Nonlinear Controllability and Observability

ROBERT HERMANN AND ARTHUR J. KRENER, MEMBER, IEEE

***Abstract*—The properties of controllability, observability, and the theory of minimal realization for linear systems are well-understood and have been very useful in analyzing such systems. This paper deals with analogous questions for nonlinear systems.**

I. Introduction

FREQUENTLY, control systems of the following form are used to model the behavior of physical, biological, or social systems,

$$\Sigma: \begin{array}{l} \dot{x}=f(x,u) \\ y=g(x) \end{array} \tag{1}$$

where $u \in \Omega$, a subset of $\boldsymbol{R}^l$, $x \in M$, a C^∞ connected manifold of dimension m, $y \in \boldsymbol{R}^n$ and f and g are C^∞ functions.

The control variable u represents the externally applied controls or exogenous inputs to the system and the output variable y represents the observable parameters of the system. The state variable x may or may not be directly measurable and is used to represent the memory of the system. The past history of Σ affects its future evolution only through information conveyed by this variable.

The state space M may be deficient in one of two ways. It may be too small to adequately represent the full variety of memory states, i.e., it may fail to distinguish between real states where some control exictes different observable effects. If this is the case, then the mathematical system Σ fails to adequately model the real system and hence, must be revised.

On the other hand, the state space may be too large. The system may not be controllable, i.e., if Σ is known to be in a given state $x^0 \in M$ at some time there may be other states $x \in M$ where the system cannot possibly get to or have come from using the given set of inputs. Or there may be distinct states which are indistinguishable from an input-output point of view, i.e., if the same input is applied in either of these states then the same output results. These problems can be caused by ignoring possible real inputs or real observables, in which case these must be added to the model. If, after all the significant input and output variables have been incorporated into Σ, the state space M is still too large, then one would like to apply a systematic technique to reduce M and still preserve the input-output structure of the model.

Using the ideas of controllability and observability, in the early 1960's Kalman and others carried out this program for linear systems. The similar questions for nonlinear systems were not effectively treated until the early 1970's. Based on the work of Chow [5], Hermann [9], Haynes–Hermes [8], and Brockett [9] and working independently, Lobry [21], [22], Sussman–Jurdjevic [25], and Krener [19], [20] developed the nonlinear analog of linear controllability in terms of the Lie algebra $\mathfrak{F}$ of vector fields on M generated by the vector fields $f(\cdot,u)$ corresponding to constant controls $u \in \Omega$.

It was shown that if the dimension of $\mathfrak{F}$ is constant or if the system Σ is analytic, then there exists a unique maximal submanifold M' of M through x^0 which carries all the trajectories of Σ passing through x^0 such that any point on this submanifold can be reached from x^0 going forward and backward along the trajectories of the system. In particular if the dimension of $\mathfrak{F}(x^0)$ is m then $M=M'$ and so the system is "controllable" in some sense, to be made precise in Section II. If it is less than m then Σ can be restricted to M' where it is "controllable." This is one half of reducing the state space M.

For linear systems

$$\begin{array}{l} \dot{x}=Ax+Bu \\ y=Cx \end{array} \tag{1}$$

where $u \in \boldsymbol{R}^l$, $x \in \boldsymbol{R}^m$, $y \in \boldsymbol{R}^n$, this reduces to the well-known criterion that the rank of the matrix

$$(B:AB:\cdots:A^{m-1}B)$$

be m.

The other half of the program of reducing the state space was supplied by Sussmann [28] for analytic or symmetric systems. This generalized the work of Brockett [3] on Lie groups. Sussmann noted that indistinguishability is an equivalence relation on M and showed that for analytic "controllable" or symmetric "controllable" systems, indistinguishability is a closed regular equivalence relation so that the quotient is another manifold. He also showed that the quotient inherits a system which has the same input–output behavior as Σ but is "controllable" and observable, i.e., has no indistinguishable states.

In this paper we take a different approach to nonlinear observability, related to that of previous authors [16]–[18] but which is more in the spirit of the approach to nonlinear controllability described above. In particular we develop results involving observability which are analogous

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R. Hermann is with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903.

A. J. Krener is with the Department of Mathematics, University of California, Davis, CA 95616.

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to the controllability results depending on the dimension of $\mathcal{F}(x)$. The relevant objects in this study are $\mathcal{G}$, the smallest linear space of functions on M which contains the observations $g_1(x),\cdots,g_n(x)$ and which is closed with respect to Lie differentiation by $\mathcal{F}$, and the differentials of $\mathcal{G}$ denoted by $d\mathcal{G}$. If the dimension of $d\mathcal{G}$ is constant over M then indistinguishability need not be regular but there is a related regular equivalence relation which we can use to factor M. On the quotient there exists a system with the same input-output behavior as Σ but which is "observable" in the sense that neighboring points are distinguishable. In particular if the dimension of $d\mathcal{G}(x)$ is always m then Σ has this property. For linear systems this reduces to the well-known criterion that the matrix

$$\begin{bmatrix} C \\ \hline CA \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline CA^{m-1} \end{bmatrix}$$

be of rank m.

In order to bring out the "duality" between "controllability" and "observability" (which is, mathematically, just the duality between vector fields and differential forms), we first review in Section II the known facts concerning nonlinear controllability and then in Section III we discuss our approach to nonlinear observability. Finally Section IV deals with the question of minimality for nonlinear systems.

II. Nonlinear Controllability

We consider the system Σ described in Section I. Recall a C^∞ (analytic) *m-dimensional manifold* M is a Hausdorff topological space with a C^∞ (analytic) structure, i.e., a countable cover of coordinate charts (U^α, x_α) where

1) U^α is an open set in M,

2) $x_\alpha = \mathrm{col}(x_{\alpha 1},\cdots,x_{\alpha m}): U^\alpha \to \boldsymbol{R}^m$ is a homeomorphism onto its range, and

3) if (U^α, x_α) and (U^β, x_β) are two such coordinate charts then the change of coordinates $x_\beta \circ x_\alpha^{-1}: x_\alpha(U^\alpha \cap U^\beta) \to x_\beta(U^\alpha \cap U^\beta)$ is C^∞ (analytic).

For further details we refer the reader to Hermann [10] or Boothby [2]. A non-Hausdorff manifold is a non-Hausdorff topological space with such a structure. If M is not Hausdorff then solutions of ordinary differential equations on M need not be unique.

Throughout this section we simplify notation by assuming M admits globally defined coordinates $x = \mathrm{col}(x_1,\cdots,x_m)$. This allows us to identify the points of M with their coordinate representations and to describe control systems in the familiar fashion (1). However when dealing with nonlinear observability in the next section we shall be forced to consider several coordinate charts.

It is with no loss of generality that we assume $y \in \boldsymbol{R}^n$. If $y \in N$, a C^∞ manifold, then by the Whitney Imbedding Theorem, N can be imbedded in $\boldsymbol{R}^n$ for some n which then can be taken as the range of g.

The assumption of infinite differentiability for M, f, and g is not essential but is only invoked to avoid counting the degree of differentiability needed in a particular argument. Occasionally we consider *analytic systems* where these are assumed to be an analytic manifold and analytic mappings. Nonautonomous systems are handled in the standard fashion by assuming time is one of the state variables.

We also assume the system is complete, i.e., for every bounded measurable control $u(t)$ and every $x^0 \in M$ there exists a solution of the differential equation $\dot{x} = f(x(t), u(t))$ satisfying $x(t^0) = x^0$ and $x(t) \in M$ for all $t \in \boldsymbol{R}$. We use the notation $(u(t),[t^0,t^1])$ to denote functions defined on $[t^0,t^1]$.

Given a subset $U \subseteq M$, x^1 is U-accessible from x^0 (denoted by $x^1 A_U x^0$) if there exists a bounded measurable control $(u(t),[t^0,t^1])$ satisfying $u(t) \in \Omega$ for $t \in [t^0,t^1]$ such that the corresponding solution $(x(t),[t^0,t^1])$ of the differential equation (1) satisfies $x(t^0) = x^0$, $x(t^1) = x^1$ and $x(t) \in U$ for all $t \in [t^0,t^1]$. M-accessibility and A_M are simply referred to as *accessibility* and A. Given any relation R on M we use the notation

$$R(x^0) = \{x^1 \in M : x^1 R x^0\}.$$

For example $A(x^0)$ is the set of *points accessible from* x^0. The system Σ is said to be *controllable at* x^0 if $A(x^0) = M$ and *controllable* if $A(x) = M$ for every $x \in M$.

If Σ is controllable at x^0 it still may be necessary to travel a considerable distance or for a long time to reach points near to x^0. As a result this type of controllability is not always of use and so we introduce a local version of this concept. Σ is *locally controllable at* x^0 if for every neighborhood U of x^0, $A_U(x^0)$ is also a neighborhood of x^0; Σ is *locally controllable* if it is locally controllable at every $x \in M$. (This is called local–local controllability by Haynes and Hermes [8].)

