Automatica 110 (2019) 108597

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Controllability of Kronecker product networks[☆]

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ARTICLE INFO

Article history:
Received 21 March 2018
Received in revised form 2 July 2019
Accepted 6 September 2019
Available online 25 September 2019

Keywords:
Network controllability
Network observability
Kronecker product graph
Multi-agent systems

ABSTRACT

A necessary and sufficient condition is derived for the controllability of Kronecker product networks, where the factor networks are general directed graphs. The condition explicitly illustrates how the controllability of the factor networks affects the controllability of the composite network. For the special case where at least one factor network is diagonalizable, an easily-verifiable condition is explicitly expressed. Furthermore, the controllability of higher-dimensional multi-agent systems is revisited, revealing that some controllability criterion reported in the literature does not hold. Consequently, a modified necessary and sufficient condition is established. The effectiveness of the new conditions is demonstrated through several examples.

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1. Introduction

Controllability is a fundamental issue to be addressed before considering how to control a dynamical system in applications (Chen, 2017). This subject has been extensively studied over more than half a century with various criteria developed, such as the PBH test, Kalman and other kinds of rank conditions, substantial graphic properties, and so on (Hautus, 1969; Kalman, 1962; Trentelman, Stoorvogel, & Hautus, 2012).

For most large-scale networked systems, these criteria cannot be applied practically because of their complex structures and heavy computational burdens. Therefore, in recent years, the notion of network controllability has received compelling attention with some efficient criteria established (Aguilar & Gharesifard, 2017; Hao, Duan, & Chen, 2018; Hao, Duan, Chen and Wu, 2019; Hsu, 2017; Liu, Slotine, & Barabasi, 2011; Nabi-Abdolyousefi & Mesbahi, 2013; Notarstefano & Parlangeli, 2013; Parlangeli & Notarstefano, 2012; Sun, Hu, & Xie, 2017; Wang, Chen, Wang, & Tang, 2016; Xue & Roy, 2017, 2018a, 2018b; Yuan, Zhao, Di, Wang, & Lai, 2013; Zhang, Cao, & Camlibel, 2014; Zhang & Zhou, 2017; Zhou, 2015). In Yuan et al. (2013), an exact controllability framework was introduced to identify the minimum set of input nodes for a general network with an arbitrary link-weight distribution. The controllability of networks with specific topologies, such as path graphs, cycle graphs, circulant graphs, multi-chain graphs and grid graphs, was explored in Hsu (2017), Nabi-Abdolyousefi and Mesbahi (2013), Notarstefano and Parlangeli (2013) and Parlangeli and Notarstefano (2012). Network controllability was investigated from a graph-theoretic perspective in Aguilar and Gharesifard (2017), Rahmani, Ji, Mesbahi, and Egerstedt (2009), Sun et al. (2017), and Zhang et al. (2014). It is noted that most of the above results are derived for the networks with one-dimensional nodes. Recently, the controllability of networks with higher-dimensional nodes has attracted a great deal of interest (Hao et al., 2018; Hao, Duan et al., 2019; Wang et al., 2016; Xue & Roy, 2017, 2018a, 2018b; Zhang et al., 2014; Zhang & Zhou, 2017; Zhou, 2015). Some controllability conditions for networked LTI systems were presented in Zhang and Zhou (2017) and Zhou (2015), which depend on the transmission zeros of every subsystem and the connection matrix. The controllability of diffusively coupled LTI systems was studied in Xue and Roy (2018a) and Zhang et al. (2014). In Wang et al. (2016), the controllability condition for networked MIMO systems was established in terms of two algebraic matrix equations. Moreover, some easily-verifiable conditions were proposed in Hao et al. (2018), Hao, Duan et al. (2019) and Xue and Roy (2017) for the controllability of networked LTI systems.

☆ This work is supported by the National Natural Science Foundation (NNSF) of China under Grants 11802006, 11932003, 11572015, U1713223, 61673026, the Beijing Natural Science Foundation, China under Grant 1194024, the Fundamental Research Funds for the Central Universities, China under Grant YWF-19-BJ-J-275, and the Hong Kong Research Grants Council under the GRF, Hong Kong Grant CityU 11200317. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Shreyas Sundaram under the direction of Editor Christos G. Cassandras.

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https://doi.org/10.1016/j.automatica.2019.108597
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Besides isolated networks, composite networks have come into play due to their broad applications in different areas of engineering (Hammack, Imrich, & Klavzar, 2011; Loan, 2000). There are many kinds of composite networks, such as Cartesian product networks, Kronecker product networks, strong product networks, lexicographic product networks and so on. In addition to their importance of constructing 'larger' networks out of 'small' ones, they are useful in the sense that one can get insights about the properties of composite networks from the factor networks. Intuitively, the controllability of a composite network might be verified by checking some properties of the factor networks. The controllability and observability of Cartesian product networks were investigated in Chapman, Nabi-Abdolyousefi, and Mesbahi (2014). In Notarstefano and Parlangeli (2013), the controllability and observability of linear dynamical systems whose dynamics are induced by the Laplacian of a grid graph were studied. Note that many real-world networks are similar to stochastic Kronecker product graphs. For example, the Kronecker power of a simple generating matrix can yield a Kronecker product graph that fits the Internet (at the autonomous systems level) fairly well (Leskovec, 2010). Large online social networks, web and blog graphs, peer-to-peer networks, etc., can also be modeled by Kronecker product networks. Moreover, every non-trivial graph has a prime factorization over the Kronecker product (Hammack et al., 2011; Loan, 2000). It has lower computational complexity to check the controllability of a large-scale network by examining some properties of the smaller factor networks. Therefore, the controllability analysis for Kronecker product networks has brought about renewed interest recently (Asavathiratham, Roy, Lesieutre, & Verghese, 2001; Chapman & Mesbahi, 2014; Horn & Johnson, 1991; Xue & Roy, 2012). The eigenanalysis for the Kronecker product of two matrices was presented in Horn and Johnson (1991), which shows that the Kronecker products of the factor matrices' eigenvectors are the eigenvectors of the composite matrix. However, not all the eigenvectors of the composite matrix are formulated therein. In Xue and Roy (2012), the Kronecker product of defective matrices was revisited, characterizing the number of the eigenvectors. Nevertheless, the explicit expressions of the eigenvectors, which are the cornerstone of controllability analysis, are not presented. Recently, a sufficient condition was established in Chapman and Mesbahi (2014) for the controllability of Kronecker product networks, where the topology matrices are required to be diagonalizable.

