Emory University

MATH 112Z - Calculus II Learning Notes

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1 Review - Pre-Calculus & Derivative

1.1 Inverse Functions

Inverse Functions

Functions that ”undo” each other.

Examples of Inverse Functions

$$f(x) = x+1 \Longleftrightarrow f^{-1}(x) = x-1$$
$$f(x) = 2x \Longleftrightarrow f^{-1}(x) = \frac{1}{2}x$$
$$f(x) = \frac{1}{x} \Longleftrightarrow f^{-1}(x) = \frac{1}{x}$$

Note: Not all functions have inverses.

One-to-one function

f is an one-to-one function if:

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

In df words, f never have the same value twice.

To testify if a function f is a one-to-one function, we apply the Horizontal Line Test (HLT):

Horizontal Line Test (HLT)

f is one-to-one if and only if (*iff*) every horizontal line intersects the graph of f in **at most one** point.

After knowing the Horizontal Line Test, we can have a look at the precise definition of inverse functions.

Precise Definition of Inverse Functions

Suppose f is a function with a domain$=D_f$ and a range$=R_f$. Now consider a function g with a domain$=D_g$ and a range$=R_g$. Then, f and g are inverses of each other *iff*:

1. $D_f = R_g$ and $D_g = R_f$;
2. $f(g(x)) = x \; \forall x \in D_g$ and $g(f(x)) = x \; \forall x \in D_f$.

Here are some notes to that formal definition:

1. In order for f to have an inverse, it must be one-to-one.
2. If f has an inverse g, then we usually write $f^{-1}(x) = g(x)$.
3. If the point (a,b) is on the graph of $f(x)$, then the point (b,a) is on the graph of $f^{-1}(x)$.

Symmetry of Inverse Functions

$\because (a,b)$ on $f(x)$

$\therefore f(a) = b$

Take the inverse on both sides, and we get:

$f^{-1}(f(a)) = f^{-1}(b)$

$\therefore a = f^{-1}(b)$, i.e., (b,a) on $f^{-1}(x)$

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Extending from the third point of the note, we come to the graph property of inverses.

Graph Property of Inverses

If $f(x)$ has an inverse, then the graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other through the line $y = x$.

1.2 Exponentials and Logs

Recall the graph of $y = a^x$, where $a > 0$ and $a \neq 1$. When $0 < a < 1$, the exponential function decreases as x increases. When $a > 1$, the exponential function increases as x increases.

Exponentials and Logs as Inverse Functions

The inverse of $f(x) = a^x$ is $f^{-1}(x) = g(x) = \log_a x$, and:

$$D_f = (-\infty, \infty),\ D_{f^{-1}} = (0, \infty);$$

$$R_f = (0, \infty),\ R_{f^{-1}} = (-\infty, \infty).$$

Extension for the Relationship

$$\log_a x = y \iff a^y = x:$$

$$\begin{cases} a^{\log_a x} = x & \forall x \in (0, \infty) \\ \log_a(a^x) = x & \forall x \in \mathbb{R} \end{cases}$$

Here, what makes the domain of x different is the which function x firstly go through. In the first expression, x firstly goes through the log function, hence having a limited domain. However, going through the exponential function first will not limit the domain of x.

There are several properties of Logs, and the next chunk lists some of them.

Properties of Logs

$$\log_a x + \log_a y = \log_a (x+y) \qquad \log_a x^r = r\log_a x$$

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right) \qquad \log_a x = \frac{\log_b x}{\log_b a}$$

Natural Log is another essential concept.

Natural Log

Natural Logs are logarithms with base $e \approx 2.71828\cdots$

We write $\log_e x = \ln x$.

1.3 Derivatives

Recall some basic rules of derivatives:

Rules of derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x\ln a}$$

Find the derivative of

$$f(x) = \ln\left(\frac{\sqrt[3]{x^5-4}}{(x^2+3)(7x-10)}\right)$$

$\boxed{\text{Method 1}}$ We can simply differentiate the function using chain and quotient rules. However, this will make the process overcomplicated.

$\boxed{\text{Method 2}}$ Use properties of logarithms to simply the expression first:

$$\begin{aligned} f(x) = \ln\left[\frac{(x^5-4)^{1/3}}{(x^2+3)(7x-10)}\right] &= \ln\left[(x^5-4)^{\frac{1}{3}}\right] - \ln\left[(x^2+3)(7x-10)\right] \\ &= \frac{1}{3}\ln(x^5-4) - \ln(x^2+3) - \ln(7x-10). \end{aligned}$$

Now, we can easily compute the derivative of $f(x)$:

$$\begin{aligned} f'(x) &= \frac{1}{3}\cdot\frac{1}{x^5-4}(x^5-4)' - \frac{1}{x^2+3}(x^2+3)' - \frac{1}{7x-10}(7x-10)' \\ &= \frac{5x^4}{3(x^5-4)} - \frac{2x}{x^2+3} - \frac{7}{7x-10} \end{aligned}$$

1.4 Inverse Trigonometric Functions

Recall the definition of inverse trigonometric functions and the restrictions on their domains.

Domain of inverse sine function

The sine function $y = \sin x$ with domain$= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range$= [-1, 1]$ has the inverse function of $y = \sin^{-1}(x) = \arcsin(x)$ with domain$= [-1, 1]$ and range$= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Because they are inverse of each other, we have the following properties holding true:

$$\begin{cases} \sin^{-1}(\sin(x)) = x & \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \sin(\sin^{-1}(x)) = x & \forall x \in [-1, 1] \end{cases}$$

Moreover, we have:

$$y = \sin^{-1}(x) \iff \sin(y) = x,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Similarly, we can define the restrictions on the domain of other inverse trigonometric functions. Also recall derivatives of inverse trigonometric functions.

2 Integration Technics

2.1 Integration by Parts

Integration by Parts

The formula of integration by parts is given by:

$$\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u$$

Integration by Parts

To begin with, we consider the product rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

The proof begins with attempting to take the integral of the both sides of this product rule.

$$\begin{aligned}
\mathrm{d}\left(f(x)g(x)\right) &= \left(f(x)g'(x)+g(x)f'(x)\right)\,\mathrm{d}x \\
\therefore \int \mathrm{d}\left(f(x)g(x)\right) &= \int \left(f(x)g'(x)+g(x)f'(x)\right)\,\mathrm{d}x \\
\Rightarrow f(x)g(x) &= \boxed{\int f(x)g'(x)\,\mathrm{d}x} + \int g(x)f'(x)\,\mathrm{d}x \\
\therefore \int f(x)g'(x)\,\mathrm{d}x &= f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x
\end{aligned}$$

If we let $\begin{cases} u = f(x) \\ \mathrm{d}v = g'(x) \end{cases}$, we also have $\begin{cases} \mathrm{d}u = f'(x) \\ v = g(x) \end{cases}$, and the formula becomes:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

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Example 1

$$\int x^3 \ln x\,\mathrm{d}x$$

Let $\begin{cases} u = \ln x \\ \mathrm{d}v = x^3 \end{cases}$, so we have $\begin{cases} \mathrm{d}u = \frac{1}{x} \\ v = \frac{1}{4}x^4 \end{cases}$:

$$\begin{aligned}
\therefore \int x^3 \ln x\,\mathrm{d}x &= \ln x \cdot \frac{1}{4}x^4 - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,\mathrm{d}x \\
&= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,\mathrm{d}x \\
&= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C
\end{aligned}$$

Example 2

$$\int (x-1)e^{5x}\,\mathrm{d}x$$

Let $\begin{cases} u = (x-1) \\ \mathrm{d}v = e^{5x} \end{cases}$, so we have $\begin{cases} \mathrm{d}u = 1 \\ v = \frac{1}{5}e^{5x} \end{cases}$:

$$\begin{aligned}
\therefore \int (x-1)e^{5x}\,\mathrm{d} &= \frac{1}{5}e^{5x}(x-1) - \int \frac{1}{5}e^{5x}\,\mathrm{d}x \\
&= \frac{1}{5}e^{5x}(x-1) - \frac{1}{25}e^{5e} + C
\end{aligned}$$

Example 3

$$\int x^3 e^{x^2}\,\mathrm{d}x$$

Method 1 Integration by Parts

Firstly, let's examine $\int e^{x^2}\,\mathrm{d}x$. To solve this integration, we can consider using u-substitution:

Let $u = x^2 \Rightarrow \frac{\mathrm{d}d}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x$.

However, there is an additional x that we have no way to eliminate. Hence, we consider to separate $x^3e^{x^2}$ into x^2 and xe^{x^2}.

Let $\begin{cases} u = x^2 \\ \mathrm{d}v = xe^{x^2} \end{cases}$, so we have $\begin{cases} \mathrm{d}u = 2x \\ v = \int xe^{x^2}\,\mathrm{d}x \end{cases}$.

We now want to compute v first: Let $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$. So we have $\int xe^{2}\,\mathrm{d}x = \frac{1}{2}\int e^u\,\mathrm{d}u = \frac{1}{2}e^{x^2}$. i.e., $v = \frac{1}{2}e^{x^2}$.

Now, using integration by parts, we can compute the original integral.

$$\begin{aligned}
\therefore \int x^3e^{x^2}\,\mathrm{d}x &= \frac{1}{2}x^2e^{x^2} - \int 2x\cdot\frac{1}{2}e^{e^2}\,\mathrm{d}x \\
&= \frac{1}{2}x^2e^{x^2} - \int xe^{e^2}\,\mathrm{d}x \\
&= \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C \\
&= \frac{1}{2}e^{x^2}(x^2-1) + C
\end{aligned}$$

Method 2 Substitution

Let $w = x^2$, $\mathrm{d}w = 2x\,\mathrm{d}x$.

$$\begin{aligned}
\therefore \int x^3e^{x^2}\,\mathrm{d}x &= \int x^2\cdot e^{x^2}\cdot x\,\mathrm{d}x \\
&= \int \frac{1}{2}w\cdot e^w\,\mathrm{d}w \\
&= \frac{1}{2}\int w\cdot e^w\,\mathrm{d}w
\end{aligned}$$

Now, we can apply integration by parts: Let $\begin{cases} u = w \\ \mathrm{d}v = e^w \end{cases}$, we have $\begin{cases} \mathrm{d}u = 1 \\ v = e^w \end{cases}$:

$$\begin{aligned} \therefore \frac{1}{2}\int we^w \,\mathrm{d}w &= \frac{1}{2}\left(we^w - \int e^w \,\mathrm{d}w\right) \\ &= \frac{1}{2}we^w - \frac{1}{2}e^w + C \\ &= \frac{1}{2}e^w(w-1) + C \\ &= \frac{1}{2}e^{x^2}(x^2-1) + C \end{aligned}$$

Example 4

$$\int \ln x \,\mathrm{d}x$$

Let $\begin{cases} u = \ln x \\ \mathrm{d}v = 1 \end{cases}$, so we have $\begin{cases} \mathrm{d}u = \frac{1}{x} \\ v = x \end{cases}$:

$$\begin{aligned} \therefore \int \ln x \,\mathrm{d}x &= x\ln x - \int \frac{1}{x}\cdot x \,\mathrm{d}x \\ &= x\ln x - x + C \\ &= x(\ln x - 1) + C \end{aligned}$$

Example 5

$$\int \cos(\ln x) \,\mathrm{d}x$$

Let $\begin{cases} u = \cos(\ln x) \\ \mathrm{d}v = 1 \end{cases}$, so we have $\begin{cases} \mathrm{d}u = -\sin(\ln x)\cdot\frac{1}{x} \\ v = x \end{cases}$.

$$\therefore \int \cos(\ln x) \,\mathrm{d}x = x\cos(\ln x) + \int \sin(\ln x) \,\mathrm{d}x$$

Let's use integration by parts again: Let $\begin{cases} u = \sin(\ln x) \\ \mathrm{d}v = 1 \end{cases}$, now we have

$$\begin{cases} \mathrm{d}u = \cos(\ln x) \cdot \dfrac{1}{x} \\ v = x \end{cases} :$$

$$\begin{aligned}
\therefore \int \sin(\ln x)\ \mathrm{d}x &= x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x \\
\therefore \int \cos(\ln x)\ \mathrm{d}x &= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x \\
2\int \cos(\ln x)\ \mathrm{d}x &= x\left[\cos(\ln x) + \sin(\ln x)\right] + C \\
\therefore \int \cos(\ln x)\ \mathrm{d}x &= \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + C
\end{aligned}$$

Example 6

$$\int_1^4 e^{\sqrt{x}}\ \mathrm{d}x$$

Evaluate $\int e^{\sqrt{x}}\ \mathrm{d}x$ first.

Recall in Example 3, Method 2, we applied substitution first: Let $w = x^{1/2}$.