Accessibility is a reflexive and transitive relation but for nonlinear systems it need not be symmetric. For this reason we need a weaker relation. Given an open set $U \subseteq M$ there is a unique smallest equivalence relation on U which contains all U-accessible pairs. We call this relation *weak U-accessibility* and denote it by WA_U. It is easy to see that $x' WA_U x''$ iff there exists $x^0,\cdots,x^k$ such that $x^0 = x'$, $x^k = x''$ and either $x^i A_U x^{i-1}$ or $x^{i-1} A_U x^i$ for $i = 1,\cdots,k$. Weak M-accessibility and WA_M are simply referred to as weak accessibility and WA. Σ is *weakly controllable at* x^0 if $WA(x^0) = M$ in which case $WA(x) = M$ for all x and so Σ is *weakly controllable.*

Notice that weak controllability is a global concept and does not reflect the behavior of Σ restricted to a neighborhood of x^0. So again we introduce a local concept. Σ is *locally weakly controllable at* x^0 if for every neighborhood U of x^0, $WA_U(x^0)$ is a neighborhood of x^0. Σ is *locally weakly controllable* if it is locally weakly controllable at

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every $x \in M$. Clearly (local) controllability implies (local) weak controllability and it is not hard to show using the transitivity of (weak) accessibility and the connectivity of M that local (weak) controllability implies (weak) controllability, i.e., we have the following implications:

$$\begin{array}{ccc} \Sigma \text{ locally controllable} & \Rightarrow & \Sigma \text{ controllable} \\ \Downarrow & & \Downarrow \\ \Sigma \text{ locally weakly controllable} & \Rightarrow & \Sigma \text{ weakly controllable.} \end{array}$$

In general there are no other implications but for autonomous linear systems it can be shown that all four concepts are equivalent.

The following result gives an intuitive interpretation of local weak controllability. Loosely put it shows that Σ is locally weakly controllable iff one needs local coordinates of dimension m to distinguish the trajectories of Σ from any initial point.

Theorem 2.1: Σ is locally weakly controllable iff for every $x \in M$ and every neighborhood U of x the interior of $A_U(x) \neq \varnothing$.

Proof: Suppose Σ is weakly controllable. Given any $x^0 \in M$ and any neighborhood U of x^0 we can choose $u^1 \in \Omega$ such that $f^1(x) = f(x,u^1)$ is not zero at x^0 (assuming $m>0$, if $m=0$ then the result is immediate). Let $s \rightarrow \gamma_s^1(x)$ denote the flow of f^1, i.e., the family of solutions of the differential equation

$$\frac{d}{ds}\gamma_s^1(x) = f^1\left(\gamma_s^1(x)\right)$$

satisfying the initial conditions

$$\gamma_0^1(x) = x.$$

For some $\epsilon > 0$ the set $V^1 = \{\gamma_s^1(x^0): 0 < s < \epsilon\}$ is a submanifold of U of dimension 1.

Suppose inductively V^{j-1} is a $j-1$ dimensional submanifold of U defined by

$$V^{j-1} = \left\{\gamma_{s_{j-1}}^{j-1} \circ \cdots \circ \gamma_{s_1}^1(x^0) : (s_1, \cdots, s_{j-1}) \right.$$
$$\left. \text{in some open subset of the positive orthant of } \boldsymbol{H}^{j-1} \right\}$$

where $\gamma_{s_i}^i(x)$ is the flow of $f^i(x) = f(x,u^i)$ for some $u^i \in \Omega$. Clearly $V^{j-1} \subseteq A_U(x^0)$. If $j \leqslant m$ we construct V^j by choosing a $u^j \in \Omega$ and $x^{j-1} \in V^{j-1}$. This is always possible for if not then every trajectory of Σ starting on V^{j-1} would remain on V^{j-1} for a while. This contradicts the local weak controllability of Σ.

It follows that we can choose an open subset of the positive orthant of $\boldsymbol{R}^j$ such that the map $(s_1, \cdots, s_j) \mapsto \gamma_{s_j}^j \circ \cdots \circ \gamma_{s_1}^1(x^0)$ is an imbedding of the subset into U. Call the range V^j. We can continue until $j = m$, V^m is a nonempty open subset of $A_U(x^0)$ and so the interior of $A_U(x^0) \neq \varnothing$.

As for the converse suppose the interior of $A_U(x^0) \neq \varnothing$, we choose a control $(u(t),[t^0,t^1])$ such that the corresponding trajectory $(x(t),[t^0,t^1])$ satisfies $x(t^0) = x^0$, $x(t^1) = x^1 \in$ interior of $A_U(x^0)$ and $x(t) \in U$ for $t \in [t^0,t^1]$. Let $\gamma_t(x,t^0)$ denote the time dependent vector field $f_t(x) = f(x,u(t))$, i.e.,

$$\frac{d}{dt}\gamma_t(x,t^0) = f_t\left(\gamma_t(x,t^0)\right)$$
$$\gamma_{t^0}(x,t^0) = x.$$

Then $\gamma_{t^0}(\cdot,t^1)$ is a diffeomorphism of a neighborhood V of x^1 onto a neighborhood of x^0. Moreover we can choose $V \subseteq A_U(x^0)$ sufficiently small so that $\gamma_{t^0}(V,t^1) \subseteq WA_U(x^0)$. □

The advantage of local weak controllability over the other forms of controllability discussed above is that it lends itself to a simple algebraic test. First we introduce some additional mathematical concepts.

The set of all C^∞ vector fields on M is an infinite dimensional real vector space denoted by $\mathfrak{X}(M)$ and also a Lie algebra under the multiplication defined by the Jacobi bracket $[h_1,h_2]$ given by

$$[h_1,h_2](x) = \frac{\partial h_2}{\partial x}(x)h_1(x) - \frac{\partial h_1}{\partial x}(x)h_2(x)$$

where h_1, h_2 and $[h_1,h_2] \in \mathfrak{X}(M)$. Elements of $\mathfrak{X}(M)$ are represented by column m-vector valued functions of x. For any fixed $h_1 \in \mathfrak{X}(M)$ the real linear transformation from $\mathfrak{X}(M)$ into itself which sends $h_2 \mapsto [h_1,h_2]$ is called Lie differentiation with respect to h_1 and is denoted by L_{h_1}.

Each constant control $u \in \Omega$ defines a vector field $f(x,u) \in \mathfrak{X}(M)$, we denote by $\mathfrak{F}^0$ the subset of all these vector fields. Let $\mathfrak{F}$ denote the smallest subalgebra of $\mathfrak{X}(M)$ which contains $\mathfrak{F}^0$. A typical element of $\mathfrak{F}$ is a finite linear combination of elements of the form

$$\left[f^1\left[f^2\left[\cdots\left[f^{k-1},f^k\right]\cdots\right]\right]\right]$$

where $f^i(x) = f(x,u^i)$ for some constant $u^i \in \Omega$. We denote by $\mathfrak{F}(x)$ the space of tangent vectors spanned by the vector fields of $\mathfrak{F}$ at x. Σ is said to satisfy the *controllability rank condition at* x^0 if the dimension of $\mathfrak{F}(x^0)$ is m; Σ satisfies the *controllability rank condition* if this is true for every $x \in M$.

Theorem 2.2: If Σ satisfies the controllability rank condition at x^0 then Σ is locally weakly controllable at x^0.

Proof: The proof is very similar to Theorem 2.1. We start by choosing a neighborhood of U of x^0 small enough so that Σ satisfies the controllability rank condition at every $x \in U$. We construct a sequence of submanifolds as before but this time there is a different reason why one can always choose $u^j \in \Omega$ and $x^{j-1} \in V^{j-1}$ such that $f^j(x) = f(x,u^j)$ is not tangent to V^{j-1} at x^{j-1}. If this is not possible then $\mathfrak{F}$ restricted to V^{j-1} is a subalgebra of $\mathfrak{X}(V^{j-1})$ which implies the dimension of $\mathfrak{F}(x) \leqslant j-1 < m$ on $V^{j-1} \subseteq U$ which is a contradiction.

It follows that for every neighborhood U of x^0, the interior of $A_U(x^0)$ is not empty. The second half of the proof of Theorem 2.1 implies $WA_U(x^0)$ is a neighborhood of x^0. □

From the above we see that if Σ satisfies the controllability rank condition then it is locally weakly controllable. The converse is almost true as we shall see later on in Theorems 2.5 and 2.6.

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Suppose the trajectories of Σ are required to satisfy the initial condition

$$x(t^0)=x^0,$$

then Σ defines a map from inputs to outputs as follows. Each admissible input $(u(t),[t^0,t^1])$ gives rise to a solution $(x(t),[t^0,t^1])$ of $\dot{x}=f(x,u(t))$ satisfying the initial condition. This, in turn, defines an output $(y(t),[t^0,t^1])$ by $y(t)=g(x(t))$. We denote this map by

$$\Sigma_{x^0}:(u(t),[t^0,t^1])\mapsto(y(t),[t^0,t^1])$$

and call it the *input–output map of* Σ *at* x^0. Given a map

$$\beta:(u(t),[t^0,t^1])\mapsto(y(t),[t^0,t^1])$$

from inputs to outputs, the pair (Σ,x^0) is said to be a realization of β if $\Sigma_{x^0}=\beta$.

Now suppose Σ is neither locally weakly controllable nor weakly controllable. Given the input–output map Σ_{x^0} we would like to find another realization (Σ',z^0) of this map which is weakly controllable in some sense. The obvious way to proceed is to find a submanifold M' of M which contains x^0 and all the trajectories of Σ passing through x^0 then let Σ' be the restriction of Σ to M' and $z^0=x^0$. If M' is chosen small enough then hopefully Σ' will be weakly controllable in some sense.

To carry out this program we introduce some mathematical tools. A connected submanifold M' of M is an *integral submanifold* of $\mathcal{F}$ if at each $x\in M'$ the tangent space to M' at x is contained in $\mathcal{F}(x)$. M' is a *maximal integral submanifold* of $\mathcal{F}$ if it is not properly contained in any other integral submanifold of $\mathcal{F}$. There are two important cases when $\mathcal{F}$ has maximal integral submanifolds.

Frobenius Theorem [10]: If the dimension of $\mathcal{F}(x)=k$ for every $x\in M$, then there exists a partition of M into maximal integral submanifolds of $\mathcal{F}$ all of dimension k.

