Optimization applied II: Consumption-saving problem

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The simplest consumption-saving problem:

Let's assume that agents:

- Have a CRRA utility function over consumption:

$$u(c) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \\ \log(c) & \text{if } \sigma = 1 \end{cases}$$

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- Live for 2 periods (young and old)
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- (Can) enter their life with some initial level of assets $a_0 \neq 0$.
- And cannot die with debt $a_2 \geq 0$!

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The Lifetime utility is:

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Here β is the discount factor.

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Agent maximize equation (1) subject to constraints:

$$\max_{c_1, c_2, a_1, a_2} \quad \underbrace{U(c_1, c_2)}_{\text{Lifetime utility function}}$$

$$c_1 + a_1 = a_0 R + y_1$$

$$c_2 + a_2 = a_1 R + y_2$$

Note: $R = 1 + r$. How to solve this problem?

The simplest consumption-saving problem:

Let's look at the budget constraints. First note that $a_2 = 0$ is optimal.

$$\begin{aligned} c_1 =& a_0 R + y_1 - a_1 \\ c_2 =& a_1 R + y_2 \end{aligned}$$

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$$\begin{aligned} U\left(c_1, c_2\right) &= u(c_1) + \beta u(c_2) \\ &= u(a_0 R + y_1 - a_1) + \beta u(a_1 R + y_2) \end{aligned}$$

The simplest consumption-saving problem:

Note that we can define a function:

$$f(a_1) = u(a_0 R + y_1 - a_1) + \beta u(a_1 R + y_2)$$

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For example assume: $R = 1, \beta = 1, a_0 = 1, y = [3, 1]$.

$$\begin{aligned} c_1 =& a_0 R + y_1 - a_1 \\ c_2 =& a_1 R + y_2 \end{aligned}$$

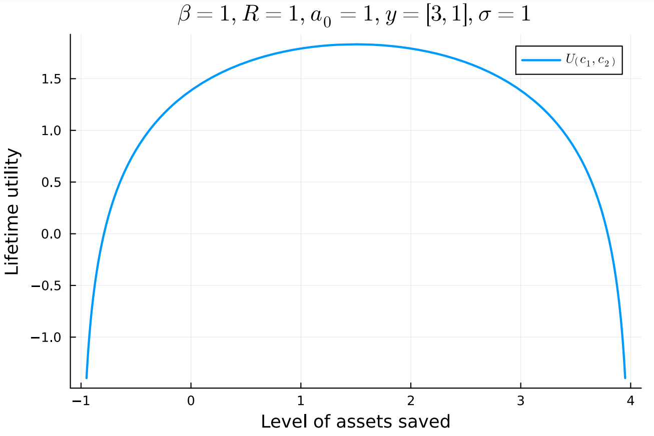
Becomes:

$$\begin{aligned} c_1 =& 1 + 3 - a_1 \\ c_2 =& a_1 + 1 \end{aligned}$$

The simplest consumption-saving problem:

$$f(a_1) = u(4 - a_1) + \beta u(a_1 + 1)$$

The lifetime utility assuming: $R = 1, \beta = 1, a_0 = 1, y = [3, 1]$.



How to find the maximum of $f(a_1) = u(a_0 R + y_1 - a_1) + \beta u(a_1 R + y_2)$?

The first order condition is:

$$\frac{\partial f}{\partial a_1} = 0$$

$$u'(\underbrace{Ra_0 + y_1 - a_1}_{c_1}) = \beta R u'(\underbrace{Ra_1 + y_2}_{c_2})$$

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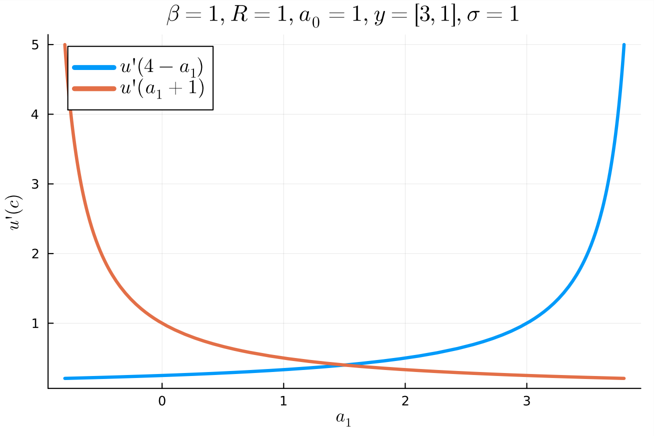