So we have $\mathrm{d}w = \frac{1}{2}x^{-1/2}\ \mathrm{d}x = \frac{1}{2x^{1/2}}\ \mathrm{d}x = \frac{1}{2w}\ \mathrm{d}x$.

$$\therefore \mathrm{d}x = 2w\ \mathrm{d}w$$

$$\begin{aligned}
\therefore \int e^{\sqrt{x}}\ \mathrm{d}x &= \int e^w 2w\ \mathrm{d}w \\
&= 2\int e^w w\ \mathrm{d}w
\end{aligned}$$

Now, we can apply integration by parts: Let $\begin{cases} u = w \\ \mathrm{d}v = e^w \end{cases}$, so we have $\begin{cases} \mathrm{d}u = 1 \\ v = e^w \end{cases}$:

$$\begin{aligned}
\therefore \int e^{\sqrt{x}}\ \mathrm{d}x &= we^w - \int e^w\ \mathrm{d}w \\
&= we^w - ew + C \\
&= (w-1)e^w + C \\
&= (\sqrt{x}-1)e^{\sqrt{x}} + C
\end{aligned}$$

$$\begin{aligned}
\therefore \int_1^4 e^{\sqrt{x}}\ \mathrm{d}x &= \left[(\sqrt{x}-1)e^{\sqrt{x}} + C\right]_1^4 \\
&= (2-1)e^2 - (1-1)e \\
&= e^2
\end{aligned}$$

Example 7

Use integration by parts to derive the following formula:

$$\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$$

$$\begin{aligned} \text{LHS} &= \int \cos^n x \, dx \\ &= \boxed{\int \cos^{n-1} x \cdot \cos x \, dx} \end{aligned}$$

Now we can use integration by parts: Let $\begin{cases} u = \cos^{n-1} x \\ dv = \cos x \end{cases}$, so we have $\begin{cases} du = -(n-1)\cos^{n-2} x \sin x \\ v = \sin x \end{cases}$

$$\therefore \int \cos^{n-1} x \cdot \cos x \, dx = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x \, dx$$

Use trigonometric identity: $\sin^2 x = 1 - \cos^2 x$

$$\begin{aligned} \therefore \boxed{\int \cos^n x \, dx} &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - \boxed{(n-1)\int \cos^n x \, dx} \end{aligned}$$

$$\begin{aligned} \therefore \int \cos^n x \, dx + (n-1)\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx \\ (1+n-1)\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx \\ \int \cos^n x \, dx &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx \\ &= \text{RHS} \end{aligned}$$

$$\therefore \text{LHS} = \text{RHS}.$$

Remark from Example 7

1. Split $\cos^n x$ to $\cos^{n-1} x \cos x$
2. Use trigonometric identities

At this time, we can introduce the hyperbolic trigonometric functions: sinh and cosh.

Hyperbolic Trigonometric Functions

Hyperbolic trigonometric functions are defined in the following ways:

$$\sinh x = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

We can compute derivatives and integrals of hyperbolic trigonometric functions by definition of them.

Basic Calculus of Hyperbolic Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right) = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right) = \frac{1}{2}e^x - \frac{1}{2}e^{-x} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\int \sinh x \,\mathrm{d}x = \cosh x + C$$

$$\int \cosh x \,\mathrm{d}x = \sinh x + C$$

Remark

There is no sign change as the ordinary trigonometric functions.

Example 8

$$\int x \sinh x \,\mathrm{d}x$$

Use integration by parts: Let $\begin{cases} u = x \\ \mathrm{d}v = \sinh x \end{cases}$, so we have $\begin{cases} \mathrm{d}u = 1 \\ v = \cosh x \end{cases}$

$$\begin{aligned} \therefore \int x \sinh x \,\mathrm{d}x &= x\cosh x - \int \cosh x \,\mathrm{d}x \\ &= x\cosh x - \sinh x + C \end{aligned}$$

Example 9

$$\int \frac{xe^{2x}}{(1+2x)^2}\,\mathrm{d}x$$

Firstly, we use substitution: Let $t = 1+2x \Rightarrow 2x = t-1$. So we have $x = \frac{1}{2}(t-1) \Rightarrow \mathrm{d}x = \frac{1}{2}\,\mathrm{d}t$.

$$\begin{aligned}
\therefore \int \frac{xe^{2x}}{(1+2x)^2}\,\mathrm{d}x &= \int \frac{\frac{1}{2}(t-1)e^{t-1}}{t^2} \cdot \frac{1}{2}\,\mathrm{d}t \\
&= \frac{1}{4}\int \frac{(t-1)e^{t-1}}{t^2}\,\mathrm{d}t \\
&= \frac{1}{4}\left(\int \frac{1}{t}e^{t-1}\,\mathrm{d}t - \int \frac{1}{t^2}e^{t-1}\,\mathrm{d}t\right)
\end{aligned}$$

Now, we can use integration by parts to compute $\int \frac{1}{t}e^{t-1}$: Let $\begin{cases} u = \dfrac{1}{t} \\ \mathrm{d}v = e^{t-1} \end{cases}$, so we have $\begin{cases} \mathrm{d}u = \dfrac{1}{t^2} \\ v = e^{t-1} \end{cases}$

$$\therefore \int \frac{1}{t}e^{t-1}\,\mathrm{d}t = \frac{1}{t}e^{t-1} + \int \frac{1}{t^2}e^{t-1}\,\mathrm{d}t$$

$$\begin{aligned}
\therefore \int \frac{xe^{2x}}{(1+2x)^2}\,\mathrm{d}x &= \frac{1}{4}\left(\frac{1}{t}e^{t-1} + \int \frac{1}{t^2}e^{t-1}\,\mathrm{d}t - \int \frac{1}{t^2}e^{t-1}\,\mathrm{d}t\right) \\
&= \frac{1}{4t}e^{t-1} + C
\end{aligned}$$

Substitute $t = 2x+1$ back:

$$\begin{aligned}
\int \frac{xe^{2x}}{(1+2x)^2}\,\mathrm{d}x &= \frac{1}{4(2x+1)}e^{2x+1-1} + C \\
&= \frac{1}{4(2x+1)}e^{2x} + C
\end{aligned}$$

Example 10

Use integration by parts to establish the "reduction formula":

$$\int x^n e^x\,\mathrm{d}x = x^n e^x - n\int x^{n-1}e^x\,\mathrm{d}x$$

Let $\begin{cases} u = x^n \\ \mathrm{d}v = e^x \end{cases}$, so we have $\begin{cases} \mathrm{d}u = nx^{n-1} \\ v = e^x \end{cases}$

$$\begin{aligned}
\therefore \mathrm{LHS} = \int x^n e^x\,\mathrm{d}x &= x^n e^n - \int nx^{n-1}e^x\,\mathrm{d}x \\
&= x^n e^x - n\int x^{n-1}e^x\,\mathrm{d}x \\
&= \mathrm{RHS}
\end{aligned}$$

2.2 Trigonometric Integrals

In this section, we will use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 1

$$\int \sin^3 x \cos x \, dx$$

We use simple substitution to do this integration: Let $u = \sin x$, so we have $du = \cos x \, dx$:

$$\begin{aligned} \therefore \int \sin^3 x \cos x \, dx &= \int u^3 \, du \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}\sin^4 x + C \end{aligned}$$

Example 2

$$\int \sin^3 x \cos^2 x \, dx$$

Let $u = \cos x$, so we have $du = -\sin x \, dx \ \Rightarrow \ -du = \sin x \, dx$

$$\begin{aligned} \therefore \int \sin^2 x \cos^2 x \cdot \sin x \, dx &= \int \left(1 - \cos^2 x\right) \cos^2 x (-du) \\ &= \int \cos^2 x - \cos^4 x (-du) \\ &= \int u^4 - u^2 \, du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C \end{aligned}$$

Example 3

$$\int \sin^2 x \cos^2 x \, dx$$

Here, it is not clear that the identity $\sin^2 x + \cos^2 x = 1$ will help.
Let's try half angle formula:

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta); \ \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\begin{aligned}\therefore \int \sin^2 x \cos^2 x \, dx &= \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x) \, dx \\ &= \frac{1}{4}\int 1-\cos^2 2x \, dx\end{aligned}$$

Use the half angle formula again:

$$\begin{aligned}\therefore \int \sin^2 x \cos^2 x \, dx &= \frac{1}{4}\int 1-\cos^2 x \, dx \\ &= \frac{1}{4}\int 1-\frac{1}{2}(1+\cos 4x) \, dx \\ &= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos 4x \, dx \\ &= \frac{1}{8}x+\frac{1}{32}\sin 4x + C\end{aligned}$$

In general, to evaluate $\int \sin^m x \cos^n x \, dx$

- If either m or n is odd:
  - Separate out part of the integral as "du"
  - Use trigonometric identities to rewrite remaining parts in terms of an appropriate u substitution.
- If both m and n are even, then use half angle formulas.

Example 4

$$\int \sin^3 5x \cos^3 5x \, dx$$

Both $m=3,\ n=3$ are odd, so we can use either for du: Let's use du in the form of $\cos 5x \, dx$

$$\begin{aligned}\therefore \int \sin^3 5x \cos^3 5x \, dx &= \int \sin^3 5x \cos^2 5x \cos 5x \, dx \\ &= \int \sin^3 5x(1-\sin^2 5x)\cos 5x \, dx\end{aligned}$$

Let $u=\sin 5x \ \Rightarrow \ du = 5\cos 5x \, dx$

$$\begin{aligned}\therefore \int \sin^3 5x \cos^3 5x \, dx &= \int u^3\left(1-u^2\right)\frac{1}{5} \, du \\ &= \frac{1}{5}\int u^3 - u^5 \, du \\ &= \frac{1}{5}\left(\frac{1}{4}u^4-\frac{1}{6}u^6\right)+C \quad = \frac{1}{20}\sin^4 x - \frac{1}{30}\sin^6 x + C\end{aligned}$$

Example 5

$$\int \sec^4 x \tan^4 x \, \mathrm{d}x$$

Recall:

1. $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$$

2. If $u = \tan x, \ \mathrm{d}u = \sec^2 x \, \mathrm{d}x$

$$\begin{aligned}
\int \sec^4 x \tan^4 x \, \mathrm{d}x &= \int \sec^2 x \tan^4 x \sec^2 x \, \mathrm{d}x \\
&= \int \left(\tan^2 x + 1\right) \tan^4 x \sec^2 x \, \mathrm{d}x \\
&= \int \left(\tan^6 x + \tan^4 x\right) \sec^2 x \, \mathrm{d}x
\end{aligned}$$

Let $u = \tan x, \ \mathrm{d}u = \sec^2 x \, \mathrm{d}x$:

$$\begin{aligned}
\therefore \int \sec^4 x \tan^4 x \, \mathrm{d}x &= \int \left(u^6 + u^4\right) \, \mathrm{d}u \\
&= \frac{1}{7}u^7 + \frac{1}{5}u^5 + C \\
&= \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C
\end{aligned}$$

In general, to evaluate $\int \tan^m x \sec^n x \, \mathrm{d}x$

- If either m is odd or n is even:
  - Separate out part of the integrand as "$\mathrm{d}u$".
  - Use trigonometric identities to rewrite remaining parts in terms of an appropriate u substitution.
- If m is even and n is odd, then will likely need integration by parts.

Example 6

$$\int \sec x \tan^3 x \, \mathrm{d}x$$

Let $u = \sec x,\ \mathrm{d}u = \sec x \tan x\ \mathrm{d}x$

$$\begin{aligned}
\therefore \int \sec x \tan^3 x\ \mathrm{d}x &= \int \tan^2 x \sec x \tan x\ \mathrm{d}x \\
&= \int \left(\sec^2 x - 1\right) \sec x \tan x\ \mathrm{d}x \\
&= \int \left(u^2 - 1\right)\ \mathrm{d}u \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}\sec^3 x - \sec x + C
\end{aligned}$$

Example 7

$$\int \tan^2 x \sec x\ \mathrm{d}x$$

$$\begin{aligned}
\int \tan^2 x \sec x\ \mathrm{d}x &= \int \left(\sec^2 x - 1\right) \sec x\ \mathrm{d}x \\
&= \int \sec^3 x\ \mathrm{d}x - \int \sec x\ \mathrm{d}x \\
&= \int \sec^3 x\ \mathrm{d}x - \ln\left|\sec x + \tan x\right| + C
\end{aligned}$$

To find $\int \sec^3 x\ \mathrm{d}x$, we use integration by parts: Let $\begin{cases} u = \sec x \\ \mathrm{d}v = \sec^2 x \end{cases}$, so we have $\begin{cases} \mathrm{d}u = \sec x \tan x \\ v = \tan x \end{cases}$.

$$\begin{aligned}
\therefore \boxed{\int \tan^2 x \sec x\ \mathrm{d}x} &= \int \sec^3 x\ \mathrm{d}x - \ln\left|\sec x + \tan x\right| + C \\
&= \sec x \tan x - \boxed{\int \sec x \tan^2 x\ \mathrm{d}x} - \ln\left|\sec x + \tan x\right| + C
\end{aligned}$$

$$\begin{aligned}
\therefore 2\int \tan^2 x \sec x\ \mathrm{d}x &= \sec x \tan x - \ln\left|\sec x + \tan x\right| + C \\
\int \tan^2 x \sec x\ \mathrm{d}x &= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C
\end{aligned}$$

To evaluate integrals involving $\sin(mx)$ and $\cos(nx)$

- $\int \sin(mx)\cos(nx)\,\mathrm{d}x$, use

$$\sin A\cos B = \frac{1}{2}\left(\sin(A-B)+\sin(A+B)\right)$$

- $\int \sin(mx)\sin(nx)\,\mathrm{d}x$, use

$$\sin A\sin B = \frac{1}{2}\left(\cos(A-B)-\cos(A+B)\right)$$

- $\int \cos(mx)\cos(nx)\,\mathrm{d}x$, use

$$\cos A\cos B = \frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)$$

Example 8

$$\int \sin 3x\sin 2x\,\mathrm{d}x$$

Here $A=3x,\ B=2x\ \Rightarrow\ A-b=x,\ A+b=5x$:

$$\begin{aligned}\therefore \sin 3x\sin 2x\,\mathrm{d}x &= \int \frac{1}{2}(\cos x-\cos 5x)\,\mathrm{d}x\\ &= \frac{1}{2}\sin x-\frac{1}{2}\cdot\frac{1}{5}\sin 5x+C\\ &= \frac{}{2}\sin x-\frac{1}{10}\sin 5x+C\end{aligned}$$

2.3 Trigonometric Substitutions

In some cases, a u substitution is obvious.

