DE GRUYTER OLDENBOURG at – Automatisierungstechnik 2020; 68(6): 395–409

Foundations and Methods

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On global and local observability of nonlinear polynomial systems: a decidable criterion

Über die globale und lokale Beobachtbarkeit nichtlinearer polynomialer Systeme: Ein entscheidbares Kriterium

https://doi.org/10.1515/auto-2020-0027
Received February 28, 2020; accepted April 28, 2020

Abstract: It is very difficult to check the observability of nonlinear systems. Even for local observability, the observability rank condition provides only a sufficient condition. Much more difficult is the verification of global observability. This paper deals with the local and global observability analysis of polynomial systems based on algebraic geometry. In particular, we derive a decidable criterion for the verification of global observability of polynomial systems. Our framework can also be employed for local observability analysis.

Keywords: nonlinear observability, polynomial systems, algebraic geometry

Zusammenfassung: Im Gegensatz zu linearen Systemen stellt sich die Beobachtbarkeitsanalyse für nichtlineare Systeme als schwierig heraus. Während die Beobachtbarkeitsmatrix ein immerhin hinreichendes Kriterium für lokale Beobachtbarkeit zulässt, ist noch keine Aussage über die globale Beobachtbarkeit möglich. Dieser Aufsatz beschäftigt sich mit der Analyse der lokalen und globalen Beobachtbarkeit polynomialer Systeme auf der Basis algebraischer Geometrie. Schließlich wird ein entscheidbares Kriterium sowohl für lokale als auch für globale Beobachtbarkeit derartiger Systeme hergeleitet.

Schlagwörter: nichtlineare Beobachtbarkeit, polynomiale Systeme, algebraische Geometrie

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1 Introduction

The concepts of controllability and observability for linear time-invariant state space systems have been introduced by Kalman [22]. For this class of systems, the observability can be easily decided using the observability rank condition [22] or several equivalent formulations [17, 19]. In addition to the observer based controller design, the property of observability also plays an important role in system supervision and fault detection [13].

For nonlinear systems, controllability and observability were introduced in [20]. In the nonlinear case, the notion of observability is based on the indistinguishability of states. In contrast to the linear case there are even different observability concepts, such as local, weak and global observability [3, 45]. Unfortunately, easy criteria for observability do not exist for nonlinear systems.

A sufficient condition for local observability is the observability rank condition [45]. Using the global inverse function theorem [51], this condition can be extended to global observability [8]. A semi-global approach is discussed in [36, 37] using interval arithmetics.

For the special class of polynomial systems there are some sufficient and necessary observability conditions [3, 4, 40, 41, 48, 49], which rely on the equality of polynomial ideals. These criteria are constructive if one wants to prove a system to be observable. However, one cannot prove a system to be not (globally) observable since these criteria include an infinite number of polynomials and there is no condition that tells when enough polynomials have been considered. Thus, observability is not decidable with these criteria, which is unsatisfactory. In addition, the approach presented in [40, 41] requires extremely high computational effort. The idea in [24] allows to decide observability for a specific initial system state, but cannot test all states at once. The global observability criterion in [23] investigates zero sets of ideals. Another work [26] of these authors discusses a necessary geometrical criterion for local observability according the definition in [20].

We use the idea from the works in the previous paragraph and give an algebraic criterion that can decide if a system is observable or not by means of polynomial ideals. This includes an algorithm to compute the previously noted ideals such that their equality can be tested. In addition, the set of locally observable points, following the definition according to [3, 45] instead of [20], can be computed. For this purpose the well-known Lie derivative is extended to polynomial ideals.

This article is organized as follows: In section 2 some basic concepts of differential equations and their solutions as well as the nonlinear observability concepts are recalled. Afterwards a short introduction to polynomial ideals and their zero sets is given before the condition from [48] is discussed. In order to fill the missing part for a decidable criterion and to formulate an algorithm to test observability, the well-known Lie derivative of scalar functions is generalized to polynomial ideals in section 3. In this section also a few properties are derived that are necessary to formulate the observability criterion. Afterwards, the main result is stated in section 4, namely a decidable global and local observability criterion. These criteria are applied to some examples. Finally, possible extensions and improvements of the proposed criteria are discussed in section 6 before a summary is given in section 7.

2 Preliminaries

2.1 Differential equations and their solution

In this article we consider a special class of system equations in the form

$$\dot{x}(t) = f(x(t)), \quad x(0) \in \mathcal{X} \subseteq \mathbb{R}^n \tag{1a}$$

$$y(t) = h(x(t)), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ denote the (possibly redundant) system coordinates and output,[1] respectively. The functions f and h are assumed to be polynomials in their arguments. Further the set $\mathcal{X}$ is defined by polynomial equations: $\mathcal{X} = \{x \mid g(x) = 0\}$ and any solution of (1a) fulfills

$$g(x(t)) = 0. \tag{1c}$$

1 The extension from a single output to multiple outputs is a trivial step, see subsection 6.1. Hence, the concept is explained on, but not restricted to systems with scalar output.

This ensures that for any $x(0) = x_0 \in \mathcal{X}$ the solution $x(t)$ of (1a) will be contained in $\mathcal{X}$, too. Such a formulation allows to handle systems that cannot be written by polynomial functions of minimal coordinates and may avoid introducing singularities [28, 50]. The vector-valued function g may consist of no component, effectively removing the constraint such that $\mathcal{X} = \mathbb{R}^n$. In this case (1) is a state space representation with system state x.

Note that (1c) is fulfilled, if the initial condition x_0 fulfills $g(x_0) = 0$. Thus, (1c) is not a restriction on $x(t)$ and the solution $x(t)$ can be studied solely on (1a). Since f is analytic, for any initial value $x(0) = x_0 \in \mathcal{X}$ one has a unique solution $x(t)$ for an interval $t \in [0, T)$ that is locally analytic. For any fixed t the map $\varphi_t(\cdot)$ maps the initial value x_0 to the solution $x(t)$ at time t and is called the *flow* of system (1a), see [1].

The time derivative $\dot{y}(t)$ of the output $y(t) = h(x(t))$, where $x(t)$ is a solution of (1a), is the *Lie derivative*

$$\mathrm{L}_f h(x) = h'(x) f(x)$$

of h along the vector field f, where h' denotes the gradient of h. Repeated Lie derivatives are defined recursively by

$$\mathrm{L}_f^{i+1} h(x) = \mathrm{L}_f \mathrm{L}_f^i h(x)$$

with $\mathrm{L}_f^0 h(x) = h(x)$, see [21, 38]. Thus, we have

$$\tfrac{\mathrm{d}^i y}{\mathrm{d}t^i}(t) = \mathrm{L}_f^i h(x(t)).$$

Since the output $y(t)$ is locally analytic, it equals its Taylor series, which, in this case, equals the *Lie series*

$$y(t) = \sum_{i=0}^{\infty} \tfrac{t^i}{i!} \tfrac{\mathrm{d}^i y}{\mathrm{d}t^i}(0) = \sum_{i=0}^{\infty} \tfrac{t^i}{i!} \mathrm{L}_f^i h(x_0)$$

that is convergent in an interval containing $t = 0$ [18]. Using this notation one can write condition (1c) as

$$\forall x \in \mathcal{X} : \mathrm{L}_f^i g_j(x) = 0 \tag{2}$$

for any component g_j of g and any derivative order $i \in \mathbb{Z}_{\geq 0}$. Therefore, the constraint defined by (1c) fulfilling (2) describes an invariant subset $\mathcal{X}$ of the superset $\mathbb{R}^n$.

2.2 Observability

Two points $x, z \in \mathcal{X}$ are called *indistinguishable* on the interval $[0, T] = \mathcal{T}$ if

$$\forall t \in \mathcal{T} : h(\varphi_t(x)) = h(\varphi_t(z)).$$

Thus, for a system (1) one can define the set of points

$$\mathcal{I}(x) = \{z \in \mathcal{X} \mid h(\varphi_t(x)) = h(\varphi_t(z))\}$$

that are indistinguishable from a point x. If for all $x \in \mathcal{X}$ this set fulfills $\mathcal{I}(x) = \{x\}$, namely all points different from x are distinguishable from x, system (1) is called *globally observable* [20]. A system (1) is called *locally observable at point* $x_0 \in \mathcal{X}$ if in an open neighborhood U_{x_0} of x_0

$$\forall x \in U_{x_0} : U_{x_0} \cap \mathcal{I}(x) = \{x\}. \tag{3}$$

If system (1) is locally observable for any $x_0 \in \mathcal{X}$, (1) is called *locally observable*[2] [3, 45].

