (1858).

[Ueber eine gewisse Determinante. *Zeitschrift f. Math. u. Phys.*, iii. pp. 298–301.]

From two sets of simultaneous linear equations

$$\begin{aligned} a_r x + b_r y + c_r z + d_r w &= 0 \quad (r=1,2,3,4) \\ \mu_r x' + \nu_r y' &= 0 \quad (r=1,2) \end{aligned}$$

there arises by multiplication a set of eight equations in

$$xx',\ \ yx',\ \ zx',\ \ wx',\ \ xy',\ \ yy',\ \ zy',\ \ wy',$$

whose determinant, Δ say, must contain as factors the determinants D_4, D_2 of the original sets. This observation leads Zehfuss to consider generally such determinants as

$$\begin{vmatrix} a_1\mu_1 & a_1\nu_1 & b_1\mu_1 & b_1\nu_1 & c_1\mu_1 & c_1\nu_1 & d_1\mu_1 & d_1\nu_1 \\ a_1\mu_2 & a_1\nu_2 & b_1\mu_2 & b_1\nu_2 & c_1\mu_2 & c_1\nu_2 & d_1\mu_2 & d_1\nu_2 \\ a_2\mu_1 & a_2\nu_1 & b_2\mu_1 & b_2\nu_1 & c_2\mu_1 & c_2\nu_1 & d_2\mu_1 & d_2\nu_1 \\ a_2\mu_2 & a_2\nu_2 & b_2\mu_2 & b_2\nu_2 & c_2\mu_2 & c_2\nu_2 & d_2\mu_2 & d_2\nu_2 \\ a_3\mu_1 & a_3\nu_1 & b_3\mu_1 & b_3\nu_1 & c_3\mu_1 & c_3\nu_1 & d_3\mu_1 & d_3\nu_1 \\ a_3\mu_2 & a_3\nu_2 & b_3\mu_2 & b_3\nu_2 & c_3\mu_2 & c_3\nu_2 & d_3\mu_2 & d_3\nu_2 \\ a_4\mu_1 & a_4\nu_1 & b_4\mu_1 & b_4\nu_1 & c_4\mu_1 & c_4\nu_1 & d_4\mu_1 & d_4\nu_1 \\ a_4\mu_2 & a_4\nu_2 & b_4\mu_2 & b_4\nu_2 & c_4\mu_2 & c_4\nu_2 & d_4\mu_2 & d_4\nu_2 \end{vmatrix} \quad \textit{i.e.}\ \ D_{4\cdot 2}$$

where the elements of D_2 are repeated 4^2 times and each time are associated with a different element of D_4. Multiplying the first and second columns by $\partial D_4/\partial a_1$, the third and fourth by $\partial D_4/\partial b_1$, the fifth and sixth by $\partial D_4/\partial c_1$, the seventh and eighth by $\partial D_4/\partial d_1$, and performing two additions we obtain

$$\Delta_{4\cdot 2} = \begin{vmatrix} D_4\mu_1 & D_4\nu & b_1\mu_1 & \dots\dots & d_1\nu_1 \\ D_4\mu_2 & D_4\nu_2 & b_1\mu_2 & \dots\dots & d_1\nu_2 \\ . & . & b_2\mu_1 & \dots\dots & d_2\nu_1 \\ . & . & b_2\mu_2 & \dots\dots & d_2\nu_2 \\ . & . & b_3\mu_1 & \dots\dots & d_3\nu_1 \\ . & . & b_3\mu_2 & \dots\dots & d_3\nu_2 \\ . & . & b_4\mu_1 & \dots\dots & d_4\nu_1 \\ . & . & b_4\mu_2 & \dots\dots & d_4\nu_2 \end{vmatrix} \div \left(\frac{\partial D_4}{\partial a_1}\right)^2$$

$$= \frac{(D_4)^2 D_2}{\left(\frac{\partial D_4}{\partial a_1}\right)^2} \begin{vmatrix} b_2\mu_1 & \dots\dots & d_2\nu_1 \\ b_2\mu_2 & \dots\dots & d_2\nu_2 \\ . & . \quad . \quad . \quad . & . \\ b_4\mu_2 & \dots\dots & d_4\nu_2 \end{vmatrix}.$$

But since the six-line determinant here is formed from $\partial D_4/\partial a_1$ and D_2 just as $D_{4\cdot 2}$ is formed from D_4 and D_2, it follows that

$$\Delta_{4\cdot 2} = \frac{(D_4)^2 D_2}{\left(\frac{\partial D_4}{\partial a_1}\right)^2} \cdot \frac{\left(\frac{\partial D_4}{\partial a_1}\right)^2 D_2}{\left(\frac{\partial^2 D_4}{\partial a_1\,\partial b_2}\right)^2} \cdot \begin{vmatrix} c_3\mu_1 & \dots\dots & d_3\nu_1 \\ c_3\mu_2 & \dots\dots & d_3\nu_2 \\ c_4\mu_1 & \dots\dots & d_4\nu_1 \\ c_4\mu_2 & \dots\dots & d_4\nu_2 \end{vmatrix},$$

and ultimately

$$\Delta_{4\cdot 2} = \frac{(D_4)^2 D_2}{\left(\frac{\partial D_4}{\partial a_1}\right)^2} \cdot \dots\dots \frac{\left(\frac{\partial^3 D_4}{\partial a_1\,\partial b_2\,\partial c_3}\right)^2 D_2}{\left(\frac{\partial^4 D_4}{\partial a_1\,\partial b_2\,\partial c_3\,\partial d_4}\right)^2},$$

$$= (D_4)^2 D_2{}^4.$$

Had the orders of the original determinants been other than the 4th and 2nd the result which would have been reached by an exactly similar process is readily foreseen. The general result we may formulate for ourselves as follows:—*If P and Q be determinants of the* p*th* *and* q*th* *orders respectively; and if P's array of elements be written* q^2 *times, namely,* q *times in each of* q *rows, thus forming a grand array of* pq *rows and* pq *columns; and if every element of each of the sub-arrays be multiplied by one and the same element of Q, the multiplier in the case of the sub-array in the place* h, k *being the element* *which occupies the corresponding place in Q; then the determinant of the grand array is equal to $P^q Q^p$.*

SIMERKA, W. (1858).

[Bestimmte Gleichungen des ersten Grades mit n Unbekannten gelöst mittels der Permutationslehre. *Sitzungsb. ... Acad. d. Wiss.* (Wien), xxxiii. pp. 277–281.]

The contents of this simply-written paper are quite in accord with the title. The author writes as if nothing had ever previously been done on the subject. The common denominator of the value of the x's in

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = g_r \Big\}_{r=1}^{r=n}$$

he denotes by $\quad \mathfrak{P}(a_1\, a_2 \ldots a_n).$

CASORATI, F. (1858, September).

[Intorno ad alcuni punti della teoria dei minimi quadrati. *Annali di Mat.*, i. pp. 329–343.]

The title here refers only to the latter half of the paper, the other half being concerned with an auxiliary series of theorems on the product-determinant. The first of these theorems is avowedly old, being that which concerns the so-called product C of two non-quadrate arrays

$$\begin{array}{lllll} a_{11} & a_{12} & a_{13} & \ldots & a_{1n}, \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n}, \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn}, \end{array} \qquad \begin{array}{lllll} b_{11} & b_{12} & b_{13} & \ldots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \ldots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & b_{m3} & \ldots & b_{mn}, \end{array}$$

where $n > m$. The second, though not so spoken of, is only new in form, and concerns any primary minor of C. Unfortunately, Casorati does not observe that any primary minor of C is a determinant formed exactly like C after omitting a row from the first array and a row from the second, and that therefore his second theorem is unnecessary. Further, his mode of procedure leads him to an expression for a multiple of the minor, namely, for

$$(n-m+1)\frac{\partial C}{\partial c_{rs}},$$

and making an oversight similar to Hesse's of 1853, he does not divide both sides by $n-m+1$.

His third theorem,

$$C\frac{\partial C}{\partial c_{rs}} = \frac{\partial C}{\partial a_{r1}}\frac{\partial C}{\partial b_{s1}} + \frac{\partial C}{\partial a_{r2}}\frac{\partial C}{\partial b_{s2}} + \dots + \frac{\partial C}{\partial a_{rm}}\frac{\partial C}{\partial b_{sm}},$$

is more worthy of note. The proof of it depends essentially on substituting for C in the first factor of each term of the right-hand member its equivalent,

$$c_{r1}\frac{\partial C}{\partial c_{r1}} + c_{r2}\frac{\partial C}{\partial c_{r2}} + \dots + c_{rm}\frac{\partial C}{\partial c_{rm}},$$

in which, it is important to note, the differential-quotients are necessarily all independent of $a_{r1}, a_{r2}, \dots$ The said right-hand member can then be transformed into

$$\begin{aligned} &\frac{\partial C}{\partial b_{s1}}\left(b_{11}\frac{\partial C}{\partial c_{r1}} + b_{21}\frac{\partial C}{\partial c_{r2}} + \dots + b_{m1}\frac{\partial C}{\partial c_{rm}}\right) \\ &+ \frac{\partial C}{\partial b_{s2}}\left(b_{12}\frac{\partial C}{\partial c_{r1}} + b_{22}\frac{\partial C}{\partial c_{r2}} + \dots + b_{m2}\frac{\partial C}{\partial c_{rm}}\right) \\ &+ \dots\dots\dots\dots \\ &+ \frac{\partial C}{\partial b_{sn}}\left(b_{1n}\frac{\partial C}{\partial c_{r1}} + b_{2n}\frac{\partial C}{\partial c_{r2}} + \dots + b_{mn}\frac{\partial C}{\partial c_{rm}}\right), \end{aligned}$$

which, if addition be performed columnwise, becomes

$$0 + 0 + \dots + C\frac{\partial C}{\partial c_{rs}} + 0 + \dots + 0,$$

because of the fact that the theorem

$$c_{r1}\frac{\partial C}{\partial c_{s1}} + c_{r2}\frac{\partial C}{\partial c_{s2}} + \dots + c_{rn}\frac{\partial C}{\partial c_{sn}} = \left.\begin{matrix} C \\ 0 \end{matrix}\right\} \quad \text{when} \quad \left\{\begin{matrix} r = s \\ r \neq s \end{matrix}\right.$$

holds in reference to the a's and b's as well as to the c's—a fact which should be noted for other purposes, and which is readily seen to be justifiable if we view C in its composite form AB and bear in mind that the operation

$$a\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial c} + \dots,$$

when performed on a homogeneous linear function of $a, b, c, \ldots$ is equivalent to a substitution.

The case where the two given arrays are identical is formulated, due care being taken with the differential-quotients because of C becoming axisymmetric.

We have only to add that the form in which this new theorem of Casorati's is stated obscures to some extent its significance. If we write the case of $AB=C$ where $m=3$, $n=4$ in the form

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{Vmatrix} \cdot \begin{Vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{Vmatrix} = \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ \Sigma bl & \Sigma bm & \Sigma bn \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix},$$

then, freed from all reference to differentiation, the theorem for the case $r=2$, $s=3$ is

$$-\begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ \Sigma bl & \Sigma bm & \Sigma bn \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am \\ \Sigma cl & \Sigma cm \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_1 & m_1 & n_1 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_1 \\ \Sigma bl & \Sigma bm & b_1 \\ \Sigma cl & \Sigma cm & c_1 \end{vmatrix} + \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_2 & m_2 & n_2 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_2 \\ \Sigma bl & \Sigma bm & b_2 \\ \Sigma cl & \Sigma cm & c_2 \end{vmatrix}$$

$$+ \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_3 & m_3 & n_3 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_3 \\ \Sigma bl & \Sigma bm & b_3 \\ \Sigma cl & \Sigma cm & c_3 \end{vmatrix} + \begin{vmatrix} \Sigma al & \Sigma am & \Sigma an \\ l_4 & m_4 & n_4 \\ \Sigma cl & \Sigma cm & \Sigma cn \end{vmatrix} \cdot \begin{vmatrix} \Sigma al & \Sigma am & a_4 \\ \Sigma bl & \Sigma bm & b_4 \\ \Sigma cl & \Sigma cm & c_4 \end{vmatrix}.$$

Further, no change but substitution is necessary on passing to the case where the two original arrays are identical.

SALMON, G. (1859).

[LESSONS INTRODUCTORY TO THE MODERN HIGHER ALGEBRA. By the Rev. George Salmon, A.M. . . . xii + 147 pp. Dublin.]

The first three lessons (pp. 1–18) of this historically interesting text-book are devoted to an elementary exposition of determinants. The only fresh matter (§ 20) concerns the determinant formed from

$$\begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{Vmatrix}, \qquad \begin{Vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{Vmatrix}$$

by row-by-row multiplication. This is shown to vanish, not by pointing out that it contains at least one zero determinant of the third order as a factor, but by partitioning it into eight determinants with monomial elements, and showing that all the eight vanish.*

Unfortunately, for *terms* of a determinant the word "elements" is used, and for *adjugate* the word "reciprocal," although the elements of the adjugate are spoken of as the "inverse constituents."

SPERLING, J. F. DE (1860, April).

[Note sur un théorème de M. Sylvester relatif à la transformation du produit de déterminants du même ordre. *Journ. (de Liouville) de Math.* . . . (2), v. pp. 121–126.]

This is a carefully formulated proof of Sylvester's theorem of 1839 and the extended theorem of 1851, the lines followed being those suggested and illustrated by Cayley in 1843. Unfortunately, however, instead of extending Cayley's method to prove directly and at once the generalisation of 1851, Sperling repeats Cayley's proof of the simpler theorem, and then uses the method of so-called mathematical induction to arrive at the generalisation.

The two determinants whose product is the subject of discus-

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* In using the notation $\| \quad \|$ he is not more explicit than its author, Cayley. If it were explained that

$$\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right\|$$

stands for $\qquad |a_1 b_2|, \quad |a_1 b_3|, \quad |a_2 b_3|,$

it would readily follow that the statement

$$\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right\| = 0$$

was short for $\qquad |a_1 b_2|, \quad |a_1 b_3|, \quad |a_2 b_3| = 0, \quad 0, \quad 0;$

and that

$$\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right\| \cdot \left\| \begin{matrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \end{matrix} \right\|$$

was short for

$$\big(|a_1 b_2|, \quad |a_1 b_3|, \quad |a_2 b_3| \big)\!\!\!\chi\big(|a_1 \beta_2|, \quad |a_1 \beta_3|, \quad |a_2 \beta_3| \big),$$

sion being $|a_{11}\, a_{22} \ldots a_{nn}|$ and $|b_{11}\, b_{22} \ldots b_{nn}|$, or, say, A and B, he forms the determinant

$$\begin{vmatrix} a_{11} & a_{12} \ldots a_{1,n-m} & \cdot & \cdot & \ldots\ldots & \cdot & a_{1n} & b_{11} & b_{12} \ldots b_{1n} \\ a_{21} & a_{22} \ldots a_{2,n-m} & \cdot & \cdot & \ldots\ldots & \cdot & a_{2n} & b_{21} & b_{22} \ldots b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} \ldots a_{n,n-m} & \cdot & \cdot & \ldots\ldots & \cdot & a_{nn} & b_{n1} & b_{n2} \ldots b_{nn} \\ \cdot & \cdot \ldots & \cdot & a_{1,n-m+1} & a_{1,n-m+2} & \ldots\ldots a_{1,n-1} & a_{1n} & b_{11} & b_{12} \ldots b_{1n} \\ \cdot & \cdot \ldots & \cdot & a_{2,n-m+1} & a_{2,n-m+2} & \ldots\ldots a_{2,n-1} & a_{2n} & b_{21} & b_{22} \ldots b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot \ldots & \cdot & a_{n,n-m+1} & a_{n,n-m+2} & \ldots\ldots a_{n,n-1} & a_{nn} & b_{n1} & b_{n2} \ldots b_{nn} \end{vmatrix},$$

which, he says, is seen to vanish on trying to find Laplace's expansion of it in terms of minors formed from the last $n+1$ columns and the minors that are complementary of those. Then, noting that the like outcome is not met with when the boundary-line necessary for the application of the said expansion-theorem is horizontal and bisects the determinant, he sets about obtaining the terms of the latter development in orderly fashion. Clearly, the first factors of those terms are all alike as regards their first $n-m$ columns, but the remaining m columns may include another column of a's or may not. Making a separation of the terms in accordance with this distinction, and calling the one aggregate Σ_{m-1} and the other Σ_m, where the suffix corresponds with the number of columns of b's appearing in each first factor, and therefore also with the number of columns of a's appearing in each second factor, Sperling gives evidence that Σ_{m-1} is Sylvester's expansion for $|a_{11}\, a_{22} \ldots a_{nn}| \cdot |b_{11}\, b_{22} \ldots b_{nn}|$ when in the formation of it there is an interchange of $m-1$ columns, that Σ_m is the corresponding expansion due to an interchange of m columns, and that the two Σ's occur with different signs. The conclusion is thus reached that, if we have previously proved the identity $\mathrm{AB} = \Sigma_{m-1}$, the identity $\mathrm{AB} = \Sigma_m$ must follow.

It is important to note in passing that if Sperling had put zeros for $a_{1n}, a_{2n}, \ldots, a_{nn}$ in the second half of his $2n$-line determinant, its value then would have been, when obtained in one way, $(-1)^{m-1}\mathrm{AB}$, and in another, $(-1)^{m-1}\Sigma_m$. He would thus have made the natural extension of Cayley's simple proof.

CHAPTER III.

AXISYMMETRIC DETERMINANTS, FROM 1841 TO 1860.

UNDER the heading of axisymmetric determinants in our first volume a reminder ought to have been given that the determinant of the set of linear equations reached by Bezout in his so-called abridged method for eliminating the unknown between two equations of the same degree is axisymmetric, the first appearance of this special form being consequently thrown back to the year 1764 (see p. 317 of Bezout's *Recherches* . . .). Further, it should naturally thereafter have been recalled that the said axisymmetry had been discussed by Jacobi in 1835 (see *History*, i. p. 214, pp. 485–487) and by Cauchy in 1840 (see *History*, i. pp. 242–243).

CAYLEY, A. (1841, May).

[On a theorem in the geometry of position. *Cambridge Math. Journ.*, ii. pp. 267–271; or *Collected Math. Papers*, i. pp. 1–4.]

A general account has already been given of this interesting paper—interesting as regards the subject, and interesting as being the author's first 'prentice effort. All that remains to be noticed here is what may be called Cayley's series of vanishing axisymmetric determinants. These we may write in the short form

$$\begin{vmatrix} \cdot & (x)_{12} & (x)_{13} & 1 \\ (x)_{21} & \cdot & (x)_{23} & 1 \\ (x)_{31} & (x)_{32} & \cdot & 1 \\ 1 & 1 & 1 & \cdot \end{vmatrix},$$

$$\begin{vmatrix} . & (xy)_{12} & (xy)_{13} & (xy)_{14} & 1 \\ (xy)_{21} & . & (xy)_{23} & (xy)_{24} & 1 \\ (xy)_{31} & (xy)_{32} & . & (xy)_{34} & 1 \\ (xy)_{41} & (xy)_{42} & (xy)_{43} & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix},$$

$$\begin{vmatrix} . & (xyz)_{12} & (xyz)_{13} & (xyz)_{14} & (xyz)_{15} & 1 \\ (xyz)_{21} & . & (xyz)_{23} & (xyz)_{24} & (xyz)_{25} & 1 \\ (xyz)_{31} & (xyz)_{32} & . & (xyz)_{34} & (xyz)_{35} & 1 \\ (xyz)_{41} & (xyz)_{42} & (xyz)_{43} & . & (xyz)_{45} & 1 \\ (xyz)_{51} & (xyz)_{52} & (xyz)_{53} & (xyz)_{54} & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . \end{vmatrix},$$

. . . .

if we put

$$\begin{aligned} &(x)_{rs} \text{ for } (x_r - x_s)^2, \\ &(xy)_{rs} \text{ for } (x_r - x_s)^2 + (y_r - y_s)^2, \\ &(xyz)_{rs} \text{ for } (x_r - x_s)^2 + (y_r - y_s)^2 + (z_r - z_s)^2, \end{aligned}$$

. . . .

The fact that they are identically equal to zero is established by showing that each one is resolvable into two factors, of which one or both vanish; for example, that the first is equal to

$$\begin{vmatrix} x_1^2 & -2x_1 & . & 1 \\ x_2^2 & -2x_2 & . & 1 \\ x_3^2 & -2x_3 & . & 1 \\ 1 & . & . & . \end{vmatrix} \times_{rr} \begin{vmatrix} 1 & x_1 & . & x_1^2 \\ 1 & x_2 & . & x_2^2 \\ 1 & x_3 & . & x_3^2 \\ . & . & . & 1 \end{vmatrix},$$

where, for the moment, we use $\times_{rr}$ to indicate row-by-row multiplication.

It should be noted that not more than three of the series are contemplated by Cayley, as he viewed the identities mainly on their geometrical side, namely, as giving the relation between the distances of three points in a straight line, four points in a plane, and five points in three-dimensional space.*

* To one taking this point of view Sylvester's paper "On Staudt's theorems. . . ." will be of interest. See *Philos. Magazine*, iv. (1852), pp. 335–345; or *Collected Math. Papers*, i. pp. 382–391.

MOON, R. (1842).

[On elimination. *Cambridge Math. Journ.*, iii. pp. 183–184.]

What is here given is a rule-of-thumb for obtaining the development of any one of Bezout's condensed eliminants from the preceding one, the exposition being continued so far as to give the eliminant, R_{44} say, of

$$\left.\begin{array}{l} ax^4 + bx^3 + cx^2 + dx + e = 0 \\ \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon = 0 \end{array}\right\}$$

in the form

$$\{f(a\epsilon)\}^4 + \{f(a\gamma)\}\{f(c\epsilon)\}^2 + \ldots\ldots,$$

where $f(a\epsilon), f(a\gamma), \ldots\ldots$ stand for $a\epsilon - \alpha e, a\gamma - \alpha c, \ldots\ldots$

In a note covering the same ground but free of anything empirical, Salmon drew attention ten years later to the existence of three or four incorrect terms in Moon's last expansion. (See *Higher Plane Curves*, pp. 293–294.)

HESSE, O. (1844, Jan.).

[Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln. *Crelle's Journal*, xxviii. pp. 68–96; or *Werke*, pp. 89–122.]

In this paper there appears the first reference to the special form of determinant which has for its elements the second differential-quotients of a function, and which consequently is axisymmetric. On account of its importance this form falls to be dealt with separately: we merely note here that Hesse himself viewed it as a special form of Jacobian, namely, the Jacobian whose originating functions are the first differential-quotients of a single function, and that he called it *the determinant of this single function,* a practice which, when the function is quadratic, is not at variance with that introduced by Gauss.

For full information the reader is referred to Chapter XIII. of the present volume.

CAYLEY, A. (1846).

[Note on the maxima and minima of functions of three variables. *Cambridge and Dub. Math. Journ.*, i. pp. 74–75; or *Collected Math. Papers*, i. pp. 228–229.]

Using Δ to stand for the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

we may formulate Cayley's first theorem by saying that if $(a+b+c)\Delta$ and $A+B+C$ be positive, then $a\Delta$, $b\Delta$, $c\Delta$, A, B, C are all positive. By way of proof it is noted that the equation

$$\begin{vmatrix} A-x & H & G \\ H & B-x & F \\ G & F & C-x \end{vmatrix} = 0,$$

i.e. $\qquad x^3 - (A+B+C)x^2 + (a+b+c)\Delta x - \Delta^2 = 0,$

has by reason of the data all its roots positive: that therefore the roots of the equations

$$\begin{vmatrix} A-x & H \\ H & B-x \end{vmatrix} = 0, \quad \begin{vmatrix} B-x & F \\ F & C-x \end{vmatrix} = 0, \quad \begin{vmatrix} C-x & G \\ G & A-x \end{vmatrix} = 0,$$

i.e. $\qquad x^2 - (A+B)x + c\Delta = 0, \quad x^2 - (B+C)x + a\Delta = 0,$

$$x^2 - (C+A)x + b\Delta = 0,$$

on account of Cauchy's localisation of them in relation to the roots of the previous equation, must also be positive; and consequently that

$$A+B, \quad B+C, \quad C+A, \quad c\Delta, \quad a\Delta, \quad b\Delta$$

must be positive. This last is essentially what was to be proved; because, for example, to say that $A+B$ and $c\Delta$ are positive implies that $A+B$ and AB are positive, and therefore that A and B are positive.

Cayley also puts on record the theorem that the equation in x

$$\begin{vmatrix} A-xa & H-xh & G-xg \\ H-xh & B-xb & F-xf \\ G-xg & F-xf & C-xc \end{vmatrix} = 0$$

has all its roots real if either of the quadrics

$$\begin{gathered} Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy, \\ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy, \end{gathered}$$

remain constant as to sign.

CAYLEY, A. (1846).

[Problème de géométrie analytique. *Crelle's Journal*, xxxi. pp. 227–230; or *Collected Math. Papers*, i. pp. 329–331.]

The problem in question depends on an algebraic identity which, after a little examination, is seen to be a property of axisymmetric determinants. Cayley writes the identity in the form

$$F_{pp}(U+V^2)\cdot K(U) - F_{pp}(U)\cdot K(U+V^2) = \{F_{po}(U)\}^2$$

where

$$U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Lxw + 2Myw + 2Nzw + Pw^2,$$
$$V = ax + \beta y + \gamma z + \delta w,$$

$$K(U) = \begin{vmatrix} A & H & G & L \\ H & B & F & M \\ G & F & C & N \\ L & M & N & P \end{vmatrix}, \qquad P_{po}(U) = \begin{vmatrix} \cdot & \xi & \eta & \zeta & \omega \\ a & A & H & G & L \\ \beta & H & B & F & M \\ \gamma & G & F & C & N \\ \delta & L & M & N & P \end{vmatrix},$$

and $P_{pp}(U)$ is what is obtained from $P_{po}(U)$ on changing a, β, γ, δ into ξ, η, ζ, ω respectively: but freed of all fresh notation it is nothing more nor less than

$$\begin{vmatrix} \cdot & \xi & \eta & \zeta & \omega \\ \xi & A+a^2 & H+a\beta & G+a\gamma & L+a\delta \\ \eta & H+\beta a & B+\beta^2 & F+\beta\gamma & M+\beta\delta \\ \zeta & G+\gamma a & F+\gamma\beta & C+\gamma^2 & N+\gamma\delta \\ \omega & L+\delta a & M+\delta\beta & N+\delta\gamma & P+\delta^2 \end{vmatrix} \cdot \begin{vmatrix} A & H & G & L \\ H & B & F & M \\ G & F & C & N \\ L & M & N & P \end{vmatrix}.$$

$$-\begin{vmatrix} \cdot & \xi & \eta & \zeta & \omega \\ \xi & A & H & G & L \\ \eta & H & B & F & M \\ \zeta & G & F & C & N \\ \omega & L & M & N & P \end{vmatrix} \cdot \begin{vmatrix} A+\alpha^2 & H+\alpha\beta & G+\alpha\gamma & L+\alpha\delta \\ H+\beta\alpha & B+\beta^2 & F+\beta\gamma & M+\beta\delta \\ G+\gamma\alpha & F+\gamma\beta & C+\gamma^2 & N+\gamma\delta \\ L+\delta\alpha & M+\delta\beta & N+\delta\gamma & P+\delta^2 \end{vmatrix}$$

$$= \begin{vmatrix} \cdot & \xi & \eta & \zeta & \omega \\ \alpha & A & H & G & L \\ \beta & H & B & F & M \\ \gamma & G & F & C & N \\ \delta & L & M & N & P \end{vmatrix}^2 .$$

Nothing is said about the mode of proving it.*

"H (1)" (1846, Nov.).

[Mathematical notes, ii. *Camb. and Dublin Math. Journ.*, i. p. 286.]

A correspondent signing himself as above puts on record without proof two identities which, when the notation of determinants is used, may be written in the form

$$\begin{vmatrix} aa'-bb'-cc' & ab'+ba' & ca'+ac' \\ ab'+ba' & bb'-cc'-aa' & bc'+cb' \\ ca'+ac' & bc'+cb' & ca'-aa'-bb' \end{vmatrix} = (a^2+b^2+c^2)\cdot(aa'+bb'+cc') \cdot(a'^2+b'^2+c'^2)$$

* If we note that the first determinant can be written in the form

$$-\begin{vmatrix} -1 & \cdot & \alpha & \beta & \gamma & \delta \\ \cdot & \cdot & \xi & \eta & \zeta & \omega \\ \alpha & \xi & A & H & G & L \\ \beta & \eta & H & B & F & M \\ \gamma & \zeta & G & F & C & N \\ \delta & \omega & L & M & N & P \end{vmatrix},$$

and the fourth in the form

$$-\begin{vmatrix} -1 & \alpha & \beta & \gamma & \delta \\ \alpha & A & H & G & L \\ \beta & H & B & F & M \\ \gamma & G & F & C & N \\ \delta & L & M & N & P \end{vmatrix}$$

light dawns at once, for the last three determinants of the identity are then seen to be principal minors of the first, and the identity itself to be a case of Jacobi's theorem regarding a minor of the adjugate.

and

$$\begin{vmatrix} 2aa' & ab'+ba' & ca'+ac' \\ ab'+ba' & 2bb' & bc'+cb' \\ ca'+ac' & bc'+cb' & 2cc' \end{vmatrix} = 0.$$

It is seen that both determinants are axisymmetric, that the second is expressible as the product of two vanishing determinants, and that the first is formable from the second by subtracting $aa' + bb' + cc'$ from each diagonal element,—a fact which, taken along with the vanishing of the second, shows that $aa' + bb' + cc'$ is a factor of the first.

CAYLEY, A. (1847).

[Note sur les hyperdéterminants. *Crelle's Journal*, xxxiv. pp. 148–152; or *Collected Math. Papers*, i. pp. 352–355.]

The second paragraph of this note concerns the expression

$$6abcd + 3b^2c^2 - a^2d^2 - 4ac^3 - 4b^3d, \quad \text{or } \nabla \text{ say,}$$

soon afterwards (1851) to be called the "discriminant" of the binary cubic

$$ax^3 + 3bx^2y + 3cxy^2 + y^3;$$

and Cayley's proposition is that the determinant whose elements are the second differential-quotients of ∇ with respect to a, b, c, d, namely, the axisymmetric determinant

$$\begin{vmatrix} -2d^2 & 6cd & 6bd-12c^2 & 6bc-4ad \\ 6cd & 6c^2-24bd & 6ad+12bc & 6ac-12b^2 \\ 6bd-12c^2 & 6ad+12bc & 6b^2-24ac & 6ab \\ 6bc-4ad & 6ac-12b^2 & 6ab & -2a^2 \end{vmatrix},$$

is a numerical multiple of ∇^2. As a matter of fact he says the multiplier is 3; but this is because, instead of writing the determinant as here, he removes from it the factors 2, 6, 6, 2. A verificatory proof, unsatisfactory to himself, is given, the determinant being as a preliminary again altered into a multiple (by 6^4) of

$$\begin{vmatrix} a^2 & ab & 2b^2-ac & 3bc-2ad \\ ab & \frac{4}{3}ac-\frac{1}{3}b^2 & \frac{2}{3}bc+\frac{1}{3}ad & 2c^2-bd \\ 2b^2-ac & \frac{2}{3}bc+\frac{1}{3}ad & \frac{4}{3}bd-\frac{1}{3}c^2 & cd \\ 3bc-2ad & 2c^2-bd & cd & d^2 \end{vmatrix}$$

or

$$\begin{vmatrix} a^2 & ab & ac-3r & ad+9q \\ ba & b^2+2r & bc-q & bd-3p \\ ca-3r & cb-q & c^2+2p & cd \\ da+9q & db-3p & dc & d^2 \end{vmatrix},$$

where

$$p = \tfrac{2}{3}(bd-c^2), \quad q = \tfrac{1}{3}(bc-ad), \quad r = \tfrac{2}{3}(ac-b^2),$$

the last change being probably due to the fact that it was known that

$$\nabla = 9(pr-q^2)$$

and that verification would thus be easier.

CAYLEY, A. (1848).

[On geometrical reciprocity. *Cambridge and Dub. Math. Journ.*, iii. pp. 173–179; or *Collected Math. Papers*, i. pp. 377–382.]

Incidentally Cayley gives the identity

$$-\begin{vmatrix} \cdot & \xi & \eta & \zeta \\ \xi & 2a & a'+b & a''+c \\ \eta & a'+b & 2b' & b''+c' \\ \zeta & a''+c & b''+c' & 2c'' \end{vmatrix}$$

$$= 4\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} \cdot \left\{ax^2+b'y^2+c''z^2+(b''+c')yz+(c+a'')zx+(a'+b)xy\right\} \\ -\big[x(ab''-a''b+a'c-ac')+y(b'c-bc'+b''a'-b'a'') \\ +z(c''a'-c'a''+cb''-c''b)\big]^2,$$

where

$$\xi = ax+a'y+a''z, \quad \eta = bx+b'y+b''z, \quad \zeta = cx+c'y+c''z.$$

No proof is adduced, and it is not noted that, when the determinant $|ab'c''|$ is axisymmetric, the expression in rectangular brackets vanishes, and the identity becomes in later notation

$$-\begin{vmatrix} \cdot & \xi & \eta & \zeta \\ \xi & a & h & g \\ \eta & h & b & f \\ \zeta & g & f & c \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \cdot \begin{array}{ccc|c} x & y & z & \\ \hline a & h & g & x \\ h & b & f & y \\ g & f & c & z \end{array},$$

where
$$\xi,\ \eta,\ \zeta = (x,\ y,\ z \between \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

This particular result, we may note, is easily verified by performing on the four-line determinant the operation

$$\text{row}_1 - x\,\text{row}_2 - y\,\text{row}_3 - z\,\text{row}_4.$$

SYLVESTER, J. J. (1850, Aug.).

[On the intersections, contacts and other correlations of two conics expressed by indeterminate co-ordinates. *Cambridge and Dublin Math. Journ.*, v. pp. 262–282; or *Collected Math. Papers*, i. pp. 119–137.]

In this paper an important property of axisymmetric determinants is incidentally brought to notice; but, unfortunately, the author's intended statement is almost hidden through want of care. He says (p. 270) that the determinant

$$\begin{vmatrix} \mathrm{A} & \mathrm{C}' & \mathrm{B}' & l \\ \mathrm{C}' & \mathrm{B} & \mathrm{A}' & m \\ \mathrm{B}' & \mathrm{A}' & \mathrm{C} & n \\ l & m & n & 0 \end{vmatrix},$$

where

$$\mathrm{A} = \quad bc - a'^2, \quad \mathrm{B} = \quad ca - b'^2, \quad \mathrm{C} = \quad ab - c'^2,$$
$$\mathrm{A}' = -b'c' + aa', \quad \mathrm{B}' = -c'a' + bb', \quad \mathrm{C}' = -a'b' + cc',$$

is "the product of the determinant

$$\begin{vmatrix} a & c' & b' \\ c' & b & a' \\ b' & a' & c \end{vmatrix}$$

by the quantity

$$al^2 + bm^2 + cn^2 - 2a'mn - 2b'ln - 2c'lm."$$

Now the said four-line determinant is not resolvable unless A′, B′, C′ be changed in sign, and even then the second factor is not as printed, but is

$$-(al^2 + bm^2 + cn^2 + 2a'mn + 2b'nl + 2c'lm).$$

The theorem, in fact, may be viewed as giving an expression for the product of a ternary quadric by its discriminant; and at a later date might have been written*

$$\begin{array}{ccc|c} x & y & z & \\ \hline a & h & g & x \\ h & b & f & y \\ g & f & c & z \end{array} \cdot \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = - \begin{vmatrix} \cdot & x & y & z \\ x & \mathrm{A} & \mathrm{H} & \mathrm{G} \\ y & \mathrm{H} & \mathrm{B} & \mathrm{F} \\ z & \mathrm{G} & \mathrm{F} & \mathrm{C} \end{vmatrix}.$$

No proof is given by Sylvester; but in a footnote we are told that it depends on "a theorem given by M. Cauchy, and which is included as a particular case in a theorem of my own, relating to compound determinants." What theorem of Cauchy's is thus referred to it is not easy to say. One would think that the most natural proceeding would be to show that the coefficients of x^2, y^2, . . . on the one side are identical with those on the other; and, in this case, the names to be mentioned would be Lagrange and Jacobi.

In a postscript Sylvester enunciates a theorem connected with the linear transformation of an n-ary quadric; and as this concerns the "determinant" of the quadric, or what a year later he named the "discriminant," it necessarily involves a property of axisymmetric determinants. His wording (p. 281) is:—"Let U be a quadratic function of any number of letters $x_1, x_2, \ldots, x_n$, and let any number r of linear equations of the general form

$$a_{1r}x_1 + a_{2r}x_2 + \ldots + a_{nr}x_n = 0$$

be instituted between them; and by means of these equations let U be expressed as a function of any $n-r$ of the given letters, say of $x_{r+1}, x_{r+2}, \ldots, x_n$, and let U so expressed be called M. Let

$$a_{1r}r_1 + a_{2r}x_2 + \ldots + a_{nr}x_n$$

* A comparison of this with a result just obtained from Cayley (1848) gives the curious identity

$$\begin{vmatrix} \cdot & x & y & z \\ x & \mathrm{A} & \mathrm{H} & \mathrm{G} \\ y & \mathrm{H} & \mathrm{B} & \mathrm{F} \\ z & \mathrm{G} & \mathrm{F} & \mathrm{C} \end{vmatrix} = \begin{vmatrix} \cdot & \xi & \eta & \zeta \\ \xi & a & h & g \\ \eta & h & b & f \\ \zeta & g & f & c \end{vmatrix}$$

where $\xi, \eta, \zeta = \left(x, y, z \between \begin{array}{|ccc|} a & h & g \\ h & b & f \\ g & f & c \end{array}\right)$ and, as usual, $\mathrm{A} = bc - f^2$, ...

be called L_r. Then the determinant of M in respect to the $n-r$ letters above given is equal to the determinant of

$$U + L_1 x_{n+1} + L_2 x_{n+2} + \dots + L_r x_{n+r}$$

considered as a function of the $n+r$ letters

$$x_1, x_2, \dots, x_{n+r}$$

divided by the square of the determinant

$$\left.\begin{matrix} a_{11} & a_{21} & \dots & a_{r1} \\ a_{12} & a_{22} & \dots & a_{r2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1r} & a_{2r} & \dots & a_{rr} \end{matrix}\right\}."$$

As regards this we have to remark (1) that again no proof is offered, and (2) that the discriminant of

$$U + L_1 x_{n+1} + \dots + L_r x_{n+r}$$

is easily got by "bordering" the discriminant of U. Taking the case where U is

$$ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2pxw+2qyw+2rzw$$

with the discriminant

$$\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix},$$

and where the linear equations serving to eliminate x, y from U are

$$\mu_1 x + \mu_2 y + \mu_3 z + \mu_4 w = 0$$
$$\nu_1 x + \nu_2 y + \nu_3 z + \nu_4 w = 0,$$

we have, according to Sylvester, the discriminant of the altered U equal to

$$\begin{vmatrix} a & h & g & p & \mu_1 & \nu_1 \\ h & b & f & q & \mu_2 & \nu_2 \\ g & f & c & r & \mu_3 & \nu_3 \\ p & q & r & d & \mu_4 & \nu_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdot & \cdot \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 & \cdot & \cdot \end{vmatrix} \div |\mu_1 \nu_2|^2.$$

SYLVESTER, J. J. (1852, July).

[A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares. *Philos. Magazine* (4), iv. pp. 138–142; or *Collected Math. Papers*, i. pp. 378–381.]

What is really proved here is the important proposition in the theory of orthogonants regarding the reality of the roots of Lagrange's determinantal equation, or, as it was then called, the equation of the secular inequalities. Our present interest in the demonstration, however, lies merely in the fact that it is based on two properties of axisymmetric determinants which it is desirable to isolate and to have more carefully formulated than it was Sylvester's wont to do. They are—

(1) If $|(11)\ (22) \ldots . (nn)|$ be axisymmetric, and the result of multiplying it by itself be $|[11]\ [22] \ldots . [nn]|$; and if $f(x)$ be the determinant got from the former by adding x to each element of the principal diagonal, and $\mathrm{F}(x)$ the determinant got similarly from the latter; then

$$f(x) \cdot f(-x) = \mathrm{F}(-x^2).$$

(2) If $\mathrm{F}(x)$ be expanded and arranged according to descending powers of x, so that

$$\mathrm{F}(x) = x^n + \mathrm{C}_1 x^{n-1} + \mathrm{C}_2 x^{n-2} + \ldots + \mathrm{C}_n,$$

then C_r is the sum of the squares of all the r-line minors of the original determinant, it being understood that the one-line minors are the elements and the n-line minor the determinant itself.

Both are taken for granted,—a liberty which is not so defensible in the second case as in the first; for C_r is at the outset obtained merely as the sum of the r-line coaxial minors of $|[11]\ [22] \ldots . [nn]|$, and use must thus be latently made of the not quite self-evident theorem that *if Δ be an axisymmetric determinant, the sum of the* r*-line coaxial minors of Δ^2 is the sum of the squares of all the* r*-line minors of Δ.* As an illustration of the whole, let us take the case where the given determinant and its square are

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} L & R & Q \\ R & M & P \\ Q & P & N \end{vmatrix}$$

and where therefore

$$\left.\begin{matrix} L & R, & Q \\ & M & P \\ & & N \end{matrix}\right\} \equiv \left\{\begin{matrix} a^2+h^2+g^2 & ah+hb+gf & ag+hf+gc \\ & h^2+b^2+f^2 & hg+bf+fc \\ & & g^2+f^2+c^2. \end{matrix}\right.$$

We have then

$$\begin{vmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{vmatrix} \cdot \begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c+x \end{vmatrix} = \begin{vmatrix} L-x^2 & R & Q \\ R & M-x^2 & P \\ Q & P & N-x^2 \end{vmatrix},$$

$$-x^6 + x^4(L+M+N) - x^2\left(\begin{vmatrix} L & R \\ R & M \end{vmatrix} + \begin{vmatrix} L & Q \\ Q & N \end{vmatrix} + \begin{vmatrix} M & P \\ P & N \end{vmatrix}\right) + \begin{vmatrix} L & R & Q \\ R & M & P \\ Q & P & N \end{vmatrix},$$

$$-x^6 + x^4\left\{\begin{matrix} a^2+h^2+g^2 \\ +h^2+b^2+f^2 \\ +g^2+f^2+c^2 \end{matrix}\right\} - x^2\left\{\begin{matrix} \begin{vmatrix} a & h \\ h & b \end{vmatrix}^2 + \begin{vmatrix} a & g \\ h & f \end{vmatrix}^2 + \begin{vmatrix} h & g \\ b & f \end{vmatrix}^2 \\ + \begin{vmatrix} a & g \\ h & f \end{vmatrix}^2 + \begin{vmatrix} a & g \\ g & c \end{vmatrix}^2 + \begin{vmatrix} h & g \\ f & c \end{vmatrix}^2 \\ + \begin{vmatrix} h & g \\ b & f \end{vmatrix}^2 + \begin{vmatrix} h & g \\ f & c \end{vmatrix}^2 + \begin{vmatrix} b & f \\ f & c \end{vmatrix}^2 \end{matrix}\right\} + \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2.$$

The fact that the coefficients here are negative and positive alternately is what Sylvester utilises for his main purpose, application being made of Descartes' rule of signs.

SYLVESTER, J. J. (1852, Oct.).

[On Staudt's theorems concerning the contents of polygons and polyhedrons, with a note on a new and resembling class of theorems. *Philos. Magazine* (4), iv. pp. 335–345; or *Collected Math. Papers*, i. pp. 382–391.]

As an illustration of his mode of expressing the product of two determinants of the n^{th} order as a determinant of the $(n+1)^{\text{th}}$ order, Sylvester gives the identity

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \cdot \begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 & 1 \\ \xi_2 & \eta_2 & \zeta_2 & 1 \\ \xi_3 & \eta_3 & \zeta_3 & 1 \\ \xi_4 & \eta_4 & \zeta_4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma x_1\xi_1 & \Sigma x_1\xi_2 & \Sigma x_1\xi_3 & \Sigma x_1\xi_4 & 1 \\ \Sigma x_2\xi_1 & \Sigma x_2\xi_2 & \Sigma x_2\xi_3 & \Sigma x_2\xi_4 & 1 \\ \Sigma x_3\xi_1 & \Sigma x_3\xi_2 & \Sigma x_3\xi_3 & \Sigma x_3\xi_4 & 1 \\ \Sigma x_4\xi_1 & \Sigma x_4\xi_2 & \Sigma x_4\xi_3 & \Sigma x_4\xi_4 & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

where $\Sigma x_r\xi_s$ is put for $x_r\xi_s + y_r\eta_s + z_r\zeta_s$. Then performing on the last determinant the operations which we may denote by

$$\begin{array}{lll} \text{row}_1 - \tfrac{1}{2}\Sigma x_1{}^2\cdot\text{row}_5, & \text{row}_2 - \tfrac{1}{2}\Sigma x_2{}^2\cdot\text{row}_5, & \dots \\ \text{col}_1 - \tfrac{1}{2}\Sigma \xi_1{}^2\cdot\text{col}_5, & \text{col}_2 - \tfrac{1}{2}\Sigma \xi_2{}^2\cdot\text{col}_5, & \dots \end{array}$$

he obtains

$$-\tfrac{1}{8}\begin{vmatrix} \Sigma(x_1-\xi_1)^2 & \Sigma(x_1-\xi_2)^2 & \Sigma(x_1-\xi_3)^2 & \Sigma(x_1-\xi_4)^2 & 1 \\ \Sigma(x_2-\xi_1)^2 & \Sigma(x_2-\xi_2)^2 & \Sigma(x_2-\xi_3)^2 & \Sigma(x_2-\xi_4)^2 & 1 \\ \Sigma(x_3-\xi_1)^2 & \Sigma(x_3-\xi_2)^2 & \Sigma(x_3-\xi_3)^2 & \Sigma(x_3-\xi_4)^2 & 1 \\ \Sigma(x_4-\xi_1)^2 & \Sigma(x_4-\xi_2)^2 & \Sigma(x_4-\xi_3)^2 & \Sigma(x_4-\xi_4)^2 & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix};$$

so that, if x_r, y_r, z_r and ξ_r, η_r, ζ_r be rectangular co-ordinates of points in space, the result reached gives an expression for thirty-six times the product of the volumes of two tetrahedrons in terms of the distances of the angular points of the one from the angular points of the other. By proceeding to the case where the two tetrahedrons are coincident, and thence to the case where the four remaining points are situated in the same plane, we reach Cayley's relation connecting the mutual distances of four such points.

It is thus seen that whereas Cayley's vanishing axisymmetric determinant was originally got as a multiple of a peculiarly obtained square of the determinant

$$\begin{vmatrix} \Sigma x_1{}^2 & x_1 & y_1 & 0 & 1 \\ \Sigma x_2{}^2 & x_2 & y_2 & 0 & 1 \\ \Sigma x_3{}^2 & x_3 & y_3 & 0 & 1 \\ \Sigma x_4{}^2 & x_4 & y_4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Sylvester arrives at it by squaring

$$\begin{vmatrix} x_1 & y_1 & 0 & 1 \\ x_2 & y_2 & 0 & 1 \\ x_3 & y_3 & 0 & 1 \\ x_4 & y_4 & 0 & 1 \end{vmatrix}$$

in a special fashion, and then performing certain transformations on the result.

SYLVESTER, J. J. (1852, Nov.).

[Sur une propriété nouvelle de l'équation qui sert à déterminer les inégalités séculaires des planètes. *Nouv. Annales de Math.*, xi. pp. 434–440; or *Collected Math. Papers*, i. pp. 364–366.]

This paper of composite authorship probably originated in a letter from Sylvester giving his theorem and demonstration, with a remark or two additional. To these, which were made §§ 7, 7′, 8, the editor prefixed an introduction (§§ 1–6) on determinants and determinant-multiplication.*

The theorem is an extension of one which is the basis of his paper in the *Philosophical Magazine* of the same year, and may be shortly enunciated as follows: *If* $|(11)\,(22)\ldots(nn)|$ *be axisymmetric and have* $|[11]\,[22]\ldots[nn]|$ *for its* p^{th} *power, then the roots of the equation*

$$\begin{vmatrix} [11]-x & [12] & \ldots & [1n] \\ [21] & [22]-x & \ldots & [2n] \\ \ldots & \ldots & \ldots & \ldots \\ [n1] & [n2] & \ldots & [nn]-x \end{vmatrix} = 0$$

are the p^{th} *powers of the roots of the equation*

$$\begin{vmatrix} (11)-x & (12) & \ldots & (1n) \\ (21) & (22)-x & \ldots & (2n) \\ \ldots & \ldots & \ldots & \ldots \\ (n1) & (n2) & \ldots & (nn)-x \end{vmatrix} = 0.$$

* In the *Coll. Math. Papers* §§ 1–6 are omitted, and §§ 7, 7′, 8 are numbered §§ 6, 7, 8. The theorem of the original § 6 is incorrect.

The "demonstration" leaves a good deal to be desired. In effect it amounts to saying that if $\zeta_1, \zeta_2, \ldots, \zeta_p$ be the p^{th} roots of unity, and D_m the determinant got from $|(11)\,(22)\ldots(nn)|$ by subtracting $\zeta_m x$ from each of the diagonal elements, then

$$D_1 D_2 D_3 \ldots D_p \equiv \begin{vmatrix} [11]-x^p & [12] & \ldots\ldots & [1n] \\ [21] & [22]-x^p & \ldots\ldots & [2n] \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ [n1] & [n2] & \ldots\ldots & [nn]-x^p \end{vmatrix}.$$

Now it is well known that the multiplication of $D_1, D_2, \ldots, D_p$ enables us to arrive at the equation whose roots are the p^{th} powers in question, but this and Sylvester's statement are not by any means identical. The separate points to be established are (1) that the element in the r^{th} row and s^{th} column of the determinant which is the product of $D_1, D_2, \ldots, D_p$ consists of $[rs]$ and a tail of other terms, (2) that this tail vanishes in the case of every non-diagonal element, (3) that in the case of the diagonal elements it reduces to $-x^p$; and Sylvester's only justificatory statements are that the product of the p determinants is independent of the order in which they are taken, and that all the terms containing a ζ in any other power than the p^{th} will vanish.

Another true proposition made on insufficient foundation is that *the p^{th} power of an axisymmetric determinant is itself axisymmetric.* The foundation here is the incorrect proposition of § 6.

CAYLEY, A. (1852, Dec.).

[On the rationalisation of certain algebraical questions. *Cambridge and Dub. Math. Journ.*, viii. pp. 97–101; or *Collected Math. Papers*, ii. pp. 40–44.]

The equations first considered are of the type

$$a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}+\ldots\ldots=0,$$

and the fresh departure consists in viewing such an equation as the outcome of the set of equations

$$x+y+z+\ldots=0,\quad x^2=a,\quad y^2=b,\quad z^2=c,\quad \ldots.$$

Taking the case where the number of variables in the set is three, Cayley operates on the equation $x+y+z=0$ with the multipliers 1, yz, zx, xy, thus obtaining with the help of the other equations a set from which the variables x, y, z, xyz may be eliminated, with the result*

$$\begin{vmatrix} . & 1 & 1 & 1 \\ 1 & . & c & b \\ 1 & c & . & a \\ . & b & a & . \end{vmatrix} = 0.$$

Also and, so to say, conversely he operates with the multipliers x, y, z, xyz, and eliminates 1, yz, zx, xy, with the result

$$\begin{vmatrix} . & a & b & c \\ a & . & 1 & 1 \\ b & 1 & . & 1 \\ c & 1 & 1 & . \end{vmatrix} = 0.$$

Similarly, when there are four variables, the multipliers are

$$1, \; yz, \; zx, \; xy, \; xw, \; yw, \; zw, \; xyzw,$$

and the eliminands

$$x, \; y, \; z, \; w, \; yzw, \; zwx, \; wxy, \; xyz,$$

or *vice versa*; but in this case the two resultants are not essentially distinct, the one being derivable from the other by mere transference of lines. Cayley then adds, "And in general for any *even* number of quadratic radicals the two forms are not essentially distinct,† but may be derived from each other by interchanging lines and columns, while for an *odd* number of quadratic radicals the two forms cannot be so derived from each other, but are essentially distinct."

The equations next dealt with are of the type

$$a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}+\ldots. = 0,$$

---

* Already thus formed by Cayley in his first paper of all (1841).

† Observe that, although Cayley considers the two determinants of the previous case to be essentially distinct, the second is derivable from the first by multiplying the columns by abc, a, b, c respectively, and then dividing the rows by 1, bc, ca, ab respectively.

Sylvester having suggested the extension of the process. Taking the case of three variables, that is to say, when the set of equations is

$$x+y+z=0, \quad x^3=a, \quad y^3=b, \quad z^3=c,$$

he first uses the multipliers

$$1,\ xyz,\ x^2y^2z^2,\ x^2z,\ y^2x,\ z^2y,\ x^2y,\ y^2z,\ z^2x,$$

the eliminands then being

$$x,\ y,\ z,\ y^2z^2,\ x^2yz,\ y^2zx,\ z^2xy,\ z^2x^2,\ x^2y^2\,;$$

next he uses the said eliminands as multipliers, the new eliminands being

$$x^2,\ y^2,\ z^2,\ yz,\ zx,\ xy,\ xy^2z^2,\ yz^2x^2,\ zx^2y^2\,;$$

and finally using the new eliminands as multipliers, he eliminates

$$1,\ xyz,\ x^2y^2z^2,\ x^2z,\ y^2x,\ z^2y,\ x^2y,\ y^2z,\ z^2x,$$

that is to say, the first set of multipliers. Only in the case of the second elimination is the determinant axisymmetric, namely,

$$\begin{vmatrix}
. & c & b & . & . & . & 1 & . & . \\
c & . & a & . & . & . & . & 1 & . \\
b & a & . & . & . & . & . & . & 1 \\
. & . & . & a & . & . & . & 1 & 1 \\
. & . & . & . & b & . & 1 & . & 1 \\
. & . & . & . & . & c & 1 & 1 & . \\
1 & . & . & . & 1 & 1 & . & . & . \\
. & 1 & . & 1 & . & 1 & . & . & . \\
. & . & 1 & 1 & 1 & . & . & . & .
\end{vmatrix}$$

which must thus be equal to $(a+b+c)^3-27abc$. In the two other cases the determinants are not essentially distinct,* the rows of the one being columns of the other; and this is said to be true of two of the three forms whatever be the number of variables.

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* Cayley fails to notice, however, that each of these is readily transformable into the second. Thus, taking his first form, we have only to multiply the 4th, 8th, 9th columns of it by a, and divide the 2nd, 4th, 7th rows by a, when we obtain a determinant which by mere permutation of the rows $\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\7\,4\,1\,9\,3\,6\,8\,5\,2\end{pmatrix}$ and subsequently of the columns $\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\1\,6\,5\,4\,9\,8\,7\,3\,2\end{pmatrix}$ becomes identical with the axisymmetric form.

SYLVESTER, J. J. (1853, March).

[On the relation between the volume of a tetrahedron and the product of the sixteen algebraical values of its superficies. *Cambridge and Dub. Math. Journ.*, viii. pp. 171–178; or *Nouv. Annales de Math.*, xiii. pp. 203–209; or *Collected Math. Papers*, i. pp. 404–410.]

Denoting the vertices of a tetrahedron by a, b, c, d, its volume in terms of the edges by V, and the areas of its faces by $\frac{1}{4}\sqrt{F}$, $\frac{1}{4}\sqrt{G}$, $\frac{1}{4}\sqrt{H}$, $\frac{1}{4}\sqrt{K}$, we know that

$$\begin{aligned}144V^2 = {} & (bc)^2(da)^2\left\{(ca)^2+(db)^2+(ab)^2+(cd)^2-(bc)^2-(da)^2\right\} \\ & +(ca)^2(db)^2\left\{(ab)^2+(cd)^2+(bc)^2+(da)^2-(ca)^2-(db)^2\right\} \\ & +(ab)^2(cd)^2\left\{(bc)^2+(da)^2+(ca)^2+(db)^2-(ab)^2-(cd)^2\right\} \\ & -(bc)^2(ca)^2(ab)^2-(bc)^2(db)^2(cd)^2-(ca)^2(cd)^2(da)^2-(ab)^2(da)^2(db)^2 \\ = {} & \text{W say,}\end{aligned}$$

and

$$\begin{aligned}F &= -(bc)^4-(cd)^4-(db)^4+2(cd)^2(db)^2+2(db)^2(bc)^2+2(bc)^2(cd)^2 \\ G &= -(ac)^4-(cd)^4-(da)^4+2(cd)^2(da)^2+2(da)^2(ac)^2+2(ac)^2(cd)^2 \\ H &= -(ab)^4-(bd)^4-(da)^4+2(bd)^2(da)^2+2(da)^2(ab)^2+2(ab)^2(bd)^2 \\ K &= -(ab)^4-(bc)^4-(ca)^4+2(bc)^2(ca)^2+2(ca)^2(ab)^2+2(ab)^2(bc)^2.\end{aligned}$$

With this notation Sylvester points out that the condition for the vanishing of the surface of the tetrahedron is

$$\sqrt{F}+\sqrt{G}+\sqrt{H}+\sqrt{K} = 0,$$

and that this when freed of root-signs is

$$\sum F^4 - 4\sum F^3G + 6\sum F^2G^2 + 4\sum F^2GH - 40FGHK = 0,$$

or say

$$N = 0,$$

where N consequently is of the eighth degree in the squared edges. His reasoning then is that as the vanishing of the surface and the vanishing of the volume are necessarily coincident, it follows that W, having no rational factors, must itself be a factor of N; and that, W being of the third degree in the squared edges, the quotient N/W must be of the fifth degree.

Relying on this he proceeds to determine the quotient by considering the cases where (1) $ab=0=cd$, (2) $ab=0=ac$, (3) $ab=ac=ad=bc=bd=cd=1$, his result being

$$\sum(ab)^2(bc)^2(ca)^2\left[\begin{array}{c}(da)^4+(db)^4+(dc)^4\\ -\left\{(da)^2+(db)^2+(dc)^2\right\}\left\{(ab)^2+(bc)^2+(ca)^2\right\}\\ +(ab)^2(bc)^2+(bc)^2(ca)^2+(ca)^2(ab)^2\end{array}\right]$$

$$+2\sum(ab)^2(bc)^2(cd)^2(da)^2(ac)^2,$$

where there are four expressions under the first Σ and six under the second; or

$$\sum(ab)^2(bc)^2(ca)^2\left[\begin{array}{c}(da)^4+(db)^4+(dc)^4\\ -\left\{(da)^2+(db)^2+(dc)^2\right\}\left\{(ab)^2+(bc)^2+(ca)^2\right\}\\ +(da)^2(db)^2+(db)^2(dc)^2+(dc)^2(da)^2\\ +(ab)^2(bc)^2+(bc)^2(ca)^2+(ca)^2(ab)^2\end{array}\right]$$

where there are four expressions under the Σ, one corresponding to each face.

Although there is no explicit mention here of determinants, it being unnecessary, it has now to be noted that Sylvester had the determinant-form of W before him throughout: he even says that he had tried to express the quotient as a determinant, but had been unsuccessful. Without further restriction, then, as to form, his proposition is *If* F, G, H, K *be the complementary minors of the elements in the places* 11, 22, 33, 44 *of the determinant*

$$\left|\begin{array}{ccccc} . & (ab)^2 & (ac)^2 & (ad)^2 & 1\\ (ab)^2 & . & (bc)^2 & (bd)^2 & 1\\ (ac)^2 & (bc)^2 & . & (cd)^2 & 1\\ (ad)^2 & (bd)^2 & (cd)^2 & . & 1\\ 1 & 1 & 1 & 1 & . \end{array}\right| \text{ or } 2\mathrm{W} \text{ say},$$

then the result of rationalising

$$\sqrt{\mathrm{F}}+\sqrt{\mathrm{G}}+\sqrt{\mathrm{H}}+\sqrt{\mathrm{K}}$$

is divisible by W. This is not all, however; for Sylvester having noted the analogous case connected with the relation between the perimeter and area of a triangle, namely, the fact that

$$\begin{vmatrix} \cdot & (ab)^2 & (ac)^2 & 1 \\ (ab)^2 & \cdot & (bc)^2 & 1 \\ (ac)^2 & (bc)^2 & \cdot & 1 \\ 1 & 1 & 1 & \cdot \end{vmatrix} \quad i.e. \quad \sum(ab)^4 - 2\sum(ab)^2(bc)^2$$

is a divisor of the result of rationalising

$$\sqrt{2(bc)^2} + \sqrt{2(ca)^2} + \sqrt{2(ab)^2}$$

where the radicands are the complementary minors of the elements in the places 11, 22, 33 of the determinant, boldly extends the proposition (without proof) to any triangular number of arbitrary quantities, taking occasion also to point out that when we leave geometry (ab), (ac), ... may be written for $(ab)^2$, $(ac)^2$, ...

HESSE, O. (1853, April).

[Ueber Determinanten und ihre Anwendung in der Geometrie, insbesondere auf Curven vierter Ordnung. *Crelle's Journal*, xlix. pp. 243–264; or *Werke*, pp. 319–343.]

The main subject of the first half of Hesse's paper (pp. 243–253) is a property of axisymmetric determinants required for the establishment of the geometrical results contained in the second half. In the first three pages he considers the relations between the minors of two general determinants A, B, and the minors of their product C; or, as he unfortunately feels himself compelled to put it, "wie die partiellen Differential-quotienten der Determinante C nach ihren Elementen c genommen durch die partiellen Differential-quotienten der Factoren A und B nach ihren Elementen genommen sich ausdrücken lassen." What follows thereafter may be described as the establishment of the simple identity

$$\begin{vmatrix} u_{11} & u_{12} & \alpha_1 \\ u_{21} & u_{22} & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 \end{vmatrix} \begin{vmatrix} u_{11} & u_{12} & \gamma_1 \\ u_{21} & u_{22} & \gamma_2 \\ \gamma_1 & \gamma_2 & 0 \end{vmatrix} - \begin{vmatrix} u_{11} & u_{12} & \alpha_1 \\ u_{21} & u_{22} & \alpha_2 \\ \gamma_1 & \gamma_2 & 0 \end{vmatrix}^2 = \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{vmatrix}^2,$$

where $u_{21}=u_{12}$, by multiplying together

$$\begin{vmatrix} a_1 & -a_2 \\ \gamma_1 & -\gamma_2 \end{vmatrix}, \quad \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix}, \quad \begin{vmatrix} -a_2 & a_1 \\ -\gamma_2 & \gamma_1 \end{vmatrix}$$

in row-by-row fashion; and then the generalisation of this identity in two different directions.

The first generalisation consists in the proposition that the two-line axisymmetric determinant

$$\begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} & a_1 \\ u_{21} & u_{22} & \dots & u_{2n} & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \dots & u_{nn} & a_n \\ a_1 & a_2 & \dots & a_n & 0 \end{vmatrix} \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} & \gamma_1 \\ u_{21} & u_{22} & \dots & u_{2n} & \gamma_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \dots & u_{nn} & \gamma_n \\ \gamma_1 & \gamma_2 & \dots & \gamma_n & 0 \end{vmatrix} - \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} & a_1 \\ u_{21} & u_{22} & \dots & u_{2n} & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \dots & u_{nn} & a_n \\ \gamma_1 & \gamma_2 & \dots & \gamma_n & 0 \end{vmatrix}^2$$

where $u_{\kappa\lambda} = u_{\lambda\kappa}$, contains

$$\begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{vmatrix}$$

as a factor, and that the cofactor is an integral homogeneous function of the a's and likewise of the γ's. The case where n is equal to 3 is treated as follows. The determinant

$$\begin{vmatrix} u_{11} & u_{12} & u_{13} & a_1 \\ u_{21} & u_{22} & u_{23} & a_2 \\ u_{31} & u_{32} & u_{33} & a_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta \end{vmatrix},$$

having $u_{\kappa\lambda}=u_{\lambda\kappa}$, is introduced and denoted by B, with the result that the two-line determinant in question is representable by

$$\begin{vmatrix} \frac{\partial B}{\partial \gamma_1}a_1+\frac{\partial B}{\partial \gamma_2}a_2+\frac{\partial B}{\partial \gamma_3}a_3 & \frac{\partial B}{\partial a_1}a_1+\frac{\partial B}{\partial a_2}a_2+\frac{\partial B}{\partial a_3}a_3 \\ \frac{\partial B}{\partial \gamma_1}\gamma_1+\frac{\partial B}{\partial \gamma_2}\gamma_2+\frac{\partial B}{\partial \gamma_3}\gamma_3 & \frac{\partial B}{\partial a_1}\gamma_1+\frac{\partial B}{\partial a_2}\gamma_2+\frac{\partial B}{\partial a_3}\gamma_3 \end{vmatrix}$$

and its predicated factor by $\partial B/\partial\beta$. It is then pointed out that the former is the differential-quotient of the product of the two determinants

$$\begin{vmatrix} \dfrac{\partial B}{\partial \gamma_1} & \dfrac{\partial B}{\partial \gamma_2} & \dfrac{\partial B}{\partial \gamma_3} \\ \dfrac{\partial B}{\partial a_1} & \dfrac{\partial B}{\partial a_2} & \dfrac{\partial B}{\partial a_3} \\ m_1 & m_2 & m_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ n_1 & n_2 & n_3 \end{vmatrix},$$

called M and N, taken with respect to $m_1n_1+m_2n_2+m_3n_3$; and that consequently it is equal to

$$\frac{\partial M}{\partial m_1}\frac{\partial N}{\partial n_1}+\frac{\partial M}{\partial m_2}\frac{\partial N}{\partial n_2}+\frac{\partial M}{\partial m_3}\frac{\partial N}{\partial n_3}. \qquad (\varpi)$$

Since, however, we have

$$\frac{\partial B}{\partial \gamma_1}u_{11}+\frac{\partial B}{\partial \gamma_2}u_{12}+\frac{\partial B}{\partial \gamma_3}u_{13}+\frac{\partial B}{\partial \beta}a_1 = 0,$$

and other similar identities, it follows that

$$M\cdot\frac{\partial B}{\partial \beta} \quad i.e. \quad \begin{vmatrix} \dfrac{\partial B}{\partial \gamma_1} & \dfrac{\partial B}{\partial \gamma_2} & \dfrac{\partial B}{\partial \gamma_3} \\ \dfrac{\partial B}{\partial a_1} & \dfrac{\partial B}{\partial a_2} & \dfrac{\partial B}{\partial a_3} \\ m_1 & m_2 & m_3 \end{vmatrix} \cdot \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{\partial B}{\partial \beta}a_1 & -\dfrac{\partial B}{\partial \beta}\gamma_1 & u_{11}m_1+u_{12}m_2+u_{13}m_3 \\ -\dfrac{\partial B}{\partial \beta}a_2 & -\dfrac{\partial B}{\partial \beta}\gamma_2 & u_{21}m_1+u_{22}m_2+u_{23}m_3 \\ -\dfrac{\partial B}{\partial \beta}a_3 & -\dfrac{\partial B}{\partial \beta}\gamma_3 & u_{31}m_1+u_{32}m_2+u_{33}m_3 \end{vmatrix},$$

and therefore

$$M = \frac{\partial B}{\partial \beta}\begin{vmatrix} a_1 & \gamma_1 & u_{11}m_1+u_{12}m_2+u_{13}m_3 \\ a_2 & \gamma_2 & u_{21}m_1+u_{22}m_2+u_{23}m_3 \\ a_3 & \gamma_3 & u_{31}m_1+u_{32}m_2+u_{33}m_3 \end{vmatrix} = \frac{\partial B}{\partial \beta}\cdot P \text{ say.}$$

The expression (ϖ) then becomes

$$\frac{\partial B}{\partial \beta}\left\{\frac{\partial P}{\partial m_1}\frac{\partial N}{\partial n_1}+\frac{\partial P}{\partial m_2}\frac{\partial N}{\partial n_2}+\frac{\partial P}{\partial m_3}\frac{\partial N}{\partial n_3}\right\},$$

and all that remains is the evaluation of the bracketed factor after the equivalents of P and N have been substituted therein. The final result is

$$\begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{vmatrix} \cdot \left\{ \begin{array}{l} u_{11}(a_2\gamma_3 - a_3\gamma_2)^2 + u_{22}(a_3\gamma_1 - a_1\gamma_3)^2 + u_{33}(a_1\gamma_2 - a_2\gamma_1)^2 \\ \qquad + (u_{23}+u_{32})(a_3\gamma_1 - a_1\gamma_3)(a_1\gamma_2 - a_2\gamma_1) \\ \qquad + (u_{31}+u_{13})(a_1\gamma_2 - a_2\gamma_1)(a_2\gamma_3 - a_3\gamma_2) \\ \qquad + (u_{12}+u_{21})(a_2\gamma_3 - a_3\gamma_2)(a_3\gamma_1 - a_1\gamma_3) \end{array} \right\}.$$

The case where $n=4$ is similarly dealt with; but as it is necessarily more complicated, it is not carried quite so far, the cofactor of $|u_{11}\,u_{22}\,u_{33}\,u_{44}|$ being merely stated to be of the desired form and easily calculable. "Sie hat aber zu viele Glieder, um sie berechnet hinzuschreiben." The general proposition, as above given, is then formally enunciated.

The other generalisation made of the case where $n=2$ is to the effect that *the product of an axisymmetric determinant by the square of any other determinant is expressible as an axisymmetric determinant.* In connection with this the interesting point is the notation used for the elements of the product-determinants. Since the differential-quotient of

$$\tfrac{1}{2}\left\{ u_{11}x_1{}^2 + u_{22}x_2{}^2 + \ldots + u_{nn}x_n{}^2 + 2u_{12}x_1x_2 + \ldots \right\},$$

$$\text{or} \quad \mathrm{F}(x_1, x_2, \ldots, x_n) \quad \text{say,}$$

with respect to x_p is

$$u_{1p}x_1 + u_{2p}x_2 + \ldots + u_{pp}x_p + \ldots + u_{np}x_n$$

the result of annexing q as a second suffix to the x's in this may be suitably denoted by

$$\mathrm{F}'(x_{pq});$$

so that in accordance with this the product of

$$\begin{vmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{vmatrix}_{u_{\kappa\lambda} = u_{\lambda\kappa}} \quad \text{and} \quad \begin{vmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{vmatrix}$$

will be

$$\begin{vmatrix} \mathrm{F}'(x_{11}) & \mathrm{F}'(x_{21}) & \ldots & \mathrm{F}'(x_{n1}) \\ \mathrm{F}'(x_{12}) & \mathrm{F}'(x_{22}) & \ldots & \mathrm{F}'(x_{n2}) \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{F}'(x_{1n}) & \mathrm{F}'(x_{2n}) & \ldots & \mathrm{F}'(x_{nn}) \end{vmatrix}$$

"Multiplicirt man diese Determinante nochmals mit der vorhergehenden und setzt:

$$F_{pq} = x_{1p}F'(x_{1q}) + x_{2p}F'(x_{2q}) + \ . \ . \ . \ + x_{np}F'(x_{nq}),$$

so erhält man

$$\begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{vmatrix}^2 = \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} \\ F_{21} & F_{22} & \cdots & F_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ F_{n1} & F_{n2} & \cdots & F_{nn} \end{vmatrix}.$$

Da aber $F_{pq} = F_{qp}$ ist, so ist die letzte Determinante wieder *symmetrisch.*"

BRIOSCHI, F. (1853, July).

[Sur une propriété d'un produit de facteurs linéaires. *Cambridge and Dub. Math. Journ.*, ix. pp. 137–144; or *Opere mat.*, v. pp. 121–128.]

This paper of Brioschi's is avowedly an outcome of Cayley's "On the rationalisation of certain algebraical equations," published two months earlier. Starting with the equation $x+y+z=0$, and using on it the multipliers 1, yz, zx, xy in succession, he obtains

$$\left.\begin{array}{rl} x + y + z &= 0 \\ xyz + \quad\ + z^2.y + y^2.z &= 0 \\ xyz + z^2.x \quad\ + x^2.z &= 0 \\ xyz + y^2.x + x^2.y \quad\ &= 0 \end{array}\right\}$$

whence on elimination of xyz, x, y, z there results

$$0 = \begin{vmatrix} \cdot & 1 & 1 & 1 \\ 1 & \cdot & z^2 & y^2 \\ 1 & z^2 & \cdot & x^2 \\ 1 & y^2 & x^2 & \cdot \end{vmatrix} = \Delta \text{ say};$$

and as this equation has its origin in the equation $x+y+z=0$, he concludes that the determinant Δ must have $x+y+z$ for a factor. Then, since any one of the three other equations

$$x+y-z=0, \quad x-y+z=0, \quad -x+y+z=0$$

gives rise to the same result, it is easily seen how he reaches the identity

$$\begin{vmatrix} . & 1 & 1 & 1 \\ 1 & . & z^2 & y^2 \\ 1 & z^2 & . & x^2 \\ 1 & y^2 & x^2 & . \end{vmatrix} = (x+y+z)(x+y-z)(x-y+z)(-x+y+z).$$

An alternative form of Δ is introduced by the words "Observons qu'on a évidemment*

$$\Delta = \frac{1}{x^2y^2z^2}\begin{vmatrix} . & x & y & z \\ x & . & z^2xy & y^2xz \\ y & z^2xy & . & x^2yz \\ z & y^2xz & x^2yz & . \end{vmatrix} = \begin{vmatrix} . & x & y & z \\ x & . & z & y \\ y & z & . & x \\ z & y & x & . \end{vmatrix}\text{"}.$$

Similarly from the equation $x+y+z+w=0$ or any one of its seven relatives by using the multipliers

$$1,\ zw,\ yw,\ xw,\ zy,\ zx,\ yx,\ xyzw$$

and eliminating

$$xyz,\ xyw,\ xzw,\ yzw,\ x,\ y,\ z,\ w$$

there is obtained the result

$$\begin{vmatrix} . & . & . & . & 1 & 1 & 1 & 1 \\ . & . & 1 & 1 & . & . & w^2 & z^2 \\ . & 1 & . & 1 & . & w^2 & . & y^2 \\ . & 1 & 1 & . & w^2 & . & . & x^2 \\ 1 & . & . & 1 & . & z^2 & y^2 & . \\ 1 & . & 1 & . & z^2 & . & x^2 & . \\ 1 & 1 & . & . & y^2 & x^2 & . & . \\ w^2 & z^2 & y^2 & x^2 & . & . & . & . \end{vmatrix}$$

$$= (x+y+z+w)\cdot(-x+y+z+w)(x-y+z+w)(x+y-z+w)(x+y+z-w)$$
$$\cdot(-x-y+z+w)(-x+y-z+w)(-x+y+z-w),$$

* It would seem preferable to multiply the columns of Δ in order by xyz, x, y, z, the quantities just eliminated; and then divide the rows by the multipliers 1, yz, zx, xy used in obtaining the set of equations. The advantage of this method would be still greater in the next case.

an alternative form being stated to be the axisymmetric determinant

$$\begin{vmatrix} . & . & . & . & x & y & z & w \\ . & . & x & y & . & . & w & z \\ . & x & . & z & . & w & . & y \\ . & y & z & . & w & . & . & x \\ x & . & . & w & . & z & y & . \\ y & . & w & . & z & . & x & . \\ z & w & . & . & y & x & . & . \\ w & z & y & x & . & . & . & . \end{vmatrix}.$$

The general theorem is then formulated as follows: "Si en général on considère n éléments $x_1, x_2, \ldots, x_n$, en posant

$$X_n = x_1+x_2+\ldots+x_n,$$

et en désignant par

$$|X_n(1, 2, \ldots, m)|$$

le produit des facteurs linéaires qu'on deduit de X_n en changeant les signes à m des éléments $x_1, x_2, \ldots, x_n$; en aura pour n impair

$$X_n \cdot |X_n(1)| \cdot |X_n(1, 2)| \ldots\ldots |X_n(1, 2, 3, \ldots, \tfrac{1}{2}(n-1)| = -\Delta,$$

et pour n pair

$$X_n \cdot |X_n(1)| \cdot |X_n(1,2)| \ldots\ldots |X_n(1,2,\ldots, \tfrac{1}{2}(n-2)| \cdot |X_n(\bar{1},2,3,\ldots,\tfrac{1}{2}n)| = \Delta$$

où le symbole

$$X_n(1, 2, 3, \ldots, \tfrac{1}{2}n)$$

dénote que dans ce produit l'élément x_1 entre toujours parmi les éléments auxquels on a changé de signe. Le déterminant Δ résultera de la multiplication successive de l'équation $X_n = 0$ par l'unité, et par chacune des combinaisons deux à deux, quatre à quatre, ..., $(n-1)$ à $(n-1)$ si n est impair: n à n si n est pair, des éléments $x_1, x_2, \ldots, x_n$."

In addition, it is shown under this head that the number of equations in the set which originates the determinant is 2^{n-1}, and, a little unnecessarily, that the number of linear factors in the product is the same. It is also noted that if $x_1, x_2, \ldots, x_n$ be quadratic radicals, the product of the linear factors is rational.

Following Cayley's paper still further, Brioschi similarly makes clear that one of the nine-line determinants there obtained, namely

$$\begin{vmatrix} . & . & . & . & . & . & 1 & 1 & 1 \\ . & . & 1 & . & . & 1 & . & z^3 & . \\ . & . & 1 & . & 1 & . & z^3 & . & . \\ . & 1 & . & . & . & 1 & . & . & y^3 \\ . & 1 & . & 1 & . & . & y^3 & . & . \\ 1 & . & . & . & 1 & . & . & . & x^3 \\ 1 & . & . & 1 & . & . & . & x^3 & . \\ . & . & . & z^3 & y^3 & x^3 & . & . & . \\ 1 & 1 & 1 & . & . & . & . & . & . \end{vmatrix},$$

may be viewed as originating in any one of the nine equations

$$\begin{array}{lll} x + y + z = 0, & x + y + az = 0, & x + y + \beta z = 0, \\ x + ay + \beta z = 0, & x + ay + z = 0, & x + \beta y + z = 0, \\ x + \beta y + az = 0, & ax + y + z = 0, & \beta x + y + z = 0, \end{array}$$

where a, β are the imaginary cube roots of unity, and could thus be shown to be equal to the product of the nine left-hand members of those equations.

SPOTTISWOODE (1851, 1853).

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journal*, li. pp. 209–271, 328–381.]

The information given by Spottiswoode regarding axisymmetric determinants appears under a variety of headings. What little the first edition contained (pp. 33–34) as a part of § vi. on "Inverse Systems" is placed in the second edition under "Compound Determinants" (pp. 368–372). Sylvester's mode of reaching Cayley's determinants connected with the mutual distances of points is given under "Multiplication" (pp. 250–53); and the chapter or "section" (§ iv.) on "Homogeneous Functions," which of course has to deal with quadrics, goes so far as to assign the name *determinant of a quadratic form* to any determinant possessed of axisymmetry.* (See pp. 329, 336.)

That the mere number of available coefficients in a quantic may suggest a form of determinant which has no connection with the properties of the said quantic is enforced when in the same chapter (pp. 331–333) the ternary quantic is reached, and it is pointed out that the number of coefficients, $\frac{1}{2}(n+1)(n+2)$, is the same as in the case of the $(n+1)$-ary quadric. Thus the axisymmetric determinant

$$\begin{vmatrix} a & h & g & f' \\ h & b & f & g' \\ g & f & c & h' \\ f' & g' & h' & k \end{vmatrix}$$

which is formed from the coefficients of the ternary cubic

$$ax^3+by^3+cz^3+3(fy^2z+gz^2x+hx^2y+f'yz^2+g'zx^2+h'xy^2)+6kxyz$$

is "foreign to the nature of that function" although intimately associated with the properties of the quaternary quadric

$$ax^2+by^2+cz^2+kw^2+2(fyz+gzx+hxy+f'xw+g'yw+h'zw).$$

The most interesting matter, however, is found in the last section of all, § 11, the contents of which are miscellaneous. There, on pages 376–380, the determinants

$$\begin{vmatrix} 1 & . & a_1 & a_2 \\ . & 1 & b_1 & b_2 \\ a_1 & b_1 & 1 & . \\ a_2 & b_2 & . & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & . & . & a_1 & a_2 & a_3 \\ . & 1 & . & b_1 & b_2 & b_3 \\ . & . & 1 & c_1 & c_2 & c_3 \\ a_1 & b_1 & c_1 & 1 & . & . \\ a_2 & b_2 & c_2 & . & 1 & . \\ a_3 & b_3 & c_3 & . & . & 1 \end{vmatrix},$$

are considered, but, to one's regret, only with reference to the case where $|a_1b_2c_3|$ is an orthogonant. The first of the two determinants is given in the form

$$1-a_1^2-b_1^2-a_2^2-b_2^2+(a_1b_2-a_2b_1)^2;$$

* With this view in one's mind, the occurrence of particular axisymmetric determinants might suggest the construction of fresh quadratic forms. The form known as a Bezoutiant is actually defined in this way, the axisymmetric determinant then being Bezout's condensed eliminant of two equations of like degree. (See above, p. 109, and below, p. 138.)

and similar non-determinant forms are given for the whole of the thirty-six primary minors and for the first fifteen of the secondary minors of the second determinant. Thus, the primary minors which are the cofactors of the elements in the places (1, 1), (1, 3), (1, 5) are

$$1-b_1^2-b_2^2-b_3^2-c_1^2-c_2^2-c_3^2+(b_2c_3-b_3c_2)^2+(b_3c_1-b_1c_3)^2+(b_1c_2-b_2c_1)^2,$$
$$(a_1c_1+a_2c_2+a_3c_3)(1-b_1^2-b_2^2-b_3^2)+(b_1c_1+b_2c_2+b_3c_3)(a_1b_1+a_2b_2+a_3b_3),$$
$$|a_1b_2c_3|\cdot|b_1c_3|-a_1(1-b_1^2-b_2^2-b_3^2-c_1^2-c_2^2-c_3^2)-b_1(a_1b_1+a_2b_2+a_3b_3)$$
$$-c_1(a_1c_1+a_2c_2+a_3c_3),$$

all the others being similar in form to one or other of these three; and in like manner the secondary minors are exemplified by

$$1-c_1^2-c_2^2-c_3^2,$$
$$-(b_1c_1+b_2c_2+b_3c_3),$$
$$-b_1(1-c_1^2-c_2^2-c_3^2)+c_1(b_1c_1+b_2c_2+b_3c_3),$$
$$a_1(b_1c_1+b_2c_2+b_3c_3)-b_1(c_1a_1+c_2a_2+c_3a_3),$$
$$-c_1|a_1b_2c_3|+|a_2b_3|.$$

CAYLEY, A. (1853*).

[Note sur la méthode d'élimination de Bezout. *Crelle's Journ.*, liii. pp. 366–367; or *Collected Math. Papers*, iv. pp. 38–39.]

The determinant, to which Bezout's so-called 'abridged method' leads, we have already seen dealt with by Jacobi (1835) and Cauchy (1840).† Cayley's noteworthy contribution to it is contained in a single sentence, namely, "Pour éliminer les variables x, y entre deux équations du $n^{\text{ième}}$ degré

$$\left.\begin{array}{r}(a,\ \ldots\between x,y)^n=0\\(a',\ \ldots\between x,y)^n=0\end{array}\right\}$$

on n'a qu'à former l'équation identique

$$\frac{(a,\ \ldots\between x,y)^n\cdot(a',\ \ldots\between\lambda,\mu)^n-(a',\ \ldots\between x,y)^n\cdot(a,\ \ldots\between\lambda,\mu)^n}{\mu x-\lambda y}$$

* The author's date is April 1855; but the rule was published as Cayley's by Sylvester in 1853. See § 62 of the memoir 'On a theory of the syzygetic relations . . .'; and note also that Sylvester there first introduced the closely related idea of the *Bezoutiant*. (See footnote on p. 137 above.)

† See *History*, i. pp. 214, 243, 485–7.

$$= \left(\begin{array}{cccc} a_{00} & a_{01} & \cdots & a_{0,n-1} \\ a_{10} & a_{11} & \cdots & a_{1,n-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{array}\right| \!\! \left(x, y)^{n-1}(\lambda, \mu)^{n-1};\right.$$

le résultat de l'élimination sera

$$\begin{vmatrix} a_{00} & a_{01} & \cdots & a_{0,n-1} \\ a_{10} & a_{11} & \cdots & a_{1,n-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix} = 0."$$

Instead of paraphrasing this by means of a more familiar notation, as the editor of *Crelle* found it necessary to do, it will be better to apply it to a simple case, namely, where the given equations are

$$\left.\begin{array}{r} a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \\ b_3x^3 + b_2x^2 + b_1x + b_0 = 0 \end{array}\right\},$$

being careful to proceed in such a way as also to justify the 'rule.' Beginning with

$$\begin{vmatrix} a_0+a_1x+a_2x^2+a_3x^3 & a_0+a_1y+a_2y^2+a_3y^3 \\ b_0+b_1x+b_2x^2+b_3x^3 & b_0+b_1y+b_2y^2+b_3y^3 \end{vmatrix} \div (y-x),$$

which vanishes for all values of y, we change it into

$$\left.\begin{array}{l} \quad |a_0b_0| \;\; + |a_0b_1|\,y \;\; + |a_0b_2|\,y^2 + |a_0b_3|\,y^3 \\ + |a_1b_0|\,x + |a_1b_1|\,xy + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{array}\right\} \div (y-x),$$

then into

$$\begin{array}{l} |a_0b_1| + |a_0b_2|(y+x) + |a_0b_3|(y^2+yx+x^2) \\ \qquad\quad + |a_1b_2|\,xy \qquad\quad + |a_1b_3|\,xy(y+x) \\ \qquad\qquad\qquad\qquad\quad\;\; + |a_2b_3|\,x^2y^2, \end{array}$$

and finally into

$$\begin{array}{ccc|l} 1 & x & x^2 & \\ \hline |a_0b_1| & |a_0b_2| & |a_0b_3| & 1 \\ |a_0b_2| & |a_0b_3| + |a_1b_2| & |a_1b_3| & y \\ |a_0b_3| & |a_1b_3| & |a_2b_3| & y^2, \end{array}$$

where the elements of the square array are those of Bezout's condensed eliminant, as we have already found in a footnote when dealing with Jacobi's paper of 1835.

BRIOSCHI, F. (1854, March).

[La Teorica dei Determinanti, e le sue principali Applicazioni. viii+116 pp. Pavia.]

Unlike Spottiswoode, Brioschi in methodical manner defines "un determinante simmetrico," and gives four known properties expressed in clear language, all within the space of one page (p. 70).

BRIOSCHI, F. (1854, Dec.; 1855, Apr.).

[Sur quelques questions de la géométrie de position. *Crelle's Journal*, l. pp. 233–238; or *Opere mat.*, v. pp. 259–265.]

[Relations de distance entre des points. *Nouv. Annales de Math.*, xiv. pp. 172–173; or *Opere mat.*, v. pp. 123–124.]

The first title here recalls that of Cayley's maiden effort; and, as a matter of fact, the paper of 1854 had its origin in the paper of 1841. Cayley, it will be remembered, obtained the relation connecting the mutual distances of five points in space by multiplying the two determinants

$$\begin{vmatrix} \Sigma x_1^2 & -2x_1 & -2y_1 & -2z_1 & -2w_1 & 1 \\ \Sigma x_2^2 & -2x_2 & -2y_2 & -2z_2 & -2w_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Sigma x_5^2 & -2x_5 & -2y_5 & -2z_5 & -2w_5 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \quad \begin{vmatrix} 1 & x_1 & y_1 & z_1 & w_1 & \Sigma x_1^2 \\ 1 & x_2 & y_2 & z_2 & w_2 & \Sigma x_2^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_5 & y_5 & z_5 & w_5 & \Sigma x_5^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix},$$

the first of which is -16 times the second, and then putting the w's equal to zero. Brioschi now follows on the same lines, but with an interesting difference. Having shown that the determinant

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2 & (x_6-x_1)^2+(y_6-y_1)^2+(z_6-z_1)^2 & 1 & -2x_1 & -2y_1 & -2z_1 \\ x_2^2+y_2^2+z_2^2 & (x_6-x_2)^2+(y_6-y_2)^2+(z_6-z_2)^2 & 1 & -2x_2 & -2y_2 & -2z_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_5^2+y_5^2+z_5^2 & (x_6-x_5)^2+(y_6-y_5)^2+(z_6-z_5)^2 & 1 & -2x_5 & -2y_5 & -2z_5 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

vanishes identically, the simple fact being that the second column is a sum of multiples of the other columns, he multiplies it by $\frac{1}{8}$ of itself, namely by

$$\begin{vmatrix} 1 & \Sigma(x_6-x_1)^2 & \Sigma x_1^2 & x_1 & y_1 & z_1 \\ 1 & \Sigma(x_6-x_2)^2 & \Sigma x_2^2 & x_2 & y_2 & z_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \Sigma(x_6-x_5)^2 & \Sigma x_5^2 & x_5 & y_5 & z_5 \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \end{vmatrix}$$

and, putting $d_{61}, \ldots$ for $\Sigma(x_6-x_1)^2, \ldots$, obtains the relation

$$\begin{vmatrix} d_{61}^2 & d_{61}d_{62}+d_{12} & \ldots & d_{61}d_{65}+d_{15} & d_{61}+1 \\ d_{61}d_{62}+d_{12} & d_{62}^2 & \ldots & d_{62}d_{65}+d_{25} & d_{62}+1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{61}d_{65}+d_{15} & d_{62}d_{65}+d_{25} & \ldots & d_{65}^2 & d_{65}+1 \\ d_{61}+1 & d_{62}+1 & \ldots & d_{65}+1 & 1 \end{vmatrix},$$

which degenerates into Cayley's result when we put

$$x_6,\ y_6,\ z_6 = x_1,\ y_1,\ z_1,$$

and make certain easy transformations.

In a similar manner the relation between the distances of five points on an ellipsoid is found, and the relation "entre les plus courtes distances respectives et les inclinaisons mutuelles de sept lignes quelconques."

The second paper contains nothing new.

FERRERS, N. M. (1855, Dec.).

[Two elementary theorems in determinants. *Quarterly Journ. of Math.*, i. p. 364; or *Nouv. Annales de Math.*, xvi. pp. 402–403, xvii. pp. 190–191.]

The first theorem referred to is

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1+a_1 & 1 & \ldots & 1 \\ 1 & 1 & 1+a_2 & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \ldots & 1+a_n \end{vmatrix} = a_1a_2\ldots a_n,$$

the proof being dependent on the fact that, if any one of the a's be put equal to 0, the determinant vanishes. The second is

$$\begin{vmatrix} 1+a_1 & 1 & 1 & \dots & 1 \\ 1 & 1+a_2 & 1 & \dots & 1 \\ 1 & 1 & 1+a_3 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \dots & 1+a_n \end{vmatrix}$$
$$= a_1 a_2 \dots a_n \left(1+\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right),$$

which is made to rest mainly on the fact that if any one of the a's be put equal to 0 the determinant takes the form of the preceding determinant. Note is taken that Sylvester's theorem on p. 55 of the same volume is a special case of this second result.

BRUNO, F. FAA DI (1855, Dec.).

[Addizione alla nota inserita nel fascicolo di ottobre ultimo. *Annali di Sci. mat. e fis.*, vi. pp. 476–479; or § vi. of his THÉORIE GÉNÉRALE DE L'ÉLIMINATION, x+224 pp., Paris, 1859.]

The note referred to in the title professed to be "Sulle funzioni simmetriche delle radici di un' equazione," and contained, besides other things, the final expansions of the resultants of two quadrics, two cubics, and two quartics. The "addizione," on the other hand, draws attention to the axisymmetric determinants which represent those resultants, the author being apparently unaware that Jacobi had already done this in 1835 and Cauchy in 1840. His rule of formation is the same as Sylvester's of 1853 (June).*

In his "Théorie Générale de l'Élimination" the matter is gone into in greater detail, the rule of formation occupying a full page (pp. 55–56). He there also devotes a section (§ ix. pp. 66, 67) to an account of Jacobi's relations between the elements of Bezout's condensed eliminant.

* See our chapter on Persymmetric Determinants.

BRIOSCHI, F. (1856, Jan.).

[Sur une nouvelle propriété du resultant de deux équations algébriques. *Crelle's Journ.*, liii. pp. 372–376; or *Opere mat.*, v. pp. 277–282.]

Bezout's condensed eliminant of the equations

$$\left.\begin{aligned} a_0x^n + a_1x^{n-1} + \ldots + a_n &= 0 \\ b_0x^n + b_1x^{n-1} + \ldots + b_n &= 0 \end{aligned}\right\}$$

being denoted by $|\beta_{11}\ \beta_{22}\ \ldots\ \beta_{nn}|$, or B say, the new property referred to is

$$\sum_{r=1}^{r=n}\beta_{mr}\frac{\partial \mathrm{B}}{\partial a_r} = -(n-m+1)b_{m-1}\mathrm{B}.$$

The proof given is of no interest in connection with the theory of determinants.

CAYLEY, A. (1856, March).

[Note upon a result of elimination. *Philos. Magazine* (4), xi. pp. 378–379; or *Collected Math. Papers*, iii. pp. 214–215.]

If the quadric

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

have a linear factor, $\xi x+\eta y+\zeta z$ say, it must vanish identically where we make the substitution

$$x,\ y,\ z = \beta\zeta-\gamma\eta,\quad \gamma\xi-\alpha\zeta,\quad \alpha\eta-\beta\xi,$$

where α, β, γ are any quantities whatever; and consequently the co-factors of α^2, β^2, ... in the result must vanish,—that is to say, we must have

$$\left.\begin{array}{rrrrrrl} & c\eta^2 & +\ b\zeta^2 & -\ 2f\eta\zeta & & & = 0 \\ c\xi^2 & & +\ a\zeta^2 & & -\ 2g\zeta\xi & & = 0 \\ b\xi^2 & +\ a\eta^2 & & & & -\ 2h\xi\eta & = 0 \\ -2f\xi^2 & & & -\ 2a\eta\zeta & +\ 2h\zeta\xi & +\ 2g\xi\eta & = 0 \\ & -\ 2g\eta^2 & & +\ 2h\eta\zeta & -\ 2b\zeta\xi & +\ 2f\xi\eta & = 0 \\ & & -\ 2h\zeta^2 & +\ 2g\eta\zeta & +\ 2f\zeta\xi & -\ 2c\xi\eta & = 0 \end{array}\right\}$$

and therefore

$$\begin{vmatrix} . & c & b & -2f & . & . \\ c & . & a & . & -2g & . \\ b & a & . & . & . & -2h \\ -2f & . & . & -2a & 2h & 2g \\ . & -2g & . & 2h & -2b & 2f \\ . & . & -2h & 2g & 2f & -2c \end{vmatrix} = 0,$$

$$\text{or, say,}\quad 8\Delta = 0.$$

But on account of the breaking up of the quadric into linear factors the discriminant must likewise vanish. It is thus suggested that the discriminant is a factor of Δ; and by actual trial it is found that

$$\Delta = -2\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2.$$

It has only in addition to be recalled that Δ originally made its appearance in Sylvester's second paper on dialytic elimination.

BRUNO, F. FAÀ DI (1857, April).

[Sopra il volume della piramide triangolare. *Annali di Sci. mat. e fis.*, viii. pp. 77–78.]

With an eye on Sylvester's paper of October 1852, Faà di Bruno first expresses the volume (V) of a tetrahedron in the form

$$\tfrac{1}{6}\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \\ x_1-x_4 & y_1-y_4 & z_1-z_4 \end{vmatrix},$$

and then by squaring obtains the result

$$288V^2 = \begin{vmatrix} 2d_{12}^2 & d_{12}^2+d_{13}^2-d_{23}^2 & d_{12}^2+d_{14}^2-d_{24}^2 \\ d_{12}^2+d_{13}^2-d_{23}^2 & 2d_{13}^2 & d_{13}^2+d_{14}^2-d_{34}^2 \\ d_{12}^2+d_{14}^2-d_{24}^2 & d_{13}^2+d_{14}^2-d_{34}^2 & 2d_{14}^2 \end{vmatrix}$$

as a consequence of the relation

$$2\sum(x_1-x_2)(x_1-x_3) = \sum(x_1-x_2)^2+\sum(x_1-x_3)^2-\sum(x_2-x_3)^2.$$

Another form of the relation between the mutual distances of four points in a plane is thus brought to light.

RUBINI, R. (1857).

[Applicazione della teorica dei determinanti. *Annali di Sci. mat. e fis.*, viii. pp. 179–200.]

Rubini starts with the theorem which expresses a determinant with binomial elements as a sum of determinants with monomial elements, and then considers a long series of special cases. Among these Ferrers' theorems of the year 1855 occupy the first place (§§ 2, 3, pp. 181–184).

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinanti. *Memorie .... Istituto Veneto ....* vii. pp. 67–144.]

Besides the paragraphs (§§ 42, 43, 44) specifically devoted in the second half of the exposition to axisymmetric determinants, there are two others (§§ 9, 35) connected with the same subject in the first half. One of the latter (§ 35) draws attention to the fact that any coaxial minor of the axisymmetric determinant which is the square of a determinant is expressible as a sum of squares. What is new in the former is the definite reference to determinants which are doubly axisymmetric ("doppiamente simmetrici"), the examples given being *

$$\begin{vmatrix} a & b \\ b & a \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ b & d & b \\ c & b & a \end{vmatrix}, \quad \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix},$$

* The third, by reason of its central two-line minor, which might have been $\begin{vmatrix} \alpha & \delta \\ \delta & \alpha \end{vmatrix}$, is more specialised than a doubly axisymmetric determinant.

the last of which is noted as being equal to

$$-(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d),$$

whereas in reality it is equal to

$$(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d).$$

BALTZER, R. (1857).

[THEORIE UND ANWENDUNG DER DETERMINANTEN, mit . . . . . vi+129 pp. Leipzig, 1857.]

In five different sections (§ 3, 8, 9; § 5, 2; § 6, 2, 5; § 7, 5; § 18, 12) Baltzer gives attention to determinants whose elements a_{ik}, a_{ki} are identical. In § 3 he notes that conjugate minors are equal, and proves Jacobi's theorem regarding the differential-quotient of a determinant with respect to a non-diagonal element by using the fact that if u be a function of x and y, and y be a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx},$$

the whole matter being

$$\frac{\partial\Delta}{\partial a_{1k}} = \mathrm{A}_{ik} + \mathrm{A}_{ki}\frac{\partial a_{ki}}{\partial a_{ik}} = \mathrm{A}_{ik} + \mathrm{A}_{ki} = 2\mathrm{A}_{ik},$$

where it will be observed ∂'s only are employed. In § 7, 5 is given the result which we have already seen in Lebesgue's paper of 1837, namely, that for a vanishing axisymmetric determinant

$$\mathrm{A}_{ik} = \sqrt{\mathrm{A}_{ii}\mathrm{A}_{kk}}\,;$$

and there is thence deduced

$$\mathrm{A}_{i1} : \mathrm{A}_{i2} : \mathrm{A}_{i3} : \ldots\ldots = \sqrt{\mathrm{A}_{11}} : \sqrt{\mathrm{A}_{22}} : \sqrt{\mathrm{A}_{33}} : \ldots\ldots$$

Lastly, in § 18, 12 he applies this to Cayley's vanishing determinants of the year 1841; for example, to the determinant

$$\begin{vmatrix} \cdot & 1 & 1 & 1 & 1 \\ 1 & \cdot & d_{12} & d_{13} & d_{14} \\ 1 & d_{12} & \cdot & d_{23} & d_{24} \\ 1 & d_{13} & d_{23} & \cdot & d_{34} \\ 1 & d_{14} & d_{24} & d_{34} & \cdot \end{vmatrix}.$$

This being equal to zero, if we write $[r, s]$ for the cofactor of the element in the place (r, s), we have of course

$$[12]+[13]+[14]+[15] = 0,$$

and consequently also

$$\sqrt{[22]}+\sqrt{[33]}+\sqrt{[44]}+\sqrt{[55]} = 0,$$

—a result hitherto only obtained independently from geometry.*

BORCHARDT, C. W. (1859, May).

[Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante. *Crelle's Journ.*, lvii. pp. 111–121; or *Monatsb. d. Akad. d. Wiss.* (Berlin), pp. 376–388; or *Gesammelte Werke,* pp. 133–144; also abstract in *Annali di Mat.* . . . , ii. pp. 262–264.]

Following a suggestion of Rosenhain's, Borchardt seeks to obtain an expression for the resultant of two equations of the n^{th} degree $\phi(x) = 0$, $\psi(x) = 0$ in terms of the values† which $\phi(x)$ and $\psi(x)$ assume when x receives any $n+1$ values $a_0, a_1, \ldots a_n$.

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* Baltzer gives (p. 20) Cayley's determinant form for

$$-(\sqrt{a}+\sqrt{b}+\sqrt{c})\cdot(-\sqrt{a}+\sqrt{b}+\sqrt{c})\cdot(\sqrt{a}-\sqrt{b}+\sqrt{c})\cdot(\sqrt{a}+\sqrt{b}-\sqrt{c}),$$

placing in front of it what looks like a generalisation, namely

$$\begin{vmatrix} \cdot & a_1 & b_1 & c_1 \\ a_1 & \cdot & c_2 & b_2 \\ b_1 & c_2 & \cdot & a_2 \\ c_1 & b_2 & a_2 & \cdot \end{vmatrix},$$

but is not really such. We can easily show that if a_1, b_1, c_1 be multiplied and a_2, b_2, c_2 be divided by x, y, z respectively, the determinant is unaltered; consequently it

$$= \begin{vmatrix} \cdot & a_1a_2 & b_1b_2 & c_1c_2 \\ a_1a_2 & \cdot & 1 & 1 \\ b_1b_2 & 1 & \cdot & 1 \\ c_1c_2 & 1 & 1 & \cdot \end{vmatrix} = \begin{vmatrix} \cdot & 1 & 1 & 1 \\ 1 & \cdot & c_1c_2 & b_1b_2 \\ 1 & c_1c_2 & \cdot & a_1a_2 \\ 1 & b_1b_2 & a_1a_2 & \cdot \end{vmatrix} = \begin{vmatrix} \cdot & \sqrt{a_1a_2} & \sqrt{b_1b_2} & \sqrt{c_1c_2} \\ \sqrt{a_1a_2} & \cdot & \sqrt{c_1c_2} & \sqrt{b_1b_2} \\ \sqrt{b_1b_2} & \sqrt{c_1c_2} & \cdot & \sqrt{a_1a_2} \\ \sqrt{c_1c_2} & \sqrt{b_1b_2} & \sqrt{a_1a_2} & \cdot \end{vmatrix} = \ldots$$

† He does not mean in terms of these alone, but in terms of these and $a_0, a_1, \ldots a_n$.

The basis of his procedure is Cayley's result that

$$\begin{vmatrix} \phi(x) & \phi(y) \\ \psi(x) & \psi(y) \end{vmatrix} \div (y-x) = \begin{array}{c|l} 1 \quad x \quad \ldots \quad x^{n-1} & \\ \hline & 1 \\ & y \\ \quad D & \vdots \\ & y^{n-1} \end{array}$$

where D is the array of Bezout's condensed eliminant $|D|$. Denoting the left-hand member of this by $F(x,y)$ or $F(y,x)$, we see that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{vmatrix} \cdot |D| \cdot \begin{vmatrix} 1 & 1 & \ldots & 1 \\ y_1 & y_2 & \ldots & y^n \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{n-1} & y_2^{n-1} & \ldots & y_n^{n-1} \end{vmatrix} = \begin{vmatrix} F(x_1,y_1) & F(x_1,y_2) & \ldots & F(x_1,y_n) \\ F(x_2,y_1) & F(x_2,y_2) & \ldots & F(x_2,y_n) \\ \cdot & \cdot & \cdot & \cdot \\ F(x_n,y_1) & F(x_n,y_2) & \ldots & F(x_n,y_n) \end{vmatrix}$$

$$\text{and } \therefore \ |D| = |F(a_1,a_1) \ F(a_2,a_2) \ \ldots \ F(a_n,a_n)| \div |a_1^0 \ a_2^1 \ \ldots \ a_n^{n-1}|^2$$

which, were it not for the illusory elements $F(a_r, a_r)$ in the diagonal, might be viewed as constituting a solution of the problem.

