A Probabilistic Algorithm to Test Local Algebraic Observability in Polynomial Time

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ABSTRACT

The following questions are often encountered in system and control theory. Given an algebraic model of a physical process, which variables can be, in theory, deduced from the input-output behavior of an experiment? How many of the remaining variables should we assume to be known in order to determine all the others? These questions are parts of the *local algebraic observability* problem which is concerned with the existence of a non trivial Lie subalgebra of the symmetries of the model letting the inputs and the outputs invariant.

We present a *probabilistic seminumerical* algorithm that proposes a solution to this problem in *polynomial time*. A bound for the necessary number of arithmetic operations on the rational field is presented. This bound is polynomial in the *complexity of evaluation* of the model and in the number of variables. Furthermore, we show that the *size* of the integers involved in the computations is polynomial in the number of variables and in the degree of the system. Last, we estimate the probability of success of our algorithm.

Keywords

Observability, identifiability, seminumerical algorithm.

1. INTRODUCTION

Local algebraic observability is a structural property of a model and one of the key-concepts in control theory. Its earliest definition goes back to the work of R.E. Kalman for the linear case (see [18]) and a large literature is devoted to this subject (see [14, 16] and the references therein). We base our work on the definition given by S. Diop & M. Fliess in [7] of the observability for the class of algebraic systems.

Figure 1: Model for circadian oscillations in the Drosophila period protein [13]

$$\left\{\begin{array}{lcl}
\dot{M} & = & \frac{v_s {K_I}^4}{{K_I}^4+{P_N}^4} - \frac{v_m M}{K_m+M}, \\
\dot{P_0} & = & k_s M - \frac{V_1 P_0}{K_1+P_0} + \frac{V_2 P_1}{K_2+P_1}, \\
\dot{P_1} & = & \frac{V_1 P_0}{K_1+P_0} + \frac{V_4 P_2}{K_4+P_2} - P_1\left(\frac{V_2}{K_2+P_1} + \frac{V_3}{K_3+P_1}\right), \\
\dot{P_2} & = & \frac{V_3 P_1}{K_3+P_1} - P_2\left(\frac{V_4}{K_4+P_2} + k_1 + \frac{v_d}{K_d+P_2}\right) + k_2 P_N, \\
\dot{P_N} & = & k_1 P_2 - k_2 P_N, \\
y & = & P_N.
\end{array}\right.$$

As in the example of figure 1, such a system is usually described by means of

- a vector field, which describes the evolution of *state variables* in function of *inputs* and of *parameters*;
- some *outputs* which are algebraic functions of these variables.

The definition of observability given in [7] relies on the theory of differential algebra founded by J.F. Ritt [27] and is based on the existence of algebraic relations between the state variables and the successive derivatives of the inputs and the outputs. These relations can be considered as an obstruction to the existence of infinitely many trajectories of the state variables which are solutions of the vector field and fit the same specified input-output behavior. If there are only finitely many such trajectories, the state variables are said to be locally observable.

In order to illustrate this notion, let us consider the *local structural identifiability* problem which is a particular case of the observability problem. The question is to decide if some unknown *parameters* of a model are observable considering these parameters as a special kind of state variables Θ satisfying $\dot{\Theta} = 0$ (see [26, 32, 30]). If they are not observable, then infinitely many values of these parameters can fit the same observed data. Hence, if these parameters have a physical significance, it may be necessary to change the experimental protocol when possible. On the other hand, if the parameters are identifiable, various numerical approximation methods can be used for their estimation (see [21] and the references therein).

We consider the local algebraic observability problem under the computer algebra standpoint. The previous studies

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that enable to test observability mainly rely on characteristic set or standard bases computation [25, 22, 23, 3, 15, 24] and their complexity is, at least, exponential in the number of variables and of parameters (see [10, 28]). Some other techniques, as the local state variable isomorphism approach [30] or the conversion between characteristic set w.r.t. different ranking [2], can also be used. The complexities of these methods are not known.

We present a probabilistic polynomial-time algorithm which computes the set of observable variables of a model and gives the number of non observable variables which should be assumed to be known in order to obtain an observable system. A Maple implementation is available at the url http://www.medicis.polytechnique.fr/~sedoglav.

Example. Let us illustrate our algorithm with a model for circadian oscillations in the Drosophila period protein [13]. This model is presented in figure 1; there are seventeen parameters and no input in it. After 10 seconds of computation, our Maple implementation gives the following results:

- the variable M and the parameters $\{v_s, v_m, K_m, k_s\}$ are not observable. All the other parameters and variables are observable;
- if the non observable variable or only one of the non observable parameters are specified, all the variables and parameters of the resulting system are observable.

Our algorithm certifies that a variable is observable and the answer for a non observable one is probabilistic with high probability of success. These results allow us to focus our attention on just four of the seventeen original parameters. Thus, the search of an infinitesimal transformation which leaves the output y and the vector field invariant is simplified and we find a group of symmetries generated by $\{M, v_s, v_m, K_m, k_s\} \to \{\lambda M, \lambda v_s, \lambda v_m, \lambda K_m, k_s/\lambda\}$. Hence, there is an infinite number of possible values for non observable parameters which fit the same specified output y: this system is certainly unidentifiable.

1.1 Notations and Main Result

Hereafter, we consider a state variable representation with time invariant parameters defined by an algebraic system of the following kind:

$$\Sigma \left\{ \begin{array}{rcll} \dot{\Theta} & = & 0, & \\ \dot{X} & = & F(X,\Theta,U), & (1.1) \\ Y & = & G(X,\Theta,U). & (1.2) \end{array} \right.$$

Capital letters stand for vector-valued objects and we suppose that there are:

- ℓ parameters $\Theta := (\theta_1, \ldots, \theta_\ell)$;
- n state variables $X := (x_1, \ldots, x_n)$;
- r input variables $U := (u_1, \ldots, u_r)$;
- m output variables $Y := (y_1, \ldots, y_m)$ with $m \leq n$.