Since two output trajectories $h(\varphi_t(x_0))$ and $h(\varphi_t(z_0))$ of a system (1) are equal if and only if the coefficients of their Lie series are equal, one can study the observability properties on the *observability map*

$$q(x) = \begin{pmatrix} h(x) \\ \mathrm{L}_f h(x) \\ \mathrm{L}_f^2 h(x) \\ \mathrm{L}_f^3 h(x) \\ \vdots \end{pmatrix} \tag{4}$$

and find the equivalent definition

$$\mathcal{I}(x) = \{z \in \mathcal{X} \mid q(x) = q(z)\} \tag{5}$$

for the set of indistinguishable points. The system is globally observable if and only if the map (4) is one-to-one, i. e., injective, on $\mathcal{X}$.

Remark 1. *Consider the observability map* (4) *with a finite number of Lie derivatives. The associated Jacobian matrix*

$$O(x) = q'(x)$$

is called observability matrix. *Assume that the* observability rank condition

$$\operatorname{rank} O(x_0) = n \tag{6}$$

is satisfied at a point $x_0 \in \mathcal{X}$ under the assumption $\mathcal{X} = \mathbb{R}^n$. Then, system (1) *is locally observable at x_0. Condition* (6) *results from the inverse function theorem and is only a sufficient condition.*

The set (5) is defined by polynomial equations and can thus be studied using algebraic geometry. Before we recall a known observability criterion, a short introduction to polynomial ideals is given.

2 Note that the notion of local observability differs form [20], where a system fulfilling (3) is called *weakly observable* (at x_0). However, the local observability introduced according to [3, 45] is more intuitive as global observability implies local observability, which does not hold for the observability concepts from [20, p. 733].

2.3 Polynomial ideals

If f and h are polynomial functions, any Lie derivative $\mathrm{L}_f^i h(x)$ is a polynomial, too. Such polynomials form a commutative ring $k[x] = k[x_1, \dots, x_n]$. In our case the field k shall contain the rational numbers,[3] i. e., $\mathbb{Q} \subseteq k$. An *ideal* $I \subseteq k[x]$ is a set of polynomials with the following properties:

1. $0 \in I$,
2. $a, b \in I \implies a + b \in I$,
3. $a \in I, c \in k[x] \implies ac \in I$.

Hilbert's basis theorem asserts that every polynomial ideal with a finite number of variables is finitely generated. This means that for every polynomial ideal $I \subseteq k[x]$ there is a set $G = \{g_1, \dots, g_s\}$ of polynomials such that

$$I = \{a_1 g_1 + \ldots + a_s g_s \mid g_i \in G, a_i \in k[x]\}$$

and one writes $I = \langle g_1, \dots, g_s \rangle$. A *Gröbner basis* G of an ideal I is a special generating set where each leading term of a polynomial in I is a multiple of a leading term of a polynomial in G [5, p. 206]. This definition depends on a ordering of the monomials. While there is only one admissible ordering $1 < x_1 < x_1^2 < x_1^3 < \cdots$ in the univariate case, there are several options in the multivariate case, e. g., the lexicographic ordering

$$1 < x_1 < x_1^2 < \cdots < x_2 < x_2^2 < \cdots < x_n < x_n^2 < \cdots.$$

Gröbner bases are in particular useful to test the ideal membership of a polynomial. For a particular monomial order there is a unique *reduced Gröbner basis* for a polynomial ideal [11, p. 93] that can be used to check if ideals are equal or not.

The zero set of an ideal I is called its *variety*

$$\mathbf{var}(I) = \{x \in \bar{k}^n \mid \forall a \in I : a(x) = 0\}.$$

Here, $\bar{k}$ denotes the algebraic closure[4] of k, which contains the complex numbers. Since we are only interested in real

3 One could have set k to be the real numbers directly. However, these do not permit an exact representation on computers and a slight change of the coefficients of a polynomial can lead to a large change of its zeros. For that reason one would like to choose numbers that permit such an exact representation, where the specific choice may depend on the considered problem. The polynomial coefficients may also depend on parameters, see subsection 5.5 for an example. The restriction that k contains the rationals is imposed to exclude the case that k is a finite field.

4 A field k is *algebraically closed* if every non-constant polynomial with coefficients in k has a root in k. An *algebraic closure* $\bar{k}$ of a field k is an algebraic extension of k that is algebraically closed.

solutions, we make use of the *real variety*

$$\mathbf{var}^{\mathbb{R}}(I) = \mathbf{var}(I) \cap \mathbb{R}^n.$$

There are different ideals with the same (real) zero set. However, these ideals are all a subset of one and the same ideal: the *(real) radical* of the latter. For an ideal I these are denoted by[5] $\mathbf{rad}(I)$ and $\mathbf{rad}^{\mathbb{R}}(I)$, respectively. One has $I \subseteq \mathbf{rad}(I) \subseteq \mathbf{rad}^{\mathbb{R}}(I)$. An ideal that equals its radical is called a *radical ideal* while an ideal that equals its real radical is called a *real ideal*. By Hilbert's Nullstellensatz [11, pp. 175] there is a one-to-one correspondence between radical ideals and varieties. Similarly, the real Nullstellensatz [7, p. 84] asserts that there is a one-to-one correspondence between real ideals and real varieties. This allows us to study the (real) zero sets on the (real) radical ideals.

Consider, for example, the ideal

$$I = \left\langle \left(x^2+1\right)y^2 \right\rangle \subseteq \mathbb{Q}[x,y]$$

generated by a single element. The corresponding variety $\mathbf{var}(I)$ is composed of three lines in $\mathbb{C}^2$: the line $y = 0$ and the parallel lines $x = \pm\mathrm{i}$, where i denotes the imaginary unit. The same variety corresponds to the greater ideal

$$\mathbf{rad}(I) = \left\langle \left(x^2+1\right)y \right\rangle$$

and there is no other ideal containing $\mathbf{rad}(I)$ with the same variety. The real variety $\mathbf{var}^{\mathbb{R}}(I)$ contains only the line $y = 0$, hence $\mathbf{rad}^{\mathbb{R}}(I) = \langle y \rangle$.

In addition, we need at least the following operations on ideals: The *ideal sum* $I + J$ is the set of polynomials that are contained in I or in J. The variety of the ideal sum is the intersection of the varieties of I and J, i. e., $\mathbf{var}(I + J) = \mathbf{var}(I) \cap \mathbf{var}(J)$.

The *intersection* $I \cap J$ of ideals is an ideal that contains all polynomials that are both contained in I and J. Geometrically this operation corresponds to the union $\mathbf{var}(I \cap J) = \mathbf{var}(I) \cup \mathbf{var}(J)$ of varieties. Note that in the example above one could write

$$I = \left\langle x^2+1 \right\rangle \cap \left\langle y^2 \right\rangle$$

and

$$\mathbf{rad}(I) = \left\langle x^2+1 \right\rangle \cap \langle y \rangle.$$

5 This notation is borrowed from [5]. A more common notation for the radical or real radical of I is $\sqrt{I}$ or $\sqrt[\mathbb{R}]{I}$, respectively.

The *ideal quotient* $I : J$ of I and J contains all those polynomials, whose product with any polynomial in J is contained in I. In the example above one has

$$\begin{aligned} I : \langle y \rangle &= \left\langle \left(x^2+1\right)y \right\rangle \\ (I : \langle y \rangle) : \langle y \rangle = \mathbf{rad}(I) : \langle y \rangle &= \left\langle x^2+1 \right\rangle. \end{aligned}$$

Geometrically this operation is related to the difference set $\mathbf{var}(I) \setminus \mathbf{var}(J)$, which is in general not a variety. The variety $\mathbf{var}(\mathbf{rad}(I) : J)$ is the smallest variety containing the difference set $\mathbf{var}(I) \setminus \mathbf{var}(J)$.

2.4 An observability criterion

In [48] and [4] a necessary and sufficient observability criterion was derived that is based on polynomial ideals and their real varieties. We recall this criterion with application to the redundant formulation (1): The set

$$\{(x,z) \in \mathcal{X} \times \mathcal{X} \mid q(x) = q(z)\},$$

where q is the observability map (4) of system (1), is the set of pairs of indistinguishable points of that system. This set is the real variety of the ideal[6]

$$I = \langle g(x), g(z), q(x) - q(z) \rangle \subseteq k[x,z]. \tag{7}$$

Here, adding the components of $g(x)$ and $g(z)$ to the set of generators ensures that the points x and z are constrained on $\mathcal{X}$. The considered system is globally observable if and only if this set equals

$$\{(x,z) \in \mathcal{X} \times \mathcal{X} \mid x = z\} = \mathbf{var}^{\mathbb{R}}(J)$$

with $J = \langle g(x), x - z \rangle \subseteq k[x,z]$. If this is the case, equal observability maps $q(x)$ and $q(z)$ are equivalent to equal initial conditions x and z. This shows that the global observability of a dynamical system is equivalent to the equivalence of the real zero sets of two ideals I and J. The work [4] makes use of the real Nullstellensatz [7, p. 84] and states the observability criterion in terms of the equivalence of the real radicals of the ideals. However, it is not known how many Lie derivatives in the observability map have to be considered in order to generate the ideal I.