Hermann–Nagano Theorem [11], [12], [14], [23]: If the system is analytic then there exists a partition of M into maximal integral submanifolds of $\mathcal{F}$ of varying dimensions. The dimension of $\mathcal{F}(x)$ can vary but it will be constant on each submanifold of the partition and equal to the dimension of that submanifold.

The relationship of these two theorems with controllability is given by the following.

Chow Theorem [5]: If either of the above is satisfied and M' is the maximal integral submanifold of $\mathcal{F}$ containing x^0 then $M'=WA(x^0)$.

This leads to the following.

Theorem 2.3: Suppose (Σ,x^0) is a realization of an input–output map such that either

1) $\mathcal{F}(x)$ is of constant dimension, or

2) Σ is analytic,

then there is a locally weakly controllable realization (Σ',x^0) of the same input–output map on the maximal integral submanifold M' of $\mathcal{F}$ containing x^0. In fact, Σ' satisfies the controllability rank condition.

For completeness we mention the case when Σ does not satisfy either of the hypotheses of Theorem 2.3. It has been shown by Sussmann [29] that for any C^∞ system, $WA(x^0)$ can be given the structure of a C^∞ manifold, in general it will not be an integral submanifold of $\mathcal{F}$.

Theorem 2.4 (Sussmann [29]): Given any realization (Σ,x^0), there exists a weakly controllable realization (Σ',x^0) of the same input–output map on $M'=WA(x^0)$.

Notice that unlike Theorem 2.3 the above does not guarantee that Σ' is locally weakly controllable or satisfies the controllability rank condition. Locally weakly controllable systems "almost" satisfy the controllability rank condition.

Theorem 2.5: If Σ is locally weakly controllable then the controllability rank condition is satisfied generically, i.e., on an open dense subset of M.

Proof: For any system the controllability rank condition is satisfied on an open subset of M, possibly empty. To see that it is dense for locally weakly controllable systems suppose there exists an open subset U of M where the dimension of $\mathcal{F}(x)<m$. Without loss of generality we can assume $\dim\mathcal{F}(x)=k<m$ for all $x\in U$. For some $x^0\in U$, let U' denote the maximal integral submanifold of $\mathcal{F}(x)$ in U given by the Frobenius Theorem, then $A_U(x^0)\subseteq U'$ and so using Theorem 2.1 we see Σ is not locally weakly controllable at x^0. □

For analytic systems we can strengthen the above, in fact, weak controllability, local weak controllability, and the controllability rank condition are equivalent.

Theorem 2.6: If Σ is analytic then Σ is weakly controllable iff it is locally weakly controllable iff the controllability rank condition is satisfied.

Proof: We have already shown that for C^∞ systems the controllability rank condition implies local weak controllability which implies weak controllability. The reverse implications follow for analytic systems if we show that x^0WAx^1 implies that the dimension of $\mathcal{F}(x^0)$ and $\mathcal{F}(x^1)$ are the same. For then, if Σ is weakly controllable, the dimension of $\mathcal{F}(x)$ must be constant and hence equal to m by the Frobenius and Chow Theorems.

To show that x^0WAx^1 implies the dimension of $\mathcal{F}(x^0)$ and $\mathcal{F}(x^1)$ are the same, it suffices to consider the case $x^1=\gamma(x^0)$ where γ is the family of solutions of the vector field $f(x)=f(x,u)$ for some constant $u\in\Omega$ and $x>0$. The map $(\partial/\partial x)\gamma_{-s}(x^1)$ is a linear isomorphism from the tangent space at x^1 to the tangent space at x^0. For any $h\in\mathcal{X}(M)$, the Campbell–Baker–Hausdorff formula allows us to expand $(\partial/\partial x)\gamma_{-s}(x^1)h(x^1)$ in a convergent series

$$\frac{\partial}{\partial x}\gamma_{-s}(x^1)h(x^1)=\sum_{k=0}^{\infty}(L_f)^k h(x^0)\frac{s^k}{k!}.$$

In particular, if $h\in\mathcal{F}$ then the right side of the above is a vector in $\mathcal{F}(x^0)$ so $(\partial/\partial x)\gamma_{-s}(x^1)$ carries $\mathcal{F}(x^1)$ into $\mathcal{F}(x^0)$. Therefore, the dimension of $\mathcal{F}(x^1)\leqslant$ dimension of $\mathcal{F}(x^0)$. Reversing the argument shows that the inverse map $(\partial/\partial x)\gamma_s(x^0)$ carries $\mathcal{F}(x^0)$ into $\mathcal{F}(x^1)$ and the result follows. □

Example 2.7: Consider the linear system (2). In this

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case $\mathcal{F}^0=\{Ax+Bu:u\in\Omega\}$, so the Lie algebra is generated by the vector fields $\{Ax,B_{*1},\cdots,B_{*l}\}$ where B_{*j} denotes the jth column of B considered as a constant vector field. Computing brackets yields

$$[Ax,B_{*j}]=-AB_{*j} \qquad [B_{*j},B_{*k}]=0$$

$$[Ax[Ax,B_{*j}]]=A^2B_{*j} \qquad [B_{*j}[Ax,B_{*k}]]=0$$

and so on. The Cayley–Hamilton Theorem implies that $\mathcal{F}$ is spanned by the linear vector fields Ax and the constant vector fields A^iB_{*j} where $i=0,\cdots,m-1$ and $j=1,\cdots,l$.

This system is analytic so the Hermann–Nagano Theorem guarantees the existence of maximal integral submanifolds of $\mathcal{F}$ through each $x\in\mathbf{R}^m$. In particular if we let M' denote the maximal integral submanifold through $x=0$ then M' must contain the integral curves of all linear combinations of the constant vector fields $\{A^iB_{*j}\}$ starting at 0 and hence must include the linear subspace spanned by these vectors. In fact M' is precisely this subspace because at each $x\in M'$ the tangent space to M' contains Ax also.

Notice that in this context the controllability rank condition reduces to the well-known linear controllability condition that

$$\operatorname{rank}(B:AB:\cdots:A^{m-1}B)=m.$$

The controllability rank condition only implies local weak controllability but for linear systems it also implies controllability (see [4] for details).

If the controllability rank condition fails and the rank of the above matrix is $m'<m$, then by restricting the system to $M'=\mathbf{R}^{m'}$, the maximal integral submanifold of $\mathcal{F}$ through 0, we obtain a controllable linear system. Notice that in this case the dimensions of the maximal integral manifolds of $\mathcal{F}$ can vary, for if $Ax\notin \operatorname{span}\{A^iB_{*j}\}$ then the maximal integral manifold of $\mathcal{F}$ through x will be of dimension $>m'$. It cannot be a linear subspace of $\mathbf{R}^m$ because it does not contain 0 which must lie in a maximal integral manifold of dimension m'.

Example 2.8: Consider the linear system (2) but where $A(t)$, $B(t)$, and $C(t)$ are C^∞ matrix valued functions of time. We adjoin another state variable $x_0=t$ and rewrite the system equations

$$\begin{aligned}\dot{x}_0&=1\\ \dot{x}&=A(x_0)x+B(x_0)u\\ y_0&=x_0\\ y&=C(x_0)x. \end{aligned} \tag{3}$$

The structure of $\mathcal{F}$ is similar to that for autonomous systems, for example,

$$\left[\begin{pmatrix}1\\ A(x_0)x\end{pmatrix},\begin{pmatrix}0\\ B_{*j}(x_0)\end{pmatrix}\right]=\begin{bmatrix}0\\ \dfrac{\partial}{\partial x_0}B_{*j}(x_0)-A(x_0)B_{*j}(x_0)\end{bmatrix}$$

$$\left[\begin{pmatrix}0\\ B_{*j}(x_0)\end{pmatrix},\begin{pmatrix}0\\ B_{*k}(x_0)\end{pmatrix}\right]=\begin{pmatrix}0\\0\end{pmatrix}.$$

From this it can be shown that the controllability rank condition is equivalent to the familiar requirement that

$$\operatorname{rank}(B(t):\Delta_cB(t):\Delta_c^2B(t):\cdots)=m$$

for every $t\in\mathbf{R}$ where

$$\Delta_cB(t)=\frac{d}{dt}B(t)-A(t)B(t),$$

$$\Delta_c^rB(t)=\left(\frac{d}{dt}-A(t)\right)\Delta_c^{r-1}B(t).$$

See [4], [6], or [15].

Example 2.9: Consider the bilinear system

$$\dot{x}=Ax+\sum_{i=1}^{l}u_iB^ix,$$

where $u\in\Omega=\mathbf{R}^l$ and M is either $\mathbf{R}^m$ or an m-dimensional subgroup of $Gl(h,\mathbf{R})$, the group of invertible $h\times h$ matrices.

The Jacobi brackets of linear vector fields Dx, Ex is seen to be

$$[Dx,Ex]=[D,E]x$$

where

$$[D,E]=ED-DE$$

is the commutator of the matrices D and E. $\mathcal{F}$ is isomorphic to a Lie subalgebra of $gl(m\mathbf{R})$ if $M=\mathbf{R}^m$ and $gl(h,\mathbf{R})$ if M is a subgroup of $Gl(h,\mathbf{R})$. ($gl(m,\mathbf{R})$ is the Lie algebra of all $m\times m$ matrices with commutation as the bracket). Let F denote the group

$$F=\{\exp X:X\in\mathcal{F}\}.$$

The maximal integral manifold M' of $\mathcal{F}$ through x_0 is just the orbit of x^0 under F

$$M'=Fx^0$$

and $\mathcal{F}$ satisfies the controllability rank condition iff F acts transitively on M.