The letter $\dot{X}$ stands for the derivatives of the state variables $(\dot{x}_1, \ldots, \dot{x}_n)$ and the letter F (resp. G) represents n (resp. m) rational functions in $\mathbb{Q}(X, \Theta, U)$ which are denoted by $(f_1, \ldots, f_n)$ (resp. $(g_1, \ldots, g_m)$). The letter d (resp. h) represents a bound on the degree (resp. size of the coefficients) of the numerators and denominators of the f_i's and g_i's.

Hereafter, we use a common encoding where the expression $e := (x+1)^5$ is represented by a sequence of instructions: $t_1 := x+1, t_2 := {t_1}^2, t_3 := {t_2}^2, e := t_3 t_1$. Hence, the system Σ is represented by a *straight-line program* without division which computes its numerators and denominators and requires L arithmetic operations (see § 3.5 and § 4 in [5]).

The following theorem is the main result of this paper.

THEOREM 1. *Let Σ be a differential system as described in Section 1.1. There exists a probabilistic algorithm which determines the set of observable variables of Σ and gives the number of non observable variables which should be assumed to be known in order to obtain an observable system.*

The arithmetic complexity of this algorithm is bounded by

$$\mathcal{O}\Big(\mathcal{M}(\nu)\big(\mathcal{N}(n+\ell) + (n+m)L\big) + m\nu\mathcal{N}(n+\ell)\Big)$$

with $\nu \leq n+\ell$ and with $\mathcal{M}(\nu)$ $\big($resp. $\mathcal{N}(\nu)\big)$ the cost of power series multiplication at order $\nu+1$ (resp. $\nu \times \nu$ matrix multiplication).

Let μ be a positive integer, D be $4(n+\ell)^2(n+m)d$ and D' be $\big(2\ln(n+\ell+r+1)+\ln\mu D\big)D+4(n+\ell)^2\big((n+m)h+\ln 2nD\big)$. If the computations are done modulo a prime number $p > 2D'\mu$ then the probability of a correct answer is at least $(1-1/\mu)^2$.

For the model presented in figure 1, the choice of $\mu = 3000$ leads to a probability of success around .9993 and the computations are done modulo 10859887151. These computations take 10 seconds on a PC Pentium III (650 MHz).

Outline of the paper. In the next section, we recall some basic definitions of differential algebra and the definition of algebraic observability used by S. Diop & M. Fliess in [7]. Furthermore, we describe the relationship between this framework and the approach of H. Pohjanpalo in [26]. Then, we present an algebraic Jacobian matrix which is derived from the theory of Kähler differentials and used in the local algebraic observability test.

In the second part of this paper, we present some new results. In Section 3, we show how to compute some specializations of this matrix using power series expansion of the output and we estimate the related arithmetic complexity. Then, we study the behavior of the integers involved in the computations and we precise the probabilistic aspect.

2. DIFFERENTIAL ALGEBRA AND OBSERVABILITY

Differential algebra, founded by J.F. Ritt, is an appropriate framework for the definition of algebraic observability introduced by S. Diop & M. Fliess in [7]. For more details on differential algebra, we refer to [27] and [20]; nevertheless, we recall briefly some necessary notions.

2.1 Differential Algebraic setting

Let us denote by k a ground field of characteristic zero. The differential algebra $k\{U\}$ is the k-algebra of multivariate polynomials defined by the infinite set of indeterminates $\{U^{(j)} | \forall j \in \mathbb{N}^*\}$ and equipped with a derivation $\mathcal{L}$ such that $\mathcal{L}u^{(i)} = u^{(i+1)}$. Its fraction field is denoted by $k\langle U\rangle$.

Hypotheses. The inputs U and all their derivatives are assumed to be independent. Furthermore, we consider non singular solutions of Σ; thus, we assume that we work in an open set where the denominators present in Σ do not vanish.

2.2 Local Algebraic Observability

Following the interpretation due to M. Fliess of some algebraic control theory problems [9], we consider the differential field $\mathcal{K} := k\langle U\rangle(X, \Theta)$ equipped with the following formal Lie derivation:

$$\mathcal{L} := \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{j \in \mathrm{N}} \sum_{u \in U} u^{(j+1)} \frac{\partial}{\partial u^{(j)}}.$$

This derivation is associated with the vector field defined by the equations (1.1). Hereafter, we denote $(\mathcal{L}f_1, \ldots, \mathcal{L}f_n)$ by $\mathcal{L}F$ and $\underbrace{\mathcal{L} \circ \cdots \circ \mathcal{L}}_{j \text{ times}}$ by $\mathcal{L}^j$. Hence, $Y^{(j)} = \mathcal{L}^j G(X, \Theta, U)$.

Definition 1. [22, 7] An element z in $\mathcal{K}$ is locally algebraically observable with respect to inputs and outputs if it is algebraic over $k\langle U, Y\rangle$. So, the system Σ is locally observable if the field extension $k\langle U, Y\rangle \hookrightarrow \mathcal{K}$ is purely algebraic.