Before we state a computable criterion to determine if a system (1) is observable or not we generalize the well-known Lie derivative on polynomial ideals.

6 This is an abuse of notation since a basis for an ideal must be finite. One could think of that ideal as generated by the components of $g(x)$, $g(z)$, and the first N components of the infinite-dimensional vector $q(x) - q(z)$ such that the remaining components are contained in the ideal. The ascending chain condition [11, p. 80] asserts that there is always such a finite N.

3 The Lie derivative of polynomial ideals

We now generalize the Lie derivative on ideals. While polynomial ideals $I \subseteq k[x]$ are the set of $k[x]$-linear combinations of polynomials $g \in I$, the Lie derivative will be the set of $k[x]$-linear combinations of the Lie derivatives of g.

Definition 1. *Let $I \subseteq k[x] = k[x_1, \dots, x_n]$ be a polynomial ideal and $f \in k[x]^n$ a polynomial vector field. The set*[7]

$$\mathfrak{L}_f^m I = \left\{ \sum_i a_i \, \mathrm{L}_f^{n_i} g_i \,\middle|\, a_i \in k[x], n_i \in \mathbb{Z}_{\geq 0}, n_i \leq m, g_i \in I \right\} \quad (8)$$

is called the m-th Lie derivative of I with respect to f. The set

$$\begin{aligned} \mathfrak{L}_f^\infty I &= \left\{ \sum_i a_i \, \mathrm{L}_f^{n_i} g_i \,\middle|\, a_i \in k[x], n_i \in \mathbb{Z}_{\geq 0}, g_i \in I \right\} \quad (9) \\ &= \bigcup_{m \in \mathbb{Z}_{\geq 0}} \mathfrak{L}_f^m I \end{aligned}$$

is called the Lie derivative of I with respect to f.

Note that this definition differs from the Lie derivative of a scalar function: The Lie derivative of a particular order m of an ideal contains all Lie derivatives of the polynomials of that ideal of any order up to m (and $k[x]$-linear combinations of them).

The following proposition allows us to represent Lie derivatives of polynomial ideals by a generating set. The proofs of the propositions are given in Appendix A.

Proposition 1. *The m-th Lie derivative $\mathfrak{L}_f^m I$ of an ideal I is an ideal, too, and so is the Lie derivative $\mathfrak{L}_f^\infty I$.*

Like the Lie derivative of functions, the Lie derivative of polynomial ideals obeys a recursion relation:

Proposition 2. *Let $I \subseteq k[x]$ be a polynomial ideal, $f \in k[x]^n$ a vector field. For any $m \in \mathbb{Z}_{\geq 0}$ one has*

$$\mathfrak{L}_f^1 \mathfrak{L}_f^m I = \mathfrak{L}_f^{m+1} I.$$

This property is especially useful for computing Lie derivatives of a particular order iteratively or a sequence of Lie derivatives of different orders. In addition, it is an ingredient for the proof of the following proposition.

Proposition 3. *The ideals $\mathfrak{L}_f^m I$, $m \in \mathbb{Z}_{\geq 0}$ form an ascending chain:*

$$\begin{aligned} I = \mathfrak{L}_f^0 I \subset \mathfrak{L}_f^1 I \subset \mathfrak{L}_f^2 I \subset \cdots \\ \cdots \subset \mathfrak{L}_f^{M-1} I \subset \mathfrak{L}_I^M = \mathfrak{L}_f^{M+1} I = \cdots = \mathfrak{L}_f^\infty I. \end{aligned}$$

Note that this ascending chain condition is stronger than the ascending chain condition [11, p. 80] for arbitrary ideals. Here, only the first inclusions are strict. Once two such consecutive ideals are equal, the chain is saturated. In contrast to Definition 1, but following the same idea, the articles [23, 24, 26] introduce a sequence $\left(\mathfrak{L}_f^m I\right)$ of ideals and define $\mathfrak{L}_f^\infty I$ as the saturated ideal. By (9), however, one has a straight definition of $\mathfrak{L}_f^\infty I$.

With the help of Proposition 2 and Proposition 3 one can easily construct an algorithm to compute the Lie derivative $\mathfrak{L}_f^\infty I$ of an ideal $I \subseteq k[x]$ with respect to f as well as the smallest integer M such that $\mathfrak{L}_f^M I = \mathfrak{L}_f^\infty I$ as shown in Alg. 1. To do so, an order for the monomials of $k[x]$ has to be chosen such that the reduction in Line 9 is unique. The fact that at this point the set H is always a Gröbner basis [5] ensures that $\mathrm{L}_f h \in \langle H \rangle \iff r = 0$.

Algorithm 1 Lie derivative of an ideal

1: **function** LIE DERIVATIVE(f, $G = \{g_1, \dots, g_s\}$)
Ensure: $\mathfrak{L}_f^M \langle G \rangle = \mathfrak{L}_f^\infty \langle G \rangle = \langle H \rangle$
2:     $M \leftarrow -1$
3:     $H \leftarrow G$
4:     **repeat**
5:         $M \leftarrow M + 1$
6:         $H \leftarrow$ GRÖBNER BASIS($\langle H \rangle$)
7:         $R \leftarrow \emptyset$
8:         **for all** $h \in H$ **do**
9:             $r \leftarrow \mathrm{rem}(\mathrm{L}_f h, H)$
10:             **if** $r \neq 0$ **then**
11:                 $R \leftarrow R \cup \{r\}$
12:         $H \leftarrow H \cup R$
13:     **until** $R = \emptyset$
14:     **return** M, H

Proposition 4. *Alg. 1 returns a Gröbner basis H of the ideal $\langle H \rangle = \mathfrak{L}_f^\infty \langle G \rangle$ and the smallest number M such that $\mathfrak{L}_f^M \langle G \rangle = \mathfrak{L}_f^\infty \langle G \rangle$.*

With this algorithm one can finally compute a generating set for the ideal (7).

7 The sum is to be understood to range over tuples $(a_i, n_i, g_i) \in k[x] \times \mathbb{Z}_{\geq 0} \times I$ like the polynomials in an ideal generated by a set G can be written as a sum over products $a_i g_i$ where $(a_i, g_i) \in k[x] \times G$. There is no restriction on pairing the same element $g \in I$ with different derivative orders n_i or coefficients a_i. Although not required, the polynomials g_i may be chosen from a generating set of I only, as will be clear in Proposition 4. In this case these generators can be enumerated from g_1 to g_s and the sum in the defining set may be written $\sum_{i=1}^s \sum_{j=0}^m a_{ij} \, \mathrm{L}_f^j g_i$.

4 Decidable observability criteria

For the sake of a compact notation we use two copies of (1) and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ z \end{pmatrix}(t) = F(x(t), z(t)) = \begin{pmatrix} f(x(t)) \\ f(z(t)) \end{pmatrix} \tag{10a}$$

$$0 = G(x(t), z(t)) = \begin{pmatrix} g(x(t)) \\ g(z(t)) \end{pmatrix} \tag{10b}$$

$$d(t) = H(x(t), z(t)) = h(x(t)) - h(z(t)) \tag{10c}$$

with the residuum d as system output. Now we can formulate the main results.

4.1 Global observability

The following theorem is the algebraic version [4] of [23, 26, 48] applied to systems of the form (1), namely, the equivalence of the two ideal in subsection 2.4.

Theorem 1. *Let F, G, and H as in* (10) *and let $\Delta(x,z) = x - z$. The system* (1) *is globally observable if and only if*

$$\mathbf{rad}^{\mathbb{R}}(\mathfrak{L}_F^{\infty}\langle G, H\rangle) = \mathbf{rad}^{\mathbb{R}}(\langle G, \Delta\rangle). \tag{11}$$

Proof. Denote by

$$Q(x,z) = \begin{pmatrix} \mathrm{L}_F^0 H(x,z) \\ \mathrm{L}_F^1 H(x,z) \\ \vdots \end{pmatrix}$$

the observability map of (10). From the linearity of the Lie derivative and the structure of the vector field F follows directly that

$$Q(x,z) = q(x) - q(z)$$

with q being the observability map of (1).