In this paper, the controllability of Kronecker product networks is revisited. The contribution of the paper is four-fold. First, the factor networks considered here are general, directed and weighted. Differing from the condition given in Chapman and Mesbahi (2014), which requires the topology matrix of the composite network to be diagonalizable, this paper removes the diagonalizability requirement. Second, a new necessary and sufficient condition on the controllability of Kronecker product networks is provided in terms of eigenvectors. Compared with the classical PBH test, the new condition typically has a much lower computational cost. Third, for the special case where at least one factor network is diagonalizable, a specified condition is explicitly expressed, which is easier to verify. Finally, this paper shows that the sufficiency of the controllability criterion given in Cai and Zhang (2010) does not hold, thereby a modified necessary and sufficient condition is derived.

The remainder of this paper is organized as follows. Some notations and preliminaries are given in Section 2. The model is formulated in Section 3. Some conditions on the controllability of Kronecker product networks are developed in Section 4. The controllability of higher-dimensional multi-agent systems is reinvestigated in Section 5. Finally, conclusions are drawn in Section 6.

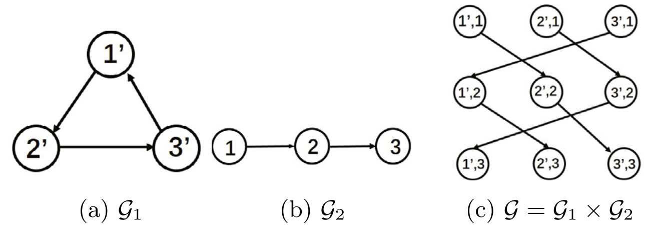


Fig. 1. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and their Kronecker product $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$.

2. Notations and preliminaries

In this section, notations and useful preliminaries are introduced.

2.1. Notations

The notations mostly follow Hao et al. (2018). Let e_i be the row vector with all zero entries except for $[e_i]_i = 1$. The linear span of row vectors $v_1, \ldots, v_k$ is a set of all the linear combinations of these vectors, i.e., $span\{v_1, v_2, \ldots, v_k\} = \left\{\sum_{i=1}^{k} c_i v_i | c_i \in \mathbb{R}\right\}$. Let $U_1 \oplus U_2$ be the direct sum of two spaces U_1 and U_2. Matrices, if their dimensions are not explicitly indicated, are assumed to be compatible for algebraic operations.

A weighted digraph $\mathcal{G} = (V, E, W)$ is characterized by a node set V with cardinality n, an edge set E comprised of ordered pairs of nodes with cardinality m, and a weight set W with cardinality m, where an edge exists from nodes i to j if $(i, j) \in E$ with edge weight $w_{ji} \in W$. $\mathcal{A}(\mathcal{G}) = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the adjacency matrix of $\mathcal{G}$ with $a_{ij} = w_{ij}$ if $(j, i) \in E$ and $a_{ij} = 0$ otherwise. The ith diagonal term of the adjacency matrix denotes the weight of the self-loop on node i.

2.2. Kronecker product network

The Kronecker product network is a kind of composite network that can be obtained by applying Kronecker product operation(s) to several smaller networks, called factor networks. Let $\mathcal{G}_1 = (V_1, E_1, W_1)$ and $\mathcal{G}_2 = (V_2, E_2, W_2)$ be two factor networks. The Kronecker product of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, has the node set $V_1 \times V_2$. There is an edge from node (i, p) to node (j, q) if and only if (i, j) is an edge of E_1 and (p, q) is an edge of E_2. If an edge exists, the corresponding weight is $w_{((i,p),(j,q))} = w_{ij} w_{pq}$. An example of the Kronecker product of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is displayed in Fig. 1.

2.3. Useful lemmas and definitions

Lemma 1 (*Roman, 2005*). *If $u_1, u_2, \ldots, u_n$ are linearly independent vectors and $v_1, v_2, \ldots, v_n$ are arbitrary vectors, then $\sum_{k=1}^{n} u_k \otimes v_k = 0$ implies that $v_k = 0$ for all $k = 1, 2, \ldots, n$. Moreover, the roles of u_k and v_k in the above statement can be exchanged.*

Definition 1 (*Roman, 2005*). A row vector x_m is called the mth-order generalized left eigenvector of matrix A corresponding to its eigenvalue λ if $x_m(A - \lambda I)^m = 0$, and $x_m(A - \lambda I)^{m-1} \neq 0$. Moreover, $x_1, \ldots, x_g$ form a left Jordan chain of A on top of x_1, where the maximum value of g is called the length of this Jordan chain.