To obtain unobjectionable expressions for these, the interpolational forms of $\phi(x)$, $\psi(x)$ are necessarily taken, namely

$$\frac{\phi(a_0)\cdot f(x)}{f'(a_0)\cdot(x-a_0)} + \ldots + \frac{\phi(a_n)\cdot f(x)}{f'(a_n)\cdot(x-a_n)} \quad \text{or} \quad \sum_{i=0}^{i=n} \frac{f(x)/(x-a_i)}{f'(a_i)}\phi(a_i),$$

$$\frac{\psi(a_0)\cdot f(x)}{f'(a_0)\cdot(x-a_0)} + \ldots + \frac{\psi(a_n)\cdot f(x)}{f'(a_n)\cdot(x-a_n)} \quad \text{or} \quad \sum_{i=0}^{i=n} \frac{f(x)/(x-a_i)}{f'(a_i)}\psi(a_i),$$

where $f(x)$ stands for $(x-a_0)(x-a_1)\ldots(x-a_n)$. The resulting expression for $F(x,y)$,

$$\frac{\sum \frac{f(x)/(x-a_i)}{f'(a_i)}\phi(a_i) \cdot \sum \frac{f(y)/(y-a_i)}{f'(a_i)}\psi(a_i) - \sum \frac{f(y)/(y-a_i)}{f'(a_i)}\phi(a_i) \cdot \sum \frac{f(x)/(x-a_i)}{f'(a_i)}\psi(a_i)}{y-x}$$

is then developed with a view to the actual performance of the division by $y-x$. Each of the two multiplications in the numerator gives $(n+1)^2$ terms; but, as the product of any term of $\phi(x)$ by the corresponding term of $\psi(y)$ is cancelled by the product of the corresponding terms of $\phi(y)$ and $\psi(x)$, the number of terms then remaining is

$$2(n+1)^2 - 2(n+1), \quad i.e. \quad 2n(n+1).$$

Next, taking the product of the $(r+1)^{\text{th}}$ term of $\phi(x)$ by the $(s+1)^{\text{th}}$ of $\psi(y)$ and subtracting the product of the $(r+1)^{\text{th}}$ of $\phi(y)$ by the $(s+1)^{\text{th}}$ of $\psi(x)$, we have

$$\frac{f(x)/(x-a_r)}{f'(a_r)}\phi(a_r)\cdot\frac{f(y)/(y-a_s)}{f'(a_s)}\psi(a_s) - \frac{f(y)/(y-a_r)}{f'(a_r)}\phi(a_r)\cdot\frac{f(x)/(x-a_s)}{f'(a_s)}\psi(a_s),$$

$$i.e. \quad \frac{f(x)\cdot f(y)\cdot \phi(a_r)\cdot\psi(a_s)}{f'(a_r)\cdot f'(a_s)}\left\{\frac{1}{(x-a_r)(y-a_s)}-\frac{1}{(y-a_r)(x-a_s)}\right\},$$

$$i.e. \quad \frac{f(x)\cdot f(y)\cdot \phi(a_r)\cdot\psi(a_s)}{f'(a_r)\cdot f'(a_s)}\cdot\frac{(y-x)(a_r-a_s)}{(x-a_r)(x-a_s)(y-a_r)(y-a_s)},$$

in which $y-x$ is a visible factor. From this, by the mere interchange of r and s, we secure the combination of another pair of terms; and by adding the two results we see that $F(x, y)$ or $F(y, x)$ can be expressed as a sum of $\frac{1}{2}n(n+1)$ terms of the form

$$\frac{f(x)\cdot f(y)}{f'(a_r)\cdot f'(a_s)}\left\{\phi(a_r)\cdot\psi(a_s)-\phi(a_s)\cdot\psi(a_r)\right\}\frac{a_r-a_s}{(x-a_r)(x-a_s)(y-a_r)(y-a_s)},$$

the individual terms of the sum being got by giving r, s the values

$$\begin{array}{llll} 0,1; & 0,2; & \ldots; & 0,n \\ & 1,2; & \ldots; & 1,n \\ & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot \\ & & & n-1,n. \end{array}$$

We thus arrive at

$$F(x, x) = \sum_{r,\,s=0,1;\ \ldots;\ n-1,n}\frac{-\{f(x)\}^2\cdot F(a_r, a_s)\cdot(a_r-a_s)^2}{f'(a_r)\cdot f'(a_s)\cdot(x-a_r)^2(x-a_s)^2},$$

and so see that if x be put equal to one of the a's, say a_i, all the terms under Σ which have r and s both different from i must vanish because of the presence of the first factor of the numerator. We have only therefore to consider the terms got by putting

$$r, s = i, 0;\ i, 1;\ \ldots;\ i, i-1;\ i, i+1;\ \ldots;\ i, n,$$

and thus have

$$F(a_i, a_i) = \sum_{r,s=i,0;\ i,1;\ \ldots;\ i,i-1}^{r,s=i,i+1;\ \ldots;\ i,n} \frac{-\{f(a_i)\}^2 \cdot F(a_r, a_s) \cdot (a_r - a_s)^2}{f'(a_r) \cdot f'(a_s) \cdot (a_i - a_s)^2 (a_i - a_s)^2}$$

$$= -\sum_{s=0,1,\ldots,i-1}^{s=i+1,\ldots,n} \frac{f'(a_i)}{f'(a_s)} F(a_i, a_s).$$

What was desired has thus been fully attained; it happens, however, that because of the peculiar constitution of this expression for $F(a_i, a_i)$, it being such that

$$\frac{F(a_i, a_i)}{f'(a_i) \cdot f'(a_i)} = -\sum_{s=0,1,\ldots,i-1}^{s=i+1,\ldots,n} \frac{F(a_i, a_s)}{f'(a_i) \cdot f'(a_s)},$$

we can proceed a little further and throw $|D|$ into a more elegant form. Thus, dividing the rows of

$$|F(a_1, a_1) \cdot F(a_2, a_2) \cdot \ldots F(a_n, a_n)|$$

by $f'(a_1)$, $f'(a_2)$, $\ldots$, $f'(a_n)$ respectively, and thereafter the columns by the same, we have, on putting

$$(rs) \quad \text{for} \quad F(a_r, a_s)/f'(a_r) \cdot f'(a_s),$$

$$|D| = \frac{|(11)\,(22)\ldots(nn)| \cdot \{f'(a_1) \cdot f'(a_2) \ldots f'(a_n)\}^2}{|a_1^0 a_2^1 \ldots a_n^{n-1}|^2}$$

$$= |(11)\,(22)\ldots(nn)| \cdot |a_0^0 a_1^1 \ldots a_n^n|^2,$$

where

$$(ii) = -\Big\{(i0) + (i1) + \ldots + (i,i-1) + (i,i+1) + \ldots + (in)\Big\},$$

that is to say, where each diagonal element (ii) with its sign changed is equal to the sum of all the other elements of its row together with the additional element $(i0)$.

Borchardt's treatment of this peculiar axisymmetric determinant is dealt with elsewhere. (See Chapter XVI. under Unisignants.) Meanwhile a remark incidentally made (p. 114) should be noted, namely, that *Any axisymmetric determinant having the sum of every row equal to zero has all its primary minors equal.*

CAYLEY, A. (1859, June).

[Note on the value of certain determinants, the terms of which are the squared distances of points in a plane or in space. *Quart. Journ. of Math.*, iii. pp. 275–277; or *Collected Math. Papers*, iv. pp. 460–462.]

The determinants referred to are those occurring in his first paper of the year 1841; but the expansions of them which are given do not assume that $\overline{12}=\overline{21}$, etc. Sylvester's related paper of March 1853 is also referred to, the 2W of which is put in the form

$$\begin{vmatrix} . & c & b & f & 1 \\ c & . & a & g & 1 \\ b & a & . & h & 1 \\ f & g & h & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix},$$

with the result that Q takes the form

$$\begin{aligned} & a^2b^2c^2\{f^4+g^4+h^4+g^2h^2+h^2f^2+f^2g^2+b^2c^2+c^2a^2+a^2b^2-(f^2+g^2+h^2)(a^2+b^2+c^2)\} \\ +\, & a^2g^2h^2\{f^4+b^4+c^4+b^2c^2+c^2f^2+f^2b^2+g^2h^2+h^2a^2+a^2g^2-(f^2+b^2+c^2)(a^2+g^2+h^2)\} \\ +\, & b^2h^2f^2\{g^4+c^4+a^4+c^2a^2+a^2g^2+g^2c^2+h^2f^2+f^2b^2+b^2h^2-(g^2+c^2+a^2)(b^2+h^2+f^2)\} \\ +\, & c^2f^2g^2\{h^4+a^4+b^4+a^2b^2+b^2h^2+h^2a^2+f^2g^2+g^2c^2+c^2f^2-(h^2+a^2+b^2)(c^2+f^2+g^2)\}. \end{aligned}$$

SALMON, G. (1859, July).

[On the relation which connects the mutual distances of five points in space. *Quart. Journ. of Math.*, iii. pp. 282–288.]

Salmon starts from the elimination of a, b, c from the set of trigonometrical equations

$$\left.\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}\right\}$$

and proceeding to similar eliminants of higher order reaches Cayley's results of 1841 and others related to them. The interest of the paper is mainly geometrical. Use is made of the proposition that *if the quadric*

$$ax^2 + by^2 + cz^2 + dw^2$$
$$+ 2lyz + 2mzx + 2nxy + 2pxw + 2qyw + 2rzw$$

be increased by $(\alpha x + \beta y + \gamma z + \delta w)^2$, *the discriminant*

$$\begin{vmatrix} a & n & m & p \\ n & b & l & q \\ m & l & c & r \\ p & q & r & d \end{vmatrix}$$

is increased by *

$$-\begin{vmatrix} \cdot & \alpha & \beta & \gamma & \delta \\ \alpha & a & n & m & p \\ \beta & n & b & l & q \\ \gamma & m & l & c & r \\ \delta & p & q & r & d \end{vmatrix}.$$

SALMON, G. (1859).

[LESSONS INTRODUCTORY TO THE MODERN HIGHER ALGEBRA. .... xii+147 pp. Dublin.]

The evectant (or first evectant) of an invariant I of a quantic $(a, b, c, \ldots \between x, y, z, \ldots)^n$ being

$$\left(\frac{\partial I}{\partial a}, \frac{\partial I}{\partial b}, \frac{\partial I}{\partial c}, \ldots \between \xi, \eta, \zeta, \ldots\right)^n,$$

* The new discriminant being

$$\begin{vmatrix} a+\alpha^2 & n+\alpha\beta & m+\alpha\gamma & p+\alpha\delta \\ n+\alpha\beta & b+\beta^2 & l+\beta\gamma & q+\beta\delta \\ m+\alpha\gamma & l+\beta\gamma & c+\gamma^2 & r+\gamma\delta \\ p+\alpha\delta & q+\beta\delta & r+\gamma\delta & d+\delta^2 \end{vmatrix}$$

is easily seen to be equal to

$$-\begin{vmatrix} -1 & \alpha & \beta & \gamma & \delta \\ \alpha & a & n & m & p \\ \beta & n & b & l & q \\ \gamma & m & l & c & r \\ \delta & p & q & r & d \end{vmatrix},$$

and therefore to be separable into the two expressions referred to.

it is readily seen that, in the special case where $n=2$ and where therefore I is the discriminant of the quantic, the evectant is expressible as an axisymmetric determinant, namely, that obtained by 'bordering' the discriminant by the contragredient variables; for example, taking the ternary quadric

$$\begin{array}{ccc|c} x & y & z & \\ \hline a & h & g & x \\ h & b & f & y \\ g & f & c & z \end{array}$$

we have for the evectant of its discriminant

$$-\begin{vmatrix} \cdot & \xi & \eta & \zeta \\ \xi & a & h & g \\ \eta & h & b & f \\ \zeta & g & f & c \end{vmatrix}.$$

In recalling this fact at the beginning of his Fifteenth Lesson and a further fact regarding the form of the evectant of a discriminant which vanishes, Salmon digresses for a moment to put on record (§ 155) an implicated property of axisymmetric determinants, namely, that *if the cofactor of the element in the place* (1, 1) *of an axisymmetric determinant vanishes, the determinant is expressible as 'a perfect square,'* his meaning being, *the square of a linear function of the elements of the first row.*

It is not uninteresting to observe here that this unpretentious but important manual of Salmon's and the very useful *Théorie Générale de l'Élimination* of Faà di Bruno above and elsewhere referred to appeared almost simultaneously in the year 1859, and that the one was dedicated to Cayley and Sylvester and the other to Cayley.

CHAPTER IV.

ALTERNANTS, FROM 1832 TO 1860.

Note ought to be taken of the fact that the first three papers here to be dealt with, namely, the papers by Murphy (1832), Binet (1837), and Haedenkamp (1841) belong to the previous period. Nothing, however, appearing in the former chapter on the subject requires modification on that account.

Further, as exaggerated statements regarding Vandermonde's contribution to the subject have been widely accepted, it seems desirable to point out the exact foundation on which such statements rest. In a paper* read in November 1770 Vandermonde says (p. 369), "Or $a^2b+b^2c+c^2a-a^2c-b^2a-c^2b$, qui égale $(a-b)(a-c)(b-c)$ a pour carré $a^4b^2+\ldots.$" This is the whole matter.†

MURPHY, R. (1832, Nov.).

[On elimination between an indefinite number of unknown quantities. *Transac. Cambridge Philos. Soc.*, v. pp. 65–76.]

Murphy's third example in illustration of his method is the set of equations

$$\left.\begin{array}{l} 1+\ x_1+\ \ x_2+\ldots.+\ \ x_n\ =\ 0 \\ 1+2x_1+2^2x_2+\ldots.+2^nx_n=0 \\ 1+3x_1+3^2x_2+\ldots.+3^nx_n=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 1+nx_1+n^2x_2+\ldots.+n^nx_n=0 \end{array}\right\}$$

* Vandermonde, N. Mémoire sur la résolution des équations. *Mém. de l'Acad. des Sci.* (Paris), Année 1771, pp. 365–416.

† See *Nouv. Annales de Math.*, xix. p. 181 footnote.

which he neatly and easily solves, giving the value of x_m, and thus in effect evaluating

$$(-)^m \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^{m-1} & 2^{m+1} & \dots & 2^n \\ 1 & 3 & 3^2 & \dots & 3^{m-1} & 3^{m+1} & \dots & 3^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & n & n^2 & \dots & n^{m-1} & n^{m+1} & \dots & n^n \end{vmatrix} \div \begin{vmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^n \\ 3 & 3^2 & \dots & 3^n \\ \cdot & \cdot & \cdot & \cdot \\ n & n^2 & \dots & n^n \end{vmatrix}.$$

His connection with our subject is thus seen to be similar to Prony's.

It should be carefully noted, however, in passing, that Prony's set of equations is not the same as Murphy's, the determinant of the one being conjugate to that of the other.* When the use of determinants is debarred or avoided, this difference is far from unimportant,—a fact which might readily be surmised from the present instance, since Murphy's mode of procedure, though strikingly effective upon his own set, is quite inapplicable to Prony's. It should also be observed that the solution of Murphy's set is not essentially different from the solution of the familiar interpolation-problem *to determine* $a_1, a_2, \dots, a_n$, *so that* $a_1+a_2x+a_3x^2+\dots+a_nx^{n-1}$ *or* y *may have the values* $y_1, y_2, \dots, y_n$ *when* x *has the values* $x_1, x_2, \dots, x_n$ *respectively*,—a problem which had been solved in one way by Newton (1687), in another way by Lagrange (1795), and in a third way to a certain extent by Cauchy (1812).†

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* The two sets of equations are

$$a_r^1x_1 + a_r^2x_2 + \dots + a_r^nx_n = u_r \Big\}_{r=1}^{r=n} \qquad \text{(I)}$$

and

$$a_1^rx_1 + a_2^rx_2 + \dots + a_n^rx_n = u_r \Big\}_{r=1}^{r=n} \qquad \text{(J)}$$

The former is substantially the set of the interpolation-problem which goes back to Newton, and which may therefore for distinction's sake be associated with his name: the latter being first found solved by Lagrange (Recherches sur les suites récurrentes . . . *Mém. de l'acad. de Berlin*, 1775, pp. 183–272; 1792, pp. 247–257: or *Œuvres complètes*, iv. pp. 149–251; v. pp. 625–641) may be called Lagrange's set, provided we remember that he also gave a solution of the other. The first to deal with both of them in more or less general form by means of determinants was Cauchy (1812)—see also his *Resumés Analytiques*, p. 19, ... 4° Turin, 1834—but in saying so a mental reservation must be made in view of Cramer's mode (1750) of continuing Newton's work.

† NEWTON, *Principia*, lib. iii. lemma v.: also *Arithmetica Universalis*, probl. lxi. LAGRANGE, *Journ. de l'éc. polyt.*, ii. cah. 8, 9, pp. 276, 277; or *Œuvres complètes*, vii. pp. 285, 286. CAUCHY, *Journ. de l'éc. polyt.*, x. cah. 17, pp. 73–74; or *Œuvres complètes*, 2ᵉ sér. i. pp. 133–134.

BINET, [J. P. M.] (1837).

[Observations sur des théorèmes de Géométrie, énoncées page 160 de ce volume et page 222 du volume précédent. *Journ.* (*de Liouville*) *de Math.*, ii. pp. 248-252; or, in abstract, *Nouv. Annales de Math.*, v. pp. 164, 165.]

The main object of this short paper of Binet's was to draw attention to the fact that a theorem regarding homofocal surfaces which Lamé had just published was originally given by Binet in 1811. He thus has occasion to say that the form under which he had considered the equation of homofocal surfaces was

$$\frac{a^2}{\mathrm{K}-\mathrm{A}} + \frac{b^2}{\mathrm{K}-\mathrm{B}} + \frac{c^2}{\mathrm{K}-\mathrm{C}} = 1,$$

where a, b, c are the co-ordinates of any point on the surface, A, B, C are positive constants such that $\mathrm{A} > \mathrm{B} > \mathrm{C}$, and K is a quantity which may be of any magnitude greater than C. And as Lamé had obtained expressions for the co-ordinates in terms of three values given to K, Binet intimates that many years before he had not only done the same but had extended the solution to the case of n equations. It is this purely algebraical problem which is of interest to us, and fortunately Binet gives it in full.

Taking the set of equations in the form

$$\begin{gathered}
\frac{a}{\mathrm{K}-\mathrm{A}} + \frac{b}{\mathrm{K}-\mathrm{B}} + \frac{c}{\mathrm{K}-\mathrm{C}} + \ldots\ldots = 1, \\
\frac{a}{\mathrm{K}_1-\mathrm{A}} + \frac{b}{\mathrm{K}_1-\mathrm{B}} + \frac{c}{\mathrm{K}_1-\mathrm{C}} + \ldots\ldots = 1, \\
\frac{a}{\mathrm{K}_2-\mathrm{A}} + \frac{b}{\mathrm{K}_2-\mathrm{B}} + \frac{c}{\mathrm{K}_2-\mathrm{C}} + \ldots\ldots = 1, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}$$

he introduces, for temporary purposes, two functions, $\mathrm{F}(x)$, $f(x)$, the former being

$$(x-\mathrm{A})(x-\mathrm{B})(x-\mathrm{C})\ldots\ldots$$

and therefore of the n^{th} degree, and the latter being any integral function of a degree less than n. He then recalls the fact that

$f(x) \div F(x)$ can be partitioned into n fractions having $x-A, x-B, x-C, \ldots$ for denominators, the result as given by Euler being

$$\frac{f(x)}{F(x)} = \frac{f(A)}{(x-A)f'(A)} + \frac{f(B)}{(x-B)f'(B)} + \frac{f(C)}{(x-C)f'(C)} + \ldots .$$

Substituting successively K, K_1, K_2, ... for x in this, a set of equations is obtained from which it is seen that the solution of the set

$$\frac{a}{K-A} + \frac{b}{K-B} + \frac{c}{K-C} + \ldots . = \frac{f(K)}{F(K)},$$

$$\frac{a}{K_1-A} + \frac{b}{K_1-B} + \frac{c}{K_1-C} + \ldots . = \frac{f(K_1)}{F(K_1)},$$

$$\frac{a}{K_2-A} + \frac{b}{K_2-B} + \frac{c}{K_2-C} + \ldots . = \frac{f(K_2)}{F(K_2)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

is

$$a = \frac{f(A)}{F'(A)} = \frac{f(A)}{(A-B)(A-C)(A-D)\ldots.},$$

$$b = \frac{f(B)}{F'(B)} = \frac{f(B)}{(B-A)(B-C)(B-D)\ldots.},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Now the set of equations here solved is more general than that with which we started, the latter being the particular case of the former where

$$\frac{f(K)}{F(K)} = \frac{f(K_1)}{F(K_1)} = \ldots . = 1.$$

To effect this specialisation it is only necessary to make the arbitrary function $f(x)$ equal to

$$(x-A)(x-B)(x-C)\ldots. \; - \; (x-K)(x-K_1)(x-K_2)\ldots.$$

or equal to

$$F(x) - \mathrm{f}(x) \text{ say};$$

(where, be it observed, the condition as to the degree of $f(x)$ is fulfilled); for then, since $\mathrm{f}(x)$ vanishes when $x = K, K_1, K_2, \ldots.$ we have

$$f(K) = F(K), \quad f(K_1) = F(K_1), \quad f(K_2) = F(K_2), \quad \ldots.$$

The corresponding change in the values of the unknowns is easily made: for example, in the case of a, we have only to substitute F(A) − f(A),—or, what is the same thing, − f(A),—for f(A) in the numerator, the result being

$$a = -\frac{(A-K)(A-K_1)(A-K_2)\ldots\ldots}{(A-B)(A-C)\ldots\ldots}$$

We have thus as Binet's theorem:—

The solution of the set of equations

$$\left.\begin{array}{l}
\dfrac{x_1}{b_1-\beta_1}+\dfrac{x_2}{b_1-\beta_2}+\dfrac{x_3}{b_1-\beta_3}+\ldots\ldots+\dfrac{x_n}{b_1-\beta_n}=1\\
\dfrac{x_1}{b_2-\beta_1}+\dfrac{x_2}{b_2-\beta_2}+\dfrac{x_3}{b_2-\beta_3}+\ldots\ldots+\dfrac{x_n}{b_2-\beta_n}=1\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\dfrac{x_1}{b_n-\beta_1}+\dfrac{x_2}{b_n-\beta_2}+\dfrac{x_3}{b_n-\beta_3}+\ldots\ldots+\dfrac{x_n}{b_n-\beta_n}=1
\end{array}\right\}$$

is

$$x_1 = -\frac{(\beta_1-b_1)(\beta_1-b_2)\ldots\ldots(\beta_1-b_n)}{(\beta_1-\beta_2)\ldots\ldots(\beta_1-\beta_n)},$$

$$x_2 = -\frac{(\beta_2-b_1)(\beta_2-b_2)\ldots\ldots(\beta_2-b_n)}{(\beta_2-\beta_1)\ldots\ldots(\beta_2-\beta_n)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

the binomial factors of the numerator in the case of x_r *being got by subtracting from* β_r *all the* b's *in succession, and the similar factors of the denominator by subtracting from* β_r *all the other* β's.

Remembering that Binet had originally been an expert in working with determinants, it is not a little curious to note that he did not compare with these expressions for $x_1, x_2, x_3, \ldots\ldots$ the expressions in terms of determinants, viz.—

$$x_1 = \begin{vmatrix} 1 & (b_1-\beta_2)^{-1} & \ldots & (b_1-\beta_n)^{-1} \\ 1 & (b_2-\beta_2)^{-1} & \ldots & (b_2-\beta_n)^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & (b_n-\beta_2)^{-1} & \ldots & (b_n-\beta_n)^{-1} \end{vmatrix} \div \begin{vmatrix} (b_1-\beta_1)^{-1} & (b_1-\beta_2)^{-1} & \ldots & (b_1-\beta_n)^{-1} \\ (b_2-\beta_1)^{-1} & (b_2-\beta_2)^{-1} & \ldots & (b_2-\beta_n)^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ (b_n-\beta_1)^{-1} & (b_n-\beta_2)^{-1} & \ldots & (b_n-\beta_n)^{-1} \end{vmatrix}$$

$$x_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Had he done so he would undoubtedly have reached a result which was not brought to light until four years later by Cauchy.

HAEDENKAMP, H. (1841).

[Ueber Transformation vielfacher Integrale. *Crelle's Journ.*, xxii. pp. 184–192.*]

The transformation referred to in the title has its origin in a special equation of the n^{th} degree in y, viz.—

$$\frac{x_1}{a_1-y}+\frac{x_2}{a_2-y}+\ldots\ldots+\frac{x_n}{a_n-y} = 1\,;$$

and, as Haedenkamp gives the values of $x_1, x_2, \ldots, x_n$ in terms of the n roots $y_1, y_2, \ldots, y_n$ of this equation he may of course be viewed as having solved the set of linear equations—

$$\left.\begin{array}{l}
\dfrac{x_1}{a_1-y_1}+\dfrac{x_2}{a_2-y_1}+\ldots\ldots+\dfrac{x_n}{a_n-y_1} = 1, \\
\dfrac{x_1}{a_1-y_2}+\dfrac{x_2}{a_2-y_2}+\ldots\ldots+\dfrac{x_n}{a_n-y_2} = 1, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right\}$$

which Binet had explicitly dealt with four years before.

BORCHARDT, C. W. (1845, Jan.).

[Neue Eigenschaft der Gleichung, mit deren Hülfe man die seculären Störungen der Planeten bestimmt. *Crelle's Journ.*, xxx. pp. 38–45; or, in extended form, *Journ. (de Liouville) de Math.*, xii. pp. 50–67; or *Gesammelte Werke*, pp. 3–13.]

For the present this paper is only noteworthy as containing the square of the difference-product in the form of a determinant of the particular type soon after to be named "persymmetric" by Sylvester. M being used to stand for

$$\left\{\sum \pm g_1^0 g_2^1 g_3^2 \cdots g_n^{n-1}\right\}^2$$

* See also *Crelle's Journ.*, xxv. pp. 178–183 (1842), and *Grunert's Archiv d. Math. u. Phys.*, xxiii. pp. 235, 236 (1854).

Borchardt says "Es wird also M die Determinante aus dem System

$$\begin{array}{cccccc} s_0 & s_1 & s_2 & \ldots\ldots & s_{n-1} \\ s_1 & s_2 & s_3 & \ldots\ldots & s_n \\ s_2 & s_3 & s_4 & \ldots\ldots & s_{n+1} \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ s_{n-1} & s_n & s_{n+1} & \ldots\ldots & s_{2n-2} \end{array}$$

wo

$$s_m = g_1^m + g_2^m + \ldots + g_n^m."$$

ROSENHAIN, G. (1845, Sept.).

[Neue Darstellung der Resultante der Elimination von z aus zwei algebraische Gleichungen $f(z)=0$ und $\phi(z)=0$, ... *Crelle's Journ.*, xxx. pp. 157–165.]

Although the subject of alternating functions is incidentally dealt with in Rosenhain's paper (p. 161), nothing of importance occurs. The identity *

$$\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_n) = \zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_m) \cdot \zeta^{\frac{1}{2}}(a_{m+1}, a_{m+2}, \ldots, a_n) . \prod_{s=1,2,\ldots,m}^{r=m+1,\ldots,n}(a_r - a_s)$$

appears in the form

$$\Pi(a_1, a_2, \ldots, a_n) = \Pi(a_1, a_2, \ldots, a_m) \cdot \Pi(a_{m+1}, a_{m+2}, \ldots, a_n) \cdot \mathrm{M}_{1,2,\ldots,m},$$

where the mode of denoting the rectangular array of differences cannot be commended.

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* A still better form for the right-hand member is

$$\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_m) \cdot \prod_{s=1,2,\ldots,m}^{r=m+1,\ldots,n}(a_r - a_s) \cdot \zeta^{\frac{1}{2}}(a_{m+1}, a_{m+2}, \ldots, a_n)$$

where the suffixes are seen to run twice from 1 to n. Another identity, just as worthy of note, is

$$\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_n) = \zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_m) \cdot \prod_{s=1,2,\ldots,m-1}^{r=m+1,\ldots,n}(a_r - a_s) \cdot \zeta^{\frac{1}{2}}(a_m, a_{m+1}, \ldots, a_n).$$

The one is exemplified by the partition

$$\begin{array}{cc:cccc} a_2 - a_1 & a_3 - a_1 & a_4 - a_1 & a_5 - a_1 & a_6 - a_1 & \\ & a_3 - a_2 & a_4 - a_2 & a_5 - a_2 & a_6 - a_2 & \\ & & a_4 - a_3 & a_5 - a_3 & a_6 - a_3 & \\ \hdashline & & & a_5 - a_4 & a_6 - a_4 & \\ & & & & a_6 - a_5, & \end{array}$$

the other when instead of this the right-to-left dotted line is made to separate the third row of differences from the second. The former is that to which we have drawn attention when dealing with Jacobi's memoir of 1841.

What the title refers to is a new variant of a form of the resultant given by Euler in 1748, namely, the variant

$$\zeta^{\frac{1}{2}}(\beta_1, \beta_2, \ldots, \beta_n, a_1, a_2, \ldots, a_n) \div \zeta^{\frac{1}{2}}(\beta_1, \beta_2, \ldots, \beta_n) \cdot \zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_n),$$

where the a's are the roots of $f(z)=0$ and the β's the roots of $\phi(z)=0$.

STURM, R. (1845); TERQUEM, O. (1846).

[Cours d'analyse de l'École Polytechnique. 4to, lithogr., Paris.*]

[Sur la résolution d'une certaine classe d'équations à plusieurs inconnues du premier degré. *Nouv. Annales de Math.*, v. pp. 67–68, 162–165.]

Employing the method of "undetermined multipliers" Sturm here supplies the want left by Prony, namely the solution of

$$a^r x_1 + a_2^r x_2 + \ldots + a_n^r x_n = b^r \Big\}_{r=0}^{r=n-1}.$$

The said method may be generally described as making the solution of a set of n equations dependent on the solution of a set of $n-1$ equations, the latter set being related to the former in having its determinant conjugate to a primary minor of the determinant of the other set. Thus the given set being

$$\left.\begin{array}{l} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = a_5 \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = b_5 \\ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = c_5 \\ d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = d_5 \end{array}\right\}$$

we conclude therefrom that the equation

$$(a_1+\lambda b_1+\mu c_1+\nu d_1)x_1 + (a_2+\lambda b_2+\mu c_2+\nu d_2)x_2 + \ldots\ldots$$
$$= a_5 + \lambda b_5 + \mu c_5 + \nu d_5$$

holds for all values of λ, μ, ν; and in order to obtain the value of x_1, we have to solve the set

$$\left.\begin{array}{l} a_2 + b_2\lambda + c_2\mu + d_2\nu = 0 \\ a_3 + b_3\lambda + c_3\mu + d_3\nu = 0 \\ a_4 + b_4\lambda + c_4\mu + d_4\nu = 0 \end{array}\right\},$$

* Not the posthumous book with this title edited by Prouhet and published in 1857.

where the determinant of the coefficients of the unknowns is the conjugate of the complementary minor of a_1 in $|a_1b_2c_3d_4|$. With this fact in view, and along with it the nature of the relation of Murphy's set to Prony's, it will be readily seen that both sets appear in Sturm's procedure.

Terquem follows Sturm, and extends his method to the set of n equations

$$\left.\begin{array}{l} x_1 + 0.x_2 \;\; + x_3 \;\; + \ldots + x_n \;\;\; = b_0 \\ a_1x_1 + 1.x_2 \;\; + a_3x_3 + \ldots + a_nx_n = b_1 \\ a_1^2x_1 + 2\,a_1x_2 + a_3^2x_3 + \ldots + a_n^2x_n = b_2 \\ a_1^3x_1 + 3\,a_1x_2 + a_3^3x_3 + \ldots + a_n^3x_n = b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

where the coefficients of x_2 are the differential-quotients of the corresponding coefficients of x_1. The possibility of this solution rests on selecting x_2 as the first unknown to be determined, and on the set being thus reducible to one of the previous type.

CAYLEY, A. (1846, Aug.).

[Note sur les fonctions de M. Sturm. *Journ. (de Liouville) de Math.*, xi. pp. 297–299; *Collected Math. Papers*, i. pp. 306–308.]

The functions referred to, which are really Sylvester's substitutes * for Sturm's functions, are introduced in the form—

$$\begin{aligned} f(x) &= (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n) \\ f_1(x) &= \Sigma\,(x-a_2)(x-a_3)(x-a_4)\ldots. \\ f_2(x) &= \Sigma\,(a_1-a_2)^2.(x-a_3)(x-a_4)\ldots. \\ f_3(x) &= \Sigma\,(a_1-a_2)^2(a_2-a_3)^2(a_3-a_1)^2.(x-a_4)\ldots. \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_m(x) &= f(x).\sum\frac{\mathrm{P}^2}{(x-a_1)(x-a_2)\ldots(x-a_m)}, \end{aligned}$$

* SYLVESTER. On rational derivation from equations of existence, ... *Philos. Mag.*, xv. (1839), pp. 428–435: *Collected Math. Papers*, i. pp. 40–46.

STURM. Démonstration d'un théorème d'algèbre de M. Sylvester. *Journ. (de Liouville) de Math.*, vii. (1842), pp. 356–368.

where P stands for the difference-product of $a_1, a_2, \ldots, a_m$, or for what Sylvester afterwards denoted by $\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_m)$: and the problem is professedly to express $f_m(x)$ " par les coefficients de $f(x)$," but in reality to express it as a series arranged according to descending powers of x.

This is accomplished by partitioning $\mathrm{P}/(x-a_1)(x-a_2)\ldots(x-a_m)$ into an aggregate of fractions having $x-a_1, x-a_2, \ldots$ for denominators,* namely,

$$(-)^{m-1}\frac{\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_m)}{(x-a_1)(x-a_2)\ldots(x-a_m)} = \frac{\zeta^{\frac{1}{2}}(a_2, a_3, \ldots, a_m)}{x-a_1} - \frac{\zeta^{\frac{1}{2}}(a_1, a_3, \ldots, a_m)}{x-a_2} + \ldots\ldots,$$

so that the coefficient of x^{-r} is seen to be

$$a_1^{r-1}\zeta^{\frac{1}{2}}(a_2, a_3, \ldots, a_m) - a_2^{r-1}\zeta^{\frac{1}{2}}(a_1, a_3, \ldots, a_m) + \quad \ldots\ldots$$

and therefore to be

$$(-)^{m-1}\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{m-2} & a_1^{r-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{m-2} & a_2^{r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_m & a_m^2 & \ldots & a_m^{m-2} & a_m^{r-1} \end{vmatrix},$$

where $a_1, a_2, \ldots, a_m$ are the first m a's chosen from $a_1, a_2, \ldots, a_n$.

Multiplying both sides by P and performing the requisite summation we find that the coefficient of x^{-r} in $f_m(x) \div f(x)$ is

$$\begin{vmatrix} s_0 & s_1 & \ldots & s_{m-2} & s_{r-1} \\ s_1 & s_2 & \ldots & s_{m-1} & s_r \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{m-1} & s_m & \ldots & s_{2m-3} & s_{r+m-2} \end{vmatrix}, \quad \text{or} \quad \mathrm{V}_{r-1} \text{ say,}$$

where s_q is the sum of the q^{th} powers of *all* the a's; in other words, that

$$f_m(x) \div f(x) = x^{-m} \cdot \mathrm{V}_{m-1} + x^{-m-1} \cdot \mathrm{V}_m + \ldots\ldots$$

---

*It may be noted in this connection that

$$\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_n) \div \phi'(a_k) = (-)^{n-k}\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$$

if $\phi(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$.

It only remains now to multiply by $f(x)$ in the form

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots\ldots$$

obtaining

$$\begin{aligned} f_m(x) = x^{n-m}\cdot \mathrm{V}_{m-1} &+ x^{n-m-1}(\mathrm{V}_m - p_1\mathrm{V}_{m-1}) \\ &+ x^{n-m-2}(\mathrm{V}_{m+1} - p_1\mathrm{V}_m + p_2\mathrm{V}_{m-1}) \\ &+ \ldots\ldots\ldots\ldots \end{aligned}$$

and then to condense the coefficients,—an easy operation, since all the V's are identical save in their last columns: for example

$$\mathrm{V}_{m+1} - p_1\mathrm{V}_m + p_2\mathrm{V}_{m-1} = \begin{vmatrix} s_0 & s_1 & \ldots & s_{m-2} & s_{m+1} - p_1 s_m + p_2 s_{m-1} \\ s_1 & s_2 & \ldots & s_{m-1} & s_{m+2} - p_1 s_{m+1} + p_2 s_m \\ \cdot & \cdot & \cdot & \cdot & \\ s_{m-1} & s_m & \ldots & s_{2m-3} & s_{2m} - p_1 s_{2m-1} + p_2 s_{2m-2} \end{vmatrix}.$$

CHELINI, D. (1846); LIOUVILLE, J. (1846).

[Determinazione geometrica in coordinate ellittiche . . . . *Raccolta sci. di Palomba*, ii. pp. 109–113, 126–131; see also v. pp. 227–263, 333–374.]

[Sur une classe d'équations du premier degré. *Journ. (de Liouville) de Math.*, xi. pp. 466–467; or *Nouv. Annales de Math.*, vi. pp. 129–131; or *Archiv d. Math. u. Phys.*, xxii. pp. 226–228.]

The set of equations referred to is that dealt with by Binet in 1837. Chelini and Liouville arrived at a new solution, much simpler than Binet's, and related to that used by Murphy in 1832 in solving other sets of linear equations.

GRUNERT, J. A. (1847).

[Vollständige independente Auflösung der n Gleichungen der ersten Grades . . . *Archiv d. Math. u. Phys.*, x. pp. 284–302.]

The equations are

$$\mathrm{A}_1 + \mathrm{A}_2 a_r + \mathrm{A}_3 a_r^2 + \ldots + \mathrm{A}_n a_r^{n-1} = \alpha_r \Big\}_{r=1}^{r=n}.$$

that is to say, are of the type to which Murphy's belong, and with which a problem in interpolation is connected; and the solution, rather tardily reached (p. 301), is

$$\begin{aligned}(-1)^m A_{n-m} = {} & \frac{\overset{m}{\mathrm{K}}(a_2, a_3, a_4, \ldots, a_n)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)} a_1 \\ & + \frac{\overset{m}{\mathrm{K}}(a_1, a_3, a_4, \ldots, a_n)}{(a_2-a_1)(a_2-a_3)(a_2-a_4)\ldots(a_2-a_n)} a_2 \\ & + \frac{\overset{m}{\mathrm{K}}(a_1, a_2, a_4, \ldots, a_n)}{(a_3-a_1)(a_3-a_2)(a_3-a_4)\ldots(a_3-a_n)} a_3 \\ & + \ldots\ldots\ldots\ldots\ldots \\ & + \frac{\overset{m}{\mathrm{K}}(a_1, a_2, a_3, \ldots, a_{n-1})}{(a_n-a_1)(a_n-a_2)(a_n-a_3)\ldots,(a_n-a_{n-1})} a_n ,\end{aligned}$$

where by $\overset{m}{\mathrm{K}}$ is denoted "die *m*te Klasse der Kombinationen ohne Wiederholungen."

ROSENHAIN, G. (1849).

[Auszug mehrerer Schreiben .... über die hyperelliptischen Transcendenten. No. IV. *Crelle's Journ.*, xl. pp. 347–360.]

In the course of an investigation regarding the relation between two Abelian integrals Rosenhain is brought up against the determinant

$$\sum \pm \frac{1}{t_1-a_1} \cdot \frac{1}{t_2-a_2} \cdot \ldots \cdot \frac{1}{t_{n-1}-a_{n-1}}$$

already dealt with by Cauchy in 1841, and afterwards known as "Cauchy's double alternant." The multiple integrals in question have to suffer transformation of the variables, and as a preliminary it is ascertained that the Jacobian [not yet so called]

$$\sum \pm \frac{\partial x_1}{\partial t_1} \cdot \frac{\partial x_2}{\partial t_2} \cdots \frac{\partial x_{n-1}}{\partial t_{n-1}} = \mathrm{C} \cdot \sum \pm \frac{1}{t_1-a_1} \cdot \frac{1}{t_2-a_2} \cdots \frac{1}{t_{n-1}-a_{n-1}}$$

and

$$\sum \pm \frac{\partial t_1}{\partial x_1} \cdot \frac{\partial t_2}{\partial x_2} \cdots \frac{\partial t_{n-1}}{\partial x_{n-1}} = \mathrm{D} \cdot \sum \pm \frac{1}{t_1-a_1} \cdot \frac{1}{t_2-a_2} \cdots \frac{1}{t_{n-1}-a_{n-1}}$$

where C and D are specified functions of the a's and t's. From this by multiplication it follows that

$$\left\{\sum \pm \frac{1}{t_1 - a_1} \cdot \frac{1}{t_2 - a_2} \cdot \cdot \cdot \cdot \frac{1}{t_{n-1} - a_{n-1}}\right\}^2 = \frac{1}{\mathrm{CD}},$$

and thence ultimately that

$$\sum \pm \frac{1}{t_1 - a_1} \cdot \frac{1}{t_2 - a_2} \cdot \cdot \cdot \cdot \frac{1}{t_{n-1} - a_{n-1}}$$
$$= \frac{(-1)^{\frac{1}{2}n(n-1)} \Pi(a_1 \cdot a_2, \ldots, a_{n-1}) \cdot \Pi(t_1, t_2, \ldots, t_{n-1})}{\phi(a_1) \cdot \phi(a_2) \ldots \phi(a_{n-1})}$$

where $$\phi(\xi) = (\xi - t_1)(\xi - t_2) \ldots (\xi - t_{n-1}).$$

There is then added a simple verificatory proof which consists in noting (1) the double alternating character of the function *

$$\phi(a_1) \cdot \phi(a_2) \ldots \phi(a_{n-1}) \cdot \sum \pm \frac{1}{t_1 - a_1} \cdot \frac{1}{t_2 - a_2} \cdot \cdot \cdot \frac{1}{t_{n-1} - a_{n-1}};$$

(2) its degree in any one of the a's or t's; (3) the sign of any one of its terms. The exact words are—

"Der Beweis der obigen Formel ergiebt sich durch die Betrachtung, dass

$$\phi(a_1) \cdot \phi(a_2) \ldots . \phi(a_{n-1}) \cdot \sum \pm \frac{1}{t_1 - a_1} \cdot \frac{1}{t_2 - a_2} \cdot \cdot \cdot \cdot \frac{1}{t_{n-1} - a_{n-1}}$$

eine ganze rationale alternirende Function sowohl in Bezug auf die $n-1$ Grössen t_p als auch in Bezug auf die $n-1$ Grössen a_m ist. Es übersteigt aber in dieser Function weder eine der Grössen t_p noch eine

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* The product $\phi(a_1) . \phi(a_2) \ldots \phi(a_{n-1})$ is arrangeable as a square array of binomial factors, being in fact, save as to sign, the product of all the denominators in the double alternant, and is thus seen to be symmetrical with respect both to the a's and to the t's. If therefore we multiply each row of the alternant by the product of the denominators of the row, or each column by the product of the denominators of the column, we multiply the alternant by

$$(-1)^{(n-1)(n-2)} \phi(a_1) \cdot \phi(a_2) \ldots \phi(a_{n-1}).$$

The two determinants thus resulting have elements which are the product of $n-2$ binomial factors, and are equal to

$$(-1)^{\frac{1}{2}(n-1)(n-2)} \Pi(a_1, a_2, \ldots, a_{n-1}) \cdot \Pi(t_1, t_2, \ldots, t_{n-1}).$$

If, on the other hand, we multiply each element $1/(a_r - t_s)$ of the alternant by $a_r^{n-1} - t_s^{n-1}$ we obtain the product of the two Π's as reached by the ordinary multiplication-theorem of determinants.

der Grössen a_m den $(n-2)$ten Grad: daher ist sie nicht bloss durch das Product

$$\Pi(a_1, a_2, \ldots, a_{n-1}) \cdot \Pi(t_1, t_2, \ldots, t_{n-1})$$

theilbar, sondern, abgesehen vom Zeichen, diesem Producte selbst gleich, da ihre einzelnen Terme keine andern Zahlencoefficienten haben, als $\pm$ 1. Da nun die Determinante positiv sein soll, so musste rechts vom Gleichheitszeichen noch der Factor $(-1)^{\frac{1}{2}n(n-1)}$ hinzugefügt werden."

CAYLEY, A. (1853).

[Note on the transformation of a trigonometrical expression. *Cambridge and Dublin Math. Journ.*, ix. pp. 61, 62; or *Collected Math. Papers*, ii. pp. 45, 46.]

In order to show that the vanishing of the alternating function

$$\begin{vmatrix} 1 & x & (a+x)\sqrt{c+x} \\ 1 & y & (a+y)\sqrt{c+y} \\ 1 & z & (a+z)\sqrt{c+z} \end{vmatrix}$$

implies the vanishing of

$$\tan^{-1}\sqrt{\frac{a-c}{c+x}} + \tan^{-1}\sqrt{\frac{a-c}{c+y}} + \tan^{-1}\sqrt{\frac{a-c}{c+z}}$$

the determinant is proved to contain the factor

$$\sqrt{\frac{a-c}{c+x}} + \sqrt{\frac{a-c}{c+y}} + \sqrt{\frac{a-c}{c+z}} - \sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}}$$

with the cofactor

$$-\frac{(c+x)^{\frac{3}{2}}(c+y)^{\frac{3}{2}}(c+z)^{\frac{3}{2}}}{(a-c)^2}, \quad \begin{vmatrix} 1 & \sqrt{\frac{a-c}{c+x}} & \frac{a-c}{c+x} \\ 1 & \sqrt{\frac{a-c}{c+y}} & \frac{a-c}{c+y} \\ 1 & \sqrt{\frac{a-c}{c+z}} & \frac{a-c}{c+z} \end{vmatrix}.$$

This is done by writing ξ, η, ζ for $\sqrt{\frac{a-c}{c+x}}$, $\sqrt{\frac{a-c}{c+y}}$, $\sqrt{\frac{a-c}{c+z}}$

respectively,* and so changing the given determinant into

$$\begin{vmatrix} 1 & (a-c)(1+\xi^{-2}) & (a-c)^{\frac{3}{2}}(\xi^{-1}+\xi^{-3}) \\ 1 & (a-c)(1+\eta^{-2}) & (a-c)^{\frac{3}{2}}(\eta^{-1}+\eta^{-3}) \\ 1 & (a-c)(1+\zeta^{-2}) & (a-c)^{\frac{3}{2}}(\zeta^{-1}+\zeta^{-3}) \end{vmatrix},$$

thence into

$$(a-c)^{\frac{5}{2}}\xi^{-3}\eta^{-3}\zeta^{-3}\begin{vmatrix} \xi^3 & \xi^3+\xi & \xi^2+1 \\ \eta^3 & \eta^3+\eta & \eta^2+1 \\ \zeta^3 & \zeta^3+\zeta & \zeta^2+1 \end{vmatrix},$$

and finally into

$$-(a-c)^{\frac{5}{2}}\xi^{-3}\eta^{-3}\zeta^{-3}\begin{vmatrix} 1 & \xi & \xi^2 \\ 1 & \eta & \eta^2 \\ 1 & \zeta & \zeta^2 \end{vmatrix}(\xi+\eta+\zeta-\xi\eta\zeta).$$

BRIOSCHI, F. (1854).

[La Teorica dei Determinanti, e le sue principali applicazioni. viii+116 pp. Pavia.]

Brioschi devotes the 9th section of his text-book (pp. 73–84) to "determinanti delle radici delle equazioni algebriche," viewing the difference-product and its allies as arising when the roots of the equation

$$x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \dots + A_1x + A_0 = 0$$

are substituted for x, and the values of $A_{n-1}, A_{n-2}, \dots$, are to

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* It is simpler still to express the given determinant in terms of alternants having $\sqrt{c+x}$, $\sqrt{c+y}$, $\sqrt{c+z}$ for variables. Thus the given determinant

$$= \begin{vmatrix} 1 & x & (c+x+a-c)\sqrt{c+x} \\ 1 & y & (c+y+a-c)\sqrt{c+y} \\ 1 & z & (c+z+a-c)\sqrt{c+z} \end{vmatrix},$$

$$= \begin{vmatrix} 1 & c+x & (c+x)\sqrt{c+x} \\ 1 & c+y & (c+y)\sqrt{c+y} \\ 1 & c+z & (c+z)\sqrt{c+z} \end{vmatrix} + (a-c)\begin{vmatrix} 1 & c+x & \sqrt{c+x} \\ 1 & c+y & \sqrt{c+y} \\ 1 & c+z & \sqrt{c+z} \end{vmatrix},$$

$$= \begin{vmatrix} 1 & \mu^2 & \mu^3 \\ 1 & \nu^2 & \nu^3 \\ 1 & \zeta^2 & \zeta^3 \end{vmatrix} - (a-c)\begin{vmatrix} 1 & \mu & \mu^2 \\ 1 & \nu & \nu^2 \\ 1 & \zeta & \zeta^2 \end{vmatrix} \text{ say,}$$

$$= \begin{vmatrix} 1 & \mu & \mu^2 \\ 1 & \nu & \nu^2 \\ 1 & \zeta & \zeta^2 \end{vmatrix}\cdot\{\mu\nu+\nu\zeta+\zeta\mu-(a-c)\}.$$

be determined from the n equations thus resulting. His proof, obtained in this way, that the common denominator of the A's is resolvable into binomial factors is not of consequence. It is more important to note that, as an alternative, he proceeds "facendo uso di sole proprieta dei determinanti," obtaining in the first place

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} a_1 - a_2 & a_2 - a_3 & \dots & a_{n-1} - a_n \\ a_1^2 - a_2^2 & a_2^2 - a_3^2 & \dots & a_{n-1}^2 - a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{n-1} - a_2^{n-1} & a_2^{n-1} - a_3^{n-1} & \dots & a_{n-1}^{n-1} - a_n^{n-1} \end{vmatrix}$$

from which he removes the factors $a_1 - a_2$, $a_2 - a_3, \dots$; then repeating the first set of operations he removes the factors $a_1 - a_3$, $a_2 - a_4, \dots$, and so on.

After this an application is made to the solution of a set of linear equations which differs from Prony's set by having z^0, z^1, $z^2, \dots$ in place of z_0, z_1, $z_2, \dots$, and where therefore, as Cauchy in 1812 had pointed out, the numerators of the unknowns, as well as the common denominator, are resolvable into binomial factors. Borchardt's and Cayley's persymmetric determinants in s_0, s_1, $s_2, \dots$, got by multiplication, are also given. The remaining pages (77–84) contain illustrations.

JOACHIMSTHAL, F. (1854, May).

[Bemerkungen über den Sturm'schen Satz. *Crelle's Journ.*, xlviii. pp. 386–416.]

In the course of his investigations Joachimsthal evaluates (§ 5) the determinant

$$\begin{vmatrix} s_0 & s_1 & s_2 & 1 \\ s_1 & s_2 & s_3 & x \\ s_2 & s_3 & s_4 & x^2 \\ s_3 & s_4 & s_5 & x^3 \end{vmatrix}$$

where $s_q = x_1^q + x_2^q + x_3^q$. Using the fact that by reason of the trinomial elements the determinant is partitionable into twenty-seven determinants with monomial elements, he shows next that all of the twenty-seven except six vanish; that the six contain the common factor

$$(x-x_1)(x-x_2)(x-x_3)\,.\,(x_3-x_1)(x_3-x_2)(x_2-x_1);$$

that the aggregate of the cofactors is

$$x_3^2x_2-x_3^2x_1+x_2^2x_1-x_2^2x_3+x_1^2x_3-x_1^2x_2$$

or

$$(x_3-x_1)(x_3-x_2)(x_2-x_1);$$

and that therefore finally the given determinant is equal to

$$\left\{(x_3-x_1)(x_3-x_2)(x_2-x_1)\right\}^2.\ (x-x_1)(x-x_2)(x-x_3).$$

This is followed by the assertion that if s_q were made to stand for $s_1^q+s_2^q+\ldots+s_n^q$ the determinant could be partitioned into n^3 determinants, of which $n(n-1)(n-2)$ would be non-evanescent; and that these could be grouped into sets of six and condensed, the ultimate result being

$$\begin{vmatrix} s_0 & s_1 & s_2 & 1 \\ s_1 & s_2 & s_3 & x \\ s_2 & s_3 & s_4 & x^2 \\ s_3 & s_4 & s_5 & x^3 \end{vmatrix} = \sum\left\{(x_3-x_2)(x_3-x_1)(x_2-x_1)\right\}^2(x-x_1)(x-x_2)(x-x_3).$$

A large generalisation is then made, the exact words being "Genau eben so beweist man folgenden allgemeinen Satz: Bezeichnet man die Potenzsumme $x_1^i+x_2^i+\ldots+s_n^i$ durch s_i; ferner das Quadrat des Productes, welches aus den $\frac{1}{2}i(i-1)$ Differenzen der i Grössen $x_1, x_2, \ldots, x_i$ gebildet ist, durch $\delta(x_1, x_2, \ldots, x_i)$, so ist

$$\begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_{a-1} & 1 \\ s_1 & s_2 & s_3 & \ldots & s_a & x \\ s_2 & s_3 & s_4 & \ldots & s_{a+1} & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_a & s_{a+1} & s_{a+2} & \ldots & s_{2a-1} & x^a \end{vmatrix} = \sum\delta(x_1, x_2, \ldots, x_a)\cdot(x-x_1)\ldots(x-x_a)$$

wo die rechte Seite eine Summe von $\dfrac{n(n-1)\ldots n-a+1)}{1.2\ldots a}$ ähnlich gebildeten Glieder enthält."

The known result (Cayley's) obtained from this by equating coefficients of x^a is pointed out: also the extension

$$\begin{vmatrix} S_0 & S_1 & S_2 & \ldots\ldots & S_{a-1} & 1 \\ S_1 & S_2 & S_3 & \ldots\ldots & S_a & x \\ S_2 & S_3 & S_4 & \ldots\ldots & S_{a+1} & x_2 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ S_a & S_{a+1} & S_{a+2} & \ldots\ldots & S_{2a-1} & x^a \end{vmatrix} = \sum \epsilon_1\epsilon_2\ldots\epsilon_a \cdot \delta(x_1, x_2, \ldots, x_a) \\ (x-x_1)(x-x_2)\ldots(x-x_a)$$

where $S_i = \epsilon_1 x_1^i + \epsilon_2 x_2^i + \ldots + \epsilon_n x_n^i$. It may be noted that the reason for discussing such determinants is that the series of them obtained by giving a the values $n, n-1, n-2, \ldots, 2, 1, 0$ is put forward (p. 400) as a substitute for Sturm's series of functions.*

Towards the end of the paper (§ 17, p. 414) the determinant

$$\begin{vmatrix} (a_1+b_1)^{-1} & (a_1+b_2)^{-1} & \ldots\ldots & (a_1+b_n)^{-1} \\ (a_2+b_1)^{-1} & (a_2+b_2)^{-1} & \ldots\ldots & (a_2+b_n)^{-1} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ (a_n+b_1)^{-1} & (a_n+b_2)^{-1} & \ldots\ldots & (a_n+b_n)^{-1} \end{vmatrix}, \quad \text{or } \Delta_n \text{ say,}$$

is evaluated. The process consists at the outset in subtracting the first column from each column after the first, removing the factor

$$\frac{(b_1-b_2)(b_1-b_3)\ldots(b_1-b_n)}{(a_1+b_1)(a_2+b_1)\ldots(a_n+b_1)},$$

and writing the cofactor in the form

$$\begin{vmatrix} 1 & (a_1+b_2)^{-1} & \ldots\ldots & (a_1+b_n)^{-1} \\ 1 & (a_2+b_2)^{-1} & \ldots\ldots & (a_2+b_n)^{-1} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ 1 & (a_n+b_2)^{-1} & \ldots\ldots & (a_n+b_n)^{-1} \end{vmatrix}.$$

The latter determinant is then transformed by subtracting the

* In this connection papers by Cayley (1846) and Borchardt (1845) are referred to by Joachimsthal, but no mention is made of Sylvester's (1839).

first row from each row after the first, when it is found that the factor

$$\frac{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}{(a_1+b_2)(a_1+b_3)\ldots(a_1+b_n)}$$

can be removed, and that the cofactor is a determinant similar to the original but of the $(n-1)^{\text{th}}$ order, namely, the determinant which is the cofactor of the element in the place (1, 1) of the original. The final result thus obtained agrees with Cauchy's save in having no sign-factor, the latter being only necessary when the b's are all made negative.

BRIOSCHI, F. (1854, Oct.).

[Intorno ad alcune formole per la risoluzione delle equazioni algebriche. *Annali di Sci. mat. e fis.*, v. pp. 416–421; also reprinted as half (§ 2) of the last note in the French translation of Brioschi's text-book; or *Opere mat.*, i pp. 157–161.]

All that occurs in this paper in connection with our subject is the statement

$$\left(\frac{\partial\Delta}{\partial x^{n-1}}\right)^2 = \begin{vmatrix} s_0 & s_1 & \ldots\ldots & s_{n-2} & 1 \\ s_1 & s_2 & \ldots\ldots & s_{n-1} & x \\ s_2 & s_3 & \ldots\ldots & s_n & x^2 \\ \cdot & \cdot & \cdot\cdot\cdot\cdot & \cdot & \cdot \\ s_{n-2} & s_{n-1} & \ldots\ldots & s_{2n-4} & x^{n-2} \\ 1 & x & \ldots\ldots & x^{n-2} & 1 \end{vmatrix}$$

where Δ is the determinant-form of the difference-product of $x_1, x_2, \ldots, x_n$. No explanation of the statement is given, nor the mode of arriving at it. All is made clear, however, if we note first that by x on the right hand is meant any x chosen at will from the set $x_1, x_2, \ldots, x_n$: second that the differential-quotient on the left is intended to stand for the cofactor of the $(n-1)^{\text{th}}$ power of that particular x in Δ, and therefore merely denotes the difference-product of a certain $n-1$ of the x's. What the statement thus gives us is an alternative form for the square of the difference-product of $n-1$ quantities.

If we use column-by-column multiplication, and put s_r for $a^r+\beta^r+\gamma^r+\delta^r$, we clearly have

$$\begin{vmatrix} 4 & s_1 & s_2 & 1 \\ s_1 & s_2 & s_3 & \beta \\ s_2 & s_3 & s_4 & \beta^2 \\ 1 & \beta & \beta^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & . \\ 1 & \beta & \beta^2 & 1 \\ 1 & \gamma & \gamma^2 & . \\ 1 & \delta & \delta^2 & . \end{vmatrix}^2,$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & \gamma & \gamma^2 \\ 1 & \delta & \delta^2 \end{vmatrix}^2,$$

and so the result is established. It will be observed that the chosen letter β which occurs most conspicuously in the new form thus obtained for $\zeta(a,\gamma,\delta)$ is one which the expression is quite independent of. Further, by performing on this new form the operations

$$\mathrm{col}_1-\mathrm{col}_4, \quad \mathrm{col}_2-\beta\,\mathrm{col}_4, \quad \mathrm{col}_3-\beta^2\,\mathrm{col}_4,$$

we return to the more natural form

$$\zeta(a,\gamma,\delta) = \begin{vmatrix} 3 & a+\gamma+\delta & a^2+\gamma^2+\delta^2 \\ a+\gamma+\delta & a^2+\gamma^2+\delta^2 & a^3+\gamma^3+\delta^3 \\ a^2+\gamma^2+\delta^2 & a^3+\gamma^3+\delta^3 & a^4+\gamma^4+\delta^4 \end{vmatrix}.$$

BORCHARDT, C. W. (1855, March).

[Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden Funktion. *Monatsb. . . . . Akad. d. Wiss.* (Berlin), 1855, pp. 165–171; *Crelle's Journ.*, liii. pp. 193–198; *Gesammelte Werke*, pp. 97–105.]

The generating function in question is

$$\sum \frac{1}{t-a}\cdot\frac{1}{t_1-a_1}\cdots\frac{1}{t_n-a_n},$$

or T say, the sign of summation being meant to indicate that of the two series of elements the one is to remain unaltered and the other is to be permuted in every possible way. The development of this function according to descending powers of $t, t_1, t_2, \ldots, t_n$ leads to those simplest types of integral symmetric functions of

$a, a_1, a_2, \ldots, a_n$ which originate by permutation from a single product of integral powers of the said variables. The determination of such functions is thus reduced to the problem of transforming T so as to have no longer occurring therein the single elements $a, a_1, a_2, \ldots, a_n$, but instead those combinatory sums of them which are the coefficients of the powers of z in the development of $(z-a)(z-a_1)(z-a_2)\ldots(z-a_n)$ or $f(z)$ say.

Without further preparatory statement the announcement is made that the solution is readily reached when the relation of T to the determinants

$$\sum \pm \frac{1}{t-a}\cdot\frac{1}{t_1-a_1}\cdots\frac{1}{t_n-a_n} \quad \text{or} \quad \Delta,$$

$$\sum \pm \frac{1}{(t-a)^2}\cdot\frac{1}{(t_1-a_1)^2}\cdots\frac{1}{(t_n-a_n)^2} \quad \text{or} \quad \mathrm{D},$$

is known, namely, the relation

$$\mathrm{D} = \mathrm{T}\cdot\Delta.$$

In proof of this relation it is pointed out that

$$\left\{f(t)\cdot f(t_1)\cdot f(t_2)\ldots f(t_n)\right\}^2\cdot \mathrm{D}$$

being an integral alternating function both with respect to the elements $t, t_1, t_2, \ldots, t_n$ and with respect to the elements $a, a_1, a_2, \ldots, a_n$ is exactly divisible by the two difference-products

$$\Pi(t, t_1, t_2, \ldots, t_n), \quad \Pi(a, a_1, a_2, \ldots, a_n),$$

and that although we cannot with equal promptness tell the remaining factor, we are able to determine it from knowing a sufficient number of its special values, namely, those values got by putting each t equal to one of the a's. Since the number of ways in which the $n+1$ a's can be taken when repetitions are allowed is $(n+1)^{n+1}$, this gives us $(n+1)^{n+1}$ values, of which, however, only two are different, namely, the value $(-1)^{\frac{1}{2}n(n+1)}\cdot f'(a)\cdot f'(a_1)\cdot f'(a_2)\ldots f'(a_n)$ obtained in the $n\,!$ cases where all the a's used are different, and the value 0 obtained in every other case. The determination, we are told, can be made by using an extension of Lagrange's interpolation-formula, the outcome of the work being

$$D = T\cdot(-1)^{\frac{1}{2}n(n+1)}\frac{\Pi(t,\, t_1,\, t_2,\, \ldots,\, t_n)\cdot\Pi(a,\, a_1,\, a_2,\, \ldots,\, a_n)}{f(t)\cdot f(t_1)\cdot f(t_2)\,\ldots\ldots\, f(t_n)}$$

which, of course, gives us

$$D = T\cdot\Delta.$$

This relation having been established, Borchardt then proceeds in a line or two to use it for the main purpose of his paper. As the determinant D, he says, arises out of the determinant Δ by performance of successive differentiation with respect to all the variables t, t_1, t_2, ... t_n, there is obtained at once an alternative expression for T, namely,

$$T = (-1)^{n+1}\frac{f(t)\cdot f(t_1)\ldots f(t_n)}{\Pi(t,\, t_1,\ldots,\, t_n)}\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_n}\left(\frac{\Pi(t,\, t_1,\ldots,\, t_n)}{f(t)\cdot f(t)\ldots f(t_n)}\right)$$

or say rather

$$T = (-1)^{n+1}\frac{\dfrac{\partial}{\partial t}\cdot\dfrac{\partial}{\partial t_1}\cdots\dfrac{\partial}{\partial t_n}\dfrac{\Pi(t,\, t_1,\, \ldots,\, t_n)}{f(t)\cdot f(t_1)\,\ldots\, f(t_n)}}{\dfrac{\Pi(t,\, t_1,\, \ldots,\, t_n)}{f(t)\cdot f(t_1)\,\ldots\, f(t_n)}};$$

and so the transformation aimed at is accomplished.

PROUHET, E. (1856, March).

[Note sur quelques identités. *Nouv. Annales de Math.*, xv. pp. 86–91.]

In order to generalise certain algebraical identities published by O. Werner in Grunert's *Archiv*, xxii. p. 353, Prouhet first establishes the theorem in alternants foreshadowed by Prony and Cauchy, and readily derivable from Schweins' first multiplication-theorem. His mode of treatment may be concisely stated as follows:—

To say that a, b, c, d, e, f are the roots of

$$x^6 - p_1x^5 + p_2x^4 - p_3x^3 + p_4x^2 - p_5x + p_6 = 0$$

implies that

$$p_1 = \Sigma a,\quad p_2 = \Sigma ab,\quad p_3 = \Sigma abc,\quad \ldots;$$

and as it also means that

$$\left.\begin{array}{l} a^6 - p_1 a^5 + p_2 a^4 - \ldots + p_6 = 0 \\ b^6 - p_1 b^5 + p_2 b^4 - \ldots + p_6 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f^6 - p_1 f^5 + p_2 f^4 - \ldots + p_6 = 0 \end{array}\right\}$$

from which we have

$$p_1 = \begin{vmatrix} a_6 & a_4 & a_3 & a_2 & a_1 & 1 \\ b_6 & b_4 & b_3 & b_2 & b_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_6 & f_4 & f_3 & f_2 & f_1 & 1 \end{vmatrix} \div \begin{vmatrix} a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_5 & f_4 & f_3 & f_2 & f_1 & 1 \end{vmatrix},$$

$$p_2 = \begin{vmatrix} a_5 & a_6 & a_3 & a_2 & a_1 & 1 \\ b_5 & b_6 & b_3 & b_2 & b_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_5 & f_6 & f_3 & f_2 & f_1 & 1 \end{vmatrix} \div \begin{vmatrix} a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_5 & f_4 & f_3 & f_2 & f_1 & 1 \end{vmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

it follows that in later notation

$$\begin{array}{l} | a^0 b^1 c^2 d^3 e^4 f^6 | \div | a^0 b^1 c^2 d^3 e^4 f^5 | = \Sigma a, \\ | a^0 b^1 c^2 d^3 e^5 f^6 | \div | a^0 b^1 c^2 d^3 e^4 f^5 | = \Sigma ab, \\ | a^0 b^1 c^2 d^4 e^5 f^6 | \div | a^0 b^1 c^2 d^3 e^4 f^5 | = \Sigma abc, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

with similar results when the alternants are of any other order.

JOACHIMSTHAL, F. (1856, Sept.).

[De aequationibus quarti et sexti gradus quae in theoria linearum et superficierum secundi gradus occurrunt. *Crelle's Journ.*, liii. pp. 149–172.]

The problem of finding the normals drawn from an external point (ξ, η, ζ) to the surface

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$$

being dependent on the solution of the sixth-degree equation

$$\frac{a\xi^2}{(a+u)^2} + \frac{b\eta^2}{(b+u)^2} + \frac{c\zeta^2}{(c+u)^2} = 1$$

the relation between four of the six roots is evidently

$$\begin{vmatrix} \frac{1}{(a+u_1)^2} & \frac{1}{(b+u_1)^2} & \frac{1}{(c+u_1)^2} & 1 \\ \frac{1}{(a+u_2)^2} & \frac{1}{(b+u_2)^2} & \frac{1}{(c+u_2)^2} & 1 \\ \frac{1}{(a+u_3)^2} & \frac{1}{(b+u_3)^2} & \frac{1}{(c+u_3)^2} & 1 \\ \frac{1}{(a+u_4)^2} & \frac{1}{(b+u_4)^2} & \frac{1}{(c+u_4)^2} & 1 \end{vmatrix} = 0.$$

Joachimsthal knowing this, and having obtained by an entirely different process the result

$$\begin{aligned} &\sum \frac{1}{(a+u_1)(b+u_2)(c+u_3)} + \sum \frac{1}{(a+u_1)(b+u_2)(c+u_4)} \\ &+ \sum \frac{1}{(a+u_1)(b+u_3)(c+u_4)} + \sum \frac{1}{(a+u_2)(b+u_3)(c+u_4)} = 0, \end{aligned}$$

where the sign of summation refers to permutation of the u's, is naturally led to inquire into the connection between the two results, and to extend the inquiry to the higher cases of the same kind, including, therefore, the evaluation of the determinant

$$\begin{vmatrix} \frac{1}{(a_1+u_1)^2} & \frac{1}{(a_2+u_1)^2} & \cdots & \frac{1}{(a_n+u_1)^2} & 1 \\ \frac{1}{(a_1+u_2)^2} & \frac{1}{(a_2+u_2)^2} & \cdots & \frac{1}{(a_n+u_2)^2} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{(a_1+u_{n+1})^2} & \frac{1}{(a_2+u_{n+1})^2} & \cdots & \frac{1}{(a_n+u_{n+1})^2} & 1 \end{vmatrix} \quad \text{or} \quad J.$$

The investigation of the determinant occupies the fifth and sixth sections (pp. 164–169) of his paper, the third and fourth being devoted to obtaining the other form of the resultant

$$\begin{aligned} &\sum \frac{1}{(a_1+u_1)(a_2+u_2)\ldots(a_n+u_n)} + \sum \frac{1}{(a_1+u_1)\ldots(a_{n-1}+u_{n-1})(a_n+u_{n+1})} \\ &+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot + \sum \frac{1}{(a_1+u_2)(a_2+u_3)\ldots(a_n+u_{n+1})}. \end{aligned}$$

or, say,

$$[1, 2, \ldots, n+1].$$

On multiplying each row of J by the product of all the denominators occurring in the row there is obtained a determinant V whose r^{th} row consists of elements which are expressible as polynomials arranged according to descending powers of u_r, the index of the highest power of u_r being $2n-2$ in all the places except the last where it is $2n$. V, which is equal to

$$\mathrm{J}\cdot \mathrm{A}_1^2\mathrm{A}_2^2 \ldots \mathrm{A}_n^2$$

if we put

$$\mathrm{A}_s = (a_s+u_1)(a_s+u_2)\ldots(a_s+u_{n+1}),$$

can thus be partitioned into $(2n-1)^n(2n+1)$ determinants, each expressible in the form

$$\mathrm{a}\,.\begin{vmatrix} u_1^{a_1} & u_1^{a_2} & \ldots & u_1^{a_n} & u_1^{a_{n+1}} \\ u_2^{a_1} & u_2^{a_2} & \ldots & u_2^{a_n} & u_2^{a_{n+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n+1}^{a_1} & u_{n+1}^{a_2} & \ldots & u_{n+1}^{a_n} & u_{n+1}^{a_{n+1}} \end{vmatrix}$$

where a is an integral function of the a's. Further, V in this way is seen to be not of higher order with respect to the u's than the determinant

$$\begin{vmatrix} u_1^{n-1} & u_1^{n} & u_1^{n+1} & \ldots & u_1^{2n-3} & u_1^{2n-2} & u_1^{2n} \\ u_2^{n-1} & u_2^{n} & u_2^{n+1} & \ldots & u_2^{2n-3} & u_2^{2n-2} & u_2^{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n+1}^{n-1} & u_{n+1}^{n} & u_{n+1}^{n+1} & \ldots & u_{n+1}^{2n-3} & u_{n+1}^{2n-2} & u_{n+1}^{2n} \end{vmatrix},$$

that is to say, its order-number cannot exceed $\frac{1}{2}n(3n+1)$; and as it is exactly divisible by the difference-product of the u's, which is of the order $\frac{1}{2}n(n+1)$, it follows that

$$\mathrm{V} = \Delta(u_1, u_2, \ldots, u_{n+1})\cdot \mathrm{V}_1$$

where V_1 is a function whose order-number is not greater than n^2. Noting now that the other form of the resultant, namely $[1, 2, \ldots, n+1]$, can by addition be transformed into

$$\frac{\mathrm{U}}{\mathrm{A}_1\mathrm{A}_2\ldots\mathrm{A}_n}$$

where U cannot contain any of the differences of the u's, and in its order-number cannot exceed $n(n+1)-n$, i.e. n^2, Joachimsthal concludes that V_1 and U can only differ by a factor dependent on the a's. He thus has the two results

$$J \cdot A_1^2 A_2^2 \ldots A_n^2 = \Delta(u_1, u_2, \ldots, u_{n+1}) \cdot V_1$$

and
$$V_1 = \zeta \cdot U = \zeta \cdot A_1 A_2 \ldots A_n [1, 2, \ldots, n+1]$$

where ζ is a rational function of the a's: and by combining the two there is deduced

$$J = \zeta \cdot [1, 2, \ldots, n+1] \cdot \frac{\Delta(u_1, u_2, \ldots, u_{n+1})}{A_1 A_2 \ldots A_n}.$$

At this stage, we are told, the investigation rested for five years until the publication, in 1855, of Borchardt's paper in the Berlin Monatsbericht. Taking a hint from this, Joachimsthal, in order to determine ζ, multiplied both sides of this result by the product of all the denominators occurring in the diagonal of J, and then put $u_1 = -a_1,\ u_2 = -a_2, \ldots, u_n = -a_n$. The left-hand side was thus changed into

$$\begin{vmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 & 0 \\ \dfrac{1}{(a_1+u_{n+1})^2} & \dfrac{1}{(a_2+u_{n+1})^2} & \ldots & \dfrac{1}{(a_n+u_{n+1})^2} & 1 \end{vmatrix}$$

or 1; the second factor of the right-hand side, being equal to

$$(u_1-u_{n+1})(u_2-u_{n+1}) \ldots (u_n-u_{n+1}) \cdot \Delta(u_1, u_2, \ldots, u_n),$$

was changed into

$$(-1)^n(a_1+u_{n+1})(a_2+u_{n+1}) \ldots (a_n+u_{n+1}) \cdot (-1)^{\frac{1}{2}n(n-1)} \Delta(a_1, a_2, \ldots, a_n);$$

and the third factor $[1, 2, \ldots, n+1]/A_1 A_2 \ldots A_n$ into a fraction with the numerator 1 and with the denominator

$$\begin{array}{l} (a_1-a_2)(a_1-a_3) \ldots (a_1-a_n) \quad (a_1+u_{n+1}) \\ \cdot\, (a_2-a_1)(a_2-a_3) \ldots (a_2-a_n) \quad (a_2+u_{n+1}) \\ \cdot\, (a_3-a_1)(a_3-a_2) \ldots (a_3-a_n) \quad (a_3+u_{n+1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot\, (a_n-a_1)(a_n-a_2) \ldots (a_n-a_{n-1})(a_n+u_{n+1}) \end{array}$$

or

$$(-1)^{\frac{1}{2}n(n-1)}\Delta(a_1, a_2, \ldots, a_n)^2 \cdot (a_1+u_{n+1})(a_2+u_{n+1}) \ldots (a_n+u_{n+1}).$$

The result of the whole change was therefore

$$1 = \zeta \cdot \frac{(-1)^n}{\Delta(a_1, a_2, \ldots, a_n)},$$

whence it followed that

$$\zeta = (-1)^n \Delta(a_1, a_2, \ldots, a_n);$$

and so the longed-for result was reached

$$J = (-1)^n \frac{\Delta(a_1, a_2, \ldots, a_n) \cdot \Delta(u_1, u_2, \ldots, u_{n+1})}{A_1 A_2 \ldots A_n}[1, 2, \ldots, n+1].$$

Thereupon additional results came with ease. First we are told that in a similar manner the determinant got from J by changing the second power in the denominator of every element into the first power* is found equal to

$$(-1)^n \frac{\Delta(a_1, a_2, \ldots, a_n) \cdot \Delta(u_1, u_2, \ldots, u_{n+1})}{A_1 A_2 \ldots A_n}.$$

Then "E combinatione aequationum prodit

$$\frac{\text{det.}\left\{\frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \ldots, \frac{1}{(a_n+u)^2}, 1\right\}}{\text{det.}\left\{\frac{1}{a_1+u}, \frac{1}{a_2+u}, \ldots, \frac{1}{a_n+u}, 1\right\}}_{u=u_1, =u_2, = \ldots, =u_{n+1}} = [1, 2, \ldots, n+1].$$

Faciendo u_{n+1} = quantitati infinite magnae, aequatio in relationem a cl. Borchardt inventam transit, scilicet in

$$\frac{\text{det.}\left\{\frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \ldots, \frac{1}{(a_n+u)^2}\right\}}{\text{det.}\left\{\frac{1}{a_1+u}, \frac{1}{a_2+u}, \ldots, \frac{1}{a_n+u}\right\}}_{u=u_1, =u_2, = \ldots, =u_n} = \sum \frac{1}{(a_1+u_1)(a_2+u_2) \ldots (a_n+u_n)}.$$

---

* Previous suggestions of such a determinant appear in Binet's paper of 1837 and Joachimsthal's of 1854.

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinanti. *Memorie .... Istituto Veneto ...*, vii. pp. 67–144.]

Bellavitis reaches the subject of the difference-product first in §7 and then in §47 of his exposition, and his proof of the results dealt with in the preceding year by Prouhet is his own and interesting. Denoting the equation whose roots are $a_1, a_2, \ldots, a_n$ by

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots\ldots = 0$$

and the difference-product of the roots by Π, he multiplies both sides of the identity

$$(x-a_1)(x-a_2)\ldots(x-a_n) = x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots.$$

by Π; and as the result on the left-hand side is evidently* the difference-product of $a_1, a_2, \ldots, a_n, x$, he obtains

$$|a_1^0 a_2^1 \ldots a_n^{n-1} x^n| = (x^n - p_1 x^{n-1} + \ldots.)\Pi.$$

It only remains then to equate like powers of x and there results

$$|a_1^0 a_2^1 a_3^2 \ldots. a_{n-1}^{n-2} a_n^n| = p_1\Pi,$$
$$|a_1^0 a_2^1 \ldots a_{n-2}^{n-3} a_{n-1}^{n-1} a_n^n| = p_2\Pi,$$
$$\ldots\ldots\ldots\ldots$$
$$|a_1^1 a_2^2 a_3^3 \ldots. a_{n-1}^{n-1} a_n^n| = p_n\Pi.$$

He points out also that as an alternative to this we may begin with $|a_1^0 a_2^1 a_3^2 \ldots a_n^{n-1} x^n|$, express it as a determinant of the next lower order, remove the factors $(x-a_1), (x-a_2), \ldots, (x-a_n)$, change the product of these into $x^n - px^{n-1} + \ldots.$, and then equate coefficients of like powers of x as before.

Multiplying again by Π he has of course

$$|a_1^0 a_2^1 a_3^2 \ldots a_n^{n-1} x^n| \cdot \Pi = (x^n - p_1 x^{n-1} + \ldots)\Pi^2,$$

* See footnote to page 163 above.

and by changing II on the left into

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ a_1 & a_2 & a_3 & \dots & a_n & 0 \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

and twice using the multiplication-theorem there is obtained

$$\begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} & 1 \\ s_1 & s_2 & \dots & s_n & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \dots & s_{2n-2} & x^{n-1} \\ s_n & s_{n+1} & \dots & s_{2n-1} & x^n \end{vmatrix} = (x^n - p_1 x^{n-1} + \dots) \begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix},$$

a result already reached by Joachimsthal, and which by the equatement of like powers of x gives "i coefficienti p espressi da rapporti di determinanti di n^esimo grado."

BETTI, E. (1857, June).

[Sur les fonctions symétriques des racines des équations. *Crelle's Journ.*, liv. pp. 98–100; or *Opere mat.*]

Betti recalls Borchardt's result of the year 1855, namely, that the symmetric function

$$\sum x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$$

where $x_1, x_2, \dots, x_n$ are the roots of the equation

$$\begin{aligned} 0 &= x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots, \\ &= f(x) \text{ say,} \end{aligned}$$

is the coefficient of $t_1^{-(a_1+1)} t_2^{-(a_2+1)} \dots t_n^{-(a_n+1)}$ in the development of

$$(-1)^n \frac{f(t_1)\cdot f(t_2) \dots f(t_n)}{\Pi(t_1, t_2, \dots t_n)} \cdot \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \dots \frac{\partial}{\partial t_n} \left(\frac{\Pi(t_1, t_2, \dots t_n)}{f(t_1)\cdot f(t_2) \dots f(t_n)} \right)$$

according to descending powers of the t's. He then gives an observation of his own, namely, that the said symmetric function is likewise the coefficient of $t_1^{-(a_1+1)} t_2^{-(a_2+1)} \ldots t_n^{-(a_n+1)}$ in the similar development of

$$\frac{f'(t_1)\cdot f'(t_2) \ldots f'(t_n)\cdot \Pi^2(t_1, t_2, \ldots, t_n)}{f(t_1)\cdot f(t_2) \ldots f(t_n)\cdot \Pi^2(x_1, x_2, \ldots, x_n)};$$

and by comparison of the two results draws the conclusion that if Borchardt's generating function be denoted by

$$\theta(t_1, t_2, \ldots, t_n),$$

and his own after removal of $\Pi^2(x_1, x_2, \ldots, x_n)$ from the denominator be denoted by

$$\phi(t_1, t_2, \ldots, t_n)$$

the squared difference-product of the x's is equal to

$$\frac{\left\{\phi(t_1, t_2, \ldots, t_n)\right\}_{t_1^{-(a_1+1)} t_2^{-(a_2+1)} \ldots t_n^{-(a_n+1)}}}{\left\{\theta(t_1, t_2, \ldots, t_n)\right\}_{t_1^{(a_1+1)} t_2^{-(a_2+1)} \ldots t_n^{-(a\ +1)}}},$$

where the notation used is sufficiently explained by saying that in accordance with it the coefficient of x^r in the expansion of $F(x)$ is denoted by

$$\left\{F(x)\right\}_{x^r}.$$

BALTZER, R. (1857).

[THEORIE UND ANWENDUNG DER DETERMINANTEN, ... vi + 129 pp., Leipzig.]

The section (§ 12) dealing with the "Product aller Differenzen von gegebenen Grössen" belongs to the second part of Baltzer's text-book, that is to say, the part concerning "applications." It occupies eleven pages, those devoted strictly to alternants being the first three (pp. 50–53).

At the outset he establishes the determinant form for the difference-product $P(a_1, a_2, \ldots, a_n)$: then he gives two determinant-forms for $P(\alpha_1, \alpha_2, \ldots, \alpha_n) \cdot P(\beta_1, \beta_2, \ldots, \beta_n)$: passes thence to the persymmetric determinants in $s_0, s_1, s_2, \ldots$: and finally gives Cauchy's evaluation of the double alternant $|(\alpha_1-\beta_1)^{-1}(\alpha_2-\beta_2)^{-1}\ldots(\alpha_n-\beta_n)^{-1}|$. The applications, which come next, concern the solution of Lagrange's set of linear equations, Sylvester's transformation of a binary quantic of odd degree into canonical form, and the discussion of the equality of two roots of the equation $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0=0$, or say $f(x)=0$, viewed in connection with what, following Salmon, he calls the "determinant" of the equation, although Sylvester's use of the word "discriminant" is explained a page or two later.

Under this last head an interesting transformation falls to be noted. Calling the roots of the said equation $\alpha_1, \alpha_2, \ldots, \alpha_n$, and taking the determinant which is the square of their difference-product, namely,

$$\begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \ldots & s_{2n-2} \end{vmatrix}, \quad \text{or Z say,}$$

he substitutes for it a determinant of the $(2n-2)^{\text{th}}$ order

$$\begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & s_0 & s_1 & s_2 & \ldots & s_{n-1} \\ 0 & 0 & 0 & \ldots & s_1 & s_2 & s_3 & \ldots & s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & s_0 & s_1 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & \ldots & s_{2n-4} \\ s_0 & s_1 & s_2 & \ldots & s_{n-2} & s_{n-1} & s_n & \ldots & s_{2n-3} \\ s_1 & s_2 & s_3 & \ldots & s_{n-1} & s_n & s_{n+1} & \ldots & s_{2n-2} \end{vmatrix}$$

where the first $n-2$ rows do not contain an s, and the rows following contain all the s's in descending order from right to left, beginning with s_{n-1} in the last place of the $(n-1)^{\text{th}}$ row, with s_n

in the last place of the n^{th} row, and so on. He then multiplies every row by a_n, and performs the operations which we may indicate by

$$\mathrm{col}_2 + \frac{a_{n-1}}{a_n}\mathrm{col}_1,$$

$$\mathrm{col}_3 + \frac{a_{n-1}}{a_n}\mathrm{col}_2 + \frac{a_{n-2}}{a_n}\mathrm{col}_1,$$

$$\mathrm{col}_4 + \frac{a_{n-1}}{a_n}\mathrm{col}_3 + \frac{a_{n-2}}{a_n}\mathrm{col}_2 + \frac{a_{n-3}}{a_n}\mathrm{col}_1,$$

$$\cdots\cdots\cdots\cdots\cdots$$

thus obtaining

$$Z \cdot a_n^{2n-2} = \begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \cdots \\ 0 & a_n & a_{n-1} & \cdots \\ 0 & 0 & a_n & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_n s_0 & a_n s_1 + a_{n-1}s_0 & \cdots \\ a_n s_0 & a_n s_1 + a_{n-1}s_0 & a_n s_2 + a_{n-1}s_1 + a_{n-2}s_0 & \cdots \\ a_n s_1 & a_n s_2 + a_{n-1}s_1 & a_n s_3 + a_{n-1}s_2 + a_{n-2}s_1 & \cdots \end{vmatrix},$$

and by using Newton's relations

$$\begin{aligned} na_n &= a_n s_0, \\ (n-1)a_{n-1} &= a_n s_1 + a_{n-1}s_0, \\ (n-2)a_{n-2} &= a_n s_2 + a_{n-1}s_1 + a_{n-2}s_0, \\ (n-3)a_{n-3} &= a_n s_3 + a_{n-1}s_2 + a_{n-2}s_1 + a_{n-3}s_0, \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

the elements of the last n rows of the right-hand determinant, we are told, can be so changed that in each there will occur only one of the a's and that in the first power. Thereupon the conclusion is formally announced that the determinant with which we started can be expressed as a rational integral function of the $(2n-2)^{\text{th}}$ degree in the quantities

$$\frac{a_0}{a_n}, \frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n},$$

and that the said function becomes homogeneous on multiplication by a_n^{2n-2}. The actual result is not given, but in the second edition (1864) it is stated to be

$$-Z \cdot a_n^{2n-2} = \begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \dots \\ 0 & a_n & a_{n-1} & \dots \\ 0 & 0 & a_n & \dots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & na_n & (n-1)a_{n-1} & \dots \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} & \dots \end{vmatrix},$$

"eine Determinante $(2n-2)$ten Grades, bei welcher die $m-2$ ersten und die $m-1$ folgenden Zeilen in Bezug auf die nicht verschwindenden Elemente übereinstimmen."

Part of the object which Baltzer had here in view was to establish the relation between two forms of the discriminant of the given equation; namely, that obtained by squaring the determinant-form of the difference-product and that obtained as the eliminant of the equations

$$f'(x) = 0, \quad nf(x) - xf'(x) = 0,$$

or the equations

$$\left.\begin{aligned} \frac{\partial}{\partial x}(a_n x^n + a_{n-1}x^n y + \dots + a_0 y^n) &= 0 \\ \frac{\partial}{\partial y}(a_n x^n + a_{n-1}x^n y + \dots + a_0 y^n) &= 0 \end{aligned}\right\}.$$

Now a glance at the final determinant suffices to show that it is not the eliminant sought, there being in it three types of rows, whereas the two equations giving rise to the said eliminant being both of the $(n-1)^{\text{th}}$ degree, the coefficients of the one must occur in as many rows of the eliminant as the coefficients of the other. Further, since the coefficients of the equation $f'(x) = 0$ are seen to occur in their full number of rows, and those of the other equation in the last row only, it is therefore the first $n-2$ rows that need to be changed. The set of operations requisite to effect this is

$$\begin{gathered} n\cdot\text{row}_1 - \text{row}_{2n-3},\\ n\cdot\text{row}_2 - \text{row}_{2n-4},\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ n\cdot\text{row}_{n-2} - \text{row}_n. \end{gathered}$$

BRIOSCHI, F. (1857, Oct.).

[Sullo svillippo di un determinante. *Annali di Mat.*, i. pp. 9–11; or *Opere mat.*, i. pp. 273–275.]

Brioschi enunciates without proof the proposition that the even-ordered determinant

$$\left|\begin{array}{ccccccc} \frac{1}{x_1-a_1} & \frac{1}{(x_1-a_1)^2} & \frac{1}{x_1-a_2} & \frac{1}{(x_1-a_2)^2} & \cdots & \frac{1}{x_1-a_n} & \frac{1}{(x_1-a_n)^2} \\ \frac{1}{x_2-a_1} & \frac{1}{(x_2-a_1)^2} & \frac{1}{x_2-a_2} & \frac{1}{(x_2-a_2)^2} & \cdots & \frac{1}{x_2-a_n} & \frac{1}{(x_2-a_n)^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{x_{2n}-a_1} & \frac{1}{(x_{2n}-a_1)^2} & \frac{1}{x_{2n}-a_2} & \frac{1}{(x_{2n}-a_2)^2} & \cdots & \frac{1}{x_{2n}-a_n} & \frac{1}{(x_{2n}-a_n)^2} \end{array}\right|,$$

which is seen to be a function of $2n$ x's and n a's, is equal to

$$(-1)^n \frac{\Pi^4(a_1, a_2, \ldots, a_n)\cdot\Pi(x_1, x_2, \ldots, x_{2n})}{\phi^2(a_1)\cdot\phi^2(a_2)\ \ldots\ \phi^2(a_n)},$$

where $\phi(x) = (x-x_1)(x-x_2)\ldots(x-x_{2n})$. He then obtains similar expressions for the principal minors, namely, (1) for the cofactor of any element in an odd-numbered column, and (2) for the cofactor of any element in an even-numbered column, his procedure being to express the minor in question in terms of determinants like the original but of the order $2n-2$ and then to make the substitutions which are thus rendered possible.

PROUHET, E. (1857, Nov.).

[Questions 410, 411. *Nouv. Annales de Math.*, (1) xvi. pp. 403, 404; xvii. pp. 187–190.]

By reason of the existence of the identity

$$2^{s-1}\cos^s a = \cos sa + s\cos(s-2)a + \tfrac{1}{2}s(s-1)\cos(s-4)a + \ldots$$

where the number of terms on the right is s and the last term has to be halved when s is odd, it is clear that the determinant

$$\begin{vmatrix} \cos na_0 & \cos(n-1)a_0 & \cos(n-2)a_0 & \dots & \cos 0.a_0 \\ \cos na_1 & \cos(n-1)a_1 & \cos(n-2)a_1 & \dots & \cos 0.a_1 \\ \cos na_2 & \cos(n-1)a_2 & \cos(n-2)a_2 & \dots & \cos 0.a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cos na_n & \cos(n-1)a_n & \cos(n-2)a_n & \dots & \cos 0.a_n \end{vmatrix}$$

may be transformed into

$$\begin{vmatrix} 2^{n-1}\cos^n a_0 & 2^{n-2}\cos^{n-1}a_0 & 2^{n-3}\cos^{n-2}a_0 & \dots & \cos^0 a_0 \\ 2^{n-1}\cos^n a_1 & 2^{n-2}\cos^{n-1}a_1 & 2^{n-3}\cos^{n-2}a_1 & \dots & \cos^0 a_1 \\ 2^{n-1}\cos^n a_2 & 2^{n-2}\cos^{n-1}a_2 & 2^{n-3}\cos^{n-2}a_2 & \dots & \cos^0 a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2^{n-1}\cos^n a_n & 2^{n-2}\cos^{n-1}a_n & 2^{n-3}\cos^{n-2}a_n & \dots & \cos^0 a_n \end{vmatrix}$$

by increasing the 1st column by multiples of the 3rd, 5th, 7th, . . . , the 2nd column by multiples of the 4th, 6th, 8th, . . . and so forth. In this way there is deduced the result

$$\Delta_1 = 2^{\frac{1}{2}n(n-1)}\cdot D,$$

where Δ_1 is the first determinant, and D is the determinant got from Δ_1 by changing the multipliers of $a_0, a_1, a_2, \dots$ into indices of powers of $\cos a_0, \cos a_1, \cos a_2, \dots$.

Again by using the similar expansion for

$$\sin(s+1)a \div \sin a$$

on every element of the determinant

$$\begin{vmatrix} \sin(n+1)a_0 & \sin na_0 & \dots & \sin a_0 \\ \sin(n+1)a_1 & \sin na_1 & \dots & \sin a_1 \\ \cdot & \cdot & \cdot & \cdot \\ \sin(n+1)a_n & \sin na_n & \dots & \sin a_n \end{vmatrix}, \quad \text{or } \Delta_2 \text{ say,}$$

it is seen that the factors $\sin a_0, \sin a_1, \dots$ can be removed from the rows in order, and that the determinant so produced is simplifiable into a multiple of D: so that there is obtained the second result

$$\Delta_2 = 2^{\frac{1}{2}n(n-1)}\cdot \sin a_0 \sin a_1 \dots \sin a_n \cdot D.$$

The two results are Prouhet's, who set them for proof by others.

CAYLEY, A. (1858, Feb.).

[A fifth memoir upon quantics. *Philos. Transac. R. Soc.* (London), cxlviii. pp. 429–460; or *Collected Math. Papers,* ii. pp. 527–557.]

When dealing with the subject of the equivalence of two anharmonic ratios, Cayley gives (p. 538) the result

$$\begin{vmatrix} 1 & \alpha+\alpha' & \alpha\alpha' \\ 1 & \beta+\beta' & \beta\beta' \\ 1 & \gamma+\gamma' & \gamma\gamma' \end{vmatrix} \cdot \begin{vmatrix} u^2 & -u & 1 \\ v^2 & -v & 1 \\ w^2 & -w & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (u-\alpha)(u-\alpha') & (v-\alpha)(v-\alpha') & (w-\alpha)(w-\alpha') \\ (u-\beta)(u-\beta') & (v-\beta)(v-\beta') & (w-\beta)(w-\beta') \\ (u-\gamma)(u-\gamma') & (v-\gamma)(v-\gamma') & (w-\gamma)(w-\gamma') \end{vmatrix}.$$

This, if we put $\alpha', \beta', \gamma' = \alpha, \beta, \gamma$, becomes

$$\begin{vmatrix} (u-\alpha)^2 & (v-\alpha)^2 & (w-\alpha)^2 \\ (u-\beta)^2 & (v-\beta)^2 & (w-\beta)^2 \\ (u-\gamma)^2 & (v-\gamma)^2 & (w-\gamma)^2 \end{vmatrix} = 2\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & u & u^2 \\ 1 & v & v^2 \\ 1 & w & w^2 \end{vmatrix},$$

or, in later notation,

$$|(u-\alpha)^2(v-\beta)^2(w-\gamma)^2| = 2\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)\cdot\zeta^{\frac{1}{2}}(u, v, w).$$

Cayley also gives (p. 539) the result

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} \cdot \begin{vmatrix} ss' & -s' & -s & 1 \\ tt' & -t' & -t & 1 \\ uu' & -u' & -u & 1 \\ vv' & -v' & -v & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (s-\alpha)(s'-\alpha') & (s-\beta)(s'-\beta') & (s-\gamma)(s'-\gamma') & (s-\delta)(s'-\delta') \\ (t-\alpha)(t'-\alpha') & (t-\beta)(t'-\beta') & (t-\gamma)(t'-\gamma') & (t-\delta)(t'-\delta') \\ (u-\alpha)(u'-\alpha') & (u-\beta)(u'-\beta') & (u-\gamma)(u'-\gamma') & (u-\delta)(u'-\delta') \\ (v-\alpha)(v'-\alpha') & (v-\beta)(v'-\beta') & (v-\gamma)(v'-\gamma') & (v-\delta)(v'-\delta') \end{vmatrix},$$

which is seen in similar fashion to include the identity

$$|(s-\alpha)^2(t-\beta)^2(u-\gamma)^2(v-\delta)^2| = 0.$$

These specialisations, however, are not referred to by Cayley.

ZEHFUSS, G. (1859).

[Ueber die Determinante $Q_p = \Sigma \pm (a_0+b_0)^p(a_1+b_1)^p \ldots (a_n+b_n)^p$. *Zeitschrift f. Math. u. Phys.*, iv. pp. 233–236.]

Zehfuss having drawn attention to the known results when $p = -1$ and $p = -2$, and having asserted that there are no similar simple results when p is less than -2, confines himself to the cases where p is a positive integer. He first shows in the usual way that Q_p is divisible by the difference-product of $a_0, a_1, \ldots, a_n$ and by the difference-product of $b_0, b_1, \ldots, b_n$. Then he points out that Q_p is of the p^{th} degree in a_0 while the difference-product of $a_0, a_1, \ldots, a_n$ is of the n^{th} degree, and thence draws the conclusion

$$\text{if } p < n \qquad Q_p = 0,$$

giving the example deduced above from Cayley,

$$\Sigma \pm (a_0+b_0)^2(a_1+b_1)^2(a_2+b_2)^2(a_3+b_3)^2 = 0.$$

The case where $p = n$ he deals with differently, namely, as might have been suggested by Cayley's paper of the preceding year. By the multiplication-theorem he derives

$$Q_n = \begin{vmatrix} 1 & n_1 a_0 & n_2 a_0^2 & \ldots. & a_0^n \\ 1 & n_1 a_1 & n_2 a_1^2 & \ldots. & a_1^n \\ 1 & n_1 a_2 & n_2 a_2^2 & \ldots. & a_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & n_1 a_n & n_2 a_n^2 & \ldots. & a_n^n \end{vmatrix} \cdot \begin{vmatrix} b_0^n & b_0^{n-1} & \ldots. & 1 \\ b_1^n & b_1^{n-1} & \ldots. & 1 \\ b_2^n & b_2^{n-1} & \ldots. & 1 \\ \cdot & \cdot & \cdot & \cdot \\ b_n^n & b_n^{n-} & \ldots. & 1 \end{vmatrix}$$

where $n_r = n(n-1)(n-2) \ldots. (n-r+1) \big/ 1 \cdot 2 \cdot 3 \ldots r$; and from this there is readily obtained

$$Q_n = (-1)^{\frac{1}{2}n(n-1)} n_1 n_2 n_3 \ldots \zeta^{\frac{1}{2}}(a_0 a_1 \ldots a_n) \cdot \zeta^{\frac{1}{2}}(b_0 b_1 \ldots b_n)$$

the illustrating example being again that which we have deduced above.

CHAPTER V.

COMPOUND DETERMINANTS, UP TO 1860.

DETERMINANTS whose elements are themselves determinants made their appearance at a very early stage in the history of the subject, the first foreshadowing of them being contained in Lagrange's "équation identique et très remarquable" of 1773, namely,

$$\ldots'' + \eta\zeta'\xi'' + \zeta\xi'\eta'' - \xi\zeta'\eta'' - \eta\xi'\zeta'' - \zeta\eta'\xi'' = (xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'')^2,$$

where

$$\xi, \quad \eta, \quad \zeta, \ \ldots = y'z'' - y''z', \quad z'x'' - z''x', \quad x'y'' - x''y', \ \ldots$$

This, viewed as a result in determinants, is a case of Cauchy's theorem of 1812 regarding the adjugate, and the adjugate of course is an instance of the special form to which we have now come. Jacobi's theorem regarding any minor of the adjugate has a like history and may be similarly classified. Passing from the case of the adjugate, where each element is a primary minor of the original determinant, Cauchy also considered the determinants, of other "systèmes dérivés," that is to say, the determinants whose elements are the secondary, ternary, . . ., minors of the original, and gave the theorem that the product of the determinants of two "complementary derived systems" is a power of the original determinant, the index of the power being

$$n(n-1)(n-2)\ldots(n-p+1)\big/1\cdot2\cdot3\ldots p,$$

where n is the order of the original determinant and p the order of each element of one of the "derived systems." He also in the same memoir established the theorem that *the determinant of any "derived system" of a product-determinant is equal to the product of the determinants of the corresponding "derived systems" of the two factors.*

Those are all the general results that fall to be noted prior to the middle of the nineteenth century; and, as is readily seen, they all concern what at a later date came to be called the "compounds" of $|a_{1n}|$. With one exception they are due to Cauchy.*

The fact has also to be recalled, however, that compound determinants of a *special* type were considered by Jacobi in 1841, namely, those whose elements are *functional* determinants, his main theorem being

$$\sum \pm \mathrm{J}_1^{(1)}\mathrm{J}_2^{(2)}\cdots\mathrm{J}_m^{(m)} = \left\{\sum \pm \frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\right\}^{m-1}\cdot\sum \pm \frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}$$

where $f_1, f_2, \ldots, f_{n+m}$ are functions of $x_1, x_2, \ldots, x_{n+m}$, and

$$\mathrm{J}_r^{(s)} = \sum \pm \frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\cdot\frac{\partial f_{n+s}}{\partial x_{n+r}}.$$

This theorem and certain deductions therefrom have been already dealt with in another connection (see pp. 381–385 of *History*, i.).

SYLVESTER, J. J. (1850).

[On the intersections, contacts, and other correlations of two conics expressed by indeterminate co-ordinates. *Cambridge and Dub. Math. Journ.*, v. pp. 262–282; or *Collected Math. Papers*, i. pp. 119–137.]

In a footnote to this paper the name "*Compound* Determinants" first appears. The passage is (p. 270): ". . . a theorem given by M. Cauchy, and which is included as a particular case in a theorem of my own relating to Compound Determinants, *i.e.* Determinants of Determinants, which will take its place as an immediate consequence of my fundamental theorem given

* They are numbered xx., xxi., xli., xlii. in *History*, i.

in a memoir about to appear. The well-known rule for the Multiplication of Determinants is also a direct and simple consequence from my theorem on Compound Determinants, which indeed comprises, I believe, in one glance all the heretofore existing doctrine of determinants."

It will be of interest as we advance to try to identify the theorems of this perfervid statement, namely, (a) Sylvester's "fundamental theorem"; (b) his widely general "theorem on compound determinants" deduced therefrom, and including as a particular case a theorem of Cauchy's, and giving rise to the multiplication-theorem and many others as corollaries.

SYLVESTER, J. J. (1851, March).

[On the relation between the minor determinants of linearly equivalent quadratic functions. *Philos. Magazine* (4), i. pp. 295–305, 415; or *Collected Math. Papers*, i. pp. 241–250, 251.]