III. Nonlinear Observability

We consider the system Σ as described in Section I and the input–output map of the pair (Σ,x^0) as described in Section II. A pair of points x^0 and x^1 are indistinguishable (denoted x^0Ix^1) if (Σ,x^0) and (Σ,x^1) realize the same input–output map, i.e., for every admissible input $(u(t),[t^0,t^1])$

$$\Sigma_{x^0}(u(t),[t^0,t^1])=\Sigma_{x^1}(u(t),[t^0,t^1]).$$

Indistinguishability I is an equivalence relation on M. Σ is said to *be observable at* x^0 if $I(x^0)=\{x^0\}$ and Σ is *observable* if $I(x)=\{x\}$ for every $x\in M$.

Notice that the observability of Σ does not imply that every input distinguishes points of M. If, however, the output is the sum of a function of the initial state and a

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function of the input, as it is for linear systems, then it is easy to see that if any input distinguishes between two initial states then every input does.

Notice also that observability is a global concept; it might be necessary to travel a considerable distance or for a long time to distinguish between points of M. Therefore we introduce a local concept which is stronger than observability. Let U be a subset of M and $x^0, x^1 \in U$. We say x^0 is *U-indistinguishable* from x^1 ($x^0 I_U x^1$) if for every control $(u(t),[t^0,t^1])$, whose trajectories $(x^0(t),[t^0,t^1])$ and $(x^1(t),[t^0,t^1])$ from x^0 and x^1 both lie in U, fails to distinguish between x^0 and x^1, i.e., if $x^0(t)\in U$ and $x^1(t)\in U$ for $t\in[t^0,t^1]$, then

$$\Sigma_{x^0}(u(t),[t^0,t^1]) = \Sigma_{x^1}(u(t),[t^0,t^1]).$$

U-indistinguishability is not, in general, an equivalence relation on U for it fails to be transitive. This is related to the fact that Σ restricted to U is not necessarily complete. (See Sussmann [27] for a fuller discussion of this point.) However, we can still define Σ to be *locally observable at* x^0 if for every open neighborhood U of x^0, $I_U(x^0)=\{x^0\}$, and Σ is *locally observable* if it is so at every $x\in M$.

On the other hand one can weaken the concept of observability; in practice it may suffice to be able to distinguish x^0 from its neighbors. Therefore we define Σ to be *weakly observable at* x^0 if there exists a neighborhood U of x^0 such that $I(x^0)\cap U=\{x^0\}$ and Σ is *weakly observable* if it is so at every $x\in M$.

Notice once again that it may be necessary to travel considerably far from U to distinguish points of U, so we make a last definition, Σ is *locally weakly observable at* x^0 if there exists an open neighborhood U of x^0 such that for every open neighborhood V of x^0 contained in U, $I_V(x^0)=\{x^0\}$ and is *locally weakly observable* if it is so at every $x\in M$. Intuitively, Σ is locally weakly observable if one can instantaneously distinguish each point from its neighbors.

It can easily be seen that the relationships between the various forms of observability parallel that of controllability, i.e.,

$$\begin{array}{ccc} \Sigma \text{ locally observable} & \Rightarrow & \Sigma \text{ observable} \\ \Downarrow & & \Downarrow \\ \Sigma \text{ locally weakly observable} & \Rightarrow & \Sigma \text{ weakly observable.} \end{array}$$

In general there are no other implications but for autonomous linear systems it can be shown that all four are equivalent. The advantage of local weak observability over the other concepts is that it lends itself to a simple algebraic test. To describe it we need some additional tools.

Let $C^\infty(M)$ denote the infinite dimensional real vector space of all C^∞ real valued functions on M. Elements of $\mathfrak{X}(M)$ act as linear operators on $C^\infty(M)$ by Lie differentiation. If $h\in\mathfrak{X}(M)$ and $\varphi\in C^\infty(M)$ then $L_h(\varphi)\in C^\infty(M)$ is given by

$$L_h(\varphi)(x) = \frac{\partial\varphi}{\partial x}(x)h(x).$$

The gradient $d\varphi = \partial\varphi/\partial x = (\partial\varphi/\partial x_1,\cdots,\partial\varphi/\partial x_m)$ is a row vector valued function.

We denote by $\mathcal{G}^0$ the subset of $C^\infty(M)$ consisting of $g_1,\cdots,g_n$, and by $\mathcal{G}$ the smallest linear subspace of $C^\infty(M)$ containing $\mathcal{G}^0$ which is closed with respect to Lie differentiation by elements of $\mathcal{F}^0$. An element of $\mathcal{G}$ is a finite linear combination of functions of the form

$$L_{f^1}(\cdots(L_{f^k}(g_i))\cdots)$$

where $f^j(x)=f(x,u^j)$ for some constant $u^j\in\Omega$.

If $h_1,h_2\in\mathfrak{X}(M)$ and $\varphi\in C^\infty(M)$ then

$$L_{h_1}(L_{h_2}(\varphi)) - L_{h_2}(L_{h_1}(\varphi)) = L_{[h_1,h_2]}(\varphi).$$

From this it follows that $\mathcal{G}$ is closed under Lie differentiation by elements of $\mathcal{F}$ also.

Let $\mathfrak{X}^*(M)$ denote the real linear space of one forms on M, i.e., all finite $C^\infty(M)$ linear combinations of gradients of elements of $C^\infty(M)$. These are represented by row m-vector valued functions of x. The pairing between one forms and vector fields denoted by $\langle\omega,h\rangle\in C^\infty(M)$ is just multiplication of $1\times m$ and $m\times 1$ matrix valued functions of x.

We define a subset of $\mathfrak{X}^*(M)$ by $d\mathcal{G}^0=\{d\varphi:\varphi\in\mathcal{G}^0\}$ and a subspace $d\mathcal{G}=\{d\varphi:\varphi\in\mathcal{G}\}$. Just as vector fields act on functions and other vector fields by Lie differentiation, they act on one forms according to the definition

$$L_h(\omega)(x) = \left(\frac{\partial\omega^*}{\partial x}(s)h(x)\right)^* + \omega(x)\frac{\partial h}{\partial x}(x)$$

where $\omega\in\mathfrak{X}^*(M)$, $h\in\mathfrak{X}(M)$ and $*$ denotes transpose. The three kinds of Lie differentiation are related by the following Liebnitz-type formula

$$L_{h_1}\langle\omega,h_2\rangle = \langle L_{h_1}\omega,h_2\rangle + \langle\omega,[h_1,h_2]\rangle.$$

If $\omega=d\varphi$, then L_h and d commute

$$L_h(d\varphi) = d(L_h(\varphi)).$$

From this it follows that $d\mathcal{G}$ is the smallest linear space of one forms containing $d\mathcal{G}^0$ which is closed with respect to Lie differentiation by elements of $\mathcal{F}^0$ (or $\mathcal{F}$). Elements of $d\mathcal{G}$ are finite linear combinations of one forms of the form

$$d(L_{f^1}(\cdots(L_{f^k}(g_i))\cdots)) = L_{f^1}(\cdots(L_{f^k}(dg_i))\cdots)$$

where $f^j(x)=f(x,u^j)$ for some constant $u^j\in\Omega$. As before we denote by $d\mathcal{G}(x)$ the space of vectors obtained by evaluating the elements of $d\mathcal{G}$ at x. The space $d\mathcal{G}(x^0)$ determines the local weak observability of Σ at x^0. Σ is said to satisfy the *observability rank condition at* x^0 if the dimension of $d\mathcal{G}(x^0)$ is m, Σ satisfies the *observability rank condition* if this is true for every $x\in M$.

Theorem 3.1: If Σ satisfies the observability rank condition at x^0 then Σ is locally weakly observable at x^0.

The proof depends on the following.

Lemma 3.2: Let V be any open subset of M. If $x^0,x^1\in V$ and $x^0 I_V x^1$ then $\varphi(x^0)=\varphi(x^1)$ for every $\varphi\in\mathcal{G}$.

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Proof: If $x^0 I_V x^1$ then for any $k \geqslant 0$, any constant controls $u^1, \cdots, u^k \in \Omega$, small $s_1, \cdots, s_k \geqslant 0$ and g_i, $i = 1, \cdots, n$, we have

$$g_i\left(\gamma_{s_k}^k \circ \cdots \circ \gamma_{s_2}^2 \circ \gamma_{s_1}^1(x^0)\right) = g_i\left(\gamma_{s_k}^k \circ \cdots \circ \gamma_{s_2}^2 \circ \gamma_{s_1}^1(x^1)\right).$$

Here $\gamma_{s_i}^i(x)$ denotes the flow of $f^i(x) = f(x, u^i)$. Differentiating with respect to $s_k, \cdots, s_1$, at 0 yields

$$L_{f^1}\left(\cdots\left(L_{f^k}(g_i)\right)\cdots\right)(x^0) = L_{f^1}\left(\cdots\left(L_{f^k}(g_i)\right)\cdots\right)(x^1).$$

$\mathcal{G}$ is spanned by functions of this form so the lemma follows. □

Proof of Theorem 3.1: If the dimension of $d\mathcal{G}(x^0) = m$ then there exists m functions $\varphi_1, \cdots, \varphi_m \in \mathcal{G}$ such that $d\varphi(x^0), \cdots, d\varphi_m(x^0)$ are linearly independent. Define a map

$$\Phi: x \mapsto \operatorname{col}(\varphi_1(x), \cdots, \varphi_m(x)).$$

The Jacobian of Φ at x^0 is nonsingular so Φ restricted to some open neighborhood U of x^0 is one to one. If $V \subseteq U$ is an open neighborhood of x^0 then Lemma 3.2 implies $I_V(x^0) = \{x^0\}$ so Σ is locally weakly observable at x^0. □

From this we see that if Σ satisfies the observability rank condition then Σ is locally weakly observable. The converse is almost true as we shall see later on in Theorems 3.11 and 3.12.