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$$\max_{\{c_t, a_t\}_{t=1}^{5}} \underbrace{U\left(c_1, \ldots, c_5\right)}_{\text{Lifetime utility}}$$

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- Consumption cannot be negative (i.e. $\forall_{t \in \{1,\ldots,5\}} : c_t \geq 0$)
- The individual cannot die with debt (i.e. $a_5 \geq 0$)
- Borrowing and lending is possible in other periods (i.e.$\forall_{t \in \{1,\ldots,4\}} : a_t \in \mathbb{R}$)

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The budget constraint in period t :

(2)

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If we had only two periods (recall, optimal $a_2 = 0$) :

$$\left(c_1 - y_1\right) + \left(\frac{1}{R}\right)\left(c_2 - y_2\right) = Ra_0.$$

Intertemporal budget constraint for 5 periods:

$$\sum_{t=0}^{4} \left(\frac{1}{R}\right)^t c_{t+1} - \sum_{t=0}^{4} \left(\frac{1}{R}\right)^t y_{t+1} + \left(\frac{1}{R}\right)^4 a_5 = Ra_0$$

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- If individual has a perfect foresight and a final planning horizon distributing resources across periods is similar to distributing resources across various goods!

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    - lead to the same optimal consumption path, $\{c_t\}$
    - lead to two different paths of $\{a_t\}$
- This is because we were able to write a single intertemporal budget constraint (BC)
- No constraints on borrowing/lending allowed us to do so

Optimization with a single intertemporal BC:

$$\max_{\{c_t, a_t\}_{t=1}^{5}} \underbrace{U\left(c_1, \ldots, c_5\right)}_{\text{Lifetime utility}}$$

subject to

$$\sum_{t=0}^{4} \left(\frac{1}{R}\right)^t c_{t+1} + \left(\frac{1}{R}\right)^4 a_5 = \sum_{t=0}^{4} \left(\frac{1}{R}\right)^t y_{t+1} + Ra_0$$

$$a_5 \geq 0$$

$$a_0 \text{ given}$$

$$[c_1, c_2, c_3, c_4, c_5] \geq 0$$

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Let's solve the model!

$$L = \sum_{t=0}^{4} \beta^t u\left(c_{t+1}\right)$$

$$- \lambda \left[\sum_{t=0}^{4} \left(\frac{1}{R}\right)^t c_{t+1} - \underbrace{\sum_{t=0}^{4} \left(\frac{1}{R}\right)^t y_{t+1}}_{\text{Human wealth}} - \underbrace{Ra_0}_{\text{Initial financial wealth}} \right]$$

$$\frac{\partial L}{\partial c_{t+1}} : \beta^t u'(c_{t+1}) = \lambda \left(\frac{1}{R}\right)^t$$

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For $t \in \{0, 1, 2\}$:

$$u'(c_1) = \lambda$$
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From the above we get the Euler equation!

$$u'(c_1) = \beta R u'(c_2)$$
$$u'(c_2) = \beta R u'(c_3)$$

The Euler equation:

$$u'(c_t) = R\beta u'(c_{t+1})$$

This can be also written as:

$$u'(Ra_{t-1} + y_t - a_t) = R\beta u'(Ra_t + y_{t+1} - a_{t+1})$$

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- saving and consuming tomorrow (and getting $(1 + r)\beta u'(c_{t+1})$).

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And finally:

$$c_2 = (R\beta)^{\frac{1}{\sigma}} c_1$$
$$c_3 = (R\beta)^{\frac{2}{\sigma}} c_1$$

The Euler equation:

$$
\begin{aligned}
c_{t+1} &= (R\beta)^{\frac{1}{\sigma}} c_t \\
\frac{c_{t+1}}{c_t} &= (R\beta)^{\frac{1}{\sigma}}
\end{aligned}
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Note:

- The slope of the consumption $\{c_t\}$ depends on β vs. $1+r$

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- Forces: Impatience vs. returns on savings

Two motives for savings

In this simple model there are two motives for savings:

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    - When $\beta(1+r)\neq 1$
    - Present even when y_t is constant over the lifecycle

Implementing the model in NLopt

We again can define a lifetime utility function:

$$f(a_1, \ldots, a_4) = u(\underbrace{a_0 R + y_1 - a_1}_{c_1}) + \beta u(\underbrace{a_1 R + y_2 - a_2}_{c_2}) + \beta^2 u(\underbrace{a_2 R + y_3 - a_3}_{c_3}) + \beta^3 u(\underbrace{a_3 R + y_4 - a_4}_{c_4}) + \beta^4 u(\underbrace{a_4 R + y_5}_{c_5})$$

Your task is to write this objective function into Julia!