3. Model formulation

Consider a network consisting of Nn nodes with a directed and weighted topology $\mathcal{G}$ in the following form (Leskovec, 2010; Pasqualetti, Zampieri, & Bullo, 2014):

$$\dot{x}_i(t) = \sum_{j=1}^{Nn} c_{ij} x_j(t) + \delta_i u_i(t), \quad i = 1, 2, \ldots, Nn, \tag{1}$$

where $x_i \in \mathbb{R}$ is the state of node i, $c_{ij} \in \mathbb{R}$ represents the coupling strength between nodes i and j, $u_i \in \mathbb{R}$ is the control input to node i, and $\delta_i = 1$ if node i is under control, but otherwise $\delta_i = 0$, for all $i = 1, 2, \ldots, Nn$. Assume that $c_{ij} \neq 0$ if there is an edge from node j to node i, otherwise $c_{ij} = 0$, for all $i, j = 1, 2, \ldots, Nn$. Denote $A(\mathcal{G}) = \mathcal{A}(\mathcal{G}) = [c_{ij}] \in \mathbb{R}^{Nn\times Nn}$ and $B = diag\{\delta_1, \delta_2, \ldots, \delta_{Nn}\}$, which represent the topology and the external input channels of the network (1), respectively. Let $X = [x_1, x_2, \ldots, x_{Nn}]^T$ be the whole state of the network, and $U = [u_1, u_2, \ldots, u_{Nn}]^T$ be the total external control input. Then, the network (1) can be rewritten in a compact form as

$$\dot{X}(t) = A(\mathcal{G})X(t) + BU(t). \tag{2}$$

In the following, consider the controllability of the network with topology graph $\mathcal{G}$ being the Kronecker product of two factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. The dynamics of the factor networks are described by $\dot{Y}_1(t) = A(\mathcal{G}_1)Y_1(t)+B_1U_1(t)$ and $\dot{Y}_2(t) = A(\mathcal{G}_2)Y_2(t)+B_2U_2(t)$, where $Y_1 \in \mathbb{R}^N$ and $U_1 \in \mathbb{R}^N$ are the state vector and the control input of the first factor network, respectively; $Y_2 \in \mathbb{R}^n$ and $U_2 \in \mathbb{R}^n$ are the state vector and the control input of the second factor network, respectively. According to the definition of Kronecker product network, it can be easily proved that $A(\mathcal{G}_1 \times \mathcal{G}_2) = A(\mathcal{G}_1)\otimes A(\mathcal{G}_2)$. Therefore, for this Kronecker product network, its compact form can be formulated as (2) with

$$\begin{aligned} &A(\mathcal{G}) = A(\mathcal{G}_1 \times \mathcal{G}_2) = A(\mathcal{G}_1)\otimes A(\mathcal{G}_2), \\ &B = B_1 \otimes B_2. \end{aligned} \tag{3}$$

The analysis here is presented in terms of two factor networks, which can be extended to larger numbers of factor networks by sequential compositions. In the following, it will be shown that the controllability of the composite network can be revealed by examining some features of the smaller factor networks, which makes the computational complexity much lower.

Remark 1. A graph $\mathcal{G}$ is prime if it is nontrivial and cannot be decomposed into the Kronecker product of two nontrivial graphs. An expression $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_k$, with each $\mathcal{G}_i$ being prime, is called a prime factorization of $\mathcal{G}$. Note that every nontrivial graph has a prime factorization over the Kronecker product (Hammack et al., 2011). It has lower computational complexity to check the controllability of a large-scale network by examining some properties of the smaller factor networks. Therefore, exploring low-dimensional controllability conditions for Kronecker product networks is of great significance in engineering applications, which gives insight into the controllability of large-scale networks.

4. Main results

In this section, controllability conditions for the composite network (2)–(3) are considered. Firstly, the general case that both $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are non-diagonalizable is investigated. Then, a special case where at least one factor network is diagonalizable is further analyzed, with a direct and easily-verifiable condition derived.

4.1. Both $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are non-diagonalizable

The connection between the eigenspaces of the factor networks and that of the composite network provides a mechanism to establish efficient controllability conditions. Firstly, all left eigenvectors of the Kronecker product of two defective matrices are characterized.

Theorem 1. *Let $P = \lambda I_p + N_p \in \mathbb{C}^{p\times p}$ and $Q = \mu I_q + N_q \in \mathbb{C}^{q\times q}$ be two defective matrices, where N_p and N_q are nilpotent matrices. The eigenvalues of $P \otimes Q$ are $\lambda\mu$. Moreover,*

- *if $\lambda\mu \neq 0$, the corresponding left eigenvectors are $\eta^1 = e_p \otimes \xi_1$, $\eta^2 = e_{p-1} \otimes \xi_1 + e_p \otimes \xi_2$, …, $\eta^\theta = e_{p-\theta+1} \otimes \xi_1 + e_{p-\theta+2} \otimes \xi_2 + \cdots + e_p \otimes \xi_\theta$, where $\xi_1 = e_q$, $\xi_k = \frac{(-1)^{k+1}}{\lambda^{k-1}} \sum_{l=0}^{k-2} C_{k-2}^l \mu^{k-l-1} e_{q-k+l+1}$, $k = 2, \ldots, \theta$, $\theta = \min\{p, q\}$;*
- *if $\lambda = 0$ and $\mu \neq 0$, the corresponding left eigenvectors are $\eta^k = e_p \otimes e_k$, $k = 1, \ldots, q$;*
- *if $\lambda \neq 0$ and $\mu = 0$, the corresponding left eigenvectors are $\eta^k = e_k \otimes e_q$, $k = 1, \ldots, p$;*
- *if $\lambda = \mu = 0$, the corresponding left eigenvectors are $\eta^1 = e_p \otimes e_1$, $\eta^2 = e_p \otimes e_2$, …, $\eta^q = e_p \otimes e_q$, $\eta^{q+1} = e_{p-1} \otimes e_q$, …, $\eta^{q+p-1} = e_1 \otimes e_q$.*