Let us illustrate this definition with the following example:

$$\left\{ \begin{array}{lcl} \dot{x}_3 & = & \theta x_1, \\ \dot{x}_2 & = & x_3/x_2, \\ \dot{x}_1 & = & x_2/x_1, \\ y & = & x_1. \end{array} \right.$$

By successive differentiations of the output, we obtain the following differential relations:

$$y - x_1, \quad y\dot{y} - x_2, \quad y\dot{y}(\dot{y}^2 + y\ddot{y}) - x_3,$$
$$(\dot{y}^2 + y\ddot{y})^2 + y\dot{y}\big(3\dot{y}\ddot{y} + yy^{(3)}\big) - \theta y.$$

Thus, the parameter and the variables are observable according to Definition 1. Furthermore, as these relations define a unique solution, the parameter and the variables are said to be *globally* algebraically observable [22, 25].

These relations depend generically of high order derivatives of the output and so, they are not of a great practical interest for parameter estimation (see [24]). As we focus our attention on local observability, we are going to avoid their computation.

Definition 1 implies that local observability is related to the transcendence degree of the field extension $k\langle U, Y\rangle \hookrightarrow \mathcal{K}$. Thus, this property can be tested by a rank computation using Kähler differentials (see Section 2.4). As noticed in [7], this approach leads to a condition which is the formal counterpart of the R. Hermann & A. Krener rank condition in the differential geometric point of view [14].

Furthermore, the transcendence degree of the field extension $k\langle U, Y\rangle \hookrightarrow \mathcal{K}$ is the number of non observable variables which should be assumed to be known in order to obtain an observable system. Thus, Theorem 1 is based on the study of this field extension.

2.3 A Description of $k\langle U, Y\rangle \hookrightarrow \mathcal{K}$

Let us denote by $\Phi(X, \Theta, U, t)$ the formal power series with coefficients in $\mathcal{K}$ solution of $\dot{\Phi} = F(\Phi, \Theta, U)$ with initial condition $\Phi(X, \Theta, U, 0) := X$. We have:

$$\Phi(X, \Theta, U, t) = X + \sum_{j \in \mathbb{N}^*} \mathcal{L}^j F(X, \Theta, U) \frac{t^j}{j!}.$$

Let us define the formal power series with coefficients in $\mathcal{K}$ such that $Y(X, \Theta, U, t) := G(\Phi(X, \Theta, U, t), \Theta, U, t)$:

$$Y(X, \Theta, U, t) = G(X, \Theta, U) + \sum_{j \in \mathbb{N}^*} \mathcal{L}^j G(X, \Theta, U) \frac{t^j}{j!}. \quad (2)$$

In [26], H. Pohjanpalo already considers the power series Y in order to test identifiability. In [7], the authors prove that a finite number of these coefficients are necessary to *describe* the field extension $k\langle U, Y\rangle \hookrightarrow \mathcal{K}$. But in these two papers the necessary order of derivation is not bounded. This can be done using the differential algebra point of view (see § 4 in [28] for a general statement). The following proposition summarizes these results in a field extension framework.

Proposition 1. *The differential field $k\langle U, Y\rangle$ is purely algebraic over the differential field $k\langle U\rangle(Y, \ldots, Y^{(n+\ell)})$.*
Proof. The transcendence degree of $k\langle U\rangle \hookrightarrow \mathcal{K}$ is equal to $n + \ell$. Hence, the transcendence degree of $k\langle U\rangle \hookrightarrow k\langle U, Y\rangle$ is bounded by $n + \ell$. It means that, for $i = 1, \ldots, m$, there is an algebraic relation $q_i\big(y_i, \ldots, {y_i}^{(n+\ell)}\big) = 0$ and the derivative ${y_i}^{(n+\ell+1)}$ is a rational function of $y_i, \ldots, {y_i}^{(n+\ell)}$ with coefficients in $k\langle U\rangle$. □

If there is more than a single output, the necessary order of derivation can be smaller than $n + \ell$ and it is denoted by ν. This index of differentiation is a measure of the complexity of our algorithm (see Section 3.5) and generically $\nu = (n + \ell)/m$. Hereafter, we take ν equal to $n + \ell$.

In the above proof, following the hypotheses of Section 2.1, we assumed that the independent input variables U and all their derivatives were in the ground field. Furthermore, we showed that we just need the first $n + \ell$ derivatives of the output equations. In order to simplify the presentation in the next section, we assume that the ground field is $\bar{k} := k\big(U, Y, \ldots, U^{(n+\ell)}, Y^{(n+\ell)}\big)$.

We present now the properties of the module of Kähler differentials which are used to compute the transcendence degree of $\bar{k} \hookrightarrow \bar{k}(X, \Theta)$ in practice.

2.4 Rank Conditions

If $S \hookrightarrow T$ is a field extension, we use the notation $\Omega_{T/S}$ for the T-vector space which is the cokernel of the Jacobian matrix $\partial(Y^{(i)})_{0 \le i \le \nu}/\partial(X, \Theta)$ and dz stands for the image of $z \in T$ in this vector space (see § 16 in [8] for standard definition and [17] for construction in differential algebra). We recall the following result:

THEOREM 2. (§ 16 in [8]) *Let us consider S a field of characteristic zero and T a finitely generated field extension of S. If $\{x_\lambda\} \subset T$ is a collection of elements, then $\{\mathrm{d}x_\lambda\}$ is a basis of $\Omega_{T/S}$ as a vector space over T iff the $\{x_\lambda\}$ form a transcendence basis of T over S.*

Our algorithm is based on the following straightforward consequences of this theorem.