From the structure of (10) and (2) follows that $\mathfrak{L}_F^{\infty}\langle G\rangle = \langle G\rangle$. Thus,

$$\mathfrak{L}_F^{\infty}\langle G, H\rangle = \langle G\rangle + \mathfrak{L}_F^{\infty}\langle H\rangle.$$

By the real Nullstellensatz, (11) is equivalent to the equality of the real varieties

$$\mathbf{var}^{\mathbb{R}}(\langle G\rangle + \mathfrak{L}_F^{\infty}\langle H\rangle) = \mathbf{var}^{\mathbb{R}}(\langle G, \Delta\rangle),$$

which can be written as intersections

$$\mathbf{var}^{\mathbb{R}}(\langle G\rangle) \cap \mathbf{var}^{\mathbb{R}}(\mathfrak{L}_F^{\infty}\langle H\rangle) = \mathbf{var}^{\mathbb{R}}(\langle G\rangle) \cap \mathbf{var}^{\mathbb{R}}(\langle \Delta\rangle). \tag{12}$$

($\Longrightarrow$) Assume (1) is globally observable. Then for $x, z \in \mathcal{X}$ the equation $Q(x,z) = q(x) - q(z) = 0$ is equivalent to $x - z = 0$. Since

$$x, z \in \mathcal{X} \implies (x,z) \in \mathcal{X}^2 = \mathbf{var}^{\mathbb{R}}(\langle G\rangle)$$

and

$$Q(x,z) = 0 \implies (x,z) \in \mathbf{var}^{\mathbb{R}}(\mathfrak{L}_F^{\infty}\langle H\rangle),$$

the equality (12) and thus (11) follows.

($\Longleftarrow$) Assume (11), which is equivalent to (12). The variety on the left hand side of (12) contains all pairs (x,z) that satisfy $x, z \in \mathcal{X}$ and $q(x) = q(z)$. However, this variety is a subset of $\{(x,z) \mid x - z = 0\}$. Thus, (1) is globally observable. □

The ideal $I = \mathfrak{L}_F^{\infty}\langle G, H\rangle$ as well as its real radical are computable. Thus, Theorem 1 shows that the problem of global observability of a system (1) is decidable.

If a system (1) is globally observable, one might ask, how many derivatives of the system output are required in order to reconstruct the system state. Next to the Lie derivative Alg. 1 computes a number M such that

$$\mathfrak{L}_F^{M}\langle G, H\rangle = \mathfrak{L}_F^{\infty}\langle G, H\rangle = I.$$

Thus, the truncated observability map (4) after the M-th Lie derivative is one-to-one on $\mathcal{X}$. This injectivity, however, may be given for a lower number of output derivatives, too.

The computation of the real radical is a very sophisticated task [12, 32]. It is even much more elaborate than the (usual) radical of an ideal, which is itself computational expensive. Since we are only interested on testing the equality of real radical ideals, one may think of a less elaborate way to do this.

If system (1) has no constraint (1c) the ideal $J = \langle G, \Delta\rangle$ is already real. The same will be true for most well-posed systems. Further, the relation

$$I \subseteq \mathbf{rad}^{\mathbb{R}}(I) \subseteq \mathbf{rad}^{\mathbb{R}}(J).$$

holds. While the first inclusion follows directly from the definition of the real radical, the second one is easily argued with the corresponding real varieties: For each $(x,z) \in \mathcal{X}^2$ in $\mathbf{var}^{\mathbb{R}}(\langle \Delta\rangle)$ one has $Q(x,z) = 0$ and thus

$$\mathbf{var}^{\mathbb{R}}(J) = \mathbf{var}^{\mathbb{R}}(\langle G\rangle) \cap \mathbf{var}^{\mathbb{R}}(\langle \Delta\rangle) \subseteq \mathbf{var}^{\mathbb{R}}(\langle G\rangle) \cap \mathbf{var}^{\mathbb{R}}(\mathfrak{L}_F^{\infty}\langle H\rangle) = \mathbf{var}^{\mathbb{R}}(I).$$

Hence, the ideals $\mathbf{rad}^{\mathbb{R}}(I)$ and $\mathbf{rad}^{\mathbb{R}}(J)$ are equal if and only if the reverse inclusion

$$\mathbf{rad}^{\mathbb{R}}(J) \subseteq \mathbf{rad}^{\mathbb{R}}(I)$$

also holds true. Thus, the problem can be reduced to test the real radical membership and computing a much simpler real radical.

If the real radical of an ideal or the real radical membership problem is not implemented in the preferred symbolic software, one may also use quantifier elimination to test the ideal membership:

$$\forall x, z : (g(x) = 0 \wedge g(z) = 0) \implies \left(\left(\bigwedge_{l \in I} l(x,z) = 0 \right) \implies x - z = 0 \right).$$

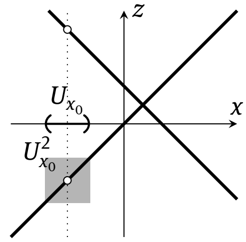
This method was already discussed in [40, 41]. Alternatively, the variety associated with the ideal can be written as a boolean combination of polynomial equations. This leads to an description by an quantifier-free formula, which can be simplified regarding to real zeros using the open source tool SLFQ [10].

4.2 Local observability

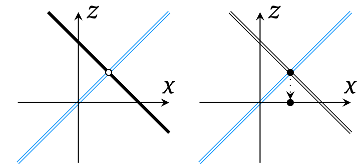
Before we state a similar theorem for local observability, the underlying idea shall be explained on an unconstrained system. Figure 1a visualizes the (real) variety of $\mathfrak{L}_F^\infty\langle H\rangle$ for a system that is not globally observable. This can be seen from the fact that this variety contains not only the (hyper-)plane $x - z = 0$. The reader is invited to verify that such a variety results from a system of the form $\dot{x} = x$ with output $y = x - x^2$.

In this diagram we consider the plane $x = x_0$. The intersection of the depicted variety with this plane is exactly the set $\mathcal{I}(x_0)$ of points indistinguishable from x_0 in that plane, visualized by the white circles. While this set contains two points for the particular x_0, there is only x_0 contained in the intersection with a neighborhood U_{x_0}. Thus, the system is locally observable at this point.

In order to state a criterion for local observability at such points, we show that in $U_{x_0} \times U_{x_0} = U_{x_0}^2$ only the plane $x - z = 0$ is contained. This is done as follows: From the real variety of $\mathfrak{L}_F^\infty\langle H\rangle$ the plane $x - z = 0$ is removed. The resulting set may not be a variety any more, but the Zariski closure [11, pp. 199] of the difference set fills the "holes" that are cut by the diagonal plane (see Figure 1b). The resulting variety is then again intersected with the plane $x - z = 0$. This leaves all points that are contained in both, the variety $\mathbf{var}(\langle x - z\rangle)$ and a component of $\mathbf{var}^{\mathbb{R}}(\mathfrak{L}_F^\infty\langle H\rangle)$ different from the plane $x - z = 0$. Since these points are contained in at least two different irreducible components in $U_{x_0}^2$, the indistinguishable set is not unique. Thus, the system is not locally observable at these points projected onto $z = 0$ (see Figure 1b).



(a) Indistinguishable set for a specific point and open neighborhood



(b) Process of eliminating all but not locally observable points: difference set (left) that Zariski closure includes the intersection points, too; intersection with diagonal plane and projection (right)

Figure 1: Variety (union of the diagonal lines) of the Lie derivative of the residuum output.

Theorem 2. *Let F, G, and H as in* (10) *and $J = \langle x - z\rangle$. System* (1) *is locally observable at a point $x_0 \in \mathcal{X}$ if and only if x_0 is not contained in the variety of*

$$\left(\left(\mathbf{rad}^{\mathbb{R}}(\mathfrak{L}_F^\infty\langle G, H\rangle) : J\right) + J\right) \cap k[x]. \tag{13}$$

Proof. Denote by I the ideal $\mathfrak{L}_F^\infty\langle G, H\rangle$.

Assume (1) is not locally observable at a point $x_0 \in \mathcal{X}$. Then in any neighborhood U_{x_0} of x_0 there is an $\bar{x} \in U_{x_0}$ such that $(\bar{x}, z) \in \mathbf{var}(I)$ is fulfilled by a $z = z_0 \in U_{x_0}$ different from $\bar{x}$. Since $z_0 \neq \bar{x}$ we must have

$$(\bar{x}, z_0) \in \mathbf{var}^{\mathbb{R}}(I) \setminus \mathbf{var}(J).$$

The latter set cannot be empty, thus its Zariski closure

$$\mathbf{var}\left(\mathbf{rad}^{\mathbb{R}}(I) : J\right) \tag{14}$$

is not empty. Using this argument on a neighborhood U_{x_0} that is an open ball with center x_0 and radius ϵ and let $\epsilon \to 0$ shows that (14) contains also (x_0, x_0). The ideal sum in (13) corresponds to the intersection of the variety (14) with $\mathbf{var}(J)$. Thus, the point (x_0, x_0) is contained in the zero set of the ideal sum. This shows that x_0 is a zero of all polynomials in (13).