As we have already had occasion to note,* there is here given, by way of illustrating the power of the umbral notation, a theorem regarding a compound determinant, namely, the theorem which Sylvester writes in the form

$$\begin{Bmatrix} \overline{a_1\, a_2 \ldots a_r\, a_{r+1}} & \overline{a_1\, a_2 \ldots a_r\, a_{r+2}} & \ldots\ldots & \overline{a_1\, a_2 \ldots a_r\, a_{r+s}} \\ a_1\, a_2 \ldots a_r\, a_{r+1} & a_1\, a_2 \ldots a_r\, a_{r+2} & \ldots\ldots & a_1\, a_2 \ldots a_r\, a_{r+s} \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1\, a_2 \ldots a_r \\ a_1\, a_2 \ldots a_r \end{Bmatrix}^{s-1} \times \begin{Bmatrix} a_1\, a_2 \ldots a_r\, a_{r+1}\, a_{r+2} \ldots a_{r+s} \\ a_1\, a_2 \ldots a_r\, a_{r+1}\, a_{r+2} \ldots a_{r+s} \end{Bmatrix},$$

but which would now be better understood in the slightly modified form

$$\left\| \begin{vmatrix} a_1\, a_2 \ldots a_r\, a_{r+1} \\ b_1\, b_2 \ldots b_r\, b_{r+1} \end{vmatrix} \quad \begin{vmatrix} a_1\, a_2 \ldots a_r\, a_{r+2} \\ b_1\, b_2 \ldots b_r\, b_{r+2} \end{vmatrix} \ldots\ldots \begin{vmatrix} a_1\, a_2 \ldots a_r\, a_{r+s} \\ b_1\, b_2 \ldots b_r\, b_{r+s} \end{vmatrix} \right\|$$

$$= \begin{vmatrix} a_1\, a_2 \ldots a_r \\ b_1\, b_2 \ldots b_r \end{vmatrix}^{s-1} \cdot \begin{vmatrix} a_1\, a_2 \ldots a_{r+s} \\ b_1\, b_2 \ldots b_{r+s} \end{vmatrix}.$$

---

* See above, pp. 59–60.

No proof of it is given. At a later date it would have been viewed as the "extensional" of the manifest identity

$$\begin{vmatrix} \begin{matrix} a_{r+1} & a_{r+1} & \ldots & a_{r+1} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{matrix} \\ \begin{matrix} a_{r+2} & a_{r+2} & \ldots & a_{r+2} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{matrix} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \begin{matrix} a_{r+s} & a_{r+s} & \ldots & a_{r+s} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{matrix} \end{vmatrix} = \begin{vmatrix} a_{r+1} & a_{r+2} & \ldots & a_{r+s} \\ b_{r+1} & b_{r+2} & & b_{r+s} \end{vmatrix}.$$

Later on in the same paper Sylvester gives for a particular purpose what he calls an "important generalisation." His words are (p. 304): "Suppose two sets of umbræ

$$\begin{matrix} a_1 & a_2 & \ldots & a_{m+n} \\ b_1 & b_2 & \ldots & b_{m+n}, \end{matrix}$$

and let r be any number less than n, and let any r-ary combination of the m numbers 1, 2, 3, . . . , m be expressed by ${}^q\theta_1, {}^q\theta_2, \ldots, {}^q\theta_m$, where q goes through all the values intermediate between 1 and μ, μ being

$$\frac{m(m-1)\ldots(m-r+1)}{1\cdot 2 \quad \ldots \quad r};$$

then I say that the compound determinant

$$\begin{matrix} \overline{\begin{matrix} a_{1_{\theta_1}} & a_{1_{\theta_2}} & \ldots & a_{1_{\theta_m}} & a_{m+1} & a_{m+2} & \ldots & a_{m+n} \\ b_{1_{\theta_1}} & b_{1_{\theta_2}} & \ldots & b_{1_{\theta_m}} & b_{m+1} & b_{m+2} & \ldots & b_{m+n} \end{matrix}} & \overline{\begin{matrix} a_{2_{\theta_1}} & a_{2_{\theta_2}} & \ldots & a_{2_{\theta_m}} & a_{m+1} & a_{m+2} & \ldots & a_{m+n} \\ b_{2_{\theta_1}} & b_{2_{\theta_2}} & \ldots & b_{2_{\theta_m}} & b_{m+1} & b_{m+2} & \ldots & b_{m+n} \end{matrix}} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \overline{\begin{matrix} a_{\mu_{\theta_1}} & a_{\mu_{\theta_2}} & \ldots & a_{\mu_{\theta_m}} & a_{m+1} & a_{m+2} & \ldots & a_{m+n} \\ b_{\mu_{\theta_1}} & b_{\mu_{\theta_2}} & \ldots & b_{\mu_{\theta_m}} & b_{m+1} & b_{m+2} & \ldots & b_{m+n} \end{matrix}} \end{matrix}$$

is equal to the following product:

$$\overbrace{\begin{matrix} a_{m+1} & a_{m+2} & \ldots & a_{m+n} \\ b_{m+1} & b_{m+2} & \ldots & b_{m+n} \end{matrix}}^{\mu'} \cdot \overbrace{\begin{matrix} a_1 & a_2 & \ldots & a_{m+n} \\ b_1 & b_2 & \ldots & b_{m+n}, \end{matrix}}^{\mu''}$$

where

$$\mu'' = \frac{(m-1)(m-2)\ldots(m-r+1)}{1\cdot 2 \quad \ldots \quad (r-1)},$$

and

$$\mu' = \frac{(m-1)(m-2)\ldots(m-r)}{1\cdot 2 \quad\ldots\quad r}\,.$$

In reference to this one must remark at the outset on the inappropriateness of the notation

$${}^q\theta_1,\ {}^q\theta_2,\ \ldots,\ {}^q\theta_m$$

for the q^{th} combination of r integers taken from $1, 2, \ldots, m$. Manifestly

$${}^q\theta_1,\ {}^q\theta_2,\ \ldots,\ {}^q\theta_r,$$

though equally awkward, would have been less misleading. Indeed, as there is one clear misprint in the enunciation, namely, "less than n" for "less than m"; and as Sylvester is known to have been inaccurate in the correction of proofs, we might suspect θ_m to be a misprint for θ_r, were it not that θ_m occurs four times in the short passage and θ_r not once.* For ${}^q\theta_p$ it would have been much more convenient to write pq, which would thus have stood for "the p^{th} integer of the q^{th} combination"; and Sylvester's theorem might then have been written

$$\left\| \begin{vmatrix} a_{11}\,a_{21}\ldots a_{r1}\,a_{m+1}\,a_{m+2}\ldots a_{m+n} \\ b_{11}\,b_{21}\ldots b_{r1}\,b_{m+1}\,b_{m+2}\ldots b_{m+n} \end{vmatrix} \begin{vmatrix} a_{12}\,a_{22}\ldots a_{r2}\,a_{m+1}\,a_{m+2}\ldots a_{m+n} \\ b_{12}\,b_{22}\ldots b_{r2}\,b_{m+1}\,b_{m+2}\ldots b_{m+n} \end{vmatrix} \ldots \begin{vmatrix} a_{1\mu}\,a_{2\mu}\ldots a_{r\mu}\,a_{m+1}\,a_{m+2}\ldots a_{m+n} \\ b_{1\mu}\,b_{2\mu}\ldots b_{r\mu}\,b_{m+1}\,b_{m+2}\ldots b_{m+n} \end{vmatrix} \right\|$$

$$= \begin{vmatrix} a_{m+1}\,a_{m+2}\ldots a_{m+n} \\ b_{m+1}\,b_{m+2}\ldots b_{m+n} \end{vmatrix}^{C_{m-1,r}} \cdot \begin{vmatrix} a_1\,a_2\ldots a_{m+n} \\ b_1\,b_2\ldots b_{m+n} \end{vmatrix}^{C_{m-1,r-1}},$$

μ being used as before for $C_{m,r}$. To help towards clearness let us illustrate by means of the case where $m=4$, $n=3$, $r=2$. We then have $\mu=6$; each set of θ's equal to a binary combination of the first four integers, that is to say,

$${}^1\theta_1{}^1\theta_2,\quad {}^2\theta_1{}^2\theta_2,\quad {}^3\theta_1{}^3\theta_2,\quad \ldots,\quad {}^6\theta_1{}^6\theta_2$$

equal to

$$12,\ 13,\ 14,\ 23,\ 24,\ 34;$$

* θ_m was actually a misprint. Sylvester himself had to draw attention to it a year later in the *Cambridge and Dub. Math. Journ.*, viii. p. 61.

and the theorem in the form

$$\left\| \begin{matrix} \begin{vmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_5 & b_6 & b_7 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ b_1 & b_3 & b_5 & b_6 & b_7 \end{vmatrix} & \Big| \quad \cdots\cdots & \begin{vmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix} \\ \begin{vmatrix} a_1 & a_3 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_5 & b_6 & b_7 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 & a_5 & a_6 & a_7 \\ b_1 & b_3 & b_5 & b_6 & b_7 \end{vmatrix} & \Big| \quad \cdots\cdots & \begin{vmatrix} a_1 & a_3 & a_5 & a_6 & a_7 \\ b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_5 & b_6 & b_7 \end{vmatrix} & \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_3 & b_5 & b_6 & b_7 \end{vmatrix} & \Big| \quad \cdots\cdots & \begin{vmatrix} a_3 & a_4 & a_5 & a_6 & a_7 \\ b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix} \end{matrix} \right\|$$

$$= \begin{vmatrix} a_5 & a_6 & a_7 \\ b_5 & b_6 & b_7 \end{vmatrix}^3 \cdot \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix}^3 .$$

No proof is given by Sylvester: attention is merely drawn by him to the fact that when r is put equal to 1 we obtain the theorem with which his paper commences. It is rather remarkable that he should not have singled out the case where $n=0$. For then the theorem becomes

$$\left\| \begin{matrix} \begin{vmatrix} a_{11} & a_{21} & \ldots & a_{r1} \\ b_{11} & b_{21} & \ldots & b_{r1} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{22} & \ldots & a_{r2} \\ b_{12} & b_{22} & \ldots & b_{r2} \end{vmatrix} & \cdots\cdots & \begin{vmatrix} a_{1\mu} & a_{2\mu} & \ldots & a_{r\mu} \\ b_{1\mu} & b_{2\mu} & \ldots & b_{r\mu} \end{vmatrix} \end{matrix} \right\|$$

$$= \begin{vmatrix} a_1 & a_2 & \ldots & a_m \\ b_1 & b_2 & \ldots & b_m \end{vmatrix}^{C_{m-1,r-1}},$$

where, as before, the subscript pq denotes the p^{th} integer of the q^{th} set of r integers taken from $1, 2, \ldots, m$, and μ stands for $C_{m,r}$; and this is the theorem well known at a later date in the form: *The* rth *compound of a determinant of the* mth *order is a power of the said determinant, the index of the power being* $C_{m-1,r-1}$. Cauchy, it will be remembered, only got the length of a similar theorem in reference to the *product* of two complementary compounds. Now, since the complementary of the r^{th} compound is the $(m-r)^{\text{th}}$ compound, the product of the two must be that power of the original determinant whose index is

$$C_{m-1,r-1} + C_{m-1,m-r-1},$$

$$i.e. \quad C_{m-1,r-1} + C_{m-1,r},$$

$$i.e. \quad C_{m,r},$$

which agrees of course with Cauchy's result.

We thus learn that Sylvester's general result may be accurately described in later phraseology as *the "extensional" of the theorem regarding the* r^{th} *compound of a determinant*, and that the discovery of both the said theorem and of its "extensional" is almost certainly due to him. At the same time it is hard to believe that this "extensional" is the all-embracing theorem referred to by him in a previous paper: for by no stretch of imagination could we see comprised in it "all the heretofore existing doctrine of determinants." His last words thereanent are: "This very general theorem is itself several degrees removed from my still unpublished Fundamental Theorem, which is a theorem for the expansion of products of determinants."

SYLVESTER, J. J. (1852, Dec.).

[On a theorem concerning the combination of determinants. *Cambridge and Dub. Math. Journ.*, viii. pp. 60–62; or *Collected Math. Papers*, i. pp. 399–401.]

The statement of the theorem referred to in the title unfortunately shows want of proper care,* with the result that it is unnecessarily lengthy. It may be recast as follows:—

If from the array

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn}, \end{matrix} \quad \textit{or } \mathrm{A}, \textit{ say,}$$

we form every possible array of r *rows* ($\mathrm{r} \not> \mathrm{m} < \mathrm{n}$), *calling the said arrays* $\mathrm{A}_1, \mathrm{A}_2, \dots, \mathrm{A}_\mu$, *where of course* $\mu = \mathrm{C}_{\mathrm{m},\mathrm{r}}$; *and if the corresponding arrays formed from*

$$\begin{matrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & \dots & b_{mn}, \end{matrix} \quad \textit{or } \mathrm{B}, \textit{ say,}$$

* See especially line 8 from bottom of p. 61, where in every case m should be $m-1$.

be denoted by $B_1, B_2, \ldots, B_\mu$; *then*

$$\begin{vmatrix} A_1\cdot B_1 & A_1\cdot B_2 & \ldots & A_1\cdot B_\mu \\ A_2\cdot B_1 & A_2\cdot B_2 & \ldots & A_2\cdot B_\mu \\ \cdot & \cdot & \cdot & \cdot \\ A_\mu\cdot B_1 & A_\mu\cdot B_2 & \ldots & A_\mu\cdot B_\mu \end{vmatrix} = (A\cdot B)^{C_{m-1,r-1}}.$$

By way of proof, Sylvester merely states that it is obtainable from his general theorem of March 1851, "by making

$$\left.\begin{matrix} a_{m+1} & a_{m+2} & \ldots & a_{m+n} \\ b_{m+1} & b_{m+2} & \ldots & b_{m+n} \end{matrix}\right\}$$

represent a determinant all whose terms (*i.e.* elements) are zeros except those which lie in one of the diagonals, these latter being all units."

His only other remark is that when $r=1$ and when $r=m$ the right-hand members are identical, and that the equating of the two left-hand members which is thus legitimised gives Cauchy's extended multiplication-theorem.

The former remark, unfortunately, is another troublesome instance of inaccuracy. The specialisation given therein is only one of two which are needed, the other being that every element of the determinant

$$\begin{vmatrix} a_1\, a_2\, a_3 & \ldots & a_m \\ b_1\, b_2\, b_3 & \ldots & b_m \end{vmatrix}$$

be made 0. To make the matter clear, let us take the case of the general theorem where $m=4$, $n=5$, $r=2$, and perform the requisite specialisations. The general theorem then is

$$\left|\begin{matrix} \begin{vmatrix} a_1\, a_2\, a_5\, a_6 \ldots a_9 \\ b_1\, b_2\, b_5\, b_6 \ldots b_9 \end{vmatrix} & \begin{vmatrix} a_1\, a_2\, a_5\, a_6 \ldots a_9 \\ b_1\, b_3\, b_5\, b_6 \ldots b_9 \end{vmatrix} & \ldots\ldots & \begin{vmatrix} a_1\, a_2\, a_5\, a_6 \ldots a_9 \\ b_3\, b_4\, b_5\, b_6 \ldots b_9 \end{vmatrix} \\ \begin{vmatrix} a_1\, a_3\, a_5\, a_6 \ldots a_9 \\ b_1\, b_2\, b_5\, b_6 \ldots b_9 \end{vmatrix} & \begin{vmatrix} a_1\, a_3\, a_5\, a_6 \ldots a_9 \\ b_1\, b_3\, b_5\, b_6 \ldots b_9 \end{vmatrix} & \ldots\ldots & \begin{vmatrix} a_1\, a_3\, a_5\, a_6 \ldots a_9 \\ b_3\, b_4\, b_5\, b_6 \ldots b_9 \end{vmatrix} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \begin{vmatrix} a_3\, a_4\, a_5\, a_6 \ldots a_9 \\ b_1\, b_2\, b_5\, b_6 \ldots b_9 \end{vmatrix} & \begin{vmatrix} a_3\, a_4\, a_5\, a_6 \ldots a_9 \\ b_1\, b_3\, b_5\, b_6 \ldots b_9 \end{vmatrix} & \ldots\ldots & \begin{vmatrix} a_3\, a_4\, a_5\, a_6 \ldots a_9 \\ b_3\, b_4\, b_5\, b_6 \ldots b_9 \end{vmatrix} \end{matrix}\right|$$

$$= \begin{vmatrix} a_5\, a_6 \ldots a_9 \\ b_5\, b_6 \ldots b_9 \end{vmatrix}^3 \cdot \begin{vmatrix} a_1\, a_2 \ldots a_9 \\ b_1\, b_2 \ldots b_9 \end{vmatrix}^3.$$

By changing now the matrix of the determinant

$$\begin{vmatrix} a_5\, a_6 \ldots a_9 \\ b_5\, b_6 \ldots b_9 \end{vmatrix}$$

into matrix unity, and the matrix of the determinant

$$\begin{vmatrix} a_1\, a_2\, a_3\, a_4 \\ b_1\, b_2\, b_3\, b_4 \end{vmatrix}$$

into matrix zero, the first element of the compound determinant becomes, if we write $a_r b_s$. in place of $\begin{matrix} a_r \\ b_s \end{matrix}$,

$$\begin{vmatrix} \cdot & \cdot & a_1b_5 & a_1b_6 & \ldots & a_1b_9 \\ \cdot & \cdot & a_2b_5 & a_2b_6 & \ldots & a_2b_9 \\ a_5b_1 & a_5b_2 & 1 & \cdot & \ldots & \cdot \\ a_6b_1 & a_6b_2 & \cdot & 1 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_9b_1 & a_9b_2 & \cdot & \cdot & \ldots & 1 \end{vmatrix}$$

which from Laplace's expansion-theorem we know to be equal to

$$\sum \begin{vmatrix} a_1b_5 & a_1b_6 \\ a_2b_5 & a_2b_6 \end{vmatrix} \cdot \begin{vmatrix} a_5b_1 & a_6b_1 \\ a_5b_2 & a_6b_2 \end{vmatrix}$$

or

$$\begin{vmatrix} a_1b_5 & a_1b_6 & \ldots & a_1b_9 \\ a_2b_5 & a_2b_6 & \ldots & a_2b_9 \end{vmatrix} \cdot \begin{vmatrix} a_5b_1 & a_6b_1 & \ldots & a_9b_1 \\ a_5b_2 & a_6b_2 & \ldots & a_9b_2 \end{vmatrix}.$$

Further, it is seen that the other elements of the compound determinant take like forms, and that in fact the said determinant is

$$\begin{vmatrix} A_1 \cdot B_1 & A_1 \cdot B_2 & \ldots & A_1 \cdot B_6 \\ A_2 \cdot B_1 & A_2 \cdot B_2 & \ldots & A_2 \cdot B_6 \\ \cdot & \cdot & \cdot & \cdot \\ A_6 \cdot B_1 & A_6 \cdot B_2 & \ldots & A_6 \cdot B_6 \end{vmatrix}$$

if A and B be taken to denote the arrays

$$\begin{matrix} a_1b_5 & a_1b_6 & \ldots & a_1b_9 \\ a_2b_5 & a_2b_6 & \ldots & a_2b_9 \\ a_3b_5 & a_3b_6 & \ldots & a_3b_9 \\ a_4b_5 & a_4b_6 & \ldots & a_4b_9 \end{matrix} \qquad \begin{matrix} a_5b_1 & a_6b_1 & \ldots & a_9b_1 \\ a_5b_2 & a_6b_2 & \ldots & a_9b_2 \\ a_5b_3 & a_6b_3 & \ldots & a_9b_3 \\ a_5b_4 & a_6b_4 & \ldots & a_9b_4. \end{matrix}$$

As for the right-hand member of the general identity, the first determinant in it becomes 1, and the second becomes

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & a_1b_5 & a_1b_6 & \dots & a_1b_9 \\ \cdot & \cdot & \cdot & \cdot & a_2b_5 & a_2b_6 & \dots & a_2b_9 \\ \cdot & \cdot & \cdot & \cdot & a_3b_5 & a_3b_6 & \dots & a_3b_9 \\ \cdot & \cdot & \cdot & \cdot & a_4b_5 & a_4b_6 & \dots & a_4b_9 \\ a_5b_1 & a_5b_2 & a_5b_3 & a_5b_4 & 1 & \cdot & \dots & \cdot \\ a_6b_1 & a_6b_2 & a_6b_3 & a_6b_4 & \cdot & 1 & \dots & \cdot \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ a_9b_1 & a_9b_2 & a_9b_3 & a_9b_4 & \cdot & \cdot & \dots & 1 \end{vmatrix},$$

which equals A·B; so that the member in question becomes

$$(\mathrm{A}\cdot\mathrm{B})^3$$

as it ought.

The important thing to note in connection with the deduction here made is the fact that Sylvester must at this date have known how to express the product of two m-by-n arrays as a determinant of the $(m+n)^{\text{th}}$ order.* (See footnote on p. 57 above.)

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* And knowing this he *might* have indicated another mode of proving Cauchy's extended multiplication-theorem. For example:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \cdot \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} \cdot & \cdot & a_1 & a_2 & a_3 \\ \cdot & \cdot & b_1 & b_2 & b_3 \\ x_1 & y_1 & 1 & \cdot & \cdot \\ x_2 & y_2 & \cdot & 1 & \cdot \\ x_3 & y_3 & \cdot & \cdot & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -a_1x_1 - a_2x_2 - a_3x_3 & -a_1y_1 - a_2y_2 - a_3y_3 & \cdot & \cdot & \cdot \\ -b_1x_1 - b_2x_2 - b_3x_3 & -b_1y_1 - b_2y_2 - b_3y_3 & \cdot & \cdot & \cdot \\ x_1 & y_1 & 1 & \cdot & \cdot \\ x_2 & y_2 & \cdot & 1 & \cdot \\ x_3 & y_3 & \cdot & \cdot & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_1y_1 + a_2y_2 + a_3y_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & b_1y_1 + b_2y_2 + b_3y_3 \end{vmatrix}.$$

SPOTTISWOODE, W. (1853).

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journal*, li. pp. 209–271, 328–381.]

In his first edition Spottiswoode had a section (§ vi.) headed *On Inverse Systems and Determinants of Determinants*; but in it, as we have seen, he dealt merely with the adjugate determinant and its minors. Now, this is supplanted by a section (§ x.) of considerably greater extent (pp. 350–372) with the short Sylvesterian title *On Compound Determinants.*

Although it is the original definition of a compound determinant which is given on starting, the name afterwards seems to be unconsciously limited to compound determinants whose elements are minors of a given determinant of the n^{th} order. This limitation leads to the introduction of the word *class* in connection with compound determinants to indicate "the degree of minority of the constituents" (*i.e.* elements), a compound determinant of the i^{th} class being one whose elements are minors of the $(n-i)^{\text{th}}$ order; thus, the adjugate determinant is a compound determinant of the n^{th} order and 1^{st} class. If a compound determinant of the i^{th} class be of the highest possible order, namely, the

$$\frac{n(n-1)\ldots(n-i+1)}{1\cdot 2\quad\ldots\quad i}\text{th},$$

—that is to say, contains *all* the minors of the $(n-i)^{\text{th}}$ order—Spottiswoode calls it the "*complete* determinant of the i^{th} class," a name equivalent therefore to the more modern "$(n-i)^{\text{th}}$ compound."

One of the notations employed is essentially the same as Sylvester's—that is to say, he uses

$$\left\{\begin{matrix} \overline{1_1\, 2_1 \ldots u_1} & \overline{1_2\, 2_2 \ldots u_2} & \ldots & \overline{1_n\, 2_n \ldots u_n} \\ 1_1\, 2_1 \ldots u_1 & 1_2\, 2_2 \ldots u_2 & & 1_n\, 2_n \ldots u_n \end{matrix}\right\}$$

for what at a later date would have been written

$$\left\| \begin{matrix} 1_1\, 2_1 \ldots u_1 \\ 1_1\, 2_1 \ldots u_1 \end{matrix} \right| \quad \left| \begin{matrix} 1_2\, 2_2 \ldots u_2 \\ 1_2\, 2_2 \ldots u_2 \end{matrix} \right| \quad \ldots\ldots \quad \left| \begin{matrix} 1_n\, 2_n \ldots u_n \\ 1_n\, 2_n \ldots u_n \end{matrix} \right\|.$$

The other notation is his own, and is worthy of careful note. It differs from Sylvester's in making use of row-numbers and column-numbers not of the retained elements but of the elements omitted, the said numbers being enclosed in brackets for the purpose of recalling this difference. "Thus," he says (p. 352), "the complete compound determinant of the first class may be written

$$\left\{\binom{1}{1}\binom{2}{2}\cdots\cdots\binom{n}{n}\right\};$$

that of the second class

$$\left\{\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}\cdots\cdots\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}\cdots\cdots\right\}$$

and generally that of the i^{th} class

$$\left\{\begin{pmatrix}1_1 & 1_2 & \ldots & 1_i\\1_1 & 1_2 & \ldots & 1_i\end{pmatrix}\quad\begin{pmatrix}2_1 & 2_2 & \ldots & 2_i\\2_1 & 2_2 & \ldots & 2_i\end{pmatrix}\cdots\cdots\begin{pmatrix}\mu_1 & \mu_2 & \ldots & \mu_i\\\mu_1 & \mu_2 & \ldots & \mu_i\end{pmatrix}\right\},$$

where $$\mu = \frac{n(n-1)\ldots(n-i+1)}{1\cdot 2\quad\ldots\quad i};\text{"}$$

and where, it should have been added, r_s stands for the s^{th} integer in the r^{th} combination of i integers taken from $1, 2, \ldots, n$.

These preliminaries having been attended to, a discussion of the properties follows. The first five pages (pp. 353–358) and two later pages (pp. 366–368) are mainly concerned with compound determinants of the first class (that is to say, with the adjugate determinant), and they do not break fresh ground. The same, however, cannot be said with reference to the next two pages (pp. 358–360), which concern those of the second class. The result first reached is that the complete determinant of this class, namely,

$$\left\{\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}\cdots\cdots\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}\cdots\cdots\right\},\quad\text{or }\Delta_2\text{, say,}$$
$$= \begin{Bmatrix}1 & 2 & \ldots & n\\1 & 2 & \ldots & n\end{Bmatrix}^{\nu},$$

where $\nu=\frac{1}{2}(n-1)(n-2)$. Although there is a semblance of reasoning, no real proof is given. Passing then to any first minor of Δ_2, say the first minor got by leaving out $\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$ from the detailed symbol for Δ_2, he finds that

$$\left\{\begin{pmatrix}1\ 3\\ 1\ 3\end{pmatrix}\begin{pmatrix}1\ 4\\ 1\ 4\end{pmatrix}\ldots\ldots\begin{pmatrix}2\ 3\\ 2\ 3\end{pmatrix}\ldots\ldots\right\}=\begin{Bmatrix}1\ 2\\ 1\ 2\end{Bmatrix}\begin{Bmatrix}1\ 2\ \ldots\ldots\ n\\ 1\ 2\ \ldots\ldots\ n\end{Bmatrix}^{\nu-1}.$$

It is next pointed out that if we proceed to the second minors of Δ_2 it becomes necessary to distinguish two cases,—to distinguish, for example, the case where we leave out $\begin{pmatrix}1\ 2\\ 1\ 2\end{pmatrix}$, $\begin{pmatrix}1\ 3\\ 1\ 3\end{pmatrix}$ from the case where we leave out $\begin{pmatrix}1\ 2\\ 1\ 2\end{pmatrix}\begin{pmatrix}3\ 4\\ 3\ 4\end{pmatrix}$. In the latter case the row-numbers 1, 2, 3, 4 are all different, and the result is

$$\left\{\begin{pmatrix}5\ 6\\ 5\ 6\end{pmatrix}\begin{pmatrix}7\ 8\\ 7\ 8\end{pmatrix}\ldots\ldots\begin{pmatrix}1\ 3\\ 1\ 3\end{pmatrix}\begin{pmatrix}2\ 4\\ 2\ 4\end{pmatrix}\ldots\ldots\right\}=\begin{Bmatrix}1\ 2\ 3\ 4\\ 1\ 2\ 3\ 4\end{Bmatrix}\begin{Bmatrix}1\ 2\ \ldots\ n\\ 1\ 2\ \ldots\ n\end{Bmatrix}^{\nu-2};$$

in the former case the row-number 1 occurs twice, and the result is

$$\left\{\begin{pmatrix}1\ 4\\ 1\ 4\end{pmatrix}\begin{pmatrix}1\ 5\\ 1\ 5\end{pmatrix}\ldots\ldots\begin{pmatrix}2\ 3\\ 2\ 3\end{pmatrix}\ldots\ldots\right\}=\begin{Bmatrix}1\\ 1\end{Bmatrix}\begin{Bmatrix}1\ 2\ 3\\ 1\ 2\ 3\end{Bmatrix}\begin{Bmatrix}1\ 2\ \ldots\ n\\ 1\ 2\ \ldots\ n\end{Bmatrix}^{\nu-2}.$$

Generally, if $\begin{pmatrix}1\ 2\\ 1\ 2\end{pmatrix}, \begin{pmatrix}3\ 4\\ 3\ 4\end{pmatrix}, \begin{pmatrix}5\ 6\\ 5\ 6\end{pmatrix}, \ldots, \begin{pmatrix}2i-1\ 2i\\ 2i-1\ 2i\end{pmatrix}$ be left out, we have

$$\left\{\begin{pmatrix}2i+1\ 2i+2\\ 2i+1\ 2i+2\end{pmatrix}\begin{pmatrix}2i+3\ 2i+4\\ 2i+3\ 2i+4\end{pmatrix}\ldots\ldots\begin{pmatrix}1\ 3\\ 1\ 3\end{pmatrix}\begin{pmatrix}2\ 4\\ 2\ 4\end{pmatrix}\ldots\ldots\right\}=\begin{Bmatrix}1\ 2\ldots 2i\\ 1\ 2\ldots 2i\end{Bmatrix}\begin{Bmatrix}1\ 2\ldots n\\ 1\ 2\ldots n\end{Bmatrix}^{\nu-i};$$

and if $\begin{pmatrix}1\ 2\\ 1\ 2\end{pmatrix}, \begin{pmatrix}1\ 3\\ 1\ 3\end{pmatrix}, \ldots, \begin{pmatrix}1\ i\\ 1\ i\end{pmatrix}$ be left out, we have

$$\left\{\begin{pmatrix}1\ i+1\\ 1\ i+1\end{pmatrix}\begin{pmatrix}1\ i+2\\ 1\ i+2\end{pmatrix}\ldots\ldots\begin{pmatrix}2\ 3\\ 2\ 3\end{pmatrix}\ldots\ldots\right\}=\begin{Bmatrix}1\\ 1\end{Bmatrix}^{i-2}\begin{Bmatrix}1\ 2\ \ldots\ i\\ 1\ 2\ \ldots\ i\end{Bmatrix}\begin{Bmatrix}1\ 2\ \ldots\ n\\ 1\ 2\ \ldots\ n\end{Bmatrix}^{\nu-i+1}.$$

Again, a fresh variety of minor may be got by leaving out $\begin{pmatrix}1\ 2\\ 1\ 2\end{pmatrix}, \begin{pmatrix}1\ 3\\ 1\ 3\end{pmatrix}, \begin{pmatrix}2\ 3\\ 2\ 3\end{pmatrix}$, the row-numbers being then two 1's, two 2's, and two 3's, and the result is

$$\left\{\begin{pmatrix}1\ 4\\ 1\ 4\end{pmatrix}\begin{pmatrix}1\ 5\\ 1\ 5\end{pmatrix}\ldots\ldots\begin{pmatrix}2\ 4\\ 2\ 4\end{pmatrix}\ldots\ldots\right\}=\begin{Bmatrix}1\ 2\ 3\\ 1\ 2\ 3\end{Bmatrix}\begin{Bmatrix}1\ 2\ \ldots\ n\\ 1\ 2\ \ldots\ n\end{Bmatrix}^{\nu-3},$$

"and so on."

From these cases of compound determinants of the second class, the author passes to those of the i^{th} class, but contents himself with stating only two results. The first is that the complete determinant of this class, denoted as above by

$$\left\{\begin{pmatrix}1_1\ 1_2 \ldots 1_i\\ 1_1\ 1_2 \ldots 1_i\end{pmatrix} \begin{pmatrix}2_1\ 2_2 \ldots 2_i\\ 2_1\ 2_2 \ldots 2_i\end{pmatrix} \ \ldots\ldots\ \begin{pmatrix}\mu_1\ \mu_2 \ldots \mu_i\\ \mu_1\ \mu_2 \ldots \mu_i\end{pmatrix}\right\}, \text{ or } \Delta_i, \text{ say,}$$

$$= \begin{Bmatrix}1\ 2 \ldots n\\ 1\ 2 \ldots n\end{Bmatrix}^{\nu'},$$

where $\nu' = \dfrac{(n-1)(n-2)\ldots(n-i+1)}{1\cdot 2\ldots i-1}$, a result which agrees with that obtained on putting $n=0$ in Sylvester's general theorem of March 1851; and the other is that any first minor of Δ_i, say the minor

$$\left\{\begin{pmatrix}2_1\ 2_2 \ldots 2_i\\ 2_1\ 2_2 \ldots 2_i\end{pmatrix} \begin{pmatrix}3_1\ 3_2 \ldots 3_i\\ 3_1\ 3_2 \ldots 3_i\end{pmatrix} \ \ldots\ldots\ \begin{pmatrix}\mu_1\ \mu_2 \ldots \mu_i\\ \mu_1\ \mu_2 \ldots \mu_i\end{pmatrix}\right\}$$

$$= \begin{Bmatrix}1_1\ 1_2 \ldots 1_i\\ 1_1\ 1_2 \ldots 1_i\end{Bmatrix} \begin{Bmatrix}1\ 2 \ldots n\\ 1\ 2 \ldots n\end{Bmatrix}^{\nu'-1}.$$

He appends, however, the words "and so on," and tells us that "other formulæ may be written as required."

Two theorems of Sylvester's are next given, the one being the general theorem just alluded to, and the other that contained in the paper of 16th December, 1852. In the case of the former he varies the notation, and probably by reason of the above-mentioned serious misprint of an m for an r in the original he misses Sylvester's meaning, and makes an incorrect statement. In the case of the other no risk of this kind is incurred, because he takes the unusual course of reproducing Sylvester's words letter for letter to the extent of almost two pages (pp. 361–363). The original two pages, however, being, as we have seen, not without evidence of Sylvester's carelessness, this course also was unsafe.

Lastly, he shows how the multiplication-theorem may be deduced from Sylvester's first theorem of March 1851 regarding compound determinants, by taking as a example of the latter the identity

$$\left\|\begin{matrix} \begin{vmatrix} 1 & . & a \\ . & 1 & a' \\ \alpha & \beta & . \end{vmatrix} & \begin{vmatrix} 1 & . & b \\ . & 1 & b' \\ \alpha & \beta & . \end{vmatrix} \\ \begin{vmatrix} 1 & . & a \\ . & 1 & a' \\ \alpha' & \beta' & . \end{vmatrix} & \begin{vmatrix} 1 & . & b \\ . & 1 & b' \\ \alpha' & \beta' & . \end{vmatrix} \end{matrix}\right\| = \begin{vmatrix} 1 & . \\ . & 1 \end{vmatrix}^{2-1} \cdot \begin{vmatrix} 1 & . & a & b \\ . & 1 & a' & b' \\ \alpha & \beta & . & . \\ \alpha' & \beta' & . & . \end{vmatrix},$$

and pointing out that this is evidently the same as saying

$$\begin{vmatrix} a\alpha + a'\beta & b\alpha + b'\beta \\ a\alpha' + a'\beta' & b\alpha' + b'\beta' \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix} \cdot \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}.$$

In later phraseology, it may be said that the specialisation necessary for the purpose is

$$\left.\begin{aligned} s &= r, \\ \text{matrix of } &\begin{vmatrix} a_1\, a_2 \ldots a_r \\ b_1\, b_2 \ldots b_r \end{vmatrix} = 1, \\ \text{matrix of } &\begin{vmatrix} a_{r+1}\, a_{r+2} \ldots a_{2r} \\ b_{r+1}\, b_{r+2} \ldots b_{2r} \end{vmatrix} = 0.* \end{aligned}\right\}$$

BRIOSCHI, FR. (1854, March).

[LA TEORICA DEI DETERMINANTI, E LE SUE PRINCIPALI APPLICAZIONI. viii+116 pp. Pavia.]

After proving (p. 100) Jacobi's theorem, above referred to, regarding a compound determinant whose elements are functional determinants, Brioschi bids the reader make the functions f_1, $f_2, \ldots$ linear, say

$$f_r = a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{r,n+m}x_{n+m},$$

and note that the outcome is

$$\sum \pm \mathrm{A}^{(1)}\mathrm{A}_2^{(2)} \ldots \mathrm{A}_m^{(m)} = \left\{\sum \pm a_{11}a_{22} \ldots a_{nn}\right\}^{m-1} \cdot \sum \pm a_{11}a_{22} \ldots a_{n+m,n+m},$$

* Some of the pages of Spottiswoode dealt with in the foregoing are, by reason of misprints and other neglects, not easy reading. On p. 360 there are at least *nine* misprints.

where

$$A_r^{(s)} = \sum \pm a_{11}a_{22}\dots a_{nn}a_{n+s,n+r},$$

that is to say, is Sylvester's first result of March 1851.

The same course is followed by Bellavitis in his *Sposizione* of 1857 (see §§ 71, 72, p. 56).

BAZIN, H. (1854, April).

[Sur une question relative aux déterminants. *Journ. (de Liouville) de Math.*, xvi. pp. 145–160.]

Bazin's problem being to find an array of n rows and $n+h$ columns such that its n-line minors shall have given values, he notes at the outset that the said values cannot in general be independent of one another. There thus arises the preliminary task of finding the connecting equations, and his first conclusion is (pp. 148–149) that *if the value of one of the minors be given, and also the values of the* nh *other minors obtainable from the said minor by interchanging any one of its* n *columns with any one of the remaining columns, then the values of all the other minors are known.* This is established by actually showing how to obtain the expression for any one of the unknown minors in terms of a number of the known. As the underlying theorem is one of considerable importance in regard to compound determinants, we separate it from its surroundings and enunciate it in our own way, namely, *If the first of the* n*-line minors of an array of* n *rows and* n+h *columns* (h$\not>$n) *be denoted by* D, *and the minor got from* D *by supplanting its first* h *columns by the last* h *columns of the array be denoted by* E, *and if further there be formed a square array of new determinants obtained by supplanting separately and in order each of the first* h *columns of* D *by each of the first* h *columns of* E, *the determinant of this square array is equal to* $D^{h-1}E$.* Thus, using *rstu* to stand for

* In connection with this see the footnote on pp. 384, 385 of *Hist.*, i. Note also how from the second example with the help of the first we have

$$1274(1234\cdot5634) - 1734(1324\cdot5624) + 7234(2134\cdot5614) = (1234)^2\cdot5674,$$

the determinant whose columns are the r^{th}, s^{th}, t^{th}, u^{th} columns of the array, if the array consist of four rows and six columns, we have

$$\begin{vmatrix} 5234 & 6234 \\ 1534 & 1634 \end{vmatrix} = 1234 \cdot 5634;$$

and if it consist of four rows and seven columns, we have

$$\begin{vmatrix} 5234 & 6234 & 7234 \\ 1534 & 1634 & 1734 \\ 1254 & 1264 & 1274 \end{vmatrix} = (1234)^2 \cdot (5674).$$

K being the compound determinant involved, Bazin's mode of proof consists in expressing K/D^h in the form of a determinant of the same order as D, then multiplying D by K/D^h, and with the help of a lemma finding the product to be E. This at once gives

$$K = D^{h-1}E$$

as desired. Thus, taking the first of the two examples just given, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} 5234/D & 6234/D & . & . \\ 1534/D & 1634/D & . & . \\ 1254/D & 1264/D & 1 & . \\ 1235/D & 1236/D & . & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_5 & a_6 & a_3 & a_4 \\ b_5 & b_6 & b_3 & b_4 \\ c_5 & c_6 & c_3 & c_4 \\ d_5 & d_6 & d_3 & d_4 \end{vmatrix},$$

that is to say,

$$D \cdot \begin{vmatrix} 5234 & 6234 \\ 1534 & 1634 \end{vmatrix} \cdot D^{-2} = 5634,$$

and practically nothing more is wanted.

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and thence

$$1274 \cdot 3564 + 1734 \cdot 2564 - 7234 \cdot 5614 = 1234 \cdot 5674,$$

or, on removing the 'extension,'

$$127 \cdot 356 - 137 \cdot 256 - 237 \cdot 156 = 123 \cdot 567.$$

A relation is thus established between two very unlike theorems.

The lemma referred to we may formulate for ourselves as follows: *If from any determinant* D *a row of new determinants be formed by supplanting in order each column of* D *by one and the same new column, the product of this row of determinants by any row of* D *is equal to the product of* D *by the corresponding element of the repeatedly substituted new column.* This is little else than the elementary fact regarding a determinant with two rows identical. Thus

$$a_1 \cdot 5234 + a_2 \cdot 1534 + a_3 \cdot 1254 + a_4 \cdot 1235$$

$$= - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdot \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{vmatrix} = a_5 \cdot 1234.$$

From what has been said above it will be understood that Bazin views the theorem $K = D^{h-1}E$ in the form $E = K/D^{h-1}$, that is to say, as expressing the determinant E in terms of h^2+1 other determinants, namely, D and the h^2 elements of K.

For the present the rest of the paper is not of interest.

C(AYLEY,) A. (1854).

[Mathematical Notes. (No. 5.) *Cambridge and Dub. Math. Journ.*, ix. p. 171.]

To eliminate x, y from the equations

$$\frac{ax+by}{\alpha x+\beta y} = \frac{a'x+b'y}{\alpha' x+\beta' y} = \frac{a''x+b''y}{\alpha'' x+\beta'' y},$$

$-\lambda$ is introduced to stand for the equivalent of each fraction, μ for the negative reciprocal of λ, and ξ, η for any quantities whatever, with the result that from the four equations

$$\left.\begin{aligned} ax + by + \lambda(\alpha x + \beta y) &= 0 \\ a'x + b'y + \lambda(\alpha' x + \beta' y) &= 0 \\ a''x + b''y + \lambda(\alpha'' x + \beta'' y) &= 0 \\ \xi x + \eta y + \lambda(\mu\xi x + \mu\eta y) &= 0 \end{aligned}\right\},$$

there is obtained

$$\begin{vmatrix} \xi & \eta & \mu\xi & \mu\eta \\ a & b & \alpha & \beta \\ a' & b' & \alpha' & \beta' \\ a'' & b'' & \alpha'' & \beta'' \end{vmatrix} = 0,$$

or, say $$\mathrm{A}\xi+\mathrm{B}\eta+\mathrm{C}\mu\xi+\mathrm{D}\mu\eta = 0.$$

As this holds for all values of ξ, η, we have

$$\mathrm{A}+\mathrm{C}\mu = 0, \quad \mathrm{B}+\mathrm{D}\mu = 0,$$

and $\therefore$ $$\mathrm{AD}-\mathrm{BC} = 0,$$

that is,

$$\begin{vmatrix} |\,ab'\alpha''\,| & |\,ab'\beta''\,| \\ |\,a'\beta''a\,| & |\,\alpha'\beta''b\,| \end{vmatrix} = 0.$$

CHAPTER VI.

RECURRENTS, FROM 1841 TO 1860.

LIKE Wronskians, and for the same reason, *recurrents* were dealt with in our first volume among "Miscellaneous Special Forms": their previous history is thus to be found under Wronski 1812, Scherk 1825, and Schweins 1825 in the chapter so entitled. (*History*, i. pp. 472–474, 478–481.)

The name is quite recent, having apparently been first used by E. Pascal in 1907 in a paper published in the *Rendiconti* ... *Ist. Lombardo* (2), xl. pp. 293-305.

SPOTTISWOODE, W. (1853, August).

[Elementary theorems relating to determinants. Second edition, ... *Crelle's Journ.*, li. pp. 209–271, 328–381.]

In the last chapter or section (§ xi.), which is headed "Miscellaneous Instances of Determinants," Spottiswoode gives (pp. 373–374) an expression for the n^{th} differential-quotient of u/v in terms of the n^{th} and lower differential-quotients of u and of v. The first four cases are

$$\left(\frac{u}{v}\right)' = \begin{vmatrix} v & v' \\ u & u' \end{vmatrix} v^{-2}, \qquad \left(\frac{u}{v}\right)'' = -\begin{vmatrix} \cdot & v & 2v' \\ v & v' & v'' \\ u & u' & u'' \end{vmatrix} v^{-3},$$

$$\left(\frac{u}{v}\right)''' = \begin{vmatrix} \cdot & \cdot & v & 3v' \\ \cdot & v & 2v' & 3v'' \\ v & v' & v'' & v''' \\ u & u' & u'' & u''' \end{vmatrix} v^{-4}, \qquad \left(\frac{u}{v}\right)^{\text{iv}} = -\begin{vmatrix} \cdot & \cdot & \cdot & v & 4v' \\ \cdot & \cdot & v & 3v' & 6v'' \\ \cdot & v & 2v' & 3v'' & 4v''' \\ v & v' & v'' & v''' & v^{\text{iv}} \\ u & u' & u'' & u''' & u^{\text{iv}} \end{vmatrix} v^{-5},$$

where the arithmetical coefficients appearing in the elements of the determinants are those of the binomial theorem.

He also notes that any binary quantic may be expressed as a determinant: thus $a_0x^n + a_1x^{n-1}y + x_2a^{n-2}y^2 + \ldots.$ is written by him in the form

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ y & x & . & . & \ldots & . & . \\ . & y & x & . & \ldots & . & . \\ . & . & y & x & \ldots & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & \ldots & y & x \end{vmatrix}$$

BRIOSCHI, F. (1854, 1855, February).

[Sur deux formules relatives à la théorie de la décomposition des fractions rationnelles. *Crelle's Journ.*, l. pp. 239–242; or *Opere mat.*, vol. v. pp. 267–276. See also *Nouv. Annales de Math.*, xiii. p. 352.]

In a review of the second edition of Serret's *Cours d'Algèbre Supérieure*, Terquem, the editor of the *Nouvelles Annales*, takes occasion to enunciate the theorem that if s_r denote the sum of the r^{th} powers of the roots of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots. + a_n = 0,$$

then

$$s_r = (-1)^r \begin{vmatrix} a_1 & 1 & . & \ldots. & . \\ 2a_2 & a_1 & 1 & \ldots. & . \\ 3a_3 & a_2 & a_1 & \ldots. & . \\ . & . & . & . & . \\ ra_r & a_{r-1} & a_{r-2} & \ldots. & a_1 \end{vmatrix},$$

attributing it to Brioschi, and indicating that it had been arrived at by the solution of a "suite indéfinie d'équations périodiques du premier degré."* The origin of the determinant is thus exactly

*By this, of course, is meant the set of identities known as "Newton's formulæ,"—

$$s_1 + a_1 = 0$$
$$s_2 + a_1s_1 + 2a_2 = 0$$
$$\ldots\ldots\ldots$$

(See NEWTON, *Arith. Univ.*, Tom. ii., cap. iii., §8.)

similar to that of the first determinant of like kind, namely, that occurring in the statement of Wronski's "loi générale des séries."

A few months later we find on p. 240 of vol. l. of *Crelle's Journal* Brioschi himself enunciating and proving with some trouble the theorem that if

$$\phi(x) = c_0x^n + c_1x^{n-1} + \ldots + c_n,$$

and

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n,$$
$$= a_0(x-x_1)(x-x_2)\ldots(x-x_n),$$

then

$$\frac{x_1{}^r\phi(x_1)}{f'(x_1)} + \frac{x_2{}^r\phi(x_2)}{f'(x_2)} + \ldots\ldots + \frac{x_n\phi(x_n)}{f'(x_n)}$$

$$= (-1)^{r+1}\frac{1}{a_0^{r+2}}\begin{vmatrix} c_0 & a_0 & \cdot & \cdots\cdots & \cdot \\ c_1 & a_1 & a_0 & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\ c_r & a_r & a_{r-1} & \cdots\cdots & a_0 \\ c_{r+1} & a_{r+1} & a_r & \cdots\cdots & a_1 \end{vmatrix}.$$

The subject, however, is not pursued further.

FAURE, [H.] (1855, March).

[Théorème sur la somme des puissances semblables des racines. *Nouv. Annales de Math.* (1), xiv. pp. 94–97; or pp. 172–175 of Combescure's translation of Brioschi's *Teorica dei Determinanti.*]

Having seen Brioschi's result regarding s_r, Faure takes up the subject and succeeds in throwing on it fresh light. His fundamental proposition is not connected with the roots of equations at all, being to the effect that if

$$\phi(x) = c_0x^m + c_1x^{m-1} + \ldots\ldots + c_m,$$

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$$

and

$$\phi(x) \div f(x) = A_0x^{m-n} + A_1x^{m-n-1} + A_2x^{m-n-2} + \ldots\ldots,$$

then

$$A_r = (-1)^r \frac{1}{a_0^{r+1}} \begin{vmatrix} c_0 & a_0 & . & \cdots & . \\ c_1 & a_1 & a_0 & \cdots & . \\ c_2 & a_2 & a_1 & \cdots & . \\ . & . & . & \cdots & . \\ c_{r-1} & a_{r-1} & a_{r-2} & \cdots & a_0 \\ c_r & a_r & a_{r-1} & \cdots & a_1 \end{vmatrix}.$$

Having stated this he recalls the theorem * that if

$$f(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n)$$

we have

$$\psi(x_1)+\psi(x_2)+\ldots+\psi(x_n) = \text{coeff. of } x^{-1} \text{ in } \frac{f'(x)\psi(x)}{f(x)},$$

and therefore as a special case

$$x_1^r + x_2^r + \ldots + x_n^r = \text{coeff. of } x^{-r-1} \text{ in } \frac{f'(x)}{f(x)}.$$

It is thus seen that to obtain a determinant expression for s_r we have only to make $\phi(x)$ identical with $f'(x)$,—in other words, put $m=n-1$, $c_0=na_0$, $c_1=(n-1)a_1$, $c_2=(n-2)a_2$, ...,—and find the coefficient of x^{-r-1}. Doing this we obtain

$$(-1)^r \frac{1}{a_0^{r+1}} \begin{vmatrix} na_0 & a_0 & . & \cdots & . \\ (n-1)a_1 & a_1 & a_0 & \cdots & . \\ (n-2)a_2 & a_2 & a_1 & \cdots & . \\ . & . & . & \cdots & . \\ (n-r)a_r & a_r & a_{r-1} & \cdots & a_1 \end{vmatrix}$$

which is readily reduced to

$$(-1)^r \frac{1}{a_0^{\,r}} \begin{vmatrix} a_1 & a_0 & . & \cdots & . \\ 2a_2 & a_1 & a_0 & \cdots & . \\ . & . & . & \cdots & . \\ ra & a_{r-1} & a_{r-2} & \cdots & a_1 \end{vmatrix},$$

and so agrees with the result obtained from Newton's relations between the a's and s's.

* Said to be first given by Cauchy in his *Exercices de Math.* for 1826.

Here Faure leaves the subject, but he might equally easily have established Brioschi's more general result. Instead of specialising by putting $\psi(x)=x^r$ he might have made $\psi(x)=x^r\phi(x)f'(x)$ and so have got

$$\frac{x_1{}^r\phi(x_1)}{f'(x_1)}+\frac{x_2{}^r\phi(x_2)}{f'(x_2)}+\ldots\ldots+\frac{x_n{}^r\phi(x_n)}{f'(x_n)} = \text{coeff. of } x^{-r-1} \text{ in } \frac{\phi(x)}{f(x)}.$$

Bearing in mind that $\phi(x)$ as used by Brioschi was of the n^{th} degree, we have from Faure's fundamental theorem the said coefficient

$$= \mathrm{A}_{r+1} = (-1)^{r+1}\frac{1}{a_0^{r+1}}\begin{vmatrix} c_0 & a_0 & . & \ldots\ldots & . \\ c_1 & a_1 & a_0 & \ldots\ldots & . \\ . & . & . & . & . \\ c_r & a_r & a_{r-1} & \ldots\ldots & a_0 \\ c_{r+1} & a_{r+1} & a_r & \ldots\ldots & a_1 \end{vmatrix}$$

as it ought to be.

BRUNO, F. FAÀ DI (1855, December).

[Note sur une nouvelle formule du calcul différentiel. *Quart. Journ. of Math.*, i. pp. 359–360; or, with a different title, *Annali di Sci. mat. e fis.*, vi. pp. 479–480.]

The formula referred to is

$$\frac{\partial^{n+1}}{\partial x^{n+1}}\phi\{\psi(x)\} = \begin{vmatrix} \psi'\phi & n\psi''\phi & \frac{1}{2}n(n-1)\psi'''\phi & \ldots\ldots & \psi^{(n+1)}\phi \\ -1 & \psi'\phi & (n-1)\psi''\phi & \ldots\ldots & \psi^{(n)}\phi \\ . & -1 & \psi'\phi & \ldots\ldots & \psi^{(n-1)}\phi \\ . & . & -1 & \ldots\ldots & \psi^{(n-2)}\psi \\ . & . & . & . & . \\ . & . & . & \ldots\ldots & \psi'\phi \end{vmatrix}$$

where the coefficients in the r^{th} row are those of the expansion of $(a+b)^{n-r+1}$, and where after development of the determinant ϕ^r is to be taken as meaning the r^{th} differential-quotient of ϕ with respect to ψ.*

* An opportunity was here lost by Bruno of noting that a recurrent with the elements in its zero-bordered diagonal all negative has all its terms positive.

BRUNO, F. FAÀ DI (1856, February).

[Sulle funzioni isobariche. *Annali di Sci. mat. e fis.*, vii. pp. 76–89.]

On p. 81 Bruno enunciates, as having been recently ("ultimamente") discovered by him, a theorem which is essentially Faure's, the A_r of Faure being given by Bruno as the coefficient of x^r in the expansion of

$$\frac{c_0+c_1x+c_2x^2+\dots\dots}{a_0+a_1x+a_2x^3+\dots\dots}.$$

The reason for A_r being the same in both is evident on putting $m=n=0$ in Faure's.

ALLEGRET, A. (1857).

[Solutions de quelques problèmes curieux d'arithmétique. *Nouv. Annales de Math.*, xvi. pp. 136–139.]

In the course of Allegret's work, the determinant

$$\begin{vmatrix} 1 & . & . & . & 1 \\ 1 & a_1 & . & . & . \\ 1 & 1 & a_2 & . & . \\ 1 & . & 1 & a_3 & . \\ 1 & . & . & 1 & a_4 \end{vmatrix}$$

appears, which he says

$$= a_1a_2a_3a_4 + \begin{vmatrix} 1 & a_1 & . & . \\ 1 & 1 & a_2 & . \\ 1 & . & 1 & a_3 \\ 1 & . & . & 1 \end{vmatrix} = a_1a_2a_3a_4 - \begin{vmatrix} 1 & . & . & 1 \\ 1 & a_1 & . & . \\ 1 & 1 & a_2 & . \\ 1 & . & 1 & a_3 \end{vmatrix}$$

and, the four-line determinant now reached being similar in form to the original, he concludes that the final expansion of the latter must be

$$a_1a_2a_3a_4 - a_1a_2a_3 + a_1a_2 - a_1 + 1.$$

By passing the first row over the others to occupy the last place the determinant is recognisable as a special case of recurrent, and it is seen that the expansion in terms of the elements of the last row and their cofactors leads at once to Allegret's result.

BRIOSCHI, F. (1857).

[Solution de la question 350 (Wronski). *Nouv. Annales de Math.*, xvi. pp. 248–249.]

The problem having been set to find what Wronski called the "Aleph" functions * of the roots $x_1, x_2, \ldots, x_n$ of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

in terms of the coefficients, Brioschi begins by saying that the r^{th} of the said functions, being the complete homogeneous function of degree r, is the coefficient of z^r in the product

$$(1+x_1z+x_1^2z^2+\ldots)(1+x_2z+x_2^2z^2+\ldots)\ldots(1+x_nz+x_n^2z^2+\ldots)$$

i.e.

$$\frac{1}{(1-x_1z)(1-x_2z)\ldots(1-x_nz)}, \quad \text{or} \quad \frac{1}{\phi(z)} \text{ say.}$$

He thus has

$$\frac{1}{\phi(z)} = 1+\aleph_1z+\aleph_2z^2+\ldots.,$$

and therefore by differentiation

$$-\frac{\phi'(z)}{\phi(z)} = \frac{\aleph_1+2\aleph_2z+3\aleph_3z^2+\ldots}{1+\aleph_1z+\aleph_2z^2+\ldots}.$$

But having also by a well-known theorem

$$\begin{aligned}-\frac{\phi'(z)}{\phi(z)} &= \frac{x_1}{1-x_1z}+\frac{x_2}{1-x_2z}+\ldots+\frac{x_n}{1-x_nz}, \\ &= \left.\begin{array}{l} x_1+x_1^2z+x_1^3z^2+\ldots. \\ +x_2+x_2^2z+x_2^3z^2+\ldots. \\ +\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad. \\ +x_n+x_n^2z+x_n^3z^2+\ldots. \end{array}\right\} \\ &= s_1+s_2z+s_3z^2+\ldots.,\end{aligned}$$

* WRONSKI, H. *Introduction à la Philosophie des Mathématiques* . . . . . (pp. 65, . . .) vi+270 pp., Paris, 1811.

he deduces

$$s_1+s_2z+s_3z^2+\ldots = \frac{\aleph_1+2\aleph_2z+3\aleph_3z^2+\ldots\ldots}{1+\aleph_1z+\aleph_2z^2+\ldots\ldots},$$

whence by equating like coefficients of z there results

$$\begin{aligned} \aleph_1 &= s_1, \\ 2\aleph_2 &= s_2+\aleph_1s_1, \\ 3\aleph_3 &= s_3+\aleph_1s_2+\aleph_2s_1, \\ &\cdots\cdots\cdots \\ r\aleph_r &= s_r+\aleph_1s_{r-1}+\ldots\ldots+\aleph_{r-1}s_1. \end{aligned}$$

Multiplying now by $a_{r-1}, a_{r-2}, \ldots, a_0$ and adding, he has, on using Newton's relations between the a's and s's,

$$a_{r-1}\aleph_1+2a_{r-2}\aleph_2+\ldots\ldots+ra_0\aleph_r = -\{ra_r+(r-1)a_{r-1}\aleph_1+\ldots\ldots+a_1\aleph_{r-1}\},$$

whence comes Wronski's relation

$$a_r+a_{r-1}\aleph_1+\ldots\ldots+a_0\aleph_r = 0;$$

and, on solution of the set of relations obtained from this by putting $r=1, 2, \ldots, r$,

$$\aleph_r = \frac{(-1)^r}{a_0{}^r}\begin{vmatrix} a_1 & a_0 & \cdot & \cdots & \cdot \\ a_2 & a_1 & a_0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_0 \\ a_r & a_{r-1} & a_{r-2} & \cdots & a_1 \end{vmatrix}.$$

CATALAN, E. (1857).

[Note sur la question 350 (Wronski). *Nouv. Annales de Math.*, xvi. pp. 416–417.]

Catalan, under the anagrammatic signature of "M. Ange le Taunéac," points out a simplification. Having got as far as

$$\frac{1}{(1-x_1z)(1-x_2z)\ldots(1-x_nz)} = 1+\aleph_1z+\aleph_2z^2+\ldots$$

he merely draws attention to the fact that the denominator on the left being

$$= (a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n)\frac{1}{a_0},$$

there results

$$a_0 = (1 + \aleph_1 z + \aleph_2 z^2 + \ldots)(a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n),$$

whence Wronski's set of relations follows at once.

This, of course, is much preferable to Brioschi's procedure. It has to be noted, however, that by taking a roundabout way Brioschi came across the equations connecting the $\aleph$'s and the s's.*

* After all, it is this set of equations and the two other similar sets that are worth knowing, namely,

Newton's,	$a_0 s_r + a_1 s_{r-1} + a_2 s_{r-2} + \ldots + a_{r-1} s_1 + r a_r = 0.$
Wronski's,	$a_r + \aleph_1 a_{r-1} + \aleph_2 a_{r-2} + \ldots + \aleph_r a_0 = 0.$
Brioschi's,	$s_r + \aleph_1 s_{r-1} + \aleph_2 s_{r-2} + \ldots \aleph_{r-1} s_1 = r\aleph_r.$

CHAPTER VII.

WRONSKIANS, FROM 1838 TO 1860.

THE previous history of Wronskians being not at all lengthy, was included in the chapter on "Miscellaneous Special Forms" (*History*, i. chap. xvi.), and is to be found there under Wronski 1812, Wronski 1815, Wronski 1816–17, and Schweins 1825 (pp. 472–478, 482–485).

The name dates only from 1882, being first suggested on p. 224 of my Text-book on Determinants.*

LIOUVILLE, J. (1838).

[Note sur la théorie de la variation des constantes arbitraires. *Journ.* (*de Liouville*) *de Math.*, iii. pp. 342–347.]

The Wronskian which incidentally appears here is of a special kind, namely, that in which the originating functions are in the so-called relation of being first differential-quotients of one and the same function, for example, in later notation,

$$\begin{vmatrix} \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} & \dfrac{\partial x}{\partial c} \\ \dfrac{\partial^2 x}{\partial a\partial t} & \dfrac{\partial^2 x}{\partial b\partial t} & \dfrac{\partial^2 x}{\partial c\partial t} \\ \dfrac{\partial^3 x}{\partial a\partial t^2} & \dfrac{\partial^3 x}{\partial b\partial t^2} & \dfrac{\partial^3 x}{\partial c\partial t^2} \end{vmatrix}.$$

It is worthy of note also that the expression for the differential-

* MUIR, TH. *A Treatise on the Theory of Determinants*, . . . viii + 240 pp., London.

quotient of this with respect to t is obtained in the form which accords with the case of Schweins' theorem of 1825 (*Hist.*, i. p. 484).

$$Zd \left|\left| (Zd)^a A_1 \cdot (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-1} A_n \right)\right.$$

$$= \left|\left| (Zd)^a A_1 \cdot (Zd)^{a+1} A_2 \ldots\ldots (Zd)^{a+n-2} A_{n-1} \cdot (Zd)^{a+n} A_n \right)\right.$$

where $Z=1$ and $a=0$.

MALMSTEN, C. J. (1849).

[Moyens pour trouver l'expression de la n-ième intégrale particulière de l'équation $y^{(n)}+Py^{(n-1)}+\ldots+Sy^{(1)}+Ty=0$ à l'aide des $n-1$ valeurs $y_1, y_2, \ldots, y_{n-1}$ qui satisfont à celle équation. *Crelle's Journ.*, xxxix. pp. 91–98; or abstract in *Cambridge and Dub. Math. Journ.*, iv. pp. 286–288.]

The result obtained, after an introductory note on Determinants, is

$$y_n = z_1 y_1 + z_2 y_2 + \ldots + z_{n-1} y_{n-1},$$

where

$$z_r = (-1)^{n-1} \int \frac{dR}{dy_r^{(n-2)}} \cdot e^{-\int P dx} dx, \quad \text{and} \quad R = 1 \Big/ \sum \pm y_1 y_2' y_3'' \ldots y_{n-1}^{(n-2)}.$$

Only one special property of the Wronskian is used, namely, that regarding its differential-quotient.

MAINARDI, G. (1849, December).

[Sulla integrazione dell' equazioni differenziali. *Annali di Sci. mat. e fis.*, i. pp. 50–89.]

Mainardi in speaking unapprovingly (p. 70) of Libri's expressions for the coefficients of a linear differential equation in terms of its particular integrals gives his own instead, his statement being that

$$\sum_{r=1}^{r=m} (-1)^{m-r} y^{(r)} S\left(p_1' p_2'' p_3''' \ldots p_{r-1}^{(r-1)} p_r^{(r+1)} p_{r+1}^{(r+2)} \ldots p_{m-1}^{(m)}\right) = 0$$

is the linear differential equation of the m^{th} order whose particular integrals are $p_1, p_2, p_3, \ldots, p_m$. With the explanation that the upper indices represent differentiations and that S( ) stands for the same as Cauchy's S(±) a glance suffices to raise doubts as to the accuracy of the statement. For one thing p_m does not occur in the equation, and there are other objections equally serious; but in any case it is clear that Mainardi had in his mind the correct result, his process probably being in the case of

$$y''' + Ay'' + By' + Cy = 0$$

to solve the equations

$$\left.\begin{array}{l} p_1''' + Ap_1'' + Bp_1' + Cp_1 = 0 \\ p_2''' + Ap_2'' + Bp_2' + Cp_2 = 0 \\ p_3''' + Ap_3'' + Bp_3' + Cp_3 = 0 \end{array}\right\}$$

for A, B, C and then substitute in the original. This of course might be accomplished by eliminating A, B, C from the four equations, with the result

$$| \, y''' \, p_1'' \, p_2' \, p_3 \, | = 0,$$

just as when α, β, γ are the roots of $x^3 + Ax^2 + Bx + C = 0$ we have

$$| \, x^3 \, \alpha^2 \, \beta^1 \, \gamma^0 \, | = 0.$$

In the analogy, therefore, between linear differential equations and ordinary algebraical equations the Wronskian is the analogue of the alternant.*

It may be added that the identities given on the same page by Mainardi as examples of his discoveries in the general theory of determinants are included in an old identity of Desnanot's numbered xlv. in *History*, i. p. 145.

PUISEUX, V. (1851).

[Sur la ligne dont les deux courbures ont entre elles un rapport constant. *Journ. (de Liouville) de Math.*, vi. pp. 208–211.]

At the close of his paper Puiseux remarks that his proof would have been shortened by using the theorem, "*Les lettres*

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* We may appropriately note that in the same volume of the *Annali* are two short papers by P. Tardy on Malmsten's theorem just referred to (pp. 136–139, 337–341).

t, u, v, . . . , w *désignant* n *variables, si le déterminant du système de* n *quantités*

$$\begin{array}{cccc} t & u & \dots\dots & w \\ dt & du & \dots\dots & dw \\ d^2t & d^2u & \dots\dots & d^2w \\ \cdot & \cdot & \cdot & \cdot \\ d^{n-1}t & d^{n-1}u & \dots\dots & d^{n-1}w \end{array}$$

est égal à zéro, on a necessairement l'équation

$$at+bu+cv+\dots+gw = 0$$

où a, b, c, . . . , g *sont des constantes.*" The theorem is spoken of as known, but no reference is given.

TISSOT, A. (1852).

[Sur un déterminant d'intégrales définies. *Journ.* (*de Liouville*) *de Math.*, xvii. pp. 177–185.]

Tissot incidentally asserts that if $y, y_1, y_2, \dots, y_n$ all satisfy a linear differential equation similar to that dealt with by Malmsten above, then

$$\sum(\pm y y_1' y_2'' \dots y_n^{(n)}) = \gamma \epsilon^{-\int P dx}$$

where γ is independent of x. It is further stated that Liouville proved this in vol. x. of his journal, but such is not the case, the subject being not even referred to in that volume.

PROUHET, E. (1852).

[Mémoire sur quelques formules générales d'analyse. *Journ.* (*de Liouville*) *de Math.* (2), i. pp. 321–344.]

The third and last section (§§ 40–45) of Prouhet's memoir is headed "Théorèmes sur quelques déterminants de fonctions." The first theorem proved is that just mentioned as having been used by Malmsten. He then takes the set of $m+1$ equations,

$$\begin{array}{l} x_0\cdot d^0(u\phi^0) + x_1\cdot d^1(u\phi^0) + \dots + x_m\cdot d^m(u\phi^0) = d^{m+1}(u\phi^0) \\ x_0\cdot d^0(u\phi^1) + x_1\cdot d^1(u\phi^1) + \dots + x_m\cdot d^m(u\phi^1) = d^{m+1}(u\phi^1) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_0\cdot d^0(u\phi^m) + x_1\cdot d^1(u\phi^m) + \dots + x_m\cdot d^m(u\phi^m) = d^{m+1}(u\phi^m), \end{array}$$

and with the help of the said theorem obtains at once

$$x_m = \frac{d\Delta}{\Delta} = d(\log \Delta),$$

where Δ stands for

$$|\, d^0(u\phi^0) \cdot d^1(u\phi^1) \cdot \; \ldots \; \cdot d^m(u\phi^m) \,|.$$

Next, by using in connection with the same sets of equations the multipliers

$$\pm \phi^m, \quad \mp m\phi^{m-1}, \quad \ldots, \quad \tfrac{1}{2}m(m+1)\phi^2, \quad -m\phi^1, \quad \phi^0$$

and performing addition, the coefficients of $x_0, x_1, \ldots, x_{m-1}$ are found to vanish, with the result

$$\left\{x_m m!\, u(d\phi)^m\right\} = \tfrac{1}{2}m(m+1)\cdot m!\, u(d\phi)^{m-1}d^2\phi + (m-1)!\, du\cdot(d\phi^m),$$

this being due to the theorem in differentiation * that the expression of $m+1$ terms

$$d^r(u\phi^m) - m\phi\,.\,d^r(u\phi^{m-1}) + \tfrac{1}{2}m(m+1)\phi^2\,.\,d^r(u\phi^{m-2}) - \ldots.$$

has the values

$$0, \quad m!\, u(d\phi)^m, \quad \tfrac{1}{2}m(m+1)\cdot m!\, u(d\phi^{m-1})d^2\phi + (m+1)!\, du\cdot(d\phi^m)$$

according as $r < m$, $= m$, or $= m+1$. An alternative value for x_m is thus found, namely,

$$\tfrac{1}{2}m(m+1)\frac{d^2\phi}{d\phi} + (m+1)\frac{du}{u} \quad \text{i.e.} \quad d\log\left[u^{m+1}(d\phi)^{\frac{1}{2}m(m+1)}\right];$$

and from the two values it follows that

$$\Delta = u^{m+1}(d\phi)^{\frac{1}{2}m(m+1)} \times \text{a constant},$$

the constant being determined to be

$$\Sigma[\pm 1^0 2^1 3^2 \ldots (m+1)^m], \text{ or } 1!\,2!\,3!\ldots m!$$

by considering the particular case where $u = \phi = e^x$. The final result thus is

$$\begin{vmatrix} u\phi^0 & d(u\phi^0) & \ldots. & d^m(u\phi^0) \\ u\phi^1 & d(u\phi^1) & \ldots. & d^m(u\phi^1) \\ \cdot & \cdot & \cdot & \cdot \\ u\phi^m & d(u\phi^m) & \ldots. & d^m(u\phi^m) \end{vmatrix} = 1!\,2!\,3!\ldots m!\, u^{m+1}(d\phi)^{\frac{1}{2}m(m+1)},$$

* Attributed in part to Lexell (1772) and to Arbogast (1800).

which on putting $u=1$ becomes Wronski's result of the year 1816, and therefore a case of Schweins' generalisation of 1825.

A[BADIE, T.] (1852).

[Sur la différentiation des fonctions de fonctions: séries de Burmann, de Lagrange, de Wronski. *Nouv. Annales de Math.*, xi. pp. 376–383; or French translation of Brioschi's *Teorica dei Determinanti*, pp. 182–193.]

By a method similar to Prouhet's, namely, by solving a set of equations in two different ways, Abadie obtains

$$\frac{\dfrac{d^{n-1}}{dh^{n-1}}\Big[\theta^{-n}\mathrm{F}'\big(n+h\big)\Big]_{h=0}}{1\cdot 2\cdot 3\ldots n} = \frac{\sum[\pm \mathrm{D}^1\phi\cdot \mathrm{D}^2\phi^2\cdot\ \ldots\ \cdot \mathrm{D}^{n-1}\phi^{n-1}\cdot \mathrm{D}^n\mathrm{F}]}{\sum[\pm \mathrm{D}^1\phi\cdot \mathrm{D}^2\phi^2\cdot\ \ldots\ \cdot \mathrm{D}^{n-1}\phi^{n-1}\cdot \mathrm{D}^n\phi^n]}$$

where θ and $\mathrm{D}^r\phi^s$ stand for

$$\frac{\phi(x+h)-\phi(x)}{h} \quad \text{and} \quad \frac{d^r}{dx^r}\Big\{\phi(x)\Big\}^s$$

respectively. Thence, by equating coefficients of $\mathrm{D}^n\mathrm{F}$, Wronski's result

$$\sum[\pm \mathrm{D}^1\phi\cdot \mathrm{D}^2\phi^2\cdot\ \ldots\ \cdot \mathrm{D}^n\phi^n] = 1!\,2!\,3!\ldots n!\,\{\phi'(x)\}^{\frac{n(n+1)}{2}}$$

is reached; and this in its turn is then used to simplify the result which has just originated it, another of Wronski's formulæ being thus arrived at.

BRIOSCHI, F. (1855).

[Sur une propriété d'un déterminant fonctionnel. *Quart. Journ. of Math.*, i. pp. 365–367; or *Opere mat.*, v. pp. 389–392.]

In later phraseology the property in question is that *if the Wronskian,* W *say, of* $y_1, y_2, \ldots y_n$ *be a known function of the independent variable* x, *then any one of the* y's, y_r *say,* can be *expressed in terms of the others and* W. Denoting the s^{th} differential-quotient of y_r by $y_r^{(s)}$ we have of course

$$W \equiv \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1^{(1)} & y_2^{(1)} & \dots & y_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

and using "cof" for "cofactor of" we readily see that since

$$\frac{d}{dx}(\operatorname{cof} y_s^{(n-1)}) = -\operatorname{cof} y_s^{(n-2)}$$

we have

$$\frac{d}{dx}\left(\frac{\operatorname{cof} y_s^{(n-1)}}{\operatorname{cof} y_r^{(n-1)}}\right) = \frac{-\operatorname{cof} y_r^{(n-1)} \operatorname{cof} y_s^{(n-2)} + \operatorname{cof} y_s^{(n-1)} \operatorname{cof} y_r^{(n-2)}}{(\operatorname{cof} y_r^{(n-2)})^2}$$

$$= \frac{W.\operatorname{cof}(y_s^{(n-1)} y_r^{(n-2)})}{(\operatorname{cof} y_r^{(n-1)})^2}.$$

Integration of both sides with respect to x then gives

$$\operatorname{cof} y_s^{(n-1)} = \operatorname{cof} y_r^{(n-1)} \int \frac{W \,.\, \operatorname{cof}(y_s^{(n-1)} y_r^{(n-2)})}{(\operatorname{cof} y_r^{(n-1)})^2} dx;$$

and it is seen that for the cofactor of any element in the last row of W except the r^{th} there is an expression in which the function y_r never explicitly occurs. It only remains then to take the well-known identity

$$0 = y_1 \operatorname{cof} y_1^{(n-1)} + y_2 \operatorname{cof} y_2^{(n-1)} + \dots + y_r \operatorname{cof} y_r^{(n-1)} + \dots + y_n \operatorname{cof} y_n^{(n-1)},$$

write therein for $\operatorname{cof} y_s^{(n-1)}$ the substitute thus provided, and divide both sides by $\operatorname{cof} y_r^{(n-1)}$; for, this being done, there results

$$0 = y_1 I_1 + y_2 I_2 + \dots + y_r + \dots + y_n I_n$$

where I stands for the integral above written.

Brioschi then proceeds to establish Malmsten's theorem of 1849.

CHRISTOFFEL, E. B. (1857).

[Ueber die lineare Abhängigkeit von Functionen einer einzigen Veränderlichen. *Crelle's Journ.*, lv. pp. 281–299.]

The results directly bearing on the subject specified in the title of the paper are summed up in five propositions (pp. 293–294), the one comparable with Puiseux's of 1851 being that *If* $\Sigma \pm f(x) \cdot f_1'(x) \ldots f_n^{(n)}(x)$ *vanishes for all values of* x *from* $x = x_0$ *to* $x = x_1$, *where* $x_0 < x_1$, *then the functions* $f(x), f_1(x), \ldots, f_n(x)$ *are for those values linearly dependent.* In a concluding section are brought together (pp. 297–299) the properties of the determinants which had been used in reaching the said results. The first theorem is

$$\sum \pm u u_1' u_2'' \ldots u_n^{(n)} = \sum \pm u \cdot \Delta u' \cdot \Delta^2 u'' \ldots \Delta^n u^{(n)} \qquad (1)$$

where

$$\Delta u_\nu^{(\mu)} = u_{\nu+1}^{(\mu)} - u_\nu^{(\mu)}, \quad \Delta^2 u_\nu^{(\mu)} = \Delta u_{\nu+1}^{(\mu)} - \Delta u_\nu^{(\mu)}, \quad \ldots .$$

This, it is stated, can be proved in the same way as the theorem

$$\sum \pm f(m) \cdot f_1(m+1) \ldots f_n(m+n) = \sum \pm f(m) \cdot \Delta f_1(m) \ldots . \Delta^n f_n(m) \qquad (1')$$

given earlier in the paper (p. 293), namely, by evolving the left-hand side from the right-hand side after substituting for the differences in the latter their equivalents obtainable from the identity

$$\Delta^i f_k(m) = f_k(m+i) - i f_k(m+i-1) + \frac{i(i-1)}{1.2} f_k(m+i-2) - \ldots .$$

There is next derived the theorem

$$\sum \pm vu \cdot \Delta(vu') \cdot \Delta^2(vu'') \ldots \Delta^n(vu^{(n)})$$
$$= v v_1 v_2 \ldots v_n \cdot \sum \pm u \cdot \Delta u' \cdot \Delta^2 u'' \ldots . \Delta^n u^{(n)} \quad (2)$$

and from this again by putting $v_r = 1/u_r$ there is obtained

$$\sum \pm u \cdot \Delta u' \cdot \Delta^2 u'' \ldots \Delta^n u^{(n)} = u u_1 u_2 \ldots u_n \cdot \sum \pm \Delta\left(\frac{u'}{u}\right) \cdot \Delta^2\left(\frac{u''}{u}\right) \ldots \Delta^n\left(\frac{u^{(n)}}{u}\right).$$

Then, by using this last theorem on itself, and continuing in like manner, there is finally reached the result

$$\sum \pm u \cdot \Delta u' \cdot \Delta^2 u'' \ldots \Delta^n u^{(n)}$$
$$= (u u_1 \ldots u_n)(u^{10} u_1^{10} \ldots u_{n-1}^{10})(u^{21} u_1^{21} \ldots u_{n-2}^{21}) \ldots . (u^{n-1, n-2} u_1^{n-1, n-2}) u^{n, n-1} \quad (3)$$

where

$$u^{\mu 0} = \Delta \frac{u^{(\mu)}}{u}, \quad u^{\mu, 1} = \Delta \frac{u^{\mu, 0}}{u^{1, 0}}, \quad u^{\mu, 2} = \Delta \frac{u^{\mu, 1}}{u^{2, 1}}, \quad \ldots .$$

In order to pass from differences to differentials Christoffel then puts

$$u_\mu^{(\nu)} = f_\nu(m\epsilon+\mu\epsilon)\,, \quad v_\mu = \phi(m\epsilon+\mu\epsilon)\,, \quad m\epsilon = x\,, \quad \epsilon = \partial x\,,$$

the result obtained from (2) being

$$\sum \pm \phi f \cdot \frac{\partial \phi f_1}{\partial x} \cdot \frac{\partial^2 \phi f_2}{\partial x^2} \cdots \frac{\partial^n \phi f_n}{\partial x^n} = \phi^{n+1} \sum \pm f \cdot \frac{\partial f_1}{\partial x} \cdot \frac{\partial^2 f_2}{\partial x^2} \cdots \frac{\partial^n f_n}{\partial x^n} \quad (4)$$

and from (3) being

$$\sum \pm f \cdot \frac{\partial f_1}{\partial x} \cdot \frac{\partial^2 f_2}{\partial x^2} \cdots \frac{\partial^n f_n}{\partial x^n} = f^{n+1} \cdot f_{1,0}^n \cdot f_{2,1}^{n-1} \cdots f_{n,n-1} \quad (5)$$

where

$$f_{\mu,0} = \frac{\partial}{\partial x}\left(\frac{f_\mu}{f}\right), \quad f_{\mu,1} = \frac{\partial}{\partial x}\left(\frac{f_{\mu,0}}{f_{1,0}}\right), \quad f_{\mu,2} = \frac{\partial}{\partial x}\left(\frac{f_{\mu,1}}{f_{2,1}}\right), \quad \ldots .$$

There is next given a result dealing with change of variable, namely,

$$\sum \pm f \cdot \frac{\partial f_1}{\partial x} \cdot \frac{\partial^2 f_2}{\partial x^2} \cdots \frac{\partial^n f_n}{\partial x^n} = \left(\frac{\partial t}{\partial x}\right)^{\frac{1}{2}n(n+1)} \sum \pm f \cdot \frac{\partial f_1}{\partial t} \cdot \frac{\partial^2 f_2}{\partial t^2} \cdots \frac{\partial^n f_n}{\partial t^n}$$

which is proved like (1) and (1′), the identity used for substitution purposes being now

$$\frac{\partial^r f}{\partial x^r} = \frac{\partial^r f}{\partial t^r}\left(\frac{\partial t}{\partial x}\right)^r + a_1^{(r)} \frac{\partial^{r-1} f}{\partial t^{r-1}} + a_2^{(r)} \frac{\partial^{r-2} f}{\partial t^{r-2}} + \ldots$$

where $a_1^{(r)}$, $a_2^{(r)}$, . . . do not involve differential-quotients of f.

Lastly, denoting the cofactors of $f^{(n)}$, $f_1^{(n)}$, . . . , $f_n^{(n)}$ in

$$\sum \pm f \cdot \frac{\partial f_1}{\partial x} \cdot \frac{\partial^2 f_2}{\partial x^2} \cdots \frac{\partial^n f_n}{\partial x^n}$$

by W_0, W_1, . . . , W_n he announces the set of results

$$\begin{aligned}
\sum \pm W_0 W_1' W_2'' \ldots W_n^{(n)} &= W^n, \\
\sum \pm W_0 W_1' W_2'' \ldots W_{n-1}^{(n-1)} &= (-1)^n W^{n-1} f_n, \\
\sum \pm W_0 W_1' W_2'' \ldots W_{n-2}^{(n-2)} &= (-1)^{2n} W^{n-2} \cdot \sum \pm f_{n-1} f_n', \\
\sum \pm W_0 W_1' W_2'' \ldots W_{n-3}^{(n-3)} &= (-1)^{3n} W^{n-3} \cdot \sum \pm f_{n-2} f'_{-1} f_n'', \\
\cdots\cdots & \cdots\cdots \\
\sum \pm W_0 W_1' &= (-1)^{(n-1)n} W \cdot \sum \pm f_2 f_3' \ldots f_n^{(n-2)}, \\
W_0 &= (-1)^{nn} \sum \pm f_1 f_2' f_3'' \ldots f_n^{(n-1)}.
\end{aligned}$$

In regard to these we may note in passing that as

$$W_0, \quad W_1, \quad \ldots, \quad W_n$$

are themselves Wronskians, the name "Compound Wronskian" would not be inappropriate for the determinants on the left. Also, that in form the results bear a resemblance to those included in Jacobi's theorem regarding any minor of the adjugate of a general determinant.

HESSE, O. (1857).

[Ueber die Criterien des Maximums und Minimums der einfachen Integrale. *Crelle's Journ.*, liv. pp. 227–273; or *Werke*, pp. 413–467.]

In the course of his investigation Hesse pauses (p. 249) to enunciate the result which we have numbered (4) in dealing with Christoffel's paper. He also gives Christoffel's fifth result, having arrived at it, however, by a different method, namely, by the repeated use of the immediately preceding result. Thus, in the case of the 3rd order, the first part of his procedure would be

$$\begin{vmatrix} u & u' & u'' \\ v & v' & v'' \\ w & w' & w'' \end{vmatrix} = \begin{vmatrix} u\cdot 1 & (u\cdot 1)' & (u\cdot 1)'' \\ u\cdot\frac{v}{u} & \left(u\cdot\frac{v}{u}\right)' & \left(u\cdot\frac{v}{u}\right)'' \\ u\cdot\frac{w}{u} & \left(u\cdot\frac{w}{u}\right)' & \left(u\cdot\frac{w}{u}\right)'' \end{vmatrix} = u^3 \begin{vmatrix} 1 & (1)' & (1)'' \\ \frac{v}{u} & \left(\frac{v}{u}\right)' & \left(\frac{v}{u}\right)'' \\ \frac{w}{u} & \left(\frac{w}{u}\right)' & \left(\frac{w}{u}\right)'' \end{vmatrix}$$

$$= u^3 \begin{vmatrix} \left(\frac{v}{u}\right)' & \left(\frac{v}{u}\right)'' \\ \left(\frac{w}{u}\right)' & \left(\frac{w}{u}\right)'' \end{vmatrix};$$

and he would next treat this two-line determinant in similar fashion, the result being

$$\sum \pm uv'w'' = u^3\cdot\left\{\left(\frac{v}{u}\right)'\right\}^2\cdot\left\{\frac{\left(\frac{w}{u}\right)'}{\left(\frac{v}{u}\right)'}\right\}'.$$

MONTFERRIER, A. S. DE (1858).

[ENCYCLOPÉDIE MATHÉMATIQUE, ou exposition complète de toutes les branches des mathématiques d'après les principes . . . . de Hoëne Wronski. Première Partie: Mathématiques Pures. Tomes i.–iv., Paris.*]

Although, from the nature of this work, it cannot be expected to contain fresh results, it would be a mistake to undervalue it, as in the matter of exposition the disciple had more skill than his master. Whether, therefore, as a substitute for, or a commentary on, the original, it deserves attention. It is only in the *third* volume that the "Schin" functions appear, a short general account being given in §§ 1041, 1042 (pp. 423–428), and special instances dealt with under the headings "Les Séries" (§§ 953–960, pp. 267–276) and "La Loi Suprême" (§§ 1006–1023, pp. 358–391).

* None of the four volumes is dated, but they appeared in 1856-59: they do not complete the First Part.

CHAPTER VIII.

JACOBIANS, FROM 1841 TO 1860.

WHEN one recalls the exceptional fullness of Jacobi's memoir on this special form and the fact that it belongs really to the first year of the period now under discussion, one cannot be surprised that during the remainder of the period little was added save commentaries and suggested amendments. Even Bertrand's contribution, the longest and most interesting, is practically of this character.

JACOBI, C. G. J. (1844–1845).

[Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi. *Crelle's Journ.*, xxvii. pp. 199–268; xxix. pp. 218–279, 333–376; or *Math. Werke* (1846), i. pp. 47–226; or *Gesammelte Werke*, iv. pp. 317–509.]

The portion of this long memoir which is of interest to us in the present connection is the first section (pp. 201–209) of the first chapter, the heading being "Lemma fundamentale eiusque varii usus: de determinantibus functionalibus partialibus." Passing over the treatment of the first two cases of the lemma we come upon the general enunciation of it, which is—

If $\mathrm{A}, \mathrm{A}_1, \mathrm{A}_2, \ldots . \mathrm{A}_n$ *be the cofactors of* $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \ldots, \dfrac{\partial f}{\partial x_n}$ *in the determinant* $\sum\left(\pm\dfrac{\partial f}{\partial x}\dfrac{\partial f_1}{\partial x_1}\dfrac{\partial f_2}{\partial x_2}\ldots\dfrac{\partial f_n}{\partial x_n}\right)$, *then*

$$\frac{\partial \mathrm{A}}{\partial x}+\frac{\partial \mathrm{A}_1}{\partial x_1}+\frac{\partial \mathrm{A}_2}{\partial x_2}+\ldots+\frac{\partial \mathrm{A}_n}{\partial x_n} = 0.$$

Preparatory for the proof it is pointed out that since

$$\frac{\partial f}{\partial x}\mathrm{A} + \frac{\partial f}{\partial x_1}\mathrm{A}_1 + \ldots + \frac{\partial f}{\partial x_n}\mathrm{A}_n = \sum\left(\pm\frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}\right)$$

an alternative form of the lemma is

$$\frac{\partial(f\mathrm{A})}{\partial x} + \frac{\partial(f\mathrm{A}_1)}{\partial x_1} + \ldots + \frac{\partial(f\mathrm{A}_n)}{\partial x_n} = \sum\left(\pm\frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}\right).$$

Then calling the given determinant R, and noting that A, $\mathrm{A}_1, \ldots, \mathrm{A}_n$ are themselves functional determinants, A_i being the determinant of $f_1, f_2, \ldots, f_n$ with respect to $x, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, Jacobi seeks to prove the lemma true in the case of R from assuming it true in the case of A_i. To be able to formulate it in the latter case he takes each element of the first row of R along with each element of the second row, thus forming $(n+1)^2$ products whose cofactors in the determinant he denotes by

$$\begin{array}{ccccc} (00) & (01) & (02) & \ldots & (0n) \\ (10) & (11) & (12) & \ldots & (1n) \\ (20) & (21) & (22) & \ldots & (2n) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (n0) & (n1) & (n2) & \ldots & (nn), \end{array}$$

that is to say, he puts

$$(\iota\kappa) \text{ for the cofactor of } \frac{\partial f}{\partial x_i}\frac{\partial f_1}{\partial x_k} \text{ in R},$$

a notation which necessitates

$$(ik) = -(ki) \quad \text{and} \quad (ii) = 0.$$

It follows on this that the cofactors of

$$\frac{\partial f_1}{\partial x}, \ \frac{\partial f_1}{\partial x_1}, \ \ldots, \ \frac{\partial f_1}{\partial x_{i-1}}, \ \frac{\partial f_1}{\partial x_{i+1}}, \ \ldots, \ \frac{\partial f_1}{\partial x_n}$$

in A_i are $(i, 0), (i, 1), \ldots, (i, n)$, and thus the assumption above made takes the form

$$\frac{\partial(i, 0)}{\partial x} + \frac{\partial(i, 1)}{\partial x_1} + \ldots. + \frac{\partial(i, n)}{\partial x_n} = 0,$$

$$\text{or} \quad \frac{\partial\{(i, 0)f_1\}}{\partial x} + \frac{\partial\{(i, 1)f_1\}}{\partial x_1} + \ldots. + \frac{\partial\{(i, n)f_1\}}{\partial x_n} = \mathrm{A}_i.$$

Since, however, $(i, k)f_1 = -(k, i)f_1$ and $(i, i)f_1 = 0$, we can apply to the latter the general proposition that *if* a_{ik} *be any quantities whatever such that* $a_{ik} = -a_{ki}$, $a_{ii} = 0$, *and* H_i *stand for*

$$\frac{\partial a_{i0}}{\partial x} + \frac{\partial a_{i1}}{\partial x_1} + \ldots + \frac{\partial a_{in}}{\partial x_n}$$

then

$$\frac{\partial H}{\partial x} + \frac{\partial H_1}{\partial x_1} + \ldots + \frac{\partial H_n}{\partial x_n} = 0,*$$

and so reach the first of our aims as desired. There only then remains to show that the lemma holds in the case of two variables, and this is unnecessary because it is then identical with the familiar proposition

$$\frac{\partial f_1}{\partial x\, \partial y} = \frac{\partial f_1}{\partial y\, \partial x}.$$

In addition to this gradational proof Jacobi gives one of a different kind. Since A_i, he says, does not involve differential-coefficients with respect to x_i, it follows that $\frac{\partial A_i}{\partial x_i}$ and $\sum \frac{\partial A_i}{\partial x_i}$ cannot involve differential-coefficients taken twice with respect to any one variable. Further, second differential-coefficients taken with respect to different variables x_i, x_k cannot occur anywhere save in †

$$\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k}.$$

All we have got to show therefore is that the cofactor of $\frac{\partial^2 f_m}{\partial x_i \partial x_k}$ in $\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k}$ vanishes. To do this we express A_i in terms of the elements of one column and their cofactors, say

$$A_i = a_1 \frac{\partial f_1}{\partial x_k} + a_2 \frac{\partial f_2}{\partial x_k} + \ldots + a_n \frac{\partial f_n}{\partial x_k},$$

---

* The reason for this, of course, is that $\dfrac{\partial \frac{\partial a_{ik}}{\partial x_k}}{\partial x_i} + \dfrac{\partial \frac{\partial a_{ki}}{\partial x_i}}{\partial x_k} = 0.$

† It would have been well to make clear here that every term of the final expansion of $\sum \frac{\partial A_i}{\partial x_i}$ contains one and only one second differential-coefficient.

and thus know as above that

$$A_k = -\alpha_1 \frac{\partial f_1}{\partial x_i} - \alpha_2 \frac{\partial f_2}{\partial x_i} - \ldots\ldots - \alpha_n \frac{\partial f_n}{\partial x_i},$$

where $\alpha_1, \alpha_2, \ldots$ involve no differential-coefficients taken with respect to x_i or with respect to x_k.

The observation made in the course of the first proof that $A, A_1, \ldots, A_n$ are themselves functional determinants leads Jacobi to the conception of "partial functional determinants" on the analogy of partial differential-quotients. The fundamental lemma then becomes viewable as the analogue of

$$\frac{\partial \frac{\partial f_1}{\partial y}}{\partial x} - \frac{\partial \frac{\partial f_1}{\partial x}}{\partial y} = 0,$$

or, in Jacobi's words, "gravissimam manifestat analogiam determinantium functionalium et quotientium differentialium partialium."

Apparently this recalls to Jacobi another analogy of the same kind, which he had omitted to draw attention to in his paper of 1830, when the first two cases of the lemma had been originally enunciated by him. The proposition involving the said analogy he now generalises thus:—*If* $\mathrm{f}, \mathrm{f}_1, \mathrm{f}_2, \ldots, \mathrm{f}_n$ *be expressible as series the terms of which involve only powers of the variables* $\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$, *the functional determinant does not involve a term in* $\mathrm{x}^{-1}\mathrm{x}_1^{-1}\mathrm{x}_2^{-1}\ldots\mathrm{x}_n^{-1}$. In support of it he has only to point out that the functional determinant is equal to

$$\frac{\partial(fA)}{\partial x} + \frac{\partial(fA_1)}{\partial x_1} + \ldots + \frac{\partial(fA_n)}{\partial x_n},$$

and that the development of the k^{th} term of this expansion cannot contain a term in $1/x_{k-1}$.

After referring to a possible application of the lemma in connection with definite multiple integrals, Jacobi concludes § 2 by returning to the lemma itself and throwing it into a third form originally announced in 1841 (*De determ. funct.* § 9).

Viewing $x, x_1, x_2, \ldots. x_n$ as functions of $f, f_1, f_2, \ldots, f_n$ he obtains of course (*De determ. funct.* § 8)

$$\frac{\partial x}{\partial f}=\frac{\mathrm{A}}{\mathrm{R}},\quad \frac{\partial x_1}{\partial f}=\frac{\mathrm{A}_1}{\mathrm{R}},\quad \ldots.,\quad \frac{\partial x_n}{\partial f}=\frac{\mathrm{A}_n}{\mathrm{R}},$$

so that by substitution the lemma becomes

$$\begin{aligned}
0 &= \frac{\partial\left(\mathrm{R}\frac{\partial x}{\partial f}\right)}{\partial x}+\frac{\partial\left(\mathrm{R}\frac{\partial x_1}{\partial f}\right)}{\partial x_1}+\ldots.+\frac{\partial\left(\mathrm{R}\frac{\partial x_n}{\partial f}\right)}{\partial x_n},\\
&= \left.\begin{matrix}\frac{\partial x}{\partial f}\frac{\partial \mathrm{R}}{\partial x}\\ +\mathrm{R}\frac{\partial\frac{\partial x}{\partial f}}{\partial x}\end{matrix}\right\}+\left.\begin{matrix}\frac{\partial x_1}{\partial f}\frac{\partial \mathrm{R}}{\partial x_1}\\ \mathrm{R}\frac{\partial\frac{\partial x_1}{\partial f}}{\partial x_1}\end{matrix}\right\}+\ldots.+\left.\begin{matrix}\frac{\partial x_n}{\partial f}\frac{\partial \mathrm{R}}{\partial x_n}\\ \mathrm{R}\frac{\partial\frac{\partial x_n}{\partial f}}{\partial x_n}\end{matrix}\right\},\\
&= \frac{\partial \mathrm{R}}{\partial f}+\mathrm{R}\left\{\frac{\partial\frac{\partial x}{\partial f}}{\partial x}+\frac{\partial\frac{\partial x_1}{\partial f}}{\partial x_1}+\ldots.+\frac{\partial\frac{\partial x_n}{\partial f}}{\partial x_n}\right\},
\end{aligned}$$

or

$$0 = \frac{\partial \log \mathrm{R}}{\partial f}+\frac{\partial\frac{\partial x}{\partial f}}{\partial x}+\frac{\partial\frac{\partial x_1}{\partial f}}{\partial x_1}+\ldots.+\frac{\partial\frac{\partial x_n}{\partial f}}{\partial x_n},$$

where firstly R and the x's have to be viewed as functions of the f's and all differentiated with respect to f, and secondly the differential-coefficients thus obtained have to be expressed in terms of the x's preparatory to performing the final set of differentiations.

Here the consideration of functional determinants would have come to an end in the present memoir, had it not been that a theorem on the subject which had been given incorrectly in 1841 (*De determ. funct.* § 14) was now wanted in § 3 for use. Two and a half pages (pp. 215–217) of matter are consequently intercalated in order to enunciate the theorem correctly, to prove it, and to elucidate it by a commentary. The enunciation is—*If* $\mathrm{f}_1=a_1$, $\mathrm{f}_2=a_2$, $\ldots.$, $\mathrm{f}_n=a_n$, *where the a's are constants, the functional determinant*

$$\sum\left(\pm\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\right)$$

will not be altered in value if before performing the differentiations every function f_i *be transformed in any way whatever by means of the equations* $f_{i+1}=a_{i+1}$, $f_{i+2}=a_{i+2}$, ..., $f_n=a_n$. On looking back it will be seen that Jacobi had previously not excluded the use of the first $i-1$ equations in making transformation of f_i. His proof now depends on taking the a's to the same side as the f's; denoting the resulting equations by $\phi_1=0$, $\phi_2=0$, ..., $\phi_n=0$; applying his theorem (*De determ. funct.* § 10) to obtain the desired determinant in the form

$$(-1)^n\frac{\sum\left(\pm\frac{\partial\phi_1}{\partial x_1}\frac{\partial\phi_2}{\partial x_2}\cdots\frac{\partial\phi_n}{\partial x_n}\right)}{\sum\left(\pm\frac{\partial\phi_1}{\partial a_1}\frac{\partial\phi_2}{\partial a_2}\cdots\frac{\partial\phi_n}{\partial a_n}\right)};$$

and then showing that the denominator of this is $(-1)^n$. His commentary closes by assuming as allowable that

$$\phi_i = \lambda_1^{(i)}f_1 + \lambda_2^{(i)}f_2 + \ldots + \lambda_n^{(i)}f_n,$$

and having thus obtained

$$\sum\left(\pm\frac{\partial\phi_1}{\partial x_1}\frac{\partial\phi_2}{\partial x_2}\cdots\frac{\partial\phi_n}{\partial x_n}\right) = \sum(\pm\lambda_1^{(1)}\lambda_2^{(2)}\ldots\lambda_n^{(n)})\cdot\sum\left(\pm\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\right)$$

he concludes that the condition for the equality of the two functional determinants is that the determinant of the λ's shall be equal to 1.

HESSE, O. (1844).

[Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln. *Crelle's Journ.*, xxviii. pp. 68–96; or *Werke*, pp. 89–122.]

Having demonstrated that for the elimination of x_1, x_2, x_3 from the three quadrics f_1, f_2, f_3 it was important to discover a function of the third degree possessing certain properties, Hesse proceeds (§ 11) to show how such a function may be obtained.

From a well-known theorem regarding the differentiation of homogeneous functions he has, on putting $u_\kappa^{(\lambda)}$ for $\dfrac{\partial f_\kappa}{\partial x_\lambda}$,

$$\left.\begin{array}{l} x_1u_1^{(1)} + x_2u_1^{(2)} + x_3u_1^{(3)} = 2f_1 \\ x_1u_2^{(1)} + x_2u_2^{(2)} + x_3u_2^{(3)} = 2f_2 \\ x_1u_3^{(1)} + x_2u_3^{(2)} + x_3u_3^{(3)} = 2f_3 \end{array}\right\} ;$$

and, ϕ being Jacobi's determinant of f_1, f_2, f_3 with respect to x_1, x_2, x_3, there thus follows

$$(a)\ x_1\phi = 2f_1(u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2f_2(u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2f_3(u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)}),$$

with similar expressions for $x_2\phi$, $x_3\phi$; so that any set of values of x_1, x_2, x_3 which makes f_1, f_2, f_3 vanish will make ϕ vanish also. The formal enunciation of this result is then given, and it is pointed out that the like theorem holds when there are n homogeneous functions all of n variables and of the r^{th} degree.

From (a) by differentiation there is next obtained

$$x_1\frac{\partial\phi}{\partial x_1} + \phi$$

$$= 2u_1^{(1)}(u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2u_2^{(1)}(u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2u_3^{(1)}(u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)})$$

$$+ 2f_1\frac{\partial}{\partial x_1}(u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2f_2\frac{\partial}{\partial x_1}(u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2f_3\frac{\partial}{\partial x_1}(u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)})$$

and thence

$$x_1\frac{\partial\phi}{\partial x_1} - \phi$$

$$= 2f_1\frac{\partial}{\partial x_1}(u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2f_2\frac{\partial}{\partial x_1}(u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2f_3\frac{\partial}{\partial x_1}(u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)}$$

so that any set of values x_1, x_2, x_3 which makes the ternary quadrics f_1, f_2, f_3 vanish, will make the first differential-quotients of the determinant of f_1, f_2, f_3 vanish also,—a second theorem which is asserted to hold when the number of homogeneous functions is n, the number of variables n, and each function of the r^{th} degree in the said variables.

Combining the two results, and using later phraseology, we may say that *When* n n-*ary* r$^{\text{th}}$-*ics vanish, their Jacobian and each of its first differential-quotients will vanish also.*

The connection of this with the problem of elimination can be indicated in a few words. Jacobi's determinant ϕ being of the third degree in x_1, x_2, x_3, its first differential-quotients are like f_1, f_2, f_3 linear in x_1^2, x_2^2, x_3^2, x_2x_3, x_3x_1, x_1x_2; and consequently the resultant is at once obtained as a six-line determinant.

CAYLEY, A. (1847, February).

[On the differential equations which occur in dynamical problems. *Cambridge and Dub. Math. Journ.*, ii. pp. 210–219; or *Collected Math. Papers*, i. pp. 276–284.]

This is a short exposition of Jacobi's elaborate memoir of 1844 with considerable variation in the details. The portion (§ 1) which concerns us is of course that referring to the "fundamental lemma." This is established in its third form, the proof, like that originally given by Jacobi, being dependent on the theorem

$$\frac{\partial \mathrm{R}}{\partial \dfrac{\partial f_i}{\partial x_k}} = \mathrm{R}\frac{\partial x_k}{\partial f_i},$$

but differing in appearance, mainly because of the use of differentials.

BERTRAND, J. (1851, February).

[Mémoire sur le déterminant d'un système de fonctions. *Journ. (de Liouville) de Math.*, xvi. pp. 212–227; abstract in *Comptes Rendus .... Acad. des Sci.* (Paris), xxxii. pp. 134–135.]

Recalling how Jacobi had insisted on the marked analogy between a functional-determinant and a differential-coefficient, Bertrand at once intimates the adoption of a new definition of the former, which in his opinion makes the analogy still more striking, and from which the properties of the determinant are deducible like mere corollaries.

Save that Δ and δ are used where Bertrand without distinction uses d, the following is the definition:—If f_1, f_2, ..., f_n be functions of $x_1, x_2, \ldots, x_n$, and the latter receive n distinct sets of increments

$$\begin{array}{cccc} \Delta_1 x_1 & \Delta_1 x_2 & \ldots\ldots & \Delta_1 x_n \\ \Delta_2 x_1 & \Delta_2 x_2 & \ldots\ldots & \Delta_2 x_n \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \Delta_n x_1 & \Delta_n x_2 & \ldots\ldots & \Delta_n x_n \end{array}$$

with the result that the corresponding increments of the functions are

$$\begin{array}{cccc} \Delta_1 f_1 & \Delta_1 f_2 & \ldots\ldots & \Delta_1 f_n \\ \Delta_2 f_1 & \Delta_2 f_2 & \ldots\ldots & \Delta_2 f_n \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \Delta_n f_1 & \Delta_n f_2 & \ldots\ldots & \Delta_n f_n , \end{array}$$

then the limiting value of the ratio of the determinant of the second array to the determinant of the first array as the elements of the latter array are indefinitely diminished is called the determinant of the n functions. Since in the circumstances mentioned

$$\Delta_k f_i = \frac{\partial f_i}{\partial x_1} \Delta_k x_i + \frac{\partial f_i}{\partial x_2} \Delta_k x_2 + \ldots\ldots + \frac{\partial f_i}{\partial x_n} \Delta_k x_n$$

for all values of k and i not greater than n, it follows from the multiplication-theorem that the aforesaid limiting value is equal to

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \ldots\ldots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \ldots\ldots & \dfrac{\partial f_2}{\partial x_n} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \ldots\ldots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix};$$

and, this determinant being independent of the increments given to the independent variables, it is held that the definition is legitimised. It might have been added that the name assigned to the limiting value is also thereby justified.