Example 1

$$\int x\sqrt{a^2-x^2}\,\mathrm{d}x,\ \text{for } a>0$$

Let $u = a^2 - x^2$, so we have $\mathrm{d}u = -2x\,\mathrm{d}x \;\Rightarrow\; -\frac{1}{2}\,\mathrm{d}u = x\,\mathrm{d}x$.

$$\begin{aligned}
\therefore \int x\sqrt{a^2-x^2}\,\mathrm{d}x &= \int \sqrt{u}\left(-\frac{1}{2}\right)\,\mathrm{d}u \\
&= -\frac{1}{2}\int u^{\frac{1}{2}}\,\mathrm{d}u \\
&= -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C \\
&= -\frac{1}{3}\left(a^2-x^2\right)^{\frac{3}{2}} + C
\end{aligned}$$

However, in other cases, an appropriate u substitution is not so obvious.

Example 2

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x, \text{ for } a > 0$$

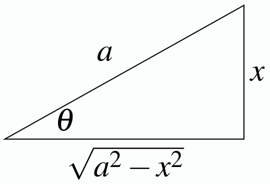
Let $x = a\sin\theta$, so we have $\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$.

$$\begin{aligned}
\therefore \int \sqrt{a^2-x^2}\,\mathrm{d}x &= \int \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,\mathrm{d}\theta \\
&= a\int \sqrt{a^2\left(1-\sin^2\theta\right)}\cdot\cos\theta\,\mathrm{d}\theta \\
&= a\int \sqrt{a^2\cos^2\theta}\cdot\cos\theta\,\mathrm{d}\theta \\
&= a^2\int \cos^2\theta\,\mathrm{d}\theta
\end{aligned}$$

Use half angle formula:

$$\begin{aligned}
\therefore a^2\int\cos^2\theta\,\mathrm{d}\theta &= a^2\cdot\frac{1}{2}\int(1+\cos 2\theta)\,\mathrm{d}\theta \\
&= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C
\end{aligned}$$

Now, we need to get x:
$x = a\sin\theta \;\Rightarrow\; \sin\theta = \frac{x}{a} \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{x}{a}\right)$ To find $\cos\theta$, we use a right angle triangle.



$$\therefore \cos\theta = \frac{\sqrt{a^2-x^2}}{a}$$

$$\begin{aligned}\therefore \int \sqrt{a^2-x^2}\,\mathrm{d}x &= \frac{a^2}{2}(\theta+\sin\theta\cos\theta)+C\\ &= \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a}+\frac{x\sqrt{a^2-x^2}}{a^2}\right)+C\end{aligned}$$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta,\ 0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$	$\sec^2\theta-1=\tan^2\theta$

Example 3

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x$$

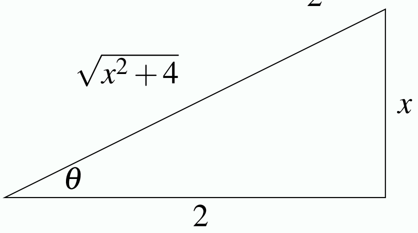
Let $x=2\tan\theta$, so we have $\mathrm{d}x=2\sec^2\theta\ \mathrm{d}\theta$:

$$\begin{aligned}\therefore \int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x &= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta\ \mathrm{d}\theta\\ &= \int \frac{1}{4\tan^2\theta\sqrt{4\left(\tan^2\theta+1\right)}}\cdot 2\sec^2\theta\ \mathrm{d}\theta\\ &= \int \frac{1}{4\tan^2\theta\sqrt{4\sec^2\theta}}\cdot 2\sec^2\theta\ \mathrm{d}\theta\\ &= \int \frac{1}{4\tan^2\theta\cdot 2\sec\theta}\cdot 2\sec^2\theta\ \mathrm{d}\theta\\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,\mathrm{d}\theta\\ &= \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,\mathrm{d}\theta\\ &= \frac{1}{4}\int \frac{\cos^2\theta}{\cos\theta\sin^2\theta}\,\mathrm{d}\theta\\ &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta\end{aligned}$$

Let $u = \sin\theta$, $\mathrm{d}u = \cos\theta \ \mathrm{d}\theta$:

$$\begin{aligned}\therefore \int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x &= \frac{1}{4}\int \frac{1}{u^2}\,\mathrm{d}u = \frac{1}{4}\int u^{-2}\,\mathrm{d}u \\ &= -\frac{1}{4}u^{-1} + C = -\frac{1}{4u} + C \\ &= -\frac{1}{4\sin\theta} + C\end{aligned}$$

We need to go back in terms of x:
$x = 2\tan\theta \ \Rightarrow \ \tan\theta = \dfrac{x}{2}$



$$\therefore \sin\theta = \frac{x}{\sqrt{x^2+4}}$$

$$\begin{aligned}\therefore \int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x &= -\frac{1}{4\sin\theta} + C \\ &= -\frac{\sqrt{x^2+4}}{4x} + C\end{aligned}$$

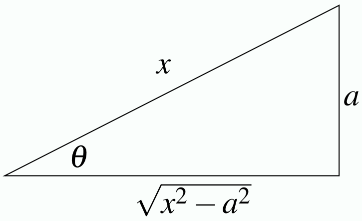
Example 4

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}, \text{ where } a > 0$$

Let $x = a\sec\theta$, then we have $\mathrm{d}x = a\sec\theta\tan\theta \ \mathrm{d}\theta$:

$$\begin{aligned}\therefore \int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta \ \mathrm{d}\theta}{\sqrt{a^2\sec^2\theta - a^2}} \\ &= \int \frac{a\sec\theta\tan\theta \ \mathrm{d}\theta}{\sqrt{a^2(\sec^2\theta - 1)}} \\ &= \int \frac{a\sec\theta\tan\theta \ \mathrm{d}\theta}{\sqrt{a^2\tan^2\theta}} \\ &= \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\,\mathrm{d}\theta \\ &= \int \sec\theta \ \mathrm{d}\theta \\ &= \ln|\tan\theta + \sec\theta| + C\end{aligned}$$

Now, we want the answer in terms of x:
$x = a\sec\theta \ \Rightarrow \ \sec\theta = \dfrac{x}{a}$, $\cos\theta = \dfrac{a}{x}$. Using a right angle triangle, we can find $tan\theta$:



$$\therefore \tan\theta = \frac{a}{\sqrt{x^2-a^2}}$$

$$\therefore \int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln\left|\frac{a}{\sqrt{x^2-a^2}+\frac{x}{a}}\right| + C$$

2.4 Partial Fractions

Recall that we know how to find common denominators:

$$\frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1-x+1}{(x-1)(x+1)} = \frac{2}{x^2-1}$$

Now, let's compare which side of the equation, the left-hand side or the right-hand side, is easier to integrate. The answer is the left handed side is easier to integrate:

$$\begin{aligned}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\,\mathrm{d}x &= \int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{1}{x+1}\,\mathrm{d}x \\ &= \ln|x-1| - \ln|x+1| + C \\ &= \ln\left|\frac{x-1}{x+1}\right| + C\end{aligned}$$

The Goal of Partial Fractions

Given something like $\frac{2}{x^2-1}$, "undo" the common denominator. That is, if possible, find constants A and B such that

$$\frac{2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Then,

$$\int \frac{2}{x^2-1}\,\mathrm{d}x = \int \frac{2}{(x-1)(x+2)}\,\mathrm{d}x = \int \frac{A}{x-1}\,\mathrm{d}x + \int \frac{B}{x+1}\,\mathrm{d}x$$

Basic Idea of Partial Fraction Decomposition

Given: a rational function, $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with degree$(P(x))$ <degree$(Q(x))$.

Goal: write $f(x)$ as

$$f(x) = \frac{P(x)}{Q(x)} = F_1 + F_2 + \cdots + F_k,$$

where each F_i looks like $\dfrac{A}{(ax+b)^n}$ or $\dfrac{Ax+B}{(ax^2+bx+c)^n}$

Approach to get Partial Fraction Decomposition

1. If degree$(P(x)) \geq$degree$(Q(x))$, use long division to get

$$\text{polynomial} + \frac{\text{new } P(x)}{\text{new } Q(x)}$$

2. Factorize $Q(x)$ as much as possible to get terms like

$$(ax+b)^n \text{ (linear terms)}$$
$$(ax^2+bx+c)^n \text{(irreducible quadratic terms)}$$

3. For each $(ax+b)^n$, the decomposition contains:

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

For each irreducible quadratic $(ax^2+bx+c)^n$, the decomposition contains:

$$\frac{Ax_1+B_1}{(ax^2+bx+c)} + \frac{Ax_2+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{Ax_n+B_n}{(ax^2+bx+c)^n}$$

4. Use algebra to find all A_i and B_i

Example 1

$$\int \frac{x^3-6x^2+5x-3}{x^2-1}\,\mathrm{d}x$$

Because degree(numerator)>degree(denominator), we need to run a long division

$$\begin{array}{r}
x-6 \\
x^2+0-1\;\overline{)x^3-6x^2+5x-31} \\
\underline{x^3+\;0x^2-\;\;x} \\
-6x^2+6x-\;\;3 \\
\underline{-6x^2-0x+\;\;6} \\
6x-9
\end{array}$$

$$\begin{aligned}
\therefore \frac{x^3-6x^2+5x-3}{x^2-1} &= \frac{(x-6)(x^2-1)+(6x-9)}{x^2-1} \\
&= (x-6)+\frac{6x-9}{x^2-1}
\end{aligned}$$

Factorize the denominator:

$$(x-6)-\frac{6x-9}{x^2-1}=(x-6)-\frac{6x-9}{(x-1)(x+1)}$$

Partial fraction decomposition:

$$\frac{6x-9}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+1}$$

Find A and B:

$$\frac{A}{x-1}+\frac{B}{x+1}=\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}=\frac{6x-9}{(x-1)(x+1)}$$

$\boxed{\text{Method 1}}$ Solving system of equation:

$$\therefore\begin{cases}(A+B)x=6x\\A-B=-9\end{cases}\Rightarrow\begin{cases}A+B=6\\A-B=-9\end{cases}\Rightarrow\begin{cases}A=-3/2\\B=15/2\end{cases}$$

$\boxed{\text{Method 2}}$ Plug-in roots (Does not always apply)

$$A(x+1)+B(x-1)=6x-9$$

Let $x=1$:

$$A(1+1)+B(1-1)=6(1)-9\ \Rightarrow\ 2A=6-9=-3$$

$$\therefore A=-\frac{3}{2}$$

Let $x=-1$:

$$A(-1+1)+B(-1-1)=6(-1)-9\ \Rightarrow\ -2B=-6-9=-15$$

$$\therefore B=\frac{15}{2}$$

$$\therefore\frac{6x-9}{(x-1)(x+2)}=-\frac{3}{2}\cdot\frac{1}{x-1}+\frac{15}{2}\cdot\frac{1}{x+1}$$

Find the integral:

$$\begin{aligned}\therefore\int\frac{x^3-6x^2+5x-3}{x^2-1}\,\mathrm{d}x&=\int(x-6)+\left(-\frac{3}{2}\cdot\frac{1}{x-1}+\frac{15}{2}\cdot\frac{1}{x+1}\right)\,\mathrm{d}x\\&=\int(x-6)\,\mathrm{d}x-\frac{3}{2}\int\frac{1}{x-1}\,\mathrm{d}x+\frac{15}{2}\int\frac{1}{x+1}\,\mathrm{d}x\\&=\frac{1}{2}x^2-6x-\frac{3}{2}\ln|x-1|+\frac{15}{2}\ln|x+1|+C\end{aligned}$$

Example 2

$$\int \frac{x+2}{(x^2+2x+1)(x-1)}\,\mathrm{d}x$$

Because degree(numerator)<degree(denominator): Nothing to do here.