Proof. According to the structure of matrix $P \otimes Q$, it is easy to verify that the eigenvalues of $P \otimes Q$ are $\lambda\mu$. In the following, the aforementioned four cases are proved respectively.

Case 1: $\lambda\mu \neq 0$. Since $\eta^1(P \otimes Q) = (e_p \otimes e_q)(P \otimes Q) = (\lambda e_p) \otimes (\mu e_q) = \lambda\mu\eta^1$, it follows that η^1 is the left eigenvector of $P \otimes Q$. Assume that η^1, …, η^{k-1} are the left eigenvectors of $P \otimes Q$. Then, η^k is the left eigenvector of $P \otimes Q$ if and only if

$$\lambda\xi_k(\mu I - Q) = \xi_{k-1}Q. \tag{4}$$

Since $\lambda\xi_k(\mu I - Q) = \frac{(-1)^{k+2}}{\lambda^{k-2}} \sum_{l=0}^{k-2} C_{k-2}^l \mu^{k-l-1} e_{q-k+l+2}$ and $\xi_{k-1}Q = \frac{(-1)^k}{\lambda^{k-2}} \sum_{l=0}^{k-3} C_{k-3}^l \mu^{k-l-2}(\mu e_{q-k+l+2} + e_{q-k+l+3}) = \frac{(-1)^k}{\lambda^{k-2}} \sum_{l=0}^{k-2} C_{k-2}^l \mu^{k-l-1} e_{q-k+l+2}$, one can verify that (4) holds. Therefore, η^k is the left eigenvector of $P\otimes Q$. It has been shown in Xue and Roy (2012) that the number of the left eigenvectors for this case is $\min\{p, q\}$, thus all the left eigenvectors of $P \otimes Q$ are η^1, …, η^θ.

By using a similar method, one can prove the results in Cases 2, 3 and 4 easily, thus the detail is omitted. ■

Remark 2. The number of the eigenvectors of $P \otimes Q$ associated with the sole eigenvalue $\lambda\mu$ was given in Xue and Roy (2012). However, it does not present explicit expressions of the eigenvectors, which are characterized in the above theorem. Theorem 1 is the cornerstone of the following controllability analysis for Kronecker product networks.

In what follows, the left eigenvectors of $A(\mathcal{G})$ are expressed through the generalized eigenspaces of the factor networks.

Theorem 2. *Let $\lambda_1, \ldots, \lambda_s$ be the eigenvalues of $A(\mathcal{G}_1)$, and $\mu_1, \ldots, \mu_t$ be the eigenvalues of $A(\mathcal{G}_2)$. Then, the eigenvalues of $A(\mathcal{G})$ are $\lambda_1\mu_1, \ldots, \lambda_1\mu_t, \ldots, \lambda_s\mu_1, \ldots, \lambda_s\mu_t$. Moreover, for the eigenvalue $\lambda_i\mu_j$ with geometric multiplicity θ_{ij},*

- *if $\lambda_i\mu_j \neq 0$, then $\theta_{ij} = \min\{p_i, q_j\}$, and the corresponding left eigenvectors are $\eta_{ij}^k = v_i^k \otimes w_j^1 - \frac{\mu_j}{\lambda_i} v_i^{k-1} \otimes w_j^2 + v_i^{k-2} \otimes \left[\frac{\mu_j^2}{\lambda_i^2} w_j^3 + \frac{\mu_j}{\lambda_i^2} w_j^2\right] + \cdots + v_i^1 \otimes \left[\frac{(-1)^{k+1}}{\lambda_i^{k+1}} \sum_{l=0}^{k-2} C_{k-2}^l \mu_j^{k-l-1} w_j^{k-l}\right]$, $k = 1, \ldots, \theta_{ij}$;*
- *if $\lambda_i = 0$ and $\mu_j \neq 0$, then $\theta_{ij} = q_j$, and the corresponding left eigenvectors are $\eta_{ij}^k = v_i^1 \otimes w_j^k$, $k = 1, \ldots, \theta_{ij}$;*

- *if* $\lambda_i \neq 0$ *and* $\mu_j = 0$, *then* $\theta_{ij} = p_i$, *and the corresponding left eigenvectors are* $\eta_{ij}^k = v_i^k \otimes w_j^1$, $k = 1, \ldots, \theta_{ij}$;
- *if* $\lambda_i = 0$ *and* $\mu_j = 0$, *then* $\theta_{ij} = q_j + p_i - 1$, *and the corresponding left eigenvectors are* $\eta_{ij}^1 = v_i^1 \otimes w_j^1, \ldots, \eta_{ij}^{q_j} = v_i^1 \otimes w_j^{q_j}, \eta_{ij}^{q_j+1} = v_i^2 \otimes w_j^1, \ldots, \eta_{ij}^{\theta_{ij}} = v_i^{p_i} \otimes w_j^1$,

where $v_i^1, \ldots, v_i^{p_i}$ *is the left Jordan chain of* $A(\mathcal{G}_1)$ *associated with the eigenvalue* λ_i, *and* $w_j^1, \ldots, w_j^{q_j}$ *is the left Jordan chain of* $A(\mathcal{G}_2)$ *associated with the eigenvalue* μ_j, $i = 1, \ldots, s$, $j = 1, \ldots, t$.