The more important of Jacobi's results, eight or nine in number, are then re-established, precedence being given to those regarding the vanishing of the determinant. Supposing, first, that the functions are independent of one another, he asserts that $x_1, x_2, \ldots, x_n$ may be conceived as expressed in terms of $f_1, f_2, \ldots, f_n$; and, the latter being viewed as independent variables, the determinant of their increments can be considered as completely arbitrary and can thus have a value different from zero. Further, in relation to this determinant the determinant of the increments of $x_1, x_2, \ldots, x_n$ cannot be infinitely great, because the terms of both determinants have the same number of infinitesimal factors of the first order. It thus follows that their quotient—that is, the functional determinant—is not zero. Next, supposing that the functions are not all independent of one another, but that $f_{p+1}, f_{p+2}, \ldots, f_n$ are functions of $f_1, f_2, \ldots, f_p$, and that the latter alone are mutually independent, Bertrand asserts that we may suppose

$$\Delta_1 f_1 = 0, \quad \Delta_1 f_2 = 0, \quad \ldots, \quad \Delta_1 f_p = 0,$$

this in fact being possible in an infinite number of ways, because only p relations are thereby established between the increments $\Delta_1 x_1, \Delta_1 x_2, \ldots, \Delta_1 x_n$. It will then result that the increments of $f_{p+1}, f_{p+2}, \ldots, f_n$ being sums of multiples of the increments of $f_1, f_2, \ldots, f_p$ will also be zero; and thus the whole of the first row

$$\Delta_1 f_1 \quad \Delta_1 f_2 \quad \ldots \quad \Delta_1 f_n$$

will be composed of zeros, and the determinant to which it belongs will vanish. On the other hand the determinant of the increments of the x's can at the same time be made different from zero, the increments in the first row being not necessarily all zero and those in the other rows being what we please. The ratio of the two determinants therefore vanishes.

Following this are given the theorem regarding the relation of

$$\sum\left(\pm\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\right) \quad \text{to} \quad \sum\left(\pm\frac{\partial x_1}{\partial f_1}\frac{\partial x_2}{\partial f_2}\cdots\frac{\partial x_n}{\partial f_n}\right):$$

the theorem for finding the functional determinant when the f's are given *mediately* as functions of the x's, that is to say, as functions of

$$\phi_1(x_1, x_2, \ldots, x_n), \quad \phi_2(x_1, x_2, \ldots, x_n), \quad \ldots, \quad \phi_p(x_1, x_2, \ldots, x_n):$$

and the corresponding theorem when the functions are only *implicitly* given, that is to say, by means of connecting equations

$$\begin{array}{c} F_1(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_n) = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ F_n(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_n) = 0. \end{array}$$

The mode of treatment will be readily guessed from what has gone before.

The same cannot, however, be confidently affirmed in connection with the theorem which expresses the functional determinant as a single product. This is found grouped under the heading " Diverses formes que l'on peut donner à un déterminant" (fonctionnel), the said forms being obtainable on varying the systems of increments assigned to the variables. In the first example, the array of increments of the x's is taken to be

$$\begin{array}{cccc} \Delta_1 x_1 & 0 & \ldots\ldots & 0 \\ 0 & \Delta_2 x_2 & \ldots\ldots & 0 \\ \cdot & \cdot & \cdot\quad\cdot\quad\cdot & \cdot \\ 0 & 0 & \ldots\ldots & \Delta_n x_n \end{array}$$

which necessitates the other array being

$$\begin{array}{cccc} \dfrac{\partial f_1}{\partial x_1}\Delta_1 x_1 & \dfrac{\partial f_2}{\partial x_1}\Delta_1 x_1 & \ldots\ldots & \dfrac{\partial f_n}{\partial x_1}\Delta_1 x_1 \\ \dfrac{\partial f_1}{\partial x_2}\Delta_2 x_2 & \dfrac{\partial f_2}{\partial x_2}\Delta_2 x_2 & \ldots\ldots & \dfrac{\partial f_n}{\partial x_2}\Delta_2 x_2 \\ \cdot & \cdot & \cdot\quad\cdot\quad\cdot & \cdot \\ \dfrac{\partial f_1}{\partial x_n}\Delta_n x_n & \dfrac{\partial f_2}{\partial x_n}\Delta_n x_n & \ldots\ldots & \dfrac{\partial f_n}{\partial x_n}\Delta_n x_n \end{array}$$

and the ratio of the determinants of the arrays to be

$$\Sigma\left(\pm\frac{\partial f_1}{\partial x_1}\,\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\right).$$

To this there is added "C'est l'expression donnée comme définition par M. Jacobi,"—a remark, however, equally applicable when, as at the outset, the increments of the x's were taken in their most general form. Preparatory to the next example it is pointed out that any n of the variables

$$x_1, x_2, \ldots, x_n, \quad f_1, f_2, \ldots, f_n$$

may have arbitrary values, the other n being then determinable; and that therefore if $n-1$ of them be taken to be invariable, the ratios of the increments of the others may be considered known. We thus see that the two arrays

$$\begin{array}{ccccc|ccccc}
\Delta_1 x_1 & \Delta_1 x_2 & \Delta_1 x_3 & \ldots & \Delta_1 x_n & \Delta_1 f_1 & 0 & 0 & \ldots & 0 \\
0 & \Delta_2 x_2 & \Delta_2 x_3 & \ldots & \Delta_2 x_n & \Delta_2 f_1 & \Delta_2 f_2 & 0 & \ldots & 0 \\
0 & 0 & \Delta_3 x_3 & \ldots & \Delta_3 x_n & \Delta_3 f_1 & \Delta_3 f_2 & \Delta_3 f_3 & \ldots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \ldots & \Delta_n x_n & \Delta_n f_1 & \Delta_n f_2 & \Delta_n f_3 & \ldots & \Delta_n f_n
\end{array}$$

of the second example are simultaneously possible, the n independent variables in the case of the first row being $x_1, f_2, f_3, \ldots, f_n$, in the case of the second row $x_1, x_2, f_3, \ldots, f_n$, and so on: and we consequently learn that the functional determinant may be written in the form

$$\left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right)\cdots\left(\frac{\partial f_n}{\partial x_n}\right)$$

on the understanding that the brackets enclosing $\partial f_r/\partial x_r$ imply that f_r is there viewed as a function of $x_1, x_2, \ldots, x_r, f_{r+1}, f_{r+2}, \ldots, f_n$. A third pair of possible arrays is

$$\begin{array}{ccccccccc}
\Delta_1 x_1 & 0 & 0 & \ldots\ldots & 0 & \Delta_1 x_{m+1} & \Delta_1 x_{m+2} & \ldots\ldots & \Delta_1 x_n \\
0 & \Delta_2 x_2 & 0 & \ldots\ldots & 0 & \Delta_2 x_{m+1} & \Delta_2 x_{m+2} & \ldots\ldots & \Delta_2 x_n \\
0 & 0 & \Delta_3 x_3 & \ldots\ldots & 0 & \Delta_3 x_{m+1} & \Delta_3 x_{m+2} & \ldots\ldots & \Delta_3 x_n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \ldots\ldots & \Delta_m x_m & \Delta_m x_{m+1} & \Delta_m x_{m+2} & \ldots\ldots & \Delta_m x_n \\
0 & 0 & 0 & \ldots\ldots & 0 & \Delta_{m+1} x_{m+1} & 0 & \ldots\ldots & 0 \\
0 & 0 & 0 & \ldots\ldots & 0 & 0 & \Delta_{m+2} x_{m+2} & \ldots\ldots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 & \ldots\ldots & \Delta_n x_n
\end{array}$$

and

$$\begin{array}{ccccccccc}
\Delta_1 f_1 & \Delta_1 f_2 & \Delta_1 f_3 & \dots & \Delta_1 f_m & 0 & 0 & \dots & 0 \\
\Delta_2 f_1 & \Delta_2 f_2 & \Delta_2 f_3 & \dots & \Delta_2 f_m & 0 & 0 & \dots & 0 \\
\Delta_3 f_1 & \Delta_3 f_2 & \Delta_3 f_3 & \dots & \Delta_3 f_m & 0 & 0 & \dots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Delta_m f_1 & \Delta_m f_2 & \Delta_m f_3 & \dots & \Delta_m f_m & 0 & 0 & \dots & 0 \\
\Delta_{m+1} f_1 & \Delta_{m+1} f_2 & \Delta_{m+1} f_3 & \dots & \Delta_{m+1} f_m & \Delta_{m+1} f_{m+1} & \Delta_{m+1} f_{m+2} & \dots & \Delta_{m+1} f_n \\
\Delta_{m+2} f_1 & \Delta_{m+2} f_2 & \Delta_{m+2} f_3 & \dots & \Delta_{m+2} f_m & \Delta_{m+2} f_{m+1} & \Delta_{m+2} f_{m+2} & \dots & \Delta_{m+2} f_n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Delta_n f_1 & \Delta_n f_2 & \Delta_n f_3 & \dots & \Delta_n f_m & \Delta_n f_{m+1} & \Delta_n f_{m+2} & \dots & \Delta_n f_n,
\end{array}$$

from which the functional determinant is obtained in the form

$$\sum\left[\pm\left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right)\cdots\left(\frac{\partial f_m}{\partial x_m}\right)\right]\cdot\sum\left(\pm\frac{\partial f_{m+1}}{\partial x_{m+1}}\frac{\partial f_{m+2}}{\partial x_{m+2}}\cdots\frac{\partial f_n}{\partial x_n}\right)$$

where, from looking as before at corresponding rows of the two arrays, we see that in the first determinant $f_1, f_2, \dots, f_m$ are to be viewed as functions of $x_1, x_2, \dots, x_m, f_{m+1}, f_{m+2}, \dots, f_n$, and in the second determinant $f_{m+1}, f_{m+2}, \dots, f_n$ are to be viewed as functions of $x_1, x_2, \dots, x_n$.

The next section is still more interesting, as it concerns the proposition which Jacobi stated incorrectly in his original memoir of 1841 and returned to in 1844. The data according to Bertrand are the usual n functions $f_1, f_2, \dots, f_n$ each dependent on $x_1, x_2, \dots, x_n$, with the addition that the said functions when expressed in terms of $x_1, x_2, \dots, x_n, f_1, f_2, \dots, f_n$ become $\phi_1, \phi_2, \dots, \phi_n$: and the problem he sets himself is to find the relation between

$$\sum\left(\pm\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\right) \quad \text{and} \quad \sum\left(\pm\frac{\partial \phi_1}{\partial x_1}\frac{\partial \phi_2}{\partial x_2}\cdots\frac{\partial \phi_n}{\partial x_n}\right),$$

the differentiations in the latter determinant being performed on the understanding that the f's there occurring in the ϕ's are to be viewed as constants. The equations

$$\phi_1 - f_1 = 0, \quad \phi_2 - f_2 = 0, \quad \dots, \quad \phi_n - f_n = 0,$$

may be held to give implicitly the f's as functions of the x's, and therefore by a previous result

$$\sum\left(\pm\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\right) = (-1)^n \frac{\sum\pm\left(\frac{\partial\phi_1}{\partial x_1}\frac{\partial\phi_2}{\partial x_2}\cdots\frac{\partial\phi_n}{\partial x_n}\right)}{\begin{vmatrix} \frac{\partial\phi_1}{\partial f_1}-1 & \frac{\partial\phi_1}{\partial f_2} & \cdots & \frac{\partial\phi_1}{\partial f_n} \\ \frac{\partial\phi_2}{\partial f_1} & \frac{\partial\phi_2}{\partial f_2}-1 & \cdots & \frac{\partial\phi_2}{\partial f_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial\phi_n}{\partial f_1} & \frac{\partial\phi_n}{\partial f_2} & \cdots & \frac{\partial\phi_n}{\partial f_n}-1 \end{vmatrix}}$$

which is the relation desired. If ϕ_1 does not involve f_1, if ϕ_2 does not involve f_1 or f_2, if ϕ_3 does not involve f_1 or f_2 or f_3, and so on,* the determinant in the denominator takes the value $(-1)^n$, and the relation becomes one of equality.

The last section deals with the theorem regarding the change of variables in multiple integrals,—a theorem which in the ten years from Jacobi's memoir to Bertrand's had been discussed by Boole† and Dienger.‡

SPOTTISWOODE, W. (1851, 1853).

[ELEMENTARY THEOREMS RELATING TO DETERMINANTS, ... viii + 63 pp., London. Second edition, as an article in *Crelle's Journ.*, li. pp. 209–271, 328–381.]

Spottiswoode has a special chapter (§ x. pp. 51–57) headed "On Functional Determinants," its contents being a selection of Jacobi's theorems in unimproved form and a reprint of the first three paragraphs of Cayley's paper of 1847. In the second edition (§ ix. pp. 338–343) there is no change, save that the extract from Cayley is left out.

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* Or if f_n be not involved in ϕ_n, and neither f_n nor f_{n-1} involved in ϕ_{n-1}, and so on.

† *Cambridge Math. Journ.*, iv. (1843), pp. 20–28.

‡ *Archiv d. Math. u. Phys.*, x. (1847), pp. 417–421.

SYLVESTER, J. J. (1853, June).

[On a theory of the syzygetic relations .... *Philos. Transac. R. Soc.* (London), cxliii. pp. 407–548 ; or *Collected Math. Papers*, i. pp. 429–586.]

In the present connection the only interest of this long and important memoir lies in the fact that Sylvester at page 476 of it uses for the first time the term *Jacobian* and the symbolism $\mathrm{J}(f, g)$. His words are "J indicates the Jacobian of the given functions $f, g, \ldots$, meaning thereby the functional determinant of Jacobi."

CAUCHY, A. L. (1853, July).

[Mémoire sur les différentielles et les variations employées comme clefs algébriques. *Comptes rendus .... Acad. des Sci.* (Paris), xxxvii. pp. 38–45, 57–64 ; or *Œuvres complètes* (1), xii. pp. 46–63.]

The first section of the memoir opens, like his memoir of 1841, with the theorem

$$\mathrm{S}(\pm \mathrm{D}_{\mathrm{x}}x\ \mathrm{D}_{\mathrm{y}}y\ \mathrm{D}_{\mathrm{z}}z \ldots) \cdot \mathrm{S}(\pm \mathrm{D}_{x}\mathrm{x}\ \mathrm{D}_{y}\mathrm{y}\ \mathrm{D}_{z}\mathrm{z} \ldots) = 1,$$

and then proceeds to the consideration of the arrays

$$\begin{matrix} (aa) & (ab) & (ac) & \ldots \\ (ba) & (bb) & (bc) & \ldots \\ (ca) & (cb) & (cc) & \ldots \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\ , \end{matrix} \qquad \begin{matrix} [aa] & [ab] & [ac] & \ldots \\ [ba] & [bb] & [bc] & \ldots \\ [ca] & [cb] & [cc] & \ldots \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\ , \end{matrix}$$

in which any element (hk) of the first array stands for

$$\begin{vmatrix} \mathrm{D}_x h & \mathrm{D}_x k \\ \mathrm{D}_u h & \mathrm{D}_u k \end{vmatrix} + \begin{vmatrix} \mathrm{D}_y h & \mathrm{D}_y k \\ \mathrm{D}_v h & \mathrm{D}_v k \end{vmatrix} + \begin{vmatrix} \mathrm{D}_z h & \mathrm{D}_z k \\ \mathrm{D}_w h & \mathrm{D}_w k \end{vmatrix} + \ldots\ldots$$

and any element $[hk]$ of the second array stands for

$$\begin{vmatrix} \mathrm{D}_h x & \mathrm{D}_k x \\ \mathrm{D}_h u & \mathrm{D}_k u \end{vmatrix} + \begin{vmatrix} \mathrm{D}_h y & \mathrm{D}_k y \\ \mathrm{D}_h v & \mathrm{D}_k v \end{vmatrix} + \begin{vmatrix} \mathrm{D}_h z & \mathrm{D}_k z \\ \mathrm{D}_h w & \mathrm{D}_k w \end{vmatrix} + \ldots\ldots$$

and $a, b, c, \ldots.$ are $2n$ functions of two sets of n variables $x, y, z, \ldots, u, v, w, \ldots.$ Of course both arrays are recognised to be zero-axial and skew, and under certain conditions it is shown that the elements of the one are expressible in terms of those of the other, and that the product of the determinant of the two arrays is 1.

The language and notation of 'clefs algébriques' are used throughout, but nothing is brought forward as justification for so doing.

DONKIN, W. F. (1854, February).

[On a class of differential equations, including those which occur in dynamical problems, Part I. *Philos. Transac. R. Soc.* (London), cxliv. pp. 71–113.]

It is only the first four pages of Donkin's memoir that concern us, these being introductory and referring to properties of a set of n functions of n variables without any regard to possible connections with dynamics. Drawing attention at the outset to the analogy remarked on by Jacobi and Bertrand, he proposes to signalise it by denoting the determinant of $f_1, f_2, \ldots, f_n$ with respect to $x_1, x_2, \ldots, x_n$ by

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

Further, he views the numerator and denominator here as standing for the determinants of Bertrand's arrays of differences, remarking pointedly that the fraction indicated "is a real fraction, provided its numerator and denominator be interpreted in a manner exactly analogous to that in which the numerator and denominator of an ordinary total or partial differential-coefficient are interpreted."

Having thus explained his notation he proceeds to generalise the proposition that

$$\frac{\partial f_r}{\partial x_1}\frac{\partial x_1}{\partial f_s} + \frac{\partial f_r}{\partial x_2}\frac{\partial x_2}{\partial f_s} + \ldots + \frac{\partial f_r}{\partial x_n}\frac{\partial x_n}{\partial f_s} = 1 \text{ or } 0$$

according as r and s are equal or unequal. He recalls Jacobi's theorem (*De determin. funct.* § 11) that if $u_1, u_2, \ldots, u_m$ be functions of $y_1, y_2, \ldots, y_n$, n being greater than m, and the y's be functions of $x_1, x_2, \ldots, x_n$, then

$$\frac{\partial(u_{a_1} u_{a_2} \ldots u_{a_m})}{\partial(x_{\gamma_1} x_{\gamma_2} \ldots x_{\gamma_m})} = \sum_\beta \left\{ \frac{\partial(u_{a_1} u_{a_2} \ldots u_{a_m})}{\partial(y_{\beta_1} y_{\beta_2} \ldots y_{\beta_m})} \cdot \frac{\partial(y_{\beta_1} y_{\beta_2} \ldots y_{\beta_m})}{\partial(x_{\gamma_1} x_{\gamma_2} \ldots x_{\gamma_m})} \right\}$$

where $a_1, a_2, \ldots, a_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_m$ are fixed sets of m integers chosen from $1, 2, \ldots, n$, and $\beta_1, \beta_2, \ldots, \beta_m$ as any such set whatever. Taking then what he considers to be a particular case of this, namely,

$$\frac{\partial(y_{a_1} y_{a_2} \ldots y_{a_m})}{\partial(y_{\gamma_1} y_{\gamma_2} \ldots y_{\gamma_m})} = \sum_\beta \left\{ \frac{\partial(y_{a_1} y_{a_2} \ldots y_{a_m})}{\partial(x_{\beta_1} x_{\beta_2} \ldots x_{\beta_m})} \cdot \frac{\partial(x_{\beta_1} x_{\beta_2} \ldots x_{\beta_m})}{\partial(y_{\gamma_1} y_{\gamma_2} \ldots y_{\gamma_m})} \right\},$$

he points out that if the sets $a_1, a_2, \ldots, a_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_m$ be identical the determinant on the left is equal to 1, and that on the other hand if even one of the γ's be not included among the a's the determinant will have a column of zeros and therefore be zero itself. The generalisation aimed at thus is, that *If* $\mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{y}_\mathrm{n}$ *be functions of* $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}$, *and* m *be less than* n, *then*

$$\sum_\beta \left\{ \frac{\partial(y_{a_1} y_{a_2} \ldots y_{a_m})}{\partial(x_{\beta_1} x_{\beta_2} \ldots x_{\beta_m})} \cdot \frac{\partial(x_{\beta_1} x_{\beta_2} \ldots x_{\beta_m})}{\partial(y_{\gamma_1} y_{\gamma_2} \ldots y_{\gamma_m})} \right\} = 1 \text{ or } 0$$

according as the a *set of* m *integers chosen from* 1, 2, . . . , n *is identical or not with the* γ *set.* The illustrative example taken is the case where $n=2$, and the mode of stating it is that

$$\sum \left\{ \left(\frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} - \frac{\partial y_p}{\partial x_j} \frac{\partial y_q}{\partial x_i} \right) \left(\frac{\partial x_i}{\partial y_a} \frac{\partial x_j}{\partial y_\beta} - \frac{\partial x_i}{\partial y_\beta} \frac{\partial x_j}{\partial y_a} \right) \right\} = 1 \text{ or } 0$$

according as $a, \beta = p, q$ or not, it being understood that i, j is in succession

$$\begin{array}{llll} 1, 2; & 1, 3; & \ldots; & 1, n \\ & 2, 3; & \ldots; & 2, n \\ & & \cdots & \\ & & & n-1, n. \end{array}$$

DONKIN, W. F. (1854, February).

[Demonstration of a theorem of Jacobi's relative to functional determinants. *Cambridge and Dub. Math. Journ.*, ix. pp. 161–163.]

The theorem or identity in question is that of the year 1844. The functions being $u_1, u_2, \ldots, u_n$ and the independent variables $x_1, x_2, \ldots, x_n$, Donkin says that the functional determinant may be represented by

$$\begin{vmatrix} \frac{\partial_1}{\partial x_1} & \frac{\partial_1}{\partial x_2} & \cdots\cdots & \frac{\partial_1}{\partial x_n} \\ \frac{\partial_2}{\partial x_1} & \frac{\partial_2}{\partial x_2} & \cdots\cdots & \frac{\partial_2}{\partial x_n} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \frac{\partial_n}{\partial x_1} & \frac{\partial_n}{\partial x_2} & \cdots\cdots & \frac{\partial_n}{\partial x_n} \end{vmatrix} u_1 u_2 \ldots u_n,$$

it being understood that each symbol of differentiation is operative only upon that one of the functions which has the same suffix as the upper ∂ of the symbol. As a consequence of this he considers that the non-zero member of the identity sought to be established would be

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ \frac{\partial_2}{\partial x_1} & \frac{\partial_2}{\partial x_2} & \cdots\cdots & \frac{\partial_2}{\partial x_n} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \frac{\partial_n}{\partial x_1} & \frac{\partial_n}{\partial x_2} & \cdots\cdots & \frac{\partial_n}{\partial x_n} \end{vmatrix} u_2 u_3 \ldots u_n \qquad (A)$$

where the upper ∂'s of the first row being now without a suffix are supposed to be no longer restricted in their effect. As, however, the unrestricted symbol $\partial/\partial x_i$ is held to be equivalent to

$$\frac{\partial_2}{\partial x_i} + \frac{\partial_3}{\partial x_i} + \cdots\cdots + \frac{\partial_n}{\partial x_i}$$

the determinant operating on $u_2 u_3 \ldots u_n$ has the first element of each column equal to the sum of all the other elements of the column, and therefore vanishes. The identity is thus thought to be established.

In regard to this so-called demonstration we need only remark in passing (1) that the subject operated on is written in too product-like a form; (2) that an appropriate substitute for it would be $(u_1, u_2, \ldots, u_n)$, this being explained to be such that

$$\frac{\partial_r}{\partial x_s}(u_1, u_2, \ldots, u_n) = \frac{\partial u_r}{\partial x_s},$$

and

$$\frac{\partial}{\partial x_s}(u_1, u_2, \ldots, u_n) = \frac{\partial u_1}{\partial x_s} + \frac{\partial u_2}{\partial x_s} + \ldots + \frac{\partial u_n}{\partial x_s};$$

(3) that the assertion (A) is unsubstantiated, the fact

$$|a_1 b_2 c_3| = a_1|b_2 c_3| - a_2|b_1 c_3| + a_3|b_1 c_2|$$

being nothing more than a suggestion that

$$\begin{vmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

may be a suitable *abridged notation* for

$$\frac{\partial}{\partial x_1}|b_2 c_3| - \frac{\partial}{\partial x_2}|b_1 c_3| + \frac{\partial}{\partial x_3}|b_1 c_2|.$$

BRIOSCHI, F. (1854, March).

[La Teorica dei Determinanti, e le sue principali Applicazioni. viii+116 pp. Pavia.]

Like Spottiswoode, Brioschi devotes his tenth chapter or section (§ 10) to "determinanti delle funzioni"; but his exposition is much more extensive (pp. 84–106), and, although of course he follows in Jacobi's footsteps, he does so less closely than Spottiswoode.

Thus the fact that the cofactor of $\dfrac{\partial f_i}{\partial x_k}$ in R is $R\dfrac{\partial x_k}{\partial f_i}$, a fact which we may write temporarily in the form

$$\left[\frac{\partial f_i}{\partial x_k}\right] = R\frac{\partial x_k}{\partial f_i},$$

he obtains by solving $n+1$ sets of equations like

$$\left.\begin{array}{l} \frac{\partial x}{\partial f}\frac{\partial f}{\partial x}+\frac{\partial x}{\partial f_1}\frac{\partial f_1}{\partial x}+\ldots\ldots+\frac{\partial x}{\partial f_n}\frac{\partial f_n}{\partial x} = 1 \\ \frac{\partial x}{\partial f}\frac{\partial f}{\partial x_1}+\frac{\partial x}{\partial f_1}\frac{\partial f_1}{\partial x_1}+\ldots\ldots+\frac{\partial x}{\partial f_n}\frac{\partial f_n}{\partial x_1} = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \frac{\partial x}{\partial f}\frac{\partial f}{\partial x_n}+\frac{\partial x}{\partial f_1}\frac{\partial f_1}{\partial x_n}+\ldots\ldots+\frac{\partial x}{\partial f_n}\frac{\partial f_n}{\partial x_n} = 0 \end{array}\right\}$$

and by using the same sets of equations after row-by-row multiplication he obtains

$$\sum\left(\pm\frac{\partial x}{\partial f}\frac{\partial x_1}{\partial f_1}\cdots\frac{\partial x_n}{\partial f_n}\right)\cdot\sum\left(\pm\frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}\right) = 1,$$

$$\text{or, say,}\quad \mathrm{S}\cdot\mathrm{R} = 1.$$

Further, he notes that as a consequence of these two theorems there results

$$\left[\frac{\partial f_i}{\partial x_k}\right]\cdot\left[\frac{\partial x_i}{\partial f_k}\right] = \frac{\partial x_k}{\partial f_i}\cdot\frac{\partial f_k}{\partial x_i},$$

which he might well have generalised by changing the i, k of the second factor of both sides into r, s.

In dealing with the "fundamental lemma" his order of procedure is the reverse of Jacobi's, that is to say, he deduces the form of 1844 from the original form of 1841. Thus, using the latter in regard to S, he has

$$\frac{\partial \mathrm{S}}{\partial f_k} = \mathrm{S}\cdot\sum_{i=0}^{i=n}\frac{\partial\frac{\partial x_i}{\partial f_k}}{\partial x_i},$$

whence, because of R being the reciprocal of S, he obtains

$$\frac{\partial \mathrm{R}}{\partial f_k} + \mathrm{R}\cdot\sum_{i=0}^{i=n}\frac{\partial\frac{\partial x_i}{\partial f_k}}{\partial x_i} = 0;$$

so that on substituting

$$\sum_{i=0}^{i=n}\frac{\partial \mathrm{R}}{\partial x_i}\frac{\partial x_i}{\partial f_k}\quad\text{for}\quad\frac{\partial \mathrm{R}}{\partial f_k}$$

there results

$$\sum_{i=0}^{i=n} \frac{\partial\left(\mathrm{R}\dfrac{\partial x_i}{\partial f_k}\right)}{\partial x_i} = 0,$$

which on further substituting $\left[\dfrac{\partial f_k}{\partial x_i}\right]$ for $\mathrm{R}\dfrac{\partial x_i}{\partial f_k}$ becomes the form desired.

The next fresh paragraph (p. 91) appears, although unnecessarily, as an addendum to Jacobi's solution of a set of simultaneous linear equations whose determinant is a functional determinant (*De determ. funct.* § 8). If the square of R be obtained by row-by-row multiplication, and the square of S by column-by-column multiplication it is easily verified that

$$(h^{\text{th}} \text{ row of } \mathrm{S}^2) \times (k^{\text{th}} \text{ column of } \mathrm{R}) = \frac{\partial x_k}{\partial f_h},$$

$$\text{i.e.} \qquad = (k, h)^{\text{th}} \text{ element of } \mathrm{S},$$

thus incidentally giving $\mathrm{S}^2\mathrm{R} = \mathrm{S}$ as it should do. From this it is deduced that

$$(h^{\text{th}} \text{ row of } \mathrm{S}^2) \times (k^{\text{th}} \text{ row of } \mathrm{R}^2) = 1 \text{ or } 0$$

according as h and k are equal or unequal,* and that therefore R^2 and S^2 as just defined are in the matter of their primary minors related as R and S have been shown to be.

In the remaining fourteen pages (pp. 92–106) the only matter calling for attention concerns Jacobi's theorem

$$\sum(\pm b b_1^{(1)} \ldots b_m^{(m)}) = \mathrm{B}^m \cdot \sum\left(\pm \frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_{n+m}}{\partial x_{n+m}}\right)$$

where

$$\mathrm{B} = \sum\left(\pm \frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_{n-1}}{\partial x_{n-1}}\right) \text{ and } b_k^{(i)} = \sum\left(\pm \frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_{n-1}}{\partial x_{n-1}}\frac{\partial f_n}{\partial x_n}\right)$$

From this Brioschi, by taking the f's to be linear functions of the x's, obtains Sylvester's theorem of March 1851 regarding a compound determinant.

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* Viewing R and S as matrices of which the conjugates are $\overline{\mathrm{R}}$ and $\overline{\mathrm{S}}$ we have as an equivalent of this

$$\overline{\mathrm{S}}\mathrm{S} \cdot \overline{\mathrm{R}}\mathrm{R} = \overline{\mathrm{S}} \cdot \mathrm{SR} \cdot \overline{\mathrm{R}}$$
$$= \overline{\mathrm{S}}\,\overline{\mathrm{R}} = 1.$$

PEIRCE, B. (1855).

[A SYSTEM OF ANALYTICAL MECHANICS. (Chap. X. i.: Determinants and Functional Determinants, pp. 172–198.) xl+496 pp., Boston (U.S.A.).]

At the outset of his Tenth Chapter, which deals with the integration of the differential equations of motion, Peirce feels the need for making his reader acquainted with the properties of functional determinants. He accordingly gives as a preparation a brief account (§§ 327–348, pp. 172–183) of determinants in general, and then expounds within the space of sixteen broad-margined pages the main theorems of Jacobi's 'De determinantibus functionalibus.' The treatment of the original is free and masterly, the order being altered with good effect. For example, Jacobi's incorrectly stated proposition is brought forward to occupy the second place, the enunciation being *If either* (i.e. *any one*) *of the given functions contains any of the other functions, these* (*latter*) *functions may be regarded as constant in finding the functional determinant.* There is thence deduced Jacobi's last proposition of all, namely, that expressing the determinant as a single product: and this in turn is used to discuss the connection between the vanishing of the determinant and the interdependence of the functions.

Had Peirce's exposition been less condensed and been published as part of an ordinary text-book of determinants, its value at that time to English-speaking students would have been considerable.

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinanti. *Memorie .... Istituto Veneto* .... vii. pp. 67–144.]

To the subject of a "Determinante formato colle derivate-prime di alquante funzioni di altrettanti variabili" Bellavitis devotes nine and a half pages (pp. 52–61, §§ 65–78), that is to say, about the same as Spottiswoode, though his selection of theorems is not quite the same. In substance he gives nothing fresh. His symbolism for the determinant of $u, v, \ldots$ with respect to $x, y, \ldots$ resembles Cauchy's of 1841, being

$$| \mathrm{D}_x u, \ \mathrm{D}_y v, \ldots | ;$$

other changes made by him in notation are less satisfactory.

BALTZER, R. (1857).

[Theorie und Anwendung der Determinanten, mit . . . . . vi+129 pp. Leipzig, 1857.]

"Die Functionaldeterminante" is the heading of Baltzer's thirteenth chapter or section (§ 13, pp. 61–72). Though the exposition is neither so full nor so fresh as Brioschi's, it has the advantage in arrangement, concision and clearness. Jacobi's last theorem (*De determ. funct.* § 18), expressing the determinant as a single product,

$$\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right)\cdots\left(\frac{\partial f_n}{\partial x_n}\right),$$

Baltzer makes his first, the proof being readily altered to suit. He then, following Peirce, uses it effectively in dealing with the proposition regarding the vanishing of the determinant. For example, if the determinant vanishes, he can assert that one of the factors of the said product must vanish; and thence step-by-step can infer the vanishing of the succeeding factors including the last,—a result which entails f_n being expressible in terms of the other f's.

A footnote recalls the fact, which we should have noted before this, that Möbius had given in *Crelle's Journ.*, xii. p. 116, in the year 1834, the equation

$$(t_x u_y - t_y u_x)(v_t w_u - v_u w_t)(x_v y_w - x_w y_v) = 1,$$

where t_x stands for $\partial t/\partial x$.

MALMSTEN, C. J. (1858, October).

[Om differential-eqvationers integrering. *K. Svenska Vet.-Akad. Handl.* (Stockholm), iii. No. 2, 94 pp.]

On pp. 9–11 Malmsten enunciates and proves Jacobi's "fundamental lemma" of 1844 without contributing any improvement.

SALMON, G. (1859).

[LESSONS INTRODUCTORY TO THE MODERN HIGHER ALGEBRA. xii + 147 pp., Dublin.]

Salmon gives little, and certainly nothing fresh, on the subject; but his unreserved adoption of Sylvester's word "Jacobian" (§§ 53, 54; p. 37) doubtless helped greatly to spread the usage.

CHAPTER IX.

SKEW DETERMINANTS, FROM 1846 TO 1860.

UNLIKE the special form of the preceding chapter, Skew Determinants received little attention in our first volume, even although in their case the period was extended to 1845 in order to include the whole of Jacobi's work. Unless by implication, indeed, they do not belong to that volume at all, the chapter there assigned to them being really occupied with the related functions afterwards named Pfaffians when the connection between the two came to be recognised. The new form is thus strictly viewable as one of the products of the Cayley-Sylvester period.

CAYLEY, A. (1846).

[Sur quelques propriétés des déterminants gauches. *Crelle's Journ.*, xxxii. pp. 119–123; or *Collected Math. Papers*, i. pp. 332–336.]

This paper, with its author's usual directness, starts at once with a definition, the first words being—

"Je donne le nom de *déterminant gauche* à un déterminant formé par un système de quantités $\lambda_{r,s}$ qui satisfont aux conditions

$$\lambda_{r,s} = - \lambda_{s,r} \quad (r \neq s).$$

J'appelle aussi un tel système, *système gauche.*"

So far as can be ascertained, the English equivalent '*skew*,' although it probably was the first of the two in order of thought, did not appear in print until a few years later.

As has been pointed out elsewhere, the title of the paper is quite misleading, the real subject being *the construction of a linear substitution for the transformation of* $x_1^2+x_2^2+x_3^2+\ldots$ *into* $\xi_1^2+\xi_2^2+\xi_3^2+\ldots$ All that can be said in defence of the inaccuracy is that skew determinants are made use of in obtaining the desired substitution. The proper place for giving an account of the contents of the paper is thus under the heading of '*orthogonants*,' if we may so name the *determinants of an orthogonal substitution*.

CAYLEY, A. (1847).

[Sur les déterminants gauches. *Crelle's Journ.*, xxxviii. pp. 93–96; or *Collected Math. Papers*, i. pp. 410–413.]

Here the title and contents agree. At the outset the former definition is repeated, and then for a particular kind of skew determinant, viz., those in which the condition

$$\lambda_{r,s} = -\lambda_{s,r} \tag{1}$$

is to hold even in the case where s and r are equal, "ou pour lesquels on a

$$\lambda_{r,s} = -\lambda_{s,r} \quad (r \neq s), \qquad \lambda_{r,r} = 0,\text{"} \tag{2}$$

the name 'skew symmetric' ("gauche et symétrique") is set apart. The reason for this is evident on the statement of the first theorem, which is to the effect that any skew determinant is expressible in terms of skew symmetric determinants and those elements of the original determinant which are not included in the latter. "En effet," he explains,

"soit Ω le déterminant gauche dont il s'agit, cette fonction peut être présentée sous la forme

$$\Omega = \Omega_0 + \Omega_1\lambda_{11} + \Omega_2\lambda_{22} + \ldots + \Omega_{12}\lambda_{11}\lambda_{22} + \ldots$$

où Ω_0 est ce que devient Ω si $\lambda_{11}, \lambda_{22}, \ldots$ sont réduits à zéro, Ω_1 est ce que devient le coefficient de λ_{11} sous la même condition, et ainsi de suite; c'est à dire, Ω_0 est le déterminant formé par les quantités $\lambda_{r,s}$ en supposant que ces quantités satisfassent aux conditions (2) et en donnant à r, s le valeurs 1, 2, 3, ..., n; Ω_1 est le déterminant formé pareillement en donnant à r, s les valeurs 2, 3, ..., n; Ω_2

s'obtient en donnant à r, s les valeurs 1, 3, . . . , n; et ainsi de suite; cela est aisé de voir si l'on range les quantités $\lambda_{r,s}$ en forme de carré."

At this point a digression is made in order to establish a theorem regarding skew determinants of odd order, and another regarding skew determinants of even order, and thus be enabled to make certain substitutions for the Ω's in the development here announced. Further, as the said substitutions for the Ω's of even order involve the functions dealt with by Jacobi in his paper on the "Pfaffsche Methode,"—functions which Cayley here calls "les fonctions de M. Jacobi," but which at a later date he designated "*Pfaffians*,"—the digression is lengthened by having prefixed to it an account of these functions.

So curious is this account and so likely to be misrepresented by condensation, that the best way of treating it is to reproduce it in the original words.* It stands thus:—

"On obtient ces fonctions (dont je reprends ici la théorie) par les propriétés générales d'un déterminant défini. Car en exprimant par $(1, 2, \ldots, n)$ une fonction quelconque dans laquelle entrent les nombres symboliques $1, 2, \ldots, n$, et par $\pm$ le signe correspondant à une permutation quelconque de ces nombres, la fonction

$$\sum \pm (1\ 2 \ldots n)$$

où $\sum$ désigne la somme de tous les termes qu'on obtient en permutant ces nombres d'une manière quelconque est ce qu'on nomme *Déterminant*. On pourrait encore généraliser cette définition en admettant plusieurs systèmes de nombres $1, 2 \ldots, n$; $1', 2' \ldots, n'$; . . . qui alors devroient être permutés independamment les uns des autres; on obtiendrait de cette manière une infinité d'autres fonctions, mentionnées (T. xxx. p. 7). Dans le cas des déterminants ordinaires, auquel je ne m'arrêterai pas ici, on aura $(1, 2 \ldots n) = \lambda_{\alpha,1}\lambda_{\beta,2} \ldots \lambda_{\kappa,n}$. Pour les cas des fonctions dont il s'agit (les fonctions de M. Jacobi), on supposera n *pair*, et l'on écrira

$$(1\ 2 \ldots n) = \lambda_{1,2}\lambda_{3,4} \ldots \lambda_{n-1,n},$$

où $\lambda_{r,s}$ sont des quantités quelconques qui satisfont aux équations (1). La fonction sera composée d'un nombre $1 . 2 \ldots n$ de termes; mais parmi eux il n'y aura que $1 . 3 \ldots (n-1)$ termes différents qui se trouveront répétés $2^{\frac{1}{2}n}(1 . 2 \ldots \frac{1}{2}n)$ fois, et qu'on obtiendra en permutant cycliquement d'abord les $n-1$ derniers nombres, puis les

* The paper, as it appears in *Crelle's Journal*, is disfigured by misprints, which have not been fully corrected in the *Collected Math. Papers*.

$n-3$ derniers nombres de chaque permutation, et ainsi de suite; le signe étant toujours $+$. Il pourra être démontré, comme pour les déterminants, que ces fonctions changent de signe en permutant deux quelconques des nombres symboliques, et qu'elles s'evanouissent si deux de ces nombres deviennent identiques. De plus, en exprimant par $[12\ldots n]$ la fonction dont il s'agit, la règle qui vient d'être énoncé, donnera pour la formation de ces fonctions:

$$[1\ 2\ldots n] = \lambda_{12}[3\ 4\ldots n]+\lambda_{13}[4\ldots n,\ 2] + \ldots\ldots\ldots + \lambda_{1n}[2\ 3\ldots n-1].\text{''}$$

Dismissing, as not of present interest, the sentence regarding the generalisation obtained by admitting more than one system of symbolic numbers, we note first of all the peculiar general use of $(1\,2\ldots n)$ for any function the expression of which involves* as suffixes or otherwise the numbers $1, 2, 3, \ldots, n$. Then we are struck with the fact that the use of this along with $\Sigma\pm$ gives a notation for a genus of functions of which determinants, as understood up to the date of the paper, formed a species: thus

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

is the case of $\Sigma\pm(123)$, where $(123)=a_1b_2c_3$. In the third place we are surprised to find that Cayley seems to propose to extend the meaning of the word *determinant* by transferring the name of the species to the genus, and to call by the name of "ordinary determinants" the functions formerly known as "determinants" merely.

All this is in itself comparatively unimportant, serving perhaps only to recall to us Cauchy's famous paper of 1812, where we have K, the originating term of an alternating function to compare and contrast with Cayley's $(12\ldots n)$, and 'alternating function' to compare and contrast with Cayley's extended meaning of 'determinant.' But what follows by way of second example is very noteworthy, because the originating term taken, viz., $\lambda_{12}\lambda_{34}\ldots\lambda_{n-1,n}$, is one that could not possibly have been used by Cauchy, with whom Σ denoted an operation of a much less simple character than permutation of the integers $1, 2, \ldots, n$.

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*Apparently it is meant to be implied that each of the numbers occurs only once in the expression.

Unfortunately the example is not fully exploited.* We are only told that in a certain special case, viz., where the elements are such that rs is always equal to $-sr$, there are only $1.3.5\ldots(2n-1)$ different terms in

$$\sum \pm \lambda_{12}\lambda_{34}\ldots\lambda_{2n-1,2n};$$

---

* Supplying this want we see that in strict accordance with Cayley's definition

$$\begin{array}{rllll}
\sum \pm 12\cdot34 = & 12\cdot34 & +\ 21\cdot43 & +\ 31\cdot24 & +\ 41\cdot32 \\
 & -\ 12\cdot43 & +\ 23\cdot14 & -\ 31\cdot42 & +\ 42\cdot13 \\
 & -\ 13\cdot24 & -\ 23\cdot41 & -\ 32\cdot14 & -\ 42\cdot31 \\
 & +\ 13\cdot42 & -\ 24\cdot13 & +\ 32\cdot41 & -\ 43\cdot12 \\
 & +\ 14\cdot23 & +\ 24\cdot31 & +\ 34\cdot12 & +\ 43\cdot21 \\
 & -\ 14\cdot32 & & -\ 34\cdot21 & \\
 & -\ 21\cdot34 & & -\ 41\cdot23 &
\end{array}$$

$$\begin{aligned}
= 2\{\, & 12\cdot34 - 12\cdot43 - 13\cdot24 + 13\cdot42 \\
& + 14\cdot23 - 14\cdot32 - 21\cdot34 + 21\cdot43 \\
& - 23\cdot41 + 24\cdot31 - 31\cdot42 + 32\cdot41\},
\end{aligned}$$

—a function of twelve variables which is not a determinant in the acceptation either of the present time or of the time preceding Cayley.

It is instructive, in connection with the matter in hand, to note that this function is expressible in terms of four Pfaffians, namely, we have

$$\begin{aligned}
\sum \pm 12\cdot34 = 2\Bigg\{ & \left|\begin{matrix} 12 & 13 & 14 \\ & 23 & 24 \\ & & 34 \end{matrix}\right| - \left|\begin{matrix} 12 & 13 & 14 \\ & 32 & 42 \\ & & 43 \end{matrix}\right| \\
& + \left|\begin{matrix} 21 & 31 & 41 \\ & 32 & 42 \\ & & 43 \end{matrix}\right| - \left|\begin{matrix} 21 & 31 & 41 \\ & 23 & 24 \\ & & 34 \end{matrix}\right| \Bigg\};
\end{aligned}$$

where, be it also observed, the third and fourth Pfaffians are obtainable from the first and second by changing rs in every case into sr, and where, if the condition $rs=-sr$ be introduced, the result is

$$\sum_{rs=-sr} \pm 12\cdot34 = 8\cdot \left|\begin{matrix} 12 & 13 & 14 \\ & 23 & 24 \\ & & 34 \end{matrix}\right|;$$

so that the Pfaffian on the right may be defined as the eighth part of a certain Cayleyan determinant; or, in Cayley's symbols,

$$[1\,2\,3\,4] = \tfrac{1}{8}\sum_{rs=-sr} \pm 12\cdot34,$$

where the 8 is the value of $2^{\frac{1}{2}n}(1.2\ldots.\tfrac{1}{2}n)$ when $n=4$.

Before leaving this it deserves to be noted that when Cayley came in 1889 to re-edit his writings, he appended to this paper a note in which it is stated that part of his purpose was to show "that the definition of a determinant may be so extended as to include within it the Pfaffian" (see *Collected Math. Papers*, i. p. 589).

that the aggregate of these is also got without repetition in a particular way already announced by Jacobi; and that it is this aliquot part of $\Sigma \pm \lambda_{12}\lambda_{34} \ldots \lambda_{2n-1,2n}$ which constitutes 'la fonction de M. Jacobi.' Jacobi's theorem regarding the effect, on the function, of interchanging two indices, is then restated; and a step further is taken in affirming that the function vanishes when two indices are equal. Finally, another law of formation—the recurrent law—is given in the form

$$[12\ldots 2n] = 12[345\ldots 2n]+13[45\ldots 2n,2]+14[5\ldots 2n,2,3]+\ldots$$

which, of course, is in substance not different from Jacobi's

$$\mathrm{R} = a_{1s}\frac{\partial \mathrm{R}}{\partial a_{1s}} + a_{2s}\frac{\partial \mathrm{R}}{\partial a_{2s}} + \ldots.$$

The digression on 'les fonctions de M. Jacobi' being exhausted, Cayley returns to skew symmetric determinants with the requisite material for proving the two theorems above referred to. The first of them, which is not new, is, in later phraseology that "*Any zero-axial skew determinant of odd order vanishes*"; and the second, which is Cayley's own, is that "*Any zero-axial skew determinant of even order is the square of a Pfaffian.*" In both cases the method of proof is the gradational or so-called 'mathematical induction'; and in both cases the main auxiliary theorem used is Cauchy's regarding the expansion of a determinant according to binary products of the elements of a row and the elements of a column.

When n is odd and the elements of the first row and those of the first column are $0,\lambda_{12},\lambda_{13},\ldots,\lambda_{1n}$ and $0,\lambda_{21},\lambda_{31},\ldots,\lambda_{n1}$ respectively, he says it is easy to see that for each term having $\lambda_{1\alpha}\lambda_{\beta 1}$ for a factor, where $\alpha \neq \beta$, there exists an equal term of opposite sign having $\lambda_{1\beta}\lambda_{\alpha 1}$ for a factor; and that therefore, since $\lambda_{1\alpha}\lambda_{\beta 1} = \lambda_{1\beta}\lambda_{\alpha 1}$, these two terms must cancel each other. As for the terms which have $\lambda_{1\alpha}\lambda_{\alpha 1}$ for a factor, the cofactor is a determinant of exactly the same form as the original, but of the order $n-2$; consequently the theorem is seen to hold for any one case if it hold for the case immediately preceding. But for the case where $n=3$, the theorem is self-evident; therefore, "Tout déterminant gauche et symétrique d'un ordre *impair* est zéro."

When n is even, the determinant dealt with is purposely taken more general than one with skew symmetry, although, strange to say, Cayley calls it 'gauche et symétrique,' the elements of the first row and those of the first column being $\lambda_{\alpha\beta}, \lambda_{\alpha 2}, \lambda_{\alpha 3}, \ldots, \lambda_{\alpha n}$ and $\lambda_{\alpha\beta}, \lambda_{2\beta}, \lambda_{3\beta}, \ldots, \lambda_{n\beta}$, and his aim being to prove that such a determinant is equal to the product of two of the functions treated of in the digression, viz., $[\alpha\, 2\, 3 \ldots n]$ and $[\beta\, 2\, 3 \ldots n]$. Developing as in the preceding case, there has this time to be considered the element common to the first row and first column, viz., $\lambda_{\alpha\beta}$, the cofactor of which is seen to be a skew symmetric determinant of odd order $n-1$, and therefore, as has just been shown, is equal to zero. As for the cofactor of $-\lambda_{\alpha\alpha'}\lambda_{\beta'\beta}$, where $\lambda_{\alpha\alpha'}$ is any element of the first row except the first, and $\lambda_{\beta'\beta}$ is any element of the first column except the first, it will be found to be a determinant which Cayley again mistakenly but consistently calls 'gauche et symétrique,' obtained by giving to r all the values $2, 3, \ldots, n$ with the exception of α', and to s all the values $2, 3, \ldots, n$, with the exception of β'. This determinant of the $(n-2)^{\text{th}}$ order is expected to be seen to be of the same kind as that with which we started, and to be temporarily admitted to be equal to

$$[\alpha'+1, \ldots, n, 2, \ldots, \alpha'-1] \cdot [\beta'+1, \ldots, n, 2, \ldots, \beta'-1].$$

The typical term of the expansion will thus be

$$\lambda_{\alpha\alpha'}[\alpha'+1, \ldots, n, 2, \ldots, \alpha'-1] \cdot \lambda_{\beta'\beta}[\beta'+1, \ldots, n, 2, \ldots, \beta'-1];$$

and the sum of all such terms

$$= \left\{\lambda_{\alpha 2}[34 \ldots n] + \lambda_{\alpha 3}[4 \ldots n2] + \ldots + \lambda_{\alpha n}[23 \ldots (n-1)]\right\}$$
$$\cdot \left\{\lambda_{\beta 2}[34 \ldots n] + \lambda_{\beta 3}[4 \ldots n2] + \ldots + \lambda_{\beta n}[23 \ldots (n-1)]\right\}$$

and therefore

$$= [\alpha\, 2\, 3 \ldots . n] \cdot [\beta\, 2\, 3 \ldots n].$$

This means, of course, that if the theorem holds for a determinant of order $n-2$ it will hold for the next succeeding case. But in the simplest case, viz., where $n=2$, it is self-evident that the theorem holds, for the determinant then

$$= \lambda_{\alpha\beta}\lambda_{22} - \lambda_{2\beta}\lambda_{\alpha 2},$$
$$= \lambda_{\beta 2}\lambda_{\alpha 2},$$
$$= [\beta 2]\cdot[\alpha 2];$$

consequently "*Le déterminant gauche et symétrique qu'on obtient en donnant à* r *les valeurs* a,2,3, . . . , n, *et à* s *les valeurs* β,2,3, . . . , n (*où* n *est* pair) *se réduit à*

$$[\alpha\, 2\, 3 \ldots \mathrm{n}]\cdot[\beta\, 2\, 3 \ldots \mathrm{n}];$$

et en particulier, en donnant à r, s *les valeurs* 1,2, . . . , n *ce déterminant se réduit à* [1 2 3 . . . n]²."

Going back now to the expansion of the skew determinant Ω with which the paper opened, and taking for simplicity's sake* $\lambda_{rr} = 1$ in every case, Cayley readily obtains,

for n even,
$$\begin{aligned}\Omega = \ & [123\ldots n]^2 \\ & + [34\ldots n]^2 + [24\ldots n]^2 + \ldots \\ & + [56\ldots n]^2 + \ldots \\ & + \ldots\ldots \\ & + 1,\end{aligned}$$

and, for n odd,
$$\begin{aligned}\Omega = \ & [23\ldots n]^2 + [13\ldots n]^2 + \ldots \\ & + [45\ldots n]^2 + \ldots \\ & + \ldots\ldots \\ & + 1.\end{aligned}$$

A special example of each identity is given, namely, the examples in which $n = 4$ and 3 respectively. If we make a slight change in the left member, viz., write Ω in Cayley's vertical-line notation (which, by the way, considering the help it would have given, and the fact that it had been introduced six years previously, it is surprising not to find employed in this paper), these examples take the form,—

$$\begin{vmatrix} 1 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ -\lambda_{12} & 1 & \lambda_{23} & \lambda_{24} \\ -\lambda_{13} & -\lambda_{23} & 1 & \lambda_{34} \\ -\lambda_{14} & -\lambda_{24} & -\lambda_{34} & 1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 12 & 13 & 14 \\ -12 & 1 & 23 & 24 \\ -13 & -23 & 1 & 34 \\ -14 & -24 & -34 & 1 \end{vmatrix}$$

$$= (\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23})^2 + \lambda^2_{12} + \lambda^2_{13} + \lambda^2_{14} + \lambda^2_{34} + \lambda^2_{24} + \lambda^2_{23} + 1,$$
$$= [1234]^2 + [12]^2 + [13]^2 + [14]^2 + [34]^2 + [24]^2 + [23]^2 + 1;$$

* And of course without loss of generality, as Cayley might have said.

and

$$\begin{vmatrix} 1 & \lambda_{12} & \lambda_{13} \\ -\lambda_{12} & 1 & \lambda_{23} \\ -\lambda_{13} & -\lambda_{23} & 1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 12 & 13 \\ -12 & 1 & 23 \\ -13 & -23 & 1 \end{vmatrix}$$

$$= \lambda^2_{23} + \lambda^2_{13} + \lambda^2_{12} + 1$$

$$= [23]^2 + [13]^2 + [12]^2 + 1 .$$

SPOTTISWOODE, W. (1851, 1853).

[ELEMENTARY THEOREMS RELATING TO DETERMINANTS. . . . . . viii+63 pp. London, 1851. Second edition, as an article in *Crelle's Journ.*, li. pp. 209–271, 328–381.]

In this the earliest of modern text-books on Determinants, a special section (§ ix. pp. 46–51; or § vi. pp. 260–266 in second edition) is set apart with the heading "On Skew Determinants." As a matter of fact, however, it is only the latter half of the section which at present concerns us, as the other half deals in reality with Cayley's determinant solution of the problem of orthogonal transformation.

In a sense the mode of treatment is indirect, the general skew determinant being viewed, not as a separate entity, but in its relation to a set of linear equations, the coefficients of which are its elements. The set of equations is

$$\left.\begin{array}{l} (11)x_1 + (12)x_2 + \ldots + (1n)x_n = u_1 \\ (21)x_1 + (22)x_2 + \ldots + (2n)x_n = u_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (n1)x_1 + (n2)x_2 + \ldots + (nn)x_n = u_n \end{array}\right\} ,$$

where it has to be remembered that in every instance $(rr)=0$ and $(rs)+(sr)=0$. The right-hand members of what he calls the "derived" set are $v_1, v_2, \ldots, v_n$; that is to say, there exists simultaneously with the original the set

$$\left.\begin{array}{l} (11)x_1 + (21)x_2 + \ldots + (n1)x_n = v_1 \\ (12)x_1 + (22)x_2 + \ldots + (n2)x_n = v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (1n)x_1 + (2n)x_2 + \ldots + (nn)x_n = v_n \end{array}\right\}$$

whose determinant is got from the determinant of the former set by the change of rows into columns, and may therefore be denominated by the same symbol Δ. Solving the two sets of equations, we have

$$\left.\begin{array}{l} x_1\Delta = [11]u_1 + [12]u_2 + \ldots + [1n]u_n \\ x_2\Delta = [21]u_1 + [22]u_2 + \ldots + [2n]u_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n\Delta = [n1]u_1 + [n2]u_2 + \ldots + [nn]u_n \end{array}\right\}$$

and

$$\left.\begin{array}{l} x_1\Delta = [11]v_1 + [21]v_2 + \ldots + [n1]v_n \\ x_2\Delta = [12]v_1 + [22]v_2 + \ldots + [n2]v_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n\Delta = [1n]v_1 + [2n]v_2 + \ldots + [nn]v_n \end{array}\right\}^{*},$$

where, be it remarked, it would have been much better if in every case the coefficients of u_r and v_r had been interchanged, for then $[rs]$ would have stood for the cofactor of (rs) in Δ. From these by addition and subtraction and by utilising the fact that $u_r + v_r = 0$ † two others are obtained, viz.,

$$\left.\begin{array}{l} 2x_1\Delta = \quad 0 \quad + ([12]-[21])u_2 + \ldots + ([1n]-[n1])u_n \\ 2x_2\Delta = ([21]-[12])u_1 + \quad 0 \quad + \ldots + ([2n]-[n2])u_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 2x_n\Delta = ([n1]-[1n])u_1 + ([n2]-[2n])u_2 + \ldots + \quad 0 \end{array}\right\}$$

and

$$\left.\begin{array}{l} 0 = \quad 2[11]u_1 \quad + ([12]+[21])u_2 + \ldots + ([1n]+[n1])u_n \\ 0 = ([21]+[12])u_1 + \quad 2[22]u_2 \quad + \ldots + ([2n]+[n2])u_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = ([n1]+[1n])u_1 + ([n2]+[2n])u_2 + \ldots + \quad 2[nn]u_n \end{array}\right\}.$$

Then follows the very curious sentence—curious, that is to say, from a logical point of view—

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* There is herein used the fact, first noted by Rothe in 1800, that the cofactor of rs in any determinant is equal to the cofactor of sr in the conjugate determinant.

† Along with this fact Spottiswoode associates the statements that

$$u_1 + u_2 + \ldots + u_n = 0, \quad v_1 + v_2 + \ldots + v_n = 0,$$

which are manifestly incorrect.

"The comparison of these three systems gives either

$$\left.\begin{array}{cccc} & \Delta = 0 & & \\ * & [12]=[21] & \ldots & [1n]=[n1] \\ [21]=[12] & * & \ldots & [2n]=[n2] \\ \cdot & \cdot & \cdot & \cdot \\ [n1]=[1n] & [n2]=[2n] & \ldots & * \end{array}\right\}$$

or

$$\left.\begin{array}{cccc} [11]=0 & [12]+[21]=0 & \ldots & [1n]+[n1]=0 \\ [21]+[12]=0 & [22]=0 & \ldots & [2n]+[n2]=0 \\ \cdot & \cdot & \cdot & \cdot \\ [n1]+[1n]=0 & [n2]+[2n]=0 & \ldots & [nn]=0 \end{array}\right\};$$

and consequently either a symmetrical skew determinant of an even order or a determinant of an odd order vanishes."

What the first half of the sentence asserts to be proved is the proposition that *If* Δ *be a zero-axial skew determinant, then either*

$$(1) \quad \Delta=0 \ \ \textit{and} \ \ [rs]=[sr],$$

$$\textit{or} \ \ (2) \ [rr]=0 \ \ \textit{and} \ \ [rs]=-[sr].$$

In this there is evidently included the assertion that *A zero-axial skew determinant either vanishes itself, or all its principal coaxial minors vanish*: but what Spottiswoode finds in it is the much wider assertion that *Either all even-ordered or all odd-ordered zero-axial skew determinants vanish.* If however his accuracy be granted up to this point, there is little objection to the cogency of the next step in the reasoning, which is worded as follows:—

"But since it is found on trial that for $n=1, 3, \ldots, \Delta$ vanishes, while for $n=2, 4, \ldots,$ it does not, the following theorems may be enunciated:—

"Theorem XIV. *A symmetrical skew determinant of an odd order in general vanishes, and the system has for its inverse an unsymmetrical skew system.*

"Theorem XV. *A symmetrical skew determinant of an even order does not in general vanish, but the system has for its inverse a symmetrical skew system.*"

The only difficulty to be raised is in regard to the name given to the "inverse system" in the first case. "Unsymmetric skew" is clearly inappropriate when, as we have seen, $[rs]=[sr]$; and it is not improved in the second edition by alteration into "quadratic skew," the fact being that the system is not skew at all, but is symmetric with respect to the principal diagonal, or, in later phraseology, is *axisymmetric*.

The treatment of the next theorem taken up is happier than the foregoing. and is after the outset no less fresh. Taking an even-ordered skew determinant with zeros in the principal diagonal he develops it according to products of an element of the first row and an element of the first column, the result being written in the form

$$\begin{vmatrix} * & 12 & \dots & 1n \\ 21 & * & & 2n \\ \cdot & \cdot & \cdot & \cdot \\ n1 & n2 & & * \end{vmatrix} = (12)^2 \begin{vmatrix} * & 34 & \dots & 3n \\ 43 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & \dots & * \end{vmatrix} + 2(12)(13) \begin{vmatrix} 34 & 35 & \dots & 32 \\ * & 45 & \dots & 42 \\ \cdot & \cdot & \cdot & \cdot \\ n4 & n5 & \dots & n2 \end{vmatrix} + \dots$$

where, be it observed, the second typical term on the right has been altered from

$$-2(12)(13) \begin{vmatrix} 32 & 34 & \dots & 3n \\ 42 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \dots & * \end{vmatrix}$$

by the translation of the first column to the last place. The determinant in this typical term is then further transformed into the square root of the product of two determinants like that in the term preceding it, the steps of the reasoning being—

$$\begin{vmatrix} 32 & 34 & \dots & 3n \\ 42 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \dots & * \end{vmatrix}^2 = \begin{vmatrix} 23 & 24 & \dots & 2n \\ 43 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & \dots & * \end{vmatrix} . \begin{vmatrix} 32 & 34 & \dots & 3n \\ 42 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \dots & * \end{vmatrix},$$

$$= \begin{vmatrix} * & 24 & \dots & 2n \\ 43 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & \dots & * \end{vmatrix} . \begin{vmatrix} * & 34 & \dots & 3n \\ 42 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n2 & n4 & \dots & * \end{vmatrix},$$

the deletion of 23 and 32 in the last step being warranted by the fact that their cofactors are determinants similar to the original but of odd order $n-3$, and therefore have the value zero. The development as thus changed has the form of the square of a polynominal; and consequently by extracting the square root there results

$$\begin{vmatrix} * & 12 & \dots & 1n \\ 21 & * & \dots & 2n \\ \cdot & \cdot & \cdot & \cdot \\ n1 & n2 & \dots & * \end{vmatrix}^{\frac{1}{2}} = 12 \cdot \begin{vmatrix} * & 34 & \dots & 3n \\ 43 & * & \dots & 4n \\ \cdot & \cdot & \cdot & \cdot \\ n3 & n4 & \dots & * \end{vmatrix}^{\frac{1}{2}} + 13 \cdot \begin{vmatrix} * & 45 & \dots & 42 \\ 54 & * & \dots & 52 \\ \cdot & \cdot & \cdot & \cdot \\ 24 & 25 & \dots & * \end{vmatrix}^{\frac{1}{2}} + \dots$$

This, according to the point of view, will be recognised either as Cayley's theorem that an even-ordered skew determinant with zeros in the principal diagonal is a *square*, or as the theorem in Pfaffians formulated by Cayley and which in Jacobi's notation would be written

$$[123\dots n] = 12[34\dots n] + 13[45\dots n2] + 14[56\dots n23] + \dots$$

The rest of the section or chapter deals with Cayley's extension of this to skew determinants whose principal elements are not zeros, the notation employed being the same.

In the Second Edition, when dealing with Cayley's orthogonal substitution, Spottiswoode gives (p. 261) without proof an important theorem on determinants of the present kind. This will be found formulated under *Orthogonants* (see p. 315 below), and need not be repeated here.

CAYLEY, A. (1851).

[On the theory of permutants. *Camb. and Dub. Math. Journ.*, vii. pp. 40–51; or *Collected Math. Papers*, ii. pp. 16–26.]

By this time the widened definition of a determinant which Cayley had given in his paper of 1847 had been exploited to a certain extent, and had been found profitable both by himself and his fellow-worker Sylvester. The paper we have now come to, however, is the only one of the series that for the present concerns us. In it he implicitly discards his former usage of the word "determinant" in any wider sense than that employed by his predecessors; adopts instead the word "*permutant*" as suggested by Sylvester, and in working out the theory of the general functions under this name assigns to determinants and Pfaffians their proper niches in the new structure, the scheme of classification being

$$\text{Permutants}\begin{cases}\text{(A) (no name)} & \begin{cases}(a)\ \text{Pfaffians}\\ \\ \end{cases}\\ \\ \text{(B) Intermutants (or hyperdeterminants)} & \begin{cases}(b)\ \text{Commutants}\begin{cases}(\beta)\ \text{Determinants.}\\ \\ \end{cases}\\ \\ \end{cases}\end{cases}$$

CAYLEY, A. (1854).

[Recherches ultérieures sur les déterminants gauches. *Crelle's Journ.*, l. pp. 299–313; or *Collected Math. Papers*, ii. pp. 202–2~~0~~5.]
215

The development with which this paper of 1847 closes is here recalled and repeated for the case where the skew determinant is of the 5th order and the elements of the diagonal are specialised, the form in which the identity appears being

$$
\begin{aligned}
\overline{12345 \mid 13345} = \quad & 11 \cdot 22 \cdot 33 \cdot 44 \cdot 55 \\
& + 11 \cdot 22 \cdot 33 \cdot (45)^2 \\
& + 11 \cdot 22 \cdot 44 \cdot (35)^2 \\
& + 11 \cdot 22 \cdot 55 \cdot (34)^2 \\
& + 11 \cdot 33 \cdot 44 \cdot (25)^2 \\
& + 11 \cdot 33 \cdot 55 \cdot (24)^2 \\
& + 11 \cdot 44 \cdot 55 \cdot (23)^2 \\
& + 22 \cdot 33 \cdot 44 \cdot (15)^2 \\
& + 22 \cdot 33 \cdot 55 \cdot (14)^2 \\
& + 22 \cdot 44 \cdot 55 \cdot (13)^2 \\
& + 33 \cdot 44 \cdot 55 \cdot (12)^2 \\
& + 11 \cdot (2345)^2 \\
& + 22 \cdot (1345)^2 \\
& + 33 \cdot (1245)^2 \\
& + 44 \cdot (1235)^2 \\
& + 55 \cdot (1234)^2,
\end{aligned}
$$

where the symbol on the left stands for the determinant whose elements are 11, 12, . . . , 21, 22, . . . and the peculiarity of skewness is understood but not expressed. Had the specialisation of the elements of the diagonal been as before, the development would clearly have been

$$\begin{gathered}1\\+(45)^2+(35)^2+(34)^2+(25)^2+(24)^2+(23)^2+(15)^2+(14)^2+(13)^2+(12)^2\\+(2345)^2+(1345)^2+(1245)^2+(1235)^2+(1234)^2,\end{gathered}$$

which, if the order be reversed, agrees exactly with the result of putting $n=5$ in the identity towards the end of the paper of 1846. By way of explanation Cayley adds the sentence "Les expressions 12, 1234, etc., à droite sont ici des *Pfaffiens*," which is noteworthy as being the first intimation that he desired "les fonctions de M. Jacobi," as he had formerly called them, to be known by the name of the mathematician whose integration-method had led Jacobi to the discovery of them. The change is easily accounted for by the fact that it was more appropriate to attach Jacobi's name to another class of determinants which were of greater importance and to which Jacobi had given far more attention.

Immediately following this there comes the announcement:—

"J'ai trouvé récemment une formule analogue pour le développement d'un *déterminant gauche* bordé, tel que

$$\overline{\alpha 1234 \mid \beta 1234} = \begin{vmatrix} \alpha\beta & \alpha 1 & \alpha 2 & \alpha 3 & \alpha 4 \\ 1\beta & 11 & 12 & 13 & 14 \\ 2\beta & 21 & 22 & 23 & 24 \\ 3\beta & 31 & 32 & 33 & 34 \\ 4\beta & 41 & 42 & 43 & 44 \end{vmatrix};$$

Cette formule est:—

$$\overline{\alpha 1234 \mid \beta 1234} = \begin{array}{l} \quad\alpha\beta\cdot 11\cdot 22\cdot 33\cdot 44 \\ \left.\begin{array}{l} +\alpha\beta\cdot 12\cdot 12\cdot 33\cdot 44 \\ +\alpha\beta\cdot 13\cdot 13\cdot 22\cdot 44 \\ +\alpha\beta\cdot 14\cdot 14\cdot 22\cdot 33 \\ +\alpha\beta\cdot 23\cdot 23\cdot 11\cdot 44 \\ +\alpha\beta\cdot 24\cdot 24\cdot 11\cdot 33 \\ +\alpha\beta\cdot 34\cdot 34\cdot 11\cdot 22 \end{array}\right\} \end{array}$$

$$\left.\begin{array}{l}+\alpha\beta 1234 \cdot 1234^*\\ \left.\begin{array}{l}+\alpha 1 \cdot \beta 1 \cdot 22 \cdot 33 \cdot 44\\ +\alpha 2 \cdot \beta 2 \cdot 11 \cdot 33 \cdot 44\\ +\alpha 3 \cdot \beta 3 \cdot 11 \cdot 22 \cdot 44\\ +\alpha 4 \cdot \beta 4 \cdot 11 \cdot 22 \cdot 33\end{array}\right\}\\ \left.\begin{array}{l}+\alpha 123 \cdot \beta 123 \cdot 44\\ +\alpha 124 \cdot \beta 124 \cdot 33\\ +\alpha 134 \cdot \beta 134 \cdot 22\\ +\alpha 234 \cdot \beta 234 \cdot 11\end{array}\right\}.\text{''}\end{array}\right.$$

Naturally enough it is noted by Cayley that the writing of $\alpha=\beta=5$ gives us the less general theorem with which he started; but he does not explain why a third way of arranging the terms of the development is adopted. Stranger still, he does not remark on the fact that by making 11, 22, 33, 44 all vanish there is obtained the identity

$$\overline{\alpha 1234 \mid \beta 1234}_{rs=-sr,\ rr=0} = \alpha\beta 1234 \cdot 1234,$$

which is the twin theorem to one given in his previous paper regarding a bordered skew symmetrical determinant of *even* order. It will be remembered, however, that in the statement of this latter theorem, the peculiar narrow use of the word 'bordé' did not occur.

Although what may be called Part Second of the paper (pp. 301, 302) may seem at first sight to concern something else, it really only draws attention to the fact that *the minors* (by which he means those afterwards named *primary* minors) *of a skew determinant are themselves skew, being "gauches ordinaires" when their cofactor in the original determinant is of the form* rr, *and "gauches bordés" when their cofactor is of the form* rs. Considerable space is occupied in verifying by two examples that the same result will be reached whether we apply the theorem of Part First directly to

$$\overline{123 \ldots n \mid 123 \ldots n}$$

or to the primary minors in its equivalent

$$11 \cdot \overline{23 \ldots n \mid 23 \ldots n} - 12 \cdot \overline{23 \ldots n \mid 13 \ldots n} + \ldots\ldots$$

* A serious misprint in the original is here corrected.

What may be called Part Third (pp. 303–305) is very forbidding, by reason of the defective mode of exposition and of the awkwardness of the notation employed. Probably this accounts for the fact that the interesting theorem which it contains has never emerged until now from its place of sepulture. A portion of it must of necessity be given verbatim, if only for the purpose of preserving historical colour. It commences—

"Je remarque que le nombre des termes du développement (p. 299) du déterminant gauche est toujours une *puissance de* 2, et que de plus, ce nombre se réduit à la moitié, en réduisant à zéro un terme quelconque *aa*. Mais outre cela, le déterminant prend dans cette supposition la forme de déterminant [gauche] d'un ordre inférieur de l'unité. Je considère par exemple le déterminant gauche $\overline{123 \mid 123}$. En y faisant $33=0$ et en accentuant, pour y mettre plus de clarté, tous les symboles, on trouve

$$\overline{123 \mid 123'} = 11' \cdot (23')^2 + 22' \cdot (13')^2.$$

De là, en écrivant

$$11 = 13' \cdot 11', \qquad 12 = 11' \cdot 23',$$
$$22 = 13' \cdot 22',$$

on obtient

$$\begin{aligned}\overline{12 \mid 12} &= 11 \cdot 22 + (12)^2,\\ &= 11' \cdot \{22' \cdot (13')^2 + 11' \cdot (23')^2\},\end{aligned}$$

c'est à dire

$$\overline{12 \mid 12} = 11' \cdot \overline{123 \mid 123'}.$$

On a de même

$$\overline{1234 \mid 1234'} = 11' \cdot 22' \cdot (34')^2 + 11' \cdot 33' \cdot (24')^2 + 22' \cdot 33' \cdot (14')^2 + (1234')^2,$$

et delà, en écrivant

$$\begin{array}{lll} 11 = 14' \cdot 11', & 12 = 11' \cdot 24', & 23 = 1234', \\ 22 = 14' \cdot 22', & 13 = 11' \cdot 34', & \\ 33 = 14' \cdot 33', & & \end{array}$$

on obtient

$$\begin{aligned}\overline{123 \mid 123} &= 11 \cdot 22 \cdot 33 + 11 \cdot (23)^2 + 22 \cdot (31)^2 + 33 \cdot (12)^2,\\ &= 11' \cdot 14' \left\{ \begin{array}{l} 22' \cdot 33' \cdot (14')^2 + (1234')^2 \\ \qquad + 11' \cdot 22' \cdot (34')^2 + 11' \cdot 33' \cdot (24')^2 \end{array} \right\}\end{aligned}$$

c'est à dire

$$\overline{123 \mid 123} = 11' \cdot 14' \cdot \overline{1234 \mid 1234'}."$$

The remainder is devoted to the next two cases, the verification of which, of course, occupies still more space. The theorem thus dealt with may be roughly described as giving *the transformation of a skew determinant, having one zero element in its main diagonal, into a skew determinant of the next lower order;* and in a notation which needs no explanation and which was perfectly familiar to Cayley at the time, the four examples may be written thus:—

$$\begin{vmatrix} 11 & 12 & 13 \\ -12 & 22 & 23 \\ -13 & -23 & \end{vmatrix} = \begin{vmatrix} 11\cdot13 & 11\cdot23 \\ -11\cdot23 & 22\cdot13 \end{vmatrix} \div 11,$$

$$\begin{vmatrix} 11 & 12 & 13 & 14 \\ -12 & 22 & 23 & 24 \\ -13 & -23 & 33 & 34 \\ -14 & -24 & -34 & . \end{vmatrix} = \begin{vmatrix} 11\cdot14 & 11\cdot24 & 11\cdot34 \\ -11\cdot24 & 22\cdot14 & [1234] \\ -11\cdot34 & -[1234] & 33\cdot14 \end{vmatrix} \div 11\cdot14,$$

$$\begin{vmatrix} 11 & 12 & 13 & 14 & 15 \\ 12 & 22 & 23 & 24 & 25 \\ 13 & -23 & 33 & 34 & 35 \\ 14 & -24 & -34 & 44 & 45 \\ 15 & -25 & -35 & -45 & . \end{vmatrix} = \begin{vmatrix} 11\cdot15 & 11\cdot25 & 11\cdot35 & 11\cdot45 \\ -11\cdot25 & 22\cdot15 & [1235] & [1245] \\ -11\cdot35 & -[1235] & 33\cdot15 & [1345] \\ -11\cdot45 & -[1245] & -[1345] & 44\cdot15 \end{vmatrix} \div 11\cdot(15)^2,$$

$$\begin{vmatrix} 11 & 12 & \ldots & 15 & 16 \\ 12 & 22 & \ldots & 25 & 26 \\ . & . & . & . & . \\ 15 & -25 & \ldots & 55 & 56 \\ 16 & -26 & \ldots & -56 & . \end{vmatrix} = \begin{vmatrix} 11\cdot16 & 11\cdot26 & 11\cdot36 & 11\cdot46 & 11\cdot56 \\ -11\cdot26 & 22\cdot16 & [1236] & [1246] & [1256] \\ -11\cdot36 & -[1236] & 33\cdot16 & [1346] & [1356] \\ -11\cdot46 & -[1246] & -[1346] & 44\cdot16 & [1456] \\ -11\cdot56 & -[1256] & -[1356] & -[1456] & 55\cdot16 \end{vmatrix} \div 11\cdot(16)^3.$$

Of course, this mode of writing does not at once suggest any better mode of proof, but it makes clear the general theorem, which consequently may be enunciated as follows:

"*A skew determinant of the* n^{th} *order which has a zero for the last element of its main diagonal may, if multiplied by* $11\cdot(1n)^{n-3}$ *be transformed into a skew determinant of the* $(n-1)^{th}$ *order, which has for its first row the last column of the original determinant multiplied by* 11, *for its main diagonal the main diagonal of the original determinant multiplied by* 1n, *and for the element in every other place* rs *situated between these two lines the Pfaffian* [1rsn]."

The rest of the paper deals with *inverse matrices*, and with the application of them to the problem afterwards known as the *automorphic transformation of a quadric*.

BRIOSCHI, F. (1854).

[LA TEORICA DEI DETERMINANTI, E LE SUE PRINCIPALI APPLICAZIONI. viii + 116 pp. Pavia.]

In this, the second text-book, the same importance is given to skew determinants as in Spottiswoode, the first part of the eighth section (pp. 55–72) being devoted to them under the heading "Dei determinanti *gobbi*," which Schellbach translates by *überschlagene*. The arrangement and treatment of the matter, however, are much more logical, zero-axial skew determinants being taken first, then the functions connected with these, namely, Pfaffians, then skew determinants which are not zero-axial, and lastly the use of skew determinants in the consideration of the problem of orthogonal transformation.

The precedence given to determinants which are "gobbi simmetrici" over those which are "puramente gobbi" is explained at the outset by reference to Cayley's theorem regarding the expressibility of the latter in terms of the former, the quite general theorem from which Cayley's immediately follows being carefully enunciated thus:

"Indicando con P_o il determinante nel quale si pongano equali a zero gli elementi principali; e con $({}^mP_{tt})_o$ un determinante minore principale delle' m-esimo ordine del determinante P nel quale siensi annullati gli elementi principali si ha:—

$$P = P_o + \sum_r a_{rr}({}^1P_{tt})_o + \sum_r \sum_s a_{rr}a_{ss}({}^2P_{tt})_o + \ldots + a_{11}a_{22}\ldots a_{nn}."$$

The proof given of Jacobi's theorem regarding the value of an odd-ordered skew determinant with zeros in the principal diagonal is essentially the same as Cayley's proof (1847), but fuller and clearer. The proof of the corresponding theorem for a determinant of even order resembles Spottiswoode's, the difference lying mainly in the use of the notation of differential-quotients in specifying the minors of the determinant.

Denoting the determinant of even order by P, he starts with the development—

$$P = -a_{1r}^2 \frac{\partial^2 P}{\partial a_{1r} \partial a_{r1}} - a_{1s}^2 \frac{\partial^2 P}{\partial a_{1s} \partial a_{s1}} \pm 2a_{1r}a_{1s} \frac{\partial^2 P}{\partial a_{1r} \partial a_{s1}} - \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

Then since a previously obtained general identity, originally due to Jacobi, viz.,

$$P \frac{\partial^2 P}{\partial a_{rs} \partial a_{pq}} = \frac{\partial P}{\partial a_{rs}} \cdot \frac{\partial P}{\partial a_{pq}} - \frac{\partial P}{\partial a_{ps}} \cdot \frac{\partial P}{\partial a_{rq}},$$

gives in this special case the identities

$$P \frac{\partial^2 P}{\partial a_{1r} \partial a_{r1}} = \frac{\partial P}{\partial a_{1r}} \cdot \frac{\partial P}{\partial a_{r1}}, \qquad P \frac{\partial^2 P}{\partial a_{1s} \partial a_{s1}} = \frac{\partial P}{\partial a_{1s}} \cdot \frac{\partial P}{\partial a_{s1}},$$

$$P \frac{\partial^2 P}{\partial a_{1r} \partial a_{s1}} = \frac{\partial P}{\partial a_{1r}} \cdot \frac{\partial P}{\partial a_{s1}},$$

because the cofactor, awkwardly denoted by $\partial P/\partial a_{ii}$, of any vanishing element a_{ii} in the principal diagonal is zero in accordance with the preceding theorem of Cayley's. From the first two of these we have

$$P^2 \cdot \frac{\partial^2 P}{\partial a_{1r} \partial a_{r1}} \cdot \frac{\partial^2 P}{\partial a_{1s} \partial a_{s1}} = \frac{\partial P}{\partial a_{1r}} \cdot \frac{\partial P}{\partial a_{r1}} \times \frac{\partial P}{\partial a_{1s}} \cdot \frac{\partial P}{\partial a_{s1}},$$

the right side of which can be changed into

$$\left(\frac{\partial P}{\partial a_{1r}} \cdot \frac{\partial P}{\partial a_{s1}}\right)^2$$

by reason of the fact that for a determinant such as P we have in every case

$$\frac{\partial P}{\partial a_{rs}} = - \frac{\partial P}{\partial a_{sr}}.$$

But from the third identity above, by squaring, we obtain on the right the same expression; so that there thus results

$$\left(\frac{\partial P}{\partial a_{1r} \partial a_{s1}}\right)^2 = \frac{\partial^2 P}{\partial a_{1r} \partial a_{r1}} \cdot \frac{\partial^2 P}{\partial a_{1s} \partial a_{s1}},$$

an equation which, in connection with the above development of P, implies the property that the determinant P is a square ("nella quale equazione trovasi appunto espressa la proprietà che il determinante P è un quadrato").

On looking now to the development with which the demonstration opened, Brioschi is led to an expression for the square in question, viz.:

$$\mathrm{P} = \left\{ \pm a_{12}\left(\frac{\partial^2 \mathrm{P}}{\partial a_{11}\partial a_{22}}\right)^{\frac{1}{2}} \pm a_{13}\left(\frac{\partial^2 \mathrm{P}}{\partial a_{11}\partial a_{33}}\right)^{\frac{1}{2}} \pm \ldots\ldots \pm a_{1n}\left(\frac{\partial^2 \mathrm{P}}{\partial a_{11}\partial a_{nn}}\right)^{\frac{1}{2}} \right\}^2,$$

or, more generally,

$$\mathrm{P} = \left\{ \sum\nolimits_s \pm a_{rs}\left(\frac{\partial^2 \mathrm{P}}{\partial a_{rr}\partial a_{ss}}\right)^{\frac{1}{2}} \right\}^2,$$

where he notes that in every case $a_{rr}=0$ and $\partial^2\mathrm{P}/\partial a_{rr}\partial a_{ss}$, being a determinant of the same kind as P, is a square. The example added is

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = \left\{ a_{12}\begin{vmatrix} 0 & a_{34} \\ a_{43} & 0 \end{vmatrix}^{\frac{1}{2}} - a_{13}\begin{vmatrix} 0 & a_{24} \\ a_{42} & 0 \end{vmatrix}^{\frac{1}{2}} + a_{14}\begin{vmatrix} 0 & a_{23} \\ a_{32} & 0 \end{vmatrix}^{\frac{1}{2}} \right\}^2,$$

$$= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2,$$

where the difficulty of the ambiguous sign, although presenting itself more prominently than in the general demonstration, is not referred to.

The new function H, which is the square root of P, is next studied. Differentiating both sides of the equation of relationship Brioschi obtains

$$\frac{\partial \mathrm{P}}{\partial a_{rs}} = \mathrm{H}\frac{\partial \mathrm{H}}{\partial a_{rs}},\text{*}$$

where the inconvenience of the differential notation comes out more strikingly than before, the differential-quotient on the left

* Since the left member is what Cayley called a "bordered skew symmetric determinant"; and since, as Jacobi noted, a differential-quotient of H with respect to one of its elements is a function of the same kind as H, we have here unnoticed one half of Cayley's proposition that *a bordered skew symmetric determinant is expressible as the product of two Pfaffians.*

being used conventionally to denote a certain minor of P, and the differentiation on the right being real. By squaring we have

$$\left(\frac{\partial \mathrm{P}}{\partial a_{rs}}\right)^2 = \mathrm{P}\left(\frac{\partial \mathrm{H}}{\partial a_{rs}}\right)^2,$$

and since, as we have seen, it is permissible to substitute

$$\mathrm{P}\frac{\partial^2 \mathrm{P}}{\partial a_{rr}\partial a_{ss}} \text{ for } \left(\frac{\partial \mathrm{P}}{\partial a_{rs}}\right)^2$$

there results

$$\left(\frac{\partial^2 \mathrm{P}}{\partial a_{rr}\partial a_{s}}\right)^{\frac{1}{2}} = \pm\frac{\partial \mathrm{H}}{\partial a_{rs}};$$

so that the expansion for P above obtained may be altered into

$$\mathrm{P} = \left\{\sum\nolimits_s \left(a_{rs}\frac{\partial \mathrm{H}}{\partial a_{rs}}\right)\right\}^2,$$

from which by extraction of the square root we have

$$\mathrm{H} = \sum\nolimits_s \left(a_{rs}\frac{\partial \mathrm{H}}{\partial a_{rs}}\right).$$

This will be recognised as a third mode of writing an already well-known result, and, as Brioschi notes, gives a property of the function H similar to a property of determinants ("la quale equazione contiene una proprietà della funzione H analoga ad una nota dei determinanti").

From this he passes to what he calls the characteristic property of H, viz., its change of sign consequent upon the transposition of two indices. Calling H′ what H becomes when r and s are interchanged, he notes that in those terms of H in which the element a_{rs} occurs there can be no other element with the same indices, and that therefore

$$\frac{\partial \mathrm{H}}{\partial a_{rs}} = -\frac{\partial \mathrm{H}'}{\partial a_{rs}}.$$

Then since the same interchange made in P leaves P in reality unaltered,—that is to say, since $\mathrm{H}^2 = \mathrm{H}'^2$,—he obtains

$$\mathrm{H}\frac{\partial \mathrm{H}}{\partial a_{rs}} = \mathrm{H}'\frac{\partial \mathrm{H}'}{\partial a_{rs}};$$

and, it having been shown that the two differential-quotients here appearing are of opposite signs, it follows that so also are H and H′.

Lastly, he passes on to skew determinants in general; and, using the theorem and notation introduced at the outset, he writes Cayley's propositions in the form—

$$n \text{ even,} \qquad \mathrm{P} = \mathrm{P}_o + \sum_r \sum_s a_{rr} a_{ss} ({}^2p_{ii})_o + \ldots + a_{11} a_{22} \ldots a_{nn},$$

$$n \text{ odd,} \qquad \mathrm{P} = \sum_r a_{rr} ({}^1p_{ii})_o + \ldots + a_{11} a_{22} \ldots a_{nn},$$

which, he says, when the principal elements are all unity, become

$$n \text{ even,} \qquad \mathrm{P} = \mathrm{P}_o + \sum_i ({}^2p_{ii})_o + \ldots + 1,$$

$$n \text{ odd,} \qquad \mathrm{P} = \sum_i ({}^1p_{ii})_o + \sum_i ({}^3p_{ii})_o + \ldots + 1,$$

the development now being in each case a sum of squares, as all the minors appearing in it are even-ordered.

BRIOSCHI, F. (1855, March).

[Sur l'analogie entre une classe de déterminants d'ordre pair et les déterminants binaires. *Crelle's Journ.*, lii. pp. 133–141; or *Opere mat.*, v. pp. 511–520. See also *Annali di Sci. mat. e fis.*, vi. pp. 430–432.]

After explaining that his purpose is to generalise a result of Hermite's (*Comptes rendus . . . Acad. des Sci.*, Paris, xl. pp. 249–254) regarding determinants of the fourth order, Brioschi sets out by establishing a necessary lemma regarding determinants of any even order whatever. It is this lemma which is of importance to us in the present connection. Taking the determinant

$$\sum (\pm a_{11} a_{22} \ldots . a_{2m,2m}), \quad \text{or} \quad \mathrm{A} \text{ say,}$$

he multiplies it by the equivalent determinant

$$\begin{vmatrix} a_{12} & -a_{11} & a_{14} & -a_{13} & \ldots & a_{1,2m} & -a_{1,2m-1} \\ a_{22} & -a_{21} & a_{24} & -a_{23} & \ldots & a_{2,2m} & -a_{2,2m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{2m,2} & -a_{2m,1} & a_{2m,4} & -a_{2m,3} & \ldots & a_{2m,2m} & -a_{2m,2m-1} \end{vmatrix}$$

obtaining the result

$$A^2 = \begin{vmatrix} \cdot & l_{12} & l_{13} & \cdots\cdots & l_{1,2m} \\ l_{21} & \cdot & l_{23} & \cdots\cdots & l_{2,2m} \\ l_{31} & l_{32} & \cdot & \cdots\cdots & l_{3,2m} \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\ l_{2m,1} & l_{2m,2} & l_{2m,3} & \cdots\cdots & \cdot \end{vmatrix}$$

where

$$l_{rs} \equiv a_{r1}a_{s2} - a_{r2}a_{s1} + a_{r3}a_{s4} - a_{r4}a_{s3} + \cdots\cdots$$

and where therefore

$$l_{rs} = -l_{sr}.$$

Using then Cayley's theorem regarding a determinant which is "gauche symétrique," he concludes that A is expressible as a rational function of the l's. This result he might have put in the form *Any even-ordered determinant is expressible as a Pfaffian*: and at a later date it would have been written

$$\sum(\pm a_{11}a_{22}\ldots a_{2m,2m}) = \begin{vmatrix} l_{12} & l_{13} & l_{14} & \cdots\cdots & l_{1,2m} \\ & l_{23} & l_{24} & \cdots\cdots & l_{2,2m} \\ & & \cdot & \cdots\cdots & \cdot \\ & & & & l_{2m-1,2m} \end{vmatrix}.$$

The rest of the paper is occupied with the consideration of the special case where

$$l_{12} = l_{34} = l_{56} = \cdots\cdots = l_{2m-1,2m},$$

and all the other l's vanish.

BELLAVITIS, G. (1857).

[Sposizione elementare della teorica dei determinanti. *Memorie ... Istituto Veneto* ... vii. pp. 67–144.]

For the determinant which Cayley named "gauche," Bellavitis introduces the term *pseudosimmetrico*, and for "gauche symétrique" he introduces *emisimmetrico* (§ 41). In the matter of notation also he suggests a change, denoting (§ 54) the Pfaffian which is the square root of

$$\begin{vmatrix} 0 & b_a & c_a & d_a \\ a_b & 0 & c_b & d_b \\ a_c & b_c & 0 & d_c \\ a_d & b_d & c_d & 0 \end{vmatrix}$$

by

$$\text{Pf.}\,(a, b, c, d).$$

Nothing else is worth noting save the carefulness of the exposition (§§ 51–54, 59). Part of Cayley's theorem regarding a bordered skew symmetric determinant appears as a theorem regarding a non-coaxial primary minor of a skew symmetric determinant (§ 59).

CAYLEY, A. (1857).

[Théorème sur les déterminants gauches. *Crelle's Journ.*, lv. pp. 277, 278; or *Collected Math. Papers*, iv. pp. 72, 73.]

This is practically a note to rectify the oversight made in the paper of 1854, where, as has been pointed out, he omitted to draw attention to the case in which the skew determinant submitted to the operation of 'bordering' has zeros for the elements of the principal diagonal.

"Un déterminant," he now says, "de cette espèce se réduit toujours au produit de deux *Pfaffiens.* En effet en écrivant dans les exemples $11 = 22 = 33 = 44 = 0$, on obtient:

$$\overline{\alpha 123 \mid \beta 123} = \alpha 123 \cdot \beta 123,$$

$$\overline{\alpha 1234 \mid \beta 1234} = \alpha\beta 1234 \cdot 1234,$$

et de même pour un déterminant gauche et symétrique bordé quelconque, suivant que l'ordre du déterminant est pair ou impair."

To this there is added the suggestive commentary:—

"Je remarque à propos de cela, que dans le cas d'un déterminant d'ordre pair, le terme $\alpha\beta$ est multiplié par un mineur premier lequel (comme déterminant gauche et symétrique d'ordre impair) se réduit à zero; le déterminant ne contient donc pas ce term $\alpha\beta$, et sera par conséquent fonction lineo-linéaire des quantités $\alpha 1$, $\alpha 2$, etc., et 1β, 2β, etc.; de manière qu'on ne saurait être surpris de voir ce déterminant se présenter sous la forme d'un produit de deux facteurs, dont l'un est fonction linéaire de $\alpha 1$, $\alpha 2$, etc., et l'autre fonction linéaire de 1β, 2β, etc. Mais pour un déterminant d'ordre impair, le coefficient du terme $\alpha\beta$ ne se réduit pas à zéro ; en supposant donc que le déterminant puisse s'exprimer comme produit de deux facteurs, il est nécessaire que l'un de ces facteurs soit (comme le déterminant même) fonction linéaire de $\alpha\beta$ et linéo-linéaire de $\alpha 1$, $\alpha 2$, etc., et 1β, 2β, etc.: de cette manière on se rend compte de la différence de la forme des facteurs, qui a lieu dans les deux cas dont il s'agit."

It is finally pointed out that by writing $\beta = \alpha$ we are brought back to

$$\overline{\alpha 123 \mid \alpha 123} = (\alpha 123)^2,$$
$$\overline{\alpha 1234 \mid \alpha 1234} = 0:$$

—"la propriété fondamentale des déterminants gauches et symétriques." There is again, however, an oversight here, for the element $\alpha\alpha$ is taken to be equal to 0, whereas it is only necessarily so in the second case.

BALTZER, R. (1857).

[Theorie und Anwendung der Determinanten, mit . . . . . vi + 129 pp. Leipzig, 1857.]

Following his two predecessors Baltzer also assigned a separate section of his text-book to skew determinants, but without giving them any special designation of his own or even taking over that used by Schellbach. The title of the section (§ 8, pp. 29–34) is thus a little lengthy, viz., "*Determinante eines Systems von Elementen, unter denen die correspondirenden* a_{ik} *und* a_{ki} *entgegengesetzt gleich sind.*"

It must be noted, however, that before this section is reached some theorems which strictly belong to the subject of the section have been already dealt with. These are in the first place (§ 3, 8; p. 12) Jacobi's theorem regarding the vanishing of a zero-axial skew determinant of odd order, and Spottiswoode's theorems regarding conjugate elements of the adjugate or inverse of a zero-axial skew determinant, the mode of proof for all being that used by Jacobi for his own theorem, viz., the multiplication of all rows or all columns by -1, and then comparing the resulting determinant with the original. In the second place

(§ 3, 10; p. 13) we have Brioschi's theorem regarding the differential-quotient of a zero-axial skew determinant of even order, and a suggestive proof of the same which it is desirable to note. It is as follows:—Let the determinant

$$\begin{vmatrix} a_{11} & \dots\dots & a_{1n} \\ \cdot & \cdot\cdot\cdot\cdot & \cdot \\ a_{n1} & \dots\dots & a_{nn} \end{vmatrix}$$

be denoted by Δ, and the cofactor of a_{rs} in Δ by A_{rs}. Then, bearing in mind that Δ is a function of a_{rs} and that a_{sr} is not independent of a_{rs}, we have

$$\begin{aligned} \frac{\partial\Delta}{\partial a_{rs}} &= A_{rs} + A_{sr}\frac{\partial a_{sr}}{\partial a_{rs}}, \\ &= A_{rs} - A_{sr}. \end{aligned}$$

But when n is even we know from Spottiswoode, as above, that $A_{rs} = -A_{sr}$; consequently we have in this case

$$\frac{\partial\Delta}{\partial a_{rs}} = 2A_{rs}.$$

as Brioschi affirmed.* In the third place (§ 7, 5; pp. 28, 29) he applies Jacobi's general theorem

$$\begin{vmatrix} A_{rr} & A_{rs} \\ A_{sr} & A_{ss} \end{vmatrix} = \Delta\frac{\partial^2\Delta}{\partial a_{rr}\partial a_{ss}},$$

as Brioschi did, to the case where Δ is zero-axial skew and of odd order to obtain the result

$$A^2_{rs} = A_{rr}\cdot A_{rs};$$

and he takes the further step of deducing from it the result

$$A_{r1} : A_{r2} : A_{r3} : \dots\dots = \sqrt{A_{11}} : \sqrt{A_{22}} : \sqrt{A_{33}} : \dots$$

* It ought to be noticed also that Baltzer uses the equation

$$\frac{\partial\Delta}{\partial a_{rs}} = A_{rs} - A_{sr}$$

to verify Spottiswoode's theorem for the case where Δ is odd-ordered, the reasoning being that as Δ is then known to be identically zero, so also must $\partial\Delta/\partial a_{rs}$, and that therefore $A_{rs} = A_{sr}$.

thus showing, as he says (1) that the ratios on the left are independent of r, and (2) that, when the sign of one of the roots has been fixed, the others are known ("dass durch das Zeichen einer unter diesen Wurzeln die Zeichen der übrigen Wurzeln bestimmt sind)."

Turning now to the section specially set apart for the consideration of skew determinants, we find that it opens with Cayley's theorem regarding a zero-axial determinant of even order, the requirement being, as here worded, to prove that such a determinant is the square of *a rational integral function of the elements.* The proof is essentially the same as Spottiswoode's and Brioschi's, and differs from Cayley's merely in that it does not begin with a determinant of a more general form than is necessary,—a point which it is desirable to insist upon, as Baltzer ignores the fact, and then does not hesitate to say in a footnote that Cayley's proof "leaves manifold doubts unrelieved." In fact the theorem which Cayley proves is, that *if a zero-axial skew determinant of odd order be 'bordered' the resulting determinant is the product of two Pfaffians*: whereas what the three others prove, is the particular case of this in which the skewness extends to the bordering elements.

The development with which the proof begins Baltzer writes in the form

$$\Delta = a_{11}\mathrm{A}_{11} - \sum_{rs} a_{r1}a_{1s}\mathrm{A}'_{rs},$$

where A$'$ is the cofactor of a_{rs} in A_{11}, and r and s have the values $2, 3, \ldots, n$. He then uses the fact that A_{11} is a zero-axial skew determinant of odd order, and that therefore by a preceding result

$$\mathrm{A}'_{rs} = \mathrm{A}'_{sr} = \sqrt{\mathrm{A}'_{rr}\,\mathrm{A}'_{ss}};$$

so that there is obtained

$$\Delta = \sum_{rs} a_{1r}\,a_{1s}\sqrt{\mathrm{A}'_{rr}\,\mathrm{A}'_{ss}};$$

and since in this aggregate the values possible for r are exactly those possible for s, he concludes (without knowing the signs of the terms of the aggregate, be it observed) that it is resolvable into two factors, viz.

$$\Big(\sum_r a_{1r}\sqrt{\mathrm{A}'_{rr}}\Big)\Big(\sum_s a_{1s}\sqrt{\mathrm{A}'_{ss}}\Big).$$

It is then argued that the two factors are identical even in the signs of their various terms "da durch das Zeichen einer Wurzel die Zeichen der übrigen bestimmt sind"; and that therefore

$$\Delta = \Big(\sum_r a_{1r}\sqrt{\mathrm{A}'_{rr}}\Big)^2,$$

$$\text{and} \quad \sqrt{\Delta} = \sum_r a_{1r}\sqrt{\mathrm{A}'_{rr}},$$

—an aggregate of $n-1$ terms, since the values to be given to r are $2, 3, \ldots, n$. The next step consists in pointing out that A'_{rr} being a determinant similar to Δ but of order $n-2$, it must follow that $\sqrt{\mathrm{A}'_{rr}}$ can in the same way be expressed as an aggregate of $n-3$ terms, and that this process can be continued until the minor under the root-sign is of the 2nd order, when manifestly its value is the square of one of its elements. The final result thus is that $\sqrt{\Delta}$ is expressible as an aggregate of $(n-1)(n-3)\ldots 3.1$ terms, each of which is the product of $\frac{1}{2}n$ elements whose collected suffixes form a permutation of $1, 2, \ldots, n$.

By way of corollary to this it is pointed out that

$$\pm\, a_{12}a_{34}\ldots a_{n-1,n}$$

is one of the terms of the aggregate; and the same is proved by showing that the square of this is a term of Δ, the reasoning being as follows:—Since in every case $a_{rs} = -a_{sr}$ we have

$$(a_{12}a_{34}\ldots a_{n-1,n})^2 = (a_{12}a_{34}\ldots a_{n-1,n})\cdot(-)^{\frac{1}{2}n}(a_{21}a_{43}\ldots a_{n,n-1}),$$

$$\text{and } \therefore \quad = (-)^{\frac{1}{2}n}(a_{12}a_{21}a_{34}a_{43}\ldots a_{n-1,n}a_{n,n-1}),$$

which clearly contains n elements, one from every row and one from every column of Δ, and will therefore be a term of Δ if only we can show that the number of inversions of order in

$$2,1,\ 4,3,\ 6,5,\ \ldots,\ n,\ n-1$$

is $\frac{1}{2}n$, a fact which is self-evident.

Baltzer's proof that the rational integral function H, which is the square root of Δ, changes signs when two suffixes, r and s, are interchanged is a simplification of Brioschi's, the operation and even the notion of differentiation being dispensed with. The function resulting from the change being H′ he concludes like Brioschi that

$$\mathrm{H}^2 = \mathrm{H}'^2;$$

also the aggregate of the terms in H which contain a_{rs} being $a_{rs}\mathrm{B}$, say, he infers as Brioschi does that B cannot be affected by the change, and that therefore $a_{rs}\mathrm{B}$ will be altered into $a_{sr}\mathrm{B}$ or $-a_{rs}\mathrm{B}$. Here, however, he brings the demonstration quickly to a satisfactory end by saying that since some of the terms of H′ are thus seen to differ in sign only from the corresponding terms of H, the equation $\mathrm{H}^2=\mathrm{H}'^2$ shows all of them must so differ; and this is what was to be proved.