Factorization:

$$\frac{x+2}{(x^2+2x+1)(x-1)} = \frac{x+2}{(x+1)^2(x-1)}$$

Decomposition:

$$\frac{x+2}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

Find A, B, C: Start by Substituting roots:

$$x+2 = A(x+1)(x-2) + B(x-1) + C(x+1)^2$$

Let $x=1$:

$$1+2 = A(1+1)(1-1) + B(1-1) + C(1+1)^2 \;\Rightarrow\; 3 = 0\cdot A + 0\cdot B + 4\cdot C$$

$$\therefore C = \frac{3}{4}$$

Let $x=-1$:

$$-1+2 = A(-1+1)(-1-1) + B(-1-1) + C(-1+1)^2 \;\Rightarrow\; 1 = 0\cdot A - 2B + 0\cdot C$$

$$\therefore B = -\frac{1}{2}$$

To find A, we need another equation that does not cancel the A term. Let's try some numbers other than the roots:

Let $x=0$:

$$0+2 = A(0+1)(0-1) + B(0-1) + C(0+1)^2 \;\Rightarrow\; 2 = -A - B + C$$

$$\therefore A = -\left(-\frac{1}{2}\right) + \frac{3}{4} - 2 = \frac{2+3-8}{4} = -\frac{3}{4}$$

$$\therefore \int \frac{x+2}{(x^2+2x+1)(x-1)} = -\frac{3}{4}\cdot\frac{1}{x+1} + \left(-\frac{1}{2}\right)\cdot\frac{1}{(x+1)^2} + \frac{3}{4}\cdot\frac{1}{x-1}$$

Integration:

$$\begin{aligned}\therefore \int \frac{x+2}{(x^2+2x+1)(x-1)}\,\mathrm{d}x &= \int -\frac{3}{4}\cdot\frac{1}{x+1}+\left(-\frac{1}{2}\right)\cdot\frac{1}{(x+1)^2}+\frac{3}{4}\cdot\frac{1}{x-1}\,\mathrm{d}x \\ &= -\frac{3}{4}\int\frac{1}{x+1}\,\mathrm{d}x-\frac{1}{2}\int\frac{1}{(x+1)^2}\,\mathrm{d}x+\frac{3}{4}\int\frac{1}{x-1}\,\mathrm{d}x \\ &= -\frac{3}{4}\ln|x+1|-\frac{1}{2}\int\frac{1}{(x+1)^2}\,\mathrm{d}x+\frac{3}{4}\ln|x-1|\end{aligned}$$

To find $\int \frac{1}{(x+1)^2}$, use u-substitution: Let $u = x+1$, so we have $\mathrm{d}u = \mathrm{d}x$

$$\begin{aligned}\therefore \int \frac{1}{(x+1)^2}\,\mathrm{d}x &= \int = frac1u^2\,\mathrm{d}u \\ &= -u^{-1} \\ &= -\frac{1}{x+1}\end{aligned}$$

$$\begin{aligned}\therefore \int \frac{x+2}{(x^2+2x+1)(x-1)}\,\mathrm{d}x &= -\frac{3}{4}\ln|x+1|-\frac{1}{2}\cdot\left(-\frac{1}{x+1}\right)+\frac{3}{4}\ln|x-1|+C \\ &= -\frac{3}{4}\ln|x+1+\frac{1}{2(x+1)}+\frac{3}{4}\ln|x-1|+C\end{aligned}$$

Example 3

$$\int \frac{4x^3-3x^2+6x-27}{x^4+9x^2}\,\mathrm{d}x$$

Because degree(numerator)¡degree(denominator), nothing to do here.

Factorization:

$$x^4+9x^2 = x^2(x^2+9)$$

Terms in this Example

x^2 is a linear term, whereas x^2+9 is an irreducible quadratic term.

Decomposition:

$$\frac{4x^3-3x^2+6x-27}{x^4+9x^2} = \frac{4x^3-3x^2+6x-27}{x^2(x^2+9)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+9}$$

Find A, B, C, D:

$$\begin{aligned}\therefore 4x^3-3x^2+6x-27 &= Ax(x^2+9)+B(x^2+9)+(Cx+D)x^2 \\ &= Ax^3+9Ax+Bx^2+9B+Cx^3+Dx^2 \\ &= (A+C)x^3+(B+D)x^2+9Ax+9B\end{aligned}$$

Matching the coefficient:

$$\begin{cases} A+C=4 \\ B+D=-3 \\ 9A=6 \\ 9B=-27 \end{cases} \Rightarrow \begin{cases} A=2/3 \\ B=-3 \\ C=4-A=10/3 \\ D=-3-B=0 \end{cases}$$

$$\therefore \frac{4x^3-3x^2+6x-27}{x^2(x^2+9)} = \frac{2}{3}\cdot\frac{1}{x} - \frac{3}{x^2} + \frac{10}{3}\cdot\frac{x}{x^2+9}$$

Integration:

$$\begin{aligned} \therefore \int \frac{4x^3-3x^2+6x-27}{x^4+9x^2}\,\mathrm{d}x &= \int \frac{2}{3}\cdot\frac{1}{x} - \frac{3}{x^2} + \frac{10}{3}\cdot\frac{x}{x^2+9}\,\mathrm{d}x \\ &= \frac{2}{3}\int\frac{1}{x}\,\mathrm{d}x - 3\int\frac{1}{x^2}\,\mathrm{d}x + \frac{10}{3}\int\frac{x}{x^2+9}\,\mathrm{d}x \\ &= \frac{2}{3}\ln|x| - 3(-1)\frac{1}{x} + \frac{10}{3}\int\frac{x}{x^2+9}\,\mathrm{d}x \end{aligned}$$

Use u-substitution to find $\int \frac{x}{x^2+9}\,\mathrm{d}x$: Let $u=x^2+9$, we have $\frac{\mathrm{d}u}{\mathrm{d}x}=2x \Rightarrow \mathrm{d}u=2x\,\mathrm{d}x$

$$\therefore \int\frac{x}{x^2+9}\,\mathrm{d}x = \frac{1}{2}\int\frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+9|$$

$$\begin{aligned} \therefore \int \frac{4x^3-3x^2+6x-27}{x^4+9x^2}\,\mathrm{d}x &= \frac{2}{3}\ln|x| - 3(-1)\frac{1}{x} + \frac{10}{3}\cdot\frac{1}{2}\ln|x^2+9| + C \\ &= \frac{2}{3}\ln|x| + \frac{3}{x} + \frac{5}{3}\ln|x^2+9| + C \end{aligned}$$

2.5 Improper Integrals

Before starting the discussion on improper integrals, we need to first define proper integrals.

Proper Integrals

An integral $\int_a^b f(x)\,\mathrm{d}x$ is considered to be proper *iff*:

1. $[a,b]$ is a finite interval, and
2. $f(x)$ is continuous on $[a,b]$.

If either of the two properties is not satisfied, the integral is called ***improper***.

By examining the definition of proper integrals, we know there are two types of improper integrals. **Type I** improper integrals fail to satisfy the first condition, and is called **infinite intervals**; **Type II** improper integrals do not satisfy the second condition and is called **discontinuous integrals**.

2.5.1 Type I Improper Integrals: Infinite Intervals

General Procedure to Solve Type I improper Integrals

1. To solve $\int_a^\infty f(x)\,\mathrm{d}x$, we go through the following procedure:
   If $\int_a^t f(x)\,\mathrm{d}x$ exists $\forall t \geq a$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x.$$

2. To solve $\int_{-\infty}^b f(x)\,\mathrm{d}x$, we go through the following procedure:
   If $\int_t^b f(x)\,\mathrm{d}x$ exists $\forall t \leq b$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x.$$

3. Provided the limits exist as finite numbers, we define the integral **converges**.
4. If both $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ are convergent, then we define:

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x$$

Example 1

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x$$

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,\mathrm{d}x &= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,\mathrm{d}x \\ &= \lim_{t\to\infty}\Big[\ln|x|\Big]_1^t \\ &= \lim_{t\to\infty}(\ln(t)-\ln(1)) \\ &= \lim_{t\to\infty}\ln(t) = \infty\end{aligned}$$

$\therefore$ Since the limit D.N.E. as a finite number, $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ is divergent.

Example 2

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$$

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,\mathrm{d}x\\ &= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t\\ &= \lim_{t\to\infty}\left(-\frac{1}{t}-(-1)\right)\\ &= \lim_{t\to\infty}\left(-\frac{1}{t}+1\right)\\ &= 0+1=1\end{aligned}$$

$\therefore$ Since the limit exists as a finite number, $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ is convergent.

Generalization

For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ convergent?

We have already done $p=1$, in which we know the integral is divergent.
So let's assume $p\neq 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t x^{-p}\,\mathrm{d}x &= \lim_{t\to\infty}\left[\frac{1}{-p+1}\cdot x^{(-p+1)}\right]_1^t\\ &= \lim_{t\to\infty}\left(\frac{1}{1-p}\cdot t^{(1-p)}-\frac{1}{1-p}\right)\\ &= \lim_{t\to\infty}\frac{1}{1-p}\left(t^{(1-p)}-1\right)\end{aligned}$$

- If $p>1$, then $1-p<0$: we have $t^{(1-p)}\to 0$ as $t\to\infty$.

  $$\therefore \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\frac{1}{1-p}\left(t^{(1-p)}-1\right) = \frac{-1}{1-p} = \frac{1}{p-1}.$$

  As the limit exists as a finite number, the integral is convergent.

- If $p<1$, then $1-p>0$: we have $t^{(1-p)}\to\infty$ as $t\to\infty$.

  $$\therefore \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\frac{1}{1-p}\left(t^{(1-p)}-1\right) = \infty.$$

  Since the limit D.N.E. as a finite number, the integral is divergent.

Summary

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x \quad \text{is} \begin{cases} \text{convergent if} & p > 1 \\ \text{divergent if} & p \leq 1 \end{cases}$$

Example 3

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{0} \frac{1}{1+x^2}\,\mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$$

Note: We can do both of these separately, but in the case, the graph of $y = \dfrac{1}{1+x^2}$ is symmetric about the y-axis.

$$\therefore \int_{-\infty}^{0} \frac{1}{1+x^2}\,\mathrm{d}x = \int_{0}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$$

To evaluate $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$, recall the following properties:

1. $$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + C$$

2. The graph of $t = \tan\theta$ and the graph of $t = \tan^{-1}\theta$.

3. $$\tan^{-1}(0) = 0; \theta \to \infty, \tan^{-1}\theta = \frac{\pi}{2}$$

$$\begin{aligned} \therefore \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,\mathrm{d}x &= \lim t \to \infty \left[\tan^{-1} x\right]_0^t \\ &= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0\right) \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

Because the limit exists as a finite number, the integral is convergent.

$$\begin{aligned} \therefore \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x &= 2\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x \\ &= 2 \cdot \frac{\pi}{2} \\ &= \pi \end{aligned}$$

2.5.2 Type II improper Integrals: Discontinuous Integrals

General Procedure to Solve Type II improper Integrals

1. If f is continuous on $[a,b)$ and discontinuous at b:

$$\int_a^b f(x)\,\mathrm{d}x = \boxed{\lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x}$$

   Note: we only cares about the left side limit in the boxed formula.

2. If f is continuous on $(a,b]$ and discontinuous at a:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x$$

3. Provide the limit exists as finite numbers, the integral converges.

4. If f is continuous on $[a,b]$ but has a discontinuity at c, where $a<c<b$, and if both $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ converges, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

Example 4

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x$$

It can be easily determined that this is a type II improper derivative because

$$\begin{cases} x-2\geq 0 \\ x-2\neq 0 \end{cases} \quad \Rightarrow\ x>2.$$

Hence, $f(x)=\dfrac{1}{\sqrt{x-2}}$ is not continuous at $x=2$.

$$\begin{aligned} \therefore \int_2^5 \frac{1}{x-2}\,\mathrm{d}x &= \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x \\ &= \lim_{t\to 2^+}\left[2\sqrt{x-2}\right]_t^5 \\ &= \lim_{t\to 2^+}\left(2\sqrt{3}-2\sqrt{t-2}\right) \\ &= 2\sqrt{3}-0 = 2\sqrt{3} \end{aligned}$$

Because the limit exists as a finite number, the integral is convergent.

Example 5

$$\int_0^3 \frac{1}{x-1}\,\mathrm{d}x$$

$f(x) = \dfrac{1}{x-1}$ is discontinuous at $x = 1$.

$$\therefore \int_0^3 \frac{1}{x-1}\,\mathrm{d}x = \int_0^1 \frac{1}{x-1}\,\mathrm{d}x + \int_1^3 \frac{1}{x-1}\,\mathrm{d}x$$

Evaluate $\displaystyle\int_0^1 \frac{1}{x-1}\,\mathrm{d}x$:

$$\begin{aligned}\int_0^1 \frac{1}{x-1}\,\mathrm{d}x &= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,\mathrm{d}x \\ &= \lim_{t\to 1^-} \Big[\ln|x-1|\Big]_0^t \\ &= \lim_{t\to 1^-} (\ln|t-1| + \ln 1) \\ &= \ln|0| + 0 = -\infty\end{aligned}$$

Because the limit D.N.E. as a finite number, the integral $\displaystyle\int_0^1 \frac{1}{x-1}\,\mathrm{d}x$ diverges.

Hence, $\displaystyle\int_0^3 \frac{1}{x-1}\,\mathrm{d}x$ is also divergent.

Comparison Test for Improper Integrals

Purpose: Test to see if integral converges, without actually computing the result.

Procedure:

Suppose f and g are continuous functions with $f(x) \geq g(x)\ \ \forall x \geq a$. Then

1. If $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is also convergent.
2. If $\displaystyle\int_a^b g(x)\,\mathrm{d}x$ is divergent, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ is also divergent.

Example 6

$$\int_1^\infty \frac{1+e^{-x}}{x}\,\mathrm{d}x$$

$$\because e^{-x} > 0$$

$$\therefore 1 + e^{-x} > 1$$

$$\therefore \frac{1+e^{-x}}{x} > \frac{1}{x}$$

As $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ is divergent, $\int_1^\infty \frac{1+e^{-x}}{x}\,\mathrm{d}x$ is also divergent.