Suppose (Σ, x^0) is a realization of an input–output map which is not observable in any of the above senses. We turn our attention to finding such a realization (Σ', z^0) of the same input–output map. To understand the difficulties involved consider the following.

Example 3.3: Let $u \in \Omega = \boldsymbol{R}$, $x \in M = \boldsymbol{R}$, $y \in \boldsymbol{R}^2$, $x(0) = x^0 = 0$, and

$$\dot{x} = u \quad y_1 = \cos x \quad y_2 = \sin x.$$

Clearly the system satisfies both controllability and observability rank conditions. Therefore, it is locally weakly controllable and locally weakly observable. It is not observable because x^0 and $x^k = x^0 + 2k\pi$ are indistinguishable for any x^0 and any integer k.

To obtain an observable system with the same input–output behavior as the original we must identify x^0 and x^k, that is, define a system Σ' on $M' = S^1$, the unit circle by $\theta(0) = \theta^0 = 0$ and

$$\dot{\theta} = u \quad y_1 = \cos\theta \quad y_2 = \sin\theta.$$

Note that (Σ, x^0) and (Σ', θ^0) realize the same input–output map.

This example seems to imply that one can obtain an observable realization from one that it is not by "factoring" by the relation I of indistinguishability, the new system lives on the state space $M' = M/I$. However, given an equivalence relation R on M it is not always true that M/R with the quotient topology is Hausdorff and admits a C^∞ structure in such a way that the canonical projection $\pi: M \to M/R$ is a *submersion*, i.e., a C^∞ map of maximal rank. A necessary and sufficient condition for the quotient topology on M/R to be Hausdorff is that R be a *closed equivalence relation*, that is, the graph of R be a closed subset of $M \times M$. An equivalence relation R which admits a C^∞ structure on M/R compatible with the projection is called *regular*. A necessary and sufficient condition for regularity is that the graph of R be a regularly imbedded submanifold of $M \times M$ and that the map $(x^0, x^1) \mapsto x^0$ from the graph of R onto M be a submersion [23].

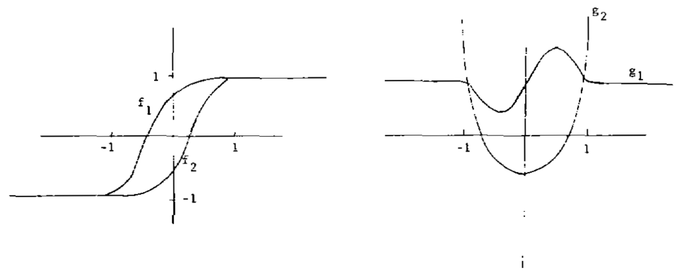

Fig. 1. Dynamics and observers of Example 3.4.

It is not hard to see using the continuity of solutions of differential equations with respect to initial conditions that for any C^∞ system Σ the relation I is closed. However, the following example due to Sussmann shows that it need not be regular even when the controllability and observability rank conditions are satisfied.

Example 3.4: Let $u \in \Omega = \{(u_1, u_2): u_i \geqslant 0\}$ $x \in M = \boldsymbol{R}$, $y \in \boldsymbol{R}^2$ and

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x)$$
$$y_1 = g_1(x), \quad y_2 = g_2(x).$$

We choose $f_i, g_i: \boldsymbol{R} \to \boldsymbol{R}$ to be C^∞ functions with the following graphs. See Fig. 1.

Since f_1 and f_2 have no common zeros the system satisfies the controllability rank condition and is locally weakly controllable. Since dg_1 and dg_2 have no common zeros the system satisfies the observability rank condition and is locally weakly observable.

From the graphs of g_1 and g_2 we see the only possible pairs of indistinguishable points are x and $-x$ where $|x| \geqslant 1$. Since $f_i(x) = 1$ if $x \geqslant 1$ and $f_i(x) = -1$ if $x \leqslant -1$ it can be seen that these pairs are indistinguishable.

If we quotient the state space $M = \boldsymbol{R}$ by the equivalence relation of indistinguishability, the result is clearly not a manifold for it looks like a circle with a ray attached. At this point one can ask what additional assumptions on Σ are needed in order to insure that I be a closed and regular equivalence relation. Sussmann has proved the following.

Theorem 3.5 [27]: If (Σ, x^0) is a weakly controllable realization and either

a) Σ is analytic or

b) Σ is symmetric (i.e., $\forall u \in \Omega\ \exists v \in \Omega$ such that $f(x,u) = -f(x,v)\ \forall x \in M$), then I is a closed and regular equivalence relation. Moreover, there exists a system Σ' on $M' = M/I$ such that $(\Sigma', I(x^0))$ is an observable and

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weakly controllable realization of the same input–output map. If Σ is analytic, then Σ' is also locally observable.

In some sense Theorem 2.3b is the dual of Theorem 3.5a and Theorem 2.4 is the dual of Theorem 3.5b. What we would like to discuss now is the dual of Theorem 2.3a. Proceeding analogously we might expect that I is a regular equivalence relation if the dimension of $d\mathcal{G}(x)$ is constant over M but Example 3.4 shows this not to be the case. Just as we have used the relation WA in addition to A when studying nonlinear controllability so must we introduce another relation for nonlinear observability.

We call this new relation strong indistinguishability, x^0 and x^1 are *strongly indistinguishable* (denoted by x^0SIx^1) if there exists a continuous curve α: $[0,1]\to M$ such that $\alpha(0)=x^0$, $\alpha(1)=x^1$ and $x^0I\alpha(s)$ for all $s\in[0,1]$. Clearly SI is an equivalence relation, x^0SIx^1 implies x^0Ix^1, and Σ weakly observable at x^0 implies $SI(x^0)=\{x^0\}$.

As we shall demonstrate in a moment, if the dimension of $d\mathcal{G}(x)$ is constant over M then SI is a regular equivalence relation and the quotient M', possibly non-Hausdorff, inherits a locally weakly observable system Σ' which realizes the same input–output map as Σ. Before proving this we must introduce some more machinery. Let $\mathcal{H}(x)$ denote the space of all tangent vectors at x which annihilate every element of $d\mathcal{G}(x)$

$$\mathcal{H}(x)=\{\tau\in T_xM:\langle d\varphi(x),\tau\rangle=0, \qquad \forall\varphi\in\mathcal{G}\}.$$

If the dimension of $d\mathcal{G}(x)$ is constant, say k, over M then the dimension of $\mathcal{H}(x)$ is constant, $m-k$. The importance of $\mathcal{H}(x)$ is explained by the corollaries to the following lemma.

Lemma 3.6: Suppose the dimension of $d\mathcal{G}(x)$ is k for every $x\in M$. Let $f_t(x)$ be a time-dependent vector field on M such that $f_t(\cdot)\in\mathcal{F}$ for every t [for example, if $u(t)$ is an admissible control and $f_t(x)=f(x,u(t))$]. Let $\gamma_t(x^0,t^0)$ be the flow of f_t, i.e.,

$$\frac{d}{dt}\gamma_t(x^0,t^0)=f_t(\gamma_t(x^0,t^0))$$
$$\gamma_0(x^0,t^0)=x^0.$$

Then

$$d\mathcal{G}(\gamma_t(x^0,t^0))\frac{\partial}{\partial x}\gamma_t(x^0,t^0)=d\mathcal{G}(x^0).$$

Proof: Since $\gamma_{t+s}(x^0,t^0)=\gamma_s(\gamma_t(x^0,t^0),t)$, it suffices to consider only small $|t-t^0|$. Choose $d\varphi_1,\cdots,d\varphi_k\in d\mathcal{G}$ which are linearly independent on a neighborhood U of x^0. A straightforward calculation yields

$$\frac{d}{dt}\left(d\varphi_i(\gamma_t(x^0,t^0))\frac{\partial\gamma_t}{\partial x}(x^0,t^0)\right)$$
$$=L_{f_t}(d\varphi_i)(\gamma_t(x^0,t^0))\frac{\partial\gamma_t}{\partial x}(x^0,t^0).$$

On the other hand we can choose functions $\lambda_{ij}(t)$ such that

$$d\varphi_i(\gamma_t(x^0,t^0))\frac{\partial\gamma_t}{\partial x}(x^0,t^0)=\sum_{j=1}^m\lambda_{ij}(t)dx_j$$

for $i=1,\cdots,k$. Moreover if $f\in\mathcal{F}$ and $d\varphi\in d\mathcal{G}$ then $L_f(d\varphi)\in d\mathcal{G}$ so there exists functions $\mu_{ir}(t)$ such that

$$L_{f_t}(d\varphi_i)(\gamma_t(x^0,t^0))=\sum_{r=1}^k\mu_{ir}(t)d\varphi_r(\gamma_t(x^0,t^0))$$

for $i=1,\cdots,k$.

Combining these three equations we obtain

$$\frac{d}{dt}\left(\sum_{j=1}^m\lambda_{ij}(t)\right)=\sum_{r=1}^k\sum_{j=1}^m\mu_{ir}(t)\lambda_{rj}(t)$$

for $i=1,\cdots,k$. This is a linear homogeneous differential equation so there exist invertible linear transformations $\Lambda(t):\boldsymbol{R}^k\to\boldsymbol{R}^k$ such that

$$(\lambda_{ij}(t))=\Lambda(t)(\lambda_{ij}(t^0)).$$

The lemma follows since $d\mathcal{G}(x^0)$ is spanned by

$$d\varphi_i(x^0)=\sum_{j=1}^m\lambda_{ij}(t^0)dx_j$$

for $i=1,\cdots,k$ and for small $|t-t^0|$, $d\mathcal{G}(\gamma_t(x^0,t^0))\partial/\partial x\gamma_t(x^0,t^0)$ is spanned by

$$d\varphi_i(\gamma_t(x^0,t^0))\frac{\partial\gamma_t}{\partial x}(x^0,t^0)=\sum_j\lambda_{ij}(t)dx_j$$

for $i=1,\cdots,k$. □

Remark: The above depends heavily on the fact that the dimension of $d\mathcal{G}(x)$ is constant so that $d\mathcal{G}$ is "locally finitely generated." For similar results see Hermann [10].