Jacobi's notation for the function H is then introduced, the formal intimation being that $(1,2,3,\ldots,n)$ *is used to denote the aggregate whose first term is* $a_{12}a_{34},\ldots,a_{n-1,n}$ *and whose square is* Δ. The other value of $\surd\Delta$ is thus of course representable by $(2,1,3,\ldots,n)$, or $(2,3,\ldots,n,1)$, or .... As this implies also that

$$\surd\mathrm{A}'_{rr} = \pm(2,3,\ldots,r-1,\,r+1,\ldots,n)$$

we have now the means, so far as symbolism is concerned, of removing the ambiguity from the various terms of the identity

$$\surd\Delta = a_{12}\surd\mathrm{A}'_{22}+a_{13}\surd\mathrm{A}_{33}+\;\ldots\;+a_{1n}\surd\mathrm{A}'_{nn}.$$

As for the knowledge necessary to use the symbolism aright, Baltzer's dictum is *that the sign taken to precede*

$$(2,3,\ldots,r-1,\,r+1,\ldots,n)$$

in substituting for $\surd\mathrm{A}'_{rr}$ *must be such that the equation*

$$\surd\mathrm{A}'_{rr}\cdot\surd\mathrm{A}'_{ss} = \mathrm{A}'_{rs}$$

will be satisfied; and this he proves will take place when the sign-factor of $(2,3,\ldots,r-1,\,r+1,\ldots,n)$ is $(-1)^r$. By hypothesis, he says, the left-hand side

$$=(-1)^r(2,3,\ldots,r-1,r+1,\ldots,n)\cdot(-1)^s(2,3,\ldots,s-1,s+1,\ldots,n)$$
$$=(-1)^{r+s}(2,3,\ldots,r-1,r+1,\ldots,n)(2,3,\ldots,s-1,s+1,\ldots,n),$$

and therefore by a previous theorem

$$= -(-1)^{r+s}(2,3,\ldots,r-1,r+1,\ldots,n)(3,\ldots,s-1,s+1,\ldots,n,2),$$

the first term of which is

$$-(-1)^{r+s}a_{23}\ldots a_{n-1,n}\cdot a_{34}\ldots a_{n,2},$$

or
$$-(-1)^{r+s}a_{23}a_{34}\ldots a_{n-1,n}a_{n2};$$

and the right-hand side

$$= \text{cofactor of } a_{rs} \text{ in } \begin{vmatrix} a_{22} & a_{23} & \ldots & a_{2n} \\ a_{32} & a_{33} & \ldots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix},$$

$$= (-1)^{r+s}\begin{vmatrix} a_{22} & a_{23} & \ldots\ldots & a_{2,s-1} & a_{2,s+1} & \ldots\ldots & a_{2n} \\ a_{32} & a_{33} & \ldots\ldots & a_{3,s-1} & a_{3,s+1} & \ldots\ldots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r-1,2} & a_{r-1,3} & \ldots\ldots & a_{r-1,s-1} & a_{r-1,s+1} & \ldots\ldots & a_{r-1,n} \\ a_{r+1,2} & a_{r+1,3} & \ldots\ldots & a_{r+1,s-1} & a_{r+1,s+1} & \ldots\ldots & a_{r+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,2} & a_{n,3} & \ldots\ldots & a_{n,s-1} & a_{n,s+1} & \ldots\ldots & a_{n,n} \end{vmatrix},$$

and therefore, on account of the translation of the first column to the last place,

$$= -(-1)^{r+s}\begin{vmatrix} a_{23} & \ldots\ldots & a_{2,s-1} & a_{2,s+1} & \ldots\ldots & a_{2,n} & a_{22} \\ a_{33} & \ldots\ldots & a_{3,s-1} & a_{3,s+1} & \ldots\ldots & a_{3,n} & a_{32} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r-1,3} & \ldots\ldots & a_{r-1,s-1} & a_{r-1,s+1} & \ldots\ldots & a_{r-1,n} & a_{r-1,2} \\ a_{r+1,3} & \ldots\ldots & a_{r+1,s-1} & a_{r+1,s+1} & \ldots\ldots & a_{r+1,n} & a_{r+1,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,3} & \ldots\ldots & a_{n,s-1} & a_{n,s+1} & \ldots\ldots & a_{n,n} & a_{n,2} \end{vmatrix},$$

the first term of which is

$$-(-1)^{r+s}a_{23}a_{34}\ \ldots\ a_{n-1,n}a_{n,2},$$

exactly as before.

To the proof no note is appended drawing attention to the fact that the very same result would have been reached by taking

$(-1)^{r-1}$, or indeed $(-1)^{r-t}$, instead of $(-1)^r$ for the sign-factor of $(2,3,\ldots,r-1,r+1,\ldots,n)$.

The very next step taken, in accordance with the above mentioned dictum, is to make the substitution in the right-hand side of the equation

$$\sqrt{\Delta} = a_{12}\sqrt{A'_{22}} + a_{13}\sqrt{A'_{33}} + \ldots + a_{1n}\sqrt{A'_{nn}},$$

the first term being used to decide whether $(1,2,3,\ldots,n)$ or $-(1,2,3,\ldots,n)$ has to be substituted for the left-hand side, and the final result being

$$(1,2,3,\ldots,n) = a_{12}(3,\ldots,n) + a_{13}(4,\ldots,n,2) + \ldots + a_{1n}(2,\ldots,n-1).$$

Since $(3,4\ldots,n)$ is the cofactor of a_{12} in $(1,2,3,\ldots,n)$ and the differential-quotient of the latter with respect to a_{12} is the same, it immediately follows from this that

$$\sqrt{\Delta} = a_{12}\frac{\partial\sqrt{\Delta}}{\partial a_{12}} + a_{13}\frac{\partial\sqrt{\Delta}}{\partial a_{13}} + \ldots + a_{1n}\frac{\partial\sqrt{\Delta}}{\partial a_{1n}}.$$

Baltzer, however, obtains a more general result by going back to the corresponding more general theorem in determinants, viz., the theorem

$$\Delta = a_{r1}A_{r1} + a_{r2}A_{r2} + \ldots + a_{rn}A_{rn},$$

with which he associates

$$0 = a_{r1}A_{s1} + a_{r2}A_{r2} + \ldots + a_{rn}A_{sn};$$

substituting $\sqrt{\Delta}\dfrac{\partial\sqrt{\Delta}}{\partial a_{rs}}$ for A_{rs}; and then dividing both sides by $\sqrt{\Delta}$. In the results,

$$\sqrt{\Delta} = a_{r1}\frac{\partial\sqrt{\Delta}}{\partial a_{r1}} + \ldots + a_{rn}\frac{\partial\sqrt{\Delta}}{\partial a_{rn}},$$

$$0 = a_{r1}\frac{\partial\sqrt{\Delta}}{\partial a_{s1}} + \ldots + a_{rn}\frac{\partial\sqrt{\Delta}}{\partial a_{sn}},$$

it has to be noticed that there is no term in $\partial\sqrt{\Delta}\big/\partial a_{rr}$.

By comparison of the first of these with the immediately preceding result (the recurring law of development) he deduces the quite general identity regarding the two forms of the cofactor of a_{rs} in $\surd\Delta$—the identity, that is to say, with which we were inclined to start. His words are—

"Setzt man

$$\begin{aligned}\surd\Delta &= (r,1,2,\ldots,r-1,r+1\ldots,n)\\ &= a_{r1}(2,\ldots,n) + a_{r2}(3,\ldots,n,1) + \ldots\end{aligned}$$

so findet man

$$\frac{\partial\surd\Delta}{\partial a_{rs}} = (s+1,\ldots,n,1,\ldots,s-1),$$

in welchem Cyclus die Suffixe r and s fehlen."

In regard to this the reader has, of course, to note that $(r,1,2,\ldots,r-1,r+1\ldots,n)$ being only one of the two values of $\surd\Delta$, the differential-quotient obtained is also only one of two; in other words, that the result reached is really

$$\partial(r,1,2,\ldots,r-1,r+1,\ldots,n)\big/\partial a_{rs} = (s+1,\ldots,n,1,\ldots,s-1),$$

where from 1 to $s-1$ and from $s+1$ to n the integers appear in natural order, save that r is omitted.

The remainder of the chapter or section, which contains no new feature, refers to Cayley's expansion of a determinant arranged according to products of elements of the principal diagonal, and the application of this to skew determinants whose diagonal elements are each equal to z.

SCHEIBNER, W. (1859, July).

[Ueber Halbdeterminanten. *Berichte . . . Ges. d. Wiss.* (Leipzig): *math.-phys. Cl.*, xi. pp. 151–159.]

This paper is not put forward by its author as containing new matter, being in fact such an exposition of the theory of Pfaffians as would suitably have formed a chapter, and a good one, of a text-book like Brioschi's or Baltzer's.

From the vanishing of a zero-axial skew determinant of odd order Scheibner reaches the already known fact that the product of two of its coaxial primary minors is equal to the square of a non-coaxial primary minor. In a quite fresh manner it is then shown that the square root of this is a rational and integral expression (Pfaffian), whose law of formation is thereafter established. Naturally following on this comes the proof (p. 156) that each of the non-coaxial primary minors is the product of two Pfaffians, the result being written in the form

$$A_{pq} = (p+1, \ldots, 2m, 0, \ldots, p-1)(q+1, \ldots, 2m, 0, \ldots, q-1),$$

where the suffixes of the elements of the original determinant are $0, 1, \ldots, 2m$. On a later page (p. 158) it is shown that a similar proposition holds when the original determinant is of even order, namely,

$$A_{pq} = (-1)^p(0,1,2, \ldots, 2m+1)(q+1, \ldots, p-1, p+1, \ldots, q-1).$$

Cayley's theorem regarding a "bordered" skew symmetric determinant thus appears broken up into two parts.

The paper concludes with the suggestions that a skew symmetric determinant should be called a *Wechseldeterminante*, that its square root should be called a *Halbdeterminante*, and that the latter should be denoted by

$$\begin{vmatrix} a_{01} & a_{02} & a_{03} & \cdots & a_{0p} \\ & a_{12} & a_{13} & \cdots & a_{1p} \\ & & a_{23} & \cdots & a_{2p} \\ & & & \cdots & \cdot \\ & & & & a_{p-1,p} \end{vmatrix},$$

an expression which would thus be an alternative for $(0,1,2,\ldots,p)$ and which would vanish for even values of p.

SOUILLART, C. (1860, Sept.).

[Note sur la question 405 et sur une composition de carrés. *Nouv. Annales de Math.*, xix. pp. 320–322.]

Souillart's subject is the skew determinant

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix},$$

and his observations are (1) that it is equal to

$$(a^2+b^2+c^2+d^2)^2,$$

and (2) that if it be multiplied by the similar determinant which is equal to

$$(p^2+q^2+r^2+s^2)^2$$

the result is a determinant of the same form, whether the multiplication be row-by-row or column-by-column. The object, of course, is to prove Euler's theorem * that the product of two sums of four squares is a sum of four squares.

CAYLEY, A. (1860, Dec.).

[Note on the theory of determinants. *Philos. Magazine*, xxi. pp. 180–185; or *Collected Math. Papers*, v. pp. 45–49.]

After expounding his, or rather Cauchy's last, mode of partitioning the ordinary expansion of a determinant, and giving his own diagrammatic representation of the partition, Cayley applies it to the expansion of a zero-axial skew determinant, showing, of course, that when of odd order it vanishes, and that when of even order it is expressible as a rational integral function of the elements.

TRUDI, N. (1862).

[TEORIA DE' DETERMINANTI, E LORO APPLICAZIONI, di Nicola Trudi. xii + 268 pp. Napoli.]

To "determinanti gobbi" Trudi devotes sixteen pages (pp. 78–94) of his text-book, the exposition, which is not a little influenced by Brioschi and Baltzer, being full and simple. There are only one or two points in it worth noting. In the first place, there is his opening proposition that *in any zero-axial skew determinant, conjugate minors, if of even order, are equal, and if of odd order differ only in sign*: this is a slight generalisation of a previously known result. In the second place (p. 80), Jacobi's general theorem

$$\begin{vmatrix} A_{rr} & A_{rs} \\ A_{sr} & A_{ss} \end{vmatrix} = \Delta \times \text{compl. minor of } \begin{vmatrix} a_{rr} & a_{rs} \\ a_{sr} & a_{ss} \end{vmatrix}$$

* *Novi Commentarii Acad. Petropolitanae*, xv. (1770), pp. 75–106. For the conclusion reached see *Nouv. Annales de Math.*, xv. pp. 403–407.

is applied to the case where Δ is a zero-axial skew determinant of *even* order, Δ_{2m} say, and where, therefore, $A_{rr}=0=A_{ss}$ and $A_{sr}=-A_{rs}$, and the said complementary minor is a determinant of the same kind as Δ_{2m} but of the order $2m-2$: and it is thus seen that if Δ_{2m-2} be a square, so also must Δ_{2m}. The use to which this is put is evident.

JANNI, G. (1863).

[Teorica di determinanti simmetrici gobbi. *Giornale di Mat.*, i. pp. 275–278.]

Janni's final result is a troublesome rule for finding the expression whose square is a skew determinant of even order, the line of thought, so far as it goes, being similar to Scheibner's (1859).

CREMONA, L. (1864); D'OVIDIO, TORELLI, MAGNI (1865).

[Quistione 32. *Giornale di Mat.*, ii. p. 62; iii. pp. 5–7, 7–10, 10–14.]

The theorem proposed by Cremona is Spottiswoode's of the year 1853, namely, *if* Δ *be a skew determinant having its diagonal elements* $a_{11}, a_{22}, \ldots, a_{nn}$ *each equal to* z, *then the product of any two rows or any two columns of the adjugate determinant contains* Δ *as a factor, and the determinant of the* n^2 *cofactors equals* Δ^{n-2}. Proofs are given by E. D'Ovidio, G. Torelli, and A. Magni; but the second alone need be attended to here, as the two others are less direct, being connected, as the theorem originally was, with the subject of orthogonal substitution.

Starting with the known result

$$a_{r1}A_{s1}+\ldots+a_{rr}A_{sr}+\ldots+a_{rn}A_{sn} \begin{array}{l} =\Delta \quad \text{when } s=r \\ =0 \quad \text{when } s\neq r \end{array}\Bigg\},$$

Torelli by subtraction of $2a_{rr}A_{sr}$ and change of signs obtains

$$a_{1r}A_{s1}+a_{2r}A_{s2}+\ldots+a_{rr}A_{sr}+\ldots+a_{nr}A_{sn} \begin{array}{l} =-\Delta+2zA_{sr} \quad \text{when } s=r \\ =\qquad\quad 2zA_{sr} \quad \text{when } s\neq r \end{array}\Bigg\}.$$

But $\left.\begin{aligned} a_{1r}A_{1s}+a_{2r}A_{2s}+\ldots+a_{rr}A_{rs}+\ldots+a_{nr}A_{ns} &= \Delta \quad \text{when } s=r \\ &= 0 \quad \text{when } s\neq r \end{aligned}\right\},$

and thence by addition, whatever s may be,

$$a_{1r}(A_{s1}+A_{1s})+\ldots+a_{nr}(A_{sn}+A_{ns}) = 2zA_{sr}.$$

Writing ω_{sr} for $(A_{sr}+A_{rs})\div 2z$ he thus has the set of n equations in $\omega_{s1}, \omega_{s2}, \ldots, \omega_{sn}$,

$$\left.\begin{array}{l} a_{11}\omega_{s1}+\ldots\ldots+a_{1n}\omega_{sn} = A_{1s} \\ a_{21}\omega_{s1}+\ldots\ldots+a_{2n}\omega_{sn} = A_{2s} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}\omega_{s1}+\ldots\ldots+a_{nn}\omega_{sn} = A_{ns} \end{array}\right\},$$

the peculiarity of which is that the right-hand members are cofactors of a column of elements of the determinant formed from the coefficients on the left. The solution thus is

$$\omega_{sr} = \frac{A_{1s}A_{1r}+A_{2s}A_{2r}+\ldots\ldots+A_{ns}A_{nr}}{\Delta},$$

whence

$$\Big(A_{1s}, A_{2s}, \ldots, A_{ns} \;\big)\!\!\big(\; A_{1r}, A_{2r}, \ldots, A_{nr}\Big) = \frac{A_{sr}+A_{rs}}{2z}\Delta,$$

as desired. Using this n^2 times, he, of course, obtains for the square of the adjugate the expression

$$\Delta^n . \begin{vmatrix} \omega_{11} & \omega_{12} & \ldots & \omega_{1n} \\ \omega_{21} & \omega_{21} & \ldots & \omega_{2n} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \omega_{n1} & \omega_{n2} & \ldots & \omega_{nn} \end{vmatrix},$$

and, it being known otherwise that the square of the adjugate is Δ^{2n-2}, it follows that

$$|\,\omega_{11}\ \omega_{22}\ \ldots\ \omega_{nn}\,| = \Delta^{n-2},$$

which is the other result wanted.

In regard to the elements ω one fact is noted, and is worth noting. Since A_{sr} may be expressed as an aggregate of terms in $z^0, z^1, z^2, \ldots$, namely, say

$$A_{sr} = \Theta_0 + \Theta_1 z + \Theta_2 z^2 + \ldots.$$

and since A_{rs} is got from A_{sr} by altering the signs of all the $(n-1)^2$ elements and then changing $-z$ into z, there results when n is even,

$$A_{sr} + A_{rs} = 2\Theta_1 z + 2\Theta_3 z^3 + \ldots ;$$

in other words, $A_{sr}+A_{rs}$ is then divisible by $2z$.

Two "observations" are added, the first in regard to the case where $z=0$, and the second in regard to an alternative proof of the first part of the foregoing. The latter is interesting in that the expression for $(A_{sr}+A_{rs})\,\Delta$ is not found at once as a whole, but is viewed as consisting of two parts corresponding to $A_{sr}\Delta$ and $A_{rs}\Delta$, the reason being the known existence* of a general theorem of determinants to the effect that if the product of $|a_{11} \ldots a_{nn}|$ and $|b_{11} \ldots b_{nn}|$, obtained in row-by-row fashion, be $|c_{11} \ldots c_{nn}|$, then

$$A_{rs} \cdot |b_{11} \ldots . b_{nn}| = b_{1s}C_{r1} + \ldots + b_{ns}C_{rn}.$$

This is seen to be immediately applicable on making $|a_{11} \ldots a_{nn}|$ identical with Δ above and the b's identical with the a's; and it, of course, implies that if the product obtained in column-by column fashion be $|c'_{11} \ldots c'_{nn}|$, then

$$A_{sr} \cdot |b_{11} \ldots b_{nn}| = b_{s1}C'_{1r} + \ldots + b_{sn}C'_{nr}.$$

Making the said necessary specialisations and noting that the two differently formed axisymmetric products are then identical (in other words, that $C_{rs}=C_{sr}=C'_{rs}=C'_{sr}$), Torelli obtains

$$A_{rs}\cdot\Delta = a_{1s}\cdot\sum_{i=1}^{i=n} A_{ri}A_{1i} + \ldots + a_{ss}\sum_{i=1}^{i=n} A_{ri}A_{si} + \ldots + a_{ns}\sum_{i=1}^{i=n} A_{r}\,A_{ni},$$

$$A_{sr}\cdot\Delta = a_{si}\cdot\sum_{i=1}^{i=n} A_{ri}A_{1i} + \ldots + a_{ss}\sum_{i=1}^{i=n} A_{ri}A_{si} + \ldots + a_{sn}\sum_{i=1}^{i=n} A_{ri}A_{ni};$$

whence by addition

$$(A_{rs}+A_{sr})\Delta = 2a_{ss}\sum_{i=1}^{i=n} A_{ri}A_{si}.$$

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* Rubini's *Elementi d'Algebra*, p. 277, is referred to.

CAYLEY, A. (1865, Oct.).

[A supplementary memoir on the theory of matrices. *Philos. Transac. R. Soc.* (London), clvi. pp. 25–35; or *Collected Math. Papers*, v. pp. 438–448.]

The expression of an even-ordered determinant, Δ_{2m}, as a Pfaffian being necessary for the second of the two investigations contained in his paper, Cayley effects the transformation (§§ 15–17) in substantially the same way as that devised by Brioschi ten years previously, the one point of difference being that the form of Δ_{2m} which is employed as a multiplier is got from Δ_{2m} by reversing the order of the columns and then changing the signs of the elements in the last m columns. Thus, Δ_4 being

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix},$$

the square found for it by row-by-row multiplication is, in Cayley's notation,

$$\begin{array}{r|cccc} & (d, c, -b, -a) & (h, g, -f, -e) & (l, k, -j, -i) & (p, o, -n, -m) \\ \hline (a, b, c, d) & ,, & ,, & ,, & ,, \\ (e, f, g, h) & ,, & ,, & ,, & ,, \\ (i, j, k, l) & ,, & ,, & ,, & ,, \\ (m, n, o, p) & ,, & ,, & ,, & ,, \end{array}$$

which is readily seen to be zero-axial skew.

Another expression is, of course, got by treating the conjugate of Δ_4 in the same manner.

Brioschi's paper of 1855 is not referred to.

HORNER, J. (1865, Oct.).

[Notes on determinants. *Quart. Journ. of Math.*, viii. pp. 157–162.]

The second of Horner's three notes consists of a fresh proof that a zero-axial skew determinant of even order, Δ_{2m} say, is the square of a rational function of the elements.

Δ_{2m} multiplied by the square of the product of the non-zero elements of the first row is evidently equal to a zero-axial skew determinant of the same order, Δ'_{2m} say, having 0, 1, 1, . . . , 1 for its first row. But by performing in order the operations which we may conveniently specify by

$$\begin{array}{cccc} \mathrm{row}_{2m} - \mathrm{row}_{2m-1}, & \mathrm{row}_{2m-1} - \mathrm{row}_{2m-2}, & \ldots & \mathrm{row}_3 - \mathrm{row}_2, \\ \mathrm{col}_{2m} - \mathrm{col}_{2m-1}, & \mathrm{col}_{2m-1} - \mathrm{col}_{2m-2}, & \ldots & \mathrm{col}_3 - \mathrm{col}_2, \end{array}$$

it is seen that for Δ'_{2m} we may substitute a zero-axial skew determinant of the next lower even order, Δ_{2m-2} say. The factor thus shown to connect Δ_{2m} and Δ_{2m-2} being a square, the little that needs to be added is evident.

CHAPTER X.

ORTHOGONANTS, FROM 1841 TO 1860.

NOTWITHSTANDING the generalisations made by Jacobi and Cauchy, the special case with which the whole theory originated continued from time to time to attract attention. In 1843 William Thomson, afterwards known as Lord Kelvin, published under the signature "T." in the *Cambridge Math. Journ.*, iii. pp. 247–248, a short note in which he proved the detached theorem that if $l_1, m_1, n_1, l_2, \ldots$ be nine quantities such that

$$\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1, & l_1 l_2 + m_1 m_2 + n_1 n_2 &= 0, \\ l_2^2 + m_2^2 + n_2^2 &= 1, & l_2 l_3 + m_2 m_3 + n_2 n_3 &= 0, \\ l_3^2 + m_3^2 + n_3^2 &= 1, & l_3 l_1 + m_3 m_1 + n_3 n_1 &= 0, \end{aligned}$$

then it follows that

$$\begin{aligned} l_1^2 + \ l_2^2 + \ l_3^2 &= 1, & l_1 m_1 + \ l_2 m_2 + \ l_3 m_3 &= 0, \\ m_1^2 + m_2^2 + m_3^2 &= 1, & m_1 n_1 + m_2 n_2 + m_3 n_3 &= 0, \\ n_1^2 + n_2^2 + n_3^2 &= 1, & n_1 l_1 + \ n_2 l_2 + \ n_3 l_3 &= 0. \end{aligned}$$

This led to a short paper by A. Göpel in the *Archiv d. Math. u. Phys.*, iv. (1843), pp. 244–246. The subject was again taken up in 1848 by L. Schläfli in the *Mitteilungen d. naturf. Ges. in Bern*, Nos. 112, 113, pp. 27–33,* and in 1850 by V.-A. Lebesgue in the *Nouv. Annales de Math.*, ix. pp. 46–51. Details of these papers need not be given. We may take the opportunity

* Published also in *Archiv d. Math. u. Phys.*, xiii. pp. 276-281.

to note, however, that after the appearance of Cayley's paper on matrices in 1857 the known general theorem embracing the one just mentioned might have been briefly formulated by saying that—*If* MM′ = 1, *where* M *is any square matrix and* M′ *its conjugate, then also* M′M = 1.

KUMMER, E. E. (1843).

[Bemerkungen über die cubische Gleichung, durch welche die Haupt-Axen der Flächen zweiten Grades bestimmt werden. *Crelle's Journ.*, xxvi. pp. 268-272.]

To prove the reality of all the roots of the equation mentioned in the title of his paper—a problem first solved by Lagrange in 1773—Kummer sought to show that the expression for the product of their squared differences was inherently positive. This he succeeded in doing by transforming the said expression into a sum of squares, the result being reached by proceeding from particular to general, and by a combined process of guess and test. The equation being

$$\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0,$$

or, say,

$$x^3 - Px^2 + Qx - R = 0,$$

the expression referred to is*

$$P^2Q^2 - 4P^3R + 18PQR - 4Q^3 - 27R^2;$$

and Kummer's equivalent for it is

$$\begin{aligned} &15\mathring{\sum}\left[gh(b-c) + f(g^2-h^2)\right]^2 \\ &+\mathring{\sum}\left[2(b-c)(c-a)h + (2c-a-b)fg + (2h^2-f^2-g^2)h\right]^2 \\ &+ \quad \left[(b-c)(c-a)(a-b) + (b-c)f^2 + (c-a)g^2 + (a-b)h^2\right]^2, \end{aligned}$$

* *I.e.* $-\frac{1}{3}$ of what afterwards came to be called the *discriminant* of

$$x^3 - Px^2y + Qxy^2 - Ry^3.$$

where we use $\mathring{\Sigma}$ to indicate the summing of the expressions obtained by performing simultaneously the cyclical substitutions

$$\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \quad \begin{pmatrix} f & g & h \\ g & h & f \end{pmatrix}.$$

JACOBI, C. G. J. (1844, March).

[Sulla condizione di ugualianza di due radici dell' equazione cubica, dalla quale dipendono gli assi principali di una superficie del second' ordine. *Giornale Arcadico*, xcix. pp. 3–11; or *Crelle's Journ.*, xxx. pp. 46–50; or *Gesammelte Werke*, i. pp. 271–276.]

By using A, B, ... for $bc-f^2$, $ca-g^2$, ... Jacobi first puts Kummer's sum of squares in a neater form, namely,

$$15\mathring{\sum}(g\mathrm{H}-h\mathrm{G})^2+\mathring{\sum}(b\mathrm{F}-f\mathrm{B}+c\mathrm{F}-f\mathrm{C}-2a\mathrm{F}+2f\mathrm{A})^2$$
$$+(b\mathrm{C}-c\mathrm{B}+c\mathrm{A}-a\mathrm{C}+a\mathrm{B}-b\mathrm{A})^2,$$

and he then gives a lengthy but thorough verification of its accuracy.

From the fundamental identity

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$=\mathrm{L}(a_1x+\beta_1y+\gamma_1z)^2+\mathrm{M}(a_2x+\beta_2y+\gamma_2z)^2+\mathrm{N}(a_3x+\beta_3y+\gamma_3z)^2,$$

where L, M, N are, as in his paper of 1827, the roots whose reality is to be established, he obtains at once

$$\begin{aligned} a &= \mathrm{L}a_1^2+\mathrm{M}a_2^2+\mathrm{N}a_3^2, & f &= \mathrm{L}\beta_1\gamma_1+\mathrm{M}\beta_2\gamma_2+\mathrm{N}\beta_3\gamma_3, \\ b &= \mathrm{L}\beta_1^2+\mathrm{M}\beta_2^2+\mathrm{N}\beta_3^2, & h &= \mathrm{L}\gamma_1a_1+\mathrm{M}\gamma_2a_2+\mathrm{N}\gamma_3a_3, \\ c &= \mathrm{L}\gamma_1^2+\mathrm{M}\gamma_2^2+\mathrm{N}\gamma_3^2, & g &= \mathrm{L}a_1\beta_1+\mathrm{M}a_2\beta_2+\mathrm{N}a_3\beta_3, \end{aligned}$$

and thence derives

$$\begin{aligned} \mathrm{A} &= \mathrm{MN}a_1^2+\mathrm{NL}a_2^2+\mathrm{LM}a_3^2, & \mathrm{F} &= \mathrm{MN}\beta_1\gamma_1+\mathrm{NL}\beta_2\gamma_2+\mathrm{LM}\beta_3\gamma_3 \\ \mathrm{B} &= \mathrm{MN}\beta_1^2+\mathrm{NL}\beta_2^2+\mathrm{LM}\beta_3^2, & \mathrm{G} &= \mathrm{MN}\gamma_1a_1+\mathrm{NL}\gamma_2a_2+\mathrm{LM}\gamma_3a_3, \\ \mathrm{C} &= \mathrm{MN}\gamma_1^2+\mathrm{NL}\gamma_2^2+\mathrm{LM}\gamma_3^2, & \mathrm{H} &= \mathrm{MN}a_1\beta_1+\mathrm{NL}a_2\beta_2+\mathrm{LM}a_3\beta_3. \end{aligned}$$

From these it can be shown with more or less trouble* that

$$
\begin{aligned}
gH - hG &= \Pi \cdot a_1 a_2 a_3, \\
bF - fB + cF - fC - 2aF + 2fA &= \Pi \cdot (a_1\beta_2\beta_3 + a_2\beta_3\beta_1 + a_3\beta_1\beta_2 \\
&\qquad - a_1\gamma_2\gamma_3 - a_2\gamma_3\gamma_1 - a_3\gamma_1\gamma_2), \\
bC - cB + cA - aC + aB - bA &= \Pi \cdot (a_1\beta_2\gamma_3 + a_2\beta_3\gamma_1 + a_3\beta_1\gamma_2 \\
&\qquad + a_1\beta_3\gamma_2 + a_2\beta_1\gamma_3 + a_3\beta_2\gamma_1),
\end{aligned}
$$

where Π stands for $(L-M)(M-N)(N-L)$. Kummer's sum of squares is thus made to take the form

$$
\begin{aligned}
\Pi^2 \cdot \Big[15 \mathring{\sum} (a_1a_2a_3)^2 + \sum \Big\{ a_1(\beta_2\beta_3 - \gamma_2\gamma_3) + a_2(\beta_3\beta_1 - \gamma_3\gamma_1) + a_3(\beta_1\beta_2 - \gamma_1\gamma_2) \Big\}^2 \\
+ \Big\{ a_1(\beta_2\gamma_3 + \beta_3\gamma_2) + a_2(\beta_3\gamma_1 + \beta_1\gamma_3) + a_3(\beta_1\gamma_2 + \beta_2\gamma_1) \Big\}^2 \Big],
\end{aligned}
$$

where we use $\mathring{\Sigma}$ to indicate the sum of a set of terms produced by the cyclical substitution $a \to \beta$, $\beta \to \gamma$, $\gamma \to a$. After this the cofactor of Π^2 is shown with seeming ease to be 1, and the desired result is reached.

A knowledge of the relationships existing between the elements of the orthogonant $|a_1\beta_2\gamma_3|$ is, of course, a constant requirement throughout the demonstration; and to two of these relationships special attention is drawn by Jacobi himself. The first is

$$
\begin{aligned}
&2\{a_1^2a_2^2a_3^2 + \beta_1^2\beta_2^2\beta_3^2 + \gamma_1^2\gamma_2^2\gamma_3^2\} \\
&\qquad = a_1\beta_2\gamma_3 \cdot a_2\beta_3\gamma_1 + a_2\beta_3\gamma_1 \cdot a_3\beta_1\gamma_2 + a_3\beta_1\gamma_2 \cdot a_1\beta_2\gamma_3 \\
&\qquad\quad + a_1\beta_3\gamma_2 \cdot a_2\beta_1\gamma_3 + a_2\beta_1\gamma_3 \cdot a_3\beta_2\gamma_1 + a_3\beta_2\gamma_1 \cdot a_1\beta_3\gamma_2,
\end{aligned}
$$

and the second is

$$
a_1^2a_2^2a_3^2 + \beta_1^2\beta_2^2\beta_3^2 + \gamma_1^2\gamma_2^2\gamma_3^2 = a_1^2\beta_1^2\gamma_1^2 + a_2^2\beta_2^2\gamma_2^2 + a_3^2\beta_3^2\gamma_3^2,
$$

---

* The modern reader would do well to use Binet's theorem regarding the determinant which is viewable as the product of two rectangular arrays. Thus

$$
\begin{aligned}
A = \begin{vmatrix} b & f \\ f & c \end{vmatrix} &= \left\| \begin{matrix} L\beta_1 & M\beta_2 & N\beta_3 \\ L\gamma_1 & M\gamma_2 & N\gamma_3 \end{matrix} \right\| \cdot \left\| \begin{matrix} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{matrix} \right\|, \\
&= LM\,|\beta_1\gamma_2|^2 + MN\,|\beta_2\gamma_3|^2 + NL\,|\beta_1\gamma_3|^2 = LMa_3^2 + MNa_1^2 + NLa_2^2
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{vmatrix} g & h \\ G & H \end{vmatrix} &= \left| \begin{matrix} L & M & N \\ MN & NL & LM \end{matrix} \right\| \cdot \left\| \begin{matrix} \gamma_1a_1 & \gamma_2a_2 & \gamma_3a_3 \\ a_1\beta_1 & a_2\beta_2 & a_3\beta_3 \end{matrix} \right\| \\
&= N(L^2 - M^2) \cdot a_1a_2\,|\gamma_1\beta_2| + \ldots.
\end{aligned}
$$

and so forth.

the latter's existence being due to the fact that the right hand member of the former is not altered by the interchange

$$\begin{pmatrix} \alpha_2 & \alpha_3 & \beta_3 \\ \beta_1 & \gamma_1 & \gamma_2 \end{pmatrix}.$$

BORCHARDT, C. W. (1845, January).

[Neue Eigenschaft der Gleichung, mit deren Hülfe man die seculären Störungen der Planeten bestimmt. *Crelle's Journ.*, xxx. pp. 38–45; or, in an extended form, *Journ. (de Liouville) de Math.*, xii. pp. 50–67; or *Werke*, pp. 3–13.]

Borchardt's "new property" is the naturally desirable generalisation of Kummer's identity. In his mode of designating the equation * he does not follow Kummer and Jacobi, but goes back to Cauchy (1829), the implied reference being to Laplace's *Mécanique Céleste*, partie i., livre ii., § 56 (1799).

The set of equations, from which by elimination there is obtained the equation referred to in the title, being

$$\left.\begin{array}{l} gx_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ gx_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ gx_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{array}\right\},$$

where $a_{ik}=a_{ki}$, Borchardt multiplies the two sides of each equation of the set by g, and then on the right-hand side substitutes for $gx_1, gx_2, \ldots, gx_n$ their equivalents as given. There thus results the new set

$$\left.\begin{array}{l} g^2x_1 = a_{11}^{(2)}x_1 + a_{12}^{(2)}x_2 + \ldots + a_{1n}^{(2)}x_n \\ g^2x_2 = a_{21}^{(2)}x_1 + a_{22}^{(2)}x_2 + \ldots + a_{2n}^{(2)}x_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ g^2x_n = a_{n1}^{(2)}x_1 + a_{n2}^{(2)}x_2 + \ldots + a_{nn}^{(2)}x_n \end{array}\right\},$$

where $a_{ik}^{(2)} = a_{ki}^{(2)} = \sum\limits_{s=1}^{s=n} a_{is}a_{sk}$. Repeating this operation he finds

* The most appropriate designation would seem to be "Lagrange's determinantal equation," because of this mathematician's early (1773) and successful investigation of the cubic. See his *Œuvres complètes*, iii. pp. 600–603.

generally that

$$\left.\begin{aligned} g^m x_1 &= a_{11}^{(m)} x_1 + a_{12}^{(m)} x_2 + \ldots + a_{1n}^{(m)} x_n \\ g^m x_2 &= a_{21}^{(m)} x_1 + a_{22}^{(m)} x_2 + \ldots + a_{2n}^{(m)} x_n \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ g^m x_n &= a_{n1}^{(m)} x_1 + a_{n2}^{(m)} x_2 + \ldots + a_{nn}^{(m)} x_n \end{aligned}\right\}$$

where

$$a_{ik}^{(m)} = a_{ki}^{(m)} = \sum_{s_1=1}^{s_1=n} \sum_{s_2=1}^{s_2=n} \cdots \sum_{s_{m-1}=1}^{s_{m-1}=n} a_{is_1} a_{s_1 s_2} a_{s_2 s_3} \ldots a_{s_{m-1}k}.$$

In the next place, $g_1, g_2, \ldots, g_n$ being the roots of the resultant of the initial set of equations, it is readily seen, from the expression for the said resultant when arranged according to descending powers of g, that

$$g_1 + g_2 + \ldots + g_n = a_{11} + a_{22} + \ldots + a_{nn}.$$

Similarly, by considering the resultant of the second set of equations we learn that

$$g_1^2 + g_2^2 + \ldots + g_n^2 = a_{11}^{(2)} + a_{22}^{(2)} + \ldots + a_{nn}^{(2)},$$

and generally that

$$g_1^m + g_2^m + \ldots + g_n^m = a_{11}^{(m)} + a_{22}^{(m)} + \ldots + a_{nn}^{(m)}.$$

Consequently, if we use s_m to stand for the sum of the m^{th} powers of the g's we have

$$s_m = \sum_{i=1}^{i=n} a_{ii}^{(m)}.$$

In the third place the difference-product of the g's being

$$\sum (\pm g_1^0 g_2^1 g_3^2 \ldots g_n^{n-1}),$$

Borchardt has only to use the multiplication-theorem of determinants to obtain as an equivalent for the product of the squared differences the determinant of the system

$$\begin{matrix} s_0 & s_1 & s_2 & \ldots & s_{n-1} \\ s_1 & s_2 & s_3 & \ldots & s_n \\ s_2 & s_3 & s_4 & \ldots & s_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & s_{n+1} & \ldots & s_{2n-2}. \end{matrix}$$

It is this last determinant, therefore, which he has to aim at expressing as a sum of squares.

The process devised by him for doing so is very interesting. Returning to the original set of equations and the sets derived therefrom, he takes the μ^{th} set and multiplies both sides of each equation by g^ν and then on the right-hand side substitutes for $g^\nu x_1, g^\nu x_2, \ldots, g^\nu x_n$ their equivalents as obtainable from the ν^{th} set. A comparison of the results with the equations of the $(\mu+\nu)^{\text{th}}$ set, he says, gives the noteworthy result

$$a_{ik}^{(\mu+\nu)} = \sum_{s=1}^{s=n} a_{si}^{(\mu)} a_{sk}^{(\nu)},$$

or

$$a_{ik}^{(m)} = \sum_{s=1}^{s=n} a_{si}^{(r)} a_{sk}^{(m-r)}.$$

This includes, of course, the recurrent law of formation

$$a_{ik}^{(m)} = \sum_{s=1}^{s=n} a_{si} a_{sk}^{(m-1)},$$

if we remember that by implication $a_{si}^{(1)}$ must be viewed to be the same as a_{si}. The variety of expressions for $a_{ik}^{(m)}$ which the identity gives makes possible a like variety for

$$a_{11}^{(m)} + a_{22}^{(m)} + \ldots + a_{nn}^{(m)},$$

that is, for s_m. We may, in fact, put as an equivalent for s_m any one of the $m-1$ expressions got from

$$\sum_{i=1}^{i=n} \sum_{s=1}^{s=n} a_{si}^{(r)} a_{si}^{(m-r)}$$

by taking $r=1, 2, \ldots, m-1$. We may even obtain an m^{th} equivalent by making $r=0$ if we agree to consider $a_{si}^{(0)}=0$ or 1 according as s is different from i or the same as i: in other words, if we agree to place before the original set of equations the set

$$\left.\begin{array}{l} g^0 x_1 = 1x_1 + 0x_2 + \ldots + 0x_n \\ g^0 x_2 = 0x_1 + 1x_2 + \ldots + 0x_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ g^0 x_n = 0x_1 + 0\cdot x_2 + \ldots + 1x_n \end{array}\right\}$$

the truth of which is incontestable. As a consequence the array of s's above given may be replaced by

$$\begin{array}{lllll}
\sum\sum a_{ik}^{(0)}a_{ik}^{(0)} & \sum\sum a_{ik}^{(0)}a_{ik}^{(1)} & \sum\sum a_{ik}^{(0)}a_{ik}^{(2)} & \ldots & \sum\sum a_{ik}^{(0)}a_{ik}^{(n-1)} \\
\sum\sum a_{ik}^{(1)}a_{ik}^{(0)} & \sum\sum a_{ik}^{(1)}a_{ik}^{(1)} & \sum\sum a_{ik}^{(1)}a_{ik}^{(2)} & \ldots & \sum\sum a_{ik}^{(1)}a_{ik}^{(n-1)} \\
\sum\sum a_{ik}^{(2)}a_{ik}^{(0)} & \sum\sum a_{ik}^{(2)}a_{ik}^{(1)} & \sum\sum a_{ik}^{(2)}a_{ik}^{(2)} & \ldots & \sum\sum a_{ik}^{(2)}a_{ik}^{(n-1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\sum\sum a_{ik}^{(n-1)}a_{ik}^{(0)} & \sum\sum a_{ik}^{(n-1)}a_{ik}^{(1)} & \sum\sum a_{ik}^{(n-1)}a_{ik}^{(2)} & \ldots & \sum\sum a_{ik}^{(n-1)}a_{ik}^{(n-1)},
\end{array}$$

where s_2, for example, is represented in the 1st, 2nd, 3rd rows by

$$\sum\sum a_{ik}^{(0)}a_{ik}^{(2)},\quad \sum\sum a_{ik}^{(1)}a_{ik}^{(1)},\quad \sum\sum a_{ik}^{(2)}a_{ik}^{(0)}$$

respectively, and where by reason of the range of the two Σ's each element is the sum of n^2 binary products. Any said element may thus be represented as the product of two rows of n^2 elements each, and a little examination shows that only n rows of the latter kind are necessary for the representation of all. In other words, the array of s's can be represented by the product obtained by multiplying the array

$$\begin{array}{ccccccccccc}
a_{11}^{(0)} & a_{12}^{(0)} & \ldots & a_{1n}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & \ldots & a_{2n}^{(0)} & a_{31}^{(0)} & \ldots & a_{nn}^{(0)} \\
a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{1n}^{(1)} & a_{21}^{(1)} & a_{22}^{(1)} & \ldots & a_{2n}^{(1)} & a_{31}^{(1)} & \ldots & a_{nn}^{(1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{11}^{(n-1)} & a_{12}^{(n-1)} & \ldots & a_{1n}^{(n-1)} & a_{21}^{(n-1)} & a_{22}^{(n-1)} & \ldots & a_{2n}^{(n-1)} & a_{31}^{(n-1)} & \ldots & a_{nn}^{(n-1)}
\end{array}$$

by itself, and therefore is, by Binet's theorem, expressible as a sum of squares.

By way of illustration, Borchardt takes the case where $n=3$. The product of the squared differences of the roots is then, in later notation,

$$\left\|\begin{array}{ccccccccc}
1 & . & . & . & 1 & . & . & . & 1 \\
a & h & g & h & b & f & g & f & c \\
r_1r_1 & r_1r_2 & r_1r_3 & r_2r_1 & r_2r_2 & r_2r_3 & r_3r_1 & r_3r_2 & r_3r_3
\end{array}\right\|^2$$

where $r_\alpha r_\beta$ means the product of the α^{th} and β^{th} rows of

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

By performing on the 3-by-9 array the operation

$$\text{row}_3 - (a+b+c)\,\text{row}_2 + (ab+bc+ca-f^2-g^2-h^2)\,\text{row}_1$$

there is obtained

$$\left\| \begin{matrix} 1 & . & . & . & 1 & . & . & . & 1 \\ a & h & g & h & b & f & g & f & c \\ \mathrm{A} & \mathrm{H} & \mathrm{G} & \mathrm{H} & \mathrm{B} & \mathrm{F} & \mathrm{G} & \mathrm{F} & \mathrm{C} \end{matrix} \right\|^2$$

which is readily shown to be equal to Kummer's sum of squares.

It is a little curious that Borchardt nowhere draws attention to the fact that the determinant of the coefficients in the right-hand members of his m^{th} set of equations is the m^{th} power of the determinant of the corresponding coefficients of the original set.

JACOBI, C. G. J. (1845, August).

[Ueber ein leichtes Verfahren die in der Theorie der Säcularstörungen vorkommenden Gleichungen numerisch aufzulösen. *Crelle's Journ.*, xxx. pp. 51–94; or *Gesammelte Werke*, i. pp. 227–270; or *Nouv. Annales de Math.*, x. pp. 258–265.]

This long memoir being intended for astronomical mathematicians and computers, there is little of it that concerns us except two of the introductory sections (§§ 2, 3, pp. 52–56); and even these need not detain us, as they are in effect but a well-constructed abstract of Cauchy's paper of 1829, the starting-point being the set of $n+1$ equations

$$\left.\begin{array}{rcl} (a_{11}-\theta)x_1 + \quad a_{12}x_2 + \ldots + \quad a_{1n}x_n & = & 0 \\ a_{21}x_1 + (a_{22}-\theta)x_2 + \ldots + \quad a_{2n}x_n & = & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \cdot \\ a_{n1}x_1 + \quad a_{n2}x_2 + \ldots + (a_{nn}-\theta)x_n & = & 0 \\ x_1^2 + \quad x_2^2 + \ldots + \quad x_n^2 & = & 1 \end{array}\right\}$$

considered without any regard to the mode in which they may have originated.

CAYLEY, A. (1846).

[Sur quelques propriétés des déterminants gauches. *Crelle's Journ.*, xxxii. pp. 119–123; or *Collected Math. Papers,* i. pp. 332–336.]

There is clear evidence that Rodrigues' paper of 1840 made a strong impression upon Cayley. In a paper published in 1843* he introduces his subject by speaking of Rodrigues as having "given some very elegant formulæ for determining the position of two sets of rectangular axes with respect to each other, employing rational functions of three quantities only"; and he proceeds at once to demonstrate these formulæ as a necessary preliminary to the essential part of his paper. In another paper published in 1845,† the first part of which deals with a quaternion identity, he makes the important observation that a set of nine coefficients which occur in the identity is precisely the same as the set of nine given in Rodrigues' transformation; and he adds, "It would be an interesting question to account *à priori* for the appearance of these coefficients here." We are thus not wholly unprepared for a communication from Cayley himself on the subject of the construction of a linear substitution for the transformation of $x_1^2+x_2^2+\ldots$ into $\xi_1^2+\xi_2^2+\ldots$. The following is his procedure, four variables being used in place of his n.

With unity and any six quantities whatever there is first formed the square array

$$\begin{matrix} 1 & l_{12} & l_{13} & l_{14} \\ -l_{12} & 1 & l_{23} & l_{24} \\ -l_{13} & -l_{23} & 1 & l_{34} \\ -l_{14} & -l_{24} & -l_{34} & 1, \end{matrix} \qquad \text{or say} \qquad \begin{matrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \\ l_{41} & l_{42} & l_{43} & l_{44} \end{matrix}$$

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*Cayley, A., "On the motion of rotation of a solid body." *Cambridge Math. Journ.*, iii. pp. 224–232; or *Collected Math. Papers*, i. pp. 28–35.

†Cayley, A., "On certain results relating to quaternions." *Philos. Magazine*, xxvi. pp. 141–145; or *Collected Math. Papers*, i. pp. 123–126.

remembering that $l_{rr}=1$ and $l_{rs}=-l_{sr}$. Then taking a new set of four variables $\theta_1, \theta_2, \theta_3, \theta_4$, and using for their coefficients the quantities in the square array, firstly as disposed in rows, and secondly as disposed in columns, he puts

$$\left.\begin{array}{l} l_{11}\theta_1 + l_{12}\theta_2 + l_{13}\theta_3 + l_{14}\theta_4 = x_1 \\ l_{21}\theta_1 + l_{22}\theta_2 + l_{23}\theta_3 + l_{24}\theta_4 = x_2 \\ l_{31}\theta_1 + l_{32}\theta_2 + l_{33}\theta_3 + l_{34}\theta_4 = x_3 \\ l_{41}\theta_1 + l_{42}\theta_2 + l_{43}\theta_3 + l_{44}\theta_4 = x_4 \end{array}\right\}$$

and

$$\left.\begin{array}{l} l_{11}\theta_1 + l_{21}\theta_2 + l_{31}\theta_3 + l_{41}\theta_4 = \xi_1 \\ l_{12}\theta_1 + l_{22}\theta_2 + l_{32}\theta_3 + l_{42}\theta_4 = \xi_2 \\ l_{13}\theta_1 + l_{23}\theta_2 + l_{33}\theta_3 + l_{43}\theta_4 = \xi_3 \\ l_{14}\theta_1 + l_{24}\theta_2 + l_{34}\theta_3 + l_{44}\theta_4 = \xi_4 \end{array}\right\},$$

thereby ensuring that

$$x_1^2+x_2^2+x_3^2+\ldots=\xi_1^2+\xi_2^2+\xi_3^3\ldots$$

Solving the two sets of equations separately for each of the θ's and equating the results, he next obtains

$$\left.\begin{array}{l} L_{11}x_1 + L_{21}x_2 + L_{31}x_3 + L_{41}x_4 = L_{11}\xi_1 + L_{12}\xi_2 + L_{13}\xi_3 + L_{14}\xi_4 \\ L_{12}x_1 + L_{22}x_2 + L_{32}x_3 + L_{42}x_4 = L_{21}\xi_1 + L_{22}\xi_2 + L_{23}\xi_3 + L_{24}\xi_4 \\ L_{13}x_1 + L_{23}x_2 + L_{33}x_3 + L_{43}x_4 = L_{31}\xi_1 + L_{32}\xi_2 + L_{33}\xi_3 + L_{34}\xi_4 \\ L_{14}x_1 + L_{24}x_2 + L_{34}x_3 + L_{44}x_4 = L_{41}\xi_1 + L_{42}\xi_2 + L_{43}\xi_3 + L_{44}\xi_4 \end{array}\right\},$$

where L_{rs} is used for the cofactor of l_{rs} in the determinant (Δ say) of the initial array. It only then remains to obtain from this the x's in terms of the ξ's, or the ξ's in terms of the x's. This Cayley does by using as multipliers, in the former case the elements of any row of the original array, and in the latter case the elements of any column. Thus, multiplying by $l_{11}, l_{12}, l_{13}, l_{14}$ respectively and adding, he obtains

$$\Delta x_1 = (2l_{11}L_{11}-\Delta)\xi_1 + 2l_{11}L_{12}\xi_2 + 2l_{11}L_{13}\xi_3+2l_{11}L_{14}\xi_4,$$

the full substitution being

$$\left.\begin{array}{llll}
x_1 = \left(\frac{2L_{11}}{\Delta}-1\right)\xi_1+ & \frac{2L_{12}}{\Delta}\xi_2+ & \frac{2L_{13}}{\Delta}\xi_3+ & \frac{2L_{14}}{\Delta}\xi_4 \\
x_2 = \quad \frac{2L_{21}}{\Delta}\xi_1+ & \left(\frac{2L_{22}}{\Delta}-1\right)\xi_2+ & \frac{2L_{23}}{\Delta}\xi_3+ & \frac{2L_{24}}{\Delta}\xi_4 \\
x_3 = \quad \frac{2L_{31}}{\Delta}\xi_1+ & \frac{2L_{32}}{\Delta}\xi_2+ & \left(\frac{2L_{33}}{\Delta}-1\right)\xi_3+ & \frac{2L_{34}}{\Delta}\xi_4 \\
x_4 = \quad \frac{2L_{41}}{\Delta}\xi_1+ & \frac{2L_{42}}{\Delta}\xi_2+ & \frac{2L_{43}}{\Delta}\xi_3+ & \left(\frac{2L_{44}}{\Delta}-1\right)\xi_4
\end{array}\right\}.$$

We may add, that had the relation of the reverse substitution to this not been already known it would have been evident from the set of equations which here produce both. The result reached is that *the* n^2 *coefficients* $a_{11}, \ldots, a_{nn}$ *for the transformation of rectangular co-ordinates can be expressed rationally in terms of* $\frac{1}{2}n(n-1)$ *arbitrary quantities* l_{rs} *satisfying the conditions* $l_{rs} = -l_{sr}$, $l_{rr} = 1$ *by forming the determinant* $|l_{11}l_{22}\ldots l_{nn}|$, *or* Δ *say, and thereafter the adjugate determinant* $|L_{11}L_{22}\ldots L_{nn}|$, *and taking*

$$a_{rs} = \frac{2L_{rs}}{\Delta}, \quad a_{rr} = \frac{2L_{rr}}{\Delta}-1.$$

By way of illustration Cayley works out the cases where $n=3$ and where $n=4$. For $n=3$ he begins with three quantities

$$\begin{array}{cc} \nu & -\mu \\ & \lambda \end{array}$$

and obtains the substitution-coefficients

$$\begin{array}{ccc}
\frac{1+\lambda^2-\mu^2-\nu^2}{1+\lambda^2+\mu^2+\nu^2} & \frac{2(\lambda\mu+\nu)}{1+\lambda^2+\mu^2+\nu^2} & \frac{2(\nu\lambda-\mu)}{1+\lambda^2+\mu^2+\nu^2} \\[2ex]
\frac{2(\lambda\mu-\nu)}{1+\lambda^2+\mu^2+\nu^2} & \frac{1+\mu^2-\nu^2-\lambda^2}{1+\lambda^2+\mu^2+\nu^2} & \frac{2(\mu\nu+\lambda)}{1+\lambda^2+\mu^2+\nu^2} \\[2ex]
\frac{2(\nu\lambda+\mu)}{1+\lambda^2+\mu^2+\nu^2} & \frac{2(\mu\nu-\lambda)}{1+\lambda^2+\mu^2+\nu^2} & \frac{1+\nu^2-\lambda^2-\mu^2}{1+\lambda^2+\mu^2+\nu^2},
\end{array}$$

remarking, in passing, on Rodrigues' introduction of them (but on this point see Euler's memoir of 1770) and on their connection with the theory of quaternions. For $n=4$ he begins with the six arbitrary quantities

$$\begin{array}{ccc} a & b & c \\ & -h & g \\ & & -f \end{array}$$

and obtains for the substitution-coefficients the following quantities all divided by Δ:

$$\begin{array}{cccc}
\Delta-2(a^2+b^2+c^2+\theta^2) & 2(f\theta+a+bh-cg) & 2(g\theta+b+cf-ah) & 2(h\theta+c+ag-bf) \\
2(-f\theta-a+bh-cg) & \Delta-2(g^2+h^2+a^2+\theta^2) & 2(-c\theta-h+fg-ab) & 2(b\theta+g+hf-ca) \\
2(-g\theta-b+cf-ah) & 2(c\theta+h+fg-ab) & \Delta-2(h^2+f^2+b^2+\theta^2) & 2(-a\theta-f+gh-bc) \\
2(-h\theta-c+ag-bf) & 2(-b\theta-g+hf-ca) & 2(a\theta+f+gh-bc) & \Delta-2(f^2+g^2+c^2+\theta^2)
\end{array}$$

where $\theta=af+bg+ch$ and $\Delta=1+a^2+b^2+c^2+f^2+g^2+h^2+\theta^2$.

Before leaving Cayley's very interesting paper it should be noted that the essential part of it is contained in the first few lines, where in effect he says that *if we put*

$$\left.\begin{array}{r} \theta_1+\lambda\theta_2+\mu\theta_3+\ldots=x_1 \\ -\lambda\theta_1+\theta_2+\nu\theta_3+\ldots=x_2 \\ -\mu\theta_1-\nu\theta_2+\theta_3+\ldots=x_3 \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \textit{and} \left\{\begin{array}{l} \theta_1-\lambda\theta_2-\mu\theta_3-\ldots=\xi_1 \\ \lambda\theta_1+\theta_2-\nu\theta_3-\ldots=\xi_2 \\ \mu\theta_1+\nu\theta_2+\theta_3-\ldots=\xi_3 \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right.$$

then $x_1, x_2, x_3, \ldots$ *and* $\xi_1, \xi_2, \xi_3, \ldots$ *are orthogonally related, the coefficients of the linear substitutions connecting them being rational functions of* $\lambda, \mu, \nu, \ldots$ The rest of the paper is taken up with the finding of these coefficients, that is to say, with the elimination of $\theta_1, \theta_2, \theta_3, \ldots$ and the expression of each of the remaining variables as a linear function of all the variables of the set to which this variable does not belong.

HERMITE, C. (1849, January).

[Sur une question relative à la théorie des nombres. *Journ.* (*de Liouville*) *de Math.*, xiv. pp. 21–30; or *Œuvres* i. pp. 265–273.]

The problem here solved has only a distant connection with our subject. What is given is a set of mutually prime integers forming the first column of a determinant, and the requirement is to find all the other elements so that the square of the determinant may be 1.

SPOTTISWOODE, W. (1851).

[ELEMENTARY THEOREMS RELATING TO DETERMINANTS. . . . . .
viii + 63 pp., London.]

Following Cayley, Spottiswoode places the construction of an orthogonal substitution at the opening of his section (§ 9) on skew determinants. The mode of treatment differs from Cayley's in being verificatory rather than investigative. Starting with the two sets of equations

$$\left.\begin{array}{l} l_{11}\theta_1 + l_{12}\theta_2 + l_{13}\theta_3 + l_{14}\theta_4 = x_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}$$

and

$$\left.\begin{array}{l} l_{11}\theta_1 + l_{21}\theta_2 + l_{31}\theta_3 + l_{41}\theta_4 = \xi_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}$$

he does not seek to ascertain therefrom the linear substitutions connecting the x's with the ξ's, but bringing forward the coefficients of these substitutions as found by Cayley, namely,

$$\left.\begin{array}{cccc} \dfrac{2L_{11}}{\Delta}-1 & \dfrac{2L_{12}}{\Delta} & \dfrac{2L_{13}}{\Delta} & \dfrac{2L_{14}}{\Delta} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{array}\right\}$$

he affirms that by using as multipliers, along with the first set of equations, the elements of the various columns of this array in succession we shall have

$$\left.\begin{array}{l} \left(\dfrac{2L_{11}}{\Delta}-1\right)x_1 + \dfrac{2L_{21}}{\Delta}x_2 + \dfrac{2L_{31}}{\Delta}x_3 + \dfrac{2L_{41}}{\Delta}x_4 = \xi_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\},$$

and that by using along with the second set of equations the elements of the various *rows* of the array in succession we shall have

$$\left.\begin{array}{l} \left(\dfrac{2L_{11}}{\Delta}-1\right)\xi_1 + \dfrac{2L_{12}}{\Delta}\xi_2 + \dfrac{2L_{13}}{\Delta}\xi_3 + \dfrac{2L_{14}}{\Delta}\xi_4 = x_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}.$$

At the close of a preceding section (§ 6) he devotes three pages (pp. 35–37) to an investigation of the conditions under which

Lagrange's determinantal equation shall have all its roots positive. The result is not so interesting in connection with our present subject as a theorem made use of in the process of attaining it, namely:—*If we have given the set of equations*

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = \theta x_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = \theta x_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = \theta x_n \end{array}\right\}$$

where $a_{rs} = a_{sr}$, *and if we put* Δ *for* $|a_{11}a_{22}\ldots a_{nn}|$, *then*

$$\left.\begin{array}{l} A_{11}x_1 + A_{21}x_2 + \ldots + A_{n1}x_n = \dfrac{\Delta}{\theta}x_1 \\ A_{12}x_1 + A_{22}x_2 + \ldots + A_{n2}x_n = \dfrac{\Delta}{\theta}x_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ A_{1n}x_1 + A_{2n}x_2 + \ldots + A_{nn}x_n = \dfrac{\Delta}{\theta}x_n \end{array}\right\}.$$

No proof of this is given, but one is readily got by using the elements of the r^{th} column of the adjugate of Δ as multipliers in connection with the given set of equations and performing addition, when there results the r^{th} equation of the required set.*

*It should be noted that the theorem holds when Δ is any determinant whatever. Further, there is implied in it another of at least equal importance, namely:—*If* Δ *stand for* $|a_{11}a_{22}\ldots a_{nn}|$, *the equation whose roots are* Δ *times the reciprocals of the roots of the equation*

$$\begin{vmatrix} a_{11}-\theta & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\theta & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\theta \end{vmatrix} = 0$$

is

$$\begin{vmatrix} A_{11}-\Theta & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22}-\Theta & \ldots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \ldots & A_{nn}-\Theta \end{vmatrix} = 0.$$

An independent proof of this is readily obtained by substituting Δ/Θ for θ in the original equation, expanding the determinant in a series arranged according to descending powers of Δ/Θ, using Θ^n/Δ as a multiplier, substituting $A_{11}, A_{12}\ldots$ for their equivalents, and returning to the determinant form.

HESSE, O. (1851, April).

[Ueber die Eigenschaften der linearen Substitutionen, durch welche eine homogene ganze Function zweiten Grades, welche nur die Quadrate von vier Variabeln enthält, in eine Function von derselben Form transformirt wird. *Crelle's Journ.*, xlv. pp. 93–101; or *Werke*, pp. 307–317.]

Starting with the supposition that the substitution

$$y_k = a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \Big\}_{k=1}^{k=n}$$

makes

$$b_1y_1^2 + b_2y_2^2 + \ldots + b_ny_n^2 = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2,$$

Hesse obtains by differentiation with respect to x_1, x_2, x_3, x_4, the reverse substitution

$$a_kx_k = a_{1k}b_1y_1 + a_{2k}b_2y_2 + \ldots + a_{nk}b_ny_n \Big\}_{k=1}^{k=n};$$

and having thus found that the latter substitution will make

$$a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 = b_1y_1^2 + b_2y_2^2 + \ldots + b_ny_n^2$$

he is able by putting η_k for a_kx_k and ξ_k for b_ky_k to say that the substitution

$$\eta_k = a_{1k}\xi_1 + a_{2k}\xi_2 + \ldots + a_{nk}\xi_k \Big\}_{k=1}^{k=n}$$

will make

$$\frac{1}{a_1}\eta_1^2 + \frac{1}{a_2}\eta_2^2 + \ldots + \frac{1}{a_n}\eta_n^2 = \frac{1}{b_1}\xi_1^2 + \frac{1}{b_2}\xi_2^2 + \ldots + \frac{1}{b_n}\xi_n^2.$$

The result is the theorem that—*If the substitution*

$$y_k = a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \Big\}_{k=1}^{k=n}$$

changes

$$b_1y_1^2 + b_2y_2^2 + \ldots + b_ny_n^2 \quad \textit{into} \quad a_1x_1^2 + a_2x_2^2 + \ldots + a_na_n^2$$

the conjugate substitution will change

$$\frac{1}{a_1}y_1^2 + \frac{1}{a_2}y_2^2 + \ldots + \frac{1}{a_n}y_n^2 \quad \textit{into} \quad \frac{1}{b_1}x_1^2 + \frac{1}{b_2}x_2^2 + \ldots + \frac{1}{b_n}x_n^2.$$

The rest of the paper is occupied with theorems which hold only in the case of four variables.

SYLVESTER, J. J. (1852, July).

[A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares. *Philos. Magazine* (4), iv. pp. 138–142; or *Collected Math. Papers*, i. pp. 378–381.]

The terms "orthogonal transformation" and "orthogonal substitution" date from the year 1852, the former appearing in a paper of Sylvester's published in the February part of the *Cambridge and Dub. Math. Journ.* (see vol. vii. p. 57), and the latter in the title of the paper now reached. In the former paper, too, the word "unimodular," as applied to a transformation, is first used (see p. 52), the meaning being that the modulus—that is to say, the determinant of the coefficients of transformation—is then unity.

As has been already noted * when dealing with axisymmetric determinants, this opens with the proposition that when $a_{rs}=a_{sr}$,

$$\begin{vmatrix} a_{11}+x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}+x & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn}+x \end{vmatrix} \cdot \begin{vmatrix} a_{11}-x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-x & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} q_{11}-x^2 & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22}-x^2 & \dots & q_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ q_{n1} & q_{n2} & \dots & q_{nn}-x^2 \end{vmatrix}$$

where $$q_{rs} = (a_{r1}a_{r2}\dots a_{rn} \between a_{1s}a_{2s}\dots a_{ns}),$$

and where therefore

$$|q_{11}q_{22}\dots q_{nn}| = |a_{11}a_{22}\dots a_{nn}|^2.$$

It is then pointed out that the last determinant multiplied by $(-1)^n$ is expressible in the form

$$(x^2)^n - Q_1(x^2)^{n-1} + Q_2(x^2)^{n-2} - \dots\dots;$$

* On verifying this, see also the account of the related paper published in the *Nouv. Annales de Math.* for November 1852.

that Q_1, Q_2, ... can be shown to be sums of squares; that consequently the values of x^2 in the equation

$$(x^2)^n - Q_1(x^2)^{n-1} + Q_2(x^2)^{n-2} - \ldots = 0$$

are all positive; and therefore, finally, that the values of x in the equation

$$\begin{vmatrix} a_{11}-x & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-x & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn}-x \end{vmatrix} = 0$$

are all real.*

The remainder of the paper deals with the "Law of Inertia for Quadratic Forms," this law being "that by whatever linear substitutions, orthogonal or otherwise, a given polynomial is reduced to the form $\Sigma A_1\xi_1^2$, the number of positive and negative coefficients is invariable."

LAMÉ, G. (1852).

[LEÇONS SUR LA THÉORIE MATHÉMATIQUE DE L'ELASTICITÉ DES CORPS SOLIDES. xvi+336 pp., Paris.]

While discussing (§§ 18–22) the axes of the ellipsoid of elasticity Lamé gives in substance the theorem that *if* $|\alpha_1\beta_2\gamma_3|$ *be an orthogonant, and the ordinary multiplication-theorem produce the identity*

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \cdot \begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} P_1 & Q_3 & Q_2 \\ Q_3 & P_2 & Q_1 \\ Q_2 & Q_1 & P_3 \end{vmatrix},$$

then

$$P_1+P_2+P_3 = a+b+c,$$

$$\begin{vmatrix} P_1 & Q_3 \\ Q_3 & P_2 \end{vmatrix} + \begin{vmatrix} P_2 & Q_1 \\ Q_1 & P_3 \end{vmatrix} + \begin{vmatrix} P_1 & Q_2 \\ Q_2 & P_3 \end{vmatrix} = \begin{vmatrix} a & f \\ f & b \end{vmatrix} + \begin{vmatrix} b & d \\ d & c \end{vmatrix} + \begin{vmatrix} a & e \\ e & c \end{vmatrix},$$

* This proof, for the case where $n=3$, is given free of determinants by Grunert in the *Archiv d. Math. u. Phys.*, xxix. (1857), pp. 442–446.

and of course

$$\begin{vmatrix} P_1 & Q_3 & Q_2 \\ Q_3 & P_2 & Q_1 \\ Q_2 & Q_1 & P_3 \end{vmatrix} = \begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix}.$$

No determinant notation, however, is used, nor are determinants spoken of.

HERMITE, C. (1853, May).

[Sur la théorie des formes quadratiques ternaires indéfinies. *Crelle's Journ.*, xlvii. pp. 307–312; or *Œuvres*, i. pp. 193–199.]

[Remarques sur un mémoire de M. Cayley relatif aux déterminants gauches. *Cambridge and Dub. Math. Journ.*, ix. pp. 63–67; or *Œuvres*, i. pp. 290–295.]

In his paper of 1846 Cayley, as we have seen, gave a general solution of the problem of the transformation of $x_1{}^2+x_2{}^2+\ldots$ into $\xi_1{}^2+\xi_2{}^2+\ldots$ by means of a linear substitution. Hermite now faces a more general problem, namely, "la transformation en elle-même d'une forme quadratique *quelconque*," a problem which in itself is rather outside our subject, but which, by reason of the important modification made in the initial step of the solution, deserves attention.

The quadric being $f(x_1, x_2, \ldots)$, the problem is to find the most general linear substitution which will transform

$$f(x_1, x_2, \ldots) \quad \text{into} \quad f(\xi_1, \xi_2, \ldots);$$

and Hermite having before him Cayley's expressions, in the simpler case, for the x's and ξ's in terms of an intermediary set of variables, and observing that any member of the intermediary set is the arithmetic mean of the corresponding members of the two given sets, begins by imagining merely "que les quantités x et ξ soient exprimés par des indéterminées auxiliaires θ, de sorte qu'on ait en général

$$x_r+\xi_r = 2\theta_r."$$

There is thus obtained

$$f(x_1, x_2, \ldots) = f(2\theta_1-\xi_1, 2\theta_2-\xi_2, \ldots),$$
$$= 4f(\theta_1, \theta_2, \ldots) - 2\left(\xi_1\frac{\partial f}{\partial\theta_1}+\xi_2\frac{\partial f}{\partial\theta_2}+\ldots\right) + f(\xi_1, \xi_2, \ldots),$$

so that in order to have $f(x_1, x_2, \ldots)=f(\xi_1, \xi_2, \ldots)$ it is seen to be necessary that

$$\xi_1\frac{\partial f}{\partial\theta_1}+\xi_2\frac{\partial f}{\partial\theta_2}+\ldots = 2f(\theta_1, \theta_2, \ldots).$$

Now this condition is manifestly satisfied by putting $\xi_r=\theta_r$, but "la maniere la plus générale de la vérifier en exprimant les quantités ξ en θ sera de faire

$$\xi_r = \theta_r+\tfrac{1}{2}\sum_{s=1}^{s=n}\lambda_{rs}\frac{\partial f}{\partial\theta_s},$$

les indéterminées λ étant assujetées à la condition $\lambda_{rs} = -\lambda_{sr}$." This of course implies that

$$x_r = \theta_r-\tfrac{1}{2}\sum_{s=1}^{s=n}\lambda_{rs}\frac{\partial f}{\partial\theta_s};$$

and there have thus been obtained in their general form the two sets of equations with which Cayley started in his special case.

For those who may wish to pursue the subject of "automorphic transformation" farther than these papers of Hermite's we may note that the actual expression of the x's in terms of the ξ's was given by Cayley in a paper dated 24th May 1854,* and that he extended his result to a *bipartite* quadric function in a paper dated 10th December, 1857.†

Another problem, which in the early history of orthogonants we have seen to be of interest, namely, the simultaneous trans-

* Cayley, A., "Sur la transformation d'une fonction quadratique en elle-même par des substitutions linéaires," *Crelle's Journ.*, l. pp. 288–299; or *Collected Math. Papers*, ii. pp. 192–201. See also Brioschi in *Annali di Sci. mat. e fis.*, v. pp. 201–206.

† Cayley, A., "A Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function," *Philos. Transac. R. Soc.* (London), cxlviii. pp. 39–46: or *Collected Math. Papers*, ii. pp. 497–505.

formation of two quadrics, Cayley also dealt with, the first time in 1849 and the second in 1857.*

SYLVESTER, J. J. (1853).

[The algebraical theory of the secular-inequality determinantive equation generalised. *Philos. Magazine*, vi. pp. 214–216; or *Collected Math. Papers*, i. pp. 634–636.]

The fundamental theorem here is that if

$$X_1 = ax+a,\ X_2 = \begin{vmatrix} ax+a & bx+\beta \\ bx+\beta & cx+\gamma \end{vmatrix},\quad X_3 = \begin{vmatrix} ax+a & bx+\beta & dx+\delta \\ bx+\beta & cx+\gamma & ex+\epsilon \\ dx+\delta & ex+\epsilon & fx+\phi \end{vmatrix},$$

and the coefficients of the highest powers of x in $X_1, X_2, X_3, \ldots$ have all the same sign, then the roots of X_i will be all real and will lie respectively in the intervals comprised between $+\infty$, the successive descending roots of X_{i-1}, and $-\infty$. The mode of proof is Cauchy's (1829).

SPOTTISWOODE, W. (1853, August).

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journ.*, li. pp. 209–271, 328–381.]

In trying to insert in his second edition an alternative process for establishing Cayley's result of 1846, Spottiswoode is very unfortunate. The place selected by him is immediately after the sentence defining *skew*, and therefore immediately preceding the former process; but in making the insertion (p. 260) the predicate of the important sentence in question has suffered excision, along with a very necessary explanation regarding the diagonal elements of the initial determinant. Further, at the utmost all that is established is the fact that the determinant of Cayley's substitution is equal to $+1$. Such neglect, however, can well be overlooked in view of certain deductions which he

* *Cambridge and Dublin Math. Journ.*, iv. pp. 47–50; and *Quart. Journ. of Math.*, ii. pp. 192–195; or *Collected Math. Papers*, i. pp. 428–431, and iii. pp. 129–131.

records, and which he says can be made from Cayley's result. These may be enunciated in more modern form as follows:—

If $|a_{11}\, a_{22}\, \ldots\, a_{nn}|$ *or* Δ *be a unit-axial skew determinant,* $|A_{11}\, A_{22}\, \ldots\, A_{nn}|$ *its adjugate, and* $|\omega_{11}\, \omega_{22}\, \ldots\, \omega_{nn}|$ *Cayley's orthogonant formed therefrom, then*

$$\left.\begin{aligned} A_{1r}^2 + A_{2r}^2 + \ldots + A_{nr}^2 &= A_{rr}\,.\,\Delta, \\ A_{1r}A_{1s} + A_{2r}A_{2s} + \ldots + A_{nr}A_{ns} &= \tfrac{1}{2}(A_{rs}+A_{sr})\,.\,\Delta \end{aligned}\right\} \qquad (\alpha)$$

and

$$\left.\begin{aligned} a_{1r}\omega_{1r} + a_{2r}\omega_{2r} + \ldots + a_{nr}\omega_{nr} &= a_{rr} \\ a_{1r}\omega_{1s} + a_{2r}\omega_{2s} + \ldots + a_{nr}\omega_{ns} &= a_{rs} \end{aligned}\right\} \qquad (\beta)$$

The former, (α), belongs strictly to the theory of skew determinants, as has already been mentioned in the proper place.

CAYLEY, A. (1853, November).

[On the homographic transformation of a surface of the second order into itself. *Philos. Magazine*, vi. pp. 326–333; or *Collected Math. Papers*, ii. pp. 105–112.]

Here Cayley recalculates the general orthogonant of the 4th order, taking note in passing of the related identity

$$\begin{aligned} &(-ax-by-cz+w)^2 \\ &+(x+\nu y-\mu z+aw)^2+(-\nu x+y+\lambda z+bw)^2+(\mu x-\lambda y+z+cw)^2 \\ &= x^2+y^2+z^2+w^2+(ax+by+cz)^2 \\ &\quad +(\nu y-\mu z+aw)^2+(-\nu x+\lambda z+bw)^2+(\mu x-\lambda y+cw)^2. \end{aligned}$$

We may add that if the last eight squares be subtracted from both sides of this there remains on the left-hand side a quadric having a zero-axial discriminant.

BRIOSCHI, F. (1854, March).

[La Teorica dei Determinanti, e le sue principali Applicazioni. viii + 116 pp., Pavia.]

In Brioschi's text-book, the paragraphs dealing with a "sostituzione ortogonale" are somewhat scattered, most of them appearing among the applications (pp. 24–26, 47–51, 62–69).

The first deserving of notice (p. 49) concerns the product $\mathrm{QP\bar{Q}}$, where P and Q are determinants of the same order and $\bar{\mathrm{Q}}$ is the conjugate of Q. Viewing the product as $\mathrm{Q}\cdot(\mathrm{P\bar{Q}})$ Brioschi first uses a result of Cauchy's to express any m-line minor of $\mathrm{Q}\cdot(\mathrm{P\bar{Q}})$ in terms of m-line minors of Q and $\mathrm{P\bar{Q}}$: then for the said m-line minors of $\mathrm{P\bar{Q}}$ he substitutes with the same assistance expressions involving m-line minors of P and Q: there is thus obtained for any m-line minor of $\mathrm{QP\bar{Q}}$ an expression involving only m-line minors of P and Q. The result may be put in the form

$$(\mathrm{QP\bar{Q}})^{(m)}_{\nu,s} = \sum_r \Big[\mathrm{Q}^{(m)}_{\nu,r}\Big\{\mathrm{P}^{(m)}_{r1}\mathrm{Q}^{(m)}_{s1} + \mathrm{P}^{(m)}_{r2}\mathrm{Q}^{(m)}_{s2} + \ldots + \mathrm{P}^{(m)}_{rz}\mathrm{Q}^{(m)}_{sz}\Big\}\Big],$$

if z be put for $n(n-1)\ldots(n-m+1)/1.2.\ldots.m$, and if generally we use $\mathrm{A}^{(m)}_{sr}$ to stand for an m-line minor of an n-line determinant A, the rows of A taken to form $\mathrm{A}^{(m)}_{rs}$ being those whose numbers constitute the r^{th} combination of m of the integers 1,2, . . . , n, and the columns those whose numbers constitute the s^{th} like combination. Putting $\nu = s$ we obtain the expression of an m-line *coaxial* minor of $\mathrm{QP\bar{Q}}$, and thence for the sum of all such minors the expression

$$\sum_r \sum_s \Big[\mathrm{Q}^{(m)}_{sr}\Big\{\mathrm{P}^{(m)}_{r1}\mathrm{Q}^{(m)}_{s1} + \mathrm{P}^{(m)}_{r2}\mathrm{Q}^{(m)}_{s2} + \ldots + \mathrm{P}^{(m)}_{rz}\mathrm{Q}^{(m)}_{sz}\Big\}\Big],$$

which changes into

$$\sum_r \big\{\mathrm{P}^{(m)}_{r1}\mathrm{M}^{(m)}_{r1} + \mathrm{P}^{(m)}_{r2}\mathrm{M}^{(m)}_{r2} + \ldots + \mathrm{P}^{(m)}_{rz}\mathrm{M}^{(m)}_{rz}\big\},$$

if M be the determinant which equals Q^2. Specialising still further by making Q the determinant of an orthogonal substitution so that

$$\mathrm{M}^{(m)}_{rr} = 1 \quad \text{and} \quad \mathrm{M}^{(m)}_{rs} = 0,$$

Brioschi finally obtains the important "formula nota"

$$\sum_s (\mathrm{QP\bar{Q}})^{(m)}_{ss} = \sum_s \mathrm{P}^{(m)}_{ss},$$

which we may express in words for ourselves thus:—*If* Q *be an orthogonant and* P *any other determinant of the same order, then the sum of the* m*-line coaxial minors of* $\mathrm{QP\bar{Q}}$ *is the same as the sum of the* m*-line coaxial minors of* P.

The other paragraph requiring notice concerns the determinant arising from Cayley's of 1846 by subtracting 1 from each diagonal element. The value of this is shown (p. 65) to be 0 when n is odd, and $2^n\Delta_0/\Delta$ when n is even, Δ being the basic determinant, and Δ_0 what Δ becomes on making all its diagonal elements zero. The result is easily reached on multiplying the given determinant by Δ and showing that the product is $(-1)^n 2^n \Delta_0$.

BRIOSCHI, F. (1854, August).

[Note sur un théorème relatif aux déterminants gauches. *Journ.* (*de Liouville*) *de Math.*, xix. pp. 253–256; or in the French translation of his *Teorica dei Determinanti*, pp. 144–147; or *Opere mat.*, v. pp. 161–164.]

Brioschi's subject is really the equation

$$\begin{vmatrix} \omega_{11}-x & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22}-x & \cdots & \omega_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn}-x \end{vmatrix} = 0,$$

in which the left-hand member is the determinant of Cayley's orthogonal substitution with $-x$ affixed to each diagonal element. He notes at once, of course, that if the basic determinant be $|a_{11}a_{22}\ldots a_{nn}|$, or Δ say, the equation may be changed into

$$\begin{vmatrix} A_{11}-y & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-y & \cdots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \cdots & A_{nn}-y \end{vmatrix} = 0,$$

where y is put for $\frac{1}{2}(1+x)\Delta$. A further transformation is then effected by multiplying both sides by Δ and putting z for $1-\Delta/y$, the result being

$$\begin{vmatrix} z & a_{21} & \cdots & a_{n1} \\ a_{12} & z & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & z \end{vmatrix} = 0.$$

Using Cayley's expansion (1847) for the determinant on the left, it is seen that when n is odd the equation resolves itself into $z=0$ and an equation in z^2 with positive coefficients, and that when n is even it is already of the latter form. All values of z^2 thus obtainable must be negative, and consequently all the values of z save the value 0 must be imaginary and must occur in pairs whose sum is zero. But as

$$z = 1 - \frac{\Delta}{\frac{1}{2}(1+x)\Delta} = \frac{x-1}{x+1},$$

and

$$\therefore \quad x = \frac{1+z}{1-z},$$

it is clear that for every pair of values of z that differ only in sign there must be a pair of values of x that are reciprocals. The theorem reached by Brioschi we may thus enunciate for ourselves as follows:—*The roots of the equation*

$$\begin{vmatrix} \omega_{11}-x & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22}-x & \cdots & \omega_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn}-x \end{vmatrix} = 0,$$

where $|\omega_{11}\,\omega_{22}\cdots\omega_{nn}|$ *is Cayley's orthogonant, are arrangeable in pairs of reciprocal imaginaries, save when* n *is odd, in which case there is the single real root* 1.

When instead of the ω's we take the coefficients of the substitution which transforms a *general* quadric into itself, the words "reciprocal imaginaries" need to be changed into "reciprocals." This generalisation Brioschi published a month or two sooner (see *Annali di Sci. mat. e fis.*, v. pp. 201–206).

BRUNO, F. FAA DI (1854, September).

[Note sur un théorème de M. Brioschi. *Journ.* (*de Liouville*) *de Math.*, xix. p. 304.]

On multiplying both sides of Brioschi's equation (1854, August) by $|\omega_{11}\,\omega_{22}\cdots\omega_{nn}|$ and dividing by $(-x)^n$ an equation is obtained which differs from the original simply in having x^{-1} for x. The portion of the theorem which concerns "reciprocity" Bruno thus readily establishes.

CAUCHY, A. L. (1857, Feb.).

[Sur les fonctions quadratiques et homogènes de plusieurs variables. *Comptes rendus . . . Acad. des Sci.* (Paris), xliv, pp. 361–370, 416; or *Œuvres complètes* (1), xii. pp. 421–432. 444–445.]

The second section of this bears the title "Sur l'équation qui détermine les maxima et minima d'une fonction réelle quadratique et homogène de plusieurs variables dont les carrés donne pour somme l'unité," and at once recalls the important memoir of 1829. The subject is the same, and any additional result obtained is quite unimportant. Further, the mode of treatment is not essentially different, the language and notation of 'clefs anastrophiques' being for some obscure reason substituted for those of 'sommes alternées.'

We have only to add, as being well worthy of note in passing, that this was Cauchy's last contribution to the literature of our subject, his first and greatest, and probably the greatest of all, having been made so long before as forty-five years. Three months after the last was presented to the Academy he was dead.

BALTZER, R. (1857).

[Theorie und Anwendungen der Determinanten, mit . . . . . vi+129 pp. Leipzig.]

Baltzer devotes a whole section (§ 15) of seventeen pages (pp. 80–96) to the subject of "Die lineare, insbesondere die orthogonale Substitutionen." The section, like its fellows, is noteworthy, not for freshness of matter, but for good arrangement, clearness and compactness.

In treating of Cayley's orthogonant (§ 15, 6) he takes l, not 1, as the constant element of the basic determinant: and, when in the course of the proof he obtains the two values for each of Cayley's θ's, he does not equate them, but uses with each of them Hermite's observation

$$x_i+\xi = 2l\theta_i,$$

thus reaching the elements

$$\frac{2l\mathrm{L}_{rr}}{\Delta} - 1, \quad \frac{2l\mathrm{L}_{rs}}{\Delta},$$

of the desired substitutions without more trouble. On the other hand, he fails to note that Cayley's θ's are so introduced as to ensure from the outset the equality of $x_1^2+x_2^2+\ldots$ and $\xi_1^2+\xi_2^2+\ldots$, and thus he is led to prove propositions already established (§ 15, 5).

Brioschi's equation of August 1854 being denoted (§ 15, 9) by $f(x)=0$, he multiplies $f(x)$ by $f(-x)$. and obtains for $f(x)\cdot f(-x)/x^n$ a skew determinant having each diagonal element equal to $1/x-x$. This determinant being therefore expressible as a sum of squares when n is even, and as $1/x-x$ times a sum of squares when n is odd, the part of Brioschi's proposition which asserts the unreality of the roots follows by a *reductio ad absurdum*.

SALMON, G. (1859).

[Lessons Introductory to the Modern Higher Algebra, . . . xii + 147 pp., Dublin.]

In Salmon's treatment of the subject (§§ 118, 139, 142, 156–7, 163–4) only two points call for remark. In the first place, "orthogonal transformation" with him is not as with his predecessors a transformation which merely changes

$$x^2+y^2+z^2+\ldots \quad \text{into} \quad \xi^2+\eta^2+\zeta^2+\ldots,$$

but one which at the same time changes

$$ax^2+by^2+cz^2+\ldots+2fyz+2gzx+2hxy+\ldots \quad \text{into} \quad \mathrm{A}\xi^2+\mathrm{B}\eta^2+\mathrm{C}\zeta^2+\ldots$$

In the second place, he has a fresh mode of arriving at the equation for determining A, B, C, . . . Calling the four quadrics just mentioned V, V′, U, U′, he forms the discriminant of $\mathrm{U}-\lambda\mathrm{V}$, and asserts that the coefficient of all the several powers of λ in it must be invariants, and that, therefore, if the said discriminant be put equal to 0 and the equation so obtained be solved for λ, the roots resulting must be identical with the roots of the equation

$$\text{Discrim. } (U'-\lambda V') = 0;$$

in other words, that we must have identically

$$\begin{vmatrix} a-\lambda & h & g & \dots \\ h & b-\lambda & f & \dots \\ g & f & c-\lambda & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} A-\lambda & 0 & 0 & \dots \\ 0 & B-\lambda & 0 & \dots \\ 0 & 0 & C-\lambda & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

so that A, B, C, ... are the values of λ in the equation

$$\text{Discrim. } (U-\lambda V) = 0.$$

HESSE, O. (1859, October).

[Neue Eigenschaften der linearen Substitutionen welche gegebene homogene Functionen des zweiten Grades in andere transformiren die nur die Quadrate der Variabeln enthalten. *Crelle's Journ.*, lvii. pp. 175–182; or *Werke*, pp. 489–496.]

Hesse's object is that of Kummer (1843), Jacobi (1844, March), and Borchardt (1845, January), namely, to prove the reality of the roots of Lagrange's determinantal equation by showing that the product of their squared differences is essentially positive.

Taking the linear substitution

$$\xi_k = a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n \Big\}_{k=1}^{k=n}$$

we readily see that $\xi_1\xi_2 \dots \xi_n$ is expressible as a sum of terms of the form $Cx_1^{e_1}x_2^{e_2} \dots x_n^{e_n}$, where $e_1+e_2+\dots+e_n=n$ and C is an integral function of a's—a result which Hesse writes

$$\xi_1\xi_2 \dots \xi_n = \sum A_{e_1e_2\dots e_n}x_1^{e_1}x_2^{e_2} \dots x_n^{e_n},$$

the coefficient of any term being denoted by an A with n suffixes identical with the n exponents of the x's. Now let us suppose the substitution to be orthogonal, in which case we know that

$$x_k = a_{1k}\xi_1 + a_{2k}\xi_2 + \dots + a_{nk}\xi_n \Big\}_{k=1}^{k=n};$$

and let us thereby transform $\sum A_{e_1e_2\dots e_n}x_1^{e_1}x_2^{e_2} \dots x_n^{e_n}$ so as to have it again in terms of the ξ's. In doing this Hesse pays attention only to the term in $\xi_1\xi_2 \ldots \xi_n$, making the assertion that *the coefficient of* $\xi_1\xi_2 \ldots \xi_n$ *in* $x_1^{e_1}x_2^{e_2} \ldots x_n^{e_n}$ *is either the same as the coefficient of* $x_1^{e_1}x_2^{e_2} \ldots x_n^{e_n}$ *in* $\xi_1\xi_2 \ldots \xi_n$ *or differs from the latter coefficient by a merely arithmetical multiplier.* From this it follows that the coefficient of $\xi_1\xi_2 \ldots \xi_n$ in any term $A_{e_1e_2\ldots e_n}x_1^{e_1}x_2^{e_2} \ldots x_n^{e_n}$ is a merely arithmetical multiple of $A^2_{e_1e_2\ldots e_n}$; and, if the multiplier in question be denoted by $\Theta_{e_1e_2\ldots e_n}$, there results from the equatement of coefficients

$$1 = \sum \Theta_{e_1e_2\ldots e_n} A^2_{e_1e_2\ldots e_n}.$$

Next, let us suppose in addition that our substitution transforms an n-ary quadric

$$f_1(x_1, x_2, \ldots, x_n) \quad \text{into} \quad g_1\xi_1^2 + g_2\xi_2^2 + \ldots + g_n\xi_n^2,$$

a step which, as we know, introduces the quantities whose reality is in question. In regard to them Hesse first recalls Jacobi's proof (1833) that they are such that

$$g_1^2\xi_1^2 + g_2^2\xi_2^2 + \ldots + g_n^2\xi_n^2, \quad g_1^3\xi_1^2 + g_2^3\xi_2^2 + \ldots + g_n^3\xi_n^2, \quad \ldots$$

are also expressible as homogeneous quadric functions of the x's, and that the coefficients of these quadrics are rational integral functions of the coefficients of the original quadric f_1. It is seen to be not inappropriate therefore to use

$$f_p(x_1, x_2, \ldots, x_n) \quad \text{for} \quad g_1^p\xi_1^2 + g_2^p\xi_2^2 + \ldots + g_n^p\xi_n^2$$

and to denote the partial differential-quotient of $f_p(x_1, x_2, \ldots, x_n)$ with respect to x_k by $f'_p(x_k)$, thus giving

$$\tfrac{1}{2}f'_p(x_k) = a_{k1}g_1^p\xi_1 + a_{k2}g_2^p\xi_2 + \ldots + a_{kn}g_n^p\xi_n.$$

The next step is the deduction of an important result from the consideration of the determinant

$$\begin{vmatrix} x_1 & x_2 & \ldots & x_n \\ \tfrac{1}{2}f'_1(x_1) & \tfrac{1}{2}f'_1(x_2) & \ldots & \tfrac{1}{2}f'_1(x_n) \\ \tfrac{1}{2}f'_2(x_1) & \tfrac{1}{2}f'_2(x_2) & \ldots & \tfrac{1}{2}f'_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \tfrac{1}{2}f'_{n-1}(x_1) & \tfrac{1}{2}f'_{n-1}(x_2) & \ldots & \tfrac{1}{2}f'_{n-1}(x_n) \end{vmatrix}, \text{ or } \Delta \text{ say.}$$

Each element being linear in the x's, the determinant is of the n^{th} degree in those variables, and therefore we may put

$$\Delta = \sum B_{e_1e_2\ldots e_n} x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}.$$

On the other hand, if we substitute for each element its expression in terms of the ξ's, the result is manifestly a product-determinant, and we learn that

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \cdot \begin{vmatrix} \xi_1 & g_1\xi_1 & g_1^2\xi_1 & \cdots & g_1^{n-1}\xi_1 \\ \xi_2 & g_2\xi_2 & g_2^2\xi_2 & \cdots & g_2^{n-1}\xi_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_n & g_n\xi_n & g_n^2\xi_n & \cdots & g_n^{n-1}\xi_n \end{vmatrix}$$

$$= (\pm 1)\cdot | g_1^0 g_2^1 \ldots g_n^{n-1} | \cdot \xi_1\xi_2 \ldots \xi_n.$$

Equating these two values and substituting the expression found at the outset for $\xi_1\xi_2 \ldots \xi_n$ we obtain

$$\sum B_{e_1e_2\ldots e_n} x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} = (\pm 1)\cdot | g_1^0 g_2^1 \ldots g_n^{n-1} | \cdot \sum A_{e_1e_2\ldots e_n} x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n},$$

and thus see that

$$B_{e_1e_2\ldots e_n} = (\pm 1)\cdot | g_1^0 g_2^1 \ldots g_n^{n-1} | \cdot A_{e_1e_2\ldots e_n},$$

as Jacobi had shown in 1845 in the case of $n=3$.

With the help of this, Hesse's first result at once becomes

$$| g_1^0 g_2^1 \ldots g_n^{n-1} |^2 = \sum \Theta_{e_1e_2\ldots e_n} B^2_{e_1e_2\ldots e_n},$$

and the desired end is reached.

CHAPTER XI.

PERSYMMETRIC DETERMINANTS, FROM 1841 TO 1860.

As has already been pointed out (*History*, i. pp. 485–487 *), the special form of determinant named "persymmetric" in 1853 by Sylvester came first to light in 1835 in a paper of Jacobi's on the elimination of the unknown from two equations of the n^{th} degree, the fact being that the adjugate of Bezout's condensed eliminant —in other words, the adjugate of the determinant resulting from Bezout's "abridged method" of elimination—is there shown to be such that the elements of it whose place-numbers have the same sum are equal.

The essentials of the proof are easily made clear if we accept the fact that from the equations

$$\left.\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ b_1x + b_2y + b_3z &= 0 \\ c_1x + c_2y + c_3z &= 0 \end{aligned}\right\}$$

it can be shown for non-zero values of x, y, z that

$$\begin{aligned} x : y : z &:: A_1 : A_2 : A_3 \\ &:: B_1 : B_2 : B_3 \\ &:: C_1 : C_2 : C_3 . \end{aligned}$$

This is something more than what Jacobi had then occasion to use, but in 1841 the portion of it which holds when there is one equation fewer was stated by him in all its generality in § 7 of

* The 7th and 8th lines of p. 486 have unfortunately been transposed by the printer. Also, in the first determinant of the footnote on the same page the first b_1 should be b_0.

the *De Formatione* . . . Specialising from it in two directions, namely, (1) by taking x, x^2, x^3 instead of x, y, z, and (2) by taking the determinant of the coefficients to be axisymmetric, we can assert that if the equations

$$\left.\begin{aligned} ax + fx^2 + ex^3 &= 0 \\ fx + bx^2 + dx^3 &= 0 \\ ex + dx^2 + cx^3 &= 0 \end{aligned}\right\}$$

hold for a non-zero value of x, then

$$\begin{aligned} x : x^2 : x^3 &:: \mathrm{A} : \mathrm{F} : \mathrm{E} \\ &:: \mathrm{F} : \mathrm{B} : \mathrm{D} \\ &:: \mathrm{E} : \mathrm{D} : \mathrm{C}; \end{aligned}$$

and it will follow that $\mathrm{B} = \mathrm{E}$, and that

$$x : x^2 : x^3 : x^4 : x^5 :: \mathrm{A} : \mathrm{F} : \mathrm{E} : \mathrm{D} : \mathrm{C}.$$

Now the set of equations from which Bezout's condensed eliminant is derived is of the very special type here posited: consequently it is seen that the adjugate of the said eliminant must be persymmetric, and that its different elements form an equirational progression whose common multiplier is the root common to the original pair of equations.

ROSENHAIN, G. (1844).

[Exercitationes analyticæ in theorema Abelianum de integralibus functionum algebraicarum. *Crelle's Journ.*, xxviii. pp. 249–278.]

What concerns our subject here is a digression (§§ 5–10, pp. 263–278) on the elimination of the unknown from two equations of the n^{th} degree. The first two sections (pp. 263–268) are little else than a reproduction of part of Jacobi's paper of 1835 dealing with Bezout's so-called "abridged method," and the remainder contains a discussion of other methods. In subject, therefore, the digression resembles Cauchy's paper of 1840.

At this point we have to recall the fact already reported, that in Borchardt's paper of 1845 (January) a determinant of the special form we are now considering appeared as an expression for the square of the difference-product, and that a generalisation of this result was given by Cayley the year following. These two papers as well as four others dealt with under Alternants should be kept in view in reading the present chapter. The full list is—

1845	Borchardt, C. W., p. 159.	1854	Brioschi, F., p. 172.
1846	Cayley, A., p. 162.	1857	Bellavitis, G., p. 181.
1854	Joachimsthal, F., p. 169.	1847	Baltzer, R. p. 183.

JACOBI, C. G. J. (1845, August).

[Ueber die Darstellung einer Reihe gegebner Werthe durch eine gebrochne rationale Function. *Crelle's Journ.*, xxx. pp. 127–156; or *Gesammelte Werke*, iii. pp. 479–511.]

The subject here dealt with by Jacobi is that first considered by Cauchy in the fifth note to the *Analyse Algébrique* of 1821, namely, the extension of Lagrange's interpolation-formula, or the finding of a function u of the form $N(x)/M(x)$ which shall have the values $u_1, u_2, \ldots, u_{n+m+1}$ when x has the values $x_1, x_2, \ldots, x_{n+m+1}$, it being understood that N and M are respectively of the n^{th} and m^{th} degrees in x.

The given $n+m+1$ equations

$$u_1 M(x_1) = N(x_1), \quad u_2 M(x_2) = N(x_2), \quad \ldots\ldots$$

are first used to eliminate the $n+1$ coefficients of $N(x)$, and thereby obtain m equations for the determination of the ratios of the coefficients of $M(x)$. This is interestingly accomplished by using the multipliers $x_1^p/f'(x_1)$, $x_2^p/f'(x_2)$, ..., where $f(x)=(x-x_1)(x-x_2)\ldots(x-x_{n+m+1})$, then performing addition, and finally utilising a known theorem regarding "partial fractions." The result is that for any one value of p we have

$$\sum_{i=1}^{i=n+m+1} \frac{u_i x_i^p M(x_i)}{f'(x_i)} = \sum_{i=1}^{i=n+m+1} \frac{x_i^p N(x_i)}{f'(x_i)};$$

and that therefore when p has any one of the values 0, 1, 2, ... $m-1$, we have

$$\sum_{i=1}^{i=n+m+1}\frac{u_i x_i^p \mathrm{M}(x_i)}{f'(x_i)} = 0.$$

By putting

$$v_p \quad \text{for} \quad \frac{x_1^p u_1}{f'(x_1)} + \frac{x_2^p u_2}{f'(x_2)} + \ldots + \frac{x_{n+m+1}^p u_{n+m+1}}{f'(x_{n+m+1})}$$

and

$$a + a_1 x + a_2 x^2 + \ldots + a_m x^m \quad \text{for} \quad \mathrm{M}(x)$$

these last m equations become

$$\left.\begin{array}{llllll} v_0 a & + v_1 a_1 & + v_2 a_2 & + \ldots & + v_m a_m & = 0 \\ v_1 a & + v_2 a_1 & + v_3 a_2 & + \ldots & + v_{m+1} a_m & = 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ v_{m-1} a & + v_m a_1 & + v_{m+1} a_2 & + \ldots & + v_{2m-1} a_m & = 0 \end{array}\right\}$$

whence for $\mathrm{M}(x)$ there is obtained the expression *

$$\begin{vmatrix} 1 & x & x^2 & \ldots & x_m \\ v_0 & v_1 & v_2 & \ldots & v_m \\ v_1 & v_2 & v_3 & \ldots & v_{m+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ v_{m-1} & v_m & v_{m+1} & \ldots & v_{2m-1} \end{vmatrix},$$

* By making the observation that the v's are neatly expressible as determinants the whole matter may be put much more simply. Thus, taking the case where $u = (\beta_0 + \beta_1 x)/(\alpha_0 + \alpha_1 x + \alpha_2 x^2)$, we see at a glance that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^p(\beta_0+\beta_1 x_1) \\ 1 & x_2 & x_2^2 & x_2^p(\beta_0+\beta_1 x_2) \\ 1 & x_3 & x_3^2 & x_3^p(\beta_0+\beta_1 x_3) \\ 1 & x_4 & x_4^2 & x_4^p(\beta_0+\beta_1 x_4) \end{vmatrix} = 0 \quad \text{when } p = 0 \text{ or } 1,$$

and therefore from the data that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & u_1 x_1^p(\alpha_0+\alpha_1 x_1+\alpha_2 x_1^2) \\ 1 & x_2 & x_2^2 & u_2 x_2^p(\alpha_0+\alpha_1 x_2+\alpha_2 x_2^2) \\ 1 & x_3 & x_3^2 & u_3 x_3^p(\alpha_0+\alpha_1 x_3+\alpha_2 x_3^2) \\ 1 & x_4 & x_4^2 & u_4 x_4^p(\alpha_0+\alpha_1 x_4+\alpha_2 x_4^2) \end{vmatrix} = 0 \quad \text{when } p = 0 \text{ or } 1,$$

or, what is the same thing, that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & u_1 x_1^p \\ 1 & x_2 & x_2^2 & u_2 x_2^p \\ 1 & x_3 & x_3^2 & u_3 x_3^p \\ 1 & x_4 & x_4^2 & u_4 x_4^p \end{vmatrix} \alpha_0 + \begin{vmatrix} 1 & x_1 & x_1^2 & u_1 x_1^{p+1} \\ 1 & x_2 & x_2^2 & u_2 x_2^{p+1} \\ 1 & x_3 & x_3^2 & u_3 x_3^{p+1} \\ 1 & x_4 & x_4^2 & u_4 x_4^{p+1} \end{vmatrix} \alpha_1 + \begin{vmatrix} 1 & x_1 & x_1^2 & u_1 x_1^{p+2} \\ 1 & x_2 & x_2^2 & u_2 x_2^{p+2} \\ 1 & x_3 & x_3^2 & u_3 x_3^{p+2} \\ 1 & x_4 & x_4^2 & u_4 x_4^{p+2} \end{vmatrix} \alpha_2 = 0.$$

or, by further putting $w = v_{p+1} - xv_p$,

$$\begin{vmatrix} w_0 & w_1 & \dots & w_{m-1} \\ w_1 & w_2 & \dots & w_m \\ \cdot & \cdot & \cdot & \cdot \\ w_{m-1} & w_m & \dots & w_{2m-2} \end{vmatrix}.$$

After finding other forms for $M(x)$, and varying (§ 2) the mode of finding them, Jacobi proceeds (§ 3, pp. 140–146) to deal with $N(x)$, first remarking, of course, that the one function is immediately determinable from the other, because the problem of representing $u_1, u_2, \dots$ by $N(x)/M(x)$ is the same as the problem of representing $u_1^{-1}, u_2^{-1}, \dots$ by $M(x)/N(x)$. Instead of utilising this, however, he takes from the theory of "partial fractions" the result

$$-\frac{N(x)}{f(x)} = \sum_{i=1}^{i=n+m+1} \frac{N(x_i)}{(x_i - x)f'(x_i)},$$

whence follows

$$-\frac{N(x)}{f(x)} = \sum_{i=1}^{i=n+m+1} \frac{u_i M(x_i)}{(x_i - x)f'(x_i)};$$

so that if we put

$$R_p \quad \text{for} \quad \sum_{i=1}^{i=n+m+1} \frac{x_i^p u_i}{(x_i - x)f'(x_i)},$$

we have

$$-\frac{N(x)}{f(x)} = a R_0 + a_1 R_1 + \dots + a_m R_m,$$

From these two equations on solving for $a_0 : a_1 : a_2$ and substituting in $a_0 + a_1 x + a_2 x^2$ we obtain

$$M(x) = \begin{vmatrix} 1 & x & x^2 \\ v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where $v^p = | x_1^0 \, x_2^1 \, x_3^2 \, x_4^p u_4 |$, or

$$M(x) = \begin{vmatrix} \omega_0 & \omega_1 \\ \omega_1 & \omega_2 \end{vmatrix}$$

where $\omega_p = v_{p+1} - xv_p = | x_1^0 \, x_2^1 \, x_3^2 \, x_4^p u_4 (x_4 - x) |$.

and therefore, by substituting the already found values of $a : a_1 : a_2 : \ldots . : a_m$,

$$-\frac{\mathrm{N}(x)}{f(x)} = \begin{vmatrix} \mathrm{R}_0 & \mathrm{R}_1 & \ldots & \mathrm{R}_m \\ v_0 & v_1 & \ldots & v_m \\ v_1 & v_2 & \ldots & v_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ v_{m-1} & v_m & \ldots & v_{2m-1} \end{vmatrix}.$$

As, however, $x\mathrm{R}_p + v_p = \mathrm{R}_{p+1}$, we can change the elements of the second row here into $\mathrm{R}_1, \mathrm{R}_2, \ldots, \mathrm{R}_{m+1}$, and then the elements of the third row into $\mathrm{R}_2, \mathrm{R}_3, \ldots, \mathrm{R}_{m+2}$, and so on, thus arriving at a determinant of the same special form as in the case of $\mathrm{M}(x)$.*

Combining the two results, Jacobi is thus led to the theorem that

$$-\frac{1}{f(x)} \cdot \frac{\mathrm{N}(x)}{\mathrm{M}(x)} = \begin{vmatrix} \mathrm{R}_0 & \mathrm{R}_1 & \ldots & \mathrm{R}_m \\ \mathrm{R}_1 & \mathrm{R}_2 & \ldots & \mathrm{R}_{m+1} \\ \mathrm{R}_2 & \mathrm{R}_3 & \ldots & \mathrm{R}_{m+2} \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{R}_m & \mathrm{R}_{m+1} & \ldots & \mathrm{R}_{2m} \end{vmatrix} \div \begin{vmatrix} w_0 & w_1 & \ldots & w_{m-1} \\ w_1 & w_2 & \ldots & w_m \\ \cdot & \cdot & \cdot & \cdot \\ w_{m-1} & w_m & \ldots & w_{2m-1} \end{vmatrix},$$

—a result not easily verifiable by giving x one of its $n+m+1$ values.