3 Differential Equations

3.1 Introduction to Ordinary Differential Equations

Physical quantities often change with time (e.g. population changes with time), or position. So we model how quantities change with derivatives (Rates of change).

Ordinary Differential Equations (ODE)

An equation relating an unknown function of one variable to one or more of its derivatives.

Solutions of an ODE

A function that satisfies the ODE.

Order of an ODE

The highest order derivative in the ODE.

Example

Consider the ODE: $x^2y'' - 3xy' + 3y = 4x^7$.

Here, the variable is not explicitly stated, so we assume $y = y(x)$, $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$, $y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

1. Order of the ODE: order = 2 because y'' is the highest order derivative.

2. Verify that $y = \frac{1}{6}x^7 + x + x^3$ is an solution of the ODE.

   Substitute $y = \frac{1}{6}x^7 + x + x^3$:

$$\begin{aligned}\therefore y' &= \frac{7}{6}x^6 + 1 + 3x^2 \\ y'' &= 7x^5 + 6\end{aligned}$$

$$\begin{aligned}\therefore \text{LHS} = x^2y'' - 3xy' + 3y &= x^2(7x^5+6) - 3x(\frac{7}{6}x^6 + 1 + 3x^2) + 3(\frac{1}{6}x^7 + x + x^3) \\ &= 7x^7 + 6x^2 - \frac{7}{2}x^6 - 3x - 9x^3 + \frac{1}{2}x^7 + 3x \\ &= \left(7 - \frac{7}{2} + \frac{1}{2}\right)x^7 + (6-9+3)x^3 + (3-3)x \\ &= 4x^7 = \text{RHS}\end{aligned}$$

In general, it can be difficult to find a solution with a given ODE. There's not one general method for all ODEs.

The Learning Curve

One model: The more you know, the slower you learn more about the task.
If $y(t)$ =percentage of a task learned over time, then $\frac{\mathrm{d}y}{\mathrm{d}t}$ decreases as y increases.
Suppose we are told: rate a person learns = percentage of task not yet learned. Then we can derive this differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 100 - y(t).$$

Observations:

1. $$\frac{\mathrm{d}y}{\mathrm{d}x} > 0 \Rightarrow 100 - y(t) > 0 \Rightarrow y(t) < 100$$

   That is, as long as a person has not learned everything, the rate at which they are learning is > 0.

2. $$\frac{\mathrm{d}y}{\mathrm{d}x} < 0 \Rightarrow 100 - y(t) < 0 \Rightarrow y(t) > 100$$

   This does not make sense in this application. How can a person know more than everything?

3. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 100 - y(t) = 0 \Rightarrow y(t) = 100$$

Note: $y(t) = 100$ is a solution of the ODE.

Proof

$$y(t) = 100 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

$$\therefore \text{LHS} = \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

$$\text{RHS} = 100 - y(t) = 100 - 100 = 0$$

$$\therefore \text{LHS} = \text{RHS}$$

Equilibrium Solutions

Constant solutions, like $y(t) = 100$, are called **equilibrium solutions** to the ODEs.

Show that $y(t) = 100 + Ce^{-t}$, where C is a constant, are solutions of the ODE.

$$y(t) = 100 + Ce^{-t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 0 - Ce^{-t}$$

$$\therefore \text{LHS} = \frac{\mathrm{d}y}{\mathrm{d}t} = -Ce^{-t}, \ \text{RHS} = 100 - y = 100 - \left(100 + Ce^{-t}\right) = -Ce^{-t}$$

$\therefore$ LHS = RHS, i.e., $y = 100 + Ce^{-t}$ are solutions of the ODE.

Remark

- C is a constant that depends on an initial condition.
- An initial value problem (IVP) is an ODE with an initial condition.

1. Solve the IVP: $\frac{\mathrm{d}y}{\mathrm{d}t} = 100 - y,\ y(0) = 0$
   We know the general solution of the ODE is $y(t) = 100 + Ce^{-t}$:
   $$y(0) = 0 \Rightarrow 100 + Ce^0 = 0 \Rightarrow C = -100$$
   $$\therefore y(t) = 100 - 100e^{-t}.$$
2. Solve the IVP: $\frac{\mathrm{d}y}{\mathrm{d}t} = 100 - y,\ y(0) = 10$
   We know the general solution of the ODE is $y(t) = 100 + Ce^{-t}$:
   $$y(0) = 10 \Rightarrow 100 + Ce^0 = 10 \Rightarrow C = -90$$
   $$\therefore y(t) = 100 - 90e^{-t}.$$

3.2 Direction Fields and Euler's Method to Solve ODEs

Recall from Calculus I that the derivative of $y(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the slope of $y(x)$.

Derivative as Slope

If $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y)$, then $F(x,y)$ is the slope of $y(x)$ at the point (x,y). As a result, $F(x,y)$ gives the direction of $y(x)$ at the point (x,y).

This theorem leads to our understanding of the direction fields. Basically, we draw a short line representing the direction of the function pointing to at a certain point on a grid.

Direction Fields (Slope Fields)

Draw small lines indicating slope for a bunch of points.

The **Euler's Method** is a crude way to approximate solutions of ODEs. The basic idea is the following:

- Think of direction field as a set of sign posts that point in certain directions.
- Pick a starting point (initial condition)
- Take a small step in direction indicated by the slope.
- Stop at a nearby point, and calculate a new slope.

- Take a small step in direction indicated by the slope.

A detailed version of the Euler's Method: Assume $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y)$ -

- Start at (x_0, y_0) (This is the initial condition)
- Slope is $F(x,y)$
- Choose a new $x_1 = x_0 + h$
- We need to find y. To do this, we use slopes:

$$\frac{y_1 - y_0}{x_1 - x_0} = F(x_0, y_0)$$

$$\begin{aligned} y_1 &= y_0 + (x_1 - x_0)F(x_0, y_0) \\ &= y_0 + (x_0 + h - x_0)F(x_0, y_0) \\ &= y_0 + h \cdot F(x_0, y_0) \end{aligned}$$

- New slope: $F(x_1, y_1)$
- Choose new $x_2 = x_1 + h$
- Find y_2, using slopes:

$$\frac{y_2 - y_1}{x_2 - x_1} = F(x_1, y_1)$$
$$y_2 = y_1 + h \cdot F(x_1, y_1)$$

The Euler's Method

- Given an IVP: $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y)$. $y(x_0) = y_0$
- Generate points:

$$\begin{aligned} x_1 &= x_0 + h & y_1 &= y_0 + hF(x_0, y_0) \\ x_2 &= x_1 + h & y_2 &= y_1 + hF(x_1, y_1) \\ x_3 &= x_2 + h & y_3 &= y_2 + hF(x_2, y_2) \\ &\vdots & &\vdots \end{aligned}$$

  where h is a small number.

- The curve through the points (x_i, y_i) is an approximate solution of $y(x)$ of the IVP.

Example

Consider the IVP: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, $y(0) = 1$
Use Euler's method to approximate $y(0.3)$ with $h = 0.1$.

n	x_n	y_n	$\mathrm{d}y/\mathrm{d}x$	y_n+1
0	0	1	0	$1+0.1\times 0=1$
1	0.1	1	-0.1	$1-0.1\times 0.1=0.99$
2	0.2	0.99	-0.202	$0.99-0.1\times 0.202=0.9698$
3	0.3	0.9698		

3.3 Separate Variables to Solve ODEs

In this section, we will consider techniques to solve ODEs with the following form:

$$\frac{\mathrm{d}y}{x}=F(x,y)=\boxed{g(x)f(y)}.$$

We can separate $F(x,y)$ as a product of:

- $g(x)$: a function only of x.
- $f(x)$: a function only of y.

To solve, move x-stuff to one side and y-stuff to the other side. Then, integrate.

$$\Rightarrow \int \frac{1}{f(y)}\,\mathrm{d}y=\int g(x)\,\mathrm{d}x,\ f(y)\neq 0.$$

Notes on the Process

- To do this, we need to assume $f(y)\neq 0$.
- If $y=a$ is a constant such that $f(a)=0$, then $y=a$ is a valid solution to the ODE.

Proof

$$y=a \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=0$$
$$\therefore \text{LHS}=\frac{\mathrm{d}y}{\mathrm{d}x}=0;\ \text{RHS}=f(a)g(x)=0\cdot g(x)=0$$
$$\therefore \text{LHS}=\text{RHS, i.e., a solution to the ODE.}$$

Separate Variables to Solve ODEs

To solve ***separable*** ODEs, $\frac{\mathrm{d}y}{\mathrm{d}x}=g(x)f(y)$:

- Solve $\int \frac{1}{f(y)}\,\mathrm{d}y=\int g(x)\,\mathrm{d}x$.
- Find constants $y=a$ where $f(a)=0$ (these are also solutions).

Example 1

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, $y(0) = 1$.

$$\begin{aligned} \int y\,\mathrm{d}y &= \int -x\,\mathrm{d}x \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + C \\ y^2 &= -x^2 + C \end{aligned}$$

Substitute $x = 0,\ y = 1$:

$$1 = -0 + C \;\Rightarrow\; C = 1$$

$$\therefore y^2 = -x^2 + 1 \;\Rightarrow\; x^2 + y^2 = 1.$$

Example 2

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y^2$

$$\begin{aligned} \int \frac{1}{y^2}\,\mathrm{d}y &= \int x^2\,\mathrm{d}x,\ y^2 \neq 0 \;\Rightarrow\; y \neq 0 \\ \int y^{-2}\,\mathrm{d}y &= \int x^2\,\mathrm{d}x \\ -y^{-1} &= \frac{1}{3}x^3 + C \\ -\frac{1}{y} &= \frac{1}{3}x^3 + C \\ y &= -\frac{1}{\frac{1}{3}x^3 + C} \end{aligned}$$

$$\therefore \text{ all solutions are: } y = -\frac{1}{\frac{1}{3}x^3 + C} \text{ and } y = 0$$

Example 3

Sometimes, we need to be creative. Solve the following ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy - 2y + x - 2.$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= y(x-2) + (x-2) \\ &= (x-2)(y+1) \end{aligned}$$

Now, we can do the separation of variable easily:

$$\int \frac{1}{y+1}\,\mathrm{d}y = \int (x-2)\,\mathrm{d}x,\ y+1 \neq 0,\ y \neq -1$$
$$\ln|y+1| = \frac{1}{2}x^2 - 2x + C$$
$$|y+1| = e^{\frac{1}{2}x^2-2x+C} = e^C \cdot e^{\frac{1}{2}x^2-2x}$$
$$y+1 = \pm\, e^c \cdot e^{\frac{1}{2}x^2-2x}$$
$$y = \pm\, A \cdot e^{\frac{1}{2}x^2-2x},$$

where A can be any positive or negative constant; A cannot be 0.

$\therefore$ All solutions are given by: $y = pm\,A \cdot e^{\frac{1}{2}x^2-2x}$ and $y=-1$.

Note: In this case, if we allow A to also be 0, we can write all solutions as

$$y = \pm\, Ae^{\frac{1}{2}x^2-2x} - 1.$$

Example 4

Newton's Law of Cooling states that the rate of change of temperature of an object is proportional to the difference between the temperature of the object and its surroundings.
Set up and solve an ODE for this application.

---

1. Set up the ODE:
   Let $T(t)$ be the temperature of the object, t is the time passed by, and T_s
2. Solve the ODE:
$$\frac{\mathrm{d}T}{\mathrm{d}t} = -K(T-T_s)$$
   Assume T_s is a constant (maybe a big assumption)

$$\int \frac{1}{T-T_s}\,\mathrm{d}T = \int -K\,\mathrm{d}t,\ T-T_s \neq 0 (T \neq T_s)$$
$$\ln|T-T_s| = -Kt + C$$
$$T - T_s = e^{-Kt+C} = e^C \cdot e^{-Kt} = Ae^{-Kt}$$

where A can be any positive or negative constant.$T \quad = Ae^{-Kt} + T_s$

$\therefore$ All solutions are: $T = Ae^{-Kt} + T_s$ and $T = T_s$

If we allow A to be 0, we can write all solutions as

$$T(t) = Ae^{-Kt} + T_s.$$

Example 5

Solve the IVP:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{t(t^2+1)}{4u^3},\ u(0) = \frac{-1}{\sqrt{2}}.$$

$$\int 4u^3\ \mathrm{d}u = \int t^3 + t\ \mathrm{d}t$$
$$4 \cdot \frac{1}{4}u^4 = \frac{1}{4}t^5 + \frac{1}{2}t^2 + C$$
$$u^4 = \frac{1}{4}t^5 + \frac{1}{2}t^2 + C$$

Substitute $t = 0,\ u = -\frac{1}{\sqrt{2}}$:

$$\left(-\frac{1}{\sqrt{2}}\right)^4 = C \Rightarrow C = \frac{1}{4}$$
$$\therefore u^4 = \frac{1}{4}t^4 + \frac{1}{2}t^2 + \frac{1}{4}$$
$$u = \sqrt[4]{\frac{1}{4}t^4 + \frac{1}{2}t^2 + \frac{1}{4}}.$$

3.4 ODE Models for Population Growth

In this section, we consider some ODEs that are used to model population growth under certain assumptions.