COROLLARY 1. *If ϕ is the transcendence degree of the field extension $\bar{k} \hookrightarrow \bar{k}(X, \Theta)$ then we have the equality*

$$\phi = (n + \ell) - \mathrm{rank}_{\bar{k}(X,\Theta)}\left(\partial\left(Y^{(i)}\right)_{0\leq i\leq\nu} / \partial(X, \Theta)\right).$$

If the rank of the Jacobian matrix $\partial(Y^{(j)})_{0\leq j\leq\nu}/\partial(X\backslash\{x_i\},\Theta)$ (resp. $\partial(Y^{(j)})_{0\leq j\leq\nu}/\partial(X, \Theta\backslash\{\theta_i\})$) is equal to $n+\ell-\phi$, then the transcendence degree of the field extension $\bar{k} \hookrightarrow \bar{k}(x_i)$ (resp. $\bar{k} \hookrightarrow \bar{k}(\theta_i)$) is zero and the variable x_i (resp. the parameter θ_i) is observable.

The computation of ϕ is mainly based on the construction and the evaluations of a *straight-line program* which allows to compute the power series expansion of $Y(X, \Theta, U, t)$. We present the necessary notions in the next section.

2.5 Data Encoding and Complexity Model

The above results can be expressed considering a polynomial f as an element of a vector space; hereafter, we consider an algebraic expression as a function. This classical point of view in numerical analysis is also used in computer algebra for complexity statements or practical algorithms (see [12, 29, 31] and the references therein). We refer to § 4 in [5] for more details about this model of computation.

Definition 2. Let $\mathcal{A}$ be a finite set of variables. A straight-line program is a finite sequence of assignments $b_i \leftarrow b' \circ_i b''$ with $\circ_i \in \{+, -, \times, \div\}$ and $\{b', b''\} \subset \bigcup_{j=1}^{i-1}\{b_j\} \cup \mathcal{A} \cup k$. Its complexity of evaluation is measured by its length L, which is the number of its arithmetic operations. Hereafter, we use the abbreviation SLP for straight-line program.

A SLP representing a rational expression f is a program which computes the value of f from any values of the ground field such that every division of the program is possible. It is possible to determine a SLP representing the gradient of f. The following constructive results allows us to handle these two aspects.

THEOREM 3 (W. BAUR & V. STRASSEN [1]). *Let us consider a* SLP *computing the value of a rational expression f in a point of the ground field and let us denote by L_f its complexity of evaluation. One can construct a* SLP *of length $5L_f$ which computes the value of* grad(f).

Furthermore, one can construct a SLP of length $4L_f$ which computes two polynomials f_1 and f_2 such that $f = f_1/f_2$.

Following our presentation, one can construct formally all the expressions introduced in Sections 2.3 and 2.4 with its favourite computer algebra system. But, in order to compute the formal expressions $Y^{(\nu)} = \mathcal{L}^\nu G$ and the associated Jacobian matrix, one has to differentiate ν times the output equations (1.2). As noticed in [19], the arithmetic complexity of computing multiple partial derivatives is likely exponential in ν. If the evaluation complexity of the output equations (1.2) is L, by Theorem 3, the computation of $Y^{(\nu)}$ requires at least $(5m)^\nu L$ arithmetic operations. This strategy cannot lead to a polynomial time algorithm.

The rank computations defined in the previous section are also cumbersome because they are mainly performed on the field $\bar{k}(X, \Theta)$. Nevertheless, in order to determine ϕ efficiently, the variables X, Θ and U can be specialized to some generic values in the Jacobian matrix and so, its generic rank can be computed numerically with high probability of success (see Section 3.7).

Thus, the main problem is to avoid the formal computation of $(Y^{(i)})_{0\leq i\leq\nu}$. In fact, our strategy is to specialize a linearized system derived form Σ first and to recover the value of ϕ just using numerical computations on a finite field.

3. A PROBABILISTIC POLYNOMIAL TIME ALGORITHM

In Section 3.1, we present the *linear variational system* derived from Σ which allows to compute directly the Jacobian matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X, \Theta)$ with X, Θ and U specialized on some given values. Then, we show how this matrix can be determined in polynomial time and we give an estimation of the arithmetic complexity of our algorithm. The purpose of the Sections 3.6 and 3.7 is to study the growth of the integers involved in the computations and to estimate the probability of success of our algorithm.

3.1 Variational System Derived From Σ

As shown in Section 2.4, our goal is to compute the generic rank of the Jacobian matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X, \Theta)$. Using relation (2), we conclude that:

$$\frac{\partial(Y^{(j)})_{0\leq j\leq\nu}}{\partial(X,\Theta)} = \mathsf{coeffs}\left(\frac{\partial G}{\partial X}\frac{\partial \Phi}{\partial X}, \frac{\partial G}{\partial X}\frac{\partial \Phi}{\partial \Theta} + \frac{\partial G}{\partial \Theta}\right).$$

The above equalities leads to the following relation:

$$\frac{\partial(Y^{(j)})_{0\leq j\leq\nu}}{\partial(X,\Theta)} = \mathsf{coeffs}\left(\nabla Y\left(\Phi, \frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial \Theta}\right), t^j, j = 0, \ldots, \nu\right), \quad (3)$$

where ∇Y denote the following $n \times (n + \ell)$ matrix:

$$\nabla Y(\Phi, \Gamma, \Lambda, \Theta, U) := \left(\frac{\partial G}{\partial X}\Gamma, \frac{\partial G}{\partial X}\Lambda + \frac{\partial G}{\partial \Theta}\right)(\Phi, \Gamma, \Lambda, \Theta, U).$$

Hence, we have to determine the first $\nu = n + \ell$ terms of the power series expansion of Φ, $\Gamma := \partial\Phi/\partial X$ and $\Lambda := \partial\Phi/\partial\Theta$.