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* Continuing the case of the previous footnote we should prefer to begin with

$$\frac{\mathrm{N}(x)}{f(x)} \,|\, x_1^0 x_2^1 x_3^2 x_4^3 \,| = \begin{vmatrix} 1 & x_1 & x_1^2 & \mathrm{N}(x_1)/(x-x_1) \\ 1 & x_2 & x_2^2 & \mathrm{N}(x_2)/(x-x_2) \\ 1 & x_3 & x_3^2 & \mathrm{N}(x_3)/(x-x_3) \\ 1 & x_4 & x_4^2 & \mathrm{N}(x_4)/(x-x_4) \end{vmatrix},$$

and then proceeding exactly as before we should arrive at

$$\frac{\mathrm{N}(x)}{f(x)} \,|\, x_1^0 x_2^1 x_3^2 x_4^3 \,| = \begin{vmatrix} \rho_0 & \rho_1 & \rho_2 \\ \rho_1 & \rho_2 & \rho_3 \\ \rho_2 & \rho_3 & \rho_4 \end{vmatrix},$$

where $\rho_p = |\, x_1^0 \; x_2^1 \; x_3^2 \; x_4^p u_4/(x-x_4) \,|$.

The function sought would then be

$$\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4) \begin{vmatrix} \rho_0 & \rho_1 & \rho_2 \\ \rho_1 & \rho_2 & \rho_3 \\ \rho_2 & \rho_3 & \rho_4 \end{vmatrix}}{|\, x_1^0 x_2^1 x_3^2 x_4^3 \,| \cdot \begin{vmatrix} \omega_0 & \omega_1 \\ \omega_1 & \omega_2 \end{vmatrix}}.$$

For ourselves we may add that the theorem becomes still more interesting when it is pointed out that, by reason of the identity

$$|y_1^0 y_2^1 \ldots y_{n-1}^{n-2} \mathrm{Y}_n| \div |y_1^0 y_2^1 \ldots y_{n-1}^{n-2} y_n^{n-1}| = \frac{\mathrm{Y}_1}{\phi'(y_1)} + \frac{\mathrm{Y}_2}{\phi'(y_2)} + \ldots + \frac{\mathrm{Y}_n}{\phi'(y_n)}$$

where $\phi(y) = (y-y_1)(y-y_2) \ldots (y-y_n)$, the R's like the w's are all expressible as determinants of the order $n+m+1$, that these determinants in both cases belong to the special type known as alternants, and that R_p differs from w_p in the last column only;—in fact, that

$$\mathrm{R}_p = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n+m-1} & x_1^p u_1/(x_1-x) \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n+m-1} & x_2^p u_2/(x_2-x) \\ 1 & x_3 & x_3^2 & \ldots & x_3^{n+m-1} & x_3^p u_3/(x_3-x) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \div \zeta^{\frac{1}{2}},$$

$$w_p = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n+m-1} & x_1^p u_1(x_1-x) \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n+m-1} & x_2^p u_2(x_2-x) \\ 1 & x_3 & x_3^2 & \ldots & x_3^{n+m-1} & x_3^p u_3(x_3-x) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \div \zeta^{\frac{1}{2}},$$

where $\zeta^{\frac{1}{2}}$ is the difference-product of $x_1, x_2, \ldots, x_{n+m+1}$.

BORCHARDT, C. W. (1847, February).

[Développements sur l'équation à l'aide de laquelle on détermine les inégalités séculaires du mouvement des planètes. *Journ. (de Liouville) de Math.*, xii. pp. 50–67; *Gesammelte Werke*, pp. 15–30.]

The new section of this paper, which is an extension of Borchardt's of 1845 (January), is the third (pp. 54–60), and explains at length how, for the purpose of ascertaining the total number of real roots of the equation of the n^{th} degree $f(x)=0$, the coefficients of highest powers in the series of Sturm's functions $f(x)$, $f_1(x)$, $f_2(x)$, ... may be replaced, according to Sylvester, by

$$1,\quad n,\quad \sum(x_2-x_1)^2,\quad \sum(x_2-x_1)^2(x_3-x_1)^2(x_2-x_1)^2,\quad \ldots .$$

where $x_1, x_2, \ldots$ are the roots, and therefore by

$$1,\quad s_0,\quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix},\quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix},\quad \ldots\ldots$$

where $s_r = x_1^r + x_2^r + \ldots + x_n^r$. All this, however, is practically implied in Cayley's paper of 1846 (August).*

SYLVESTER, J. J. (1851, May).

[ESSAY ON CANONICAL FORMS: Supplement to a "Sketch of a Memoir on Elimination, Transformation, and Canonical Forms," 36 pp., London. Or *Collected Math. Papers*, i. pp. 203–216.]

In giving a preliminary notice of his general method for reducing odd-degreed functions to their canonical form, Sylvester says he based his method on the proposition that every one of the n-line minor determinants of the array

$$\begin{matrix} T_1 & T_2 & T_3 & \ldots & T_{n+1} \\ T_2 & T_3 & T_4 & \ldots & T_{n+2} \\ T_3 & T_4 & T_5 & \ldots & T_{n+3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ T_n & T_{n+1} & T_{n+2} & \ldots & T_{2n} \end{matrix}$$

vanishes if

$$T_i = a_1^{r-i} b_1^{s+1} + a_2^{r-i} b_2^{s+i} + \ldots + a_{n-1}^{r-i} b_{n-1}^{s+i}.$$

This, which he hastily calls "a beautiful and striking theorem," and which he generalises in Note B of an Appendix, arises from the simple fact that each determinant is the product of two zeros, T_i being

$$\left(a_1^{r-i},\ a_2^{r-i},\ \ldots,\ a_{n-1}^{r-i},\ 0 \between b_1^{s+i},\ b_2^{s+i},\ \ldots,\ b_{n-1}^{s+1},\ 0\right).$$

It is of more importance, therefore, to recall that it was in

* The proposition Borchardt is concerned with is of course that *The equation* $f(x) = 0$ *has as many pairs of imaginary roots as there are changes of sign in any one of the three series mentioned.*

this year that Sylvester made the fruitful observation, already chronicled,* that the persymmetric determinants

$$ac-b^2, \quad ace+2bcd-ae^2-bd^2-c^3 , \quad \ldots .$$

are expressible as "commutants," or rather that these special determinants could be represented in the umbral notation by using umbræ not wholly unconnected with one another. Thus, while

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} \text{ stands for the general determinant } \begin{vmatrix} 00 & 01 & 02 \\ 10 & 11 & 12 \\ 20 & 21 & 22 \end{vmatrix}$$

so long as the umbræ are understood to be entirely independent, it might also be used to stand for the special determinant

$$\begin{vmatrix} 00 & 01 & 02 \\ 01 & 02 & 03 \\ 02 & 03 & 04 \end{vmatrix}$$

if some mark were added to indicate that in the development 01 is to be put for 10, 02 for 20 or 11, 03 for 12 or 21, and 04 for 22.

SYLVESTER, J. J. (1851, October).

[On a remarkable discovery in the theory of canonical forms and of hyperdeterminants. *Philos. Magazine*, ii. pp. 391–410; or *Collected Math. Papers*, i. pp. 265–283.]

The consideration of the problem of the canonisation of the binary quintic led Sylvester to the more general problem of determining the p's and q's in

$$(p_1x+q_1y)^{2n+1} + (p_2x+q_2y)^{2n+1} + \ldots + (p_{n+1}x+q_{n+1}y)^{2n+1}$$

so as to make this expression identical with

$$a_0x^{2n+1}+(2n+1)a_1x^{2n}y^1+\tfrac{1}{2}(2n+1)2na_2x^{2n+1}y^2+\ldots+a_{2n+1}y^{2n+1}.$$

* See above, pp. 68, ...

This is at once seen to depend on the solution of the peculiar set of $2n+2$ equations

$$\left.\begin{array}{lllll}\pi_1 & + \pi_2 & + \dots\dots + & \pi_{n+1} & = a_0 \\ \pi_1\lambda_1 & + \pi_2\lambda_2 & + \dots\dots + & \pi_{n+1}\lambda_{n+1} & = a_1 \\ \pi_1\lambda_1^2 & + \pi_2\lambda_2^2 & + \dots\dots + & \pi_{n+1}\lambda_{n+1}^2 & = a_2 \\ \cdot & \cdot & \cdot\quad\cdot\quad\cdot & \cdot & \cdot \\ \pi_1\lambda_1^{2n+1} & + \pi_2\lambda_2^{2n+1} & + \dots\dots + & \pi_{n+1}\lambda_{n+1}^{2n+1} & = a_{2n+1}\end{array}\right\}$$

where the new unknowns $\pi_1, \pi_2, \dots\dots, \pi_{n+1}, \lambda_1, \lambda_2, \dots\dots, \lambda_{n+1}$ are introduced merely for shortness' sake, namely

$$\pi_r \text{ for } p_r^{2n+1} \quad \text{and} \quad \lambda_r \text{ for } q_r \div p_r .$$

Taking $n+2$ consecutive equations beginning with the first, and eliminating the π's, there is obtained

$$\begin{vmatrix} 1 & 1 & \dots\dots & 1 & a_0 \\ \lambda_1 & \lambda_2 & \dots\dots & \lambda_{n+1} & a_1 \\ \lambda_1^2 & \lambda_2^2 & \dots\dots & \lambda_{n+1}^2 & a_2 \\ \cdot & \cdot & \cdot\quad\cdot\quad\cdot & \cdot & \cdot \\ \lambda_1^{n+1} & \lambda_2^{n+1} & \dots\dots & \lambda_{n+1}^{n+1} & a_{n+1} \end{vmatrix} = 0,$$

which, if division by the difference-product of the λ's be effected, gives

$$a_{n+1} - a_n \sum\lambda_1 + a_{n-1}\sum\lambda_1\lambda_2 - \dots\dots = 0.$$

A similar result is evidently reached by taking *any* $n+2$ consecutive equations, so that altogether we shall have

$$\left.\begin{array}{l} a_{n+1} - a_n \sum\lambda_1 + a_{n-1}\sum\lambda_1\lambda_2 - \dots\dots = 0 \\ a_{n+2} - a_{n+1} \sum\lambda_1 + a_n\sum\lambda_1\lambda_2 - \dots\dots = 0 \\ a_{n+3} - a_{n+2} \sum\lambda_1 + a_{n+1}\sum\lambda_1\lambda_2 - \dots\dots = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{2n+1} - a_{2n} \sum\lambda_1 + a_{2n-1}\sum\lambda_1\lambda_2 - \dots\dots = 0 \end{array}\right\}$$

—that is to say, a set of $n+1$ equations in the $n+1$ unknowns $\Sigma\lambda_1, \Sigma\lambda_1\lambda_2, \dots\dots, \lambda_1\lambda_2\dots\lambda_{n+1}$, the solution of which is

$$\frac{1}{A_0} = \frac{\sum\lambda_1}{A_1} = \frac{\sum\lambda_1\lambda_2}{A_2} = \dots\dots$$

where A_r is the determinant whose array is got by deleting the $(r+1)^{\text{th}}$ column from the array

$$\begin{array}{lllll} a_{n+1} & a_n & a_{n-1} & \ldots & a_0 \\ a_{n+2} & a_{n+1} & a_n & \ldots & a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{2n+1} & a_{2n} & a_{2n-1} & \ldots & a_n . \end{array}$$

From this it follows that the λ's are the roots of the equation

$$A_0\lambda^{n+1} - A_1\lambda^n + A_2\lambda^{n-1} - \ldots = 0,$$

$$i.e. \qquad \begin{vmatrix} \lambda^{n+1} & \lambda^n & \lambda^{n-1} & \ldots\ldots & \lambda^0 \\ a_{n+1} & a_n & a_{n-1} & \ldots\ldots & a_0 \\ a_{n+2} & a_{n+1} & a_n & \ldots\ldots & a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{2n+1} & a_{2n} & a_{2n-1} & \ldots\ldots & a_n \end{vmatrix} = 0,$$

$$i.e. \qquad \begin{vmatrix} a_{n+1} - a_n\lambda & a_n - a_{n-1}\lambda & \ldots\ldots & a_1 - a_0\lambda \\ a_{n+2} - a_{n+1}\lambda & a_{n+1} - a_n\lambda & \ldots\ldots & a_2 - a_1\lambda \\ \cdot & \cdot & \cdot & \cdot \\ a_{2n+1} - a_{2n}\lambda & a_{2n} - a_{2n-1}\lambda & \ldots\ldots & a_{n+1} - a_n\lambda \end{vmatrix} = 0.$$

On substituting in the first $n+1$ equations of the original set the values of $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ thus found, the values of $\pi_1, \pi_2, \ldots, \pi_{n+1}$ are obtainable from a set of linear equations of the type associated with the name of Lagrange.

The latter part of this procedure is not given by Sylvester, who on reaching the set of equations in $\Sigma\lambda_1$, $\Sigma\lambda_1\lambda_2, \ldots$ suddenly draws the seemingly irrelevant conclusion "that

$$(x+\lambda_1 y)(x+\lambda_2 y)\ldots\ldots(x+\lambda_{n+1}y)$$

is a constant multiple of the determinant

$$\begin{vmatrix} x^{n+1} & -x^n y & x^{n+1}y^2 & \ldots\ldots \\ a_{n+1} & a_n & a_{n-1} & \ldots\ldots \\ a_{n+2} & a_{n+1} & a_n & \ldots\ldots \\ \cdot & \cdot & \cdot & \cdot \\ a_{2n+1} & a_{2n} & a_{2n-1} & \ldots\ldots \end{vmatrix}\text{''}, \quad \text{or } \Delta, \text{ say.}$$

As a matter of fact $(p_1x+q_1y)(p_2x+q_2y)\ldots.(p_{n+1}x+q_{n+1}y)$

$$= p_1p_2\ldots p_{n+1}(x+\lambda_1y)(x+\lambda_2y)\ldots.(x+\lambda_{n+1}y),$$

$$= p_1p_2\ldots p_{n+1}\left(x^{n+1}+\sum\lambda_1\cdot x^ny+\sum\lambda_1\lambda_2\cdot x^{n-1}y^2+\ldots.\right),$$

$$= \frac{p_1p_2\ldots p_{n+1}}{A_0}(A_0x^{n+1}+A_1x^ny+A_2x^{n-1}y^2+\ldots),$$

$$= \frac{p_1p_2\ldots p_{n+1}}{A_0}\Delta,$$

$$= \frac{p_1p_2\ldots p_{n+1}}{A_0}\begin{vmatrix} a_{n+1}y+a_nx & a_ny+a_{n-1}x & \ldots. \\ a_{n+2}y+a_{n+1}x & a_{n+1}y+a_nx & \ldots. \\ \cdot & \cdot & \cdot \\ a_{2n+1}y+a_{2n}x & a_{2n}y+a_{2n-1}x & \ldots. \end{vmatrix},$$

from which we see (1) the point which Sylvester wished to make, namely, that p_1x+q_1y, p_2x+q_2y, ... being viewed as the original unknowns, it is important to know that their values are multiples of the linear factors of Δ, and (2) that

$$\Delta = A_0(x+\lambda_1y)(x+\lambda_2y)\ldots.$$

Of course the conclusion drawn is that the transformation of a binary $(2n+1)$-ic into the sum of $n+1$ powers depends on the solution of a determinantal equation of the $(n+1)^{\text{th}}$ degree. As examples, the quintic and septimic are taken, the latter mainly for the purpose of drawing attention to the fact that the conditions of "catalecticism," that is, of $(a, b, \ldots, h \between x, y)^7$ being expressible in the form of the sum of three seventh powers—instead of four, as the general rule provides—require that the cofactors of the elements of the first row of the determinant

$$\begin{vmatrix} y^4 & -y^3x & y^2x^2 & -yx^3 & x^4 \\ a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g \\ d & e & f & g & h \end{vmatrix}$$

must all vanish, or, what by the homaloidal law is the same thing, that *two* of them vanish.

The analogous problem for even-degreed functions is next taken up, a beginning being made with the transformation of the quartic $(a, b, \ldots, e \between x, y)^4$ into the form

$$(p_1x+q_1y)^4 + (p_2x+q_2y)^4 + 6\epsilon(p_1x+q_1y)^2(p_2x+q_2y)^2.$$

On putting

$$q_1 = p_1\lambda_1, \quad p_2 = q_2\lambda_2, \quad \epsilon p_1^2p_2^2 = \mu, \quad \lambda_1+\lambda_2 = s_1, \quad \lambda_1\lambda_2 = s_2$$

there is obtained by equatement of like powers of x and y

$$\left.\begin{aligned} a &= p_1^4 + p_2^4 + 6\mu \\ b &= p_1^4\lambda_1 + p_2^4\lambda_2 + 3\mu s_1 \\ c &= p_1^4\lambda_1^2 + p_2^4\lambda_2^2 + \mu s_1^2 + 2\mu s_2 \\ d &= p_1^4\lambda_1^3 + p_2^4\lambda_2^3 + 3\mu s_1 s_2 \\ e &= p_1^4\lambda_1^4 + p_2^4\lambda_2^4 + 6\mu s_2^2 \end{aligned}\right\},$$

and from these by operations which lead to the elimination of p_1^4, p_2^4 from every consecutive triad of equations

$$\left.\begin{aligned} as_2 - bs_1 + c - \mu(8s_2-2s_1^2) &= 0 \\ bs_2 - cs_1 + d - \mu(4s_2-s_1^2)s_1 &= 0 \\ cs_2 - ds_1 + e - \mu(8s_2-2s_1^2)s_2 &= 0 \end{aligned}\right\}.$$

or, if we put ν for $-\mu(8s_2-2s_1^2)$,

$$\left.\begin{aligned} as_2 - bs_1 + (c+\nu) &= 0 \\ bs_2 - (c-\tfrac{1}{2}\nu)s_1 + d &= 0 \\ (c+\nu)s_2 - ds_1 + e &= 0 \end{aligned}\right\},$$

From the resulting cubic equation

$$\begin{vmatrix} a & b & c+\nu \\ b & c-\frac{1}{2}\nu & d \\ c+\nu & d & e \end{vmatrix} = 0$$

ν can be determined, and thence in backward order s_1, s_2; μ; λ_1, λ_2; p_1, p_2; q_1, q_2; m.

In passing, note is taken of the fact that the said cubic when arranged according to powers of ν is

$$\nu^3 - (ae-4bd+3c^2)\nu + 2\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = 0,$$

and that $ae - 4bd + 3c^2$ and the determinant here appearing are the two invariants * of the quartic under investigation.

The reduction of the octavic $(a_0, a_1, \ldots, a_8 \between x, y)^8$ to the form

$$u_1{}^8 + u_2{}^8 + u_3{}^8 + u_4{}^8 + 70\epsilon u_1{}^2 u_2{}^2 u_3{}^2 u_4{}^2,$$

where $u_r = p_r x + q_r y$, is shown in similar fashion to depend on the solution of the quintic equation

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 - \nu \\ a_1 & a_2 & a_3 & a_4 + \frac{1}{4}\nu & a_5 \\ a_2 & a_3 & a_4 - \frac{1}{6}\nu & a_5 & a_6 \\ a_3 & a_4 + \frac{1}{4}\nu & a_5 & a_6 & a_7 \\ a_4 - \nu & a_5 & a_6 & a_7 & a_8 \end{vmatrix} = 0,$$

where

$$\nu = 72\epsilon p_1{}^2 p_2{}^2 p_3{}^2 p_4{}^2 \mathrm{I} \quad \text{and} \quad \mathrm{I} = s_4 - \tfrac{1}{4}s_1 s_3 + \tfrac{1}{12}s_2{}^2,$$

I being the quadratic invariant of

$$x^4 + s_1 x^3 y + s_2 x^2 y^2 + s_3 x y^3 + s_4 y^4 \quad \text{or} \quad (x+\lambda_1 y)(x+\lambda_2 y)(x+\lambda_3 y)(x+\lambda_4 y).$$

The fact that the coefficients of ν^3, ν^2, ν^1, ν^0 are invariants of the octavic is insisted on, and generalisations are effected for functions of the degree $4m$ and the degree $4m+2$.

Further, it is pointed out that when the said even-degreed functions after transformation are without the last (or unique) term,—that is to say, are in Sylvester's phraseology "meio-catalectic,"—the last of the series of invariants must vanish: for example, the condition that $(a_0, a_1, \ldots, a_6 \between x, y)^6$ may be expressible as the sum of three sixth powers is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix} = 0.$$

This, of course, may be proved independently, but is seen to be a conclusion from putting $\epsilon = 0$ in the foregoing.

* The term "invariant" is first used in this paper.

SYLVESTER, J. J. (1852, April).

[On the principles of the calculus of forms. *Cambridge and Dub. Math. Journ.*, vii. pp. 52–97, 179–217; or *Collected Math. Papers*, i. pp. 284–327, 328–363.]

Here the same subjects and the same special determinants are dealt with as in the preceding; and the determinant whose vanishing has been seen to be the condition for "meio-catalecticism" is denominated (p. 62) the *catalecticant** of the even-degreed function in question, while the determinant whose resolution into linear factors furnishes Sylvester's canonical form of an odd-degreed function is called the *canonizant*† of the said function. As the former is an invariant of its function, so the latter is a covariant.

BRUNO, F. FAÀ DI (1852, May).

[Démonstration d'un théorème relatif à la réduction des fonctions homogènes à deux lettres à leur forme canonique. *Journ. (de Liouville) de Math.* . . . xvii. pp. 193–201.]

The subject of the whole of this paper is simply the solution of the set of equations dealt with in Sylvester's paper of 1851 (October). The process is lengthy and uninviting, the sole point of interest being that the equation in λ comes out in the form

$$\begin{vmatrix} a_0\lambda - a_1 & a_1\lambda - a_2 & \dots & a_n\lambda - a_{n+1} \\ a_0\lambda^2 - a_2 & a_1\lambda^2 - a_3 & \dots & a_n\lambda^2 - a_{n+2} \\ \cdot & \cdot & \cdot & \cdot \\ a_0\lambda^{n+1} - a_{n+1} & a_1\lambda^{n+1} - a_{n+2} & \dots & a_n\lambda^{n+1} - a_{2n+1} \end{vmatrix} = 0,$$

where the determinant is easily shown to be the same as one of Sylvester's forms by diminishing each row in order, beginning with the last, by λ times the row immediately preceding.

* "Meicatalecticizant," Sylvester truly says, would have been the more correct word, but even he took alarm sometimes.

† The name would have been equally appropriate for the determinants of the preceding paper which have ν in their diagonal.

CHIO, F. (1853, June).

[Mémoire sur les fonctions connues sous le nom de résultantes ou de déterminans. 32 pp., Turin.]

The second part (pp. 23–32) of Chio's memoir, which is headed "Exemples," mainly concerns Sylvester's set of equations of 1851 (October). His procedure is much more interesting than Faà di Bruno's. Using any multipliers $A_0, A_1, \ldots.$ with the first $n+2$ equations he obtains by addition

$$\begin{aligned}
&x_0(A_0+A_1\lambda_0+A_2\lambda_0^2+\ \ldots.\ +A_{n+1}\lambda_0^{n+1}) \\
+\ &x_1(A_0+A_1\lambda_1+A_2\lambda_1^2+\ \ldots.\ +A_{n+1}\lambda_1^{n+1}) \\
+\ &\ldots\ldots\ldots\ldots\ldots \\
+\ &x_n(A_0+A_1\lambda_n+A_2\lambda_n^2+\ \ldots.\ +A_{n+1}\lambda_n^{n+1}) = A_0a_0+A_1a_1+\ \ldots.\ +A_{n+1}a_{n+1};
\end{aligned}$$

and, the ratios of $A_0, A_1, \ldots.$ being supposed to be determined so as to make the coefficients of $x_0, x_1, \ldots, x_n$ vanish, there results

$$A_0a_0 + A_1a_1 + \ldots. + A_{n+1}a_{n+1} = 0.$$

If each succeeding set of $n+2$ consecutive equations be treated in the same manner, it will be found that the *same* multipliers will make the coefficients of the x's vanish in every case: consequently there is obtained

$$\begin{aligned}
A_0a_1 + A_1a_2 \quad + \ldots. + A_{n+1}a_{n+2} &= 0, \\
A_0a_2 + A_1a_3 \quad + \ldots. + A_{n+1}a_{n+3} &= 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots & \\
A_0a_n + A_1a_{n+1} + \ldots. + A_{n+1}a_{2n+1} &= 0.
\end{aligned}$$

This derived set of $n+1$ equations suffices to give the values of the ratios of $A_0, A_1, \ldots, A_{n+1}$ in terms of the a's, and the substitution of the said values in

$$A_0 + A_1\lambda + A_2\lambda^2 + \ldots. + A_{n+1}\lambda^{n+1} = 0$$

gives the equation for the determination of the λ's.

It is not noted by the author that having $n+2$ equations linear and homogeneous in the A's he could at once deduce

$$\begin{vmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^{n+1} \\ a_0 & a_1 & a_2 & \dots & a_{n+1} \\ a_1 & a_2 & a_3 & \dots & a_{n+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n+1} & a_{n+2} & \dots & a_{2n+1} \end{vmatrix} = 0.$$

The other forms of the equation, however, he gives full attention to.

SYLVESTER, J. J. (1853, June).

[On a theory of the syzygetic relations of two rational integral functions, ... *Philos. Transac. R. Soc.* (London). cxliii. pp. 407–548; or *Collected Math. Papers*, i. pp. 429–586.]

When dealing in art. 7 with Bezout's condensed eliminant of two equations of the n^{th} degree, Sylvester illustrates by the case of $n=5$, that is to say, where the equations are

$$\left.\begin{aligned} a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5 &= 0 \\ b_0x^5+b_1x^4+b_2x^3+b_3x^2+b_4x+b_5 &= 0 \end{aligned}\right\},$$

pointing out that the eliminant may be constructed by first forming the array

$$\begin{matrix} |a_0b_1| & |a_0b_2| & |a_0b_3| & |a_0b_4| & |a_0b_5| \\ |a_0b_2| & |a_0b_3| & |a_0b_4| & |a_0b_5| & |a_1b_5| \\ |a_0b_3| & |a_0b_4| & |a_0b_5| & |a_1b_5| & |a_2b_5| \\ |a_0b_4| & |a_0b_5| & |a_1b_5| & |a_2b_5| & |a_3b_5| \\ |a_0b_5| & |a_1b_5| & |a_2b_5| & |a_3b_5| & |a_4b_5| \end{matrix}$$

and then, as it were, superposing the array

$$\begin{matrix} |a_1b_2| & |a_1b_3| & |a_1b_4| \\ |a_1b_3| & |a_1b_4| & |a_2b_4| \\ |a_1b_4| & |a_2b_4| & |a_3b_4| \end{matrix}$$

and next the array

$$|a_2b_3|.$$

In regard to these arrays he says in a footnote (p. 424), "A square arrangement having this kind of symmetry, namely, such as obtains in the so-called Pythagorean addition-table as distinguished from that which obtains in the multiplication-table, may be universally called *persymmetric*." This is apparently the first use of the word.

SPOTTISWOODE, W. (1853, August).

[Elementary theorems relating to determinants. Second edition, ... *Crelle's Journ.*, li. pp. 209–271, 328–381.]

Just as Spottiswoode viewed an axisymmetric determinant as the determinant of an n-ary quadric, so he closely associated a persymmetric determinant with an even-ordered binary quantic. Taking, for example, the binary quartic which Cayley would a year later have denoted by $(a_0, a_1, \ldots, a_4 \between x, y)^4$, namely,

$$a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4,$$

Spottiswoode writes it in the form *

$$\begin{aligned} &(a_0x^2+2a_1xy+a_2y^2)x^2 \\ +2&(a_1x^2+2a_2xy+a_3y^2)xy \\ +\ &(a_2x^2+2a_3xy+a_4y^2)y^2; \end{aligned}$$

* A preferable form, because making the "catalecticant" still more prominent, is

$$\begin{array}{ccc|l} x^2 & 2xy & y^2 & \\ \hline a_0 & a_1 & a_2 & x^2 \\ a_1 & a_2 & a_3 & 2xy \\ a_2 & a_3 & a_4 & y^2. \end{array}$$

Similarly an odd-degreed function may be represented so as to bring the "canonizant" into prominence: for example $(a, b, \ldots, f \between x, y)^5$ may be written $(ax+by, bx+cy, \ldots, ex+fy \between x, y)^4$, or

$$\begin{array}{ccc|l} x^2 & 2xy & y^2 & \\ \hline ax+by & bx+cy & cx+dy & x^2 \\ bx+cy & cx+dy & dx+ey & 2xy \\ cx+dy & dx+ey & ex+fy & y^2. \end{array}$$

and calling it U points out that

$$\left.\begin{aligned}\frac{\partial^2 U}{\partial x^2} &= 12(a_0x^2+2a_1xy+a_2y^2)\\ \frac{\partial^2 U}{\partial x\,\partial y} &= 12(a_1x^2+2a_2xy+a_3y^2)\\ \frac{\partial^2 U}{\partial y^2} &= 12(a_2x^2+2a_3xy+a_4y^2)\end{aligned}\right\},$$

and thus like Sylvester concludes that the evanescence of

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

is the condition that the second differential-quotients of U shall simultaneously vanish, or, say, that we shall have

$$\frac{\partial^2 U}{\partial (x,y)^2} = 0.$$

BRIOSCHI, F. (1854, March).

[LA TEORICA DEI DETERMINANTI, E LE SUE PRINCIPALI APPLICAZIONI. viii+116 pp. Pavia.]

Denoting by s_r the sum of the r^{th} powers of the roots of the equation

$$a_n \quad + a_{n-1}x + \ldots + a_1x^{n-1} + x^n \quad = 0,$$

Brioschi recalls the n known relations

$$\begin{array}{llllll} a_ns_0 & + a_{n-1}s_1 & + \ldots & + a_1s_{n-1} & + s_n & = 0 \\ a_ns_1 & + a_{n-1}s_2 & + \ldots & + a_1s_n & + s_{n+1} & = 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_ns_{n-1} & + a_{n-1}s_n & + \ldots & + a_1s_{2n-2} & + s_{2n-1} & = 0, \end{array}$$

and thus derives by elimination

$$\begin{vmatrix} s_0 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \ldots & s_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix} = 0,$$

and therefore

$$\begin{vmatrix} s_1 - s_0x & s_2 - s_1x & \dots & s_n - s_{n-1}x \\ s_2 - s_1x & s_3 - s_2x & \dots & s_{n+1} - s_nx \\ \cdot & \cdot & \cdot & \cdot \\ s_n - s_{n-1}x & s_{n+1} - s_nx & \dots & s_{2n-1} - s_{2n-2}x \end{vmatrix} = 0.$$

Further, he points out that if the last determinant be denoted by V_n, and the cofactor of its last element by V_{n-1}, and so on, then V_n being axisymmetric it follows from Cauchy's theorem of 1829 that $V_n, V_{n-1}, \dots, V_1, 1$ possess the characteristic property of Sturm's remainders.

It is not noted that the set of n relations used gives each of the a's in terms of the s's, and that substitution in the original equation then gives

$$\begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix} = (x^n + a_1x^{n-1} + \dots + a_n) \begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix},$$

as may be otherwise seen.

BRIOSCHI, F. (1854, February).

[Sur les fonctions de Sturm. *Nouv. Annales de Math.*, xiii. pp. 71–80; or *Opere mat.*, v. pp. 89–97.]

Brioschi in effect here recalls that if $f, f_1, f_2, \dots$ be the series of Sturm's functions originating in the consideration of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a^n = 0, \quad \text{or, say,} \quad f(x) = 0,$$

and $q_1, q_2, \dots$ be the linear functions of x which are the quotients obtained in the process of finding $f_2, f_3, \dots$ then

(1) $f = q_1f_1 - f_2, \; f_1 = q_2f_2 - f_3, \; \dots, \; f_{r-2} = q_{r-1}f_{r-1} - f_r$;

(2) f_r is of the $(n-r)^{\text{th}}$ degree in x;

(3) from (1) $\dfrac{f_1}{f} = \dfrac{1}{q_1-}\,\dfrac{1}{q_2-}\,\dfrac{1}{q_3-}\ldots\ldots;$

(4) the successive convergents $\dfrac{1}{q_1}$, $\dfrac{q_2}{q_1q_2-1}$, ... to this continued fraction being N_1/D_1, N_2/D_2, ...

$$N_r = \begin{vmatrix} q_2 & 1 & . & \ldots & . \\ 1 & q_3 & 1 & \ldots & . \\ . & 1 & q_4 & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & q_r \end{vmatrix}, \quad D_r = \begin{vmatrix} q_1 & 1 & . & \ldots & . \\ 1 & q_2 & 1 & \ldots & . \\ . & 1 & q_3 & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & q_r \end{vmatrix}.$$

(5) from (1) after eliminating $f_2, f_3, \ldots, f_{r-1}$ by repeated substitution or otherwise

$$f_r = D_{r-1}f_1 - N_{r-1}f;$$

(6) from (1) after eliminating $f_1, f_2, \ldots, f_{r-2}$

$$f = D_{r-1}f_{r-1} - D_{r-2}f_r;$$

With these facts before him he seeks to find expressions for f_r, D_r, N_r, or, say, for the coefficients in

$$\begin{aligned} &A_{r,1}x^{n-r} + A_{r,2}x^{n-r-1} + \ldots, \\ &B_{r,1}x^{r} \quad + B_{r,2}x^{r-1} \quad + \ldots, \\ &C_{r,1}x^{r-1} + C_{r,2}x^{r-2} \quad + \ldots, \end{aligned}$$

failing to note that, Cayley having in 1846 found such an expression for Sylvester's substitute for f_r, the annexure of a known multiplier to Cayley's result would have given him the most important of the three expressions sought.

In the first place he deduces from (4) that the coefficient of the highest power of x in D_{r-1} is always $\dfrac{1}{n}$ of the coefficient of the highest power of x in N_{r-1}, because $q_1 = \dfrac{1}{n}x + \dfrac{a_1}{n^2}$; and from (6), by equating coefficients of x^n, that the coefficient of the highest power of x in f_{r-1} is the reciprocal of the highest power of x in D_{r-1}: in other words, that

$$C_{r,1} = nB_{r,1} = n/A_{r,1}.$$

In the next place, denoting the roots of the given equation by $x_1, x_2, \ldots, x_n$ he has from the theory of "partial fractions"

$$\sum_{i=1}^{i=n}\frac{f_r(x_i)}{f_1(x_i)} = 0, \quad \sum_{i=1}^{i=n}\frac{x_i f_r(x_i)}{f_1(x_i)} = 0, \quad \sum_{i=1}^{i=n}\frac{x_i^2 f_r(x_i)}{f_1(x_i)} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \;, \quad \sum_{i=1}^{i=n}\frac{x_i^{r-2} f_r(x_i)}{f_1(x_i)} = 0, \quad \sum_{i=1}^{i=n}\frac{x_i^{r-1} f_r(x_i)}{f_1(x_i)} = \mathrm{A}_{r,1};$$

and therefore from (5)

$$\sum_{i=1}^{i=n}\mathrm{D}_{r-1}(x_i) = 0, \quad \sum_{i=1}^{i=n}x_i\mathrm{D}_{r-1}(x_i) = 0, \quad \sum_{i=1}^{i=n}x_i^2\mathrm{D}_{r-1}(x_i) = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \;, \quad \sum_{i=1}^{i=n}x_i^{r-2}\mathrm{D}_{r-1}(x_i) = 0, \quad \sum_{i=1}^{i=n}x_i^{r-1}\mathrm{D}_{r-1}(x_i) = \mathrm{A}_{r,1};$$

and consequently on putting

$$\mathrm{B}_{r-1,1}x^{r-1} + \mathrm{B}_{r-1,2}x^{r-2} + \ldots + \mathrm{B}_{r-1,r} \quad \text{for} \quad \mathrm{D}_{r-1}(x)$$

$$\text{and} \quad s_m \quad \text{for} \quad x_1^m + x_2^m + \ldots + x_n^m$$

there results

$$\left.\begin{array}{llll}
\mathrm{B}_{r-1,r}s_0 & + \mathrm{B}_{r-1,r-1}s_1 & + \ldots + \mathrm{B}_{r-1,1}s_{r-1} & = 0 \\
\mathrm{B}_{r-1,r}s_1 & + \mathrm{B}_{r-1,r-1}s_2 & + \ldots + \mathrm{B}_{r-1,1}s_r & = 0 \\
\cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\
\mathrm{B}_{r-1,r}s_{r-2} & + \mathrm{B}_{r-1,r-1}s_{r-1} & + \ldots + \mathrm{B}_{r-1,1}s_{2r-3} & = 0 \\
\mathrm{B}_{r-1,r}s_{r-1} & + \mathrm{B}_{r-1,r-1}s_r & + \ldots + \mathrm{B}_{r-1,1}s_{2r-2} & = \mathrm{A}_{r,1}
\end{array}\right\}.$$

The solution of this set of equations gives the B's in terms of $\mathrm{A}_{r,1}$ and the s's, so that the finding of $\mathrm{A}_{r,1}$ in terms of the s's is the next desideratum. This with the help of the relation $\mathrm{A}_{r,1}\,\mathrm{B}_{r,1}=1$ is easily obtained; for from the set of equations it is seen that Δ_r being written for the persymmetric determinant of s's

$$\mathrm{B}_{r-1,1} = \frac{\mathrm{A}_{r,1}\Delta_{r-1}}{\Delta_r}$$

and therefore

$$\mathrm{A}_{r,1} = \frac{\Delta_r}{\Delta_{r-1}}\cdot\frac{1}{\mathrm{A}_{r-1,1}};$$

whence,

$$\left.\begin{aligned} &\text{for } r \text{ even} \qquad A_{r,1} = \left(\frac{\Delta_2\Delta_4 \ldots \Delta_{r-2}}{\Delta_1\Delta_3 \ldots \Delta_{r-1}}\right)^2 \Delta_r \\ \text{and} \\ &\text{for } r \text{ odd} \qquad A_{r,1} = \left(\frac{\Delta_1\Delta_3 \ldots \Delta_{r-2}}{\Delta_2\Delta_4 \ldots \Delta_{r-1}}\right)^2 \Delta_r \end{aligned}\right\},$$

—a result in agreement, as far as it goes, with Sturm's of 1842, Sturm's non-determinant p_r being the equivalent of Brioschi's Δ_r.

The obtaining of Δ_r in terms of the coefficients of $f(x)$ is next illustrated by changing Δ_4 into the form *

$$\begin{vmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & s_0 & s_1 & s_2 & s_4 \\ . & s_0 & s_1 & s_2 & s_3 & s_4 \\ s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{vmatrix}$$

and performing operations which we may denote by

$$\begin{aligned} &\mathrm{col}_6 + a_1\mathrm{col}_5 + a_2\mathrm{col}_4 + \ldots . + a_5\mathrm{col}_1, \\ &\mathrm{col}_5 + a_1\mathrm{col}_4 + \ldots . + a_4\mathrm{col}_1, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\mathrm{col}_2 + a_1\mathrm{col}_1, \end{aligned}$$

* Brioschi (and afterwards Baltzer, § 12, 9) would have done better to change into the similar form of the *seventh* order, for then the result would have been

$$\Delta_4 = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ . & 1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ . & . & 1 & a_1 & a_2 & a_3 & a_4 \\ . & . & . & n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 \\ . & . & n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 \\ . & n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 & (n-5)a_5 \\ n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 & (n-5)a_5 & (n-6)a_6 \end{vmatrix},$$

in which the determinant on the right has a simpler law of formation than Brioschi's and yet is readily reducible to the latter, and which, as we see on putting $n=4$, has the further merit of showing that Δ_n equals the dialytic eliminant of $f(x)=0$, $f'(x)=0$.

the result being

$$\Delta_4 = \begin{vmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & 5a_5 & 6a_6 \\ 1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ . & 1 & a_1 & a_2 & a_3 & a_4 \\ . & . & n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 \\ . & n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 \\ n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 & (n-5)a_5 \end{vmatrix}.$$

To obtain the desired expression for $D_{r-1}(x)$ Brioschi takes the set of equations in the x's together with the equation from which the set was derived, and eliminates the B's, the result in the case of $r=3$ being

$$\begin{vmatrix} 1 & x & x^2 & -D_2 \\ s_0 & s_1 & s_2 & . \\ s_1 & s_2 & s_3 & . \\ s_2 & s_3 & s_4 & A_{3,1} \end{vmatrix} = 0,$$

whence of course he deduces for D_2 the expression

$$\frac{A_{3,1}}{\Delta_3}\begin{vmatrix} 1 & x & x^2 \\ s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \end{vmatrix} \quad \text{or} \quad \left(\frac{\Delta_1}{\Delta_2}\right)^2 \begin{vmatrix} 1 & x & x^2 \\ s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \end{vmatrix}$$

and the alternative form

$$\frac{n^2}{\begin{vmatrix} a_1 & 2a_2 \\ n & (n-1)a_1 \end{vmatrix}^2} \cdot \begin{vmatrix} a_1 & 2a_2 & 3a_3 \\ n & (n-1)a_1 & (n-2)a_2 \\ 1 & x+a_1 & x^2+a_1x+a_2 \end{vmatrix}.$$

The process of finding f_r is quite similar to this but much more troublesome, the equation taken along with the set of equations in the s's preparatory for elimination being

$$\frac{f_r(x)}{f(x)} = B_{r-1,r}u_0 + B_{r-1,r-1}u_1 + \ldots + B_{r-1,1}u_{r-1}$$

where $$u_m = \frac{x_1^m}{x-x_1} + \frac{x_2^m}{x-x_2} + \ldots + \frac{x_n^m}{x-x_n}.$$

The two previous steps necessary to reach this are

$$\frac{f_r(x)}{f(x)} = \sum_{i=1}^{i=n} \frac{f_r(x_i)}{(x-x_i)f_1(x_i)} = \sum_{i=1}^{i=n} \frac{D_{r-1}(x_i)}{x-x_i};$$

and the result of the elimination is

$$\frac{f_r(x)}{f(x)} = \frac{A_r}{\Delta_r}\begin{vmatrix} s_0 & s_1 & \dots & s_{r-1} \\ s_1 & s_2 & \dots & s_r \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ s_{r-2} & s_{r-1} & \dots & s_{2r-3} \\ u_0 & u_1 & \dots & u_{r-1} \end{vmatrix}$$

which in the case of $r=4$ is changed by means of the substitutions

$$u_m = xu_{m-1} - s_{m-1}, \quad u_0 = f_1(x) \div f(x),$$

into

$$f_4(x) = \left(\frac{\Delta_2}{\Delta_1\Delta_3}\right)^2 \begin{vmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & 5a_5 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ \cdot & n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 \\ n & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 \\ \cdot & f_1(x) & Z_1 & Z_2 & Z_3 \end{vmatrix}$$

where

$$\begin{aligned} Z_1 &= (x+a_1)f_1(x) - n\,.\,f(x), \\ Z_2 &= (x^2+a_1x+a_2)f_1(x) - [nx+(n-1)a_1]f(x), \\ Z_3 &= (x^3+a_1x^2+a_2x+a_3)f_1(x)-[nx^2+(n-1)a_1x+(n-2)a_2]f(x). \end{aligned}$$

The worthlessness of this in itself is apparent as soon as we note the presence of $f_1(x)$ and $f(x)$: when, however, the determinant is partitioned into two, one having $f_1(x)$ for a factor and the other $f(x)$, and the result compared with $f_4 = D_3f_1 - N_3f$, we obtain for N_3 an expression similar to that for D_3.

BRIOSCHI, F. (1854, August).

[Intorno ad alcune questioni d' algebra superiore. *Annali di Sci. mat. e fis.*, v. pp. 301–312; or French translation of Brioschi's *Teorica dei Determinanti*, pp. 151–170; or *Opere mat.*, i. pp. 127–142.]

The questions referred to are much the same as those of his paper on Sturm's functions (1854, February), the first function

$$a_0x^n + a_1x^{n-1} + \ldots + a_n \quad \text{or} \quad a_0(x-x_1)(x-x_2)\ldots(x-x_n) \quad \text{or} \quad f(x)$$

being, however, no longer connected with the second

$$b_0x^m + b_1x^{m-1} + \ldots + b_m \quad \text{or} \quad \phi(x) \quad \text{where} \quad m < n.$$

Putting

$$\mathrm{S}_r \quad \text{for} \quad \frac{x_1^r\phi(x_1)}{f'(x_1)} + \frac{x_2^r\phi(x_2)}{f'(x_2)} + \ldots + \frac{x_n^r\phi(x_n)}{f'(x_n)}$$

he first proves the interesting theorem

$$a_0\mathrm{S}_r + a_1\mathrm{S}_{r-1} + \ldots + a_r\mathrm{S}_0 = b_{r+m-n+1} \qquad (r < n).$$

Then temporarily denoting

$$\left\{\phi(x_r) \div f'(x_r)\right\}^{\frac{1}{2}} \quad \text{by} \quad \mathrm{A}_r$$

he squares in two ways the determinant

$$\begin{vmatrix} \mathrm{A}_1 & \mathrm{A}_2 & \ldots & \mathrm{A}_n \\ \mathrm{A}_1x_1 & \mathrm{A}_2x_2 & \ldots & \mathrm{A}_nx_n \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{A}_1x_1^{n-1} & \mathrm{A}_2x_2^{n-1} & \ldots & \mathrm{A}_nx_n^{n-1} \end{vmatrix},$$

thus obtaining

$$\begin{vmatrix} \mathrm{S}_0 & \mathrm{S}_1 & \ldots & \mathrm{S}_{n-1} \\ \mathrm{S}_1 & \mathrm{S}_2 & \ldots & \mathrm{S}_n \\ \cdot & \cdot & \cdot & \cdot \\ \mathrm{S}_{n-1} & \mathrm{S}_n & \ldots & \mathrm{S}_{2n-2} \end{vmatrix} = \mathrm{A}_1^2\mathrm{A}_2^2 \ldots \mathrm{A}_n^2 \begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_2 & \ldots & s_{2n-2} \end{vmatrix},$$

from which, on putting $(-1)^{\frac{1}{2}n(n-1)}f'(x_1) . f'(x_2) \ldots f'(x_n)$ for the determinant on the right* and substituting for the A's, he deduces

$$\begin{vmatrix} S_0 & S_1 & \dots & S_{n-1} \\ S_1 & S_2 & \dots & S_n \\ \cdot & \cdot & \cdot & \cdot \\ S_{n-1} & S_n & \dots & S_{2n-2} \end{vmatrix} = (-1)^{n(n-1)}\phi(x_1)\,.\,\phi(x_2)\,.\,.\,.\,\phi(x_n).$$

This result, be it noted, is not given in the original paper, but appears first in Combescure's translation (1856), which contains six pages (pp. 153–159) more than the original. Brioschi does not point out its significance in connection with Euler's first form of the resultant of $f(x)=0$, $\phi(x)=0$.

The remainder of the paper is of little interest in the present connection.

BRIOSCHI, F. (1855, January).

[Sur les questions 241 et 141. *Nouv. Annales de Math.*, xiv. pp. 20–24; or *Opere mat.*, v. pp. 107–111.]

If for all positive integral values of r and s we have

$$A_{r+s} = a_1A_{r+s-1} + a_2A_{r+s-2} + \dots + a_sA_r\,,$$

—in other words, if this last be a "recurrence-formula,"—it is readily seen that the last column of the persymmetric determinant

$$\begin{vmatrix} A_r & A_{r+1} & \dots & A_{r+s-1} \\ A_{r+1} & A_{r+2} & \dots & A_{r+s} \\ \cdot & \cdot & \cdot & \cdot \\ A_{r+s-1} & A_{r+s} & \dots & A_{r+2s-2} \end{vmatrix}, \quad \text{or } \Delta_{rs} \text{ say,}$$

may be legitimately changed into

$$a_sA_{r-1}\,,\ a_sA_r\,,\ \dots,\ a_sA_{r+s-2}$$

so that there is deducible

$$\Delta_{r,s} = (-1)^{s-1}a_s\Delta_{r-1,s}\,,$$

* Brioschi unfortunately neglects the sign-factor. See *History*, i. p. 345, where the footnote might have made mention of the fact that the identity there spoken of as used by Jacobi had already appeared in one of Cauchy's own memoirs of the year 1813. (See *Journ. de l'éc. polyt.*, x. cah. 17, p. 485.)

and thence

$$\Delta_{r,s} = (-1)^{r(s-1)} a_s^r \Delta_{0,s},$$

thus implying that $\Delta_{r,s}/(-1)^{r(s-1)}a_s^r$ is independent of r,—a result suggested to Brioschi by an old proposition of Euler's which is referred to in our chapter on Continuants.

CAYLEY, A. (1856, March).

[A third memoir on quantics. *Philos. Transac. R. Soc.* (London), cxlvi. pp. 627–647; or *Collected Math. Papers,* ii. pp. 310–335.]

Among Cayley's tables of invariants there naturally appear the catalecticants of the binary quartic, sextic, and octavic; so that we have from him the final expansions of the persymmetric determinants of the 3rd, 4th, 5th orders. Of canonizants only that of the quintic is given. The four results are those which he numbers 10, 34, 35, 16. The first and last we need not reproduce. The second is

$$\begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{vmatrix} = \begin{cases} aceg - acf^2 - ad^2g + 2adef - ae^3 - b^2eg + b^2f^2 \\ + 2bcdg - 2bcef - 2bd^2f + 2bde^2 - c^3g + 2c^2df \\ + c^2e^2 - 3cd^2e + d^4, \end{cases}$$

where the terms are arranged, as with Cayley, in alphabetical order. The third, altered in form, is

$$\begin{vmatrix} a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g \\ d & e & f & g & h \\ e & f & g & h & i \end{vmatrix} = \begin{aligned}[t] & e^5 - e^3(ai + 3cg + 4df) \\ & + e^2\left\{2(afh + bdi) + (ag^2 + c^2i) + 4(bfg + cdh) + 3(cf^2 + d^2g)\right\} \\ & - e\left\{(ach^2 + b^2gi) + 2(adgh + bcfi) + 3(af^2g + cd^2i)\right. \\ & \quad + 4(bdg^2 + c^2fh) + 2(bf^3 + d^3h) - acgi \\ & \quad \left. - 2adfi - b^2h^2 - 2bcgh - 2c^2g^2 + 2cdfg + 3d^2f^2\right\} \\ & + \left\{-(acf^2i + ad^2gi) + 2(acfgh + bcdgi) - (acg^3 + c^3gi)\right. \\ & \quad + (ad^2h^2 + b^2f^2i) - 2(adf^2h + bd^2fi) + 2(adfg^2 + c^2dfi) \\ & \quad + (af^4 + d^4i) - 2(b^2fgh + bcdh^2) + (b^2g^3 + c^3h^2) \\ & \quad + 2(bcf^2h + bd^2gh) - 2(bcfg^2 + c^2dgh) + 2(bdf^2g + cd^2fh) \\ & \quad \left. + (c^2f^2g + cd^2g^2) - 2(cdf^3 + d^3fg)\right\}. \end{aligned}$$

It is the term, L, independent of λ in the almost persymmetric determinant which Cayley * on Sylvester's suggestion calls the *lamdaic*, namely,

$$\begin{vmatrix} a & b & c & d & e-12\lambda \\ b & c & d & e+3\lambda & f \\ c & d & e-2\lambda & f & g \\ d & e+3\lambda & f & g & h \\ e-12\lambda & f & g & h & i \end{vmatrix},$$

and which, if I, J, K be the other invariants of the octavic, is equal to

$$-2592\lambda^5 + 18\mathrm{I}\lambda^3 + 3\mathrm{J}\lambda^2 + 2\mathrm{K}\lambda + \mathrm{L}.$$

The expansions of I, J, K are those numbered 39, 43, 44 in Cayley's collection.

SCHEIBNER, W. (1856, May).

[Ueber die Auflösung eines gewissen Gleichungsystems. *Berichte .... Ges. d. Wiss.* (Leipzig): *math.-phys. Cl.*, viii. pp. 65–76.]

This is still another attempt to deal with Sylvester's set of equations of 1851 (October); but any interest which the process of solution possesses is unconnected with determinants.

BRUNO, F. FAA DI (1856, August).

[Sopra i resti di Sturm. *Annali di Sci. mat. e fis.*, vii. pp. 313–317.]

Beginning with two unrelated functions, P, Q, of the n^{th} and $(n-1)^{\text{th}}$ degrees, Bruno gives an expression for any one of the Sturmian series of functions thence derived, the coefficients of x

* CAYLEY, A. Mémoire sur la forme canonique des fonctions binaires. *Crelle's Journ.*, liv. pp. 48–58, 292; or *Collected Math. Papers*, iv. pp. 43–52. If "lamdaic" be not used as a noun, "lamdaic canonizant" would be better than "lamdaic determinant."

in this expression being determinants which resemble in outward form those of Cayley's analogous expression of 1846 (August), but which have for elements the coefficients of $x^{-1}, x^{-2}, \ldots.$ in Q/P. He says that the expression "salvo qualche modificazione" was found by Cauchy, but gives no reference.

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinanti. *Memorie ... Istituto Veneto* .... vii. pp. 67–144.]

Although the persymmetric determinant

$$\begin{vmatrix} s_0 & s_1 & s_2 & \ldots. \\ s_1 & s_2 & s_3 & \ldots. \\ s_2 & s_3 & s_4 & \ldots. \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot \end{vmatrix}$$

where s_r is the sum of the r^{th} powers of the roots of the equation $x^n+a_1x^{n-1}+\ldots+a_n = 0$ has repeatedly come before us, it has always been with the understanding that no two of the roots were equal, and the order of the determinant has never been greater than the n^{th}. Bellavitis takes up the subject (§§ 45–50) with these conditions removed. He affirms that *when the number of rows exceeds the number of different roots, the persymmetric determinant*

$$\begin{vmatrix} s_{0+r} & s_{1+r} & s_{2+r} & \ldots. \\ s_{1+r} & s_{2+r} & s_{3+r} & \ldots. \\ s_{2+r} & s_{3+r} & s_{4+r} & \ldots. \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot \end{vmatrix}$$

vanishes, and when the numbers are the same the determinant is a multiple of the square of the difference-product of the said roots. By way of proof of the second part of the proposition the special case is taken where the roots are a, a, a, b, b, c, and where, since

$$s_0 = 6, \quad s_1 = 3a + 2b + c, \quad s_2 = 3a^2 + 2b^2 + c^2, \; \ldots\ldots$$

we have

$$\begin{vmatrix} s_5 & s_6 & s_7 \\ s_6 & s_7 & s_8 \\ s_7 & s_8 & s_9 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ 3a & 2b & c \\ 3a^2 & 2b^2 & c^2 \end{vmatrix} \cdot \begin{vmatrix} a^5 & b^5 & c^5 \\ a^6 & b^6 & c^6 \\ a^7 & b^7 & c^7 \end{vmatrix},$$

$$= 6\,|\,a^0b^1c^2\,| \cdot a^5b^5c^5\,|\,a^0b^1c^2\,|,$$

$$= 6a^5b^5c^5\,|\,a^0b^1c^2\,|^2.$$

The other part of the proposition rests on the statement that a similar procedure leads in that case to factors having at least one column of zeros.

The case where the number of rows is *less* than the number of different roots is not considered; but the first part of the proposition is used to obtain the modification which it is possible to make in the relation

$$s_{n+r} + a_1s_{n+r-1} + a_2s_{n+r-2} + \ldots + a_ns_r = 0$$

when the roots cease to be all different.

ROUCHÉ, E. (1858, Dec.).

[Sur les fonctions X_n de Legendre. *Comptes rendus . . . Acad. des Sci.* (Paris), xlvii. pp. 917–921.]

X_n being the n^{th} differential-quotient of $(x^2-1)^n$ Rouché first proves that every rational integral function of x of the n^{th} degree $x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, or V_n say, which for all integral values of $k < n$ satisfies the equation

$$\int_{-1}^{+1} x^kV_n dx = 0$$

does not differ from X_n save by a constant multiplier. To determine V_n,—in other words, to determine its coefficients $a_1, a_2, \ldots, a_n$,—there is thus available the set of n equations

$$\int_{-1}^{+1} V_n dx = 0, \quad \int_{-1}^{+1} xV_n dx = 0, \quad \ldots, \quad \int_{-1}^{+1} x^{n-1}V_n dx = 0,$$

which if we put i_r for $\frac{1}{2}\int_{-1}^{+1} x^r dx$ take the form

$$\left.\begin{array}{llllll} a_n i_0 & + a_{n-1} i_1 & + \ldots + & a_1 i_{n-1} & + i_n & = 0 \\ a_n i_1 & + a_{n-1} i_2 & + \ldots + & a_1 i_n & + i_{n+1} & = 0 \\ a_n i_2 & + a_{n-1} i_3 & + \ldots + & a_1 i_{n+1} & + i_{n+2} & = 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n i_{n-1} & + a_{n-1} i_n & + \ldots + & a_1 i_{2n-2} & + i_{2n-1} & = 0 \end{array}\right\}.$$

Solving for $a_1, a_2, a_3, \ldots, a_n$ and substituting in V_n Rouché of course obtains

$$V_n = \begin{vmatrix} i_0 & i_1 & i_2 & \ldots & i_{n-1} & i_n \\ i_1 & i_2 & i_3 & \ldots & i_n & i_{n+1} \\ i_2 & i_3 & i_4 & \ldots & i_{n+1} & i_{n+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ i_{n-1} & i_n & i_{n+1} & \ldots & i_{2n-2} & i_{2n-1} \\ 1 & x & x^2 & \ldots & x^{n-1} & x^n \end{vmatrix} \div \varpi,$$

where ϖ stands for the cofactor of x^n in the determinant, and being independent of x may be left out of account; and, since i_r is $1/(r+1)$ or 0 according as r is even or odd, it follows that $X_1, X_2, X_3, \ldots$ are proportional to

$$\begin{vmatrix} 1 & 0 \\ 1 & x \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ 1 & x & x^2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 \\ 1 & x & x^2 & x^3 \end{vmatrix}, \quad \ldots$$

respectively.

By reason of the peculiar distribution of the zero-elements these determinants are changeable into

$$x, \quad \frac{1}{3}\begin{vmatrix} 1 & \frac{1}{3} \\ 1 & x \end{vmatrix}, \quad \begin{vmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{5} \end{vmatrix} . \begin{vmatrix} \frac{1}{3} & \frac{1}{5} \\ x & x^3 \end{vmatrix}, \quad \begin{vmatrix} \frac{1}{3} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{7} \end{vmatrix} . \begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ 1 & x^2 & x^4 \end{vmatrix}, \quad \ldots$$

and the result may consequently be put in an alternative form, namely,

$$X_{2n} = \begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \cdots\cdots & \frac{1}{2n+1} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots\cdots & \frac{1}{2n+3} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots\cdots & \frac{1}{2n+5} \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot & \cdot \\ \frac{1}{2n-1} & \frac{1}{2n+1} & \frac{1}{2n+3} & \cdots\cdots & \frac{1}{4n-1} \\ 1 & x^2 & x^4 & \cdots\cdots & x^{2n} \end{vmatrix} \times \text{ a constant,}$$

and

$$X_{2n+1} = \begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots\cdots & \frac{1}{2n+3} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots\cdots & \frac{1}{2n+5} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \cdots\cdots & \frac{1}{2n+7} \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot & \cdot \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots\cdots & \frac{1}{4n+1} \\ x & x^3 & x^5 & \cdots\cdots & x^{2n+1} \end{vmatrix} \times \text{ a constant.}$$

Lastly it is pointed out that by throwing X_n into the form

$$\begin{vmatrix} i_1 - i_0 x & i_2 - i_1 x & \cdots\cdots & i_n - i_{n-1} x \\ i_2 - i_1 x & i_3 - i_2 x & \cdots\cdots & i_{n+1} - i_n x \\ \cdot & \cdot & \cdot\cdot\cdot\cdot & \cdot \\ i_n - i_{n-1} x & i_{n+1} - i_n x & \cdots\cdots & i_{2n-1} - i_{2n-2} x \end{vmatrix} \times \text{ a constant}$$

it is possible in the manner of Cauchy (1829) and Sylvester (1853) to draw conclusions regarding the situation and character of the roots of the equation $X_n = 0$.

SALMON, G. (1859).

[LESSONS INTRODUCTORY TO THE MODERN HIGHER ALGEBRA, ... xii+147 pp., Dublin.]

In Salmon's treatment of the foregoing subjects (p. 14, §§ 119–126, 162–165, p. 146) there are several points of freshness. His proof, for example, that *if*

$$(a,b,c,d,e,f \between x,y)^5 = (l_1x+m_1y)^5 + (l_2x+m_2y)^5 + (l_3x+m_3y)^5,$$

the persymmetric form of the canonizant is equal to

$$|l_1m_2|^2\,|l_1m_3|^2\,|l_2m_3|^2\cdot(l_1x+m_1y)(l_2x+m_2y)(l_3x+m_3y),$$

is accomplished (§ 119) by using four differentiations to show that

$$\begin{aligned} ax+by &= l_1^4(l_1x+m_1y)+l_2^4(l_2x+m_2y)+l_3^4(l_3x+m_3y) \\ &= l_1^4u+l_2^4v+l_3^4w, \text{ say,} \\ bx+cy &= l_1^3m_1u+l_2^3m_2v+l_3^3m_3w, \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

substituting these trinomials for $ax+by$, $bx+cy$, . . . in the canonizant, and then examining * the twenty-seven determinants into which the latter can be partitioned. He also gives (§ 121) a new mode of arriving at the canonizant in the form

$$\left(\left\|\begin{matrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{matrix}\right\| \between x,y\right)^3.$$

In the course of a short "Note on Commutants" he suggests (p. 146) that the two rows of umbræ

$$\left\langle\begin{matrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{matrix}\right\rangle \text{ should stand for } \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix},$$

in which case the three suffixes of any term of the developed determinant are got by *adding* the first row of umbræ to a permutation of the second row.

* It is better to note at this stage that the determinant is the product of

$$\begin{vmatrix} l_1^2u & l_2^2v & l_3^2w \\ l_1m_1u & l_2m_2v & l_3m_3w \\ m_1^2u & m_2^2v & m_3^2w \end{vmatrix} \text{ and } \begin{vmatrix} l_1^2 & l_2^2 & l_3^2 \\ l_1m_1 & l_2m_2 & l_3m_3 \\ m_1^2 & m_2^2 & m_3^2 \end{vmatrix}$$

and therefore is equal to $|\,l_1^2\;l_2m_2\;m_3^2\,|^2\,.\,uvw$.

CHAPTER XII.

BIGRADIENTS, UP TO 1860.

As we have already pointed out (*History*, i. p. 487), bigradients were first brought to light by Sylvester in 1840 in the paper in which he made known his so-called "dialytic" method of eliminating the unknown from two equations of the same or different degrees. Shortly afterwards Richelot and Cauchy recalled attention to Euler's and Bezout's method of 1764, as giving substantially the same result as Sylvester's, the fact being that the determinant obtained by Sylvester differs from that obtainable in the other case merely by being its conjugate. The details of these papers and of others related to them have already been given.

CAYLEY, A. (1844).

[Note sur deux formules données par MM. Eisenstein et Hesse. *Crelle's Journ.*, xxix. pp. 54–57; or *Collected Math. Papers*, i. pp. 113–116.]

Although Eisenstein's property * of the discriminant

$$a^2d^2-3b^2c^2+4ac^3+4b^3d-6abcd, \quad \text{or } \Delta \text{ say,}$$

of the binary cubic $ax^3+3bx^2y+3cxy^2+dy^3$,—namely, the property that

$$\begin{aligned} A^2D^2-3B^2C^2+4AC^3+4B^3D-6ABCD \\ = (a^2d^2-3b^2c^2+4ac^3+4b^3d-6abcd)^3 \end{aligned}$$

* *Crelle's Journ.*, xxvii. pp. 105-106, 319–321.

when

$$\mathrm{A,\ B,\ C,\ D} = -\frac{1}{2}\frac{\partial\Delta}{\partial d},\quad \frac{1}{6}\frac{\partial\Delta}{\partial c},\quad -\frac{1}{6}\frac{\partial\Delta}{\partial b},\quad \frac{1}{2}\frac{\partial\Delta}{\partial a},$$

—can be expressed in the form

$$\begin{vmatrix} \mathrm{A} & 2\mathrm{B} & \mathrm{C} & . \\ . & \mathrm{A} & 2\mathrm{B} & \mathrm{C} \\ \mathrm{B} & 2\mathrm{C} & \mathrm{D} & . \\ . & \mathrm{B} & 2\mathrm{C} & \mathrm{D} \end{vmatrix} = \begin{vmatrix} a & 2b & c & . \\ . & a & 2b & c \\ b & 2c & d & . \\ . & b & 2c & d \end{vmatrix}^3,$$

it is not as a relation between two four-line determinants that it has been studied.

Cayley in effect says that if we wish to find substitutes A, B, C, . . . for a, b, c, . . . so that

$$\phi(\mathrm{A, B, C}, \ldots) = \left\{\phi(a, b, c, \ldots)\right\}^p,$$

we must (1) find a quantic u of which $\phi(a, b, c, \ldots)$ is an invariant; (2) express $\phi(a, b, c, \ldots)$ as a determinant of the same number of lines as u has facients; and (3) transform u into U by a linear substitution of which the said determinant is the modulus. The coefficients of U will then be the substitutes required.

For example, Δ being an invariant of the binary cubic and being expressible in the form

$$\begin{vmatrix} bc-ad & 2(c^2-bd) \\ 2(b^2-ac) & bc-ad \end{vmatrix},$$

we should have to transform the said cubic by the substitution

$$\left.\begin{aligned} x &= (bc-ad)\xi + 2(c^2-bd)\eta \\ y &= 2(b^2-ac)\xi + (bc-ad)\eta \end{aligned}\right\},$$

and the discriminant of the new cubic thus obtained being Δ multiplied by a power of the modulus must be a power of Δ. Unfortunately, in this case it would be Δ^7, whereas in Eisenstein's case the power-index is 3. Instead of the binary cubic, therefore, Cayley takes the binary trilinear

$$ax_1y_1z_1 + bx_1y_1z_2 + cx_1y_2z_1 + dx_1y_2z_2 + ex_2y_1z_1 + fx_2y_1z_2 + gx_2y_2z_1 + hx_2y_2z_2,$$

of which a generalisation of Δ, namely,

$$\left.\begin{array}{r} a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2 + 4adfg + 4bceh \\ - 2ahbg - 2ahcf - 2ahde - 2bgcf - 2bgde - 2cfde \end{array}\right\} \text{ or Q say,}$$

is an invariant; expresses Q as a two-line determinant

$$\begin{vmatrix} ah - bg - cf + de & -2(eh-fg) \\ -2(ad-bc) & ah - bg - cf + de \end{vmatrix}$$

makes the substitution

$$\left.\begin{array}{l} x_1 = (ah-bg-cf+de)\xi_1 - 2(eh-fg)\xi_2 \\ y_1 = \qquad -2(ad-bc)\xi_1 + (ah-bg-cf+de)\xi_2 \end{array}\right\};$$

and, as the multiplier connecting the new Q and the old is now the *second* power of the modulus, he obtains what was wanted.*

The substitutes found turn out to be

$$\frac{1}{2}\frac{\partial Q}{\partial a}, \quad \frac{1}{2}\frac{\partial Q}{\partial b}, \quad \frac{1}{2}\frac{\partial Q}{\partial c}, \quad \ldots .$$

but no explanation of this is vouchsafed.

Eisenstein's case is the degeneration reached by putting

$$\begin{array}{l} a,\ b,\ c,\ d,\ e,\ f,\ g,\ h \\ = a,\ b,\ b,\ c,\ b,\ c,\ c,\ d. \end{array}$$

Had the two other sets of variables been at the same time transformed with the same determinant for modulus, we should have had the new Q equal to Q^7.

BOOLE, G. (1844, June).

[Notes on linear transformation. *Cambridge Math. Journ.*, iv. pp. 167–171.]

The third note being devoted to the proof of his well-known theorem of 1841 regarding what afterwards came to be known

* This short paper of Cayley's teems with misprints, both in the original and in the *Collected Math. Papers.*

as 'the invariance of the discriminant,' the fourth gives at full length and correctly the discriminant of

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

—that is to say, the eliminant of

$$\left.\begin{aligned} ax^3 + 3bx^2y + 3cxy^2 + dy^3 &= 0 \\ bx^3 + 3cx^2y + 3dxy^2 + ey^3 &= 0 \end{aligned}\right\}.$$

HEILERMANN, [H.] (1845).

[Ueber die Verwandlung der Reihen in Kettenbrüche. *Crelle's Journ.*, xxxiii. pp. 174–188.]

The determinant which here appears for the first time is different from but resembles those to which Sylvester's dialytic method of elimination leads, being exemplified for the 4th and 5th orders by

$$\begin{vmatrix} a_3 & a_2 & b_2 & b_3 \\ a_2 & a_1 & b_1 & b_2 \\ a_1 & a_0 & b_0 & b_1 \\ a_0 & . & . & b_0 \end{vmatrix}, \quad \begin{vmatrix} a_4 & a_3 & a_2 & b_3 & b_4 \\ a_3 & a_2 & a_1 & b_2 & b_3 \\ a_2 & a_1 & a_0 & b_1 & b_2 \\ a_1 & a_0 & . & b_0 & b_1 \\ a_0 & . & . & . & b_0 \end{vmatrix}.$$

Calling these Δ_3, Δ_4 Heilermann writes his main result in the form

$$\frac{a_0 + a_1x + \ldots + a_nx^n}{b_0 + b_1x + \ldots + b_mx^m} = \frac{\Delta_0}{b_0} - \frac{\Delta_1 x}{\Delta_0} - \frac{\Delta_2 x}{\Delta_1} - \frac{\Delta_0\Delta_3 x}{\Delta_2} - \frac{\Delta_1\Delta_4 x}{\Delta_3 - \ldots}$$

the end on the right being

$$-\frac{\Delta_{2n-3}\Delta_{2n}x}{\Delta_{2n-1}} \quad \text{or} \quad -\frac{\Delta_{2m-4}\Delta_{2m-1}}{\Delta_{2m-2}}$$

according as $m <$ or $> n$.

CAYLEY, A. (1848, August).

[Nouvelles recherches sur les fonctions de M. Sturm. *Journ.* (*de Liouville*) *de Math.*, xiii. pp. 269–274; or *Collected Math. Papers*, i. pp. 392–396.]

Recalling his former paper on the same subject in *Liouville's Journal*, xi. (1846), pp. 297–299, where Sturm's functions had been expressed in terms of sums of powers of the roots of the original function, he intimates now the discovery of more simple expressions in terms of the *coefficients* of the said function. At the same time he draws attention to the fact that his result may be viewed as unconnected with Sturm's division-process, and it is in this general light that he prefers to state it. Beginning with two functions V and V′ of the n^{th} degree, namely,

$$ax^n + bx^{n-1} + \dots \qquad a'x^n + b'x^{n-1} + \dots$$

and forming therefrom the series of functions

$$\left|\begin{matrix} V & V' \\ a & a' \end{matrix}\right|, \quad \left|\begin{matrix} xV & V & xV' & V' \\ a & . & a' & . \\ b & a & b' & a' \\ c & b & c' & b' \end{matrix}\right|, \quad \left|\begin{matrix} x^2V & xV & V & x^2V' & xV' & V' \\ a & . & . & a' & . & . \\ b & a & . & b' & a' & . \\ c & b & a & c' & b' & a' \\ d & c & b & d' & c' & b' \\ e & d & c & e' & d' & c' \end{matrix}\right|,$$

etc., which he denotes by $-F_1, F_2, -F_3, \dots$, he affirms that there is a homogeneous linear relation connecting every consecutive three of the latter functions,* namely,

$$P_1^2F_3 + (xP_1P_2 + P_1P_2' + P_1'P_2)F_2 + P_2^2F_1 = 0,$$
$$P_2^2F_4 + (xP_2P_3 + P_2P_3' + P_2'P_3)F_3 + P_3^2F_2 = 0,$$
$$\dots\dots\dots\dots\dots\dots$$

* The signs require verification.

where by $P_1, P_2, P_3, \ldots.$ are meant the determinants

$$\begin{vmatrix} a & a' \\ b & b' \end{vmatrix}, \quad \begin{vmatrix} a & . & a' & . \\ b & a & b' & a' \\ c & b & c' & b' \\ d & c & d' & c' \end{vmatrix}, \quad \begin{vmatrix} a & . & . & a' & . & . \\ b & a & . & b' & a' & . \\ c & b & a & c' & b' & a' \\ d & c & b & d' & c' & b' \\ e & d & c & e' & d' & c' \\ f & e & d & f' & e' & d' \end{vmatrix}, \ldots.$$

and by $P'_1, P'_2, P'_3, \ldots.$ the determinants got from $P_1, P_2, P_3, \ldots.$ by altering the last rows into

$$c\ c'; \quad e\ d\ e'\ d'; \quad g\ f\ e\ g'\ f'\ e'; \quad \ldots.$$

No proof is given of the relations; indeed, after pointing out that they involve the proposition that the first and last of three consecutive functions are of opposite sign for every value of x that makes the intermediate function vanish, Cayley adds: "Je n'ai pas encore réussi à démontrer dans toute la généralité l'équation identique d'où dépend cette propriété."

The case which brings him into closer contact with Sturm, namely, where V′ is the differential-quotient of V, is dealt with in some detail.

HEILERMANN, [H.] (1852, December).

[Independente Berechnung der Sturm'schen Reste. *Crelle's Journ.*, xlviii. pp. 190–206.]

The subject of this paper is of course closely connected with that of the author's previous work (1845). Like Cayley, he begins with two functions that are unrelated, and subsequently passes to the special case where the one is the derivate of the other; but, unlike Cayley, he makes the said functions of different degrees, namely,

$$c_{00}x^n \ + c_{10}x^{n-1} + c_{20}x^{n-2} + \ \ldots. \ + c_{n0},$$
$$c_{01}x^{n-1} + c_{11}x^{n-2} + c_{21}x^{n-3} + \ \ldots. \ + c_{n-1,1}.$$

Following the ordinary division-process for expressing the ratio of the second function to the first as a continued fraction of the form

$$\frac{p_0}{x+}\frac{p_1}{1+}\frac{p_2}{x+}\frac{p_3}{1+}\cdots$$

and denoting the remainders in order by

$$\frac{1}{c_{01}}\left\{c_{02}x^{n-1}+c_{12}x^{n-2}+c_{22}x^{n-3}+\ldots\right\},$$
$$\frac{1}{c_{02}}\left\{c_{03}x^{n-2}+c_{13}x^{n-3}+c_{23}x^{n-4}+\ldots\right\},$$
$$\frac{1}{c_{01}c_{03}}\left\{c_{04}x^{n-3}+c_{14}x^{n-4}+c_{24}x^{n-5}+\ldots\right\},$$
$$\cdots\cdots\cdots\cdots$$

he finds that

$$p_0=\frac{c_{01}}{c_{00}},\quad p_{r+1}=\frac{c_{0,r+2}}{c_{0,r}c_{0,r+1}},\quad p_{2n-1}=\frac{c_{1,2n-2}}{c_{0,2n-2}}.$$

The second suffix of any one of the new c's is seen to indicate the remainder-function to which the c belongs, and the first suffix the position of the c in that remainder. To obtain expressions for these in terms of the original two sets of c's it is taken for granted, and with reason, that as a result of the process we have generally

$$c_{r,s}=c_{0,s-1}c_{r+1,s-2}-c_{0,s-2}c_{r+1,s-1}=\begin{vmatrix} c_{0,s-1} & c_{0,s-2} \\ c_{r+1,s-1} & c_{r+1,s-2}\end{vmatrix}. \qquad (1)$$

By using this twice upon itself, so as to lower the second suffixes of the first column, there is found

$$c_{r,s}=\begin{vmatrix} c_{0,s-2} & c_{0,s-3} & \cdot \\ c_{1,s-2} & c_{1,s-3} & c_{0,s-2} \\ c_{r+2,s-2} & c_{r+2,s-3} & c_{r+1,s-2}\end{vmatrix}, \qquad (2)$$

where the second suffixes are now $s-2$, $s-3$. A page is then occupied in ridding (2) in the same way of the elements which have $s-3$ for a suffix, the result being

$$c_{r,s} = c_{0,s-3} \begin{vmatrix} c_{0,s-3} & c_{0,s-4} & \cdot & \\ c_{1,s-3} & c_{1,s-4} & c_{0,s-3} & c_{0,s-4} \\ c_{2,s-3} & c_{2,s-4} & c_{1,s-3} & c_{1,s-4} \\ c_{r+3,s-3} & c_{r+3,s-4} & c_{r+2,s-3} & c_{r+2,s-4} \end{vmatrix} \qquad (3)$$

With increasing tediousness a five-line determinant is reached having elements with $s-4$ and $s-5$ for second suffixes, a six-line determinant having elements with $s-5$ and $s-6$ for second suffixes, and so on. The form of the determinant of the $(2q+2)^{\text{th}}$ order is thus deduced, the factor preceding it being said to be

$$(c_{0,s-3}c_{0,s-4})(c_{0,s-5}c_{0,s-6})^2(c_{0,s-7}c_{0,s-8})^3 \ldots (c_{0,s-2q+1}c_{0,s-2q})^{q-1}(c_{0,s-2q-1})^q.$$

Sturm's division-process, in which each remainder is of a lower degree than the remainder preceding it, and for which, therefore, the corresponding continued fraction has each partial denominator a linear function of x, has close relationship with the above division-process, because the continued fraction already obtained is identical with*

$$\frac{p_0}{x+p_1} - \frac{p_1p_2}{x+p_2+p_3} - \frac{p_3p_4}{x+p_4+p_5} - \cdots$$

Expressions for the remainders corresponding to this continued fraction are thus readily obtainable, and a section (§ 3) is devoted to finding simplified substitutes for them. The remaining section concerns the strictly Sturmian case, where one of the original functions is the derivate of the other.

BRUNO, F. FAÀ DI (1855, July).

[Sulle funzioni simmetriche delle radici di un' equazione. *Annali di Sci. mat. e fis.*, vi. pp. 412–419.]

As evidence of the value of a certain theorem Bruno adduces the ease with which the expansion of the resultant of a pair of

* This identity Heilermann published again separately in 1860 (see *Zeitschrift f. Math. u. Phys.*, v. pp. 262–263). It is included, however, in a result given by Stern in 1833 (see *Crelle's Journ.*, x. p. 156); and a still more general identity will be found in the *Proceed. Edinburgh Math. Soc.*, xxiii. p. 37.

equations may be calculated, and he prints at full length the resultants $R_{2,2}$, $R_{3,3}$, $R_{4,4}$, that is to say, the final expansions of

$$\begin{vmatrix} . & a & b & c \\ a & b & c & . \\ . & p & q & r \\ p & q & r & . \end{vmatrix}, \text{ etc.,}$$

the arrangement of the terms being such as to make evident the fact that each resultant is unaltered by reversing the order of the two sets of coefficients of which it is a function: for example:—*

$$R_{2,2} = (a^2r^2+c^2p^2) - (abqr+bcpq) - 2acpr + acq^2 + b^2pr.$$

CAYLEY, A. (1856).

[A second memoir on quantics. *Philos. Transac. R. Soc.* (London), cxlvi. pp. 101–126; or *Collected Math. Papers,* ii. pp. 250–275.]

In addition to previously calculated discriminants Cayley gives (No. 26) the discriminant of the binary quintic,—that is to say, the eliminant of

$$\left.\begin{aligned} ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 &= 0 \\ bx^4 + 4cx^3y + 6dx^2y^2 + 4exy^3 + fy^4 &= 0 \end{aligned}\right\}\dagger$$

CAYLEY, A. (1856, December).

[Memoir on the resultant of a system of two equations. *Philos. Transac. R. Soc.* (London), cxlvii. pp. 703–715; or *Collected Math. Papers,* ii. pp. 440–453.]

As the resultant, $R_{3,2}$ say, of the pair of equations

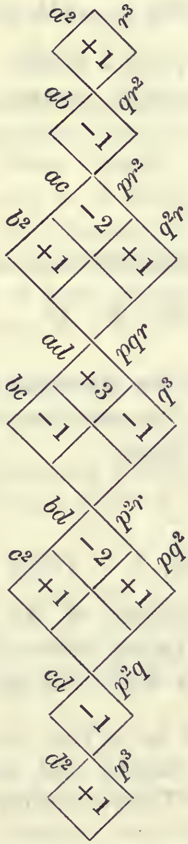
$$ax^3+bx^2y+cxy^2+dy^3 = 0, \quad px^2+qxy+ry^2 = 0,$$

is homogeneous and of the 3rd degree in the coefficients of the second equation, and at the same time homogeneous and of the 2nd degree in the coefficients of the first equation, Cayley

* The expression for $R_{4,4}$ is full of inaccuracies.

† In the reprint, p. 288, the coefficient of b^4cef^2 should be -1920, not $+1920$.

seeks a convenient form of representation in which this double homogeneity will be prominent. What he obtains is *



where the first square represents a^2r^3, the second $-abqr^2$, the third $-2acpr^2+b^2pr^2+acq^2r$, and so on. The result is reached in two ways, the second being by developing the dialytic eliminant

$$\left|\begin{array}{ccccc} . & a & b & c & d \\ a & b & c & d & . \\ . & . & p & q & r \\ . & p & q & r & . \\ p & q & r & . & . \end{array}\right|.$$

* Three misprints being corrected.

The two-line minors of the first two rows of this determinant being denoted by 12, 13, . . . , 45, and the three-line minors of the remaining rows by 123, 124, . . . , 345, it is seen that

$$
\begin{aligned}
R_{3,2} = 12{\cdot}345 - 13{\cdot}245 + 14{\cdot}235 - 15{\cdot}234 & \\
+ 23{\cdot}145 - 24{\cdot}135 + 25{\cdot}134 & \\
+ 34{\cdot}125 - 35{\cdot}124 & \\
- 45{\cdot}123 &
\end{aligned}
$$

and it will be found that

$$12{\cdot}345, \quad -13{\cdot}245, \quad 14{\cdot}235+23{\cdot}145, \quad -(15{\cdot}234+24{\cdot}135), \quad \ldots$$

correspond to the 1st, 2nd, 3rd, 4th, . . . squares of Cayley's expression.

The paper closes with the six resultants $R_{2,2}$, $R_{3,2}$, $R_{4,2}$, $R_{3,3}$, $R_{4,3}$, $R_{4,4}$ printed each in the new form as a chain of squares;* they occupy four quarto pages.

ZEIPEL, V. v. (1858, June).

[Demonstration of a theorem of Mr. Cayley's in relation to Sturm's functions. *Quart. Journ. of Math.*, iii. pp. 108–117, or *Nouv. Annales de Math.*, xix. pp. 220–224.]

The theorem referred to is that of August 1848. Zeipel's proof (pp. 108–114) is lengthy and unattractive, and scarcely warrants reproduction. The remaining pages are occupied with the curious identities

$$
\begin{vmatrix} P_{r-1} & P_r & . \\ P'_{r-1} & P'_r & P_r \\ P''_{r-1} & P''_r & P'_r \end{vmatrix} = -P^2_{r-1}P_{r+1}, \qquad
\begin{vmatrix} P_{r-1} & P_r & . \\ P'_{r-1} & P'_r & P_r \\ P'''_{r-1} & P'''_r & P''_r \end{vmatrix} = -P^2_{r-1}P'_{r+1},
$$

and the corresponding identities in which the left-hand members are of a higher odd order than the third.

* In the expression for $R_{4,4}$ there is at least one misprint, namely, a^2c^2 for a^2b^2 outside the third square of the chain.

HESSE, O. (1858, October).

[Il determinante di Sylvester ed il risultante di Eulero. *Annali di Mat.* . . . ii. pp. 5–8; or *Werke*, 475–480.]

The determinant referred to is that to which Sylvester was led in 1840 by his so-called "dialytic" method, and which, as we have already seen, Hesse himself arrived at in 1843; and the resultant coupled with it is Euler's of 1748, which takes the form of a product of differences of the roots of the two given equations. Both forms, as well as others, are treated of by Cauchy in his paper of 1840, already dealt with.

The equations being

$$\left.\begin{aligned} a_3x^3 + a_2x^2 + a_1x^1 + a_0 &= 0 \\ b_2x^2 + b_1x^1 + b_0 &= 0 \end{aligned}\right\} \text{ or say } \left\{\begin{aligned} \phi(x) &= 0 \\ \psi(x) &= 0, \end{aligned}\right.$$

Hesse multiplies Sylvester's eliminant by $s_0s_2 - s_1^2$, the squared difference-product of the roots β_1, β_2 of the second equation, with the result

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & \cdot & \cdot \\ \cdot & b_0 & b_1 & b_2 & \cdot \\ \cdot & \cdot & b_0 & b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} s_0 & s_1 & s_2 & s_3 & s_4 \\ s_1 & s_2 & s_3 & s_4 & s_5 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_0s_0 + a_1s_1 + \ldots + a_3s_3 & a_0s_1 + a_1s_2 + \ldots + a_3s_4 & a_2 & a_3 & \cdot \\ a_0s_1 + a_1s_2 + \ldots + a_3s_4 & a_0s_2 + a_1s_3 + \ldots + a_3s_5 & a_1 & a_2 & a_3 \\ b_0s_0 + b_1s_1 + b_2s_2 & b_0s_1 + b_1s_2 + b_2s_3 & b_2 & \cdot & \cdot \\ b_0s_1 + b_1s_2 + b_2s_3 & b_0s_2 + b_1s_3 + b_2s_4 & b_1 & b_2 & \cdot \\ b_0s_2 + b_1s_3 + b_2s_4 & b_0s_3 + b_1s_4 + b_2s_5 & b_0 & b_1 & b_2 \end{vmatrix},$$

where $s_p = \beta_1{}^p + \beta_2{}^p$, and where, therefore, the elements in the places 31, 32, 41, 42, 51, 52 of the right-hand member all vanish. There is thus obtained, if we denote Sylvester's eliminant by S,

$$\mathrm{S} \cdot \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = b_2{}^3 \begin{vmatrix} a_0s_0 + a_1s_1 + \ldots + a_3s_3 & a_0s_1 + a_1s_2 + \ldots + a_3s_4 \\ a_0s_1 + a_1s_2 + \ldots + a_3s_4 & a_0s_2 + a_1s_3 + \ldots + a_3s_5 \end{vmatrix}.$$

But the determinant on the right is resolvable into

$$\begin{vmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{vmatrix} \cdot \begin{vmatrix} a_0+a_1\beta_1+a_2\beta_1^2+a_3\beta_1^3 & a_0+a_1\beta_2+a_2\beta_2^2+a_3\beta_2^3 \\ a_0\beta_1+a_1\beta_1^2+a_2\beta_1^3+a_3\beta_1^4 & a_0\beta_2+a_1\beta_2^2+a_2\beta_2^3+a_3\beta_2^4 \end{vmatrix},$$

that is, into

$$\begin{vmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{vmatrix} \cdot \begin{vmatrix} \phi(\beta_1) & \phi(\beta_2) \\ \beta_1\phi(\beta_1) & \beta_2\phi(\beta_2) \end{vmatrix}$$

and therefore into

$$\begin{vmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{vmatrix}^2 \cdot \phi(\beta_1)\cdot\phi(\beta_2).$$

We thus have

$$\mathrm{S} = b_2^3\phi(\beta_1)\phi(\beta_2),$$

and, consequently, if a_1, a_2, a_3 be the roots of the first given equation,

$$\mathrm{S} = b_2^3a_3^2(\beta_1-a_1)(\beta_1-a_2)(\beta_1-a_3)$$
$$(\beta_2-a_1)(\beta_2-a_2)(\beta_2-a_3),$$

which is what was to be shown, the cofactor of $b_2^3a_3^2$ being Euler's product of differences.

ZEIPEL, V. v. (1859, April).

[Undersökningar i högre algebran, jemte några deraf beroende theoremer i determinant-theorien. *K. Svenska Vet.-Akad. Handl.* (Stockholm), iii. No. 4, 32 pp.]

This simply written and fully detailed memoir is an extension of the author's paper of the preceding year. It consists of five chapters, of which the fifth contains the freshest matter.

The first, and much the longest (pp. 3–16), is occupied in § 1 (pp. 3–11) with the application of the Sturmian division-process to

$$a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m, \quad \text{or } \mathrm{F}(x),$$
$$b_0x^{m-1} + b_1x^{m-2} + \ldots + b_{m-1}, \text{ or } f(x),$$

with the object of obtaining Cayley's expressions of August 1848 for the remainders (R_r). In this he is successful, and he adds

Brioschi's of 1854 for comparison. In § 2 (pp. 12–15) he works out for himself a third form, which, in the case of $m=4$, gives

$$R_1 = \begin{vmatrix} a_0 & b_0 & \cdot & \cdot & \cdot \\ a_1 & b_1 & b_0 & \cdot & \cdot \\ a_2 & b_2 & b_1 & -1 & \cdot \\ a_3 & b_3 & b_2 & x & -1 \\ a_4 & \cdot & b_3 & \cdot & x \end{vmatrix}, \quad R_2 = \begin{vmatrix} a_0 & \cdot & b_0 & \cdot & \cdot & \cdot \\ a_1 & a_0 & b_1 & b_0 & \cdot & \cdot \\ a_2 & a_1 & b_2 & b_1 & b_0 & \cdot \\ a_3 & a_2 & b_3 & b_2 & b_1 & \cdot \\ a_4 & a_3 & \cdot & b_3 & b_2 & -1 \\ \cdot & a_4 & \cdot & \cdot & b_3 & x \end{vmatrix}, \quad R_3 = \ldots$$

—expressions which take a more familiar appearance when developed and arranged according to descending powers of x.

He then passes to the case where $f(x)=F'(x)$, his second chapter being occupied with a short but very complete historical sketch of Sturm's functions.

In the third chapter the allied subject of the common roots of two equations is taken up, and to illustrate the advantage of his own procedure over Lagrange's and Brioschi's he takes the equations

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$
$$b_0x^3 + b_1x^2 + b_2x + b_3 = 0,$$

and (1) supposing them to have *one* common root gives

$$\begin{vmatrix} a_0 & \cdot & b_0 & \cdot \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & b_2 & b_1 \\ a_3 & a_2 & b_3 & b_2 \end{vmatrix} x + \begin{vmatrix} a_0 & \cdot & b_0 & \cdot \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & b_2 & b_1 \\ \cdot & a_3 & \cdot & b_3 \end{vmatrix} = 0$$

as the equation for determining it; (2) gives the vanishing of the two determinants in this equation as the conditions requisite for the existence of *two* common roots; and (3) gives

$$\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} x^2 + \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} x + \begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix} = 0$$

as the equation for determining the said two. In this connection Sylvester's paper of 1840 might well have been referred to (see *History*, i. pp. 236–238).

The fourth chapter is headed "Relation between any three remainders," but its main subject is Cayley's relation of August 1848, and the obtaining of others like it.

Lastly, a chapter is devoted to the bigradient compound determinants which he (Zeipel) drew attention to in his paper of 1858, and which when of the $(2r-1)^{\text{th}}$ order are equal to

$$P_n^{n-2}P_{n+r} \quad \text{and} \quad P_n^{n-2}P_{n+r}^{(r)}$$

respectively. Even to-day these excite a real interest, and a purely determinantal proof of the identities is much to be desired.

BRUNO, F. FAÀ DI (1859).

[Théorie Générale de l'Élimination. Par le Chevalier François Faà di Bruno. . . . x+224 pp. Paris.]

In his section (pp. 32–40) dealing with the dialytic eliminant, Bruno, besides reprinting $R_{3,3}$, $R_{4,4}$, gives the full expansion of the discriminant of the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

and of the discriminant of

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0 .$$

In the printing of the latter discriminant, however, there are at least seven mistakes. In the next section (pp. 40–46) he seeks to improve on what we have called Cayley's "chain of squares" by combining the last square with the first, the second from the end with the second from the beginning, and so on. For example, his expression for $R_{3,3}$, that is to say, for

$$(a^3s^3 - d^3p^3) + (-a^2brs^2 + cd^2p^2q) + \{2(-a^2cqs^2 + bd^2p^2r) + \ldots\} + \ldots$$

is

$$\begin{array}{c|c|} & \substack{s^3\\ d^3} \\ \cline{2-2} \substack{a^3\\ p^3} & \pm 1 \\ \cline{2-2} \end{array} + \begin{array}{c|c|} & \substack{rs^2\\ cd^2} \\ \cline{2-2} \substack{a^2b\\ p^2q} & \mp 1 \\ \cline{2-2} \end{array} + \begin{array}{c|c|c|} & \substack{qs^2\\ bd^2} & \substack{r^2s\\ c^2d} \\ \cline{2-3} \substack{a^2c\\ p^2r} & \mp 2 & \pm 1 \\ \cline{2-3} \substack{ab^2\\ pq^2} & \pm 1 & \\ \cline{2-3} \end{array} + \begin{array}{c|c|c|c|} & \substack{ps^2\\ ad^2} & \substack{qrs\\ bcd} & \substack{r^3\\ c^3} \\ \cline{2-4} \substack{a^2d\\ p^2s} & \mp 3 & \pm 3 & \mp 1 \\ \cline{2-4} \substack{abc\\ pqr} & \pm 3 & \mp 1 & \\ \cline{2-4} \substack{b^3\\ q^3} & \mp 1 & & \\ \cline{2-4} \end{array} + \begin{array}{c|c|c|c|} & \substack{prs\\ acd} & \substack{q^2s\\ b^2d} & \substack{qr^2\\ bc^2} \\ \cline{2-4} \substack{abd\\ pqs} & \mp 1 & \mp 2 & \pm 1 \\ \cline{2-4} \substack{ac^2\\ pr^2} & \mp 2 & \pm 1 & \\ \cline{2-4} \substack{b^2c\\ q^2r} & \pm 1 & & \\ \cline{2-4} \end{array},$$

a marked improvement on which would be

$$\begin{array}{cccccc} & s^3 & & & rs^2 & \\ & p^3 & & & p^2q & \\ \begin{matrix}a^3\\ d^3\end{matrix} & \boxed{\pm 1} & + & \begin{matrix}a^2b\\ cd^2\end{matrix} & \boxed{\mp 1} & + \;\ldots . \end{array}$$

the second term in each binomial being derived from the first term by the change of a, b, c, d, p, q, r, s, into d, c, b, a, s, r, q, p, —that is to say, by the interchange

$$\begin{pmatrix} a & b & p & q \\ d & c & s & r \end{pmatrix}.$$

Further, he improves upon Cayley's squares by so transposing their rows, where necessary, as to bring about axisymmetry.* Lastly, he tries (pp. 43–46) to justify Cayley's rule for calculating the coefficients placed inside any square of the chain.

SALMON, G. (1859).

[Lessons introductory to the Modern Higher Algebra. . . . xii + 147 pp. Dublin.]

Having stated Euler's and Bezout's method of 1764 for eliminating the unknown from two equations (see Cauchy's account of it, *History*, i. pp. 241–242), and having previously defined such eliminant as the expression whose vanishing is the condition for the two equations having a common root, Salmon naturally thinks (p. 32, § 47) of extending the method to find the conditions for the equations having *two* common roots. The equations being

$$\left.\begin{aligned} ax^4 + bx^3 + cx^2 + dx + e &= 0 \\ \alpha x^3 + \beta x^2 + \delta x + \epsilon &= 0 \end{aligned}\right\} \quad \text{or say} \quad \left\{\begin{aligned} \phi(x) &= 0 \\ \psi(x) &= 0, \end{aligned}\right.$$

he asserts that if $\phi(x)$ and $\psi(x)$ have two linear factors in common the result of multiplying $\phi(x)$ by the remaining linear factor of $\psi(x)$ must be the same as the result of multiplying

* The expression for $R_{4,4}$, though now given more accurately than before, is still disfigured by at least ten misprints.

$\psi(x)$ by the remaining quadratic factor of $\phi(x)$. Taking, therefore, these remaining factors to be

$$\mathrm{A}x + \mathrm{B} \quad \text{and} \quad \mathrm{A}'x^2 + \mathrm{B}'x + \mathrm{C}',$$

and performing the multiplications, he is entitled to equate the coefficients of like powers in the two products. He thus obtains

$$\left.\begin{array}{l}
\mathrm{A}a - \mathrm{A}'\alpha = 0 \\
\mathrm{A}b + \mathrm{B}a - \mathrm{A}'\beta - \mathrm{B}'\alpha = 0 \\
\mathrm{A}c + \mathrm{B}b - \mathrm{A}'\gamma - \mathrm{B}'\beta - \mathrm{C}'\alpha = 0 \\
\mathrm{A}d + \mathrm{B}c - \mathrm{A}'\delta - \mathrm{B}'\gamma - \mathrm{C}'\beta = 0 \\
\mathrm{A}e + \mathrm{B}d - \mathrm{B}'\delta - \mathrm{C}'\gamma = 0 \\
\mathrm{B}e - \mathrm{C}'\delta = 0
\end{array}\right\},$$

from which the deduction is

$$\left\|\begin{array}{cccccc}
a & b & c & d & e & . \\
. & a & b & c & d & e \\
\alpha & \beta & \gamma & \delta & . & . \\
. & \alpha & \beta & \gamma & \delta & . \\
. & . & \alpha & \beta & \gamma & \delta
\end{array}\right\| = 0.$$

BORCHARDT, C. W. (1859, November).

[Vergleichung zweier Formen der Eliminations-Resultante. *Crelle's Journ.*, lvii. pp. 183–186; or *Gesammelte Werke*, pp. 145–150.]

The problem here is exactly the same as Hesse's of the previous year. Instead, however, of multiplying S by $\zeta(\beta_1, \beta_2)$, he preferably multiplies $\zeta^{\frac{1}{2}}(\beta_1, \beta_2, a_1, a_2, a_3)$ by S, thus obtaining

$$\begin{vmatrix}
1 & \beta_1 & \beta_1^2 & \beta_1^3 & \beta_1^4 \\
1 & \beta_2 & \beta_2^2 & \beta_2^3 & \beta_2^4 \\
1 & a_1 & a_1^2 & a_1^3 & a_1^4 \\
1 & a_2 & a_2^2 & a_2^3 & a_2^4 \\
1 & a_3 & a_3^2 & a_3^3 & a_3^4
\end{vmatrix} \cdot \begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & . \\
. & a_0 & a_1 & a_2 & a_3 \\
b_0 & b_1 & b_2 & . & . \\
. & b_0 & b_1 & b_2 & . \\
. & . & b_0 & b_1 & b_2
\end{vmatrix}$$

$$= \begin{vmatrix}
\phi(\beta_1) & \beta_1\phi(\beta_1) & . & . & . \\
\phi(\beta_2) & \beta_2\phi(\beta_2) & . & . & . \\
. & . & \psi(a_1) & a_1\psi(a_1) & a_1^2\psi(a_1) \\
. & . & \psi(a_3) & a_2\psi(a_2) & a_2^2\psi(a_2) \\
. & . & \psi(a_3) & a_3\psi(a_3) & a_3^2\psi(a_3)
\end{vmatrix}.$$

Now, E being Euler's product of differences, the first determinant on the left is resolvable into

$$\zeta^{\frac{1}{2}}(\beta_1, \beta_2) \, . \, \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3) \, . \, \mathrm{E},$$

as was first observed by Rosenhain in 1845 (Sept.); and the determinant on the right is resolvable into

$$\left\{\zeta^{\frac{1}{2}}(\beta_1, \beta_2) \, . \, \phi(\beta_1) \, . \, \phi(\beta_2)\right\}\left\{\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3) \, . \, \psi(\alpha_1) \, . \, \psi(\alpha_2) \, . \, \psi(\alpha_3)\right\}.$$

We thus have

$$\begin{aligned} \mathrm{ES} &= \left\{\phi(\beta_1) \, . \, \phi(\beta_2)\right\}\left\{\psi(\alpha_1) \, . \, \psi(\alpha_2) \, . \, \psi(\alpha_3)\right\}, \\ &= a_3{}^2\mathrm{E} \, . \, b_2{}^3\mathrm{E}, \end{aligned}$$

and

$$\therefore \quad \mathrm{S} = a_3{}^2 b_2{}^3 \mathrm{E}, \text{ as before.}$$

CHAPTER XIII.

HESSIANS, UP TO 1860.

SPECIAL cases of the determinant

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} & \cdots \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} & \cdots \\ \frac{\partial^2 u}{\partial z \partial x} & \frac{\partial^2 u}{\partial z \partial y} & \frac{\partial^2 u}{\partial z^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

where u is a function of $x, y, z, \ldots$, may well have appeared at a very early date in the history of determinants. The case where $u = ax^2 + 2bxy + cy^2$ may be viewed as traceable to Lagrange (1773), and the case where $u = ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$ to Gauss (1801); but it is certain that in those cases the elements of the determinants were not looked on as second differential-quotients of u. The general conception first occurred to Hesse in the year 1843.

HESSE, O. (1844, January).

[Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln. *Crelle's Journal*, xxviii. pp. 68–96; or *Werke*, pp. 89–122.]

In § 15 (p. 83) Hesse passes from the direct subject of his paper to the special case in which the three functions f_1, f_2, f_3 are the first differential-quotients of the homogeneous function of the third degree

$$\Sigma a_{\kappa,\lambda,\mu} x_\kappa x_\lambda x_\mu, \quad \text{or } f \text{ say,}$$

where each of the suffixes κ, λ, μ may be 1 or 2 or 3. The determinant, afterwards called the *Jacobian*, of f_1, f_2, f_3 he says may in that case be styled "the determinant of f." This expression at once recalls that used by Gauss in 1801, namely, "determinant of a form of the second degree," the determinant of

$$ax^2 + 2bxy + cy^2,$$

according to Gauss, being $b^2 - ac$, and the determinant of

$$a_1x^2 + a_2y^2 + a_3z^2 + 2b_1yz + 2b_2zx + 2b_3xy$$

being $a_1b_1^2 + a_2b_2^2 + a_3b_3^2 - a_1a_2a_3 - 2b_1b_2b_3$. The two usages, when Hesse's is restricted to the second degree, are not so far apart: for, according to Hesse, the determinants of the same two forms are

$$\begin{vmatrix} 2a & 2b \\ 2b & 2c \end{vmatrix}, \qquad \begin{vmatrix} 2a_1 & 2b_3 & 2b_2 \\ 2b_3 & 2a_2 & 2b_1 \\ 2b_2 & 2b_1 & 2a_3 \end{vmatrix},$$

i.e. $$-\,2^2(b^2 - ac), \quad -2^3(a_1b_1^2 + a_2b_2^2 + \ldots.).$$

The elements of Hesse's "determinant of f" being evidently the second differential-quotients of f, those in conjugate places must be equal—that is to say, the determinant is axisymmetric.

The first result enunciated is (p. 85)—*Die Determinante der Determinante einer gegebenen homogenen Function dritten Grades von drei Variabeln ist gleich der Summe der gegebenen Function und ihre Determinante, jede mit einem passenden constanten Factor multiplicirt.* In symbols at a later date this would have been written

$$\mathrm{H}\left\{\mathrm{H}(u_{33})\right\} = cu_{33} + c'\mathrm{H}(u_{33}).$$

Following thereupon is a theorem of like type

$$\mathrm{H}\left\{c_1u_{33} + c_2\mathrm{H}(u_{33})\right\} = c_3u_{33} + c_4\mathrm{H}(u_{33}),$$

and this is used to solve the equation

$$\mathrm{H}(u'_{33}) = u_{33},$$

where u_{33} and u'_{33} stand for ternary cubics, and u'_{33} is the unknown.