Assume (1) is locally observable at the point $x_0 \in \mathcal{X}$. Then there is a neighborhood U_{x_0} of x_0 such that if (x, z) is contained in

$$V = \mathbf{var}^{\mathbb{R}}(I) \cap U_{x_0}^2$$

the equation $x = z$ must hold true. Thus, no irreducible component of $\mathbf{var}^{\mathbb{R}}(I)$ other than the irreducible variety $\mathbf{var}(J)$ is contained in $U_{x_0}^2$ and thus the Zariski closure

$$\mathbf{var}\left(\mathbf{rad}^{\mathbb{R}}(I) : J\right) \tag{15}$$

of $\mathbf{var}^{\mathbb{R}}(I) \setminus \mathbf{var}(J)$ has no component in $U_{x_0}^2$. Further, any point (x, z) in

$$\{(x,z) \mid x \in U_{x_0}, z \in \mathbb{R}^n\} \tag{16}$$

that is contained in the variety $\mathbf{var}(J)$ must be contained in the smaller set

$$\{(x,z) \mid x \in U_{x_0}, z \in U_{x_0}\} = U_{x_0}^2.$$

This shows that the intersection of the varieties (15) and $\mathbf{var}(J)$ has no component in (16). Thus, in the projection onto the coordinates x there is no point contained in U_{x_0}. This finally shows that the variety of (13) does not contain x_0. □

Note that Theorem 2 allows to compute the set of all locally observable points at once.

Corollary 1. *Let F, G, and H as in* (10) *and $J = \langle x - z\rangle$. System* (1) *is locally observable if and only if*

$$\left(\mathbf{rad}^{\mathbb{R}}(\mathfrak{L}_F^\infty\langle G, H\rangle) : J\right) + J = k[x,z].$$

5 Examples

Using the theorems of the previous sections, global or local observability of the following examples can be decided automatically. We have computed these results using SageMath [47] that uses Singular for computations with polynomial ideals. Nonetheless, we investigate the intermediate steps.

5.1 Academic Example 1

Consider the example $\dot{x} = f(x) = \frac{1}{2}x$ with the nonlinear output $h(x) = x^2$ from [26]. Using the notation (10) one writes

$$F(x,z) = \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}z \end{pmatrix}, \quad H(x,z) = x^2 - z^2.$$

The zeroth Lie derivative $\mathfrak{L}_F^0\langle H\rangle$ of the ideal $\langle H\rangle = \langle x^2 - z^2\rangle \subseteq \mathbb{Q}[x,z]$ is by definition $\langle H\rangle$. In order to compute $\mathfrak{L}_F^1\langle H\rangle$, the Lie derivative $\mathrm{L}_F H(x,z) = x^2 - z^2$ must be added to the generating set. This polynomial is already contained in $\mathfrak{L}_F^0\langle H\rangle$ such that the chain of ideals is saturated and one has

$$\begin{aligned} \mathfrak{L}_F^0\langle H\rangle = \mathfrak{L}_F^1\langle H\rangle = \mathfrak{L}_F^\infty\langle H\rangle &= \langle x^2 - z^2\rangle \\ &= \langle (x-z)(x+z)\rangle \\ &= \langle x - z\rangle \cap \langle x + z\rangle. \end{aligned} \tag{17}$$

This ideal is obviously real (because its variety is the union of two real lines). Since $\mathfrak{L}_F^\infty\langle H\rangle$ does not equal $J = \langle x - z\rangle$, the system is not globally observable by Theorem 1.

In order to test local observability we compute the ideal (13) and start with the ideal quotient

$$\mathfrak{L}_F^\infty\langle H\rangle : J = \langle x + z\rangle,$$

which is easily obtained from the decomposition (17). Thus, one gets

$$\begin{aligned} (\mathfrak{L}_F^\infty\langle H\rangle : J) + J &= \langle x + z\rangle + \langle x - z\rangle \\ &= \langle x + z, x - z\rangle \\ &= \langle x, z\rangle. \end{aligned}$$

for the ideal sum. Since this ideal is proper, i. e., different from $\mathbb{Q}[x,z]$, the system is not locally observable by Corollary 1. Finally the elimination ideal $\langle x, z\rangle \cap \mathbb{Q}[x]$ reads $\langle x\rangle$. Thus, the system is locally observable at any point in $\mathbb{R} \setminus \mathbf{var}(\langle x\rangle) = \mathbb{R} \setminus \{0\}$, that is, for $x \neq 0$. This is consistent with the observability rank condition, where $O(x) = 2x \neq 0$ for $x \neq 0$. The system is not locally observable at $x = 0$ because the states $x = \epsilon$ and $z = -\epsilon$ with $0 < \epsilon \ll 1$ are indistinguishable in any neighbourhood of the origin. Note that this result differs from [26] since different definitions for local observability have been used. There, the system is locally observable even at $x = 0$ since the state $x = 0$ is not indistinguishable from any $z \neq 0$.

5.2 Academic Example 2

The following example is from [4, 29, 40, 44]. Let $x(t) \in \mathbb{R}$ and $f(x) = x$ with the nonlinear output $h(x) = x^3$. Thus, one writes

$$F(x,z) = \begin{pmatrix} x \\ z \end{pmatrix}, \quad H(x,z) = x^3 - z^3$$

for two copies of this system with the output difference as a new output. Since $\mathrm{L}_F H(x,z) = 3x^3 - 3z^3 = 3H(x,z)$, one has already computed the ideal

$$\mathfrak{L}_F^\infty\langle H\rangle = \langle x^3 - z^3\rangle.$$

In order to compute the real radical write this ideal as

$$\begin{aligned} \langle x^3 - z^3\rangle &= \langle (x-z)(x^2 + xz + z^2)\rangle \\ &= \langle x - z\rangle \cap \langle x^2 + xz + z^2\rangle \\ &= \langle x - z\rangle \cap \left\langle \tfrac{3}{4}(x+z)^2 + \tfrac{1}{4}(x-z)^2\right\rangle. \end{aligned}$$

In this form the real radical

$$\begin{aligned}\mathbf{rad}^{\mathbb{R}}\left(\left\langle x^3 - z^3\right\rangle\right) &= \mathbf{rad}^{\mathbb{R}}(\langle x - z\rangle \cap \langle x + z, x - z\rangle)\\ &= \mathbf{rad}^{\mathbb{R}}(\langle x - z\rangle)\\ &= \langle x - z\rangle\end{aligned}$$

is easily computed. Since the real radical equals $\langle x - z\rangle$, the system is globally observable although the 1×1 observability matrix $O(x) = 3x^2$ is zero at $x = 0$. Note that the complex variety of $\mathfrak{L}_F^\infty\langle H\rangle$ contains two additional lines.

5.3 Academic Example 3

Consider the system (1) with vector field

$$f(x) = \begin{pmatrix} -x_1 \\ x_1^2 \\ 3x_2^2 \end{pmatrix}$$

and output map $h(x) = x_1^3 + 3x_2^2 + x_3$ from [48]. Using the notation from (10) one gets

$$\mathfrak{L}_F^0\langle H\rangle = \langle H\rangle = \left\langle x_1^3 + 3x_2^2 + x_3 - z_1^3 - 3z_2^2 - z_3\right\rangle.$$

Since Lie derivatives of higher order become rather extensive, we only note that the chain saturates after the fourth Lie derivative and directly state the radical

$$\mathbf{rad}(\mathfrak{L}_F^\infty\langle H\rangle) = \langle x - z\rangle \cap \langle x_1, z_1, x_2 + z_2, x_3 - 6z_2 - z_3\rangle,$$

which can be decomposed into primary ideals. The radical also coincides with the real radical so that this system is not globally observable, as was pointed out in [48]. Computing the ideal (13) yields $\langle x_1, x_2\rangle$ whose variety is $\{x \in \mathbb{R}^3 \mid x_1 = x_2 = 0\}$. Thus, the system is locally observable at exactly any point $x \in \mathbb{R}^3$ not on the x_3-axis.