Let us denote by $P(\dot{X}, X, \Theta, U) = 0$, the numerators of the rational relations $\dot{X} - F(X, \Theta, U) = 0$ and by ∇P the following expressions:

$$\left\{\begin{array}{lr} P(\dot{X}, X, \Theta, U), & (4.1) \\ \frac{\partial P}{\partial \dot{X}}(X, \Theta, U)\dot{\Gamma} + \frac{\partial P}{\partial X}(\dot{X}, X, \Theta, U)\Gamma, & (4.2) \\ \frac{\partial P}{\partial \dot{X}}(X, \Theta, U)\dot{\Lambda} + \frac{\partial P}{\partial X}(\dot{X}, X, \Theta, U)\Lambda + \frac{\partial P}{\partial \Theta}(\dot{X}, X, \Theta, U). & (4.3) \end{array}\right. \quad (4)$$

The power series Φ, Γ and Λ are solutions of the system of ordinary differential equations $\nabla P = 0$ with the associated initial conditions $\Gamma(X, \Theta, U, 0) := \mathrm{Id}_{n\times n}$, $\Lambda(X, \Theta, U, 0) := 0_{n\times\ell}$.

3.2 Computational Strategy

One can compute symbolically the expression of the formal Jacobian matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$. The rank computations described in Corollary 1 are sufficient to conclude. Furthermore, if X, Θ and U are specialized on some random values, these computations can be performed numerically with high probability of success. We summarize this possible strategy in the upper horizontal and the right vertical arrow of the following diagram:

$$\begin{tikzcd}
\Sigma \arrow[r, "\text{formal computation}"] \arrow[d] & \left(\frac{\partial(\mathcal{L}^i G)_{0\leq i\leq\nu}}{\partial(X,\Theta)}\right) \arrow[d, "\begin{array}{l} X \to X_0 \in \mathbb{Z}^n, \\ \Theta \to \widetilde{\Theta} \in \mathbb{Z}^\ell, \\ U \to \widetilde{U} \in (\mathbb{Z}[t])^r. \end{array}"] \\
\nabla P \arrow[r, "\text{numerical computation}"] & \left(\frac{\partial(\mathcal{L}^i G)_{0\leq i\leq\nu}}{\partial(X,\Theta)}\right)(X_0, \widetilde{\Theta}, \widetilde{U})
\end{tikzcd}$$

As the symbolic computation of the Jacobian matrix is cumbersome, we specialize the parameters on some random integers $\widetilde{\Theta}$ and the inputs U on the power series $\widetilde{U}$ which are truncated at order $n+\ell+1$ with random integer coefficients. Then, we solve the associated system ∇P for some integer initial conditions X_0 and we compute the specialization $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)(X_0,\widetilde{\Theta})$ with ∇Y. This approach is summarized by the left vertical and the lower horizontal arrow. We present an algorithm which relies on this standpoint and we give in Section 3.7 its probability of success.

The hypothesis $\partial P/\partial\dot{X} \neq 0$ assumed in Section 2.1 ensures that the differential system $\nabla P(\Phi, \Gamma, \Lambda, \widetilde{\Theta}, \widetilde{U}) = 0$ admits an unique formal solution which can be computed with the Newton operator presented in the next section.

3.3 A Quadratic Newton Operator

The aim of this section is to present the Newton operator used in our algorithm. In [11, 4], the authors show that its convergence is quadratic. We work with vector-valued expressions. Thus, the expression (4.1) (resp. (4.2), (4.3)) represents a $n \times 1$ (resp. $n \times n$ and $n \times \ell$) matrix.

From a SLP of length L which encodes Σ, Theorem 3 allows to construct another SLP of length $\mathcal{O}(\mathcal{N}(n+\ell)+nL)$ which encodes the system ∇P. For some given series Φ, Γ and Λ, this SLP computes the following $n \times (1+n+\ell)$ matrix:

$$\begin{pmatrix} p_1(\dot{\Phi},\Phi,\widetilde{\Theta},\widetilde{U}) & \frac{\partial P}{\partial \dot{X}}(\Phi,\widetilde{\Theta},\widetilde{U})\dot{\Gamma} & \frac{\partial P}{\partial \dot{X}}(\Phi,\widetilde{\Theta},\widetilde{U})\dot{\Lambda} \quad + \\ \vdots & + & \frac{\partial P}{\partial X}(\dot{\Phi},\Phi,\widetilde{\Theta},\widetilde{U})\Lambda + \\ p_n(\dot{\Phi},\Phi,\widetilde{\Theta},\widetilde{U}) & \frac{\partial P}{\partial X}(\dot{\Phi},\Phi,\widetilde{\Theta},\widetilde{U})\Gamma & \frac{\partial P}{\partial \Theta}(\dot{\Phi},\Phi,\widetilde{\Theta},\widetilde{U}) \end{pmatrix}.$$

Let us represent the approximation of Φ (resp. Λ, Γ) mod t^{2^j} by Φ_j (resp. Λ_j, Γ_j) and denote by E_{j+1} the correction term:

$$(\Phi - \Phi_j, \Gamma - \Gamma_j, \Lambda - \Lambda_j) \bmod t^{2j+1}.$$

As usually, we construct our Newton operator from the Taylor series expansion of the function ∇P. This yields the following relations:

$$\begin{aligned} \nabla P(\Phi,\Gamma,\Lambda)(X,\Theta,U,t) = \;& \nabla P(\Phi_j,\Gamma_j,\Lambda_j) + \tfrac{\partial\,\nabla P}{\partial(\dot{X},\dot{\Gamma},\dot{\Lambda})}\dot{E}_{j+1} \\ & + \tfrac{\partial\,\nabla P}{\partial(X,\Gamma,\Lambda)}E_{j+1} + \ldots = 0. \end{aligned}$$

The remaining terms are of order in t greater than 2^{j+1}. Thus, they are not necessary for the computation of E_j.