The effect of linear transformation, so strikingly brought to the front by Boole three years before, is then (§ 19) entered on, f being no longer a ternary cubic, but any function whatever of $x_1, x_2, \ldots, x_n$, and supposed to be expressed also as a function of the variables $y_1, y_2, \ldots, y_n$ by means of the equations

$$\left.\begin{array}{l} a_1^{(1)}x_1 + a_1^{(2)}x_2 + \ldots . + a_1^{(n)}x_n = y_1 \\ a_2^{(1)}x_1 + a_2^{(2)}x_2 + \ldots . + a_2^{(n)}x_n = y_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n^{(1)}x_1 + a_n^{(2)}x_2 + \ldots . + a_n^{(n)}x_n = y_n \end{array}\right\}.$$

Denoting the determinant of f when viewed as a function of the x's by ϕ, and when viewed as a function of the y's by ϕ', Hesse affirms that

$$\phi = r^2\phi',$$

where r is the determinant formed from the coefficients of the x's in the transforming equations. His proof is essentially that still followed; that is to say, he recalls that from the multiplication theorem we have

$$\sum \pm u_1^{(1)}u_2^{(2)} \ldots u_n^{(n)} = \sum \pm a_1^{(1)}a_2^{(2)} \ldots a_n^{(n)} \cdot \sum \pm w_1^{(1)}w_2^{(2)} \ldots w_n^{(n)},$$

if $\qquad u_\kappa^{(\lambda)} = a_1^{(\lambda)}w_1^{(\kappa)} + a_2^{(\lambda)}w_2^{(\kappa)} + \ldots + a_n^{(\lambda)}w_n^{(\kappa)};$

and

$$\sum \pm u_1^{(1)}u_2^{(2)} \ldots u_n^{(n)} = r^2 \cdot \sum \pm v_1^{(1)}v_2^{(2)} \ldots v_n^{(n)},$$

if in addition $\qquad u_\kappa^{(\lambda)} = a_1^{(\lambda)}v_1^{(\kappa)} + a_2^{(\lambda)}v_2^{(\kappa)} + \ldots + a_n^{(\lambda)}v_n^{(\kappa)};$

and he then merely asserts that application to the case where

$$u_\kappa^{(\lambda)} = \frac{\partial^2 f}{\partial x_\kappa \partial x_\lambda}, \qquad w_\lambda^{(\kappa)} = \frac{\partial\left(\dfrac{\partial f}{\partial x_\kappa}\right)}{\partial y_\lambda}, \qquad v_\kappa^{(\lambda)} = \frac{\partial^2 f}{\partial y_\kappa \partial y_\lambda}$$

accomplishes the desired aim. The result, as stated in later phraseology, is that "the Hessian is a covariant." The case where $f = ax^2 + 2bxy + cy^2$ was given by Lagrange in 1773, and the case $f = ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$ by Gauss in 1801.

The ternary cubic u_{33} is next returned to and shown to be transformable by a linear substitution into the form

$$y_1^3 + y_2^3 + y_3^3 + 6\pi y_1 y_2 y_3$$

and to be such that constants c_1, c_2 are determinable which make

$$c_1 u_{33} + c_2 \mathrm{H}(u_{33})$$

resolvable into linear factors.

CAYLEY, A. (1845, early).

[Note sur deux formules données par MM. Eisenstein et Hesse. *Crelle's Journ.*, xxix. pp. 54–57; or *Collected Math. Papers*, i. pp. 113–116.]

Cayley, who had, like others, been attracted by Boole's epoch-making paper on Linear Transformations, and was about to publish his own first paper on the subject (*Camb. and Dubl. Math. Journ.*, i. pp. 104–122), was naturally interested in that part of Hesse's paper which concerned the "determinant of *f*." He consequently wrote the note we have now reached, for the purpose of adding to Hesse's results and of extending an identity of Eisenstein's not distantly related to the same subject.

The "équation remarquable" of Hesse's which he starts with he writes in the form

$$\nabla(\mathrm{U}+a\nabla\mathrm{U}) = \mathrm{AU}+\mathrm{B}\nabla\mathrm{U},$$

noting that its author had not given the values of the coefficients A, B, "ce qui parait être très difficile à effectuer." He then announces the analogous theorem: "Soit U une fonction homogène et de l'ordre ν des deux variables x, y, et ∇U la déterminante

$$\frac{\partial^2\mathrm{U}}{\partial x^2}\cdot\frac{\partial^2\mathrm{U}}{\partial y^2} - \left(\frac{\partial^2\mathrm{U}}{\partial x\,\partial y}\right)^2,$$

l'on a

$$\text{-}2)(\nu-3)\cdot\nabla(\mathrm{U}+a\nabla\mathrm{U}) = \Big\{-\nu(\nu-1)(\nu-3)^2 a\mathrm{J} + \nu(\nu-1)(2\nu-5)^2 a^2\mathrm{I}\Big\}\mathrm{U}$$
$$+\Big\{(\nu-2)(\nu-3)^3 + (\nu-2)(\nu-3)(2\nu-5)a^2\mathrm{J}\Big\}\nabla\mathrm{U}.$$

En représentant par i, j, k, l, m les coefficients différentiels du quatrième ordre de U, on a

$$\mathrm{I} = ikm - il^2 - mj^2 - k^3 + 2jkl,$$
$$\mathrm{J} = 4jl - 3k^2 - mi,$$

de manière que I, J sont des fonctions de x, y des ordres $3(\nu-4)$ et $2(\nu-4)$ respectivement." To this he adds the remarkable fact, that if the binary quartic

$$i\xi^4 + 4j\xi^3\eta + 6k\xi^2\eta^2 + 4l\xi\eta^3 + m\eta^4,$$

where i, j, k, l, m are now any quantities independent of ξ, η, be transformed by the substitution

$$\left.\begin{aligned} \xi &= \lambda\xi' + \mu\eta' \\ \eta &= \lambda'\xi' + \mu'\eta' \end{aligned}\right\}$$

and I and J thus become I′ and J′, then

$$\mathrm{I}' = (\lambda\mu'-\lambda'\mu)^6\mathrm{I}, \qquad \mathrm{J}' = (\lambda\mu'-\lambda'\mu)^4\mathrm{J}.$$

In other words, he makes known for the first time the two "invariants" of a binary quartic, and notes the curious fact that expressions of exactly the same form occur in his equivalent for $\nabla(\mathrm{U}_{2,\nu}+a\nabla\mathrm{U}_{2,\nu})$.

Another remark is equally suggestive, namely, that simpler results might be reached if U were taken a homogeneous function in x', y' as well as in x, y, and $\nabla\mathrm{U}$ were defined as

$$\frac{\partial^2\mathrm{U}}{\partial x\,\partial x'}\cdot\frac{\partial^2\mathrm{U}}{\partial x\,\partial y'} - \frac{\partial^2\mathrm{U}}{\partial y\,\partial y'}\cdot\frac{\partial^2\mathrm{U}}{\partial y\,\partial x'}.$$

For example, U being a quadric in both sets of variables, namely,

$$\begin{aligned} \mathrm{U} = \quad & x_1{}^2(\mathrm{A}x^2+2\mathrm{B}xy+\mathrm{C}y^2) \\ &+ 2x_1y_1(\mathrm{A}'x^2+2\mathrm{B}'xy+\mathrm{C}'y^2) \\ &+ \quad y_1{}^2(\mathrm{A}''x^2+2\mathrm{B}''xy+\mathrm{C}''y^2), \end{aligned}$$

or in later notation

$$\mathrm{U} = \begin{array}{ccc|l} x^2 & 2xy & y^2 & \\ \hline \mathrm{A} & \mathrm{B} & \mathrm{C} & x_1{}^2 \\ \mathrm{A}' & \mathrm{B}' & \mathrm{C}' & 2x_1y_1 \\ \mathrm{A}'' & \mathrm{B}'' & \mathrm{C}'' & y_1{}^2, \end{array}$$

then we should have*

$$\nabla\nabla U = 2^{10} \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} U.$$

In connection with the first of these results of Cayley's the reader should note that on putting $\nu=4$ we obtain

$$\nabla(U+a\nabla U) = (-6aJ+54a^2I)U + (1+3a^2J)\nabla U,$$

or $$\nabla(\alpha U+\beta\nabla U) = (-6\alpha\beta J+54\beta^2 I)U + (\alpha^2+3\beta^2 J)\nabla U,$$

and $\therefore$ $$\nabla\nabla U = 54\, I\cdot U+3\, J\cdot\nabla U.$$

Further, if the particular form of U be

$$ax^2+4bx^3y+6cx^2y^2+4dxy^3+ey^4,$$

this gives

$$\nabla\nabla U = 12^3(432\, I\cdot U-J\cdot\nabla U)$$

where $$I = ace+2bcd-ad^2-eb^2-c^3,$$

and $$J = ae+3c^2-4bd.$$

In his famous first paper (February 1845) "On the Theory of Linear Transformations," of which this is merely an offshoot, Cayley states that the invariance of I had been communicated to him by Boole, along with the still more interesting fact that Boole's invariant (*i.e.* the discriminant) is equal to J^3-27I^2.

CAYLEY, A. (1846).

[On homogeneous functions of the third order with three variables. *Cambridge and Dub. Math. Journ.*, i. pp. 97–104; or *Collected Math. Papers*, i. pp. 230–233.]

One of such functions, U say, being taken in the form

$$ax^3 + by^3 + cz^3 + 3iy^2z + 3jz^2x + 3kx^2y$$
$$+ 3i_1yz^2 + 3j_1zx^2 + 3k_1xy^2 + 6lxyz,$$

Cayley calculates ∇U, and writing it in the form

$$Ax^3 + By^3 + Cz^3 + 3Iy^2z + \ldots\ldots$$

gives the values† of A, B, C, 3I, ... in terms of a, b, c, i. ....

* Instead of 2^{10} we find in the original 2^6, and in the *Collected Math. Papers* 2^8.

† In the value of $3K_1$ there is printed $-2bi_1l$ instead of $-2bj_1l$.

CAYLEY, A. (1847).

[Note sur les hyperdéterminants. *Crelle's Journal*, xxxiv. pp. 148–152; or *Collected Math. Papers*, i. pp. 352–355.]

As already noted, the second section of this short paper concerns what would, a few years later, have been called "the Hessian of the discriminant

$$6abcd + 3b^2c^2 - a^2d^2 - 4ac^3 - 4b^3d$$

of the binary cubic." The result, which is rather inelegantly verified, is that the said Hessian is a numerical multiple of the square of the discriminant.

HESSE, O. (1847, August).

[Ueber Curven dritter Classe und Curven dritter Ordnung. *Crelle's Journ.*, xxxviii. pp. 241–256; or *Werke*, pp. 193-210.]

Any homogeneous function of the m^{th} degree in the variables x_1, x_2, x_3, being denoted by u, its first differential-quotients by u_1, u_2, u_3, and its second differential-quotients by u_{11}, u_{12}, . . . , there is obtained from Euler

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= (m-1)u_1 \\ u_{21}x_1 + u_{22}x_2 + u_{23}x_3 &= (m-1)u_2 \\ u_{31}x_1 + u_{32}a_2 + u_{33}x_3 &= (m-1)u_3 \end{aligned}\right\},$$

and thence, on solving,

$$\left.\begin{aligned} \frac{\Delta}{m-1}x_1 &= \mathrm{U}_{11}u_1 + \mathrm{U}_{12}u_2 + \mathrm{U}_{13}u_3 \\ \frac{\Delta}{m-1}x_2 &= \mathrm{U}_{21}u_1 + \mathrm{U}_{22}u_2 + \mathrm{U}_{23}u_3 \\ \frac{\Delta}{m-1}x_3 &= \mathrm{U}_{31}u_1 + \mathrm{U}_{32}u_2 + \mathrm{U}_{33}u_3 \end{aligned}\right\},$$

where, evidently, Δ is used for Hesse's determinant of u, and U_{rs} for the cofactor of u_{rs} in Δ. Using in connection with the latter three equations the multipliers u_1, u_2, u_3, and adding, Hesse derives the interesting result

$$\frac{m}{m-1}u\Delta = \mathrm{U}_{11}u_1^2+\mathrm{U}_{22}u_2^2+\mathrm{U}_{33}u_3^2+2\mathrm{U}_{23}u_2u_3+2\mathrm{U}_{31}u_3u_1+2\mathrm{U}_{12}u_1u_2,$$

and this by a process of differentiation leads to six results of the type

$$u_{12}u_1u_3+u_{13}u_1u_2-u_{23}u_1^2-u_{11}u_2u_3 = \frac{m}{m-1}\mathrm{U}_{23}u - \frac{x_2x_3}{(m-1)^2}\Delta.$$

The rest of the paper is geometrical.

HESSE, O. (1849, January).

[Transformation einer beliebigen gegebenen homogenen Function 4ten Grades von zwei Variabeln. . . . . *Crelle's Journ.*, xli. pp. 243–263; or *Werke*, pp. 223–246.]

A binary quartic u_{24} being the only other homogeneous integral function whose determinant, in Hesse's sense, is of the same degree as the function, there was naturally an inclination to make a study of its properties in the same fashion as has been followed with the ternary cubic. Analogous results are reached, such, for example, as the theorem that *it is possible to determine constants* c_1, c_2 *so that*

$$c_1u_{24}+c_2u\mathrm{H}(_{24})$$

may be an exact square. Most of the matter, however, more directly concerns the quartic than its so-called determinant.

ARONHOLD, S. (1849, July).

[Zur Theorie der homogenen Functionen dritten Grades von drei Variabeln. *Crelle's Journ.*, xxxix. pp. 140–159.]

This is an inspiration from, and a striking development of, the latter part of Hesse's paper of the year 1844, and like that paper may be said to concern itself more with the ternary cubic than with the so-called determinant of that function. In regard to the latter, however, there is one very noteworthy result: for, just as Cayley in 1845 established the two invariants I and J of a binary quartic u_{24}, and used them for the expression of $\mathrm{H}\left\{a \cdot u_{24} + b \cdot \mathrm{H}(u_{24})\right\}$ in the form $\mathrm{A} \cdot u_{24} + \mathrm{B} \cdot \mathrm{H}(u_{24})$, so Aronhold here announces the two invariants S and T of a ternary cubic, and gives the similar expression for

$$\mathrm{H}\left\{a \cdot u_{33} + b \cdot \mathrm{H}(u_{33})\right\}.$$

Obtained from this by putting $a = 0$ is the result

$$\mathrm{H}\left\{\mathrm{H}(u_{33})\right\} = 3\mathrm{S}^3 \cdot u_{33} - 2\mathrm{T} \cdot \mathrm{H}(u_{33})$$

—the longed-for definite form of Hesse's theorem of the year 1844.

We may also note in passing that the result of eliminating x, y, z from the equations

$$\frac{\partial u_{33}}{\partial x} = 0, \quad \frac{\partial u_{33}}{\partial y} = 0, \quad \frac{\partial u_{33}}{\partial z} = 0$$

is expressed in terms of S and T, namely,

$$\mathrm{T}^2 - \mathrm{S}^3 = 0;$$

or, in later phraseology, that the discriminant of u_{33} is $\mathrm{T}^2 - \mathrm{S}^3$.

HESSE AND JACOBI (1849, December).

[Auszug zweier Schreiben des Prof. Hesse an den Herrn Prof. Jacobi und eines Schreibens des Prof. Jacobi an Herrn Prof. Hesse. *Crelle's Journ.*, xl. pp. 316–318.]

Hesse having communicated to Jacobi a theorem regarding a homogeneous function of three variables, Jacobi sent back a proof showing that the theorem held in the case of n variables. The function being noted by u, and being of the m^{th} degree in the variables $x_1, x_2, \ldots, x_n$, Jacobi, like Hesse himself in his paper of 1847 (August), obtains the equations

$$\left.\begin{array}{l} x_1 u_{11} + x_2 u_{21} + \ldots + x_n u_{n1} = (m-1)u_1 \\ x_1 u_{12} + x_2 u_{22} + \ldots + x_n u_{n2} = (m-1)u_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_1 u_{1n} + x_2 u_{2n} + \ldots + x_n n_{nn} = (m-1)u_n \end{array}\right\},$$

and thence

$$x_i\Delta = (m-1)\Big\{U_{i1}u_1 + U_{i2}u_2 + \ \dots \ + U_{in}u_n\Big\}. \qquad (\alpha)$$

Differentiating both sides of this with respect to x_k, we have

$$\begin{aligned} x_i\frac{\partial\Delta}{\partial x_k} &= (m-1)\Big\{U_{i1}u_{k1} + U_{i2}u_{k2} + \ \dots \ + U_{in}u_{kn}\Big\} \\ &\quad + (m-1)\left\{u_1\frac{\partial U_{i1}}{\partial x_k} + u_2\frac{\partial U_{i2}}{\partial x_k} + \ \dots \ + u_n\frac{\partial U_{in}}{\partial x_k}\right\} \\ &= (m-1)\left\{u_1\frac{\partial U_{i1}}{\partial x_k} + u_2\frac{\partial U_{i2}}{\partial x_k} + \ \dots \ + u_n\frac{\partial U_{in}}{\partial x_k}\right\} \qquad (\beta) \end{aligned}$$

if k be different from i. A second differentiation, but this time with respect to x_l, gives

$$\begin{aligned} x_i\frac{\partial^2\Delta}{\partial x_l\partial x_k} &= (m-1)\left\{u_1\frac{\partial^2 U_{i1}}{\partial x_l\partial x_k} + u_2\frac{\partial^2 U_{i2}}{\partial x_l\partial x_k} + \ \dots \ + u_n\frac{\partial^2 U_{in}}{\partial x_l\partial x_k}\right\} \\ &\quad + (m-1)\left\{u_{l1}\frac{\partial U_{i1}}{\partial x_k} + u_{l2}\frac{\partial U_{i2}}{\partial x_k} + \ \dots \ + u_{ln}\frac{\partial U_{in}}{\partial x_k}\right\}. \end{aligned}$$

But on the supposition that l is different from i we have

$$u_{l1}U_{i1} + u_{l2}U_{i2} + \ \dots \ + u_{ln}U_{in} = 0,$$

and therefore by differentiation with respect to x_k

$$\begin{aligned} & u_{l1}\frac{\partial U_{i1}}{\partial x_k} + u_{l2}\frac{\partial U_{i2}}{\partial x_k} + \ \dots \ + u_{ln}\frac{\partial U_{in}}{\partial x_k} \\ & + u_{kl1}U_{i1} + u_{kl2}U_{i2} + \ \dots \ + u_{kln}U_{in} = 0. \end{aligned}$$

Consequently by substitution

$$\begin{aligned} x_i\frac{\partial^2\Delta}{\partial x_l\partial x_k} &= (m-1)\left\{u_1\frac{\partial^2 U_{i1}}{\partial x_l\partial x_k} + u_2\frac{\partial U_{i2}}{\partial x_l\partial x_k} + \ \dots \ + u_n\frac{\partial^2 U_{in}}{\partial x_l\partial x_k}\right\} \\ &\quad -(m-1)\Big\{u_{kl1}U_{i1} + u_{kl2}U_{i2} + \ \dots \ + u_{kln}U_{in}\Big\}, \qquad (\gamma) \end{aligned}$$

where l and k are each different from i.

If now special values of $x_1, x_2, \dots x_n$ make $u_1, u_2, \dots, u_n$ all vanish, then, Jacobi says, we shall also have

$$\Delta = 0, \quad \frac{\partial\Delta}{\partial x_k} = 0, \quad U_{rs} = Nx_rx_s$$

for all values of r and s *; and consequently from (γ)

$$x_i \frac{\partial^2 \Delta}{\partial x_i \partial x_k} = -(m-1)\left\{\mathrm{N}x_i x_1 \frac{\partial u_{kl}}{\partial x_1} + \mathrm{N}x_i x_2 \frac{\partial u_{kl}}{\partial x_2} + \ldots\right\}$$
$$= -(m-1)\mathrm{N}x_i\left\{x_1 \frac{\partial u_{kl}}{\partial x_1} + x_2 \frac{\partial u_{kl}}{\partial x_2} + \ldots\right\}$$
$$= -(m-1)(m-2)\mathrm{N}x_i u_{kl},$$

whence

$$\frac{\partial^2 \Delta}{\partial x_i \partial x_k} = -(m-1)(m-2)\mathrm{N}\cdot\frac{\partial^2 u}{\partial x_i \partial x_k}.$$

This result we may formally enunciate as follows:—*If the first differential-quotients of a homogeneous rational integral function all vanish, the elements of the Hessian of the function are proportional to the elements of the Hessian of the Hessian.*

TERQUEM, O. (1851, March).

[Note sur les déterminants. *Nouv. Annales de Math.*, x. pp. 124–131.]

This is an elementary exposition of Hesse's determinant, with simple illustrations from algebraic geometry, the property of "covariance" being made prominent. A curious distinction is made between what are called the "first" and "second" determinants: for example,

$$4ac - b^2$$

* The first two of these results, which follow from (α) and (β), there is no pressing reason for mentioning: it would have been equally pertinent to note that $u=0$. The third result Jacobi probably obtained (see *Crelle's Journal*, xv. p. 304) by taking in every possible way $n-1$ of the initiatory set of equations and deducing

$$\begin{aligned} x_1 : x_2 : \ldots\ldots : x_n &= \mathrm{U}_{11} : \mathrm{U}_{21} : \ldots\ldots : \mathrm{U}_{n1}, \\ &= \mathrm{U}_{12} : \mathrm{U}_{22} : \ldots\ldots : \mathrm{U}_{n2}, \\ &\ldots\ldots\ldots\ldots\ldots \\ &= \mathrm{U}_{1n} : \mathrm{U}_{2n} : \ldots\ldots : \mathrm{U}_{nn}. \end{aligned}$$

This implies that $\mathrm{U}_{rs} : \mathrm{U}_{rs'} = x_s : x_{s'}$

and $\mathrm{U}_{s'r} : \mathrm{U}_{s'r'} = x_r : x_{r'}$;

and from these, by reason of the equality of $\mathrm{U}_{rs'}$ and $\mathrm{U}_{s'r}$ there is got by multiplication

$$\mathrm{U}_{rs} : \mathrm{U}_{s'r'} = x_s x_r : x_{s'} x_{r'}.$$

is styled "le *premier déterminant* de la fonction hexanôme du second degré à deux variables," and

$$8acf + 4bde - 2ae^2 - 2cd^2 - 2fb^2$$

the "second déterminant de la même fonction rendue homogène et ternaire." The former is, in fact, the determinant of

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2$$

or

$$ax^2 + bxy + cy^2 + dx + ey + f$$

with respect to x, y; and the latter the determinant with respect to x, y, z.

HESSE, O. (1851, March).

[Ueber die Bedingung, unter welcher eine homogene ganze Function von n unabhängigen Variabeln durch lineare Substitutionen von n andern unabhängigen Variabeln auf eine homogene Function sich zurück führen lässt, die eine Variabel weniger enthält. *Crelle's Journ.*, xlii. pp. 117–124; or *Werke*, pp. 289–296.]

Hesse here returns to the subject of § 19 of his original paper, calling $\Sigma \pm u_{11}u_{22} \ldots u_{nn}$ or Δ the determinant of u with respect to the variables $x_1, x_2, \ldots x_n$, and $\Sigma \pm u^{11}u^{22} \ldots u^{nn}$ or ∇ the determinant of u with respect to $y_1, y_2, \ldots, y_n$, and proving once more his theorem that

$$\nabla = r^2\Delta.$$

He then supposes that in the result of the transformation y_n does not appear, and says that as this implies that $u^{1n}, u^{2n}, \ldots, u^{nn}$ all vanish, it follows that $\nabla = 0$, and that therefore from the said theorem Δ also must vanish. There is thus obtained the result that, "*Wenn eine homogene ganze Function der* n *unabhängigen Variabeln* $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$, *durch*

$$\mathrm{x}_k = \mathrm{a}_1^k \mathrm{y}_1 + \mathrm{a}_2^k \mathrm{y}_2 + \ldots + \mathrm{a}_n^k \mathrm{y}_n$$

in eine Function der Variabeln $\mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{y}_n$ *übergeht, in welcher eine dieser Variabeln fehlt, so ist die Determinante dieser Function in Rücksicht auf die Variabeln* $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$, *identisch gleich* 0."

The rest of the paper is occupied with the converse theorem; but as the author himself came to be dissatisfied with his attempt at a proof and returned to the subject seven years later, it need not be entered on here.

SYLVESTER, J. J. (1851, April).

[Sketch of a memoir on elimination, transformation, and canonical forms. *Cambridge and Dub. Math. Journ.*, vi. pp. 186–200; or *Collected Math. Papers*, i. pp. 184–197.]

The expression "determinant of a function" or, more definitely, "determinant of a function in respect to certain variables" occurs repeatedly in Sylvester's writings of the year 1850, the accompanying notation being*

$$\Box(u);$$

for example, when dealing with ternary quadrics U and V, expressions like

$$\underset{\lambda\mu}{\Box}\;\underset{xyz}{\boxed{}}\;(\lambda \mathrm{U}+\mu \mathrm{V})$$

are in constant use by him. It is clear, however, that the determinant which he had in mind was not Hesse's, but that which the year following he named the "discriminant." †

The interest of the present paper lies in the fact that, amid much other matter, not only are the said two determinants clearly defined and distinguished, but are shown to be viewable as having a common parentage, being indeed two extreme members of a family group. In the first place, the determinant of any homogeneous integral function is incidentally defined as the resultant of the first partial differential coefficients of the function, when drawing attention to Boole's proposition (1843) that the said determinant "is unaltered by any linear transformation of the variables, except so far as regards the introduction of

---

* $\Box$ was used by Cayley in 1846 as the symbol of hyperdeterminant derivation. See *Collected Math. Papers*, i. p. 97.

† See *Philos. Magazine*, ii. (1851), p. 406, and *Cambridge and Dub. Math. Journ.*, vii. (1852), p. 52; or Sylvester's *Collected Math. Papers*, i. pp. 280, 284.

a power of the modulus of transformation." It is spoken of later in the paper as the "common constant determinant" or the "ordinary determinant" of the function, the word *discriminant* not being proposed until a later date in the same year. In the second place, there is brought into notice in connection with any homogeneous integral function $\phi(x, y, \ldots, z)$ of the n^{th} degree the family of functions

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ \ldots\ +\zeta\frac{\partial}{\partial z}\right)^r\phi(x,\,y,\,\ldots,\,z),$$

where r has the values $1, 2, \ldots, n$. Corresponding to these there is a family of determinants (*i.e.* discriminants), namely

$$\underset{\xi,\,\eta,\,\ldots}{\boxed{}}\ \left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ \ldots\ +\zeta\frac{\partial}{\partial z}\right)^r\phi(x,\,y,\,\ldots,\,z),$$

where $r = 2, 3, \ldots, n$, the first being according to Sylvester the "Hessian" or "First Boolian" determinant* of ϕ, and the last the "Final Boolian" or "ordinary determinant" of ϕ. The reader is left in the former case to reconcile the new definition with Hesse's own definition, and in the latter case to observe that

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ \ldots\ +\zeta\frac{\partial}{\partial z}\right)^n\phi(x,\,y,\,\ldots,\,z)\ =\ \phi(\xi,\,\eta,\,\ldots,\,\zeta).$$

The notation used for the Hessian of ϕ is $\mathrm{H}(\phi)$: by "second Hessian" he says he means "Hessian of the Hessian"; by "post-Hessian" the determinant of the function got by taking $r=3$; and similarly for "præter-post-Hessian"!

We may at once remark that much of this nomenclature had a very short life, being supplanted by other coinages made by Sylvester himself. The functions

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\ldots\right)^r\phi(x,\,y,\,\ldots)$$

* On p. 194 he says the Hessian of $\mathrm{F}(x, y)$ is "the determinant *of the determinant,* in respect of ξ and η, of

$$\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)^2\mathrm{F}(x,y)\text{"}$$

—an error which is repeated in the *Collected Math. Papers.*

he soon named the *emanants* of ϕ: and thus the Hessian, post-Hessian, præter-post-Hessian, and other determinants forming the "Hessian (or Boolian) Scale" became known as the discriminants of the emanants of ϕ. To the first member of the scale, however, the word "Hessian" became permanently attached, although Sylvester's mode of defining it as the "discriminant of the quadratic (or second) emanant" * did not spread. It was introduced by Salmon into the first edition of his *Higher Plane Curves* (see p. 72) about a year after Sylvester's first use of it, and met with rapid acceptance.

SALMON, G. (1852).

[A TREATISE ON THE HIGHER PLANE CURVES. . . . . xii+316 pp. Dublin.]

In sect. ix. (pp. 181–195) Salmon deals with the "General Equation of the Third Degree" on the lines of Aronhold's paper of 1849, and the Hessian naturally comes in for attention. The ternary cubic U which constitutes the non-zero side of the equation he writes in the form

$$a_1x^3+b_2y^3+c_3z^3+3a_2x^2y+3b_3y^2z+3c_1z^2x \\ +3a_3x^2z+3b_1y^2x+3c_2z^2y+6dxyz,$$

and following Cayley gives its Hessian † as

$$\mathrm{a}_1x^3+\mathrm{b}_2y^3+\mathrm{c}_3z^3+3\mathrm{a}_2x^2y+\ \ldots\ .,$$

where

$$\begin{aligned} \mathrm{a}_1 &= a_1d^2+b_1a_3{}^2+c_1a_2{}^2-a_1b_1c_1-2da_2a_3, \\ \mathrm{b}_2 &= b_2d^2+c_2b_1{}^2+a_2b_3{}^2-a_2b_2c_2-2db_3b_1, \\ \mathrm{c}_3 &= c_3d^2+a_3c_2{}^2+b_3c_1{}^2-a_3b_3c_3-2dc_1c_2, \end{aligned}$$

* See *Philos. Magazine*, v. p. 122; or *Collected Math. Papers*, i. p. 591.

† It should be noted that this is $\frac{1}{216}$ of the Hessian as defined, that a_1 is expressible as a three-line determinant, that $3\mathrm{a}_2$ and $3\mathrm{a}_3$ are expressible as the sum of two such determinants, that $6\mathrm{d}$ is expressible as the sum of three such determinants, and that the performance of the circular substitutions

$$a_1,\ b_2,\ c_3=b_2,\ c_3,\ a_1, \quad a_2,\ b_3,\ c_1=b_3,\ c_1,\ a_2, \quad a_3,\ b_1,\ c_2=b_1,\ c_2,\ a_3,$$

on the expressions for a_1, $3\mathrm{a}_2$, $3\mathrm{a}_3$, gives us six other of the expressions.

$$
\begin{aligned}
3\mathrm{a}_2 &= c_2a_2^{\,2} + a_2c_1b_1 - 2a_2a_3b_3 - a_2d^2 + 2a_1b_3d + b_2a_3^{\,2} - a_1c_1b_2 - a_1b_1c_2,\\
3\mathrm{b}_3 &= a_3b_3^{\,2} + b_3a_2c_2 - 2b_3b_1c_1 - b_3d^2 + 2b_2c_1d + c_3b_1^{\,2} - b_2a_2c_3 - b_2c_2a_3,\\
3\mathrm{c}_1 &= b_1c_1^{\,2} + c_1b_3a_3 - 2c_1c_2a_2 - c_1d^2 + 2c_3a_2d + a_1c_2^{\,2} - c_3b_3a_1 - c_3a_3b_1,\\
3\mathrm{a}_3 &= b_3a_3^{\,2} + a_3b_1c_1 - 2a_3a_2c_2 - a_3d^2 + 2a_1c_2d + c_3a_2^{\,2} - a_1b_1c_3 - a_1c_1b_3,\\
3\mathrm{b}_1 &= c_1b_1^{\,2} + b_1c_2a_2 - 2b_1b_3a_3 - b_1d^2 + 2b_2a_3d + a_1b_3^{\,2} - b_2c_2a_1 - b_2a_2c_1,\\
3\mathrm{c}_2 &= a_2c_2^{\,2} + c_2a_3b_3 - 2c_2c_1b_1 - c_2d^2 + 2c_3b_1d + b_2c_1^{\,2} - c_3a_3b_2 - c_3b_3a_2,\\
6\mathrm{d} &= -2d^3 + 2d(b_1c_1+c_2a_2+a_3b_3) + (a_1b_3c_2+b_2c_1a_3+c_3a_2b_1) - a_1b_2c_3\\
&\qquad - 3(a_2b_3c_1+a_3b_1c_2).
\end{aligned}
$$

The invariants S and T are also printed in full, viz.

$$
\begin{aligned}
\mathrm{S} &= d^4 - 2d^2(b_1c_1+c_2a_2+a_3b_3) + 3d(a_2b_3c_1+a_3b_1c_2) - d\cdot a_1b_2c_3\\
&\quad + d(a_1b_3c_2+b_2c_1a_3+c_3a_2b_1) - (b_1c_1\cdot c_2a_2+c_2a_2\cdot a_3b_3+a_3b_3\cdot b_1c_1)\\
&\quad + (b_1^{\,2}c_1^{\,2}+c_2^{\,2}a_2^{\,2}+a_3^{\,2}b_3^{\,2}) - (a_1b_2\cdot c_1c_2+b_2c_3\cdot a_2a_3+c_3a_1\cdot b_3b_1)\\
&\quad + (b_3c_3a_2^{\,2}+c_1a_1b_3^{\,2}+a_2b_2c_1^{\,2}+b_2c_2a_3^{\,2}+c_3a_3b_1^{\,2}+a_1b_1c_2^{\,2}),\\
\mathrm{T} &= -8d^6 + 24d^4(b_1c_1+c_2a_2+a_3b_3) - \ldots\ldots
\end{aligned}
$$

As these differ from Aronhold's by numerical factors, we are prepared to find corresponding differences in the expressions for the Hessian of the Hessian and for the discriminant, namely,

$$4\mathrm{S}^2\cdot\mathrm{U} - \mathrm{T}\cdot\mathrm{H}(\mathrm{U}) \quad \text{and} \quad \mathrm{T}^2 - 64\mathrm{S}^3$$

respectively.

BRIOSCHI, F. (1852, August).

[Sur les déterminants des formes quadratiques. *Nouv. Annales de Math.*, xi. pp. 307–311; or *Opere mat.* v. pp. 81–85.]

After an introduction of two pages on determinants in general, the determinant of a quadratic form is defined as the determinant whose elements are the second differential-quotients of the form, the editor adding in a footnote the words, "c'est le déterminant *hessien* des Anglais." Starting then from the known fact that if $a_1a_2 - b_1^2 = 0$

$$a_1x_1^2 + a_2x_2^2 + 2b_1x_1x_2 = \frac{1}{a_1}(a_1x_1+b_1x_2)^2,$$

Brioschi states that similarly, if the determinants of

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_1x_2 + 2b_2x_1x_3 + 2c_1x_2x_3\,,$$
$$a_1x_1^2 + a_2x_2^2 + 2b_1x_1x_2\,,$$
$$a_1x_1^2 + a_3x_3^2 + 2b_2x_1x_3\,,$$

all vanish, the ternary quadric is equal to

$$\frac{1}{a_1}(a_1x_1+b_1x_2+b_2x_3)^2\,;$$

and if the determinants of

$$\left.\begin{array}{r} a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + 2b_1x_1x_2 + 2b_2x_1x_3 + 2b_3x_1x_4 \\ + 2c_1x_2x_3 + 2c_2x_2x_4 + 2d_1x_3x_4 \end{array}\right\}\,,$$
$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_1x_1x_2 + 2b_2x_1x_3 + 2c_1x_2x_3\,,$$
$$a_1x_1^2 + a_2x_2^2 + a_4x_4^2 + 2b_1x_1x_2 + 2b_3x_1x_4 + 2c_2x_2x_4\,,$$
$$a_1x_1^2 + a_2x_2^2 + 2b_1x_1x_2\,,$$
$$a_1x_1^2 + a_3x_3^2 + 2b_2x_1x_3\,,$$
$$a_1x_1^2 + a_4x_4^2 + 2b_3x_1x_4\,,$$

all vanish, the quaternary quadric is equal to

$$\frac{1}{a_1}(a_1x_1+b_1x_2+b_2x_3+b_3x_4)^2\,;$$

and so on generally. An alternative set of conditions is referred to, and is exemplified by the case of the ternary quadric, where the vanishing of $a_1c_1-b_1b_2$ is substituted for the vanishing of

$$a_1a_2a_3 + 2b_1b_2c_1 - a_1c_1^2 - a_2b_2^2 - a_3b_1^2\,,$$

the latter being equal to

$$\Big\{(a_1a_2-b_1^2)(a_1a_3-b_2^2) - (a_1c_1-b_1b_2)^2\Big\} \div a_1\,.$$

SYLVESTER, J. J. (1853).

[On the conditions necessary and sufficient to be satisfied in order that a function of any number of variables may be linearly equivalent to a function of any less number of variables. *Philos. Magazine*, v. pp. 119–126; or *Collected Math. Papers*, i. pp. 587–594.]

The title at once suggests a connection with Hesse's converse theorem of 1851 (March): the investigation, however, proceeds on totally different lines, and only concerns us because of the doubt thrown on the truth of the said theorem by Sylvester's assertion that the Hessian "is really foreign to the nature" of the question under discussion.

SPOTTISWOODE, W. (1853, August).

[Elementary theorems relating to determinants; second edition, rewritten and much enlarged by the author. *Crelle's Journ.*, li. pp. 209–271, 328–381.]

The latter portion (pp. 343–350) of his chapter (§ ix.) "On Functional Determinants" Spottiswoode devotes to one or two theorems connected with Hessians, and mainly to Hesse's theorem of the year 1849 (December): his proof of the latter, however, is not an improvement on Jacobi's. One of his notations for the Hessian, U being the function and $x, y, z, \ldots$ the variables, is

$$\left\{\begin{matrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \cdots \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \cdots \end{matrix}\right\} \mathrm{U},$$

suggested, doubtless, by Sylvester's general umbral notation. Further, he uses the word "Hessian" in a geometrical sense, namely, for the locus represented by

$$\mathrm{H}(\mathrm{U}) = 0.$$

SALMON, G. (1854, Feb.).

[Exercises in the hyperdeterminant calculus. *Cambridge and Dub. Math. Journ.*, ix. pp. 19–33.]

Of these exercises, which usefully served as an exposition of Cayley's so-called hyperdeterminant method of deriving invariants and covariants, it is the seventh (pp. 24–25) which here concerns us, the subject being the calculation of the Hessian of the Hessian of a binary quantic. In this method the symbol $\overline{12}^2$, primarily introduced to stand for the operator

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2} \end{vmatrix}^2,$$

comes by the adoption of additional conventions to represent the Hessian of a binary quantic: and similar considerations lead to the Hessian of the Hessian being denoted by

$$(\overline{13}+\overline{14}+\overline{23}+\overline{24})^2\overline{12}^2\cdot\overline{34}^2.$$

Adopting this expression Salmon succeeds in transposing it so as to give Cayley's binomial expression of the year 1845. The proof is reproduced with improvements in his *Modern Higher Algebra* of 1859 (§§ 170, 171: pp. 140–141), where also the special case of the binary quartic is referred to (§ 135, pp. 103–104), and Hesse's analogous theorem regarding the ternary cubic (§§ 143, 145: pp. 112–114).

BRIOSCHI, F. (1854).

[Solutions des questions 285, 286. *Nouv. Annales de Math.*, xiii. pp. 402–409.]

The theorems which had been set for proof were geometrical theorems due to Hesse, and as a foundation on which to base them and others Brioschi establishes a general result regarding Hessians. This with only slight departures from the original may be formally enunciated as follows:—*If* u *be a homogeneous integral function of the* m^{th} *degree in* $r+1$ *variables* $x_0, x_1, x_2, \ldots, x_r$, *the Hessian of which with respect to those variables is* H_{r+1} *and with respect to the variables* $x_1, x_2, \ldots, x_r$ *is* H_r, *then the determinant which is the result of bordering* H_r *by prefixing*

$$0,\ \frac{\partial u}{\partial x_1},\ \frac{\partial u}{\partial x_2},\ \ldots,\ \frac{\partial u}{\partial x_r}$$

as a first row and as a first column is equal to

$$(-1)^{r+1}\frac{m}{m-1}uH_r + \frac{{x_0}^2}{(m-1)^2}H_{r+1}.$$

The bordered Hessian, B say, being equal to

$$\frac{1}{m-1}\begin{vmatrix} \cdot & u_1 & u_2 & \dots & u_r \\ (m-1)u_1 & u_{11} & u_{12} & \dots & u_{1r} \\ (m-1)u_2 & u_{21} & u_{22} & \dots & u_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (m-1)u_r & u_{r1} & u_{r2} & \dots & u_{rr} \end{vmatrix}$$

and Euler's theorem regarding the differentiating of homogeneous functions giving

$$\left.\begin{aligned} mu &= x_0u_0 + x_1u_1 + \dots + x_ru_r \\ (m-1)u_0 &= x_0u_{00} + x_1u_{01} + \dots + x_ru_{0r} \\ (m-1)u_1 &= x_0u_{10} + x_1u_{11} + \dots + x_ru_{1r} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (m-1)u_r &= x_0u_{r0} + x_1u_{r1} + \dots + x_ru_{rr} \end{aligned}\right\},$$

there is obtained

$$B = \frac{1}{m-1}\begin{vmatrix} x_0u_0 - mu & u_1 & u_2 & \dots & u_r \\ x_0u_{10} & u_{11} & u_{12} & \dots & u_{1r} \\ x_0u_{20} & u_{21} & u_{22} & \dots & u_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_0u_{r0} & u_{r1} & u_{r2} & \dots & u_{rr} \end{vmatrix}$$

$$= -\frac{m}{m-1}u\begin{vmatrix} u_{11} & u_{12} & \dots & u_{1r} \\ u_{21} & u_{22} & \dots & u_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ u_{r1} & u_{r2} & \dots & u_{rr} \end{vmatrix} + \frac{x_0}{m-1}\begin{vmatrix} u_0 & u_1 & \dots & u_r \\ u_{10} & u_{11} & \dots & u_{1r} \\ u_{20} & u_{21} & \dots & u_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ u_{r0} & u_{r1} & \dots & u_{rr} \end{vmatrix}.$$

By similar treatment, however, the second determinant on the right of this

$$= \frac{1}{m-1}\begin{vmatrix} (m-1)u_0 & (m-1)u_1 & \dots & (m-1)u_r \\ u_{10} & u_{11} & \dots & u_{1r} \\ u_{20} & u_{21} & \dots & u_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ u_{r0} & u_{r1} & \dots & u_{rr} \end{vmatrix},$$

$$= \frac{x_0}{m-1}\begin{vmatrix} u_{00} & u_{01} & \dots & u_{0r} \\ u_{10} & u_{11} & \dots & u_{1r} \\ \cdot & \cdot & \cdot & \cdot \\ u_{r0} & u_{r1} & \dots & u_{rr} \end{vmatrix},$$

so that finally we have

$$B = -\frac{m}{m-1}uH_r + \frac{x_0^{\ 2}}{(m-1)^2}H_{r+1},$$

as desired.

As a corollary it is noted that if H_{r+1} vanishes identically, then

$$(m-1)B + muH_r = 0,$$

i.e.

$$\begin{vmatrix} mu & u_1 & \dots & u_r \\ (m-1)u_1 & u_{11} & \dots & u_{r1} \\ (m-1)u_2 & u_{21} & \dots & u_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ (m-1)u_r & u_{r1} & \dots & u_{rr} \end{vmatrix} = 0,$$

"et l'équation

$$u(x_1, x_2, \dots, x_r) = 0$$

est elle-même homogène"—a sort of converse of Euler's theorem above referred to.

BRIOSCHI, F. (1854, March).

[LA TEORICA DEI DETERMINANTI, E LE SUE PRINCIPALI APPLICAZIONI. viii+116 pp. Pavia.]

The last section (§ 11, pp. 106–116) of Brioschi's text-book is headed "Del determinante di Hesse." Opening with the definition of "l'Hessiano," it gives a clear and orderly exposition of a goodly number of the main theorems up till then discovered, with geometrical applications.*

Separated altogether, however, from these is a demonstration (pp. 20, 21) which strictly belongs to this section. Recognising that Hesse's expression (1847, August) for the product of u and its Hessian Δ is in reality obtained by eliminating x_1, x_2, x_3 from four equations, Brioschi performs this elimination openly, with the result:

* The reason given for the deduction $U_{rs} = Nx_r x_s$, which occurs in his presentation of Jacobi's proof of the year 1849, is disappointing.

$$0 = \begin{vmatrix} \frac{m}{m-1}u & u_1 & u_2 & \cdots & u_n \\ u_1 & u_{11} & u_{12} & \cdots & u_{1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ u_n & u_{n1} & u_{n2} & \cdots & u_{nn} \end{vmatrix};$$

$$= \frac{m}{m-1}u\Delta + \begin{vmatrix} \cdot & u_1 & u_2 & \cdots & u_n \\ u_1 & u_{11} & u_{12} & \cdots & u_{1n} \\ u_2 & u_{21} & u_{22} & \cdots & u_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ u_n & u_{n1} & u_{n2} & \cdots & u_{nn} \end{vmatrix}.$$

CAYLEY, A. (1856).

[A second memoir on quantics. A third memoir on quantics. *Philos. Transac. R. Soc.* (London), cxlvi. pp. 101–126, pp. 627–647; or *Collected Math. Papers*, ii. pp. 250–275, pp. 310–332.]

In his tables of invariants and covariants Cayley gives the Hessian of the binary quartic, binary quintic, binary sextic, binary octavic and ternary cubic. The results are those numbered 9, 15, 33, 42, 61.*

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinanti. *Memorie . . . Istituto Veneto* . . . . vii. pp. 67–144.]

Bellavitis (§§ 79, 80) denotes "l'*Hessiano* delle funzione ϕ" by

$$| D_xD_x \ D_yD_y \ \ldots\ldots | \phi,$$

calling it also "il determinante delle derivate-seconde." He confines himself to three of the main theorems. Hesse's theorem of 1851 (March) he amplifies, his enunciation being:—*If* u, *a homogeneous integral function of the variables* $x_1, x_2, \ldots, x_n$, *be transformed by means of the substitution*

$$x_k = a_1^{(k)}y_1 + a_2^{(k)}y_2 + \ldots + a_n^k y_n$$

*In the third column of this last in the *Collected Math. Papers*, *cij* and *fkl* should be *cfj* and *gil*, and in the eighth column *ach* should be *ack*.

into v, *and one of the new variables, say* y_1, *be absent from* v, *then* (1) *the Hessian* H *of* u *must vanish identically,* (2) *the cofactor of the elements of any row of* H *must be proportional to the coefficients of* y_1 *in the substitution,* (3) *the product of the first differential-quotients of* u *by the said column of coefficients is equal to* 0. The third of these Bellavitis reaches very easily, because generally we have

$$\frac{\partial v}{\partial y_r} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_r} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_r} + \ldots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_r},$$

and therefore when $r=1$

$$0 = \frac{\partial u}{\partial x_1}a_1^{(1)} + \frac{\partial u}{\partial x_2}a_1^{(2)} + \ldots + \frac{\partial u}{\partial x_n}a_1^{(n)}$$

or

$$0 = u_1a_1^{(1)} + u_2a_1^{(2)} + \ldots + u_na_1^{n}. \qquad (\pi)$$

As regards the second he notes that on account of the vanishing of H we have in the first place

$$U_{11} : U_{12} : \ldots : U_{1n} = U_{s1} : U_{s2} : \ldots : U_{sn},$$

and in the second place * the set of equations

$$\left.\begin{array}{l} u_1U_{11} + u_2U_{12} + \ldots + u_nU_{1n} = 0 \\ u_1U_{21} + u_2U_{22} + \ldots + u_nU_{2n} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\},$$

from which there is the evident deduction that the said set reduces to a single equation: the identity of this equation with (π) is then assumed.

Hesse's converse theorem he treats with a wise caution, deducing as before from the vanishing of H the existence of a single equation of the form

$$\alpha\frac{\partial u}{\partial x_1} + \beta\frac{\partial u}{\partial x_2} + \ldots\ldots = 0,$$

but then adding, "ma rimane da dimostrare che le $\alpha, \beta, \ldots$ sieno quantità costanti."

Lastly, he notes that if there be *two* such equations with constant coefficients, the function is transformable into one with two fewer variables, and all the primary minors of H vanish.

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* See (α) in Jacobi's proof of 1849.

BALTZER, R. (1857).

[THEORIE UND ANWENDUNGEN DER DETERMINANTEN, mit . . . . . vi+129 pp. Leipzig, 1857.]

In Hesse's converse theorem of 1851 (March) Baltzer (§ 13, 3) wisely substitutes for $\Delta=0$ the condition

$$c_1u_1 + c_2u_2 + \ldots + c_nu_n = 0$$

(which by a property of Jacobians implies $\Delta=0$), his proof being that the substitution

$$x_k = b_{k1}y_1 + b_{k2}y_2 + \ldots + b_{k,n-1}y_{n-1} + c_ky_n$$

will then give $\partial u/\partial y_n = 0$, for

$$\begin{aligned}\frac{\partial u}{\partial y_n} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_n} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_n} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial y_n}\\ &= u_1c_1 + u_2c_2 + \ldots + u_nc_n.\end{aligned}$$

In the second place, from the same $n+1$ equations, namely

$$\left.\begin{array}{l} -(m-1)\dfrac{mu}{m-1} + u_1x_1 + \ldots + u_nx_n = 0\\ \quad -(m-1)u_1 + u_{11}x_1 + \ldots + u_{n1}x_n = 0\\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \quad -(m-1)u_n + u_{1n}x_1 + \ldots + u_{nn}x_n = 0 \end{array}\right\},$$

he obtains (§ 14, 4)

$$\left.\begin{aligned} -(m-1) : x_1 : x_2 : \ldots : x_n &= \Delta : V_1 : V_2 : \ldots : V_n\\ &= V_1 : V_{11} : V_{21} : \ldots : V_{n1}\\ &= \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\ &= V_n : V_{1n} : V_{2n} : \ldots : V_{nn} \end{aligned}\right\},$$

where the V's are the cofactors of the corresponding u's in the resultant of the set of equations. The first and $(r+1)^{\text{th}}$ lines imply respectively

$$-(m-1) : x_r = \Delta : V_r,$$

and

$$-(m-1) : x_s = V_r : V_{sr},$$

the former of which gives

$$V_r = -\frac{x_r}{m-1}\Delta, \tag{a}$$

and the two together

$$V_{sr} = \frac{x_s x_r}{(m-1)^2} \Delta . \qquad (\beta)$$

The result (α) is essentially the same as the first result reached in Jacobi's proof of 1849 (December)—a proof which Baltzer restates (§ 14, 7, 8) without noting the fact; and (β) is essentially the same as the second of the two results given in Hesse's paper of 1847 (August), it being noted, however, that the case of this where $r=s$ had been established by Hesse in 1844 (see *Crelle's Journal*, xxviii. p. 103, lines 1 and 2).

HESSE, O. (1858).

[Zur Theorie der ganzen homogenen Functionen. *Crelle's Journ.*, lvi. pp. 263–269; or *Werke*, pp. 481–488.]

The first part of Hesse's attempted proof of his converse theorem of 1851 (March) was to show that the vanishing of "the determinant of u" led to the establishment of a linear relation connecting the first differential coefficients of u. In this there was an oversight, which Brioschi repeated, but which Bellavitis and Baltzer, from the course followed by them, must have been conscious of. Accordingly, Hesse now returns to the subject, the one object of his six-page paper being "diesen Lehrsatz strenger zu begründen." He first clears the ground a little by setting aside the case where one, and therefore all, of the primary minors of Δ vanish, merely stating that a linear substitution is then possible which will transform u into a function with *two* variables less than before. He then sets himself to supply the want which Bellavitis had drawn attention to, the result being a lengthy (pp. 265–268) and still unconvincing argument.

CHAPTER XIV.

CIRCULANTS, UP TO 1860.

So far as mathematical writers have as yet noted, a set of equations of the type

$$\left.\begin{array}{r} a_1x_1 + a_2x_2 + \ldots + a_nx_n \quad = u_1 \\ a_nx_1 + a_1x_2 + \ldots + a_{n-1}x_n = u_2 \\ a_{n-1}x_1 + a_nx_2 + \ldots + a_{n-2}x_n = u_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_2x_1 + a_3x_2 + \ldots + a_1x_n \quad = u_n \end{array}\right\}$$

had not made its appearance in mathematical work prior to the year 1846: and it is almost absolutely certain that before that year the *determinant* of such a set had never been considered. It is not at all unlikely, however, that the expression

$$a^3+b^3+c^3-3abc$$

which is the case of the determinant for n equal to 3 had more than once turned up in other connections, and that its divisibility by $a+b+c$ had been noted: but of this, too, there is no record.

CATALAN, E. (1846).

[Recherches sur les déterminants. *Bull. de l'Acad. roy. . . . . . de Belgique,* xiii. pp. 534–555.]

As has been already explained, about half of Catalan's paper is occupied with an elementary exposition of known properties of determinants and with the establishment of a fresh theorem of his own, which in his notation might have been written in the form

$$\text{dét.}\,(A_1+A_2+\ldots+A_n,\ A_1-A_2,\ A_2-A_3,\ \ldots,\ A_{n-1}-A_n)$$
$$=(-1)^{n-1}\cdot n\cdot\text{dét.}\,(A_1,\ A_2,\ \ldots,\ A_n),$$

and which is to the effect that *If from a determinant* Δ *of the* n^{th} *order we form another* Δ', *such that the first row of* Δ' *is the sum of all the rows of* Δ, *and every other row of* Δ' *is got by subtracting the corresponding row of* Δ *from the row preceding it in* Δ, *then*

$$\Delta'=(-1)^{n-1}n\Delta.*$$

Strange to say, almost all the examples given in illustration of this theorem (of § 13) are of the special form distinguished at a later date by the name "circulant," and consequently fall now to be considered. He says (§ 17):—

"Afin de sortir de ces généralités, considérons les équations

$$\left.\begin{array}{r} -x_1+x_2+x_3+\ \ldots\ +x_n = u_1 \\ x_1-x_2+x_3+\ \ldots\ +x_n = u_2 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_1+x_2+x_3+\ \ldots\ -x_n = u_n \end{array}\right\}.$$

Pour obtenir le déterminant Δ, je remplace d'abord les équations données par les suivantes :

$$\left.\begin{array}{r} (n-2)x_1+(n-2)x_2+\ \ldots\ +(n-2)x_n = u_1+u_2+\ \ldots\ +u_n \\ -2x_1+2x_2 = u_1-u_2 \\ -2x_2+2x_3 = u_2-u_3 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ -2x_{n-1}+2x_n = u_{n-1}-u_n \end{array}\right\}.$$

D'après ce qui précède, le déterminant Δ' du nouveau système sera $(-1)^{n-1}n\Delta$. Mais, d'un autre côté, en comparant Δ' au déterminant Δ'' du système

$$\left.\begin{array}{r} x_1+x_2+\ \ldots\ +x_3 = \ldots \\ -x_1\ \ +x_2 = \ldots \\ -x_2\ \ +x_3 = \ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ -x_{n-1}+x_n = \ldots \end{array}\right\}$$

* This result is reached in a way different from Catalan's by performing on Δ' the operation

$$\text{row}_1 + (n-1)\,\text{row}_2 + (n-2)\,\text{row}_3 + \ldots + \text{row}_n,$$

separating out the factor n, and then showing that the resulting determinant is $(-1)^{n-1}\Delta$.

on a $\Delta' = (n-2)\,2^{n-1}\Delta''$. Enfin, d'après le n° 13, et en observant que les quantités $A_1 - A_2$, $A_2 - A_3$, . . . ont ici changé de signe

$$\Delta'' = n.$$

On déduit, de ces diverses formules

$$\Delta = (n-2)(-2)^{n-1}."$$

This result, which at a later date would have been written

$$C(-1, 1, 1, \ldots, 1) = (n-2)(-2)^{n-1},$$

and which, we may point out in passing, could also be reached by the operations

$$\begin{array}{l} \text{row}_1 + \text{row}_2 + \ldots + \text{row}_n, \\ \text{removal of factor } n-2, \\ \text{row}_n - \text{row}_{n-1},\ \text{row}_{n-1} - \text{row}_{n-2},\ \ldots . \end{array}$$

is then attempted to be generalised (§ 18) by withdrawing the restriction as to the number of negative units in a row. The reasoning, however, seems to have been incautiously conducted, the extension arrived at being

$$C(-1, -1, \ldots, 1, 1)_{p,n-p} = (n-2p)(-2)^{n-1},$$

where the number of consecutive negative units in the first row is p, and the number of positive units $n-p$.

Catalan then passes (§ 19) to the consideration of the similar circulant whose first row consists of p consecutive positive units followed by $n-p$ zeros, separating the investigation into two parts, (1) the case where p and n have a common factor other than unity, (2) where they are mutually prime. In the former case he shows that the equations which have the circulant in question for determinant are "indéterminées ou incompatibles"; in the latter case he shows that the equations are determinate. He thereupon goes on to supplement the information in the second case by proving that the circulant is equal to p: he omits, however, any similar proof that in the first case the circulant is zero.*

* Catalan proposed a question on this subject in *Nouv. Annales de Math.* xv. (1856), p. 257, and returned again to it in *Nouv. Corresp. Math.* iv. (1868), p. 78.

Lastly, he attacks the general circulant, or, as he calls it, "le déterminant du système

$$\begin{matrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_3 & a_4 & \dots & a_1 \\ a_3 & a_4 & a_5 & \dots & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_1 & a_2 & \dots & a_{n-1}. \end{matrix}\text{"}$$

The procedure, however, is rather perverse, the theorem of § 13 being forced into service. This gives

$$\Delta = (-1)^{n-1}\frac{1}{n}\Delta',$$

where Δ' is the determinant of the system

$$\begin{matrix} s & a_1-a_2 & a_2-a_3 & \dots\dots & a_{n-1}-a_n \\ s & a_2-a_3 & a_3-a_4 & \dots\dots & a_n\ -a_1 \\ s & a_3-a_4 & a_4-a_5 & \dots\dots & a_1\ -a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s & a_n-a_1 & a_1-a_2 & \dots\dots & a_{n-2}-a_{n-1}, \end{matrix}$$

after which Δ'/s is partitioned into determinants with monomial elements, and certain more or less evident reductions made. The result is "Le déterminant du système proposé s'obtiendra en multipliant $a_1 + a_2 + \dots + a_n$ (i.e. s) par le déterminant du système

$$\begin{matrix} a_1-a_2 & a_2-a_3 & \dots\dots & a_{n-1}-a_n \\ a_2-a_3 & a_3-a_4 & \dots\dots & a_n\ -a_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1}-a_n & a_n-a_1 & \dots\dots & a_{n-3}-a_{n-2}, \end{matrix}\text{"}$$

a theorem which afterwards came to be written in the form

$$\begin{aligned} C(a_1, a_2, \dots, a_n) = {} & (a_1+a_2+\dots+a_n) \\ & \cdot P(a_1-a_2, \dots, a_{n-1}-a_n, a_n-a_1, \dots, a_{n-3}-a_{n-2}), \end{aligned}$$

the symbol $P(x, y, z, w, v)$ being used to stand for the "persymmetric" determinant

$$\begin{vmatrix} x & y & z \\ y & z & w \\ z & w & v \end{vmatrix}.$$

BERTRAND, J. (1850).

[TRAITÉ ELÉMENTAIRE D'ALGÈBRE, avec un grand nombre d'exercices: par Joseph Bertrand. 407 pp. Paris.]

On page 25 the student is asked to prove that if

$$\begin{aligned} A &= bc' + cb' + aa', \\ B &= ab' + ba' + cc', \\ C &= ac' + ca' + bb', \end{aligned}$$

then

$$\begin{aligned} A+B+C &= (a+b+c)(a'+b'+c'), \\ A^2+B^2+C^2 - AB-AC-BC &= (a^2+b^2+c^2-ab-ac-bc) \\ &\quad \cdot(a'^2+b'^2+c'^2-a'b'-a'c'-b'c'), \\ A^3+B^3+C^3 - 3ABC &= (a^3+b^3+c^3-3abc) \\ &\quad \cdot(a'^3+b'^3+c'^3-3a'b'c'). \end{aligned}$$

These results may possibly have been got by the multiplication of two circulants of the third order, but as against this it has to be noted that determinants are not referred to in the book.

SPOTTISWOODE, W. (1853).

[Elementary theorems relating to determinants. Rewritten and much enlarged by the author. *Crelle's Journ.*, li. pp. 209–271, 328–381.]

In the section (§ xi.) which did not appear in the first edition, and which bears the title "Miscellaneous instances of determinants," the following is given (p. 375), being the fourth of the said instances:—

"Let $1, i_1, i_2, \ldots, i_n$ be the $n+1$ roots of the equation

$$x^{n+1} - 1 = 0,$$

then, whatever be the values of $A, A_1, A_2, \ldots, A_n$

$$\begin{vmatrix} A & A_1 & \ldots & A_n \\ A_1 & A_2 & \ldots & A_1 \\ \cdot & \cdot & \cdot & \cdot \\ A_n & A_1 & \ldots & A_{n-1} \end{vmatrix} = (A+A_1+\ldots+A_n)(A+i_1A_1+\ldots+i_1^nA_n) \\ \ldots\ldots\ldots\ldots (A+i_nA_1+\ldots+i_n^nA_n)."$$

No word of proof is added: probably the result was reached by Sylvester's "dialytic" method of elimination. But, however this may be, it should be noted that resolvability into linear factors soon came to be looked on as the fundamental property of the circulant.

It has to be noted that Spottiswoode makes a slip in omitting the sign-factor $(-1)^{\frac{1}{2}n(n-1)}$ from the right-hand member; and that he writes his determinant in such a way as to have it persymmetric with respect to the principal diagonal, whereas Catalan wrote his so as to have it persymmetric with respect to the secondary diagonal. Putting C′ for the functional symbol in the former case we have

$$C(a_1, a_2, \ldots, a_n) = (-1)^{\frac{1}{2}(n-1)(n-2)} \cdot C'(a_1, a_2, \ldots, a_n).$$

If therefore Spottiswoode had followed Catalan's mode of writing, his result would have been strictly accurate.

SYLVESTER, J. J. (1855, April).

[On the change of systems of independent variables. *Quart. Journ. of Math.*, i. pp. 42–56; or *Collected Math. Papers*, ii. pp. 65–85.]

Having reached in the course of his investigation (p. 55) a determinant of the form

$$\begin{vmatrix} a_1+a_2+a_3 & -b_3 & -c_2 \\ -a_2 & b_1+b_2+b_3 & -c_3 \\ -a_3 & -b_2 & c_1+c_2+c_3 \end{vmatrix},$$

the final expansion of which, he says, contains only positive terms with the coefficient unity, Sylvester naturally notes that the number of such terms must be

$$\begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}.$$

He is thus led to the consideration of the n-line circulant

$$\begin{vmatrix} a & -1 & -1 & \dots & -1 \\ -1 & a & -1 & \dots & -1 \\ -1 & -1 & a & \dots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \dots & a \end{vmatrix},$$

to which he assigns the value

$$(a-n+1)(a+1)^{n-1}.$$

CREMONA, L. (1856).

[Intorno ad un teorema di Abel. *Annali di Sci. mat. e fis.*, vii. pp. 99–105.]

To prove the theorem of Abel referred to in the title, Cremona starts by establishing three lemmas, the first of which is Spottiswoode's theorem regarding circulants. Taking any n quantities

$$a_0, a_1, a_2, \dots, a_{n-1}$$

and denoting

$$a_0 + a_1 a_r^1 + a_2 a_r^2 + \dots + a_{n-1} a_r^{n-1} \quad \text{by} \quad \theta_r$$

where a_r stands for a^r and a for a primitive root of the equation $x^n - 1 = 0$, he multiplies the determinant

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_1 & a_2 & a_3 & \dots & a_0 \\ a_2 & a_3 & a_4 & \dots & a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \end{vmatrix}, \quad \text{or D say,}$$

by the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & a_1 & a_2 & \dots & a_{n-1} \\ 1 & a_1^2 & a_2^2 & \dots & a_{n-1}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_1^{n-1} & a_2^{n-1} & \dots & a_{n-1}^{n-1} \end{vmatrix}, \quad \text{or } \Delta \text{ say,}$$

and obtains a product-determinant from whose columns, he says, the factors $\theta_1, \theta_2, \ldots, \theta_n$ may be removed in order, so that there results

$$\mathrm{D}\Delta = \theta_1\theta_2\ldots\theta_n \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \ldots & \alpha_{n-1}^{n-1} \\ 1 & \alpha_{n-2} & \alpha_{n-2}^2 & \ldots & \alpha_{n-2}^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{n-1} \end{vmatrix}$$

$$= \theta_1\theta_2\ldots\theta_n\cdot(-1)^{\frac{1}{2}n(n-1)}\Delta,$$

and $\therefore$ $$\mathrm{D} = (-1)^{\frac{1}{2}n(n-1)}\cdot\theta_1\theta_2\ldots\theta_n.$$

The proof, which is said to be due to Brioschi, is not improved in neatness by introducing the conception of a primitive root, nor by writing the root 1 in a different form from the other roots.

The second lemma concerns the differential-quotient of D with respect to any variable of which the α's are functions. Denoting this differential-quotient by D′, and by D_r the determinant got from D by substituting for each element in the r^{th} column the differential-quotient of that element, Cremona of course has at once

$$\mathrm{D}' = \mathrm{D}_1 + \mathrm{D}_2 + \ldots + \mathrm{D}_n.$$

As, however, D_1 here can be shown by translation of a number of rows and the same number of columns to be equal to any one of the D's following it, there results

$$\mathrm{D}' = n\mathrm{D}_1 = n\mathrm{D}_2 = \ldots.$$

The third lemma is to the effect that the quotient of the determinant

$$\begin{vmatrix} m_0 & q_0 & q_1d & \ldots & q_{n-2}d^{n-2} \\ m_1d & q_1d & q_2d^2 & \ldots & q_{n-1}d^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{n-1}d^{n-1} & q_{n-1}d^{n-1} & q_0 & \ldots & q_{n-3}d^{n-3} \end{vmatrix}$$

by d is a rational function of d^n. By multiplying the 2nd, 3rd, 4th, ... columns by d^n, d^{n-1}, d^{n-2}, ... respectively, and then dividing the corresponding rows by d, d^2, d^3, ... respectively, there is obtained

$$\begin{vmatrix} m_0 & q_0 d^n & q_1 d^n & \dots & q_{n-2}d^n \\ m_1 & q_1 d^n & q_2 d^n & \dots & q_{n-1}d^n \\ m_2 & q_2 d^n & q_3 d^n & \dots & q_0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{n-1} & q_{n-1}d^n & q_0 & \dots & q_{n-3} \end{vmatrix},$$

where no power of d occurs except the n^{th}. But, if the original determinant be H, the latter is

$$\frac{\mathrm{H}\cdot d^n d^{n-1} d^{n-2} \dots d^2}{d d^2 d^3 \dots d^{n-1}}, \quad i.e. \quad \frac{\mathrm{H}}{d}\cdot d^n;$$

consequently H/d is of the form asserted.

In connection with this last lemma it is curious to find no note taken of the closely related and more attractive fact that

$$\mathrm{C}(a_1, a_2 d, a_3 d^2, \dots, a_n d^{n-1})$$

is a rational function of d^n.

BELLAVITIS, G. (1857).

[Sposizione elementare della teoria dei determinanti. *Memorie ... Istituto Veneto* .... viii. pp. 67–143.]

Circulants are practically unconsidered by Bellavitis in his exposition, all that appears (§ 85) being two of Laplace's expansions for $\mathrm{C}(a, b, c, d)$ obtained by means of Cauchy's "chiavi algebriche," namely,

$$(a^2-bd)^2-(b^2-ac)^2+(c^2-bd)^2-(d^2-ac)^2-2(ab-cd)(ad-bc)$$

and $$(a^2-c^2)^2-(b^2-d^2)^2-4(ab-cd)(ad-bc).$$

PAINVIN, L. (1858); ROBERTS, M. (1859).

[Questions 432, 465. *Nouv. Annales de Math.*, xvii. p. 185; xviii. p. 117; xix. pp. 151–153, 170–174.]

Here it is special circulants that are set for consideration, namely, by Painvin the circulant whose elements are the first n

integers, and by Michael Roberts the circulant whose elements are a, $a+d$, $a+2d$, ... , the result in regard to the former circulant being

$$C'(1, 2, \ldots, n) = (-1)^{\frac{1}{2}n(n-1)} \cdot \tfrac{1}{2}n^{n-1}(n+1),$$

and in regard to the latter

$$C'(a, a+d, \ldots, a+\overline{n-1}\cdot d) = (-1)^{\frac{1}{2}n(n-1)} \cdot (nd)^{n-1} \cdot \left(a+\frac{n-1}{2}d\right).$$

The first to offer a proof was Cremona, who, after repeating (xix. pp. 151–153) Brioschi's demonstration regarding the resolvability of a circulant, says that in Roberts' case θ_r being

$$\begin{aligned} &\equiv a\frac{1-a_r^n}{1-a_r} + d\left\{a_r\frac{1-a_r^{n-1}}{(1-a_r)^2} - \frac{na_r^n}{1-a_r}\right\}, \\ &= \frac{nd}{a_r-1} \quad \text{for } r = 1, 2, \ldots, n-1, \end{aligned}$$

and $\qquad \theta_n = na + \tfrac{1}{2}n(n-1)d,$

and that consequently

$$\theta_1\theta_2 \ldots \theta_n = \frac{(nd)^{n-1}}{(a_1-1)(a_2-1)\ldots(a_{n-1}-1)}\left\{na+\tfrac{1}{2}n(n-1)d\right\};$$

whence the desired result readily follows, because the denominator is equal to

$$(-1)^{n-1}(1-\Sigma a_1+\Sigma a_1a_2-\Sigma a_1a_2a_3+\ldots),$$

where $\Sigma a_1 = -1$, $\Sigma a_1a_2 = 1$, $\Sigma a_1a_2a_3 = -1$, ...

A proof was also given by G. F. Baehr of Groningen (xix. pp. 170–173), who changes

$$C'(a_1, a_2, \ldots, a_n) \quad \text{into} \quad (-1)^{\frac{1}{2}n(n-1)}C(a_n, a_{n-1}, \ldots, a_1),$$

performs on the latter determinant the operations

$\mathrm{col}_1 - \mathrm{col}_2$, $\mathrm{col}_2 - \mathrm{col}_3$, ...
$\mathrm{row}_1 + \mathrm{row}_2 + \ldots + \mathrm{row}_n$,
removal of factors d^n and $(-1)^{n-1}\cdot\tfrac{1}{2}n\left\{2a+(n-1)d\right\}$,

leaving as cofactor a determinant of the $(n-1)^{\text{th}}$ order whose diagonal elements are all $1-n$ and non-diagonal elements all 1. On this new determinant he then performs the operations

$$\text{row}_1 + \text{row}_2 + \;\ldots\; + \text{row}_{n-1},$$
$$\text{row}_2 + \text{row}_1, \quad \text{row}_3 + \text{row}_1, \quad \ldots\ldots$$

and so finds its value to be

$$(-1)^{n-1} \cdot n^{n-2},$$

which gives for the circulant with which he started the value

$$(-1)^{\frac{1}{2}n(n-1)} \cdot (nd)^{n-1} \cdot \left(a + \frac{n-1}{2}d\right).$$

SOUILLART, C. (1858, May).

[Solution de la question 405. *Nouv. Annales de Math.*, xvii. pp. 192–194; xix. pp. 320–321.]

Michael Roberts having in November 1857 set the problem of finding X, Y, Z as functions of x, y, z, x', y', z', so as to have

$$(x^3+y^3+z^3-3xyz)(x'^3+y'^3+z'^3-3x'y'z') = \mathrm{X}^3+\mathrm{Y}^3+\mathrm{Z}^3-3\mathrm{XYZ},$$

—in other words, having resuscitated Bertrand's exercise of the year 1850,—Souillart showed that the solution

$$\left.\begin{aligned} \mathrm{X} &= (x, y, z \between x', y', z') \\ \mathrm{Y} &= (x, y, z \between y', z', x') \\ \mathrm{Z} &= (x, y, z \between z', x', y') \end{aligned}\right\}$$

is comprised in a general theorem, namely, the theorem which at a later date would have been expressed by saying that the product of two circulants is a circulant. In 1860 he returned to the subject in order to point out that a second suitable set of values is

$$\begin{aligned} \mathrm{X} &= (x, y, z \between z', y', x'), \\ \mathrm{Y} &= (x, y, z \between x', z', y'), \\ \mathrm{Z} &= (x, y, z \between y', x', z'). \end{aligned}$$

In illustrating he uses the fourth order, that is to say, where the initial expression is

$$\begin{vmatrix} x & y & z & u \\ y & z & u & x \\ z & u & x & y \\ u & x & y & z \end{vmatrix} \quad \text{or} \quad - \begin{vmatrix} x & u & z & y \\ y & x & u & z \\ z & y & x & u \\ u & z & y & x \end{vmatrix}$$

or

$$-x^4+y^4-z^4+u^4-4y^2xz-4xu^2z+4x^2yu+4yuz^2+2x^2z^2-2u^2y^2.$$

BAEHR, G. F. (1860).

[Solution de la question 432. *Nouv. Annales de Math.*, xix. pp. 170–174.]

After dealing as we have seen with the circulant whose elements are in equidifferent progression, Baehr proceeds to the circulant whose elements are in equirational progression, namely,

$$C'(a, ar, ar^2, \ldots, ar^{n-1}).$$

This he first changes into

$$a^n \cdot C'(1, r, r^2, \ldots, r^{n-1})$$

and then into

$$(-)^{\frac{1}{2}n(n-1)} \cdot a^n \cdot C(r^{n-1}, r^{n-2}, \ldots, r, 1).$$

On the determinant thus reached the operations

$$r\,\text{row}_1 - \text{row}_2, \quad r\,\text{row}_2 - \text{row}_3, \quad \ldots .$$

are performed, with the result that its value is found to be

$$(-1)^{n-1} \cdot (1-r^n)^{n-1},$$

and thence the value of the original circulant to be

$$(-1)^{\frac{1}{2}n(n-1)} \cdot a^n (r^n-1)^{n-1}.$$

It is worth noting that instead of the last set of operations we might substitute with advantage the set

$$\text{row}_n - r\,\text{row}_{n-1}, \quad \text{row}_{n-1} - r\,\text{row}_{n-2}, \quad \ldots . ;$$

also, that Baehr's circulant is a special case of that referred to under Cremona's third lemma.

CHAPTER XV.

CONTINUANTS, UP TO 1870.

THE more or less disguised use of continued fractions has been traced back to the publication of Bombelli's *Algebra* in 1572, eighty-four years, that is to say, before the publication of Wallis' *Arithmetica Infinitorum*, in which Brouncker's discovery was announced and the fractions explicitly expressed.* The study of the numerators and denominators of the convergents viewed as functions of the partial denominators was first seriously undertaken by Euler in his *Specimen Algorithmi Singularis* of the year 1764, in which denoting by

$$(a), \quad \frac{(a,b)}{(b)}, \quad \frac{(a,b,c)}{(b,c)}, \quad \ldots .$$

the convergents to

$$a + \frac{1}{b} + \frac{1}{c} + $$

he established a long series of identities, such as

$$\begin{aligned}(a,b,c,d,\ldots) &= a(b,c,d,\ldots) + (c,d,\ldots)\\ (a,b,c,\ldots l) &= (l,\ldots,c,b,a),\\ (a,b)(b,c) - (b)(a,b,c) &= 1,\\ (a,b,c)(d,e,f) - (a,b,c,d,e,f) &= -(a,b)(e,f),\\ \ldots\ldots&\ldots\ldots\end{aligned}$$

* For the early history see Favaro's *Notizie storiche sulle frazioni continue dal secolo decimoterzo al decimosettimo* published in vol. vii. of Boncompagni's *Bollettino*; and as regards Bombelli see a paper by G. Wertheim in the *Abhandl. zur Gesch. d. Math.*, viii. pp. 147–160.

The study was pursued by Hindenburg and his followers during the last twenty years of the eighteenth century, but not with any great profit; and, although in the first half of the nineteenth century considerable attention was given to the theory of continued fractions as a whole, little advance was made in elucidating the properties of the functions referred to.* Their connection with determinants, after the awakening of interest in the latter about 1841, was sure sooner or later to be detected: there is no evidence, however, of the discovery having been made before the year 1853.

SYLVESTER, J. J. (1853, May 13).

[On a remarkable modification of Sturm's theorem. *Philos. Magazine* (4), v. pp. 446–457; or *Collected Math. Papers*, i. pp. 609–619.]

The mention of Sturm's theorem in the title of a paper renders not improbable the occurrence therein of matter connected with continued fractions. Especially likely is this in the case of a writer like Sylvester when in a characteristic mood; and, assuredly, the present communication is in structure, style, and originality redolent of its author. It must have been written in the white heat of discovery. The main part of it consists of six pages: this is followed by a "Remark" a page and a quarter long; then comes a "Postscript" of three and a half pages; and finally a small-page footnote as long as the "Remark."

It is the postscript which particularly concerns us. It begins thus:—

"Suppose that we have any series of terms, $u_1, u_2, u_3, \ldots, u_n$, where

$$u_1 = A_1, \quad u_2 = A_1A_2 - 1, \quad u_3 = A_1A_2A_3 - A_1 - A_3, \quad \ldots.$$

and in general

$$u_i = A_i u_{i-1} - u_{i-2},$$

then $u_1, u_2, u_3, \ldots, u_n$ will be the successive principal coaxal determi-

* The state of the theory in 1833 can best be gathered from Stern's monograph, published in vol. x. of *Crelle's Journal.*

nants of a symmetrical matrix. Thus suppose $n=5$; if we write down the matrix

$$\begin{matrix} A_1 & 1 & 0 & 0 & 0 \\ 1 & A_2 & 1 & 0 & 0 \\ 0 & 1 & A_3 & 1 & 0 \\ 0 & 0 & 1 & A_4 & 1 \\ 0 & 0 & 0 & 1 & A_5 \end{matrix}$$

(the mode of formation of which is self-apparent), these successive coaxal determinants will be

$$1,\quad A_1,\quad \begin{vmatrix} A_1 & 1 \\ 1 & A_2 \end{vmatrix},\quad \begin{vmatrix} A_1 & 1 & 0 \\ 1 & A_2 & 1 \\ 0 & 1 & A_3 \end{vmatrix},\quad \begin{vmatrix} A_1 & 1 & 0 & 0 \\ 1 & A_2 & 1 & 0 \\ 0 & 1 & A_3 & 1 \\ 0 & 0 & 1 & A_4 \end{vmatrix},\ \text{etc.,}$$

i.e.

$$\begin{aligned} &1,\quad A_1,\quad A_1A_2-1,\quad A_1A_2A_3-A_1-A_3, \\ &A_1A_2A_3A_4-A_1A_2-A_1A_4-A_3A_4+1, \\ &A_1A_2A_3A_4A_5-A_1A_2A_5-A_1A_4A_5-A_3A_4A_5-A_1A_2A_3 \\ &\qquad\qquad\qquad\qquad\qquad\qquad +A_5+A_3+A_1. \end{aligned}$$

It is proper to introduce the unit because it is, in fact, the value of a determinant of zero places, as I have observed elsewhere."

After using this as an aid to prove his proposition regarding Sturm's theorem, he returns to his new determinant in the following words:—

"I may conclude with noticing that the determinative [determinantal?] form of exhibiting the successive convergents to an improper continued fraction affords an instantaneous demonstration of the equation which connects any two consecutive such convergents as

$$\frac{N_{i-1}}{D_{i-1}} \quad \text{and} \quad \frac{N_i}{D_i},$$

namely, $$N_i \cdot D_{i-1} - N_{i-1}D_i = 1.$$

For if we construct the matrix which for greater simplicity I limit to five lines and columns,

$$\boxed{\begin{matrix} A & 1 & 0 & 0 & 0 \\ 1 & B & 1 & 0 & 0 \\ 0 & 1 & C & 1 & 0 \\ 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 1 & E \end{matrix}}$$

and represent umbrally as

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{matrix};$$

and if, by way of example, we take the fourth and fifth convergents, these will be in the umbral notation represented by

$$\frac{\begin{matrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{matrix}}{\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}} \quad \text{and} \quad \frac{\begin{matrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \end{matrix}}{\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{matrix}}$$

respectively. Hence

$$N_5D_4 - N_4D_5$$

$$= \begin{matrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \end{matrix} \times \begin{matrix} a_2 & a_3 & a_4 & a_1 \\ b_2 & b_3 & b_4 & b_1 \end{matrix} - \begin{matrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{matrix} \times \begin{matrix} a_2 & a_3 & a_4 & a_5 & a_1 \\ b_2 & b_3 & b_4 & b_5 & b_1 \end{matrix},$$

$$= \begin{matrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \end{matrix} \times \begin{matrix} a_2 & a_3 & a_4 & a_1 \\ b_2 & b_3 & b_4 & b_1 \end{matrix} - \overline{\begin{matrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \end{matrix}}\ \overline{\begin{matrix} a_2 & a_3 & a_4 & a_1 \\ b_2 & b_3 & b_4 & b_1 \end{matrix}}$$

$$= \begin{matrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_1 \end{matrix} \times \begin{matrix} a_2 & a_3 & a_4 & a_1 \\ b_2 & b_3 & b_4 & b_5 \end{matrix},$$

$$= \begin{matrix} a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 \end{matrix} \times \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 & b_5 \end{matrix},$$

$$= \begin{matrix} 1 & B & 1 & 0 \\ 0 & 1 & C & 1 \\ 0 & 0 & 1 & D \\ 0 & 0 & 0 & 1 \end{matrix} \times \begin{matrix} 1 & 0 & 0 & 0 \\ B & 1 & 0 & 0 \\ 1 & C & 1 & 0 \\ 0 & 1 & D & 1 \end{matrix},$$

$$= 1 \times 1 = 1,$$

as was to be proved. And the demonstration is evidently general in its nature."

In regard to this there has to be noted, first the use of

$$\begin{matrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{matrix}$$

when it would have been equally effective to use

$$\begin{matrix} 2 & 3 & 4 \\ 2 & 3 & 4 \end{matrix};$$

and, second, the use of a theorem for expressing the product of a five-line determinant and one of its secondary minors as an aggregate of products of pairs of four-line determinants.

Following on this comes the assertion that

"We may treat a proper continued fraction [*i.e.* with positive unit numerators] in precisely the same manner, substituting throughout $\sqrt{-1}$ in place of 1 in the generating matrix, and we shall thus, by the same process as has been applied to improper continued fractions, obtain

$$\begin{aligned} N_{i+1}D_i - N_iD_{i+1} &= (\sqrt{-1})^i \times (\sqrt{-1})^i \\ &= (-1)^i. \text{"} \end{aligned}$$

This would seem to imply that as yet Sylvester had not observed that an alternative mode of representation was obtainable by merely changing the sign of the units on one side of the diagonal.

The footnote contains two additional observations, the first being to the effect that the new mode of representation

"gives an immediate and visible proof of the simple and elegant rule for forming any such numerators or denominators by means of the principal terms [term?] in each; the rule, I mean, according to which the i^{th} denominator may be formed from

$$q_1q_2q_3q_4 \cdots q_i$$

($q_1, q_2, \ldots, q_i$ being the successive quotients) and the i^{th} numerator from

$$q_2q_3q_4 \cdots q_i$$

by leaving out from the above products respectively any pair or any number of pairs of consecutive quotients as $q_\rho q_{\rho+1}$. For instance, from $q_1q_2q_3q_4q_5$ by leaving out q_1q_2, q_2q_3, q_3q_4 and q_4q_5 we obtain

$$q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3 :$$

and by leaving out $q_1q_2 \cdot q_3q_4$, $q_1q_2 \cdot q_4q_5$, $q_2q_3 \cdot q_4q_5$ we obtain

$$q_5 + q_3 + q_1 ;$$

so that the total denominator becomes

$$q_1q_2q_3q_4q_5 + q_3q_4q_5 + q_1q_4q_5 + q_1q_2q_5 + q_1q_2q_3 + q_5 + q_3 + q_1 ;$$

and in like manner the numerator of the same convergent is

$$q_2q_3q_4q_5 \left\{ 1 + \frac{1}{q_2q_3} + \frac{1}{q_3q_4} + \frac{1}{q_4q_5} + \frac{1}{q_2q_3q_4q_5} \right\},$$

i.e.

$$q_2q_3q_4q_5 + q_4q_5 + q_2q_5 + q_2q_3 + 1.\text{"}$$

The "rule" here spoken of is that enunciated for the more general case of

$$a_1 + \frac{b_1}{a_2 +} \frac{b_2}{a_3 +} \cdots\cdots$$

in Stern's *Theorie der Kettenbrüche*, the fourth section of which is given up to the consideration of such rules (*Crelle's Journ.*, x. pp. 4–7).

The other observation is to the effect that

"every progression of terms constructed in conformity with the equation

$$u_n = a_n u_{n-1} - b_n u_{n-2} + c_n u_{n-3} - \dots .$$

may be represented as an ascending series of principal coaxal determinants to a common matrix. Thus if each term in such progression is to be made a linear function of the three preceding terms, it will be representable by means of the matrix

$$\begin{matrix} A & B' & C'' & 0 & 0 \\ 1 & A' & B'' & C''' & 0 \\ 0 & 1 & A'' & B''' & C'''' \\ 0 & 0 & 1 & A''' & B'''' \\ 0 & 0 & 0 & 1 & A'''' \end{matrix}$$

indefinitely continued, which gives the terms

$$1, \quad A, \quad AA' - B', \quad AA'A'' - B'A'' - AB'' + C'', \quad \dots ."$$

This exhausts the paper so far as determinants are concerned: the results announced in it, one can readily own, were such as fairly to entitle the enthusiastic author to express his belief that "the introduction of the method of determinants into the algorithm of continued fractions cannot fail to have an important bearing upon the future treatment and development of the theory of numbers."

SPOTTISWOODE, W. (1853, August).*

[Elementary theorems relating to determinants. Second edition, rewritten and much enlarged by the author. *Crelle's Journ.*, li. (1856), pp. 209–271, 328–381.]

Save the utilisation of the fact that the denominator of any convergent of the continued fraction

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} + \dots$$

* This is the author's date at the end of the paper (p. 381). The first two parts of the volume, however, are dated 1855, and the remaining two 1856.

is the differential-quotient of the numerator, Spottiswoode did nothing but report the fundamental result reached by Sylvester. The full passage (p. 374) is as follows:—

"The improper continued fraction

$$\frac{1}{A} - \frac{1}{B} - \frac{1}{C} - \cdots \qquad = \frac{d}{dA}\log_e \nabla$$

where

$$\nabla = \begin{vmatrix} A & 1 & 0 & \ldots & 0 & 0 \\ 1 & B & 1 & \ldots & 0 & 0 \\ 0 & 1 & C & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & M & 1 \\ 0 & 0 & 0 & \ldots & 1 & N \end{vmatrix},$$

in which any number of rows may be taken at pleasure, and the formula will give the corresponding convergent fraction.

The same holds good for the continued fraction

$$\frac{1}{A} + \frac{1}{B} + \cdots$$

if we write

$$\nabla = \begin{vmatrix} A & 1 & 0 & \ldots \\ -1 & B & 1 & \ldots \\ 0 & -1 & C & \ldots \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot \end{vmatrix}."$$

SYLVESTER, J. J. (1853, Sept.).

[On a fundamental rule in the algorithm of continued fractions. *Philos. Magazine* (4), vi. pp. 297–299; or *Collected Math. Papers*, i. pp. 641–644.]

Without any reference to his previous paper on the subject Sylvester here announces that if

$$(a_1, a_2, \ldots, a_i)$$

be the denominator of the i^{th} convergent to

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} +$$

then

$$(a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+n}) = (a_1, \ldots, a_m)(a_{m+1}, \ldots, a_{m+n}) \\ + (a_1, \ldots, a_{m-1})(a_{m+2}, \ldots, a_{m+n}),$$

—a possibly new result which he considers "the fundamental theorem in the theory of continued fractions." This, he says, is an immediate consequence of the fact that $(a_1, \ldots, a_{m+n})$ can be expressed as a determinant, all that is further necessary being the application of the "well-known simple rule for the decomposition of determinants. Thus, *e.g.*, the determinant

$$\begin{array}{cccccc} a & 1 & & & & \\ -1 & b & 1 & & & \\ & -1 & c & 1 & & \\ & & -1 & d & 1 & \\ & & & -1 & e & 1 \\ & & & & -1 & f \end{array}$$

is obviously decomposable into

$$\begin{array}{ccc} a & 1 & \\ -1 & b & 1 \\ & -1 & c \end{array} \times \begin{array}{ccc} d & 1 & \\ -1 & e & 1 \\ & -1 & f \end{array} + \begin{array}{cc} a & 1 \\ -1 & b \end{array} \times \begin{array}{cc} e & 1 \\ -1 & f, \end{array}$$

or into

$$\begin{array}{cc} a & 1 \\ -1 & b \end{array} \times \begin{array}{cccc} c & 1 & & \\ -1 & d & 1 & \\ & -1 & e & 1 \\ & & -1 & f \end{array} + a \times \begin{array}{ccc} d & 1 & \\ -1 & e & 1 \\ & -1 & f, \end{array}$$

or into

$$a \times \begin{array}{ccccc} b & 1 & & & \\ -1 & c & 1 & & \\ & -1 & d & 1 & \\ & & -1 & e & 1 \\ & & & -1 & f \end{array} + \begin{array}{cccc} c & 1 & & \\ -1 & d & 1 & \\ & -1 & e & 1 \\ & & -1 & f. \end{array} \quad ''$$

Following this is what is called "Corollary I.," namely,

$$(a_1, a_2, \ldots, a_m)\cdot(a_2, a_3, \ldots, a_{m+i}) - (a_2, a_3, \ldots, a_m)\cdot(a_1, a_2, \ldots, a_{m+i}) \\ = (-)^m(a_{m+i}a_{m+i-1} \ldots \text{to } i-1 \text{ factors}),$$

its connection with the expression for the difference of two convergents being illustrated by the instances $i=1, 2, 3, 4, \ldots$

The next "corollary," namely,

$$\begin{aligned}&(a_1, \ldots, a_\rho, a_{\rho+1}, \ldots, a_{\rho+f})(a_1, \ldots, a_\rho, a_{\rho+1}, \ldots, a_{\rho+k})\\ &\quad - (a_1, \ldots, a_\rho, a_{\rho+1}, \ldots, a_{\rho+g})(a_1, \ldots, a_\rho, a_{\rho+1}, \ldots, a_{\rho+h})\\ &= (-)^\rho\Big\{(a_{\rho+1}, \ldots, a_{\rho+f})(a_{\rho+1}, \ldots, a_{\rho+k}) - (a_{\rho+1}, \ldots, a_{\rho+g})(a_{\rho+1}, \ldots, a_{\rho+h})\Big\}\end{aligned}$$

is clearly incorrect, it being impossible for the value of the left-hand side to be independent of the elements $a_1, a_2, \ldots, a_\rho$. Further, as the author gives no accompanying word of comment, the difficulty of suggesting the true theorem is increased. A "sub-corollary" is appended dealing with the case where all the a's are equal, and leading up, not without some misprints or inaccuracies, to a theorem of Euler's quoted from the *Nouvelles Annales de Math.*, v. (Sept. 1851), pp. 357–358, to the effect that if $T_{n+1} = aT_n - bT_{n-1}$ be the generating equation of a recurrent series, then

$$\frac{T_{n+1}^2 - aT_nT_{n+1} + bT_n^2}{b^n}$$

is a constant with respect to n. Of course the more natural form of this expression is

$$\frac{T_{n+1}^2 - T_nT_{n+2}}{b^n},$$

the numerator of which being

$$\begin{vmatrix} T_{n+1} & T_{n+2} \\ T_n & T_{n+1} \end{vmatrix}$$

is successively transformable by means of the recursion-formula into

$$b\begin{vmatrix} T_n & T_{n+1} \\ T_{n-1} & T_n \end{vmatrix}, \quad b^2\begin{vmatrix} T_{n-1} & T_n \\ T_{n-2} & T_{n-1} \end{vmatrix}, \quad b^3\begin{vmatrix} T_{n-2} & T_{n-1} \\ T_{n-3} & T_{n-2} \end{vmatrix}, \quad \ldots,$$

so that the constant in question is

$$\begin{vmatrix} T_1 & T_2 \\ T_0 & T_1 \end{vmatrix}.$$

This, however, Sylvester does not show.*

* An interesting extension of this is given by Brioschi in the *Nouv. Annales de Math.*, xiv. (Jan. 1854), p. 20.

Finally, and to more purpose, it is noted that if we pass from $(a_1, a_2, \ldots, a_i)$ to the readily-suggested extension

$$\begin{matrix} m_1 & l_1 & & & & & \\ n_1 & m_2 & l_2 & & & & \\ & n_2 & m_3 & l_3 & & & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & & n_{i-1} & m_i & l_i & \\ & & & & n_i & m_{i+1}, & \end{matrix}$$

the corresponding fundamental theorem is

$$\begin{aligned}\begin{pmatrix} l_1 \ldots\ldots l_{i+j} \\ m_1, m_2, \ldots, m_{i+j+1} \\ n_1 \ldots\ldots n_{i+j} \end{pmatrix} &= \begin{pmatrix} l_1 \ldots\ldots l_{i-1} \\ m_1, m_2, \ldots, m_i \\ n_1 \ldots\ldots n_{i-1} \end{pmatrix}\begin{pmatrix} l_{i+1} \ldots\ldots l_{i+j} \\ m_{i+1}, m_{i+2}, \ldots, m_{i+j+1} \\ n_{i+1} \ldots\ldots n_{i+j} \end{pmatrix} \\ &\quad - l_i n_i \begin{pmatrix} l_1 \ldots\ldots l_{i-2} \\ m_1, m_2, \ldots, m_{i-1} \\ n_1 \ldots\ldots n_{i-2} \end{pmatrix}\begin{pmatrix} l_{i+2} \ldots\ldots l_{i+j} \\ m_{i+2}, m_{i+3}, \ldots, m_{i+j+1} \\ n_{i+2} \ldots\ldots n_{i+j} \end{pmatrix}.\end{aligned}$$

SYLVESTER, J. J. (1853, Oct., Nov.).

[On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure. *Philos. Transac. R. Soc.* (London), cxliii. pp. 407–548; or *Collected Math. Papers*, i. pp. 429–586.]

Although this lengthy memoir in its original form bears date "16th June, 1853," it is the equally lengthy "supplements" added later while passing through the press that claim attention in the present connection. In the first of these (§ i., p. 474) the denominator of the fraction

$$\cfrac{1}{q_1 - \cfrac{1}{q_2 - \ddots - \cfrac{1}{q_n}}}$$

is denoted by $[q_1, q_2, \ldots, q_n]$, and termed a "cumulant," and throughout the later portion of the paper this name constantly recurs. It is not, however, until we come to the second "supplement" that anything apparently new in substance is met with. There, in § a (p. 497), the following lemma occurs:—

"The roots of the cumulant $[q_1, q_2, \ldots, q_i]$, in which each element is a linear function of x, and wherein the coefficient of x for each element has the like sign, are all real: and between every two of such roots is contained a root of the cumulant $[q_1, q_2, \ldots, q_{i-1}]$ and *ex converso* a root of the cumulant $[q_2, q_3, \ldots, q_i]$: and (as an evident corollary) for all values of ζ and ζ' intermediate between 1 and i the greatest root of $[q_1, q_2, \ldots, q_i]$ will be greater, and the least root of the same will be less than the greatest and least roots respectively of

$$[q_\rho, q_{\rho+1}, \ldots, q_{\rho'-1}, q_{\rho'}]."$$

Even this, however, may be placed under the well-known theorem regarding the roots of the equation

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \ldots \\ a_{12} & a_{22}-x & a_{23} & \ldots \\ a_{13} & a_{23} & a_{33}-x & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

which had been enunciated by Cauchy in 1829.

The next noteworthy result occupies § i. (p. 502). As a preparation for it the theorem

$$\begin{aligned}[a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n] = {} & [a_1, a_2, \ldots, a_m][b_1, b_2, \ldots, b_n] \\ & - [a_1, a_2, \ldots, a_{m-1}][b_2, b_3, \ldots, b_n]\end{aligned}$$

may be recalled, the group of elements on the left being now viewed as consisting of two sub-groups. This theorem Sylvester writes in the form

$$[\Omega_1\Omega_2] = [\Omega_1][\Omega_2] - [\Omega_1'][{}'\Omega_2],$$

and he succeeds in including it in a general theorem, not explicitly formulated, in which the number of groups is i, the next two cases being

$$\begin{aligned}[\Omega_1\Omega_2\Omega_3] = {} & [\Omega_1][\Omega_2][\Omega_3] \\ & - [\Omega_1'][{}'\Omega_2][\Omega_3] - [\Omega_1][\Omega_2'][{}'\Omega_3] + [\Omega_1'][{}'\Omega'][{}'\Omega_3],\end{aligned}$$

and

$$\begin{aligned}[\Omega_1\Omega_2\Omega_3\Omega_4] = {} & [\Omega_1][\Omega_2][\Omega_3][\Omega_4] \\ & - [\Omega_1'][{}'\Omega_2][\Omega_3][\Omega_4] - [\Omega_1][\Omega_2'][{}'\Omega_3][\Omega_4] - [\Omega_1][\Omega_2][\Omega_3'][{}'\Omega_4] \\ & + [\Omega_1'][{}'\Omega_2'][\Omega_3][\Omega_4] + [\Omega_1'][{}'\Omega_2][\Omega_3'][{}'\Omega_4] + [\Omega_1][\Omega_2'][{}'\Omega'][{}'\Omega_4] \\ & - [\Omega_1'][{}'\Omega_2'][{}'\Omega_3'][{}'\Omega_4].\end{aligned}$$

The general theorem is described as giving an expression for $[\Omega_1\Omega_2 \ldots \Omega_i]$ in terms of

$$\begin{array}{l} [\Omega_1], [\Omega_2], \ldots, [\Omega_{i-1}], [\Omega_i] \\ [\Omega_1'], [\Omega_2'], \ldots, [\Omega_{i-1}'] \\ \qquad [{}'\Omega_2], \ldots, [{}'\Omega_{i-1}], [{}'\Omega_i] \\ \qquad [{}'\Omega_2'], \ldots, [{}'\Omega_{i-1}']; \end{array}$$

that is to say, in terms of all the unaltered Ω's, all the curtailed Ω's except the last, all the beheaded Ω's except the first, and all the "doubly-apocopated" Ω's except the first and the last; and it is pointed out that the number of products (or terms) in the expansion is 2^{i-1} "separable into i alternately positive and negative groups containing respectively

$$1, \quad (i-1), \quad \tfrac{1}{2}(i-1)(i-2), \quad \ldots, \quad i-1, \quad 1$$

products." Further, it is noted that "in every one of the above groups forming a product the accents enter in pairs and between contiguous factors, it being a condition that if any Ω have an accent on the right the next Ω must have one on the left, and if it have one on the left the preceding Ω must have an accent on the right, and the number of pairs of accents goes on increasing in each group from 0 to $i-1$." *

In a footnote the case where each Ω has only one element, and where, therefore, each singly-accented Ω becomes 1, and each doubly-accented Ω vanishes, is stated to be identical with the "rule"

$$[a_1, a_2, \ldots, a_i] = a_1a_2 \ldots a_i - \sum \frac{1}{a_e a_{e+1}} \cdot a_1 a_2 \ldots a_i$$
$$+ \sum \frac{1}{a_e a_{e+i} a_f a_{f+1}} \cdot a_1 a_2 \ldots a_i - \ldots$$

formerly given by him in words.

* It is to be regretted that Sylvester did not give the recurrent law of formation

$$[\Omega_1\Omega_2 \ldots \Omega_{r-1}\Omega_r] = [\Omega_1\Omega_2 \ldots \Omega_{r-1}][\Omega_r] - [\Omega_1\Omega_2 \ldots \Omega_{r-1}'][{}'\Omega_r],$$

as this would have made all his statements clear and a number of them unnecessary.

SMITH, H. J. [S.] (1854, May).

[De compositione numerorum primorum formae $4\lambda+1$ ex duobus quadratis. *Crelle's Journ.*, l. pp. 91–92; or *Collected Math. Papers*, i. pp. 33–34.]

Pointing out, as Sylvester had already done, that the writing of Euler's algorithm $(q_1, q_2, \ldots, q_n)$ as a determinant leads easily to the properties

$$\begin{aligned}(q_1, q_2, \ldots, q_i) &= (q_i, q_{i-1}, \ldots, q_1),\\ (q_1, q_2, \ldots, q_n) &= (q_1, q_2, \ldots, q_i)(q_{i+1}, \ldots q_n)\\ &\quad +(q_1, q_2, \ldots, q_{i-1})(q_{i+2}, \ldots, q_n),\end{aligned}$$

Smith therefrom deduces that for centro-symmetric series of elements

$$(q_1, q_2, \ldots, q_i, q_i, \ldots, q_2, q_1) = q_1, q_2, \ldots, q_i)^2+(q_1, q_2, \ldots, q_{i-1})^2,$$

and

$$(q_1, q_2, \ldots, q_{i-1}, q_i, q_{i-1}, \ldots, q_1) = (q_1, q_2, \ldots, q_{i-1})\left\{q_1, \ldots, q_i)+(q_1, \ldots, q_{i-2}\right\},$$

noting in regard to the former that the two numbers squared on the right are mutually prime. He then makes application to the theorem referred to in the title of his paper.

SYLVESTER, J. J. (1854, August).

[Théorème sur les déterminants de M. Sylvester. *Nouv. Annales de Math.*, xiii. p. 305; or *Collected Math. Papers.*, ii. p. 28.]

This communication in its entirety is as follows:—

"Soient les déterminants

$$\lambda, \qquad \begin{matrix}\lambda & 1\\ 1 & \lambda\end{matrix}, \qquad \begin{matrix}\lambda & 1 & 0\\ 2 & \lambda & 2\\ 0 & 1 & \lambda\end{matrix}, \qquad \begin{matrix}\lambda & 1 & 0 & 0\\ 3 & \lambda & 2 & 0\\ 0 & 2 & \lambda & 3\\ 0 & 0 & 1 & \lambda\end{matrix},$$

$$\begin{matrix}\lambda & 1 & 0 & 0 & 0\\ 4 & \lambda & 2 & 0 & 0\\ 0 & 3 & \lambda & 3 & 0\\ 0 & 0 & 2 & \lambda & 4\\ 0 & 0 & 0 & 1 & \lambda\end{matrix}, \quad \ldots\ldots$$

la loi de formation est évidente; effectuant, on trouve

$$\lambda,\ \lambda^2-1,\quad \lambda(\lambda^2-2^2),\quad (\lambda^2-1^2)(\lambda^2-3^2),\quad \lambda(\lambda^2-2^2)(\lambda^2-4^2),$$
$$(\lambda^2-1^2)(\lambda^2-3^2)(\lambda^2-5^2),\quad \lambda(\lambda^2-2^2)(\lambda^2-4^2)(\lambda^2-6^2),$$

et ainsi de suite."

That Sylvester was the author of the implied theorem may be considered proved by an entry in the index of the volume (See p. 478), and by a statement of Cayley's in the *Quarterly Journal of Mathematics*, ii., p. 163. Probably the title of the communication was prefixed by the editors, who, knowing of Sylvester's papers in the *Philosophical Magazine*, felt themselves justified in applying the name "Sylvester's determinants."

SCHLÄFLI, L. (Nov. 1855).

[Réduction d'une intégrale multiple qui comprend l'arc de cercle et l'aire du triangle sphérique comme cas particuliers. *Journ.* (*de Liouville*) *de Math.*, xx. pp. 359–394.]

Here there appears the equation

$$\frac{\Delta(\alpha,\beta,\ldots,\zeta,\eta)}{\Delta(\beta,\ldots,\zeta,\eta)} = 1 - \frac{\cos^2\alpha}{1} - \frac{\cos^2\beta}{1} - \ddots - \frac{\cos^2\zeta}{1-\cos^2\eta},$$

where, in view of the contents of a subsequent paper (see under year 1858), it would seem that $\Delta(\alpha,\beta,\ldots,\zeta,\eta)$ was used for

$$\begin{vmatrix} 1 & \cos\alpha & & & & \\ -\cos\alpha & 1 & \cos\beta & & & \\ & -\cos\beta & 1 & & & \\ & & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \\ & & & -\cos\zeta & 1 & \cos\eta \\ & & & & -\cos\eta & 1 \end{vmatrix}.$$

No properties, however, of this determinant are given.