Proof. Let $V \in \mathbb{C}^{N\times N}$ be a nonsingular matrix such that $VA(\mathcal{G}_1)V^{-1} = P = blockdiag\,\{P_1, \ldots, P_s\}$, where P is the Jordan form of $A(\mathcal{G}_1)$ and the ith Jordan block $P_i = \lambda_i I_{p_i} + N_{p_i} \in \mathbb{C}^{p_i\times p_i}$ with N_{p_i} being a nilpotent matrix, $i = 1, \ldots, s$.

Let $W \in \mathbb{C}^{n\times n}$ be a nonsingular matrix such that $WA(\mathcal{G}_2)W^{-1} = Q = blockdiag\,\{Q_1, \ldots, Q_t\}$, where Q is the Jordan form of $A(\mathcal{G}_2)$ and the jth Jordan block $Q_j = \mu_j I_{q_j} + N_{q_j} \in \mathbb{C}^{q_j\times q_j}$, $j = 1, \ldots, t$.

Since $(V\otimes W)[A(\mathcal{G}_1)\otimes A(\mathcal{G}_2)](V\otimes W)^{-1} = P\otimes Q = blockdiag\,\{P_1\otimes Q_1, \ldots, P_1\otimes Q_t, \ldots, P_s\otimes Q_1, \ldots, P_s\otimes Q_t\}$, the eigenvalues of $A(\mathcal{G})$ are $\lambda_1\mu_1, \ldots, \lambda_1\mu_t, \ldots, \lambda_s\mu_1, \ldots, \lambda_s\mu_t$. Let $V_i = [v_i^{p_iT} \;\; \cdots \;\; v_i^{1T}]^T$ and $W_j = [w_j^{q_jT} \;\; \cdots \;\; w_j^{1T}]^T$. It is easy to derive that $(V_i\otimes W_j)A(\mathcal{G}) = (P_i\otimes Q_j)(V_i\otimes W_j)$. If $\zeta \in \mathbb{C}^{1\times p_iq_j}$ is a left eigenvector of $P_i\otimes Q_j$, then $\zeta(V_i\otimes W_j)$ is the left eigenvector of $A(\mathcal{G})$, $i = 1, \ldots, s$, $j = 1, \ldots, t$. Thus, the results follow from Theorem 1 directly. ∎

The explicit expressions of the left eigenvectors given in Theorem 2 are the core of the following controllability criteria for Kronecker product networks. Let $U_{ij} = span\{\eta_{ij}^1, \ldots, \eta_{ij}^{\theta_{ij}}\}$ be the left eigenspace of $A(\mathcal{G})$ corresponding to the eigenvalue $\lambda_i\mu_j$. In what follows, a theorem is established for the controllability of the Kronecker product network (2)–(3).

Theorem 3. *Let* $\Lambda = \{\lambda_1, \;\lambda_2, \;\ldots, \;\lambda_s\}$ *be the set of the eigenvalues of* $A(\mathcal{G}_1)$, *and* $\Sigma = \{\mu_1, \ldots, \mu_t\}$ *be the set of the eigenvalues of* $A(\mathcal{G}_2)$. *The composite network* (2)–(3) *is controllable if and only if the following two conditions hold simultaneously:*

(1) $\forall \eta \in U_{ij}$ *and* $\eta \neq 0$, $\eta(B_1\otimes B_2) \neq 0$, *for* $i = 1, \ldots, s$, $j = 1, \ldots, t$;
(2) if $\lambda_{i_1}\mu_{j_1} = \lambda_{i_2}\mu_{j_2} = \cdots = \lambda_{i_r}\mu_{j_r}$, *where* $\lambda_{i_k} \in \Lambda$, $\mu_{j_k} \in \Sigma$, $k = 1, \ldots, r$, $r > 1$, *then* $\forall \eta \in \oplus_{k=1}^{r} U_{i_kj_k}$ *and* $\eta \neq 0$, $\eta(B_1\otimes B_2) \neq 0$.

Proof. Using Theorem 2 and the PBH test, one can prove this theorem easily. Thus, the detail is omitted. ∎

Remark 3. Theorem 3 provides a precise and efficient criterion for determining the controllability of a general Kronecker product network, using the generalized eigenvectors of the factor networks with low dimensions. Compared with the classical PBH test, the new condition typically has a much lower computational cost. Due to linear systems duality, the results in Theorem 3 are equally applicable to the observability of Kronecker product networks.

The effectiveness of the above condition can be illustrated by the following example.

Example 1. Consider graph $\mathcal{G}_1$ depicted in Fig. 2. It is easy to verify that $A(\mathcal{G}_1) = \begin{bmatrix} 8.5 & 4 & -0.5 \\ 3.5 & 8 & 0.5 \\ 3.5 & 3 & 5.5 \end{bmatrix}$. The eigenvalues of $A(\mathcal{G}_1)$ are $\lambda_1 = 12$, $\lambda_2 = 5$. The left eigenvector associated with λ_1 is $v_1 = -e_1 - e_2$ and the left Jordan chain corresponding to λ_2 is

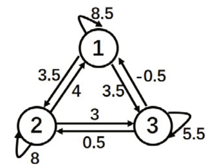


Fig. 2. Factor graph $\mathcal{G}_1$.