We consider Φ as a variable in the first column of ∇P and as a constant in the others. Thus, we have the following relations:

$$\frac{\partial\,\nabla P}{\partial(\dot{X},\dot{\Gamma},\dot{\Lambda})} = \left(\frac{\partial P}{\partial \dot{X}}, \frac{\partial P}{\partial \dot{X}}, \frac{\partial P}{\partial \dot{X}}\right), \; \frac{\partial\,\nabla P}{\partial(X,\Gamma,\Lambda)} = \left(\frac{\partial P}{\partial X}, \frac{\partial P}{\partial X}, \frac{\partial P}{\partial X}\right).$$

The above hypothesis induces a *shift* between the order of correct coefficients of Λ_j, Γ_j and Φ_j. In fact, Λ_j and Γ_j are correct modulo t^{2j-1}. Thus, we need to stop the following operator with $j+1 = \ln_2(n+\ell+1)$ and to repeat one more time the last resolution at the same order.

Newton operator. The above hypothesis leads to a Newton operator based on the resolution of the following system of linear ordinary differential equations:

$$\frac{\partial P}{\partial \dot{X}}\dot{E}_{j+1} + \frac{\partial P}{\partial X}E_{j+1} + \nabla P = 0 \bmod t^{2^{j+1}}. \tag{5}$$

This system is solved iteratively using the recurrence relations $(\Phi_{j+1}, \Gamma_{j+1}, \Lambda_{j+1}) = (\Phi_j, \Gamma_j, \Lambda_j) + E_{j+1}$ and the initial conditions $\Phi_0 \in \mathbb{Z}^n$, $\Gamma_0 := \mathrm{Id}_{n\times n}$ and $\Lambda_0 := 0_{n\times\ell}$.

The resolution of the linear ordinary differential system (5) relies on the method of integrating factors. First, we consider the Homogeneous system

$$\frac{\partial P}{\partial \dot{X}}(\Phi_j,\widetilde{\Theta},\widetilde{U})\dot{W}_j + \frac{\partial P}{\partial X}(\dot{\Phi}_j,\Phi_j,\widetilde{\Theta},\widetilde{U})W_j = 0 \bmod t^{2^{j+1}}$$

where W_j denote a $n \times n$ unknown matrix which coefficients are series truncated at order 2^j.

We consider matrices with coefficients in a series ring as series with coefficients in a matrix ring. For example, we have $A \bmod t^{2^{j+1}} = A_0 + A_1 t + \cdots + A_{2^j} t^{2^j}$ where the A_i's are matrices with coefficients in the rational field. Thus, the product, the exponential and, if A_0 is invertible, the inverse of matrices with coefficients in a series ring can be computed at precision j with the classical Newton operator (see § 5.2 in [4] for more details). For example, if A_0 is invertible and B_j denotes the inverse of A at order t^{2^j}, we have $B_{j+1} = 2B_j - B_j A B_j$.

Furthermore, it is a basic fact from the theory of linear ordinary system that if $A\dot{W} + A'W = 0$ and A is invertible then $W = \exp(\int A^{-1}A')$ is a matricial solution of this system. Hence, the above homogeneous system can be solved at precision j by a procedure called HomogeneousResolution in figure 2. With the same tools, one can check that the following formal expression deduced from the formula for variation of constants

$$W^{-1}\int\left(W\left(\frac{\partial P}{\partial \dot{X}}\right)^{-1}\nabla P\right)(\Phi_j,\Gamma_j,\Lambda_j,\widetilde{\Theta},\widetilde{U})\,dt$$

is a solution of system (5). This expression can be computed at precision j by a procedure called ConstantsVariation.

3.4 Algorithm

We summarize our algorithm in figure 2. This is a simplified presentation where the technical details are neglected.

A preprocessing is necessary to construct, from a SLP coding Σ, another SLP which encodes the associated linear variational system ∇P and the expressions used during its integration. This step relies mainly on Theorem 3.

Figure 2: Local Algebraic Observability Test

```
Input          : $\dot{X} - F(X,\Theta,U)$,  $Y - G(X,\Theta,U)$
Output         : Succeed, a boolean

Preprocessing    Construction of the SLP coding $\frac{\partial P}{\partial \dot{X}}, \frac{\partial P}{\partial X}, \frac{\partial P}{\partial \Theta}$.

Initialization   Choice of a prime number;

                 $U \leftarrow$ Random Power Series mod $t^{n+\ell+1}$;
                 $\Theta \leftarrow$ Random integers; $X \leftarrow$ Random integers;
                 Succeed$\leftarrow$ true; $\nu \leftarrow 1$; $\Lambda \leftarrow 0_{n\times\ell}$; $\Gamma \leftarrow \mathrm{Id}_{n\times n}$;

while $\nu \leq n+\ell+1$ do
    $W \leftarrow$ HomogeneousResolution $\left(\frac{\partial P}{\partial \dot{X}}\dot{W} + \frac{\partial P}{\partial X} W = 0\right)$mod $t^{\nu}$;

    $(\Phi,\Lambda,\Gamma) \leftarrow (\Phi,\Lambda,\Gamma)$ + ConstantsVariation$\left(W, \nabla P\right)$ mod $t^{\nu}$;

    $\nu \leftarrow 2\,\nu$; # Increase Order
end while

    JacobianMatrix$\leftarrow$ Coeffs$\left(\nabla Y\,(\Phi,\Gamma,\Lambda), t^{j}, j = 0,\dots,n+\ell\right)$;

Test             if $n+\ell >$ Rank(JacobianMatrix)
                     then Succeed := false
                 end if
```

The next part of the algorithm consists in the computation at order $n+\ell+1$ of the power series solution of ∇P. We recall that in one iteration, the number of correct coefficients is doubled (see Theorem 2 in [11]).