1. Law of Natural Growth:
   Here we use the simple assumption that growth rate is proportional to population size.
   Let $P(t)$ =Population size at any time t.
   Then
   $$\frac{\mathrm{d}P}{\mathrm{d}t} = k \cdot P.$$
   Note:
   $$\frac{1}{P} \cdot \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}P/\mathrm{d}t}{P} = \frac{\text{Rate of growth}}{\text{population size}} = \text{"Relative growth rate"}$$
   $$\therefore k = \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = \text{Relative growth rate.}$$
   The OED: $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is separable.
   $$\int \frac{\mathrm{d}P}{P} = \int k\ \mathrm{d}t$$
   $$\ln|P| = kt + C \quad \Rightarrow \quad |P| = e^{kt+C} = e^C \cdot e^{kt}$$
   $$P = \pm e^C \cdot e^{kt} = C_1 e^{kt}$$

where C_1 is any positive or negative constant.

$$\therefore P = C_1 e^{kt} \quad \text{and} \quad P = 0.$$

If we also allow C_1 to be 0, all solutions are $P(t) = C_1 e^{kt}$.
If $P(0) = P_0$, then $C_1 \cdot e^{k\cdot 0} = P_0 \quad \Rightarrow \quad C_1 = P_0$

Natural Growth ODE Model and Solution

$$\begin{aligned} \text{IVP:} \quad & \frac{\mathrm{d}P}{\mathrm{d}t} = kP,\ P(0) = P_0 \\ \text{Solution:} \quad & P(t) = P_0 e^{kt} \end{aligned}$$

Example 1

Suppose:

- Bacteria grows with constant relative growth rate,
- count of bacteria was 400 after 2 hours, and
- count of bacteria was $25,600$ after 6 hours.

How long did it take for the population to double P_0?

From the first condition, we know the ODE above is applicable in this example. So we assume $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.
From the second and the third condition, we get

$$\begin{cases} P(2) = 400 \\ P(6) = 25600 \end{cases} \Longrightarrow \begin{cases} P_0 e^{2k} = 400 & \mathrm{I} \\ P_0 e^{6k} = 25600 & \mathrm{II} \end{cases}$$

$$\begin{aligned} \frac{\mathrm{II}}{\mathrm{I}} : \frac{P_0 e^{6k}}{P_0 e^{2k}} &= \frac{25600}{400} \\ e^{4k} &= 64 \\ k &= \frac{1}{4}\ln(64) = \frac{1}{4}\ln(8^2) = \frac{1}{2}\ln(8) \end{aligned}$$

We know $P(2) = 400$:

$$\begin{aligned} \therefore P_0 e^{2k} &= 400 \\ P_0 e^{2\cdot\frac{1}{2}\ln(8)} &= 400 \\ P_0 e^{\ln(8)} &= 400 \\ 8P_0 &= 400 \\ P_0 &= 50 \end{aligned}$$

$$\begin{aligned}\therefore P(t) &= 50e^{1/2\ln(8)t} \\ &= 50e^{t/2\ln(8)} \\ &= 50e^{\ln(8^{t/2})} \\ &= 50\cdot 8^{t/2}\end{aligned}$$

So we need to find t such that:

$$\begin{aligned}50\cdot 8^{t/2} &= 100 \\ 8^{t/2} &= 2 \\ \ln(8^{t/2}) &= \ln(2) \\ \frac{t}{2}\ln(8) &= \ln(2) \\ t &= 2\cdot\frac{\ln(2)}{\ln(8)} = \frac{2\ln(2)}{\ln(2^3)} = \frac{2\ln(2)}{3\ln(2)} = \frac{2}{3}.\end{aligned}$$

2. The Logistic Model:
   A more realistic model is that population growth should level off, and approach a ”carrying capacity” because of limited resources.
   Assumptions:

   - If P is small, then $\frac{\mathrm{d}P}{\mathrm{d}t} \approx kP$.
   - As P grows, the relative growth rate decreases as P increases.
   - Relative growth rate should be negative if P gets too large.
     That is
     $$\frac{\mathrm{d}P}{\mathrm{d}t} > 0 \quad \text{for} \quad P < M$$
     $$\frac{\mathrm{d}P}{\mathrm{d}t} < 0 \quad \text{for} \quad P > M$$
     where M is the ”carrying capacity” of P.

   An ODE model that incorporates these assumptions is:

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{M}\right),\ k>0,\ M>0$$

   Notice:

   $$\begin{aligned}P\approx 0 \quad&\Rightarrow\quad \frac{P}{M}\approx 0 \quad\Rightarrow\quad \frac{\mathrm{d}P}{\mathrm{d}t}\approx kP \\ P>M \quad&\Rightarrow\quad \frac{P}{M}>1 \quad\Rightarrow\quad \frac{\mathrm{d}P}{\mathrm{d}t}<0 \\ P<M \quad&\Rightarrow\quad \frac{P}{M}<1 \quad\Rightarrow\quad \frac{\mathrm{d}P}{\mathrm{d}t}>0\end{aligned}$$

The equilibrium (constant) solutions are:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0 \implies kP\left(1 - \frac{P}{M}\right) = 0 \Rightarrow P = 0 \text{ or } P = M.$$

Note: This model does not take into account if the population is too small, then the population becomes extinct.

Finding all solutions of the OED:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$$

$$\Rightarrow \frac{\mathrm{d}P}{P\left(1 - \frac{P}{M}\right)} = k\ \mathrm{d}t,\ P \neq 0,\ P \neq M$$

Using partial fractions:

$$\frac{1}{P\left(1 - \frac{P}{M}\right)} = \frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

$$\Rightarrow M = A(M-P) + B \cdot P$$

$$P = 0 \Rightarrow M = A \cdot M \Rightarrow A = 1$$

$$P = M \Rightarrow M = B \cdot M \Rightarrow B = 1$$

$$\therefore \int \left[\frac{1}{P} + \frac{1}{M-P}\right]\ \mathrm{d}P = \int k\ \mathrm{d}t$$

$$\begin{aligned}
\ln|P| - \ln|M-P| &= kt + C_1 \\
\ln\left|\frac{P}{M-P}\right| &= kt + C_1 \\
\left|\frac{P}{M-P}\right| &= e^{kt+C_1} = e^{C_1}e^{kt} \\
\frac{P}{M-P} &= \pm e^{C_1} \cdot e^{kt} \\
&= Ce^{kt}
\end{aligned}$$

where C can be any positive or negative constant. Now, solve explicitly for P:

$$P = Ce^{kt}(M-P) = CMe^{kt} - Ce^{kt}P$$

$$\begin{aligned}
\left(1 + Ce^{kt}\right)P &= CMe^{kt} \\
P &= \frac{CMe^{kt}}{\left(1 + Ce^{kt}\right)} \\
&= \frac{CM}{e^{-kt} + C} \\
&= \frac{M}{\frac{1}{C}e^{-kt} + 1} = \frac{M}{Ae^{-kt} + 1} \quad \left(A = \frac{1}{C}\right)
\end{aligned}$$

Note: If $P(0) = P_0$, then

$$\frac{M}{A+1} = P_0 \Rightarrow \frac{M}{P_0} = A+1 \Rightarrow A = \frac{M}{P_0} - 1 = \frac{M-P_0}{P_0}$$

Therefore:

$$P(t) = \frac{M}{\frac{M-P_0}{P_0}e^{-kt}+1}$$

Logistic Growth ODE Model and Solution

Assume M is the "carrying capacity" of the population.

IVP: $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{M}\right),\ P(0) = P_0$

General Solution: $P(t) = \frac{M}{1+Ae^{-kt}}$, where A is a constant.

If$P(0) = P_0$, then: $A = \frac{M-P_0}{P_0}$

When does the population grow fastest?

That is, when is $\frac{\mathrm{d}P}{\mathrm{d}t}$ maximized? i.e., we want to find when $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right) = 0$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(k-P\left(1-\frac{P}{M}\right)\right) \\
&= kP'\left(1-\frac{P}{M}\right) + kP\left(0-\frac{P'}{M}\right) \\
&= kP'\left(1-\frac{P}{M}-\frac{P}{M}\right) \\
& kP'\left(1-\frac{2P}{M}\right) \qquad \left[\text{We know } P' = kP\left(1-\frac{P}{M}\right)\right] \\
&= k^2\boxed{P\left(1-\frac{P}{M}\right)\left(1-\frac{2P}{M}\right)}
\end{aligned}$$

The boxed part will be 0 if:

$$\begin{aligned}
&\text{minimum growth rate} \qquad \begin{cases} P = 0 \\ P = M \end{cases} \\
&\text{maximum growth rate} \qquad P = \frac{1}{2}M
\end{aligned}$$

3.5 Linear Equations

In section 3.3, we learned an approach to solve separable ODES. But what do we do if the ODE is not separable? The answer is that we need another approach. In this section, we consider an approach to solve **linear ODEs**, which have the general equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

where $P(x)$ and $Q(x)$ are functions of x.
We can derive an approach to solve this ODE:
Let

$$I(x) = e^{\int P(x)\ \mathrm{d}x}$$

This is called the **integrating factor**.
Note:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[I(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\int P(x)\ \mathrm{d}x}\right] \\
&= e^{\int P(x)\ \mathrm{d}x} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \\
&= e^{\int P(x)\ \mathrm{d}x} \cdot P(x) \\
&= I(x)P(x) \\
\therefore \frac{\mathrm{d}}{\mathrm{d}x}[I(x)] &= I(x)P(x)
\end{aligned}$$

Scale the ODE:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + P(x) &= Q(x) \\
\boxed{I(x)\left[\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y\right]} &= I(x)Q(x)
\end{aligned}$$

By product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[I(x)y] &= \frac{\mathrm{d}}{\mathrm{d}x}[I(x)]y + I(x)\frac{\mathrm{d}y}{\mathrm{d}x} \\
&= I(x)P(x)y + I(x)\frac{\mathrm{d}y}{\mathrm{d}x} \\
&= \boxed{I(x)\left[P(x)y + \frac{\mathrm{d}y}{\mathrm{d}x}\right]} \\
I(x)Q(x)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[I(x)y] &= I(x)Q(x) \\
\int \mathrm{d}[I(x)y] &= \int I(x)Q(x)\ \mathrm{d}x \\
I(x)y &= \int I(x)Q(x)\ \mathrm{d}x + C \\
y &= \frac{1}{I(x)}\left[\int I(x)Q(x)\ \mathrm{d}x + C\right]
\end{aligned}$$

Note: Plus the constant before dividing $I(x)$ to both sides of the equation.

Integrating Factor Method to Solve Linear ODEs

For linear ODEs:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x):$$

1. Computing the integrating factor: $I(x) = e^{\int P(x)\,\mathrm{d}x}$ and simplify

2. Set up the equation:

$$I(x)y(x) = \underbrace{\int I(x)Q(x)\,\mathrm{d}x}_{\text{integrate and simplify}}$$

3. Solve for y:

$$y(x) = \frac{1}{I(x)}\left(\int I(x)Q(x)\,\mathrm{d}x + C\right)$$

Example 1

Solve the ODE:

$$xy' - 4y = x^6e^x$$

The ODE is not in the standard form.

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x}y = x^5e^x \Rightarrow \therefore P(x) = -\frac{4}{x},\ Q(x) = x^5e^x$$

$$\therefore I(x) = e^{\int P(x)\,\mathrm{d}x} = e^{-4\int \frac{1}{x}\,\mathrm{d}x} = e^{-4\ln x} = x^{-4}$$

$$\begin{aligned}
\therefore \frac{\mathrm{d}}{\mathrm{d}x}[I(x)y] = I(x)Q(x) &= x^{-4}x^5e^x = xe^x \\
\int \mathrm{d}[I(x)y] &= \int xe^x\,\mathrm{d}x \\
I(x)y &= \int xe^x\,\mathrm{d}x
\end{aligned}$$

Let $\begin{cases} u = x \\ \mathrm{d}v = e^x\,\mathrm{d}x \end{cases}$, so we have $\begin{cases} \mathrm{d}u = \mathrm{d}x \\ v = e^x \end{cases}$

$$\begin{aligned}
\therefore \int xe^x\,\mathrm{d}x &= xe^x - \int e^x\,\mathrm{d}x \\
&= xe^x - e^x + C \\
\therefore I(x)y &= xe^x - e^x + C \\
x^{-4}y &= xe^x - e^x + C \\
y &= x^5e^x - x^4e^x + Cx^4
\end{aligned}$$

Example 2

Solve the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x, \quad y(0) = -3.$$

$$P(x) = 2x,\ Q(x) = x \Rightarrow I(x) = e^{\int P(x)\ \mathrm{d}x} = e^{\int 2x\ \mathrm{d}x} = e^{x^2}$$

$$I(x)y = \int I(x)Q(x)\ \mathrm{d}x = \int e^{x^2} x\ \mathrm{d}x$$

Let $u = x^2 \Rightarrow \mathrm{d}u = 2x\ \mathrm{d}x$

$$\therefore \int e^{x^2} x\ \mathrm{d}x = \frac{1}{2}\int e^u\ 2x\mathrm{d}x = \frac{1}{2}\int e^u\ \mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

$$\therefore e^{x^2} y = \frac{1}{2}e^{x^2} + C$$

$$y = \frac{1}{2} + Ce^{-x^2}$$

Substitute $y(0) = 3$:

$$-3 = \frac{1}{2} + Ce^0$$

$$C = -3 - \frac{1}{2} = -\frac{7}{2}$$

$$\therefore y = \frac{1}{2} - \frac{7}{2}e^{-x^2}$$

Example 3

Solve the ODE:

$$xy' + 3y = x\ln(3+\cos^2 x),\ x > 0$$

$$P(x) = \frac{3}{x},\ Q(x) = \ln(3+\cos^2 x) \Rightarrow I(x) = e^{\int P(x)\ \mathrm{d}x} = e^{3\int \frac{1}{x}\ \mathrm{d}x} = e^{3\ln x} = x^3$$

$$\therefore I(x)y = \int I(x)Q(x)\ \mathrm{d}x$$

$$x^3 y = \boxed{\int x^3 \ln(3+\cos^2 x)\ \mathrm{d}x}$$

This is not easy to integrate.