RAMUS, C. (1856, March).

[Determinanternes Anvendelse til at bestemme hoven for de convergerende Bröker. *Oversigt . . . danske Vidensk. Selsk. Forhandl.* . . . (Kjøbenhavn), pp. 106–119.]

Ramus' introduction consists in recalling the result of the application of determinants to the solution of a set of linear equations, his mode of stating the result being that given by Jacobi in the *De formatione* . . . of the year 1841,—that is to say, he takes for his set of equations

$$\left.\begin{array}{l} a_0{}^0y_0 + a_1{}^0y_1 + a_2{}^0y_2 + \ldots + a_n{}^0y_n = u_0 \\ a_0{}^1y_0 + a_1{}^1y_1 + a_2{}^1y_2 + \ldots + a_n{}^1y_n = u_1 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_0{}^ny_0 + a_1{}^ny_1 + a_2{}^ny_2 + \ldots + a_n{}^ny_n = u_n \end{array}\right\},$$

and puts the solution in the form

$$\mathrm{R}_ny_r = \mathrm{A}_r{}^0u_0 + \mathrm{A}_r{}^1u_1 + \mathrm{A}_r{}^2u_2 + \ldots + \mathrm{A}_r{}^nu_n, \qquad (\omega)$$

where

$$\begin{aligned} \mathrm{R}_n &= \sum \pm a_0{}^0a_1{}^1a_2{}^2 \ldots a_n^n, \\ \mathrm{A}_i^i &= \sum \pm a_0{}^0a_1{}^1 \ldots a_{i-1}^{i-1}a_{i+1}^{i+1} \ldots a_n^n, \\ \mathrm{A}_\kappa^i &= -\sum \pm a_0{}^0a_1{}^1 \ldots a_{i-1}^{i-1}a_i^\kappa a_{i+1}^{i+1} \ldots a_{\kappa-1}^{\kappa-1}a_{\kappa+1}^{\kappa+1} \ldots a_n^n.* \end{aligned}$$

He then recalls the further fact that if $y_0, y_1, y_2, \ldots, y_n$ be the numerators of the convergents of the continued fraction

$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +} \ldots + \frac{b_n}{a_n}$$

* It is in this mode of writing A_κ^i, namely, with the negative sign, that Jacobi's peculiarity consists. Not content with removing from R_n the row and column in which a_κ^i occurs and prefixing to the minor thus obtained the sign-factor $(-1)^{i+\kappa}$, he takes the further step of moving the row with the index κ over $\kappa - i + 1$ rows, thus arriving at

$$\mathrm{A}_\kappa^i = -\sum \pm a_0^0a_1^1 \ldots a_{i-1}^{i-1}a_{i+1}^{i+1} \ldots a_{\kappa-1}^{\kappa-1}a_i^\kappa a_{\kappa+1}^{\kappa+1} \ldots a_n^n.$$

Of course there is at this second step the option of moving the *column* with the index i over $\kappa - i + 1$ columns, and this Ramus does.

there exists the set of equations

$$\left.\begin{array}{l} y_0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad = a_0 \\ -a_1y_0 + \quad y_1 \qquad\qquad\qquad\qquad\quad = b_1 \\ -b_2y_0 - a_2y_1 + \quad y_2 \qquad\qquad\qquad = 0 \\ \qquad -b_3y_1 - a_3y_2 + y_3 \qquad\qquad\quad = 0 \\ \cdot \\ \qquad\qquad -b_ny_{n-2} - a_ny_{n-1} + y_n = 0 \end{array}\right\},$$

and he thereupon draws the natural conclusion that the previous result can be applied to the determination of $y_0, y_1, y_2, \ldots, y_n$.

Making the necessary substitution for the u's and for R_n he of course obtains

$$y_n = a_0A_n^0 + b_1A_n^1,$$

A_n^0, A_n^1 being now determinants which for want of Cayley's notation he cannot accurately specify, but which he persists in writing in the form

$$-\sum \pm a_0^n a_1^1 a_2^2 \ldots a_{n-1}^{n-1}, \quad -\sum \pm a_0^0 a_1^n a_2^2 \ldots a_{n-1}^{n-1}.$$

From this result he calculates in succession the values of y_1, y_2, y_3, y_4; but it will readily be understood that the process is neither elegant nor short.

In the remainder of the paper (§§ 4–9) no further use of the properties of determinants is made, the contents of the last ten pages being such as might appear in any ordinary exposition of continued fractions. First there is established the old "rule" for writing out the value of y_n, above referred to as being found in Stern's monograph. This is followed by the results

$$(y_n)_{\substack{a_0=a_1=\ldots=a \\ b_0=b_1=\ldots=b}} = \frac{1}{\sqrt{a^2+4b}}\left\{\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^{n+2} - \left(\frac{a-\sqrt{a^2+4b}}{2}\right)^{n+2}\right\},$$
$$= a^{n+1} + C_{n,1}a^{n-1}b + C_{n-1,2}a^{n-3}b^2 + \ldots,$$

which by putting $a=1=b$ give the number of terms in y_n, a number also obtained in the form

$$\frac{1}{2^{n+1}}\left\{C_{n+2,1} + C_{n+2,3}\cdot 5 + C_{n+2,5}\cdot 5^2 + \ldots\right\}.$$

Anything else is of small moment.

BRIOSCHI, F. (1856).

[THÉORIE DES DÉTERMINANTS, et leurs principales applications; par le Dr. F. Brioschi: traduit de l'italien par M. Edouard Combescure. xii + 216 pp. Paris.]

In the French edition of his text-book Brioschi added an expository note of two pages (pp. 142–144) on the subject, beginning at once with the general form referred to at the end of Sylvester's paper of 1853, Sept., namely,

$$\begin{vmatrix} a_1 & m_1 & \cdot & \cdots & \cdot & \cdot \\ n_1 & a_2 & m_2 & \cdots & \cdot & \cdot \\ \cdot & n_2 & a_3 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & a_{r-1} & m_{r-1} \\ \cdot & \cdot & \cdot & \cdots & n_{r-1} & a_r \end{vmatrix}, \qquad \text{or } D_r \text{ say.}$$

His first result, obtained rather clumsily, is

$$D_r = a_r D_{r-1} - m_{r-1} n_{r-1} D_{r-2};$$

his second is

$$N_r D_{r-1} - D_r N_{r-1} = m_1 m_2 \ldots m_{r-1} \cdot n_1 n_2 \ldots n_{r-1},$$

where N_r is the cofactor of a_1 in D_r, this being got by consideration of the two-line minor whose elements are the corner elements of the adjugate of D_r; and his third is that given by Sylvester at the place just mentioned.

CAYLEY, A. (1857, April).

[On the determination of the value of a certain determinant. *Quart. Journ. of Math.*, ii. pp. 163–166; or *Collected Math. Papers*, iii. pp. 120–123.]

The determinant in question is rather more general than Sylvester's of the year 1854, being

$$\begin{vmatrix} \theta & 1 & \cdot & \cdot & \cdots & \cdot & \cdot \\ x & \theta & 2 & \cdot & \cdots & \cdot & \cdot \\ \cdot & x-1 & \theta & 3 & \cdots & \cdot & \cdot \\ \cdot & \cdot & x-2 & \theta & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \theta & n-1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & x-n+2 & \theta \end{vmatrix},$$

while the other is obtained from this by putting $x=n-1$. Denoting his own form by U_n, Cayley, with Sylvester's results before him, found

$$\begin{aligned} U_2 &= (\theta^2-1) - (x-1), \\ U_3 &= \theta(\theta^2-4) - 3(x-2)\theta, \\ U_4 &= (\theta^2-1)(\theta^2-9) - 6(x-3)(\theta^2-1) + 3(x-3)(x-1); \end{aligned}$$

so that, if he put H_n for the value of U_n in Sylvester's case (viz., when $x=n-1$), he could write

$$\begin{aligned} U_2 &= H_2 - (x-1)H_0 \\ U_3 &= H_3 - 3(x-2)H_1 \\ U_4 &= H_4 - 6(x-3)H_2 + 3(x-3)(x-1)H_0, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

and thence, doubtless, divined the generalisation

$$U_n = H_n - B_{n,1}\cdot(x-n+1)\cdot H_{n-2} + B_{n,2}\cdot(x-n+1)(x-n+3)\cdot H_{n-4} - \ldots,$$

where

$$H_n = (\theta+n-1)(\theta+n-3)(\theta+n-5) \ldots \text{ to } n \text{ factors}$$

and

$$B_{n,s} = \frac{n(n-1)(n-2) \ldots (n-2s+1)}{2^s\cdot 1\cdot 2\cdot 3 \ldots s}.$$

The establishment of the truth of this is all that the paper is occupied with, the procedure being to expand U_n in terms of the elements of its last row and their complementary minors, thus obtaining

$$U_n = \theta U_{n-1} - (n-1)(x-n+2)U_{n-2},$$

and thence

$$\begin{aligned} U_n + \left\{ (n-1)(x-n+2) + (n-2)(x-n+3) - \theta^2 \right\} U_{n-2} & \\ + (n-2)(n-3)(x-n+3)(x-n+4)U_{n-4} &= 0, \end{aligned}$$

and showing that the above conjectural expression for U_n satisfies the latter equation. The process of verification is troublesome, and was not viewed with satisfaction by Cayley himself.

As a preliminary the coefficients of the H's in the value of U_n are for shortness' sake denoted by $A_{n,0}$, $-A_{n,1}$, ..., and for the same and an additional reason the coefficient of U_{n-2} in the difference-equation is denoted by

$$M_{n,s} - \left\{ \theta^2 - (n-2s-1)^2 \right\},$$

which is equivalent to putting

$$M_{n,s} \equiv (n-1)(x-n+2) + (n-2)(x-n+3) - (n-2s-1)^2.$$

The operation to be performed being thus the substitution of

$$A_{n,0}H_n - A_{n,1}H_{n-2} + \ldots . + (-)^s A_{n,s}H_{n-2s} + \ldots .$$

for U_n in the expression

$$U_n + \left[M_{n,s} - \left\{\theta^2 - (n-2s-1)^2\right\}\right]U_{n-2} + (n-2)(n-3)(x-n+3)(x-n+4)U_{n-4},$$

it is readily seen that the result will be an aggregate of expressions like

$$A_{n,s}H_{n-2s} + \left[M_{n,s} - \left\{\theta^2 - (n-2s-1)^2\right\}\right]A_{n-2,s}H_{n-2-2s}$$
$$+ (n-2)(n-3)(x-n+3)(x-n+4)A_{n-4,s}H_{n-4-2s}.$$

Now if we bear in mind that by definition

$$\left\{\theta^2 - (n-2s-1)^2\right\}H_{n-2-2s} = H_{n-2s},$$

the second of the three terms of this

$$= M_{n,s}A_{n-2,s}H_{n-2-2s} - A_{n-2,s}H_{n-2s},$$

or, if we write $s-1$ for s in one case,

$$= - M_{n,s-1}A_{n-2,s-1}H_{n-2s} - A_{n-2,s}H_{n-2s}$$
$$= - H_{n-2s}\left\{M_{n,s-1}A_{n-2,s-1} + A_{n-2,s}\right\};$$

and the third, by writing $s-2$ for s,

$$= (n-2)(n-3)(x-n+3)(x-n+4)A_{n-4,s-2}H_{n-2s}.$$

Consequently the sum of the three will vanish if

$$A_{n,s} - (M_{n,s-1}A_{n-2,s-1} + A_{n-2,s}) + (n-2)(n-3)(x-n+3)(x-n+4)A_{n-4,s-2} = 0,$$

and therefore if

$$B_{n,s}(x-n+1) - B_{n-2,s}(x-n+2s+1)$$
$$- B_{n-2,s-1}M_{n,s-1} + B_{n-4,s-2}(n-2)(n-3)(x-n+4) = 0,$$

that is, if

$$
\begin{aligned}
(x-n)\Big[\mathrm{B}_{n,s} - \mathrm{B}_{n-2,s} - (2n-3)\mathrm{B}_{n-2,s-1} + (n-2)(n-3)\mathrm{B}_{n-4,s-2}\Big] \\
+ \Big[\mathrm{B}_{n,s} - (2s+1)\mathrm{B}_{n-2,s} - \Big\{5n - 8 - (n-2s+1)^2\Big\}\mathrm{B}_{n-2,s-1} \\
+ 4(n-2)(n-3)\mathrm{B}_{n-4,s-2}\Big] = 0.
\end{aligned}
$$

But this is the case; for, as Cayley shows, both the cofactor of $x-n$ and the other similar expression following it vanish identically. The verification aimed at is thus attained.

PAINVIN, L. (1858, February).

[Sur un certain système d'équations linéaires. *Journ. (de Liouville) de Math.* (2), iii. pp. 41–46.]

The system of equations referred to in the title of Painvin's paper had presented themselves to Liouville in the course of the research which led to his "Mémoire sur les transcendantes elliptiques . . ." (*Journ. de Liouville* (1), v. pp. 441–464). Painvin's reason for taking up the subject was his belief that one of Liouville's results could be more simply arrived at by the use of determinants; and in a few lines of introduction he succeeds in showing that the result in question can be viewed as merely the resolution of the determinant

$$
\begin{vmatrix}
r & a & \cdot & \cdot & \cdots & \cdot & \cdot \\
n(a-1) & r-1 & 2a & \cdot & \cdots & \cdot & \cdot \\
\cdot & (n-1)(a-1) & r-2 & 3a & \cdots & \cdot & \cdot \\
\cdot & \cdot & (n-2)(a-1) & r-3 & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & r-n+1 & na \\
\cdot & \cdot & \cdot & \cdot & \cdots & a-1 & r-n
\end{vmatrix}
$$

into factors.

In explanation of the process followed the case of the fourth order

$$
\begin{vmatrix}
r & a & \cdot & \cdot \\
3(a-1) & r-1 & 2a & \cdot \\
\cdot & 2(a-1) & r-2 & 3a \\
\cdot & \cdot & a-1 & r-3
\end{vmatrix}
$$

will suffice. Increasing each element of the first row by the corresponding elements of the other rows,—an operation which as before we may symbolise by

$$\mathrm{row}_1 + \mathrm{row}_2 + \mathrm{row}_3 + \ldots\ldots,$$

—he removes the factor $r+3a-3$ and finds left the cofactor

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 3(a-1) & r-1 & 2a & \cdot \\ \cdot & 2(a-1) & r-2 & 3a \\ \cdot & \cdot & a-1 & r-3 \end{vmatrix}.$$

On this are performed the operations

$$\mathrm{col}_1 - \mathrm{col}_2, \quad \mathrm{col}_2 - \mathrm{col}_3, \quad \mathrm{col}_3 - \mathrm{col}_4, \quad \ldots\ldots$$

the result being a determinant of the next lower order

$$-\begin{vmatrix} 3a-r-2 & r-2a-1 & 2a \\ 2-2a & 2a-r & r-3a-2 \\ \cdot & 1-a & a-r+2 \end{vmatrix}.$$

Finally, after changing the signs of all the elements here, the operations

$$\mathrm{row}_1 + \mathrm{row}_2 + \mathrm{row}_3 + \ldots,$$
$$\mathrm{row}_2 + \mathrm{row}_3 + \ldots,$$
$$\mathrm{row}_3 + \ldots, \quad \ldots$$

are performed, the result

$$\begin{vmatrix} r-a & a & \cdot \\ 2(a-1) & r-a-1 & 2a \\ \cdot & a-1 & r-a-2 \end{vmatrix}$$

being a determinant exactly similar in form to the original, but with $r-a$ instead of r. This, therefore, in turn may be transformed into

$$(r+a-2)\begin{vmatrix} r-2a & a \\ a-1 & r-2a-1 \end{vmatrix},$$

and so on.

The value thus obtained for the above-written determinant of the $(n+1)^{\text{th}}$ order is

$$(r+na-n)(r+na-n-2a+1)(r+na-n-4a+2)\ldots(r-na),$$

each factor being less than the preceding by $2a-1$, and the whole a function of $a(a-1)$.

The special case is noted where $a=\frac{1}{2}$, and where therefore all the $n+1$ resulting factors are alike. This Painvin writes in the form

$$\begin{vmatrix} r & \frac{1}{2} & . & . & \dots & . & . \\ -\frac{n}{2} & r-1 & \frac{2}{2} & . & \dots & . & . \\ . & -\frac{n-1}{2} & r-2 & \frac{3}{2} & \dots & . & . \\ . & . & -\frac{n-2}{2} & r-3 & \dots & . & . \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & . & \dots & r-n+1 & \frac{n}{2} \\ . & . & . & . & \dots & -\frac{1}{2} & r-n \end{vmatrix} = \left(r-\frac{n}{2}\right)^{n+1};$$

but a preferable form is, evidently,

$$\begin{vmatrix} \rho & 1 & . & . & \dots & . & . \\ -n & \rho-2 & 2 & . & \dots & . & . \\ . & -n+1 & \rho-4 & 3 & \dots & . & . \\ . & . & -n+2 & \rho-6 & \dots & . & . \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & . & \dots & \rho-2n+2 & n \\ . & . & . & . & \dots & -1 & \rho-2n \end{vmatrix} = (\rho-n)^{n+1}.$$

HEINE, E. (1858, Sept.).

[Auszug eines Schreibens über die *Lamé*schen Functionen an den Herausgeber. Einige Eigenschaften der *Lamé*schen Functionen. *Crelle's Journ.*, lvi. pp. 79–86, 87–99.]

In the case of Heine the functions afterwards known as "continuants" made their appearance under totally different circumstances, namely, while he was engaged in transforming a special homogeneous function of the second degree by means of an orthogonal transformation. It will be remembered that if the quadric

$$\begin{array}{l} a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \dots \\ \qquad + \quad a_{22}x_2^2 + 2a_{23}x_2x_3 + \dots \\ \qquad\qquad\qquad + \quad a_{33}x_3^2 \quad + \dots \\ \qquad\qquad\qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

be transformed by an orthogonal transformation into

$$A_{11}\xi_1^2 + A_{22}\xi_2^2 + A_{33}\xi_3^2 + \dots$$

the coefficients of the latter expression are the roots of the equation

$$\begin{vmatrix} a_{11}-A & a_{12} & a_{13} & \dots \\ a_{12} & a_{22}-A & a_{23} & \dots \\ a_{13} & a_{23} & a_{33}-A & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

Now Heine's peculiar quadric was

$$\begin{array}{l} c_0^2x_0^2 - 2\kappa c_0c_1x_0x_1 \\ \qquad + (c_1^2+c_2^2)x_1^2 - 2\kappa c_2c_3x_1x_2 \\ \qquad\qquad\qquad + (c_3^2+c_4^2)x_2^2 - \\ \qquad\qquad\qquad\qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \qquad\qquad\qquad\qquad\qquad\qquad + (c^2_{2\sigma-1}+c^2_{2\sigma})x^2_\sigma, \end{array}$$

where in every case the coefficient of the product of two x's vanishes if their suffixes differ by more than 1, and where

$$\begin{aligned} c_0^2 &= \tfrac{1}{2}(n)(n+1), \\ c_1^2 &= \tfrac{1}{4}(n-1)(n+2), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_r^2 &= \tfrac{1}{4}(n-r)(n+r+1), \quad (r>0) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c^2_{n-1} &= \tfrac{1}{2}n, \\ \text{and } \kappa &= \frac{c^2-b^2}{c^2+b^2}. \end{aligned}$$

He was thus naturally led to the equation in z

$$\begin{vmatrix} z-c_0^2 & \kappa c_0c_1 & \cdot & \dots\dots & \cdot \\ \kappa c_0c_1 & z-c_1^2-c_2^2 & \kappa c_2c_3 & \dots\dots & \cdot \\ \cdot & \kappa c_2c_3 & z-c_3^2-c_4^2 & \dots\dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots\dots & \kappa c_{2\sigma-2}c_{2\sigma-1} \\ \cdot & \cdot & \cdot & \dots\dots & z-c^2_{2\sigma-1}-c^2_{2\sigma} \end{vmatrix} = 0,$$

where either $c^2_{2\sigma}$ is c^2_{n-1}, or $c^2_{2\sigma-1}$ is c^2_{n-1} and, if the latter, $c^2_{2\sigma}=0$. From a knowledge of Painvin's paper he recognised the left-hand side of the equation as being the numerator of the continued fraction

$$z-c_0{}^2-\frac{\kappa^2 c_0{}^2 c_1{}^2}{z-c_1{}^2-c_2{}^2-\frac{\kappa^2 c_2{}^2 c_3{}^2}{z-c_3{}^2-c_4{}^2-\ddots}}$$

but he ventured nothing in elucidation of it. Even the special case where $b=0$ and where therefore $\kappa=1$ appears to have proved at the time too troublesome, although he knew otherwise that in this case the continued fraction

$$=\frac{z(z-2^2)(z-4^2)\ldots\ldots(z-n^2)}{(z-1^2)(z-3^2)\ldots\ldots(z-\overline{n-1}^2)}\text{ if } n \text{ be even,}$$

and

$$=\frac{(z-1^2)(z-3^3)(z-5^2)\ldots\ldots(z-n^2)}{(z-2^2)(z-4^2)\ldots\ldots(z-\overline{n-1}^2)}\text{ if } n \text{ be odd;}$$

for his words are—" Einen directen Beweis für diese Summirung des Kettenbruchs habe ich noch nicht aufgefunden."

SCHLÄFLI, L. (1858).

[On the multiple integral $\int^n dx\,dy\ldots dz$ whose limits are $p_1 = a_1x + b_1y + \ldots + h_1z > 0$, $p_2 > 0, \ldots, p_n > 0$, and $x^2 + y^2 + \ldots + z^2 > 1$. *Quart. Journ. of Math.*, ii. pp. 269–301; iii. pp. 54–68, 97–108.]

The determinant which makes its appearance in the course of Schläfli's research is

$$\begin{vmatrix}
1 & -\cos\alpha & . & \ldots & . & . & . \\
-\cos\alpha & 1 & -\cos\beta & \ldots & . & . & . \\
. & -\cos\beta & 1 & \ldots & . & . & . \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
. & . & . & \ldots & 1 & -\cos\eta & . \\
. & . & . & \ldots & -\cos\eta & 1 & -\cos\theta \\
. & . & . & \ldots & . & -\cos\theta & 1
\end{vmatrix},$$

which for shortness' sake he denotes by

$$\Delta(\alpha, \beta, \gamma, \ldots, \eta, \theta)$$

and whose connection with continued fractions he therefore specifies by the equation

$$\frac{\Delta(\alpha, \beta, \gamma, \ldots, \eta, \theta)}{\Delta(\beta, \gamma, \ldots, \eta, \theta)} = 1 - \frac{\cos^2\alpha}{1} - \frac{\cos^2\beta}{1} - \cdots - \frac{\cos^2\eta}{1-\cos^2\theta}.$$

The first property noticed is, naturally,

$$\Delta(\alpha, \beta, \gamma, \ldots, \theta) = \Delta(\beta, \gamma, \ldots, \theta) - \cos^2\alpha \cdot \Delta(\gamma, \ldots, \theta).$$

Later there is given what may be viewed as an extension of this, viz.,

$$\begin{aligned}\Delta(\alpha, \ldots, \delta, \epsilon, \zeta, \eta, \theta, \ldots, \lambda) = \Delta(\alpha, \ldots, \delta, \epsilon) \cdot \Delta(\eta, \theta, \ldots, \lambda)& \\ -\cos^2\zeta \,.\, \Delta(\alpha, \ldots, \delta) \cdot \Delta(\theta, \ldots, \lambda)&,\end{aligned}$$

the proof being said to present no difficulty. The third is a little more complicated, and is logically led up to by taking four instances of the first property, namely,

$$\begin{aligned}
\Delta(\alpha, \beta, \gamma, \ldots, \zeta) &= \Delta(\beta, \gamma, \ldots, \zeta) - \cos^2\alpha \cdot \Delta(\gamma, \delta, \ldots, \zeta),\\
\Delta(\beta, \gamma, \delta, \ldots, \zeta, \eta) &= \Delta(\gamma, \delta, \ldots, \eta) - \cos^2\beta \cdot \Delta(\delta, \ldots, \zeta, \eta),\\
\Delta(\gamma, \delta, \ldots, \zeta, \eta, \theta) &= \Delta(\gamma, \delta, \ldots, \eta) - \cos^2\theta \cdot \Delta(\gamma, \delta, \ldots, \zeta),\\
\Delta(\delta, \ldots, \zeta, \eta, \theta, \alpha) &= \Delta(\delta, \ldots, \zeta, \eta, \theta) - \cos^2\alpha \cdot \Delta(\delta, \ldots, \zeta, \eta),
\end{aligned}$$

using in connection with these the multipliers

$$\Delta(\delta, \ldots, \zeta, \eta), \quad -\Delta(\delta, \ldots, \zeta), \quad \Delta(\delta, \ldots, \zeta), \quad -\Delta(\gamma, \delta, \ldots, \zeta),$$

respectively, performing addition, and then showing that the right-hand sum vanishes, the result thus being

$$\begin{aligned}\Delta(\alpha, \beta, \gamma, \delta, \ldots, \zeta) \cdot \Delta(\delta, \ldots, \zeta, \eta) - \Delta(\delta, \ldots, \zeta, \eta, \theta, \alpha) \cdot \Delta(\gamma, \delta, \ldots, \zeta)& \\ = \left\{\Delta(\beta, \gamma, \delta, \ldots, \zeta, \eta) - \Delta(\gamma, \delta, \ldots, \zeta, \eta, \theta)\right\} \cdot \Delta(\delta, \ldots, \zeta)&.\end{aligned}$$

The fourth property concerns the determinant

$$\begin{vmatrix} \Delta(\beta, \gamma, \ldots, \eta, \theta) & \Delta(\alpha, \beta, \gamma, \ldots, \eta, \theta) \\ \Delta(\beta, \gamma, \ldots, \eta) & \Delta(\alpha, \beta, \gamma, \ldots, \eta) \end{vmatrix},$$

which by reason of the first property can be shown equal to

$$\begin{vmatrix} \Delta(\beta,\gamma,\ldots,\eta,\theta) & -\Delta(\gamma,\ldots,\eta,\theta) \\ \Delta(\beta,\gamma,\ldots,\eta) & -\Delta(\gamma,\ldots,\eta) \end{vmatrix} \cos^2\alpha,$$

or

$$\begin{vmatrix} \Delta(\gamma,\ldots,\eta,\theta) & \Delta(\beta,\gamma,\ldots,\eta,\theta) \\ \Delta(\gamma,\ldots,\eta) & \Delta(\beta,\gamma,\ldots,\eta) \end{vmatrix} \cos^2\alpha,$$

and ultimately, "by repeating this sort of transformation," equal to

$$\cos^2\alpha \ \cos^2\beta \ \cos^2\gamma \ \ldots \ \cos^2\theta.$$

If we use for a moment the present-day notation for continuants, viz., where

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} + \ddots \quad = \quad \frac{K\begin{pmatrix} & b_1 & & b_2 & \\ a_1 & & a_2 & & a_3 \ldots \end{pmatrix}}{K\begin{pmatrix} & b_2 & \\ a_2 & & a_3 \ldots \end{pmatrix}}$$

Schläfli's results are seen to be

$$K\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \cdot\cdot \\ 1 & 1 & 1 & \ldots \end{pmatrix} = K\begin{pmatrix} \beta_2 & \beta_3 & \cdot\cdot \\ 1 & 1 & \ldots \end{pmatrix} + \beta_1 K\begin{pmatrix} \beta_3 & \beta_4 & \cdot\cdot \\ 1 & 1 & \ldots \end{pmatrix},$$

$$\left.\begin{aligned} K\begin{pmatrix} \beta_1 & \beta_2 \cdots & \beta_\kappa \cdots & \beta_n \\ 1 & 1 \ \ldots 1 & 1 \ldots & 1 \end{pmatrix} &= K\begin{pmatrix} \beta_2 & \beta_3 \cdot\cdot & \beta_{\kappa-1} \\ 1 & 1 \ \ldots & 1 \end{pmatrix} \cdot K\begin{pmatrix} \beta_{\kappa+1} \cdots & \beta_n \\ 1 \quad 1 \ldots & 1 \end{pmatrix}, \\ &\quad - \beta_\kappa K\begin{pmatrix} \beta_1 \cdots & \beta_{\kappa-2} \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \cdot K\begin{pmatrix} \beta_{\kappa+2} \cdots & \beta_n \\ 1 \quad 1 \ldots & 1 \end{pmatrix}, \end{aligned}\right\}$$

$$\left.\begin{aligned} &K\begin{pmatrix} \beta_1 \cdots & \beta_{n-2} \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \cdot K\begin{pmatrix} \beta_4 \cdots & \beta_{n+1} \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \\ -&K\begin{pmatrix} \beta_4 \cdots \beta_n & \beta_1 \\ 1 \quad 1 \ldots \ 1 & 1 \end{pmatrix} \cdot K\begin{pmatrix} \beta_3 \cdots & \beta_{n-2} \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \end{aligned}\right\} = \left\{\begin{aligned} &K\begin{pmatrix} \beta_2 \cdots & \beta_{n-1} \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \\ -&K\begin{pmatrix} \beta_3 \cdots & \beta_n \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \end{aligned}\right\} K\begin{pmatrix} \beta_4 \cdots & \beta_{n-2} \\ 1 \quad 1 \ldots & 1 \end{pmatrix},$$

$$\begin{vmatrix} K\begin{pmatrix} \beta \cdots & \beta_n \\ 1 \quad 1 \ldots & 1 \end{pmatrix} & K\begin{pmatrix} \beta_1 \cdots & \beta_n \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \\ K\begin{pmatrix} \beta_2 \cdots & \beta_{n-1} \\ 1 \quad 1 \ldots & 1 \end{pmatrix} & K\begin{pmatrix} \beta_1 \cdots & \beta_{n-1} \\ 1 \quad 1 \ldots & 1 \end{pmatrix} \end{vmatrix} = (-1)^n \beta_1\beta_2\beta_3 \ldots \beta_n,$$

the only change being the writing of β_1, β_2, ... for $-\cos^2\alpha$, $-\cos^2\beta$, ...

WORPITZKY, [J. D. T.] (1865, April).

[Untersuchungen über die Entwickelung der monodromen und monogenen Functionen durch Kettenbrüche. (Sch. Progr.) 39 pp., Berlin.]

Of the six sections into which the paper giving the results of Worpitzky's painstaking investigation is divided it is only the first headed "Fundamentalrelationen" which concerns us, these relations being nothing else than what we should now call "properties of continuants."

He takes his continued fraction in the same form as Schläfli, viz.,

$$1 + \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1},$$

showing of course that it equals

$$\frac{N_{1,n}}{N_{2,n}},$$

where

$$N_{\kappa,n} = \begin{vmatrix} 1 & 1 & . & \cdots & . & . & . \\ -a_\kappa & 1 & 1 & \cdots & . & . & . \\ . & -a_{\kappa-1} & 1 & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & -a_{n-1} & 1 & 1 \\ . & . & . & \cdots & . & -a_n & 1 \end{vmatrix}.$$

The first matter of interest is the expansion of $N_{\kappa,n}$ as a sum of products of $a_\kappa, a_{\kappa-1}, \ldots, a_n$, *e.g.*,

$$N_{1,3} = 1 + (a_1+a_2+a_3) + a_1a_3.$$

This is written in the form

$$1 + \overset{1}{a}_{\kappa,n} + \overset{2}{a}_{\kappa\, n} = \ldots.,$$

where, he says, "$\overset{r}{a}_{\kappa,n}$ die Summe aller möglichen (als Producte aufgefassten) Combinationscomplexionen ohne Wiederholung bedeutet, welche sich aus $a_\kappa, a_{\kappa+1}, \ldots, a_n$ so zu je r Elementen bilden lassen, dass nicht zwei neben einander stehende Elemente

a_s, a_{s+1} dieser Reihe in den einzelnen Producten zugleich vorkommen." By way of proof it is pointed out (1) that the term independent of all the a's is

$$\begin{vmatrix} 1 & 1 & & & \\ 0 & 1 & 1 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \end{vmatrix} \quad i.e. \ +1;$$

(2) that the cofactor* of $(-a_r)(-a_s)(-a_t)\ldots.$ when two of the a's are consecutive is

$$\begin{vmatrix} 1 & 1 & & & & & & & & \\ 0 & 1 & 1 & & & & & & & \\ & \cdot & \cdot & \cdot & \cdot & & & & & \\ & & 0 & 1 & 1 & & & & & \\ & & & 1 & 0 & 0 & & & & \\ & & & & 1 & 0 & 0 & & & \\ & & & & & 0 & 1 & 1 & & \\ & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & 0 & 1 & 1 \\ & & & & & & & & 0 & 1 \end{vmatrix} \quad i.e. \ 0;$$

and (3) that the cofactor of $(-a_r)(-a_s)(-a_t)\ldots.$ when no two of the a's are consecutive and their number is p, is another long-drawn-out continuant found equal to

$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^p \quad i.e. \ (-1)^p,$$

and that, therefore, the cofactor of $a_r a_s a_t \ldots$ in this case is $+1$.

In exactly similar fashion by partitioning $N_{\kappa,n}$ into terms which contain $-a_s$ and terms which do not, he finds

$$N_{k,n} = D_0 - a_s D_s,$$

where D_0 and D_s, determinants equally greedy of page space, are

* To obtain the cofactor of the product of a number of a set of elements in a determinant Worpitzky puts a 1 in the determinant in place of each element occurring in the said product, 0's in all the other places of the rows to which these elements belong, and 0's for all the other elements of the set.

shown to be equal to $N_{k,s-1} \cdot N_{s+1,n}$ and $-N_{k,s-2}N_{s+2,n}$ respectively, and he thus reaches the result

$$N_{k,n} = N_{k,s-1}N_{s+1,n} + a_s N_{k,s-2}N_{s+2,n}$$

already obtained in a different way by Schläfli.

Lastly, taking a determinant of the same form as $N_{k,n}$, but having

$$-a_s, \ -a_{s-1}, \ldots, \ -a_{k+1}, \ -a_k, \ -a_k, \ -a_{k+1}, \ldots, \ -a_{n-1}, \ -a_n$$

for its minor diagonal of a's, he obtains for it by isolating the first a_k the expression

$$N_{s,k+1}N_{k,n} + a_k N_{s,k+2}N_{k+1,n},$$

and by isolating the second a_k

$$N_{s,k}N_{k+1,n} + a_k N_{s,k+1}N_{k+2,n};$$

and thus deduces

$$N_{k,n}N_{k+1,s} - N_{k,s}N_{k+1,n} = -a_k(N_{k+1,n}N_{k+2,s} - N_{k+1,s}N_{k+2,n}).$$

It is then noted that the bracketed expression on the right differs from the expression on the left merely in having $k+1$ in place of k; so that there results

$$\begin{aligned} N_{k,n}N_{k+1,s} - N_{k,s}N_{k+1,n} &= (-1)^2 a_k a_{k+1}(N_{k+2,n}N_{k+3,s} - N_{k+2,s}N_{k+3,n}) \\ &= \cdots\cdots\cdots \\ &= (-1)^{s-k+1} a_k a_{k+1} \ldots a_{s+1} N_{s+3,n}. \end{aligned}$$

This also, it will be seen, is connected with a result of Schläfli's; for putting $s = n-1$ we have *

$$\begin{vmatrix} N_{k+1,n-1} & N_{k,n-1} \\ N_{k+1,n} & N_{k,n} \end{vmatrix} = (-1)^{n-k} a_k a_{k+1} \ldots a_n,$$

which becomes identical with Schläfli's last proposition on transposing the two rows of the determinant and (what is equally immaterial) putting $k=1$.

* In giving to $N_{s+1,s}$, $N_{s+2,s}$, $N_{s+3,s}$ the values 1, 1, 0 which are necessitated by assuming the generality of the recursion-formula

$$N_{k,n} = N_{k+1,n} + a_k N_{k+2,n},$$

Worpitzky forgets to note that in these cases the proposition $N_{k,n} = N_{n,k}$, used by him in the demonstration, does not hold.

MAZZA, F. (1866).

[ELEMENTI DI ALGEBRA, par R. Rubini. Terza edizione, accrescinta e migliorata. iv+295 pp. Napoli.]

In his chapter on determinants (Cap. x. pp. 249–292) Rubini gives (p. 270) the result

$$\begin{vmatrix} \lambda & a_1 & . & . & \cdots & . & . \\ a_n & \lambda & a_2 & . & \cdots & . & . \\ . & a_{n-1} & \lambda & a_3 & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & a_{n-1} & . \\ . & . & . & . & \cdots & \lambda & a_n \\ . & . & . & . & \cdots & a_1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & a_n \\ a_n & \lambda \end{vmatrix} \begin{vmatrix} \lambda & a_1 & . & . & \cdots \\ a_{n-1} & \lambda & a_2 & . & \cdots \\ . & a_{n-2} & \lambda & a_3 & \cdots \\ . & . & . & . & \cdots \end{vmatrix}$$

where $a_1, a_2, \ldots, a_n$ are elements increasing by the common difference a_1. This is established by performing the operations which would at a later date have been denoted by

$$\begin{gathered} \text{row}_1 + \text{row}_3 + \text{row}_5 + \ldots . \\ \text{row}_2 + \text{row}_4 + \text{row}_6 + \ldots . \\ \text{row}_3 + \text{row}_5 + \text{row}_7 + \ldots . \\ \cdots\cdots\cdots \\ \text{col}_{n+1} - \text{col}_{n-1}, \quad \text{col}_n - \text{col}_{n-2}, \quad \text{col}_{n-1} - \text{col}_{n-3}, \quad \ldots . \end{gathered}$$

The process is said to be due to Francesco Mazza, and the result to degenerate into Sylvester's of the year 1854 on putting $a_1 = 1$. It is not noticed, however, that on the other hand the result may be viewed as a special case of Sylvester's, namely, where λ/a_1 is put for λ.

THIELE, T. N. (1869, 1870).

[Bemærkninger om Kjædebrøker. *Tidsskrift for Math.* (2), v. pp. 144–146.
Den endelige Kjædebrøksfunktions Theori. *Tidsskrift for Math.* (2), vi. pp. 145–170.]

The first of the two notes comprising Thiele's first paper contains only one result, namely,

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots + \cfrac{b_n}{a_n}}} = \frac{(a_1, a_2, \ldots, a_n)}{(a_2, \ldots, a_n)},$$

where $(a_1, a_2, \ldots, a_n)$ is used to stand for

$$\begin{vmatrix} a_1 & b_1 & . & \ldots & . & . \\ -1 & a_2 & b_2 & \ldots & . & . \\ . & . & . & . & . & . \\ . & . & . & \ldots & a_{n-1} & b_{n-1} \\ . & . & . & \ldots & -1 & a_n \end{vmatrix}$$

There is nothing to indicate that this is not viewed as a fresh discovery, notwithstanding the fact that Ramus' paper of 1856 containing virtually the same identity had been published in the same city.

The other paper may be described as a careful study of finite continued fractions with the help of determinants. Instead of $b_1, b_2, \ldots$ are used $a_{12}, a_{23}, \ldots$; and

$$\begin{vmatrix} a_p & a_{p,p+1} & . & \ldots & . & . \\ 1 & a_{p+1} & a_{p+1,p+2} & & . & . \\ . & . & . & . & . & . \\ . & . & . & \ldots & a_{q-1} & a_{q-1,q} \\ . & . & . & \ldots & 1 & a_q \end{vmatrix}$$

is denoted by

$$\mathrm{K}(p,q).$$

Further, this determinant is spoken of as a "Kjædebrøksdeterminant," or, shortly, a "K-Determinant"; and a section (§ 3, pp. 149–152) is devoted to a statement of its properties.

There is no need to rehearse all of these, the last portion (D) of the section being alone that which contains fresh matter. Opening with the double use of a previous property, viz.,

$$\mathrm{K}(h,m) = \mathrm{K}(h,k-1)\cdot\mathrm{K}(k,m) - a_{k-1,k}\mathrm{K}(h,k-2)\cdot\mathrm{K}(k+1,m),$$
$$\mathrm{K}(h,n) = \mathrm{K}(h,k-1)\cdot\mathrm{K}(k,n) - a_{k-1,k}\mathrm{K}(h,k-2)\cdot\mathrm{K}(k+1,n),$$

where h, k, m, n are in ascending order of magnitude, the author eliminates $\mathrm{K}(h,k-1)$ and obtains

$$\begin{vmatrix} \mathrm{K}(h,m) & \mathrm{K}(k,m) \\ \mathrm{K}(h,n) & \mathrm{K}(k,n) \end{vmatrix} = a_{k-1,k}\cdot \mathrm{K}(h,k-2)\cdot \begin{vmatrix} \mathrm{K}(k,m) & \mathrm{K}(k+1,m) \\ \mathrm{K}(k,n) & \mathrm{K}(k+1,n) \end{vmatrix}. \qquad (a)$$

Then by taking the particular case of this where k appears in place of h and $k+1$ in place of k there results

$$\begin{vmatrix} \mathrm{K}(k,m) & \mathrm{K}(k+1,m) \\ \mathrm{K}(k,n) & \mathrm{K}(k+1,n) \end{vmatrix} = a_{k,k+1} \begin{vmatrix} \mathrm{K}(k+1,m) & \mathrm{K}(k+2,m) \\ \mathrm{K}(k+1,n) & \mathrm{K}(k+2,n) \end{vmatrix},$$

which when applied to one of the determinants occurring in itself gives

$$\begin{vmatrix} \mathrm{K}(k,m) & \mathrm{K}(k+1,m) \\ \mathrm{K}(k,n) & \mathrm{K}(k+1,n) \end{vmatrix} = a_{k,k+1}a_{k+1,k+2} \begin{vmatrix} \mathrm{K}(k+2,m) & \mathrm{K}(k+3,m) \\ \mathrm{K}(k+2,n) & \mathrm{K}(k+3,n) \end{vmatrix},$$

and finally

$$= a_{k,k+1}a_{k+1,k+2}\,\ldots\,a_{m,m+1}\cdot \begin{vmatrix} \mathrm{K}(m+1,m) & \mathrm{K}(m+2,m) \\ \mathrm{K}(m+1,n) & \mathrm{K}(m+2,n) \end{vmatrix},$$

$$= a_{k,k+1}a_{k+1,k+2}\,\ldots\,a_{m,m+1}\cdot \mathrm{K}(m+2,n). \qquad (\beta)$$

Further, by using this to make a substitution in the previous result (a) there is obtained

$$\begin{vmatrix} \mathrm{K}(h,m) & \mathrm{K}(k,m) \\ \mathrm{K}(h,n) & \mathrm{K}(k,n) \end{vmatrix} = a_{k-1,k}a_{k,k+1}\,\ldots\,a_{m,m+1}\cdot \mathrm{K}(h,k-2)\mathrm{K}(m+2,n), \quad (\gamma)$$

which on putting $k=h+1$ and $m=n-1$ becomes

$$\begin{vmatrix} \mathrm{K}(h,n-1) & \mathrm{K}(h+1,n-1) \\ \mathrm{K}(h,n) & \mathrm{K}(h+1,n) \end{vmatrix} = a_{h,h+1}a_{h+1,h+2}\,\ldots\,a_{n-1,n},$$

—a result which may be compared with one of Schläfli's and Worpitzky's, but which is more general in that the main diagonal of each "K-Determinant" does not consist of units.

CHAPTER XVI.

THE LESS COMMON SPECIAL FORMS, UP TO 1860.

THERE now only remain for consideration those special forms which, prior to 1860, had not received any noteworthy attention. These will be found to include: (α) permanents, which are touched on by three authors; (β) determinants with the typical element $a_{rs}+b_{rs}i$, which are referred to in four memoirs; (γ) two other forms, which are each dealt with in two papers; and (δ) nine others, which make their appearance only once. It will also be appropriate to collect in a note (ϵ) the facts ascertained up to 1860 regarding the census of terms in special forms of determinants.

(α) PERMANENTS.

As we have already seen, Cauchy, in his memoir of 1812, widened the ordinary meaning of the term "symmetric function," and was consequently led to call such expressions as

$$a_1b_2 + a_2b_1, \quad a_1b_2 + a_2b_3 + a_3b_1 + a_1b_3 + a_2b_1 + a_3b_2, \quad \ldots$$

"fonctions symétriques permanentes," denoting them by $S^2(a_1b_2)$, $S^3(a_1b_2)$, ...

In the same year, as we have also noted, Binet gave the identities

$$\begin{aligned} \Sigma ab' &= \Sigma a\Sigma b - \Sigma ab, \\ \Sigma ab'c'' &= \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma c\Sigma ab, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

which in Cauchy's notation would have been written

$$S^n(a_1b_2) = S^n(a_1)S^n(b_1) - S^n(a_1b_1),$$
$$S^n(a_1b_2c_3) = S^n(a_1)S^n(b_1)S^n(c_1) + 2S^n(a_1b_1c_1) - \;\; \ldots\ldots,$$

but which, in reality, are due to Waring, who, denoting the sum of the p^{th} powers of $\alpha, \beta, \gamma, \ldots$ by s_p asserted in his *Miscellanea Analytica* of the year 1762 that

$$\begin{aligned} \sum \alpha^p\beta^q &= s_p \cdot s_q - s_{p+q}, \\ \sum \alpha^p\beta^q\gamma^r &= s_p \cdot s_q \cdot s_r + 2s_{p+q+r} - \ldots \\ \ldots & \ldots\ldots\ldots \end{aligned}$$

Proofs of Waring's identities were given by Paoli in his *Supplemento agli Elementi di Algebra*, published in 1804 (See *Op.*, ii. § 28), and by Meier Hirsch in his *Sammlung von Aufgaben aus der Theorie der algebraischen Gleichungen*, published in 1809 (See pp. 34–41).

It is only symmetric functions like $S^2(a_1b_2)$, $S^3(a_1b_2c_3)$, $S^4(a_1b_2c_3d_4)$, ..., whose every term involves the full number of letters, that at the present day are spoken of as *permanents*.

BORCHARDT, C. W. (1855).

[Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden Function. *Monatsb. ... Akad. d. Wiss.* (Berlin), 1855, pp. 165–171; or *Crelle's Journ.*, liii. pp. 193–198; or *Gesammelte Werke*, pp. 97–105.]

Having already fully dealt with this paper under the heading *Alternants*, it suffices merely to recall the identity therein given, namely,

$$\sum\left(\frac{1}{t-a}\cdot\frac{1}{t_1-a_1}\cdots\frac{1}{t_n-a_n}\right)\times\sum\left(\pm\frac{1}{t-a}\cdot\frac{1}{t_1-a_1}\cdots\frac{1}{t_n-a_n}\right)$$
$$=\sum\left(\pm\frac{1}{(t-a)^2}\cdot\frac{1}{(t_1-a_1)^2}\cdots\frac{1}{(t_n-a_n)^2}\right),$$

where the first factor on the left differs from the determinant which is its cofactor merely in having the signs of all its terms positive.

JOACHIMSTHAL, F. (1856, September).

[De æquationibus quarti et sexti gradus quæ in theoria linearum et superficierum secundi gradus occurrunt. *Crelle's Journ.*, liii. pp. 149–172.]

Joachimsthal, requiring the use of the so-called "Binet's" identities, devotes section iii. of his paper to them, combining them in one proposition, and showing more or less satisfactorily, after the manner of Meier Hirsch, how the proof of each case can be made dependent on the previous case. His proposition is—*There being* m *rows each of* z *quantities*

$$\begin{array}{cccc} a_1 & a_2 & \dots & a_z \\ \beta_1 & \beta_2 & \dots & \beta_z \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \lambda_1 & \lambda_2 & \dots & \lambda_z \\ \mu_1 & \mu_2 & \dots & \mu_z \end{array}$$

and z *being not less than* m, *the sum*

$$\sum a_1\beta_2 \dots \mu_m,$$

consisting of $z(z-1)(z-2)\dots(z-m+1)$ *terms, can be expressed as an integral function of the sums arranged in the following rows:*

$$\begin{array}{llll} \Sigma a_1 & \Sigma\beta_1 & \dots\dots\ \Sigma\mu_1 & \\ \Sigma a_1\beta_1 & \Sigma a_1\gamma_1 & \dots\dots\dots\dots & \Sigma\lambda_1\mu_1 \\ \Sigma a_1\beta_1\gamma_1 & \Sigma a_1\beta_1\delta_1 & \dots\dots\dots\dots\dots\dots & \Sigma\kappa_1\lambda_1\mu_1 \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \Sigma a_1\beta_1\gamma_1\dots\mu_1, & & & \end{array}$$

each sum consisting of z *terms: further, the said function when* $z < m$ *vanishes identically.*

To prove the proposition when $m=3$ he takes the previous case

$$\Sigma a_1\beta_2 = \Sigma a_1\Sigma\beta_1 - \Sigma a_1\beta_1 \qquad (x_2)$$

and multiplies both sides by $\Sigma\gamma_1$, thus obtaining

$$\Sigma a_1\beta_2\gamma_3 + \Sigma a_1\gamma_1\beta_2 + \Sigma\beta_1\gamma_1 a_2 = \Sigma a_1\Sigma\beta_1\Sigma\gamma_1 - \Sigma a_1\beta_1\cdot\Sigma\gamma_1,$$

in which the previous case enables him to replace

$$\Sigma a_1\gamma_1\beta_2 \quad \text{by} \quad \Sigma a_1\gamma_1\cdot\Sigma\beta_1 - \Sigma a_1\beta_1\gamma_1,$$

and

$$\Sigma\beta_1\gamma_1 a_2 \quad \text{by} \quad \Sigma\beta_1\gamma_1\cdot\Sigma a_1 - \Sigma a_1\beta_1\gamma_1,$$

with the result that

$$\Sigma a_1\beta_2\gamma_3 = \Sigma a_1\Sigma\beta_1\Sigma\gamma_1 - \Sigma a_1\beta_1\cdot\Sigma\gamma_1 - \Sigma a_1\gamma_1\cdot\Sigma\beta_1 - \Sigma\beta_1\gamma_1\cdot\Sigma a_1 + 2\Sigma a_1\beta_1\gamma_1 \quad (x_3)$$

as desired. Similarly, on multiplying both sides of this by $\Sigma\delta_1$ there is obtained on the left

$$\Sigma a_1\beta_2\gamma_3\delta_4 + \Sigma a_1\delta_1\beta_2\gamma_3 + \Sigma\beta_1\delta_1 a_2\gamma_3 + \Sigma\gamma_1\delta_1 a_2\beta_3,$$

the last three terms of which have only to be replaced by expressions warranted from (x_3) in order to give the desired equivalent * for $\Sigma a_1\beta_2\gamma_3\delta_4$.

It is then pointed out that when $z=m$ the sum $\Sigma a_1\beta_2 \ldots \mu_m$ "tantum a determinante differt, quod omnes ejus termini sunt positivi," and that therefore when $z<m$ the sum vanishes.

The three sections following (iv., v., vi.) are occupied, as has been noted elsewhere, with the generalisation of Borchardt's theorem of the previous year.

CAYLEY, A. (1857).

[Note sur les normals d'une conique. *Crelle's Journ.*, lvi. pp. 182–185; or *Collected Math. Papers*, iv. pp. 74–77.]

In dealing with essentially the same geometrical subject as Joachimsthal, Cayley gives, in support of part of his demonstration, the identity

$$\left\{\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}\right\} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_2^2 & y_2^2 & z_2^2 \\ x_3^2 & y_3^2 & z_3^2 \end{vmatrix} + \begin{vmatrix} y_1z_1 & z_1x_1 & x_1y_1 \\ y_2z_2 & z_2x_2 & x_2y_2 \\ y_3z_3 & z_3x_3 & x_3y_3 \end{vmatrix}$$

where the first factor on the left is what Cauchy denoted by $S^3(x_1y_2z_3)$.

* By an oversight three terms of this are left out by Joachimsthal.

(β) DETERMINANTS WITH COMPLEX ELEMENTS.

HERMITE, C. (1854).

[Extrait d'une lettre . . . . sur le nombre des racines d'une équation algébrique comprises entre des limites données. *Crelle's Journ.*, lii. pp. 39–51; or *Œuvres*, i. pp. 397–414.]

On p. 40 it is pointed out that any determinant whose conjugate elements are of the form $a_{rs}+b_{rs}\sqrt{-1}$, $a_{rs}-b_{rs}\sqrt{-1}$, and whose diagonal elements are therefore of the form a_{rr}, must be real, for the reason that it is not altered in value by changing $\sqrt{-1}$ into $-\sqrt{-1}$.

HERMITE, C. (1855, August).

[Remarque sur un théorème de M. Cauchy. *Comptes rendus . . . . Acad. des Sci.* (Paris), xli. pp. 181–183; or *Œuvres*, i. pp. 479–481.]

The remark concerns the determinant just referred to, and is to the effect that the equation

$$\begin{vmatrix} a_{11}-x & a_{12}+b_{12}i & \dots\dots & a_{1n}+b_{1n}i \\ a_{21}+b_{21}i & a_{22}-x & \dots\dots & a_{2n}+b_{2n}i \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}+b_{n1}i & a_{n2}+b_{n2}i & \dots\dots & a_{nn}-x \end{vmatrix} = 0,$$

where $a_{rs}=a_{sr,}$ $b_{rs}=-b_{sr,}$ $i=\sqrt{-1}$, has all its roots real if the a's and b's be real,—a result which degenerates into one previously known (Lagrange, 1773; Cauchy, 1829) when all the b's vanish. No proof is given, but it is stated that one is obtainable by transforming "le déterminant en un autre à éléments réels, d'un nombre double de colonnes et symétrique par rapport à la diagonale." A rule is formulated for determining the number of roots of the equation which lie between two limits. Lastly, it is remarked that the equation arises in connection with the study of forms of the type

$$\begin{array}{cc|c} x+x'i & y+y'i & \\ \hline a_{11} & a_{12}-\beta_{12}i & x-x'i \\ a_{12}+\beta_{12}i & a_{22} & y-y'i, \end{array}$$

that is to say,

$$\left.\begin{array}{l} a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \\ + a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 \end{array}\right\} + 2\beta_{12}|xy'|.$$

RUBINI, R. (1857, May).

[Applicazione della teorica dei determinanti. *Annali di Sci. mat. e fis.*, viii. pp. 179–200.]

In treating of determinants with binomial elements Rubini's most interesting example is that in which the element in the $(r, s)^{\text{th}}$ place is $a_{rs}+b_{rs}\sqrt{-1}$. By substitution in his general result he readily obtains the expansion of the determinant in the form $C+D\sqrt{-1}$, which is seen to alter into $C-D\sqrt{-1}$ on changing the signs of the b's. The product of $|a_{1n}+b_{1n}\sqrt{-1}|$ and $|a_{1n}-b_{1n}\sqrt{-1}|$ is consequently expressible as the sum of two squares. His next point is that on using the ordinary multiplication-theorem the same product is got in the form

$$\begin{vmatrix} a_{11} & a_{12}-\beta_{12}\sqrt{-1} & \dots & a_{1n}-\beta_{1n}\sqrt{-1} \\ a_{12}+\beta_{12}\sqrt{-1} & a_{22} & \dots & a_{2n}-\beta_{2n}\sqrt{-1} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ a_{1n}+\beta_{1n}\sqrt{-1} & a_{2n}+\beta_{2n}\sqrt{-1} & \dots & a_{nn} \end{vmatrix},$$

and that a comparison of the two forms may be fruitful of results. When $n=2$, the identity resulting from such comparison is

$$(ad-bc-\alpha\delta+\beta\gamma)^2 + (a\delta-b\gamma+\alpha d-\beta c)^2$$
$$=(a^2+\alpha^2+b^2+\beta^2)(c^2+\gamma^2+d^2+\delta^2)-(ac+\alpha\gamma+bd+\beta\delta)^2-(a\gamma-\alpha c+b\delta-\beta d)^2,$$

a result which gives the product of two sums of four squares as a like sum.

In connection with this special example, however, note should be taken that Hermite in a letter to Jacobi published in 1850 (see *Crelle's Journ.*, xl. p. 297), had pointed out that it followed from the row-by-row multiplication of

$$\begin{vmatrix} a+\alpha\sqrt{-1} & b+\beta\sqrt{-1} \\ -b+\beta\sqrt{-1} & a-\alpha\sqrt{-1} \end{vmatrix} \text{ by } \begin{vmatrix} -c+\gamma\sqrt{-1} & -d+\delta\sqrt{-1} \\ d+\delta\sqrt{-1} & -c-\gamma\sqrt{-1} \end{vmatrix}.$$

CLEBSCH, A. (1859).

[Theorie der circularpolarisirenden Medien. *Crelle's Journ.*, lvii. pp. 319–358.]

In § 3 (pp. 324–330) Clebsch is led to consider the nature of the roots of the equation dealt with by Hermite in 1855, not knowing, apparently, what the latter had done. Unfortunately the proof given of the reality of the roots is not effected without the use of a set of unessential equations of which the determinant is the eliminant.

The interesting fact is noted that when $n=3$ the equation can be changed into

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{12} & a_{22}-x & a_{23} \\ a_{13} & a_{23} & a_{33}-x \end{vmatrix} - \begin{array}{c|ccc|c} & b_{23} & -b_{13} & b_{12} & \\ \hline & a_{11}-x & a_{12} & a_{13} & b_{23} \\ & a_{12} & a_{22}-x & a_{23} & -b_{13} \\ & a_{13} & a_{23} & a_{33}-x & b_{12} \end{array} = 0.$$

(γ_1) DETERMINANTS CONNECTED WITH ANHARMONIC RATIOS.

CAYLEY, A. (1854, February).

[On some integral transformations. *Quart. Journ. of Math.*, i. pp. 4–6; or *Collected Math. Papers,* iii. pp. 1–4.]

This paper opens with two statements in reference to the determinant

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix}, \quad \text{or } \Psi \text{ say.}$$

The first is to the effect that the equation

$$\Psi = 0$$

asserts the equality of the anharmonic ratios of $\alpha, \beta, \gamma, \delta$ and

$\alpha', \beta', \gamma', \delta'$: and the second that the said equation may also be expressed in the forms*

$$\mathrm{K}\alpha = -\left\{\gamma\delta(\gamma'-\delta')(\alpha'-\beta') + \delta\beta(\delta'-\beta')(\alpha'-\gamma') + \beta\gamma(\beta'-\gamma')(\alpha'-\delta')\right\},$$
$$\mathrm{K}(\alpha-\beta) = (\delta-\beta)(\beta-\gamma)(\gamma'-\delta')(\alpha'-\beta'),$$
$$\mathrm{K}(\alpha-\gamma) = (\beta-\gamma)(\gamma-\delta)(\delta'-\beta')(\alpha'-\gamma'),$$
$$\mathrm{K}(\alpha-\delta) = (\gamma-\delta)(\delta-\beta)(\beta'-\gamma')(\alpha'-\delta'),$$

if we use K to stand for

$$\beta(\gamma'-\delta')(\alpha'-\beta') + \gamma(\delta'-\beta')(\alpha'-\gamma') + \delta(\beta'-\gamma')(\alpha'-\delta').$$

Accepting the first statement, and knowing that the equality referred to is

$$\frac{(\gamma-\alpha)(\beta-\delta)}{(\alpha-\beta)(\gamma-\delta)} = \frac{(\gamma'-\alpha')(\beta'-\delta')}{(\alpha'-\beta')(\gamma'-\delta')},$$

we readily make the deduction that

$$\Psi = (\gamma-\alpha)(\beta-\delta)(\alpha'-\beta')(\gamma'-\delta') - (\alpha-\beta)(\gamma-\delta)(\gamma'-\alpha')(\beta'-\delta').$$

By accepting the second statement, like conclusions may be drawn; for then the elimination of K from any two of the

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* These may be established as follows. By separating the terms of K which involve α' from those which do not, we see that

$$\mathrm{K} = -\begin{vmatrix} . & 1 & . & \alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix},$$

a determinant differing from Ψ in the first row only, and consequently on multiplying by α and adding we obtain

$$\Psi + \mathrm{K}\alpha = \begin{vmatrix} 1 & . & \alpha' & . \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} = \begin{vmatrix} \beta & \beta'-\alpha' & \beta\beta' \\ \gamma & \gamma'-\alpha' & \gamma\gamma' \\ \delta & \delta'-\alpha' & \delta\delta' \end{vmatrix} = \dots.$$

Similarly,

$$\Psi + \mathrm{K}(\alpha-\beta) = \begin{vmatrix} 1 & \beta & \alpha' & \beta\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} = \begin{vmatrix} 1 & . & \alpha' & . \\ 1 & . & \beta' & . \\ 1 & \gamma-\beta & \gamma' & (\gamma-\beta)\gamma' \\ 1 & \delta-\beta & \delta' & (\delta-\beta)\delta' \end{vmatrix} = \dots.$$

and so of the others.

In doing this we learn, too, that

$$\Psi + \mathrm{K}\alpha = \Psi_{\alpha=0}, \quad \Psi + \mathrm{K}(\alpha-\beta) = \Psi_{\alpha=\beta}, \quad \dots.$$

equations involving it must of course lead us back to some form or other of the equation with which we started. Thus, multiplying K by a and using the first of the four derived equations we obtain by subtraction

$$\begin{aligned}0 = (a\beta+\gamma\delta)(\gamma'-\delta')(a'-\beta') \\ - (a\gamma+\beta\delta)(\beta'-\delta')(a'-\gamma') + (a\delta+\beta\gamma)(\beta'-\gamma')(a'-\delta'),\end{aligned}$$

whence we deduce in the same manner as before that the expression* on the right when changed in sign is equal to Ψ; and using any pair of the remaining equations we reach either the form of Ψ previously obtained or one of the two forms derivable from it by means of the simultaneous circular substitutions

$$\beta, \gamma, \delta = \gamma, \delta, \beta,$$
$$\beta', \gamma', \delta' = \gamma', \delta', \beta'.$$

CAYLEY, A. (1858, February).

[A fifth memoir on quantics. *Philos. Transac. R. Soc.* (London) cxlviii. pp. 429–460; or *Collected Math. Papers*, ii. pp. 527–557.]

The second part (§§ 96–114) of the memoir deals with two or more quadrics, and forming part of it is a digression (§§ 105–114) on involution and the anharmonic relation. The determinant Ψ thus again makes its appearance, and associated with it is the determinant

$$\begin{vmatrix} 1 & a+a' & aa' \\ 1 & \beta+\beta' & \beta\beta' \\ 1 & \gamma+\gamma' & \gamma\gamma' \end{vmatrix}, \quad \text{or } \Upsilon \text{ say,}$$

for the reason that, when $\delta=a'$ and $\delta'=a$, Ψ is readily shown to be equal to

$$(a'-a)\Upsilon.$$

* This second form of Ψ may be got directly from the determinant by expanding in terms of the two-line minors formable from the first and third columns, and the minors complementary to these. Of course we also have

$$\Psi = (a'\beta'+\gamma'\delta')(a-\beta)(\gamma-\delta) - (a'\gamma'+\beta'\delta')(a-\gamma)(\beta-\delta) + (a'\delta'+\beta'\gamma')(a-\delta)(\beta-\gamma).$$

To obtain the required non-determinant forms of the two the multiplication-theorem is used with pleasing effect. In the first place Υ is multiplied row-wise by

$$\begin{vmatrix} u^2 & -u & 1 \\ v^2 & -v & 1 \\ w^2 & -w & 1 \end{vmatrix},$$

the result being, of course,

$$\Upsilon \cdot (w-v)(w-u)(v-u) = \begin{vmatrix} (u-a)(u-a') & (v-a)(v-a') & (w-a)(w-a') \\ (u-\beta)(u-\beta') & (v-\beta)(v-\beta') & (w-\beta)(w-\beta') \\ (u-\gamma)(u-\gamma') & (v-\gamma)(v-\gamma') & (w-\gamma)(w-\gamma') \end{vmatrix}.$$

In this Cayley then puts $u=a$, $v=a'$, obtaining

$$\Upsilon \cdot (a'-a) = (a-\beta)(a-\beta')(a'-\gamma)(a'-\gamma') - (a'-\beta)(a'-\beta')(a-\gamma)(a-\gamma'),$$

and putting $u, v, w = a, \beta, \gamma$ obtains

$$\Upsilon = (a-\beta')(\beta-\gamma')(\gamma-a') - (a-\gamma')(\beta-a')(\gamma-\beta'),$$

a result known to Hesse in 1849 (see *Crelle's Journ.* l. p. 265).

In the next place (§ 114) Ψ is multiplied by the similar determinant

$$\begin{vmatrix} ss' & -s' & -s & 1 \\ tt' & -t' & -t & 1 \\ uu' & -u' & -u & 1 \\ vv' & -v' & -v & 1 \end{vmatrix}, \quad \text{or } \Psi' \text{ say,}$$

the result being

$$\Psi\Psi' = \begin{vmatrix} (s-a)(s'-a') & (t-a)(t'-a') & (u-a)(u'-a') & (v-a)(v'-a') \\ (s-\beta)(s'-\beta') & (t-\beta)(t'-\beta') & (u-\beta)(u'-\beta') & (v-\beta)(v'-\beta') \\ (s-\gamma)(s'-\gamma') & (t-\gamma)(t'-\gamma') & (u-\gamma)(u'-\gamma') & (v-\gamma)(v'-\gamma') \\ (s-\delta)(s'-\delta') & (t-\delta)(t'-\delta') & (u-\delta)(u'-\delta') & (v-\delta)(v'-\delta') \end{vmatrix},$$

so that on putting

$$\left.\begin{matrix} s, & t, & u, & v, \\ s', & t', & u', & v', \end{matrix}\right\} = \left\{\begin{matrix} a, & \beta, & \gamma, & \delta \\ \beta', & a', & \delta', & \gamma' \end{matrix}\right.$$

the product becomes

$$\begin{vmatrix} \cdot & \cdot & (\gamma-\alpha)(\delta'-\alpha') & (\delta-\alpha)(\gamma'-\alpha') \\ \cdot & \cdot & (\gamma-\beta)(\delta'-\beta') & (\delta-\beta)(\gamma'-\beta') \\ (\alpha-\gamma)(\beta'-\gamma') & (\beta-\gamma)(\alpha'-\gamma') & \cdot & \cdot \\ (\alpha-\delta)(\beta'-\delta') & (\beta-\delta)(\alpha'-\delta') & \cdot & \cdot \end{vmatrix},$$

and there is obtained

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} \cdot \begin{vmatrix} 1 & \alpha & \beta' & \alpha\beta' \\ 1 & \beta & \alpha' & \beta\alpha' \\ 1 & \gamma & \delta' & \gamma\delta' \\ 1 & \delta & \gamma' & \delta\gamma' \end{vmatrix} = \begin{cases} \quad (\alpha-\gamma)(\beta-\delta)(\alpha'-\delta')(\beta'-\gamma') \\ -(\alpha-\delta)(\beta-\gamma)(\alpha'-\gamma')(\beta'-\delta') \end{cases}$$

From this Cayley concludes (1) that Ψ is not changed by the transposition

$$\begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{pmatrix},$$

and (2) that either form equals

$$(\alpha-\gamma)(\beta-\delta)(\alpha'-\delta')(\beta'-\gamma') - (\alpha-\delta)(\beta-\gamma)(\alpha'-\gamma')(\beta'-\delta').$$

SARDI, C. (1864).

[Quistione 39. *Giornale di Mat.*, p. 256, pp. 315–316.]

On the determinant Ψ Sardi performs the operation which we may indicate by

$$\mathrm{col}_4 - \beta\,\mathrm{col}_3 - \delta'\,\mathrm{col}_2 + \beta\delta'\,\mathrm{col}_1,$$

thus obtaining

$$\begin{vmatrix} 1 & \alpha & \alpha' & (\alpha-\beta)(\alpha'-\delta') \\ 1 & \beta & \beta' & \cdot \\ 1 & \gamma & \gamma' & (\gamma-\beta)(\gamma'-\delta') \\ 1 & \delta & \delta' & \cdot \end{vmatrix},$$

in which the cofactor of $(\alpha-\beta)(\alpha'-\delta')$ is

$$\begin{vmatrix} \beta-\gamma & \beta'-\gamma' \\ \delta-\gamma & \delta'-\gamma' \end{vmatrix}$$

and the cofactor of $(\gamma-\beta)(\gamma'-\delta')$ is

$$\begin{vmatrix} \beta-\alpha & \beta'-\alpha' \\ \delta-\alpha & \delta'-\alpha' \end{vmatrix},$$

where, be it observed, it is the rows 1, β, β' and 1, δ, δ' that are diminished on both occasions. There is thus obtained

$$\begin{aligned}\Psi = \quad & (\alpha-\beta)(\alpha'-\delta')\Big\{(\beta-\gamma)(\delta'-\gamma') - (\delta-\gamma)(\beta'-\gamma')\Big\} \\ & -(\gamma-\beta)(\gamma'-\delta')\Big\{(\beta-\alpha)(\delta'-\alpha') - (\beta'-\alpha')(\delta-\alpha)\Big\}, \\ = \; & (\alpha-\beta)(\alpha'-\delta')(\gamma-\delta)(\beta'-\gamma') + (\gamma-\beta)(\gamma'-\delta')(\beta'-\alpha')(\delta-\alpha), \\ = \; & -(\alpha-\beta)(\gamma-\delta)(\alpha'-\delta')(\gamma'-\beta') + (\alpha'-\beta')(\gamma'-\delta')(\alpha-\delta)(\gamma-\beta).^*\end{aligned}$$

(γ_2) SYLVESTER'S UNISIGNANT.

SYLVESTER, J. J. (1855, April).

[On the change of systems of independent variables. *Quart. Journ. of Math.*, i. pp. 42–56; or *Collected Math. Papers*, ii. pp. 65–85.]

In the course of Sylvester's investigations a peculiar three-line determinant turns up, which he considers deserving of attention on its own account, namely, the determinant

$$\begin{vmatrix} a_1+a_2+a_3 & -a_2 & -a_3 \\ -b_1 & b_1+b_2+b_3 & -b_3 \\ -c_1 & -c_2 & c_1+c_2+c_3 \end{vmatrix},$$

the final expansion of which consists of 16 terms, all positive. To obtain this expansion a "simple rule" is laid down, namely, to substitute

$$\left.\begin{matrix} a & a_b & a_c \\ b_a & b & b_c \\ c_a & c_b & c \end{matrix}\right\} \quad \text{for} \quad \left\{\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right.$$

* It will be seen that merely by accident the three ways in which Ψ can be expressed as the difference of two products have turned up in succession, and that they may be written

$$|PQ'|, |QR'|, |PR'|$$

if we put

$$(\alpha-\beta)(\gamma-\delta) = P,$$
$$(\alpha-\gamma)(\beta-\delta) = Q,$$
$$(\alpha-\delta)(\beta-\gamma) = R.$$

and then multiply together the elements of the diagonal, rejecting every term such as $a_b b_a$, $a_b b_c c_a$, ... in which the letters form a cycle. Two examples are given, but no justification of the "rule" is vouchsafed. The examples are—

$$\begin{vmatrix} a+a_b+a_c & -a_b & -a_c \\ -b_a & b+b_c+b_a & -b_c \\ -c_a & -c_b & c+c_a+c_b \end{vmatrix} = \begin{aligned}[t] & abc+(c_a+c_b)ab+(a_b+a_c)bc+(b_c+b_a)ca \\ & +a(b_a c_a+b_a c_b+c_a b_c) \\ & +b(c_b a_b+c_b a_c+a_b c_a) \\ & +c(a_c b_c+a_c b_a+b_c a_b), \end{aligned}$$

$$\begin{vmatrix} a+a_b+a_c+a_d & -b_a & -c_a & -d_a \\ -a_b & b+b_c+b_d+b_a & -c_b & -d_b \\ -a_c & -b_c & c+c_d+c_a+c_b & -d_c \\ -a_d & -b_d & -c_d & d+d_a+d_b+d_c \end{vmatrix}$$

$$= abcd + \sum abc(d_a+d_b+d_c) + \sum ab(c_d d_a+ \ldots\ldots) + \sum a(b_c c_d d_a+ \ldots\ldots).$$

The arrangement of the two developments almost raises doubts as to whether the "rule" had been utilised, suggesting indeed that in the latter instance, for example, the cofactor of ab was first obtained in the form

$$\begin{vmatrix} c_d+c_a+c_b & -d_c \\ -c_d & d_a+d_b+d_c \end{vmatrix},$$

and the cofactor of a in the form of a similar determinant of the third order. The "rule," however, is noted by Cayley in *Crelle's Journal*, lii. (1855), p. 279.

The number of terms is $(n+1)^{n-1}$, n being the order-number of the determinant. This Sylvester obtains by putting a, a_b, a_c, . . . all equal to 1. It will be observed that from the form of the development we thus have

$$1 + 3\cdot2 + 3\cdot3 = 4^2$$
$$1 + 4\cdot3 + 6\cdot8 + 4\cdot16 = 5^3$$
$$1 + 5\cdot4 + 10\cdot15 + 10\cdot50 + 5\cdot125 = 6^4$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

BORCHARDT, C. W. (1859, May).

[Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante. *Crelle's Journ.*, lvii. pp. 111–121; or *Monatsb. d. Akad. d. Wiss.* (Berlin), pp. 376–388; also abstract in *Annali di Mat.*, ii. pp. 262–264.]

The representation in question is in terms of the values which the two functions $\phi(x)$ and $\psi(x)$, both of the n^{th} degree, assume for the values $a_0, a_1, a_2, \ldots, a_n$ of x. It emerges as a special determinant of the form

$$\begin{vmatrix} \sigma_1-(11) & -(12) & \ldots\ldots & -(1n) \\ -(21) & \sigma_2-(22) & \ldots\ldots & -(2n) \\ \cdot & \cdot & \cdot & \cdot \\ -(n1) & -(n2) & \ldots\ldots & \sigma_n-(nn) \end{vmatrix},$$

where

$$\sigma_r=(r0)+(r1)+\ldots\ldots+(rn) \quad \text{and} \quad (rs)=(sr),$$

a form which we readily recognise to be the axisymmetric case of Sylvester's determinant of the year 1855. To the consideration of it Borchardt, probably supposing it to be new, devotes the last six pages of his paper.

Denoting it by $(0, 1, 2, \ldots, n)$, since it is a function of the $\frac{1}{2}n(n+1)$ quantities,

$$\begin{array}{cccc} (01) & (02) & \ldots\ldots & (0n) \\ & (12) & \ldots\ldots & (1n) \\ & & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ & & & (n-1, n), \end{array}$$

he first shows with some prolixity that the cofactor of (01) in it is $(\overline{0+1}, 2, 3, \ldots, n)$, next that the cofactor of $(01)(02)\ldots(0i)$ is

$$(\overline{0+1+\ldots\ldots+i}, i+1, i+2, \ldots\ldots, n),$$

and finally that

$$\begin{aligned}(0, 1, 2, \ldots, n) = {} & \sum(01)(1, 2, \ldots, n) \\ & + \sum(01)(02)(\overline{1+2}, 3, \ldots, n) \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + \sum(01)(02)\ldots(0k)(\overline{1+2+\ldots+k}, k+1, \ldots, n) \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + (01)(02)\ldots(0n).\end{aligned}$$

Resuming consideration, but proceeding on a different tack, he arrives at Sylvester's "rule," namely, that $(0, 1, 2, \ldots, n)$ is "gleich der Summe aller nicht-cyclischen Producte, die aus je n jener $\frac{1}{2}n(n+1)$ Elemente $(i\,k)$ gebildet werden können." Unlike Sylvester, however, he is careful to give a justification of it based on four observed facts, namely, (1) that $(0, 1, 2, \ldots, n)$ is unaltered by interchanging any two of the umbræ; (2) that the coefficient of the term $(01)(02)\ldots(0n)$ is 1; (3) that none of the terms is free of the umbra 0; (4) that, as already mentioned, the cofactor of (01) is $(\overline{0+1}, 2, \ldots, n)$.* As the proof, which extends to two pages (pp. 119–120), applies only to the case of axisymmetry, it need not be given.

Lastly, the number of terms in the development of $(0, 1, 2, \ldots, n)$ is investigated, the result obtained agreeing with Sylvester's.

We may note for ourselves in passing that the first three of the basic facts of the proof are, like the last, most readily appreciated by observing the determinant form, the case where $n=3$, namely,

$$\begin{vmatrix} 10+12+13 & -12 & -13 \\ -21 & 20+21+23 & -23 \\ -31 & -32 & 30+31+32 \end{vmatrix}$$

being amply sufficient. Thus, increasing any column by all the others, and thereafter increasing the corresponding row by all the other rows, we obtain the first result, learning at the same time that it only holds when axisymmetry exists; the second is self-evident; and the third follows from the fact that the aggregate of the terms which are free of 0, being got by deleting 10, 20, 30, is expressible as a vanishing determinant.

(δ) MISCELLANEOUS SPECIAL FORMS.

CAYLEY, A. (1845).

[On certain results relating to quaternions. *Philos. Magazine*, xxvi. pp. 141–145; or *Collected Math. Papers*, i. pp. 123–126.]

Assuming that in each term of the development of a deter-

* As (01) occurs only in the element $\sigma_1 - (11)$, its cofactor is the primary minor obtained by deleting the first row and the first column, and this is seen to be $(\overline{0+1}, 2, \ldots, n)$ by definition.

minant the elements are arranged in the order of the columns from which they are taken, Cayley points out that if the elements be quaternions

$$\begin{vmatrix} \pi & \pi' \\ \pi & \pi' \end{vmatrix} = \pi\pi' - \pi\pi' = 0,$$

but

$$\begin{vmatrix} \pi & \pi \\ \pi' & \pi' \end{vmatrix} = \pi\pi' - \pi'\pi \neq 0.$$

He is thus led to inquire what the non-zero value is in this latter case and in other similar cases. Taking

$$\begin{aligned} \pi &= x + iy + jz + kw, \\ \pi' &= x' + iy' + jz' + kw', \\ \pi'' &= x'' + iy'' + jz'' + kw'', \end{aligned}$$

he says it is easy to show * that

$$\begin{vmatrix} \pi & \pi \\ \pi' & \pi' \end{vmatrix} = -2 \begin{vmatrix} i & j & k \\ y & z & w \\ y' & z' & w' \end{vmatrix},$$

$$\begin{vmatrix} \pi & \pi & \pi \\ \pi' & \pi' & \pi' \\ \pi'' & \pi'' & \pi'' \end{vmatrix} = -2 \begin{vmatrix} 3 & i & j & k \\ x & y & z & w \\ x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \end{vmatrix},$$

---

* Probably the easiest way is to express the determinant as a sum of determinants with monomial elements. In the case of the third order the number of such determinants is 64, of which 40 vanish, the sum remaining being

$$\begin{aligned} &123 + 132 + 213 + 231 + 312 + 321 \\ &+ 124 + 142 + \ldots \\ &+ 134 + 143 + \ldots \\ &+ 234 + 243 + \ldots \end{aligned}$$

where rst stands for the determinant whose columns are in order the r^{th}, s^{th}, t^{th} columns of the array

$$\begin{matrix} x & iy & jz & kw \\ x' & iy' & jz' & kw' \\ x'' & iy'' & jz'' & kw'' \end{matrix}$$

and where therefore

$$123 + 132 + \ldots = |xy'z''| \cdot (ij + ij + ij - ij - ij + ij) = 2k\,|xy'z''|,$$

and so on. The multiplication table of i, j, k, it may be recalled, is

$$\begin{vmatrix} ii & ij & ik \\ ji & jj & jk \\ ki & kj & kk \end{vmatrix} = \begin{vmatrix} -1 & k & -j \\ -k & -1 & i \\ j & -i & -1 \end{vmatrix}.$$

but that for higher orders the result is 0. He next notes the identity

$$\begin{vmatrix} \phi & \chi \\ \phi' & \chi' \end{vmatrix} + \begin{vmatrix} \chi & \phi \\ \chi' & \phi' \end{vmatrix} = \begin{vmatrix} \phi & \phi \\ \chi' & \chi' \end{vmatrix} - \begin{vmatrix} \phi' & \phi' \\ \chi & \chi \end{vmatrix},$$

adding "etc. for determinants of any order";* and then from this set of identities and the previous set he concludes that *if any four adjacent columns of a quaternion determinant be transposed in every possible manner, the sum of the determinants thus obtained vanishes*—a property which, he says, is much less simple than the analogous one for the rows, this last being the same that holds in the case of determinants with ordinary elements. Lastly, he gives the important warning that the eliminant of

$$\left.\begin{array}{l} \pi\Pi + \phi\Phi = 0 \\ \pi'\Pi + \phi'\Phi = 0 \end{array}\right\}$$

is neither $\pi\phi' - \pi'\phi$ nor $\pi\phi' - \phi\pi'$, but

$$\pi^{-1}\phi - \pi'^{-1}\phi'.$$

TISSOT, A. (1852, May).

[Sur un déterminant d'intégrales définies. *Journ. (de Liouville) de Math.*, xvii. pp. 177–185.]

The subject here is the evaluation of the determinant of the $(n+1)^{\text{th}}$ order whose $(r, s)^{\text{th}}$ element is

$$\int_{a_{s-1}}^{a_s} e^{-x} \frac{x^r dx}{(\phi)_{s-1}(x)},$$

* Very probably the next case is the identity

$$\begin{vmatrix} \phi & \chi & \psi \\ \phi' & \chi' & \psi' \\ \phi'' & \chi'' & \psi'' \end{vmatrix} + \begin{vmatrix} \phi & \psi & \chi \\ \phi' & \psi' & \chi' \\ \phi'' & \psi'' & \chi'' \end{vmatrix} + \begin{vmatrix} \chi & \phi & \psi \\ \chi' & \phi' & \psi' \\ \chi'' & \phi'' & \psi'' \end{vmatrix} + \dots + \begin{vmatrix} \psi & \chi & \phi \\ \psi' & \chi' & \phi' \\ \psi'' & \chi'' & \phi'' \end{vmatrix}$$

$$= \begin{vmatrix} \phi & \phi & \phi \\ \chi' & \chi' & \chi' \\ \psi'' & \psi'' & \psi'' \end{vmatrix} - \begin{vmatrix} \phi & \phi & \phi \\ \chi'' & \chi'' & \chi'' \\ \psi' & \psi' & \psi' \end{vmatrix} - \begin{vmatrix} \phi' & \phi' & \phi' \\ \chi & \chi & \chi \\ \psi'' & \psi'' & \psi'' \end{vmatrix} + \dots - \begin{vmatrix} \phi'' & \phi'' & \phi'' \\ \chi' & \chi' & \chi' \\ \psi & \psi & \psi \end{vmatrix},$$

where, as in the other cases, the r^{th} determinant on the left is equal to the aggregate of the r^{th} terms of all the determinants on the right.

where

$$\phi_i(x) = (x-a_0)^{m_0}(x-a_1)^{m} \ldots . (x-a_i)^{m_i}(a_{i+1}-x)^{m_{i+1}} \ldots . (a_n-x)^{m_n},$$

the m's are all less than 1, and $a_{n+1}=\infty$. The simplest example is

$$\begin{vmatrix} \int_{a_0}^{a_1} e^{-x}\frac{dx}{\phi_0(x)} & \int_{a_1}^{\infty} e^{-x}\frac{dx}{\phi_1(x)} \\ \int_{a_0}^{a_1} e^{-x}\frac{x\,dx}{\phi_0(x)} & \int_{a_1}^{\infty} e^{-x}\frac{x\,dx}{\phi_1(x)} \end{vmatrix} = \Gamma(1-m_0)\cdot\Gamma(1-m_1)\cdot(a_1-a_0)^{1-m_0-m_1}\cdot e^{-a_0-a_1},$$

In establishing the result, use is made of the fact that the determinant is expressible also as a multiple integral: for example, the two-line determinant just written is equal to

$$\int_{a_0}^{a_1}\int_{a_1}^{\infty}\frac{e^{-x-x_1}(x_1-x)\,dx\,dx_1}{(a-x_0)^{m_0}(a_1-x)^{m_1}(x_1-a_0)^{m_0}(x_1-a_1)^{m_1}}.$$

BAZIN, [H.] (1854, July).

[Démonstration d'un théorème sur les déterminants. *Journ.* (*de Liouville*) *de Math.*, xix. pp. 209–214.]

The theorem in question is to the effect that if there be two n-by-m arrays R, R′ with integral elements, and such that the ratio of any n-line minor of R to the corresponding minor of R′ is constant and integral, and if the n-line minors of R have 1 for their highest common factor, then it is possible to find a determinant S of the n^{th} order with integral elements so that the product of S by any n-line minor of R′ shall equal the corresponding minor of R. For example, it being given that

$$k\left\|\begin{matrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\| = \left\|\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right\|$$

where all the letters denote integers, and that the highest common factor of $|b_1c_2|$, $|b_1c_3|$, $|b_2c_3|$ is 1, four integers α, β, γ, δ can be found such that

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \cdot \left\|\begin{matrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\| = \left\|\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right\|.$$

BRIOSCHI, F. (1855).

[Additions à l'article No. 15, page 239 de ce tome. *Crelle's Journ.*, L. pp. 318–321 ; or *Opere mat.*, v. pp. 271–276.]

The determinant here (pp. 320–321) dealt with is, for shortness' sake, taken to be of the 4th order, namely, $|m_1\ \delta_2\ b_3\ c_4|$, in which $\delta_1, \delta_2, \delta_3, \delta_4$ stand for

$$\begin{aligned} &ax_1 + ex_2 + fx_3 + gx_4, \\ &ex_1 + bx_2 + hx_3 + kx_4, \\ &fx_1 + hx_2 + cx_3 + lx_4, \\ &gx_1 + kx_2 + lx_3 + dx_4, \end{aligned}$$

and where the δ's, b's, c's are such that

$$\begin{aligned} \delta_1x_1 + \delta_2x_2 + \delta_3x_3 + \delta_4x_4 &= 0, \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 &= 0, \\ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 &= 0. \end{aligned}$$

The cofactor of m_r in $|m_1\ \delta_2\ b_3\ c_4|$ being denoted by M_r, we see that, as an example,

$$M_4{}^2 = -\begin{vmatrix} a & e & f & \delta_1 & b_1 & c_1 \\ e & b & h & \delta_2 & b_2 & c_2 \\ f & h & c & \delta_3 & b_3 & c_3 \\ \delta_1 & \delta_2 & \delta_3 & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & \cdot & \cdot & \cdot \end{vmatrix}$$

which after performance of the operations

$$\begin{aligned} &\mathrm{col}_4 - x_1\mathrm{col}_1 - x_2\mathrm{col}_2 - x_3\mathrm{col}_3, \\ &\mathrm{row}_4 - x_1\mathrm{row}_1 - x_2\mathrm{row}_2 - x_3\mathrm{row}_3, \end{aligned}$$

becomes

$$M_4{}^2 = -\begin{vmatrix} a & e & f & g & b_1 & c_1 \\ e & b & h & k & b_2 & c_2 \\ f & h & c & l & b_3 & c_3 \\ g & k & l & d & b_4 & c_4 \\ b_1 & b_2 & b_3 & b_4 & \cdot & \cdot \\ c_1 & c_2 & c_3 & c_4 & \cdot & \cdot \end{vmatrix} x_4{}^2, \quad \text{or say } -x_4{}^2\Delta.$$

As it can be shown similarly that

$$M_3^2 = -x_3^2\Delta, \quad M_2^2 = -x_2^2\Delta, \quad M_1^2 = -x_1^2\Delta,$$

Brioschi obtains *

$$|m_1\delta_2b_3c_4| = (m_1x_1+m_2x_2+m_3x_3+m_4x_4)\sqrt{-\Delta},$$

nothing being said as to the sign to be taken in extracting the square root of x_r^2.

We have only to add for ourselves that the first of the conditioning equations is the vanishing of the quaternary quadric

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline a & e & f & g & x_1 \\ e & b & h & k & x_2 \\ f & h & c & l & x_3 \\ g & k & l & d & x_4 \end{array},$$

and that the δ's are the halved differential quotients of this with respect to x_1, x_2, x_3, x_4.

HERMITE, C. (1855, January).

[Sur la théorie de la transformation des fonctions abéliennes. *Comptes rendus .... Acad. des Sci.* (Paris), xl. pp. 249–254: or *Œuvres*, i. pp. 444–478.]

The special determinant here considered, as being auxiliary to Hermite's main purpose, is $|a_1b_2c_3d_4|$ with its elements subject to the conditions

$$\left.\begin{array}{l} |a_1d_2| + |b_1c_2| = 0 = |a_1d_3| + |b_1c_3| \\ |a_1d_4| + |b_1c_4| = k = |a_2d_3| + |b_2c_3| \\ |a_2d_4| + |b_2c_4| = 0 = |a_3d_4| + |b_3c_4| \end{array}\right\},$$

and the results in regard to it are:—(1) that it is equal to k^2; (2) that the row-by-row product of two such determinants is a determinant of the same type. No proof is given, but from the way in which Hermite writes the conditions, it would appear

* The minus sign is omitted by him throughout. If the number of x's had been odd, the sign would have been +.

that the first was obtained by multiplying the given determinant columnwise by itself in the form

$$\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ -b_1 & -b_2 & -b_3 & -b_4 \\ -a_1 & -a_2 & -a_3 & -a_4 \end{vmatrix}.$$

A generalisation by Brioschi (1855) has already been dealt with under Skew Determinants.

ZEHFUSS, G. (1858).

[Uebungsaufgaben für Schüler. *Archiv d. Math. u. Phys.*, xxxi. p. 246; or *Nouv. Annales de Math.*, xviii. p. 171; (2) ii. pp. 60–61.]

The proposition offered for proof by Zehfuss is in modern phraseology to the effect that the determinant of the difference of the two square matrices

$$\begin{matrix} a_1 & a_1 & \ldots. & a_1 \\ a_2 & a_2 & \ldots. & a_2 \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ a_n & a_n & \ldots. & a_n, \end{matrix} \qquad \begin{matrix} b_1 & b_2 & \ldots. & b_n \\ b_1 & b_2 & \ldots. & b_n \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ b_1 & b_2 & \ldots. & b_n \end{matrix}$$

vanishes for all orders higher than the second. The proof given by Gustave Harang in the *Nouvelles Annales* rests on the operations

$$\mathrm{col}_1 - \mathrm{col}_2, \quad \mathrm{col}_2 - \mathrm{col}_3, \quad \ldots..$$

When $n=2$ we have

$$\begin{vmatrix} a_1 - b_1 & a_1 - b_2 \\ a_2 - b_1 & a_2 - b_2 \end{vmatrix} = (a_1 - a_2)(b_1 - b_2).$$

CAYLEY, A. (1859, March).

[On the double tangents of a plane curve. *Philos. Transac. R. Soc.* (London), cxlix. pp. 193–212; or *Collected Math. Papers*, iv. pp. 186–206.]

The theorem on which an important part of this investigation rests is enunciated by its author as follows: *If the* 2n − 1 *columns* *of the special three-row matrix*

$$\begin{array}{cccccc|cccc} a_0 & a_1 & a_2 & \ldots & a_{n-1}; & & a'_0 & a'_1 & \ldots & a'_{n-2} \\ a_1 & a_2 & a_3 & \ldots & a_n & : & a'_1 & a'_2 & \ldots & a'_{n-1} \\ a'_0 & a'_1 & a'_2 & \ldots & a'_{n-1}; & & a''_0 & a''_1 & \ldots & a''_{n-2} \end{array}$$

be represented by

$$1 \quad 2 \quad 3 \quad \ldots \quad n\ ; \qquad (1)\,(2)\ \ldots\ (n-1)$$

respectively: the determinant whose columns are those thus represented by r, s, (t) *be denoted by* {r, s, (t)}; *and the determinant aggregates*

$$\begin{gathered} \{n, n-1, (2)\} + \{n, n-2, (3)\} + \ldots + \{n, 2, (n-1)\}, \\ -\{n, n-1, (1)\} - \{n, n-2, (2)\} - \ldots - \{n, 2, (n-2)\} - \{n, 1, (n-1)\}, \\ -\{1, 2, (n-1)\} - \{1, 3, (n-2)\} - \ldots - \{1, n-1, (2)\} - \{1, n, (1)\}, \\ \{1, 2, (n-2)\} + \{1, 3, (n-3)\} + \ldots + \{1, n-1, (1)\}, \end{gathered}$$

by I, II, III, IV; *then*

$$a_0\mathrm{I} + a_1\mathrm{II} + a_{n-1}\mathrm{III} + a_n\mathrm{IV} = 0.$$

The mode of verification suggested consists in showing that there exist six quantities (12), (13), (14), (23), (24), (34), say, such that

$$\begin{aligned} \mathrm{I} &= \quad a_0\cdot 0 + a_1(12) + a_{n-1}(13) + a_n(14), \\ \mathrm{II} &= -a_0(12) + a_1\cdot 0 + a_{n-1}(23) + a_n(24), \\ \mathrm{III} &= -a_0(13) - a_1(23) + a_{n-1}\cdot 0 + a_n(34), \\ \mathrm{IV} &= -a_0(14) - a_1(24) - a_{n-1}(34) + a_n\cdot 0; \end{aligned}$$

and then taking the sum of the requisite multiples. The six quantities in question are actually found for the cases where $n=3, 4, 6$. In the last case, the matrix being

$$\begin{array}{cccccc|ccccc} a & b & c & d & e & f & a' & b' & c' & d' & e' \\ b & c & d & e & f & g & b' & c' & d' & e' & f' \\ a' & b' & c' & d' & e' & f' & a'' & b'' & c'' & d'' & e'', \end{array}$$

their values are written by Cayley in the form

$$\begin{array}{lll}
(12) = -e''g & & +f'f', \\
(13) = b''f+c''e+d''d+e''c & & -b'f'-c'e'-d'd'-e'c'-f'b', \\
(14) = -b''e-c''d-d''c & & +b'e'+c'd'+d'c'+e'b', \\
(23) = -a''f-b''e-c''d-d''c-e''b & & +a'f'+b'e'+c'd'+d'c'+e'b'+f'a' \\
(24) = a''e+b''d+c''c+d''b & & -a'e'-b'd'-c'c'-d'b'-e'a', \\
(34) = -a''a & & +a'a'.
\end{array}$$

The final lemma used in the verification may be formulated thus:

If from the n *quantities* $x_1, x_2, \ldots, x_n$ *and the* $\frac{1}{2}n(n-1)$ *others*

$$\begin{array}{llll}
12, & 13, & \ldots, & 1n \\
 & 23, & \ldots, & 2n \\
 & & \cdot\ \cdot\ \cdot\ \cdot & \\
 & & & nn
\end{array}$$

there be formed n *lineo-linear functions of the two sets, namely,*

$$\left.\begin{array}{l}
f_1 = \quad x_1\cdot 0 \quad + x_2(12) + x_3(13) + \ldots + x_n(1n) \\
f_2 = -x_1\cdot(12) + x_2\cdot 0 \quad + x_3(23) + \ldots + x_n(2n) \\
f_3 = -x_1(13) - x_2(23) + x_3\cdot 0 \quad + \ldots + x_n(3n) \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
f_n = -x_1(1n) - x_2(2n) - x_3(3n) - \ldots + x_n\cdot 0
\end{array}\right\},$$

then

$$x_1f_1+x_2f_2+\ldots+x_nf_n=0.*$$

It may be viewed as included in the identity

$$\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & \ldots & x_n & \\
\hline
. & 12 & 13 & \ldots & 1n & x_1 \\
-12 & . & 23 & \ldots & 2n & x_2 \\
-13 & -23 & . & \ldots & 3n & x_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
-1n & -2n & -3n & \ldots & . & x_n
\end{array} = 0;$$

or in the statement that *Any quadric whose discriminant is a zero-axial skew determinant vanishes identically.*

---

* When the coefficient of x_r in f_r is not 0 but (rr), the result of course is

$$x_1f_1+x_2f_2+\ldots+x_nf_n = x_1^2(11)+x_2^2(22)+\ldots x_n^2(nn);$$

and in this connection it may be well to recall a step in Hermite's mode of effecting the automorphic transformation of a quadric (See under *Orthogonants*).

HIRST, T. A. (1859)

[Question 489. (A determinant which vanishes for every order higher than the fourth.) *Nouv. Annales de Math.*, xviii. p. 358; (2) ix. pp. 561–563.]

Hirst's theorem is that *if*

$$a_{rs} = (\alpha_r + \beta_r s)\cos s\phi + (\gamma_r + \delta_r s)\sin s\phi$$

then the determinant

$$\begin{vmatrix} a_{1,s} & a_{1,s+1} & \dots\dots & a_{1,s+n-1} \\ a_{2,s} & a_{2,s+1} & \dots\dots & a_{2,s+n-1} \\ \cdot & \cdot & \cdot\cdot\cdot\cdot & \cdot \\ a_{n,s} & a_{n,s+1} & \dots\dots & a_{n,s+n-1} \end{vmatrix}$$

vanishes when $n > 4$, *and has a non-zero value independent of* s *when* n = 4; and the real significance of it is best grasped by noting—as is not done in the *Annales*—that the determinant is the product

$$\begin{Vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_n & \beta_n & \gamma_n & \delta_n \end{Vmatrix}$$

$$\cdot \begin{Vmatrix} \cos s\phi & \cos(s+1)\phi & \dots\dots & \cos(s+n-1)\phi \\ s\cos s\phi & (s+1)\cos(s+1)\phi & \dots\dots & (s+n-1)\cos(s+n-1)\phi \\ \sin s\phi & \sin(s+1)\phi & \dots\dots & \sin(s+n-1)\phi \\ s\sin s\phi & (s+1)\sin(s+1)\phi & \dots\dots & (s+n-1)\sin(s+n-1)\phi \end{Vmatrix}.$$

The vanishing of it when $n > 4$ is then self-evident, and its value when $n = 4$ being

$$|\alpha_1\beta_2\gamma_3\delta_4| \cdot \begin{vmatrix} \cos s\phi & s\cos s\phi & \sin s\phi & s\sin s\phi \\ \cos(s+1)\phi & (s+1)\cos(s+1)\phi & \sin(s+1)\phi & (s+1)\sin(s+1)\phi \\ \cos(s+2)\phi & (s+2)\cos(s+2)\phi & \sin(s+2)\phi & (s+2)\sin(s+2)\phi \\ \cos(s+3)\phi & (s+3)\cos(s+3)\phi & \sin(s+3)\phi & (s+3)\sin(s+3)\phi \end{vmatrix},$$

we have only to show that the second determinant here is independent of s. The solver (Lucien Bignon) does this by multiplying the determinant by itself in the form

$$\begin{vmatrix} s\cos s\phi & -\cos s\phi & s\sin s\phi & -\sin s\phi \\ (s+1)\cos(s+1)\phi & -\cos(s+1)\phi & (s+1)\sin(s+1)\phi & -\sin(s+1)\phi \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{vmatrix},$$

and so finding for its square a determinant whose every element is independent of s, the element in the place i,j being in fact

$$(j-i)\cos(j-i)\phi.$$

He does not note, however, that such a determinant is zero-axial and skew, and that its value is thus readily seen, by a theorem of Cayley's, to be

$$(\cos^2\phi - 4\cos^2 2\phi + 3\cos\phi\cos 3\phi)^2,$$

i.e.

$$(-4\sin^4\phi)^2.$$

CAYLEY, A. (1859).

[Note on the value of certain determinants, the terms of which are the squared distances of points in a plane or in space. *Quart. Journ. of Math.*, iii. pp. 275–277; or *Collected Math. Papers*, iv. pp. 460–462.]

The five results given in the paper are more important than the title would imply, being true when instead of Cayley's elements $\overline{12}^2$, $\overline{13}^2$, . . . we write any elements whatever, namely, 12, 13, . . . This change being made, the fourth and fifth are

$$\begin{vmatrix} \cdot & 12 & 13 & 14 \\ 21 & \cdot & 23 & 24 \\ 31 & 32 & \cdot & 33 \\ 41 & 42 & 43 & \cdot \end{vmatrix} = \sum 12\ 21\ .\ 34\ 43 - \sum 12\ 23\ 24\ 41,$$

$$\begin{vmatrix} \cdot & 12 & 13 & 14 & 15 \\ 21 & \cdot & 23 & 24 & 25 \\ 31 & 32 & \cdot & 34 & 35 \\ 41 & 42 & 43 & \cdot & 45 \\ 51 & 52 & 53 & 54 & \cdot \end{vmatrix} = \left\{ \begin{array}{l} \ \ \sum 12\ 23\ 34\ 45\ 51 \\ -\sum 12\ 23\ 31\ .\ 45\ 54, \end{array} \right.$$

where the Σ's cover 3, 6, 24, 20 terms respectively. No commentary is added, nor any indication of a law including both results. The three of the other set are less general, namely,

$$\begin{vmatrix} . & 12 & 12 & 1 \\ 21 & . & 23 & 1 \\ 31 & 32 & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix} = \sum 12\,21 - \sum 12\,23,$$

where the Σ's cover 3 and 6 terms respectively;

$$\begin{vmatrix} . & 12 & 13 & 14 & 1 \\ 21 & . & 23 & 24 & 1 \\ 31 & 32 & . & 34 & 1 \\ 41 & 42 & 43 & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix} = \sum 12\,23\,34 - \sum 12\,34\,43 - \sum 12\,23\,31,$$

where the Σ's cover 24, 12, 8, terms respectively;

$$\begin{vmatrix} . & 12 & 13 & 14 & 15 & 1 \\ 21 & . & 23 & 24 & 25 & 1 \\ 31 & 32 & . & 34 & 35 & 1 \\ 41 & 42 & 43 & . & 45 & 1 \\ 51 & 52 & 53 & 54 & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . \end{vmatrix} = \begin{cases} -\sum 12\,23\,34\,45 & -\sum 12\,21\,.\,34\,43 \\ +\sum 12\,23\,.\,45\,54 & +\sum 12\,23\,34\,41 \\ +\sum 12\,23\,31\,.\,45, & \end{cases}$$

where the Σ's cover 120, 15, 60, 30, 40 terms respectively. Here again no generalisation is attempted.

(ϵ) CENSUS OF TERMS IN SPECIAL DETERMINANTS.

The first instance of the finding of the number of terms in the final development of a determinant of special form has already been drawn attention to, the investigator being Scherk, and the date 1825.

During the period now occupying us, the earliest suggestion on the subject occurs in 1844 in *Crelle's Journ.*, xxviii. pp. 191–192, the determinant being of the 8th order, and the specialisation consisting in having a zero in the places

$$\begin{array}{c} 16,\ 17,\ 18,\ 27,\ 28,\ 31,\ 38,\ 41,\ 42 \\ 83,\ 82,\ 81,\ 72,\ 71,\ 68,\ 61,\ 58,\ 57. \end{array}$$

The proposer of the problem, so far as it appears, received no satisfaction. The next instance occurred to Sylvester, who in 1855 having hit upon a peculiar determinant whose terms were all positive, ascertained the number of them by evaluating a special circulant. (See *Quart. Journ. of Math.*, i. pp. 42–56, or our notice of it given above, pp. 406–407.) The third instance, like the first, arose as a problem and remained long unsolved. It appeared in 1858 in the *Nouv. Annales de Math.*, xvii. p. 262, under the heading "Question 445," the requirement being to find the number of terms remaining in the case of a determinant of the n^{th} order when all those terms have been deleted which contain two or more diagonal elements. The fourth instance —which is more closely connected with the third than might at first appear—was the actual but incidental determination by Cayley in 1859 of the number of terms in a zero-axial determinant whose order is not greater than the 7^{th}. The numbers found were 9, 44, 265, for the 4^{th}, 5^{th}, 6^{th} orders respectively. (See *Quart. Journ. of Math.*, iii. pp. 275–277, or our notice of it given above, pp. 469–470.)

LIST OF AUTHORS WHOSE WRITINGS ARE REPORTED ON.

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