After the main loop, the procedure Coeffs evaluates ∇Y on the series Φ_j, Γ_j and Λ_j where $j = \ln_2(n+\ell+1)$; this furnishes the coefficients of the Jacobian matrix (see Section 3.1). Last, the rank computations described in Corollary 1 are performed to solve the local observability problem.

If there is more than one output variable, the evaluation of ∇Y and the rank computations necessary to determine ϕ can be done in the main loop: the computation can be stopped when the expected rank is reached or when the computed ranks become stationary. Thus, we can determine the order of derivation ν and avoid useless computations. We now present a upper bound for the arithmetic complexity.

3.5 Arithmetic Complexity Estimation

Hereafter, let L denote the complexity of evaluation of the system Σ and let $\mathcal{M}(j)$ represent the multiplication complexity of two series at order $j+1$. Using classical multiplication formula, we have $\mathcal{M}(j) \in \mathcal{O}(j^2)$. Furthermore, let $\mathcal{N}(j)$ denotes the number of arithmetic operations sufficient for the multiplication of two square $j \times j$ matrices. Using classical algorithms, we have $\mathcal{N}(j) \in \mathcal{O}(j^3)$.

PROPOSITION 2. *The number of arithmetic operations on the ground field used in the algorithm presented in § 3.4 is bounded by*

$$\mathcal{O}\Big(\mathcal{M}(\nu)\big(\mathcal{N}(n+\ell) + (n+m)L\big) + m\nu\mathcal{N}(n+\ell)\Big).$$

PROOF. From construction done in Section 3.1 and Theorem 3, we conclude that $\mathcal{O}(\mathcal{N}(n+\ell) + (n+m)L)$ is a upper bound for the complexity of evaluation of the SLP coding $\partial P/\partial(\dot{X}, X, \Theta), \nabla P$ and ∇Y. Hence, at each step, the number of arithmetic operations necessary to evaluate this SLP on power series truncated at order j, is bounded by

$$\mathcal{O}\big(\mathcal{M}(j)(\mathcal{N}(n+\ell) + (n+m)L)\big).$$

Furthermore, the determination of the first j terms of the solution series of a system of linear ODE (5) requires

$$\mathcal{O}\big(\mathcal{M}(j)(\mathcal{N}(n) + \mathcal{N}(n+\ell))\big)$$

arithmetic operations by the well-known method of integrating factors (see § 5.2 in [4] for more details). So, as our operator is quadratic and as $\mathcal{M}(j) + \mathcal{M}(\lfloor j/2 \rfloor) + \cdots = \mathcal{O}(\mathcal{M}(j))$, the arithmetic complexity of the computations of the Jacobian matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$ is bounded by

$$\mathcal{O}\big(\mathcal{M}(\nu)(\mathcal{N}(n+\ell) + (n+m)L)\big).$$

To conclude, we notice that the cost of a rank computation for a $i \times j$ matrix is $\mathcal{O}(j\mathcal{N}(i)/i)$ if $i \leq j$. The Corollary 1 describes the rank computations done at the end of the main loop of our algorithm. □

We have presented the complexity of our algorithm in term of arithmetic operations on $\mathbb{Q}$. Such an operation requires a time proportional to the size of its operands. Using modular techniques, we control the growth of the integers involved in the computations. We estimate now an upper bound on these integers; this bound will be used in Section 3.7 in order to estimate the probability of success of our algorithm.

3.6 Growth of the Integers

The forthcoming estimations relies on the formal definition of the Jacobian matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$ and are not dependent of the computations described in § 3.1 and 3.3. Let us introduce a measure for the size of a $(n+\ell+r)$-variate polynomial which influence the growth of the integers (see [6] for more details).

Definition 3. Let $\mathcal{A}$ be a finite set of non zero integers. The (logarithmic) *height* of $\mathcal{A}$ is defined as $ht(\mathcal{A}) := \ln|\mathcal{A}|$ with $|\mathcal{A}| := \max\{|\alpha| + 1, \alpha \in \mathcal{A}\}$. The height of a polynomial with integer coefficients is defined by the height of its set of coefficients.

We present in the following lemma some properties of height:

LEMMA 1. *Let $p_1, \dots, p_s$ be $(n+\ell+r)$-variate polynomials with integer coefficients, x an integer and ∂ a partial derivation ($\partial/\partial x$ for example).*

- $ht(\partial p) \leq ht(p) + \ln \deg p;$
- $ht(p(x)) \leq ht(x) \deg p + ht(p);$
- $ht\big(\sum_{i=1}^{s} p_i\big) \leq \max_{i=1..s} ht(p_i) + \ln s;$
- $ht(p_1 p_2) \leq \min\{\deg p_1, \deg p_2\} \ln(n+\ell+r+1)$
  $+\, ht(p_1) + ht(p_2).$

We use the notations introduced in Section 1.1 and we denote by h (resp. d) the maximum height (resp. degree) of the numerator and of the denominator of the expression involving in system Σ.