A curve $\alpha:[0,1]\to M$ is *piecewise* C^∞ if it is C^∞ at all but finitely many points of $[0,1]$ and left (right) limits of α and all its derivatives exist at every point of $(0,1]$ ($[0,1)$).

We define another equivalence relation on M, x^0Hx^1 if there exists a continuous and piecewise C^∞ curve $\alpha:[0,1]\to M$ such that $\alpha(0)=x^0$, $\alpha(1)=x^1$ and

$$\frac{d}{ds}\alpha(s)\in\mathcal{H}(\alpha(s)).$$

Corollary 3.7: Assuming the dimension of $d\mathcal{G}(x)$ is constant, if x^0Hx^1 then x^0SIx^1.

Proof: Let $(u(t),[t^0,t^1])$ be any admissible control with flow $\gamma_t(x,t^0)$. The ith component of the output at time t when the system is started at $\alpha(s)$ at time t^0 is

$$y_i(t)=g_i(\gamma_t(\alpha(s),t^0)).$$

The derivative with respect to s is given by

$$\frac{d}{ds}y_i(t)=\left\langle dg_i(\gamma_t(\alpha(s),t^0))\frac{\partial\gamma_t}{\partial x}(\alpha(s),t^0),\frac{d\alpha}{ds}(s)\right\rangle$$

which is identically zero by the last lemma.

This next result shows that $\mathcal{F}$ is a collection of symme-

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try vector fields (in the sense of Sussmann [26]) for the relation H.

Corollary 3.8: Assume that the dimension of $d\mathcal{G}(x)$ is constant. Let $f\in\mathcal{F}$ with flow $\gamma_t(x)$. If x^0Hx^1 then $\gamma_t(x^0)H\gamma_t(x^1)$ for all $t\in\boldsymbol{R}$.

Proof: Let $\alpha:[0,1]\to M$ be a continuous and piecewise C^∞ curve such that $\alpha(0)=x^0$, $\alpha(1)=x^1$ and $d/ds\,\alpha(s)\in\mathcal{H}(\alpha(s))$. Define a curve, $\beta:[0,1]\to M$ by $\beta(s)=\gamma_t(\alpha(s))$. Clearly β is continuous, piecewise C^∞, and $\beta(0)=\gamma_t(x^0)\,\beta(1)=\gamma_t(x^1)$. Moreover for $\varphi\in\mathcal{G}$

$$\begin{aligned}\langle d\varphi(\beta(s)),\frac{d}{ds}\beta(s)\rangle&=\frac{d}{ds}\varphi(\beta(s))=\frac{d}{ds}\varphi(\gamma_t(\alpha(s)))\\&=\langle d\varphi(\gamma_t(\alpha(s)))\frac{\partial\gamma_t}{\partial x}(\alpha(s)),\frac{d}{ds}\alpha(s)\rangle\\&=0.\end{aligned}$$

So $d/ds\beta(s)\in\mathcal{H}(\beta(s))$. □

Theorem 3.9: Suppose the dimension of $d\mathcal{G}(x)$ is k, then SI is a regular equivalence relation and there exists a locally weakly observable system Σ' on the k-dimensional non-Hausdorff manifold $M'=M/SI$ which has the same input–output properties as Σ. More precisely if $\pi:M\to M'$ is the canonical projection then (Σ,x^0) and $(\Sigma',\pi(x^0))$ realize the same input–output map for every $x^0\in M$. If Σ is (locally)(weakly) controllable then so is Σ'. If Σ satisfies the controllability rank condition then so does Σ' and moreover M' is Hausdorff.

Proof: An outline goes like this.

From Corollary 3.7 we see that H equivalence implies SI equivalence. We first show that H is a regular equivalence relation and that we can define a system Σ' on $M'=M/H$ with the same input–output properties as Σ. We then note that Σ' satisfies the observability rank condition, hence Σ' is locally weakly observable from which it follows that $H=SI$.

Remark: By Corollary 3.7 x^0Hx^1 implies x^0SIx^1 which by definition implies x^0Ix^1 which by Lemma 3.2 implies $\varphi(x^0)=\varphi(x^1)$ for every $\varphi\in\mathcal{G}$.

Given any $x^0\in M$ there exists $d\varphi_1,\cdots,d\varphi_k\in d\mathcal{G}$ which are linearly independent at x^0. After reordering the x coordinates we can suppose that $d\varphi_1,\cdots,d\varphi_k$, $dx_{k+1},\cdots,dx_m$ are linearly independent at x^0. Define $z_i(x)=\varphi_i(x)-\varphi_i(x^0)$, $i=1,\cdots,k$, $z_i(x)=x_i-x_i^0$, $i=k+1,\cdots,m$ and

$$U=\{x:|z_i(x)|<\epsilon\}.$$

If ϵ is chosen sufficiently small, then (U,x) is a coordinate chart around x^0.

Claim If $x^1,x^2\in H(U)$ then x^1Hx^2 iff $\varphi_i(x^1)=\varphi_i(x^2)$ for $i=1,\cdots,k$. The only if part follows immediately from the above remark. As for the converse suppose first that $x^1,x^2\in U$ and $\varphi_i(x^1)=\varphi_i(x^2)$ for $i=1,\cdots,k$. Then $z_i(x^1)=z_i(x^2)$ *for* $i=1,\cdots,k$. Clearly $\partial/\partial z_{k+1},\cdots,\partial/\partial z_m\in\mathcal{H}(x)$ for every $x\in U$ and these vector fields can be used to get from x^1 to x^2 within the z-cube U so x^1Hx^2. More generally if $x^1,x^2\in H(U)$ and $\varphi_i(x^1)=\varphi_i(x^2)$ for $i=1,\cdots,k$ then there exists $p^j\in U$ such that x^jHp^j and $\varphi_i(p^j)=\varphi_i(x^j)$ for $i=1,\cdots,k$ and $j=1,2$. From before we know p^1Hp^2 which by transitivity implies x^1Hx^2.

We next put a C^∞ structure on $M'=M/H$. Notice that $H(U)$ is open and $\pi^{-1}(\pi(H(U)))=H(U)$ so $V=\pi(H(U))$ is open in the quotient topology on M'. The functions of $\mathcal{G}$ are constant on H-equivalence classes so every $\varphi\in\mathcal{G}$ can be pulled down to M', i.e., there exists a continuous $\varphi':M'\to\boldsymbol{R}$ such that $\varphi'\circ\pi=\varphi$.

In particular we define $z_i'=\varphi_i'-\varphi_i(x^0)$ for $i=1,\cdots,k$ and $z'=(z_1',\cdots,z_k')$. The claim implies that the map $z':V\to\boldsymbol{R}^k$ is one to one. Moreover it is open because if W is an open subset of V then

$$z'(W)=\pi_k\circ z\circ(\pi^{-1}(W)\cap U)$$

where $\pi_k:\boldsymbol{R}^m\to\boldsymbol{R}^k$ is the projection on the first k factors. Therefore $z':V\to\boldsymbol{R}^k$ is a homeomorphism into and (V,z') a coordinate chart. In the coordinates (U,z) and (V,z') on M and M', the projection π is just π_k and clearly a submersion.

M is covered by a countable number of charts (U,z) and so M' is covered by the corresponding (V,x'). Verifying that changes of coordinates on M' are C^∞ reduces to checking that every $\varphi':M'\to\boldsymbol{R}$ is C^∞ on (V,z'). But if $\varphi\in\mathcal{G}$ then on (U,z)

$$d\varphi=\Sigma\lambda_i(z)\,dz_i$$

so $\varphi(z)=\varphi(z_1,\cdots,z_k)$. Clearly $\varphi'(z')=\varphi(z_1',\cdots,z_k')$ is C^∞. This shows H is a regular equivalence relation on M.

Next we define Σ' on M' locally. In the coordinate system (U,z) on M, the dynamics of Σ are given by

$$\dot z=\frac{\partial z}{\partial x}(x(z))f(x(z),u).$$

In particular for $i=1,\cdots,k$

$$\dot z_i=\frac{\partial\varphi_i}{\partial x}(x(z))f(x(z),u)=L_{f(\cdot,u)}(\varphi_i)(x(z)).$$

The right side is an element of $\mathcal{G}$ and hence pulls down to a function $f_i'(z',u)$ on V'. We define the dynamics of Σ' on (V,z') by

$$\dot z'=f'(z',u).$$

Clearly if $z(t)$ is a curve in U generated by the dynamics of Σ under the control $u(t)$ satisfying $z(t^0)=z^0$ and if $z'(t)$ is the curve in V generated by the dynamics of Σ' under the control $u(t)$ satisfying $z'(t^0)=\pi(z^0)$ then for small $|t-t^0|$

$$z'(t)=\pi(z(t)).$$

From this it follows that if Σ is (locally) (weakly) controllable so is Σ'.

Moreover each $g_i\in\mathcal{G}$ and hence pulls down to a function g_i' on M' which we use to define the output of Σ' on (V,Z') as

$$y=g'(z').$$

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The outputs of Σ from z^0 under $u(t)$ and Σ' from $\pi(z^0)$ under $u(t)$ are the same because $g = g' \circ \pi$ and so

$$g(z(t)) = g' \circ \pi(z(t)) = g'(z'(t)).$$

Notice M' is of dimension k.