$v_2^1 = e_2 - e_3$, $v_2^2 = -e_1 + e_2$. It is easy to obtain that $A(\mathcal{G}_1)$ is non-diagonalizable. Assume that the second node of $\mathcal{G}_1$ is under control. Thus, one has $B_1 = e_2^T$. Let $A(\mathcal{G}) = A(\mathcal{G}_1)\otimes A(\mathcal{G}_1)$, $B = B_1\otimes B_1$. Then $\eta_{22}^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$ and $\eta_{22}^2 = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$. If $c_1 = 0$, one has $(c_1\eta_{22}^1 + c_2\eta_{22}^2)(B_1\otimes B_1) = c_1 = 0$. It then follows from Theorem 3 that $(A(\mathcal{G}), B)$ is uncontrollable. This example fully demonstrates the effectiveness of the condition.

Remark 4. Theorem 6 in Chapman and Mesbahi (2014) has established a controllability condition for Kronecker product networks, which requires $A(\mathcal{G})$ to be diagonalizable. This assumption is conservative and the condition can only be applied to very restricted type of networks. The new condition here removes this restriction, thus is more general and flexible. This nontrivial extension is of great significance in engineering applications.

In the following, some more intuitive and easily-verifiable conditions are presented, which reveal how the controllability of the factor networks affects the controllability of the whole composite network.

Corollary 1. *If the composite network* (2)–(3) *is controllable, then the factor networks* $(A(\mathcal{G}_1), B_1)$ *and* $(A(\mathcal{G}_2), B_2)$ *are controllable.*

Corollary 2. *If* $A(\mathcal{G}_1)$ *has an eigenvalue* 0, *to ensure the controllability of the composite network* (2)–(3), *it is necessary that* $B_2 = I_n$. *Moreover, if* $A(\mathcal{G}_2)$ *has an eigenvalue* 0, *to ensure the controllability of the composite network, it is necessary that* $B_1 = I_N$.

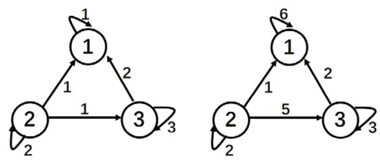
Proof. Based on Theorem 3, this corollary can be proved easily. More details can be found in Hao, Wang, Duan, Chen (2019). ∎

4.2. $A(\mathcal{G}_1)$ *Or* $A(\mathcal{G}_2)$ *is diagonalizable*

In this subsection, a controllability criterion is established for the Kronecker product network (2)–(3) with one diagonalizable factor network. Compared with the conditions for the general case, this easy-to-verify condition allows to check the network controllability more efficiently.

Corollary 3. *Assume that* $A(\mathcal{G}_1)$ *or* $A(\mathcal{G}_2)$ *is diagonalizable. Let* $\Lambda = \{\lambda_1, \ldots, \;\lambda_s\}$ *be the set of the eigenvalues of* $A(\mathcal{G}_1)$, *and* $\Sigma = \{\mu_1, \ldots, \mu_t\}$ *be the set of the eigenvalues of* $A(\mathcal{G}_2)$. *The composite network* (2)–(3) *is controllable if and only if the following four conditions hold simultaneously:*

(1) $(A(\mathcal{G}_1), B_1)$ *and* $(A(\mathcal{G}_2), B_2)$ *are controllable;*
(2) if $A(\mathcal{G}_1)$ *has an eigenvalue* 0, *then* $B_2 = I_n$;
(3) if $A(\mathcal{G}_2)$ *has an eigenvalue* 0, *then* $B_1 = I_N$;
(4) if $\lambda_{i_1}\mu_{j_1} = \cdots = \lambda_{i_r}\mu_{j_r} \neq 0$, *where* $\lambda_{i_k} \in \Lambda$, $\mu_{j_k} \in \Sigma$, $k = 1, \ldots, r$, $r > 1$, *then* $(v_{i_1}B_1)\otimes(w_{j_1}B_2), \ldots, (v_{i_r}B_1)\otimes(w_{j_r}B_2)$ *are linearly independent, where* v_{i_k} (w_{j_k}) *is the left eigenvector of* $A(\mathcal{G}_1)$ $(A(\mathcal{G}_2))$ *corresponding to the eigenvalue* λ_{i_k} (μ_{i_k}), $k = 1, \ldots, r$.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$

Fig. 3. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

Proof. To save space, a sketch of the proof is given here and more details can be found in Hao, Wang et al. (2019). The case that $A(\mathcal{G}_1)$ is diagonalizable is firstly proved. Then with a similar method, one can prove the other case easily.

Necessity: The necessity follows from Corollaries 1, 2 and Theorem 3.

Sufficiency: One needs to prove that, if the composite network (2)–(3) is uncontrollable, then at least one condition in Corollary 3 does not hold. Based on the PBH test, if the composite network (2)–(3) is uncontrollable, then there exists a left-eigenpair of $A(\mathcal{G})$, denoted as (σ, ξ), such that

$$\xi B = 0. \tag{5}$$

This will be discussed in the following 3 cases.