5.4 Academic Example 4

The example from [37] considers the linear oscillator

$$f(x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$

with the nonlinear output $h(x) = ax_1 + x_1^2$. Here, we have introduced a parameter a that in the original work takes the value 1. Thus, the polynomial ring for the following computations will be $\mathbb{Q}[x_1, x_2, z_1, z_2, a]$. Using the same notation as before, we get

$$\begin{aligned}\mathfrak{L}_F^0\langle H\rangle = \langle H\rangle &= \left\langle x_1^2 - z_1^2 + ax_1 - az_1\right\rangle\\ &= \langle x_1 + z_1 + a\rangle \cap \langle x_1 - z_1\rangle,\end{aligned}$$

if written as an intersection of primary ideals. The first Lie derivative reads

$$\mathfrak{L}_F^1\langle H\rangle = \langle x_2 - z_2, x_1 - z_1\rangle \cap \langle x_2 + z_2, x_1 + z_1 + a\rangle \cap I_1$$

with I_1 an ideal whose radical is $\langle 2z_1 + a, 2x_1 + a\rangle$. One can see that $\mathfrak{L}_F^1\langle H\rangle$ is already contained in, but is not yet equal to $\langle x - z\rangle$. The next ideal in the chain allows a decomposition

$$\mathfrak{L}_F^2\langle H\rangle = \langle x - z\rangle \cap \langle a, x_2 + z_2, x_1 + z_1\rangle \cap I_2$$

with the radical $\mathbf{rad}(I_2) = \langle 2z_1 + a, x_2 + z_2, 2x_1 + a\rangle$ of I_2. The ideal $\mathfrak{L}_F^2\langle H\rangle$ strictly contains $\mathfrak{L}_F^1\langle H\rangle$. Thus, by Proposition 3, the next Lie derivative may be an even greater set. One computes

$$\mathfrak{L}_F^3\langle H\rangle = \langle x - z\rangle \cap \langle a, x_2 + z_2, x_1 + z_1\rangle \cap I_3 \tag{18}$$

with $\mathbf{rad}(I_3) = \langle x_1, x_2, z_1, z_2, a\rangle$ containing the other ideals occurring in the decomposition (18). Therefore, one finds

$$\mathbf{rad}\left(\mathfrak{L}_F^3\langle H\rangle\right) = \langle x - z\rangle \cap \langle a, x_2 + z_2, x_1 + z_1\rangle \tag{19}$$

for the radical. If the parameter $a \neq 0$, the primary component

$$\langle a, x_2 + z_2, x_1 + z_1\rangle$$

equals $k[x, z]$. In this case the ideal (19) equals $\langle x - z\rangle$ and the system is globally observable, as was pointed out in [37, 40].

One may still ask, if observability is given for $a = 0$. Of course, the parameter can be substituted in this case. However, we continue and compute in the ring $\mathbb{Q}[x, z, a]$

$$\mathfrak{L}_F^4\langle H\rangle = \langle x_2 - z_2, x_1 - z_1\rangle \cap \langle a, x_2 + z_2, x_1 + z_1\rangle \cap I_4,$$

where $\mathbf{rad}(I_4) = \mathbf{rad}(I_3)$. The next Lie derivative in the chain obeys $\mathfrak{L}_F^4\langle H\rangle = \mathfrak{L}_F^5\langle H\rangle$ so that these ideals equal $\mathfrak{L}_F^\infty\langle H\rangle$ by Proposition 3. The (real) radical of the latter ideal equals (19). This shows that for $a = 0$ there is another component $\mathbf{var}(\langle x_1 + z_1, x_2 + z_2\rangle)$ in the real variety of $\mathfrak{L}_F^\infty\langle H\rangle$ other than $\mathbf{var}(\langle x - z\rangle)$ and thus the system is not globally observable. With the here proposed method the latter can also be proved.

5.5 The Rössler system

Consider the vector field

$$f(x) = \begin{pmatrix} -x_2 - x_3 \\ x_1 + ax_2 \\ b + x_3(x_1 - c) \end{pmatrix}$$

of the well-known Rössler system [42]. The local observability of this system using the state components as outputs was investigated in [30]. For $y = x_2$ one obtains a linear observability map, where global observability can easily be shown. The output $y = x_3$ can be used to design a nonlinear observer with linear error dynamics [35, Ex. 4], [39, Chap. 7]. We consider the output map $y = h(x) = x_1$.

The system contains three parameters a, b, c. For that reason the computations are carried out in the ring $\mathbb{Q}[a,b,c,x,z]$. In this ring one computes the ideal

$$\begin{aligned}\mathfrak{L}_F^4\langle H\rangle &= \mathfrak{L}_F^\infty\langle H\rangle = \\ &= \langle x - z\rangle \cap \langle a+b+c, z_2+z_3, z_1+b, x_2+x_3, x_1+b\rangle,\end{aligned}$$

which is already real. This ideal does not coincide with $\langle x - z\rangle$ such that the system with output x_1 is not globally observable, at least not for any parameter constellation.

In order to find the parameter set that causes a not global observable system we investigate the quotient

$$\begin{aligned}I &= \mathfrak{L}_F^\infty\langle H\rangle : \langle x - z\rangle \\ &= \langle a+b+c, z_2+z_3, z_1+b, x_2+x_3, x_1+b\rangle\end{aligned}$$

as an ideal in $\mathbb{Q}[a,b,c][x,z] = k[x,z]$. If I equals $k[x,z]$, the system is globally observable. Thus we compute the elimination ideal

$$I \cap \mathbb{Q}[a,b,c] = \langle a+b+c\rangle$$

to find that the system is globally observable if and only if $a+b+c \neq 0$.

If, on the other hand $(a,b,c) \in \mathbf{var}^{\mathbb{R}}(I \cap \mathbb{Q}[a,b,c])$, one can compute the point set where local observability is given. In contrast to (13) we have to eliminate the variables z, only. Thus, the elimination ideal is computed with respect to $\mathbb{Q}[a,b,c][x] = k[x]$. In this case we have the set

$$\mathbf{var}(\langle a+b+c, x_2+x_3, x_1+b\rangle)$$

of points where local observability is not given, provided the parameters are in the zero set, too.

5.6 The pendulum

The equations of motion of a mathematical pendulum with mass m and length l in the gravitational field with acceleration g can be described by

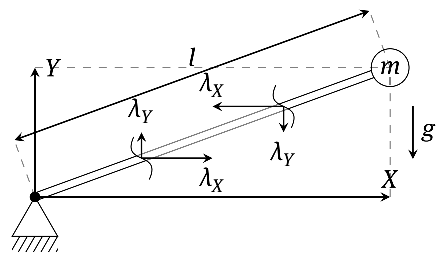


Figure 2: Schematic of the mathematical pendulum.

$$\begin{aligned}m\tfrac{\partial^2}{\partial\tau^2}X &= \lambda_X \\ m\tfrac{\partial^2}{\partial\tau^2}Y &= \lambda_Y - mg \\ 0 &= X\lambda_Y - Y\lambda_X \\ 0 &= X^2 + Y^2 - l^2,\end{aligned}$$

where X and Y denote the Cartesian coordinates of the pendulum mass and λ_X and λ_Y the vectorial components of the tensile force of the rod in direction X and Y, respectively. A schematic illustrating the physical quantities can be found in Figure 2. The algebraic equations ensure that the tension is parallel to the rod and that the rod has constant length l. We scale the coordinates $X = lx_1$, $Y = lx_2$, the forces $\lambda_X = mgx_5$, $\lambda_Y = mgx_6$, and time $\tau = t\sqrt{l/g}$ and introduce velocities x_3 and x_4 for the pendulum mass to write the equations in the form

$$\dot{x}_1 = x_3 \tag{20a}$$
$$\dot{x}_2 = x_4 \tag{20b}$$
$$\dot{x}_3 = x_5 \tag{20c}$$
$$\dot{x}_4 = x_6 - 1 \tag{20d}$$
$$0 = x_1x_6 - x_2x_5 \tag{20e}$$
$$0 = x_1^2 + x_2^2 - 1, \tag{20f}$$

which constitutes a set of *differential-algebraic equations* [9]. The system (20) has not yet the desired form (1), since explicit differential equations for x_5 and x_6 are missing. Furthermore, one would expect additional algebraic constraints, as the dimension of $\mathcal{X}$ should equal two. Further algebraic equations are obtained by index reduction [16]. Deriving (20f) with respect to time yields

$$x_1x_3 + x_2x_4 = 0 \tag{21a}$$

and after another reduction step

$$x_1x_5 + x_2x_6 - x_2 + x_3^2 + x_4^2 = 0. \tag{21b}$$

From (20e), (20f), and (21b) the normalized forces

$$x_5 = x_1\left(x_2 - x_3^2 - x_4^2\right)$$
$$x_6 = x_2\left(x_2 - x_3^2 - x_4^2\right)$$

can be computed such that one finally arrives at the form (1) with

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 - 1 \\ x_3(x_2 - x_3^2 - x_4^2 - 2x_3x_5) + x_4(3x_1 - 2x_1x_6) \\ x_4(4x_2 - x_3^2 - x_4^2 - 2x_2x_6) - 2x_2x_3x_5 \end{pmatrix}$$
$$g(x) = \begin{pmatrix} x_1x_6 - x_2x_5 \\ x_1^2 + x_2^2 - 1 \\ x_1x_3 + x_2x_4 \\ x_1x_5 + x_2x_6 - x_2 + x_3^2 + x_4^2 \end{pmatrix}. \tag{22}$$

One can verify that there are no other algebraic equations other than $g(x) = 0$ by checking the condition (2)

$$\mathfrak{L}_f^\infty\langle g\rangle = \langle g\rangle.$$

This means that (1c) with the map g from (22) defines the set of consistent initial values of the differential-algebraic equations (20). Note that there are six variables and four independent algebraic equations such that the dimension of $\mathcal{X}$ equals two, as expected.