We turn now to the relationship between $\mathcal{F}$ and $\mathcal{G}$ of Σ and $\mathcal{F}'$ and $\mathcal{G}'$ of Σ'. Note that

$$f'(\pi(x),u) = \frac{\partial \pi}{\partial x}(x) f(x,u).$$

It follows from this and the definition of $\mathcal{F}$ and $\mathcal{F}'$ that $f' \in \mathcal{F}'$ iff there exists an $f \in \mathcal{F}$ such that

$$f'(\pi(x)) = \frac{\partial \pi}{\partial x}(x) f(x).$$

In particular

$$\mathcal{F}'(\pi(x)) = \frac{\partial \pi}{\partial x}(x)\mathcal{F}(x).$$

and since π is a submersion this shows that if Σ satisfies the controllability rank condition so does Σ'.

Furthermore

$$g' \circ \pi = g$$

and so

$$L_f g_i(x) = L_{f'}(g_i')(\pi(x)).$$

Repeated applications of this formula show that $\varphi' \in \mathcal{G}'$ iff there exists on $\varphi \in \mathcal{G}$ such that

$$\varphi'(\pi(x)) = \varphi(x).$$

In particular the coordinate functions $z_i' = \varphi_i'$ are in $\mathcal{G}'$ so Σ' satisfies the observability rank condition and therefore is locally weakly observable.

This also implies that H and SI are the same relation. We have already seen that $x^0 H x^1$ implies $x^0 SI x^1$. Suppose the converse does not hold; there exists $x^0, x^1 \in M$ such that $x^0 SI x^1$ but $x^0 H x^1$. Let $\alpha(s)$ be the arc of Σ-indistinguishable points joining x^0, x^1, since Σ and Σ' have the same input-output properties, $\pi(\alpha(s))$ is an arc of Σ'-indistinguishable points joining $\pi(x^0)$ and $\pi(x^1)$. Moreover this arc is not constant because $\pi(x^0) \neq \pi(x^1)$. This contradicts the local weak observability of Σ'.

If Σ satisfies the controllability rank condition then M' can be seen to be Hausdorff. Given $x^0, x^1 \in M$ first suppose there exists a $\varphi \in \mathcal{G}$ such that $\varphi(x^0) \neq \varphi(x^1)$. Define disjoint open sets

$$U^0 = \{x \in M : |\varphi(x) - \varphi(x^0)| < |\varphi(x^1) - \varphi(x^0)|/2\}$$
$$U^1 = \{x \in M : |\varphi(x) - \varphi(x^1)| > |\varphi(x^1) - \varphi(x^0)|/2\}.$$

Since φ is constant on H equivalence classes

$$\pi^{-1}(\pi(U^i)) = U^i$$

and so $\pi(U^0)$ and $\pi(U^1)$ are disjoint open neighborhoods of $\pi(x^0)$ and $\pi(x^1)$.

On the other hand suppose $\varphi(x^0) = \varphi(x^1)$ for every $\varphi \in \mathcal{G}$. Using the controllability rank condition we choose $f_1, \cdots, f_m \in \mathcal{F}$ which are linearly independent at x^0. We also choose $d\varphi_1, \cdots, d\varphi_k \in d\mathcal{G}$ which are linearly independent at x^0. Then the $k \times m$ matrix with i–jth element

$$\langle d\varphi_i(x), f_j(x)\rangle = L_{f_j}(\varphi_i)(x)$$

is of rank k at x^0. Without loss of generality we can assume the first k columns of this matrix are linearly independent at x^0. The elements of this matrix are all in $\mathcal{G}$ so the first k columns are also linearly independent at x^1 and therefore $d\varphi_1, \cdots, d\varphi_k$ are linearly independent at x^1.

As before we can construct a coordinate cube (U^0, z) around x^0 using $\varphi_1, \cdots, \varphi_k, x_{k+1}, \cdots, x_m$. In a similar fashion using the same ϵ and $\varphi_1, \cdots, \varphi_k$ but perhaps different $x_{k+1}, \cdots, x_m$ we construct a cube (U^1, z) around x^1. Given $\delta > 0$ define $m-k$ slices S_δ^0 and S_δ^1 in U^0 and U^1 by

$$S_\delta^j = \{x \in U^j : z_i(x) = 0, \quad i = 1, \cdots, k \quad \text{and} \quad |z_i(x)| < \delta, \quad i = k+1, \cdots, m\}.$$

Let $\gamma_s^i(x)$ denote the flow of $f^i(x)$ for $i = 1, \cdots, k$ and C_δ^k denote the cube of side 2δ around 0 in $\boldsymbol{R}^k$. Define maps

$$\beta_j : C_\delta^k \times S_\delta^j \to M$$

by

$$\beta_j(s_1, \cdots, s_n, x) = \gamma_{s_k}^k \circ \cdots \gamma_{s_1}^1(x)$$

where $x \in S_\delta^j$ and $|s_i| < \delta$ for $i = 1, \cdots, k$. Of course S_δ^j is a $m-k$ dimensional cube so we can view these maps as

$$\beta^j : C_\delta^m \to M$$

where C_δ^m is an m-cube of side 2δ. More precisely if $(s_1, \cdots, s_m) \in C_\delta^m$ then

$$\beta^j(s_1, \cdots, s_m) = \beta^j(s_1, \cdots, s_k, x)$$

where $x \in S_\delta^j$ with $z_i(x) = 0$ for $i = 1, \cdots, k$ and $z_i(x) = s_i$ for $i = k+1, \cdots, m$. For sufficiently small δ these maps are diffeomorphisms onto open neighborhoods W^j of x^j contained in U^j. In fact $(W^j, s_1, \cdots, s_m)$ are coordinate charts at x^j. If δ is chosen sufficiently small then we can assume that for every $x \in W^1$ and for every $(s_1, \cdots, s_n) \in C_\delta^k$ we have

$$\gamma_{s_1}^1 \circ \cdots \circ \gamma_{s_k}^k(x) \in U^1.$$

Now suppose there exists $p^0 \in W^0$, $p^1 \in W^1$ such that $p^0 H p^1$. In particular suppose $p^0 = \gamma_{s_k}^k \circ \cdots \circ \gamma_{s_1}^1(q^0)$ where $q^0 \in S_\delta^0$. Let

$$q^1 = \gamma_{-s_1}^1 \circ \cdots \circ \gamma_{-s_k}^k(p^1),$$

then by assumption $q^1 \in U^1$ and by Corollary 3.8, $q^0 H q^1$. Because $q^0 \in S_\delta^0$, $x^0 H q^0$ so $x^0 H q^1$. This implies that $\varphi_i(x^0) = \varphi_i(q^1)$ for $i = 1, \cdots, k$ and so by assumption $\varphi_i(q^1) = \varphi_i(x^1)$ for $i = 1, \cdots, k$. By our previous claim, $q^1 H x^1$ and so $x^0 H x^1$.

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We have just shown that if $\pi(W^0)\cap\pi(W^1)\neq\emptyset$ then $\pi(x^0)=\pi(x^1)$. It follows that if $\pi(x^0)\neq\pi(x^1)$ then these are disjoint open neighborhoods of $\pi(x^0)$ and $\pi(x^1)$. □

The following simple example shows that if Σ fails to satisfy the controllability rank condition then M' need not be Hausdorff.

Example 3.10: Let $M=\boldsymbol{R}^2\setminus\{(x_1,0):x_1\leqslant 0\}\, y\in\boldsymbol{R}^1$ and

$$\dot{x}=0 \quad y=x_1.$$

The dimension of $d\mathcal{G}(x)$ is one for all x so we can form the quotient $M'=M/SI$. However this is a non-Hausdorff manifold because $\pi(0,1)$ and $\pi(0,-1)$ are distinct points without a pair of disjoint neighborhoods. Notice that factoring M by the closure of the relation SI is not a way around this problem for the closure of SI is not a regular relation.

Theorem 3.5a and b and Theorem 3.9 exhibit three situations where realizations can be made observable in some sense. If Σ is analytic then Σ' is locally observable, if Σ is symmetric then Σ' is observable and if $d\mathcal{G}(x)$ of Σ is of constant dimension then Σ' satisfies the observability rank condition and hence is locally weakly observable. The converse of this last remark is "almost" true.

Theorem 3.11: If Σ is locally weakly observable then the observability rank condition is satisfied generically.

Proof: For any system the observability rank condition is satisfied on an open subset of M, possibly empty. Suppose there exists an open subset U of M where the dimension of $d\mathcal{G}(x)<m$. Without loss of generality we can assume $\dim d\mathcal{G}(x)=k<m$ for $x\in U$. Choose $x^0\in U$, an open set V such that $x^0\in V\subseteq U$ and a C^∞ function $\varphi:M\to\boldsymbol{R}$ such that $\varphi(x)=1$ for all $x\in V$, $\varphi(x)\neq 0$ for all $x\in U$ and $\varphi(x)=0$ for all $x\in M\setminus U$. Consider the system Σ' defined by

$$\dot{x}=\varphi(x)f(x,u)$$
$$y=g(x)$$
$$x(0)=x^0.$$

The state space of Σ' is U and Σ' is complete. It is easy to see $\mathcal{F}'(x)=\mathcal{F}(x)$ for all $x\in U$. From this it follows that $d\mathcal{G}'(x)=d\mathcal{G}(x)$ for all $s\in U$, and so $\dim d\mathcal{G}(x)=k$ for all $x\in U$. We can apply Theorem 3.9 to Σ' on U so Σ' is not locally weakly observable. Since Σ and Σ' agree on V, Σ is not locally weakly observable either. □

Recall that for analytic systems, weak controllability, local weak controllability, and the controllability rank condition are equivalent. With regard to observability an analogous result holds for analytic systems which are weakly controllable.