- If $\sigma \neq 0$ is an eigenvalue with the geometric multiplicity being 1, from equality (5), it can be derived that $(A(\mathcal{G}_1), B_1)$ is uncontrollable or $(A(\mathcal{G}_2), B_2)$ is uncontrollable.
- If $\sigma \neq 0$ is an eigenvalue with the geometric multiplicity more than 1, from equality (5), one can deduce that $(v_{i_1}B_1)\otimes(w_{j_1}B_2), \ldots, (v_{i_r}B_1)\otimes(w_{j_r}B_2)$ are linearly dependent.
- If $\sigma = 0$, it follows from equality (5) that $B_1 \neq I_N$ or $B_2 \neq I_n$.

Therefore, if the composite network (2)–(3) is uncontrollable, then at least one condition in Corollary 3 does not hold. This completes the proof. ■

If $A(\mathcal{G})$ is diagonalizable, then both $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are diagonalizable. The conditions in Corollary 3 are also effective for diagonalizable Kronecker product networks, as demonstrated by the following example.

Example 2. Consider two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, which are depicted in Fig. 3. It is easy to verify that $A(\mathcal{G}_1) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $A(\mathcal{G}_2) = \begin{bmatrix} 6 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}$. The eigenvalues of $A(\mathcal{G}_1)$ are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 3$ with the corresponding left eigenvectors $v_1 = e_2$, $v_2 = e_1 - e_3$ and $v_3 = e_2 + e_3$, respectively. The eigenvalues of $A(\mathcal{G}_2)$ are $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 6$ with the corresponding left eigenvectors $w_1 = e_2$, $w_2 = 5e_2 + e_3$, $w_3 = 3/2e_1 + 13/8e_2 + e_3$, respectively. It is easy to verify that $A(\mathcal{G})$ is diagonalizable. Assume that the first two nodes of $\mathcal{G}_1$ have control inputs. Node 2 and node 3 of $\mathcal{G}_2$ are under control. Thus, one has $B_1 = [e_1^T,\ e_2^T]$ and $B_2 = [e_2^T,\ e_3^T]$. Both pairs $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable. Since $\lambda_1\mu_2 = \lambda_2\mu_3 = \lambda_3\mu_1 = 6$, the value of $[c_1(v_1 \otimes w_2) + c_2(v_2 \otimes w_3) + c_3(v_3 \otimes w_1)](B_1 \otimes B_2)$ is required to be checked, which gives $[c_1(v_1 \otimes w_2) + c_2(v_2 \otimes w_3) + c_3(v_3 \otimes w_1)](B_1 \otimes B_2) = [\frac{13}{8}c_2,\quad c_2,\quad 5c_1 + c_3,\quad c_1] \neq 0$, for any nonzero $[c_1,\ c_2,\ c_3]$. It then follows from Corollary 3 that $(A(\mathcal{G}), B)$ is controllable, which coincides with the conclusion derived by the Kalman rank condition.

Remark 5. A controllability criterion for diagonalizable Kronecker product networks was established in Chapman and Mesbahi (2014), which was claimed to be necessary and sufficient. However, Example 2 shows that the necessity of that criterion does not hold. Corollary 3 provides a new controllability condition, which is not only sufficient but also necessary.

5. Controllability of higher-dimensional multi-agent systems revisited

5.1. Problem statement

Consider a multi-agent system consisting of N agents labeled by the set $V = \{1, 2, \ldots, N\}$. Assign the roles of leaders and followers to the agents by denoting $V_L = \{v_1, v_2, \ldots, v_m\}$ and $V_F = V \backslash V_L$ as the sets of indices of the leaders and followers, respectively, where $m \leq N$. To each follower $i \in V_F$, associate a dynamical system $\dot{x}_i = z_i$, and to each leader $i \in V_L$, we associate a dynamical system $\dot{x}_i = z_i + Bu_i$, where $x_i \in \mathbb{R}^n$ is the state of agent i; $u_i \in \mathbb{R}^p$ is the external input to agent $i \in V_L$; $z_i \in \mathbb{R}^n$ is the coupling input from other agents.

Agent i is said to be a neighbor of agent j if its state is known by agent j. Here, assume that the neighboring relationships are fixed, which can be described by a directed and weighted graph $\mathcal{G} = (V, E, W)$. The coupling input z_i to each agent $i \in V$ is determined by the diffusive coupling rule based on the neighboring relations as follows: $z_i = H\sum_{(j,i)\in E} w_{ij}(x_j - x_i)$, where $H \in \mathbb{R}^{n\times n}$ is the matrix describing inner-coupling between different components.

The Laplacian matrix of $\mathcal{G}$ and the external input channels of the multi-agent system are denoted by $L \in \mathbb{R}^{N\times N}$ and $\Delta = diag\,\{d_1, d_2, \ldots, d_N\}$, respectively, where $d_i = 1$ for $i \in V_L$, but otherwise $d_i = 0$, for all $i = 1, 2, \ldots, N$. Let $X = [x_1^T,\ x_2^T,\ \ldots,\ x_N^T]^T$ be the whole state of the multi-agent system, and $U = [u_1^T,\ u_2^T,\ \ldots,\ u_N^T]^T$ be the total external control input. Then, the above multi-agent system can be rewritten in a compact form as

$$\dot{X} = FX + GU, \tag{6}$$

with

$$F = -L \otimes H, \quad G = \Delta \otimes B. \tag{7}$$

Note that matrix F has the form of the Kronecker product of two matrices and so does G. This higher-dimensional multi-agent system can be seen as a special case with $A(\mathcal{G}_1)$ being a Laplacian matrix. In the following, conditions for ensuring the controllability of the multi-agent system (6)–(7) are specified.