Consider the output $y_1 = h_1(x) = x_1x_5 + x_2x_6$, which corresponds to the force in the rod. For this output the ideal (13) reads

$$\left\langle x_1, x_2^2 - 1, x_3, x_4, x_5, x_6 - 1\right\rangle.$$

Thus, the system is locally observable except at the real variety of this ideal. This variety corresponds to equilibrium points, since $x_3 = x_4 = 0$ for the velocities. The solutions $(x_1, x_2) \in \{(0, \pm 1)\}$ for the Cartesian coordinates of the pendulum mass correspond to the straight up and down orientation of the pendulum, respectively.

If the angular velocity $y_2 = h_2(x) = x_1x_4 - x_2x_3$ is measured, one arrives, using the usual notation, at

$$\begin{aligned}
&\mathfrak{L}_F^3\langle G, H_2\rangle = \mathfrak{L}_F^\infty\langle G, H_2\rangle = \\
&= \langle z_6 - 1, z_5, z_4, z_3, z_2 - 1, z_1, x_6 - 1, x_5, x_4, x_3, x_2 + 1, x_1\rangle \\
&\cap \langle z_6 - 1, z_5, z_4, z_3, z_2 + 1, z_1, x_6 - 1, x_5, x_4, x_3, x_2 - 1, x_1\rangle \\
&\cap \langle x - z, z_3z_5 + z_4z_6, z_2z_5 - z_1z_6, z_2^3 + z_4^2 + z_1z_5 - z_2, \\
&\quad z_3^2 + z_4^2 + z_1z_5 + z_2z_6 - z_2, z_1z_3 + z_2z_4, z_1^2 + z_2^2 - 1, \\
&\quad z_4^2z_5 + z_1z_5^2 - z_3z_4z_6 + z_1z_6^2 - z_1z_6, z_2^2z_3 - z_1z_2z_4 - z_3, \\
&\quad z_2z_3z_4 - z_1z_4^2 + z_1z_2 - z_5, z_1z_2^2 + z_3z_4 - z_1z_6\rangle.
\end{aligned} \tag{23}$$

The last ideal of the intersection (23) is precisely $\langle x - z, G(x,z)\rangle$. But the variety of (23) contains other real components. Thus, the pendulum with output y_2 is not globally observable. To test local observability we compute the ideal (13), which yields $\langle 1\rangle$. This shows that this system is everywhere locally observable. Indeed, from (23) one can see that there are only two isolated points that cannot be distinguished, namely the same two configurations of the pendulum at rest pointing up or down.

6 Concluding remarks

6.1 Systems with several outputs

The proposed observability criteria are not restricted to systems with scalar output. If in (1b) one has $y(t) \in \mathbb{R}^p$, the only difference is the definition of the ideal $\langle H(x,z)\rangle = \langle h(x) - h(z)\rangle$, which is to be understood as

$$\left\langle h_1(x) - h_1(z), \dots, h_p(x) - h_p(z)\right\rangle,$$

i. e., the ideal is generated by the vectorial components of the output map H.

An interesting question regards the number of required Lie derivatives of each output y_i, $i = 1, \dots, p$ in order to generate the ideal $\mathfrak{L}_F^\infty\langle H\rangle$ or $\mathfrak{L}_F^\infty\langle G, H\rangle$ for systems with algebraic constraints. These numbers can be regarded as a generalization of the *observability indices* [6] and have an impact on the observer's structure.

6.2 Nonpolynomial systems

While not all systems are given in the form (1) with polynomial vector field and output map, quite many can be transformed into such a form [27, 43, 50]. The pendulum discussed in subsection 5.6 is such a system. This process usually introduces new variables constrained by implicit polynomial equations. However, once such a transformation is found, the proposed observability criteria can be tested algorithmically.

6.3 Discrete time systems

Global observability of a nonlinear discrete time system

$$x[j+1] = f(x[j])$$
$$y[j] = h(x[j])$$

is also equivalent to the injectivity of an observability map [25, 31, 33]

$$q(x) = \begin{pmatrix} h(x) \\ h(f(x)) = h(x) \circ f(x) \\ h(x) \circ f(x) \circ f(x) = h(x) \circ f^2(x) \\ h(x) \circ f^3(x) \\ \vdots \end{pmatrix}.$$

With this observability map global and local observability of discrete time systems is defined the same way using the set of indistinguishable points. Thus, the same theorems can be applied, if the Lie derivative is replaced by the composition:

Definition 2. *Let $I \subset k[x]$ be a polynomial ideal and $f \in k[x]^n$ a polynomial vector field. The set*

$$I \circ f^{\infty} = \left\{ \sum_i a_i g_i \circ f^{n_i} \,\middle|\, a_i \in k[x], n_i \in \mathbb{Z}_{\geq 0}, g_i \in I \right\}$$

is called the composition of I with the vector field f.

This set is also an ideal and the proofs in section 3 are essentially the same.

6.4 Systems with input

Nonautonomous systems can be handled as well. If the vector field f in (1a) depends on a control vector u, computations can be carried out in the ring $k[x,z,u]$ so that the ideal

$$I_u = \mathfrak{L}_{F_u}^{\infty} \langle G, H \rangle$$

depends on u. For systems with input two points x, z in the space $\mathcal{X}$ are called distinguishable, if there exists a control u such that the corresponding outputs differ in an interval $t \in [0, T)$ [46, p. 262]. Thus, one is interested in the intersection of the real varieties of I_u for all u, which is the set of indistinguishable pairs for any u. In the language of ideals this corresponds to the ideal sum

$$I = \sum_u I_u = \langle \operatorname{coeff}_u p \mid p \in I_u \rangle.$$

This ideal equals the ideal of the coefficients of generators of $I_u \subseteq k[x,z][u]$ as polynomials in u with coefficients in $k[x,z]$. In order to test observability of a nonautonomous system, the Lie derivative in Theorem 1 or Theorem 2 is replaced by I.

Instead of computing I_u in the ring $k[x,z,u]$ first, one can directly consider the coefficients of the vector field f as a polynomial in u as done in [23], which will reduce the computational effort significantly.

From a practical point of view one may also ask, if the system state or redundant coordinates can be computed for any control u. Such a system is called *completely observable* [14]. In this case the elimination ideal

$$I = I_u \cap k[x,z]$$

is computed and replaced in Theorem 1 or Theorem 2.

6.5 Observability normal form

If system (1) is observable, the observability map (4) induces a change of coordinates

$$\xi_1 = h(x),\ \xi_2 = \mathrm{L}_f h(x), \ldots,\ \xi_{M+1} = \mathrm{L}_f^M h(x)$$

to map the system into the *observability normal form* [52]. For $M \geq n$, this map is an embedding or an immersion into a higher dimensional space [2, 15]. To carry out the observer design we need the inverse of this map on its image. For this purpose we consider the corresponding ideal

$$\left\langle g(x), \xi_1 - h(x), \xi_2 - \mathrm{L}_f h(x), \ldots, \xi_{M+1} - \mathrm{L}_f^M h(x) \right\rangle$$

and compute a Gröbner basis with the lexicographic order $x_1 > \cdots > x_n > \xi_1 > \cdots > \xi_{M+1}$. Then, we are able to eliminate the variables $x_n, \ldots, x_1$ step by step.

6.6 Algorithmic improvements

The structure of the ideals occurring in Theorem 1 or Theorem 2, namely the symmetry with respect to the variables x and z, calls for a coordinate transformation

$$x_i = \sigma_i + \delta_i,\ z_i = \sigma_i - \delta_i.$$

This seems to reduce the number of terms of the involved polynomials.

Alg. 1 is a comparably simple algorithm, yet not designed for efficiency. If one is not interested in computing the Lie derivative of a particular order one could think of many improvements. In addition, there are many relations linking the Lie derivative, the (real) radical, and the ideal intersection that are especially useful for computing the (real) radical of the Lie derivative.