Theorem 3.12: If Σ is a weakly controllable analytic system then Σ is weakly observable iff it is locally weakly observable iff the observability rank condition is satisfied.

Proof: It suffices to show that weak observability implies the observability rank condition. By Theorem 2.6 Σ satisfies the controllability rank condition, we proceed in a similar fashion to the proof of that theorem to show the dimension of $d\mathcal{G}(x)$ is constant.

Let $x^1=\gamma_s(x^0)$ where γ denotes the flow of $f\in\mathcal{F}$. The adjoint of $(\partial/\partial x)\gamma_s(x^0)$ carries one forms at x^1 to one forms at x^0 according to the rule

$$\omega(x^1)\mapsto\omega(x^1)\frac{\partial}{\partial x}\gamma_s(x^0),$$

for $\omega\in\mathcal{X}^*(M)$.

This map also has a Campbell–Baker–Hausdorff expansion

$$\omega(x^1)\frac{\partial}{\partial x}\gamma_s(x^0)=\sum_{k=0}^{\infty}(L_f)^k\omega(x^0)\frac{s^k}{k!}.$$

In particular, if $\omega\in H$ then the adjoint of $(\partial/\partial x)\gamma_s(x^0)$ carries $d\mathcal{G}(x^1)$ into $d\mathcal{G}(x^0)$ so the dimension of $d\mathcal{G}(x^1)$ is less than or equal to the dimension $d\mathcal{G}(x^0)$. A similar argument using the adjoint of $(\partial/\partial x)\gamma_{-s}(x^1)$ shows the reverse inequality.

Therefore, if x^0WAx^1 then the dimensions of $d\mathcal{G}(x^0)$ and $d\mathcal{G}(x^1)$ are the same. Since Σ is weakly controllable, this implies that the dimension of $d\mathcal{G}(x)$ is constant.

Suppose $\dim d\mathcal{G}(x)=k$ then we apply Theorem 3.9 and identify strongly indistinguishable points. But Σ weakly observable implies there are no strongly indistinguishable points so k must equal m. □

To relate the results of this section to the well-known linear theory, we consider the following.

Example 3.11: Consider the linear system

$$\dot{x}=Ax+Bu$$
$$y=Cx.$$

In Example 2.5 we saw that $\mathcal{F}$ was spanned by $\{Ax,A^rB_{*j}:r=0,\cdots,m-1,\ j=1,\cdots,l\}$ where B_{*j} denotes the jth column of B. Let C_{i*} denote the ith row of C then for $r\geqslant 0$, $\rho\geqslant 0$

$$L_{Ax}C_{i*}yA^rx=C_{i*}A^{r+1}x$$
$$L_{A^rB_{*j}}C_{i*}A^\rho x=C_{i*}A^{r+\rho}B_{*j}$$
$$L_{Ax}C_{i*}A^\rho B_{*j}=L_{A^rB_{*j}}C_{i*}A^\rho B_{*j}=0.$$

Therefore by the Cayley–Hamilton Theorem $\mathcal{G}$ is spanned by

$$\left\{C_{i*}A^rx,C_{i*}A^rB_{*j}:i=1,\cdots,m,j=1,\cdots,l \text{ and } r=0,\cdots,m-1\right\}$$

and $d\mathcal{G}(x)$ is spanned by

$$\left\{C_{i*}A^r:i=1,\cdots,m,\ r=0,\cdots,m-1\right\}.$$

Clearly $d\mathcal{G}(x)$ is of constant dimension. The observability rank condition reduces to the well-known linear observability condition that

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$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{m-1} \end{bmatrix} = m.$$

If the observability rank condition fails then we define $\mathcal{H}(x)$ to be the constant linear space of column vectors orthogonal to $d\mathcal{G}(x)$. Two points x^0 and x^1 are H equivalent (or SI equivalent) if $x^1 - x^0 \in \mathcal{H}(x)$ because

$$\alpha(s) = x^0 + s(x^1 - x^0)$$

is a curve tangent to $\mathcal{H}(x)$ from x^0 to x^1. Factoring M by $\mathcal{H}(x)$ results in a locally weakly observable system which because of linearity is also locally observable ($SI = I$ for linear systems).

Example 3.12: Suppose the linear system is nonautonomous as in Example 2.6. As in formula 2.1 we add time as a state variable which we can observe directly. Then direct calculation shows that the observability rank condition is equivalent to

$$\operatorname{rank}\begin{bmatrix} C(t) \\ \cdot \\ \cdot \\ \cdot \\ \Delta_o C(t) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^2{}_* C(t) \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = m$$

for every $t \in \boldsymbol{R}$ where

$$\Delta_o C(t) = \frac{d}{dt} C(t) + C(t) A(t),$$

$$\Delta_o^r C(t) = \frac{d}{dt} \Delta_o^{r-1} C(t) + \Delta_o^{r-1} C(t) A(t).$$

See [3] or [5].

Example 3.13: Consider the bilinear system of Example 2.8

$$\dot{x} = Ax + \sum_{i=1}^{l} u_i B^i x.$$

If $M = \boldsymbol{R}^m$ (or a subgroup of $Gl(h, \boldsymbol{R})$), let C be a $n \times m$ ($n \times h$) matrix and

$$y = Cx,$$

then

$$\mathcal{G} = \{ C_{ix} Dx : i = 1, \cdots, n;\ Dx \in \mathcal{F} \}$$
$$d\mathcal{G} = \{ C_{ix} D : i = 1, \cdots, n;\ Dx \in \mathcal{F} \}.$$

See [3] for further details.

IV. Minimality

A linear system is said to be *minimal* if it is controllable and observable. As is well known, two minimal linear systems initialized at 0 which realize the same input–output map differ only by a linear diffeomorphism of the state spaces, [1], [4]. A nonlinear system which is observable, weakly controllable and either analytic or symmetric is called *minimal* by Sussmann [29]. He has shown that two minimal nonlinear systems which realize the same input–output map from their respective initial states differ only by a diffeomorphism of the state spaces.

A nonlinear system Σ is *locally weakly minimal* if it is locally weakly controllable and locally weakly observable. Two locally weakly minimal realizations of a given input–output map need not be diffeomorphic as is seen by Example 3.3, but the following theorem shows they must be of the same state dimension which is minimal over all possible realizations.

Let Σ, Σ' be two nonlinear systems with control set $\Omega = \Omega' \subseteq \boldsymbol{R}^l$ states spaces M and M' of dimension m and m' and output with values in $\boldsymbol{R}^n$ given by

$$\Sigma : \dot{x} = f(x, u)$$
$$y = g(x)$$
$$\Sigma' : \dot{z} = f'(z, u)$$
$$y = g'(z).$$

Theorem 4.1: Suppose (Σ, x^0) and (Σ', z^0) realize the same input–output map. If Σ' is locally weakly minimal then $m \geqslant m'$.

Proof: Without loss of generality we can assume that Σ' satisfies the controllability and observability rank condition at z^0. (If not, by Theorems 2.5 and 3.11 we can find a control $(u(t), [t^0, t^1])$ and a corresponding Σ' trajectory $(z(t), [t^0, t^1])$ with $z(t^0) = z^0$, $z(t^1) = z^1$ such that these conditions are satisfied at z^1. Moreover if x^1 is the endpoint of the corresponding Σ trajectory then (Σ, x^1) and (Σ', z^1) realize the same input–output map.)

Since Σ' satisfies the observability rank condition at z^0 we can find a neighborhood V of z^0 and functions $\varphi_1', \cdots, \varphi_{m'}' \in \mathcal{G}'$ such that the map

$$\Phi' = \operatorname{col}(\varphi_1', \cdots, \varphi_m') : V \to \boldsymbol{R}^{m'}$$

is a diffeomorphism into.

Because Σ' satisfies the controllability rank condition, as in Theorems 2.1 and 2.2 we can find controls $u^1, \cdots, u^m$ and corresponding vector fields $f'^i(z) = f'(z, u^i)$ with flows $\gamma'^i_s(z)$ such that the map

$$\Psi'(s_1, \cdots, s_m) = \gamma'^m_{s_m} \circ \cdots \gamma'^1_{s_1}(z^0)$$

is a diffeomorphism from some open subset S of the positive orthant of $\boldsymbol{R}^{m'}$ into M'.

Let φ_i, Φ, f^i, γ^i, Ψ be the corresponding objects of Σ. Since (Σ, x^0) and (Σ', z^0) realize the same input–output map $\Phi \circ \Psi = \Phi' \circ \Psi'$ and so

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$$S \subseteq \boldsymbol{R}^{m'} \xrightarrow{\Psi} M \xrightarrow{\Phi} \boldsymbol{R}^{m'}$$

is a diffeomorphism. But clearly this is only possible if $m \geqslant m'$.

Corollary 4.2: If (Σ, x^0) and (Σ', z^0) are locally weakly minimal realizations of the same input–output map then their state spaces are of the same dimension.

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Robert Hermann, for a photograph and biography see p. 25 of the February 1977 issue of this TRANSACTIONS.

Arthur J. Krener (M '76) was born in Brooklyn, NY on October 8, 1942. He received the B.S. degree in mathematics from Holy Cross College, Worcester, MA in 1964 and the M.A. and Ph.D. degrees in mathematics from the University of California, Berkeley, in 1967 and 1971, respectively.

From 1971 to 1974 he was an Assistant Professor of Mathematics at the University of California, Davis. During 1974–1975 he was a Research Fellow in decision and control at Harvard University, Cambridge, MA. He is currently an Associate Professor in the Department of Mathematics, University of California, Davis. His present research interests are in the areas of optimal control, nonlinear systems, stochastic process, and image processing.

Dr. Krener is a member of the Society for Industrial and Applied Mathematics.

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