PROPOSITION 3. *Let h_0 be the maximum of heights of the integers X_0, $\widetilde{\Theta}$ and of the integer coefficients of $\widetilde{U}$. We have,*

- $ht\big(\mathrm{denom}\, Y^{(j)}(X_0)\big) \leq (2j+1)(n+m)\Big(h + d\big(h_0$
  $+\, 2\ln(n+\ell+r+1)\big)\Big);$

- $ht(\text{numer}\, Y^{(j)}(X_0)) \leq (2j+1)(n+m)\big((2\ln(n+\ell+r+1)+h_0)d+h\big) + (j+1)\ln 2n(n+m)d + (2j+1)\ln(2j+1)$.

PROOF. As we are interested in an upper bound, we do not consider the reduced form of the fractions f_i and g_i involved in $\mathcal{L}g$ but we consider that all these fractions share the same denominator q. So, $\mathcal{L} = \left(\sum f_i\partial_i\right)/q$ and q is the common denominator of all g_i. Thus, the degree of these numerators and denominators is bounded by $(n+m)d$ and the height by $(n+m)\big(h+d\ln(n+\ell+r+1)\big)$. Let us notice that the denominator of $Y^{(j)}$ is q^{2j+1}; these facts and Lemma 1 prove the first part of our proposition.

We prove the second part by induction; let us consider the sequence $(v_j)_{j\in\mathbb{N}}$ of polynomials defined by the numerator of g as initial condition v_0 and by the recurrence relation

$$v_{j+1} := \sum f_i\big(q\partial_i v_j - (2j+1)v_j\partial_i q\big).$$

By construction, v_j is equal to the numerator of $Y^{(j)}$. Thus, the degree of v_j is bounded by $(2j+1)(n+m)d-j$ and we obtain the following recurrence relation from Lemma 1:

$$\begin{aligned} ht(v_{j+1}) \leq\ & 2(n+m)\big(2d\ln(n+\ell+r+1)+h\big) \\ & + ht(v_j) + \ln 2n(2j+1)(n+m)d. \end{aligned}$$

This is sufficient to conclude. □

Modular computation. We have shown that the size of the coefficients of the final specialized Jacobian matrix is mainly linear in the differentiation index ν. But some intermediate computations can require integers of bigger size. In order to construct a practical and efficient algorithm, we have to avoid this growth using modular techniques.

Almost all the operations used in our algorithm can be performed on a finite field $\mathbb{F}_p$. But, when we choose a prime number p, we have to avoid the cancellation of $\partial P/\partial\dot{X} \bmod t$ and of the determinant of $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$.

The cancellation of $\partial P/\partial\dot{X} \bmod t$ can be checked at the begining of our algorithm. Thus, the probabilistic aspects concern mainly the choice of specialization and of a prime number s.t. the determinant of $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$ does not vanish modulo p when this matrix is of full generic rank.

3.7 Probabilistic Aspects

Hereafter, we call a *singular point*, a set of specializations where the Jacobian matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$ is not of full generic rank. Thus, a singular point is a zero of the polynomial associated with a minor of this matrix. We estimate the probability for a specializations to be a singular point with the following proposition.

PROPOSITION 4. (R. Zippel & J. Schwartz [33]) *Let q be a s-variate polynomial of total degree D and Ω a set of integers. The worst case bound for the probability that a point in Ω^s will be a zero of q is $D/\#\Omega$.*

This result shows the relation between the choice of the size h_0 of the used specializations and the probability of success of our algorithm. In fact, as the determinant of the matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$ is a polynomial of degree bounded by $D := (n+\ell)(2\nu+1)(n+m)d$, a point in the set $\{0,\ldots,\mu_1 D\}^{(n+\ell)(r+1)}$ is not a singular point with probability at least $1-1/\mu_1$. Furthermore, we can estimate the probability that the determinant is divisible by a prime number p with the arithmetic analogue of Proposition 4.

PROPOSITION 5. (§ 18 in [31]) *For any integers a and b such that $b < a < c$, the probability that a prime number p between $b+1$ and $2b$ divides a is bounded by $2\ln c/b$.*

From Proposition 3 and Lemma 1, we can estimate the size of the coefficients of the specialization of the Jacobian matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$. Thus, using Hadamard's inequality, we find the following rough upper bound for the size of the specialized determinant:

$$ht(c) := \big(2\ln(n+\ell+r+1)+h_0\big)D + (n+\ell)(2\nu+1)\big((n+m)h + \ln 2nD\big)$$

Thus, if the computations are done modulo a prime number p greater or equal to $2ht(c)\,\mu_2$ then the probability that the specialized determinant is not divisible by p is greater than $1-1/\mu_2$. These results lead to the estimation:

PROPOSITION 6. *Let μ be a positive integer and*

$$\begin{aligned} D &:= (n+\ell)(2\nu+1)(n+m)d, \\ ht(c) &:= \big(2\ln(n+\ell+r+1)+\ln D\big)D \\ &\quad + (n+\ell)(2\nu+1)\big((n+m)h+\ln 2nD\big). \end{aligned}$$

If the matrix $\partial(Y^{(i)})_{0\leq i\leq\nu}/\partial(X,\Theta)$ is of full generic rank then the determinant of this matrix specialized on random integers in $\{0,\ldots,\mu D\}$ is not divisible by a prime number $p > 2ht(c)\,\mu$ with probability at least $(1-1/\mu)^2$.

4. CONCLUDING REMARKS

Our algorithm is mainly based on generic rank computation. As shown in Corollary 1, the local observability property is associated to the fact that the used Jacobian matrix is of full rank. Hence, when our process states that a system is observable, this answer is certainly correct.

Using the results presented here and the elimination algorithm presented in [12, 29], one can test the global observability and retrieve the relations between the state variables, the outputs and the inputs. A forthcoming paper will be devoted to this aspect.

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