Sometimes, we will stop here and write the solution as:

$$y = \frac{1}{x^3}\left[\int x^3 \ln(3+\cos^2 x)\ \mathrm{d}x\right]$$

Example 4

Solve the ODE:

$$xy' + y = -xy^2$$

Try to put it into the standard form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = -y^2$$

However, this is still not in the standard form.
In this case, we will try substitution:

$$u = y^{-1} \Rightarrow y = u^{-1},\ y^2 = u^{-2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -y^2\frac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -y^2\frac{\mathrm{d}u}{\mathrm{d}x} = -u^{-2}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$-u^{-2}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{x}u^{-1} = -u^{-2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x}u = 1$$

This is in the standard form: $P(x) = -\frac{1}{x},\ Q(x) = 1$

$$\therefore I(x) = e^{\int -\frac{1}{x}\,\mathrm{d}x} = e^{-\ln x} = x^{-1}$$

$$\begin{aligned}
\therefore I(x)u &= \int I(x)Q(x)\ \mathrm{d}x \\
x^{-1}u &= \int x^{-1}\ \mathrm{d}x \\
x^{-1}u &= \ln|x| + C \\
u &= x\ln|x| + Cx \\
\therefore y = u^{-1} &= \frac{1}{x\ln|x| + Cx}
\end{aligned}$$

General Approach of Substitution in Linear ODEs

If we have an ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n,$$

where $n \neq 0,\ n \neq 1$ Then, use substitution:

$$u = y^{1-n}$$

Example 4 - *Explained*

If $n = 2$: $u = y^{1-2} = y^{-1}$

Extension from the General Approach

If $n=1$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y&=Q(x)y\\ \frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y-Q(x)y&=0\\ \frac{\mathrm{d}y}{\mathrm{d}x}+(P(x)-Q(x))\,y&=0\end{aligned}$$

This equation can be solved by separable variable or linear ODE.

$$\begin{aligned}\boxed{\text{Separable}}\quad \frac{\mathrm{d}y}{\mathrm{d}x}&=(Q(x)-P(x))\,y\\ \int\frac{1}{y}\,\mathrm{d}y&=\int Q(x)-P(x)\,\mathrm{d}x\\ \boxed{\text{Linear ODE}}\quad I(x)&=e^{\int P(x)-Q(x)\,\mathrm{d}x}\end{aligned}$$

4 Series

4.1 Sequences

In section 4, we will study the very important topics of infinite sequences, and infinite series (i.e., adding an infinite sequence of numbers).

In section 4.1, we will introduce the concept of infinite sequences.

Sequence

A **sequence** is a function where the domain is the positive integers.

Example 1

Consider the sequence:

$$a_n=\frac{10}{n}\;\Rightarrow\;\text{OR }f(n)=\frac{10}{n},\;n\in\mathbb{Z}^+.$$

$$\begin{aligned}a_1&=10\\ a_2&=5\\ &\vdots\\ a_{10}&=1\\ &\vdots\\ a_{100}&=0.1\\ &\vdots\end{aligned}$$

Note: the trend of a_n is growing to 0.

Example 2

Consider the sequence:

$$a_n = (-1)^{n-1} n.$$

$$a_1 = 1$$
$$a_2 = -2$$
$$a_3 = 3$$
$$a_4 = -4$$
$$\vdots$$

This is an alternating sign sequence. This sequence is alternating between larger and larger positive and negative numbers.

Notations

Sequences are often written with brackets:

$$\left\{\frac{1}{n}\right\}, \ \left\{(-1)^{n-1}n\right\}, \ \{a_n\}.$$

An important question to answer in this section is that "Does the sequence $\{a_n\}$ has a definite trend, or is it indecisive, as n increases?" That is, does $\lim_{n\to\infty} a_n$ exist? If the limit exists, is it finite?

$$\lim_{n\to\infty} a_n = L \quad \text{If } L \text{ is finite} \ \Rightarrow \ \text{converges; otherwise, diverges.}$$

Convergence of a Sequence

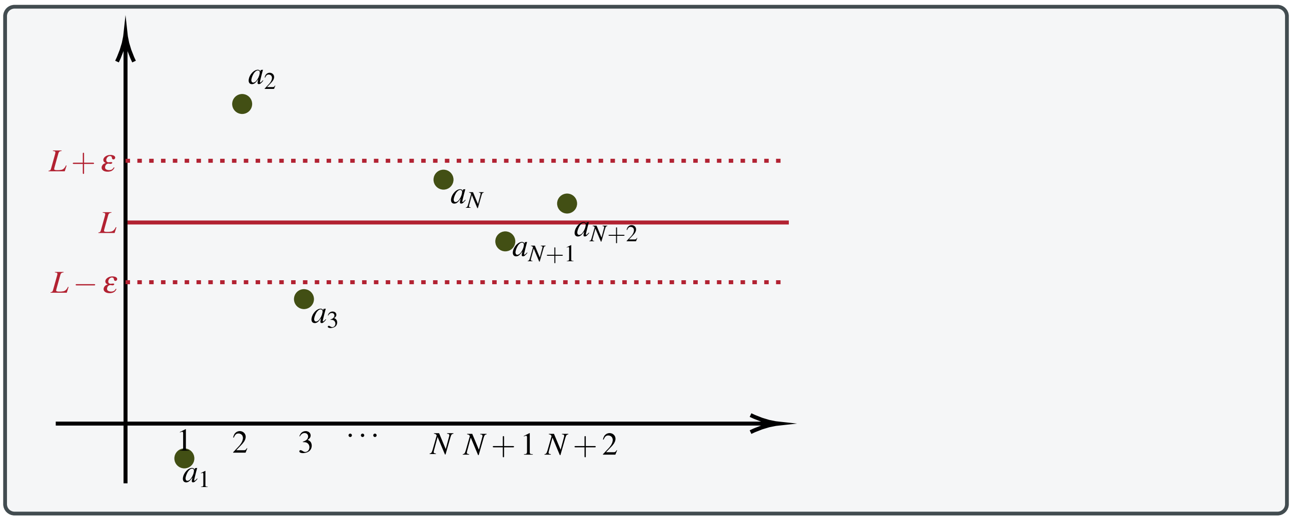
- If $\lim_{n\to\infty} a_n = L$ (i.e., exists), we say the sequence $\{a_n\}$ converges to L.
- If $\lim_{n\to\infty} a_n = \pm\infty$ or does not exist (D.N.E.), we say the sequence $\{a_n\}$ diverges to L.

Formal Definition of Convergence

$\lim_{n\to\infty} a_n = L$ if for every (small) $\varepsilon > 0$, there is an integer n such that:

$$\boxed{|a_n - L| < \varepsilon} \text{ whenever } n > N$$

The boxed inequality stands for $-\varepsilon < a_n - L < \varepsilon$ (i.e., $L - \varepsilon < a_n < L + \varepsilon$).
Note: N depends on ε. If ε is tiny, then N may need to be large.
But for $n > N$, a_n will be in the envelope $L - \varepsilon$ and $L + \varepsilon$.



Function Method to Find Sequence Limit

Suppose $f(x)$ is a function defined for all (not just integers) $x \geq 1$, and $f(n) = a_n, n = 1,\ 2, \cdots$:

- If $\lim\limits_{n\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$;
- If $\lim\limits_{n\to\infty} f(x) = \pm\infty$, then $\lim\limits_{n\to\infty} a_n = \pm\infty$.

That is, we can then use techniques from Calculus I to find limits.

Example 3

Does $\left\{\dfrac{3n^3}{e^{2n}}\right\}$ converge or diverge?

$$a_n = \frac{3n^3}{e^{2n}},\ \text{Let } f(x) = \frac{3x^3}{e^{2x}}$$

Look at the limit, and we find it in indeterminant form, so we will use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{3x^3}{e^{2x}} = \lim_{x\to\infty} \frac{9x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{18x}{4e^{2x}} = \lim_{x\to\infty} \frac{18}{8e^{2x}} = 0.$$

$$\therefore \lim_{n\to\infty} \frac{3n^3}{e^{2n}} = 0.$$

Example 4

Does $\{(-1)^n\}$ converge or diverge?

If n is odd: $a_n = -1$

If n is even: $a_n = 1$

The sequence is oscillating between -1 and 1 and never converges to a single number.

$$\therefore \lim_{n\to\infty}(-1)^n \text{ D.N.E.}$$

Absolute Value Theorem

If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
Note: The limit must be 0. If the limit is not 0, we cannot use this theorem.

Example 5

Does $\left\{\left(-\frac{1}{2}\right)^n\right\}$ converge or diverge?

It looks like the sequence converges to 0, but to state this, we need the absolute value theorem.

$$a_n = \left(-\frac{1}{2}\right)^n \Rightarrow |a_n| = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}.$$

$$\lim_{n\to\infty}\frac{1}{2^n} = 0$$

$\therefore$ By the absolute value theorem,

$$\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0.$$

Example 6

Does $\left\{(-1)^n\frac{n}{e^n}\right\}$ converge or diverge?

$$a_n = (-1)^n\frac{n}{e^n} \Rightarrow |a_n| = \frac{n}{e^n}$$

Let $f(x) = \frac{x}{e^x}$. This limit is in the indeterminant form, so we apply L'Hopital's rule:

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$$

and by absolute value theorem, $\lim_{n\to\infty}(-1)^n\frac{n}{e^n} = 0$.

Example 7

Does $\left\{(-1)^n\frac{n^2}{1+n^2}\right\}$ converge or diverge?

$$a_n = (-1)^n \frac{n^2}{1+n^2} \Rightarrow |a_n| = \frac{n^2}{1+n^2}$$

$\boxed{\text{Method 1}}$ Function and L'Hopital's Rule:

Let $f(x) = \dfrac{x^2}{1+x^2}$. This limit is in the indeterminant form, so we apply L'Hopital's rule:

$$\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{2x}{2x} = 1$$

$\boxed{\text{Method 2}}$

$$\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}(x^2)}{\frac{1}{x^2}(1+x^2)} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^2}+1}$$

Because $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$, $\lim\limits_{x\to\infty} \dfrac{x^2}{1+x^2} = 1$.
In either case,

$$\lim_{x\to\infty} \frac{x^2}{1+x^2} = 1 \neq 0,$$

so we cannot use the absolute value theorem.

Notice:

- If n is large and odd, the sequence converges to -1.
- If n is large and even, the sequence converges to 1.

$\therefore \left\{(-1)^n \dfrac{n^2}{1+n^2}\right\}$ does not converge to a single limit.

$$\therefore \lim_{n\to\infty} (-1)^n \frac{n^2}{1+n^2} \text{ D.N.E.}$$

An Important Sequence

Suppose we have a sequence $\{r^n\}$, where r is a real number.

$$\lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1 \quad \text{and} \quad \lim_{n\to\infty} |r^n| = \infty \text{ if } |r| > 1$$

Example 8

Does $\left\{(-1)^n \dfrac{2^n}{3^n}\right\}$ converge or diverge?

$$a_n = (-1)^n \frac{2^n}{3^n} = \left(-\frac{2}{3}\right)^n$$

Because $r| = \left|-\frac{2}{3}\right| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1$, the sequence $\left\{(-1)^n \frac{2^n}{3^n}\right\}$ converges to 0.

The Squeeze Theorem

If $a_n \leq b_n \leq c_n$ $\forall n$ and $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} c_n = L$.
Then, $\lim\limits_{n\to\infty} b_n = L$.

Example 10

Find

$$\lim_{n\to\infty} (-1)^n \frac{\cos n}{n^2}$$

$$a_n = (-1)^n \frac{\cos n}{n^2}, \ |a_n| = \frac{|\cos n|}{n^2}$$

If we write $f(x) = \frac{|\cos x|}{x^2}$, we cannot find the limit use L'Hopital's rule. So, we need an alternative method, which is to use the squeeze theorem.
Note: $-1 \leq \cos n \leq 1$

$$\therefore \frac{-1}{n^2} \leq (-1)^n \frac{\cos n}{n^2} \leq \frac{1}{n^2}$$

By absolute value theorem, $\lim\limits_{n\to\infty} -\frac{1}{n^2} = 0$, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.
$\therefore$ By squeeze theorem,

$$\lim_{n\to\infty} (-1)^n \frac{\cos n}{n^2} = 0.$$

Example 11

Does $\left\{\left(1+\frac{2}{n}\right)^n\right\}$ converge or diverge?