7 Summary

For polynomial systems of the form (1) it was already known that the set of polynomials in the output map is finitely generated. This fact has been used to prove global observability for such systems. With the generalisation of the Lie derivative on polynomial ideals we have an algebraic tool at hand that also allows to algorithmically prove a polynomial system to be not globally observable. This decision is made on the equality of polynomial ideals. In addition, decidable conditions for local observability were given for this class of systems in conjunction with the set of locally observable points. These methods do not require a state space representation, which greatly extends the set of polynomial systems on that the criteria can be applied.

Acknowledgment: The authors would like to thank Prof. Bernd Tibken (Bergische Universität Wuppertal) for interesting discussions, which lead to this article. In addition, the second author wants to thank Thomas Paradowski (M. Sc.).

Funding: This work has been supported by Deutsche Forschungsgemeinschaft (DFG) under research grant RO 2427/5-1.

Appendix A. Proofs of the propositions

Proof of Proposition 1. By setting all $a_i = 0$ in (8) one has $0 \in \mathfrak{L}_I^m$. Let $A, B \in \mathfrak{L}_f^m I$. Then, by Definition 1, there are $a_i \in k[x]$, $n_i \in \mathbb{Z}_{\geq 0}$ with $n_i \leq m$, and $g_i \in I$ for $i = 1, \ldots, N, N+1, \ldots, M$ and finite M such that

$$A = \sum_{i=1}^{N} a_i \,\mathrm{L}_f^{n_i} g_i, \quad B = \sum_{i=N+1}^{M} a_i \,\mathrm{L}_f^{n_i} g_i.$$

Thus,

$$A + B = \sum_{i=1}^{M} a_i \,\mathrm{L}_f^{n_i} g_i$$

is contained in $\mathfrak{L}_f^m I$. Further, if $c \in k[x]$ one can set $\bar{a}_i = c a_i \in k[x]$ and write

$$cA = c \sum_{i=1}^{N} a_i \,\mathrm{L}_f^{n_i} g_i = \sum_{i=1}^{N} c a_i \,\mathrm{L}_f^{n_i} g_i = \sum_{i=1}^{N} \bar{a}_i \,\mathrm{L}_f^{n_i} g_i$$

to see that $cA \in \mathfrak{L}_f^m I$.

The last part of the proposition can be proved the same way. □

Proof of Proposition 2. Let $h \in \mathfrak{L}_f^{m+1} I$. Then h can be written as a sum

$$h = \sum_i a_i \,\mathrm{L}_f^{n_i} g_i = \sum_{n_i \leq m} a_i \,\mathrm{L}_f^{n_i} g_i + \sum_{n_i = m+1} a_i \,\mathrm{L}_f \mathrm{L}_f^{m} g_i,$$

where $a_i \in k[x]$ and $g_i \in I$. Set

$$\bar{g}_i = \begin{cases} \mathrm{L}_f^{n_i} g_i, & n_i \leq m \\ \mathrm{L}_f^{m} g_i, & n_i = m+1 \end{cases}$$

and write

$$h = \sum_{n_i \leq m} a_i \bar{g}_i + \sum_{n_i = m+1} a_i \,\mathrm{L}_f \bar{g}_i.$$

Thus, since $\bar{g}_i \in \mathfrak{L}_f^m I$, one has $h \in \mathfrak{L}_f^1 \mathfrak{L}_f^m I$ and $\mathfrak{L}_f^{m+1} I \subseteq \mathfrak{L}_f^1 \mathfrak{L}_f^m I$.

For the reverse inclusion assume $h \in \mathfrak{L}_f^1 \mathfrak{L}_f^m I$. Then h can be written as

$$h = \sum_i a_i y_i + \sum_j b_j \,\mathrm{L}_f z_j$$

with $a_i, b_j \in k[x]$ and $y_i, z_j \in \mathfrak{L}_f^m I$. Since $z_j \in \mathfrak{L}_f^m I$, each of these polynomials can in return be written as

$$z_j = \sum_l c_l \,\mathrm{L}_f^{n_l} g_l$$

with $c_l \in k[x]$, $n_l \leq m$ and $g_l \in I$. Thus,

$$\mathrm{L}_f z_j = \sum_l \left(\mathrm{L}_f c_l \,\mathrm{L}_f^{n_l} g_l + c_l \,\mathrm{L}_f^{n_l+1} g_l \right) \in \mathfrak{L}_f^{m+1} I.$$

From Definition 1 follows that $\mathfrak{L}_f^m I \subseteq \mathfrak{L}_f^{m+1} I$ and so must $y_i \in \mathfrak{L}_f^{m+1} I$, too. This shows that $h \in \mathfrak{L}_f^{m+1} I$ and the proposition follows. □

Proof of Proposition 3. From Definition 1 follows that for $p, q \in \mathbb{Z}_{\geq 0}$

$$p < q \implies \mathfrak{L}_f^p I \subseteq \mathfrak{L}_f^q I.$$

The ascending chain condition [11, p. 80] asserts that the chain

$$I \subseteq \mathfrak{L}_f^1 I \subseteq \mathfrak{L}_f^2 I \subseteq \cdots$$

of ideals saturates and one has $\mathfrak{L}_f^M I = \mathfrak{L}_f^{M+1} I = \cdots$ for an $M \in \mathbb{Z}_{\geq 0}$. Since $\mathfrak{L}_f^\infty I$ is the union of all $\mathfrak{L}_f^m I$, $m \in \mathbb{Z}_{\geq 0}$, the ideal $\mathfrak{L}_f^\infty I$ must be equal to the saturated ideal $\mathfrak{L}_f^M I$.

Assume for some $m \in \mathbb{Z}_{\geq 0}$ one has $\mathfrak{L}_f^m I = \mathfrak{L}_f^{m+1} I$. By Proposition 2 one has then

$$\mathfrak{L}_f^{m+1} I = \mathfrak{L}_f^1 \mathfrak{L}_f^m I = \mathfrak{L}_f^1 \mathfrak{L}_f^{m+1} I = \mathfrak{L}_f^{m+2} I.$$

Thus, by induction, $\mathfrak{L}_f^m I = \mathfrak{L}_f^{m+1} I$ implies that $\mathfrak{L}_f^m I = \mathfrak{L}_f^p I$ for all $p \geq m$. Since such an m exists, there must be a smallest such $m \in \mathbb{Z}_{\geq 0}$, say $m = M$. Thus, for all $p \in \mathbb{Z}_{\geq 0}$, $p < M$ one has the strict inclusion $\mathfrak{L}_f^p I \subset \mathfrak{L}_f^{p+1} I$. □

Proof of Proposition 4. At Line 6 at the first iteration of the outer loop one has $M = 0$ and $\langle H \rangle = \mathfrak{L}_f^0 \langle G \rangle$. In addition H is obviously a Gröbner basis.

Assume that at iteration M one has $\langle H \rangle = \mathfrak{L}_f^M \langle G \rangle$ after Line 6. Thus, $\mathfrak{L}_f^M \langle G \rangle = \langle h_1, \ldots, h_s \rangle$. The remainder r_i of $\mathrm{L}_f h_i$ at the reduction step at Line 9 can be written as

$$r_i = \mathrm{L}_f h_i + \sum_{j=1}^{s} c_{ij} h_j.$$

Thus, the ideal $\langle H \cup R \rangle$ at Line 12 is the set

$$\left\{ \sum_{i=1}^{s} \left(a_i h_i + b_i \left(\mathrm{L}_f h_i + \sum_{j=1}^{s} c_{ij} h_j \right) \right) \middle| a_i, b_i \in k[x] \right\}.$$

Setting $a_i = \bar{a}_i - \sum_{j=1}^{s} b_j c_{ji}$ this set may be written as

$$\left\{ \sum_{i=1}^{s} (\bar{a}_i h_i + b_i \, \mathrm{L}_f h_i) \middle| \bar{a}_i, b_i \in k[x] \right\}$$

and equals $\mathfrak{L}_f^1 \langle H \rangle = \mathfrak{L}_f^{M+1} \langle G \rangle$. This shows by induction that at iteration M one has $\langle H \rangle = \mathfrak{L}_f^M \langle G \rangle$ after Line 6.

If all remainders r_i of $\mathrm{L}_f h_i$ for $h_i \in H$ equal zero, the set R will be empty and one has $\mathfrak{L}_f^M \langle G \rangle = \mathfrak{L}_f^{M+1} \langle G \rangle = \mathfrak{L}_f^{\infty} \langle G \rangle$. Alg. 1 will exit the outer loop returning M. If on the other hand one r_i of the remainders is different from zero, it cannot be expanded in terms of the h_i, since the set H is a Gröbner basis. In this case $\langle H \rangle \subset \langle H \cup R \rangle$ and the loop continues. This shows that the correct derivative order M is computed along. □

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