Remark 6. The controllability of networked LTI systems or multi-agent systems with linear dynamics has been investigated in Hao et al. (2018), Xue and Roy (2017, 2018a, 2018b) and Zhang et al. (2014). The state matrices for those systems have the form of $I \otimes A + L \otimes H$ rather than a pure Kronecker product. The higher-dimensional multi-agent system investigated here can be seen as a special case with nodes having no internal dynamics.

5.2. A counterexample and modified controllability criteria

Recall the controllability condition for the multi-agent system (6)–(7) given in Cai and Zhang (2010), where it was assumed that the first N_l agents are leaders. Then, L can be partitioned as $L = \begin{bmatrix} L_{ll} & L_{lf} \\ L_{fl} & L_{ff} \end{bmatrix}$, where $L_{ll} \in \mathbb{R}^{N_l\times N_l}$ and $L_{ff} \in \mathbb{R}^{(N-N_l)\times(N-N_l)}$.

Theorem 4 (*Theorem* 1 *of Cai & Zhang, 2010*)**.** *For an LTI swarm system described by (6)–(7), suppose the first N_l agents are leaders. Then, the system is completely controllable if and only if*

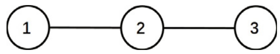


Fig. 4. A path network.

(1) (H, B) is a controllable matrix pair;
(2) (L_{ff}, L_{fl}) is a controllable matrix pair.

However, while this condition is necessary for the controllability of the network, it may not be sufficient, as shown in the following example.

Consider the undirected path network with three nodes depicted in Fig. 4. Node 1 is selected to be the leader. Let $H = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. It is easy to verify that (H, B) and (L_{ff}, L_{fl}) are controllable. From Theorem 1 in Cai and Zhang (2010), it follows that this undirected path network is controllable.

However, if one checks the controllability of this system by using the classical Kalman rank condition, one can find that $(-L\otimes H, \varDelta\otimes B)$ is actually uncontrollable. Therefore, its sufficiency does not hold.

This example is a modification of the example presented in Xue and Roy (2018a), which was used to demonstrate that the controllability condition for diffusive networks proposed in Zhang et al. (2014) is not always sufficient. The incomplete eigenanalysis of networks presented in Cai and Zhang (2010) and Zhang et al. (2014) leads to errors in controllability analysis. In the following, a modified controllability condition is proposed based on the results in Section 4, which is both necessary and sufficient.

Corollary 4. *The multi-agent system (6)–(7) is controllable if and only if the following three conditions hold simultaneously:*

(1) $(L, \varDelta)$ is controllable;
(2) $rank(B) = n$;
(3) if H has an eigenvalue 0, then $\varDelta = I_N$.

Proof. Note that L has an eigenvalue 0. If the multi-agent system (6)–(7) is controllable, then $rank(B) = n$. From Corollaries 1 and 2, it follows that the conditions (1) and (3) are necessary for the controllability of the multi-agent system (6)–(7).

For sufficiency, one needs to prove that, if the conditions (1)–(3) hold, then no left eigenvectors of F are orthogonal to G, thus the multi-agent system (6)–(7) is controllable. It is easy to verify that if the conditions (1)–(3) hold, then no left eigenvectors of F associated with eigenvalue 0 are orthogonal to G. For a nonzero eigenvalue $\sigma = -\lambda\mu$, if it is not a common eigenvalue, it follows from Theorem 2 that any corresponding left eigenvector can be expressed as $\xi = \left(\sum_{i=1}^{\theta} c_i v^i\right) \otimes w^1 + \sum_{k=2}^{\theta}\left[\left(\sum_{i=k}^{\theta}\sum_{j=1}^{i-k+1} c_i l_{ij} v^j\right) \otimes w^k\right]$, where $v^1, \ldots, v^p$ is the left Jordan chain of L associated with the eigenvalue λ, and $w^1, \ldots, w^q$ is the left Jordan chain of H associated with the eigenvalue μ, $\theta = \min\{p, q\}$, l_{ij} is a nonzero scalar about λ and μ, $c_1, \ldots, c_\theta$ are scalars, which are not all zero. Note that $rank(B) = n$ and $(L, \varDelta)$ is controllable. According to Lemma 1, one can deduce that $\xi(\varDelta\otimes B) \neq 0$, referring to Hao, Wang et al. (2019) for more details. Thus, for any nonzero eigenvalue of F, if it is not a common eigenvalue, no corresponding left eigenvectors are orthogonal to G. Similarly, one can prove that, for each nonzero common eigenvalue of F, no corresponding left eigenvectors are orthogonal to G. Consequently, no left eigenvectors of F are orthogonal to G. Therefore, the multi-agent system (6)–(7) is controllable. This completes the proof. ■

According to Corollary 4, one can easily verify that the undirected path network in the above counterexample is uncontrollable. This example fully demonstrates the effectiveness of the condition.

6. Conclusions

The controllability of Kronecker product networks has been investigated, in which the factor networks have general directed topologies. A necessary and sufficient condition for the controllability of the composite network has been derived, which is effective and has a much lower computational cost as compared to existing criteria. For the special case where at least one factor network is diagonalizable, a specified condition has also been established, which is simple and easier to verify. Moreover, the controllability of higher-dimensional multi-agent systems has been reinvestigated. It is found that the sufficiency of the controllability criterion given in Cai and Zhang (2010) does not hold. Consequently, a modified condition is derived, which is necessary and sufficient. In future studies, the controllability and observability of other types of network-of-networks will be further considered.

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