Let $f(x) = \left(1+\frac{2}{x}\right)^n$, and consider $\lim\limits_{x\to\infty} \left(1+\frac{2}{x}\right)^n$.
$\lim\limits_{x\to\infty} \left(1+\frac{2}{x}\right)^n$ is in the indeterminant form of 1^∞. To use L'Hopital's rule:

$$y = \left(1+\frac{2}{x}\right)^x \Rightarrow \ln y = x \ln\left(1+\frac{2}{x}\right) = \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}.$$

$$\therefore \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} \text{ (which is in indeterminant form of } \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = 2$$

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}(\ln y)} = e^2$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^2.$$

Monotone Sequences

- Monotone increasing sequence: $a_n \leq a_{n+1}$ for all $n \geq N$, where N is a finite number.
- Monotone decreasing sequence: $a_n \geq a_{n+1}$ for all $n \geq N$, where N is a finite number.

Extension from the Definition of Monotone Sequence

Every bounded monotone sequence converges.

Methods to show a sequence is increasing or decreasing

1. Show either

$$a_{n+1} - a_n \geq 0 \;\Rightarrow\; a_{n+1} \geq a_n \;\Rightarrow\; \text{increasing}$$

OR

$$a_{n+1} - a_n \leq 0 \;\Rightarrow\; a_{n+1} \leq a_n \;\Rightarrow\; \text{decreasing}$$

2. Show either

$$\frac{a_{n+1}}{a_n} \geq 1 \;\Rightarrow\; a_{n+1} \geq a_n \;\Rightarrow\; \text{increasing}$$

OR

$$\frac{a_{n+1}}{a_n} \leq 1 \;\Rightarrow\; a_{n+1} \leq a_n \;\Rightarrow\; \text{increasing}$$

3. Define $f(x)$ with $f(n) = a_n$.
Then

$$f'(x) > 0 \;\Rightarrow\; \text{increasing}$$

OR

$$f'(x) < 0 \;\Rightarrow\; \text{decreasing}$$

Example 12

Does $\left\{\frac{3^n}{n!}\right\}$ converge or diverge?

---

1. Find an expression for a_{n+1}:

$$a_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3^n \cdot 3}{n!(n+1)}$$

2. Find an expression for $\frac{a_{n+1}}{a_n}$:

$$\frac{a_{n+1}}{a_n} = a_{n+1} \cdot \frac{1}{a_n} = \frac{3^n \cdot 3}{n!(n+1)} \cdot \frac{n!}{3^n} = \frac{3}{n+1}$$

3. Is the sequence increasing for decreasing?

$$\frac{a_{n+1}}{a_n} \frac{3}{n+1} < 1 \; \forall n \geq 3$$

$$\therefore a_{n+1} < a_n \; \forall n \geq 3 \;\Rightarrow \text{ The sequence is decreasing for } n \geq 3.$$

4. Show $\left\{\frac{3^n}{n!}\right\}$ is bounded:

$$a_1 = \frac{3}{1} = 3$$
$$a_2 = \frac{9}{2} = 4.5$$
$$a_3 = \frac{27}{6} = 4.5$$

We know from step three, that $a_3 > a_4 > a_5 > a_6 > \cdots$, so the sequence is bounded above by 4.5.

Note also that $\frac{3^n}{n!}$ is always positive, so the sequence is bounded below by 0.

Because $\left\{\frac{3^n}{n!}\right\}$ is a bounded monotone decreasing sequence, it converges (converges to 0). $\longrightarrow n!$ as the factorial grows faster than exponential.

4.2 Infinite Series

In this section, we start to look at adding up an infinite number of terms, which is called an infinite series:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

Example 1

$$\sum_{j=1}^{\infty} \frac{1}{5j} = \frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \cdots$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Note: we could also write this as $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}$

Partial Sums

Consider

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + a_{n+1} + \cdots$$

Then

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

is called the nth partial sum.

Example 2

Consider

$$\sum_{i=1}^{\infty} (3i+2),$$

The 4th partial sum is

$$S_4 = \sum_{i=1}^{4} (3i+2) = (3+2) + (6+2) + (9+2) + (12+2) = 38.$$

Convergence/Divergence of Infinite Series

If $\sum_{i=1}^{\infty} a_i = S$ (finite), then we say the series <u>converges</u>; otherwise, we say the series <u>diverges</u>.

Partial Sum to Test Convergence

Suppose $S_n = \sum_{i=1}^{n} a_i$ is the nth partial sum.

If $\lim_{n\to\infty} S_n = S$, then $\sum_{i=1}^{\infty} a_i = S$.

Example 3

Show that

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

converges, and find its sum.

Look at S_n:

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$$

This is called a telescoping series.

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

Thus,

$$\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1.$$

Example 4

Show that

$$\sum_{k=1}^{\infty}\left(\frac{3}{k^2+3k+2}\right)$$

converges, and find its sum.

$$\sum_{k=1}^{\infty}\left(\frac{3}{k^2+3k+2}\right) = 3\sum_{k=1}^{\infty}\left(\frac{1}{(k+1)(k+2)}\right)$$

Partial fractions: Assume

$$\frac{A}{k+1} + \frac{B}{k+2} = \frac{1}{(k+1)(k+2)}$$

$$(A+B)k + (2A+B) = 1 \;\Rightarrow\; \begin{cases} A+B=0 \\ 2A+B=1 \end{cases} \;\Rightarrow\; \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$\therefore \frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$$

$$\therefore \sum_{k=1}^{\infty}\frac{3}{k^2+3k+2} = 3\sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

Find the partial sum:

$$\begin{aligned} S_n = \sum_{k=1}^{n}\frac{3}{k^2+3k+2} &= 3\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\ &= 3\left[(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \cdots + (\frac{1}{n+1} - \frac{1}{n+2})\right] \\ &= 3\left(\frac{1}{2} - \frac{1}{n+2}\right) \end{aligned}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} 3\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{3}{2}$$

Thus,

$$\sum_{k=1}^{\infty}\frac{3}{k^2+3k+2} = \frac{3}{2}.$$

Example 5: Geometric Series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$
$$= a(1 + r + r^2 + r^3 + \cdots),\ a \neq 0 \text{ and } r \text{ are constnats}$$

Find the nth partial sum:

$$S_n = \sum_{k=1}^{n} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-2}$$

We will need to simplify this to find $\lim_{n\to\infty} S_n$:

$$\begin{aligned}
rS_n &= ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\
S_n - rS_n &= a - ar + ar - ar^2 + ar^2 - ar^3 + ar^3 - ar^4 + \cdots + ar^{n-2} - ar^{n-1} + ar^{n-1} - ar^n \\
&= a - ar^n \\
\therefore (1-r)S_n &= a - ar^n \\
S_n &= \frac{a - ar^n}{1-r}\ (r \neq 1)
\end{aligned}$$

Find $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a - ar^n}{1-r} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$
$$= a(1 + r + r^2 + r^3 + \cdots),\ a \neq 0$$

- converges to $\dfrac{a}{1-r}$ if $|r| < 1$
- diverges if $|r| \geq 1$

Example 6

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^{k-1}}$$

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^{k-1}} = \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k - 1 = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots$$

Here, this is a geometric sequence with $a \neq 1,\ r = \frac{2}{3}$

$$\because |r| = \frac{2}{3} < 1 \ \Rightarrow \ \text{converges}$$

$$\therefore \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^{k-1}} = \frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = 3$$

Example 7

$$\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$$

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k &= \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots \\
&= \frac{2}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right]
\end{aligned}$$

This is a geometric sequence with $a = \frac{2}{3},\ r = \frac{2}{3}$.

$$\because |r| = \frac{2}{3} < 1 \ \Rightarrow \ \text{converges}$$

$$\therefore \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \frac{a}{1-r} = \frac{2/3}{1/3} = 2.$$

Important Series

- Harmonic Series (Diverges):

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- Alternating harmonic Series (Converges to $\ln(2)$):

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Properties of Infinite Series

1. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges. Then:

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n + \pm \sum_{n=1}^{\infty} b_n$$

2. For any positive integer k,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and

$$\sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + a_{k+3} + \cdots$$

either both converge or both diverge.
In other words, throw away the first k terms of an infinite series will not affect its convergency or divergency.

Example 8

Show that $\sum_{n=1}^{\infty} \frac{1}{n+5}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n+5} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots = \sum_{n=6}^{\infty} \frac{1}{n}$$

$\sum_{n=6}^{\infty} \frac{1}{n}$ is the harmonic series that diverges.

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n+5} \text{ also diverges.}$$

***n*th Term Test**

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
This means:

- If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ must diverge.

- If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ might converge or diverge.

Diverge: harmonic
Converge: alternating harmonic

Example 9

$$\sum_{n=1}^{\infty} \frac{n+2}{5n-3}$$

$$\lim_{n\to\infty} \frac{n+2}{5n-3} = \frac{1}{5} \neq 0 \quad \text{(By L'Hopital's rule)}$$

$$\therefore \sum_{n=1}^{\infty} \frac{n+2}{5n-3} \text{ must diverge by } n\text{th term test.}$$

Example 10

$$\sum_{n=3}^{\infty} \left(3^{-n} + (-1)^n \left(\frac{2}{3}\right)^{n-1} \right)$$

Look at the terms separately:

$$\sum_{n=3}^{\infty} \left(3^{-n} + (-1)^n \left(\frac{2}{3}\right)^{n-1} \right) = \sum_{n=3}^{\infty} (3^{-n}) + \sum_{n=3}^{\infty} \left((-1)^n \left(\frac{2}{3}\right)^{n-1} \right)$$

$$\begin{aligned} \sum_{n=3}^{\infty} (3^{-n}) = \sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n &= \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \cdots + \left(\frac{1}{3}\right)^n \\ &= \left(\frac{1}{3}\right)^3 \left[1 + \frac{1}{3} + \cdots + \left(\frac{1}{3}\right)^{n-3} \right] \end{aligned}$$

This is a geometric series with $a = \left(\frac{1}{3}\right)^3$, $r = \frac{1}{3}$

$$\because |r| = \frac{1}{3} < 1 \Rightarrow \text{ converges}$$

$$\therefore \sum_{n=3}^{\infty} (3^{-n}) = \frac{a}{1-r} = \frac{(1/3)^3}{2/3} = \frac{1}{18}$$

$$\begin{aligned}
\sum_{n=3}^{\infty}\left((-1)^n\left(\frac{2}{3}\right)^{n-1}\right) &= \sum_{n=3}^{\infty}\left((-1)^n\left(\frac{2}{3}\right)^{n}\left(\frac{2}{3}\right)^{-1}\right)\\
&= \frac{3}{2}\sum_{n=3}^{\infty}\left(-\frac{2}{3}\right)^n\\
&= \frac{3}{2}\left(\left(-\frac{2}{3}\right)^3+\left(-\frac{2}{3}\right)^4+\left(-\frac{2}{3}\right)^5+\cdots+\left(-\frac{2}{3}\right)^n\right)\\
&= \frac{3}{2}\left(-\frac{2}{3}\right)^3\left(1+\left(-\frac{2}{3}\right)+\left(-\frac{2}{3}\right)^2+\cdots+\left(-\frac{2}{3}\right)^{n-3}\right)
\end{aligned}$$

This is a geometric sequence with $a=\frac{3}{2}\left(-\frac{2}{3}\right)^3$, $r=-\frac{2}{3}$

$$\because |r|=\frac{2}{3}<1 \Rightarrow \text{converges.}$$

$$\therefore \sum_{n=3}^{\infty}\left((-1)^n\left(\frac{2}{3}\right)^{n-1}\right)=\frac{a}{1-r}=\frac{\frac{3}{2}\left(-\frac{2}{3}\right)^3}{1-\left(-\frac{2}{3}\right)}=\frac{-4/9}{5/3}=-\frac{4}{15}$$

Thus,

$$\begin{aligned}
\sum_{n=3}^{\infty}\left(3^{-n}+(-1)^n\left(\frac{2}{3}\right)^{n-1}\right) &= \sum_{n=3}^{\infty}(3^{-n})+\sum_{n=3}^{\infty}\left((-1)^n\left(\frac{2}{3}\right)^{n-1}\right)\\
&= \frac{1}{18}-\frac{4}{15}\\
&= -\frac{18}{90} \quad \left(=-\frac{2}{45}\right)
\end{aligned}$$

4.3 Integral Test and Estimates of Sums

Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a "positive term" series, and let $f(x)$ be a function with $f(n)=a_n$.
If

- $f(x)\geq 0\ \forall x\geq 1$
- $f(x)$ is continuous $\forall x\geq 1$
- $f(x)$ is decreasing $\forall x\geq 1$

Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,\mathrm{d}x$$

either both converge or both diverge

Proof: Integral Test

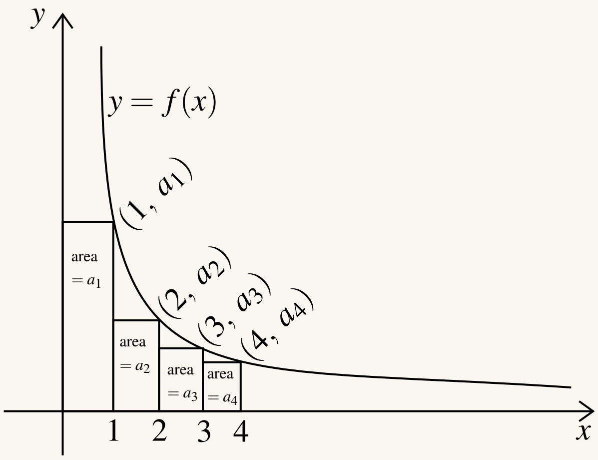


Fig.1

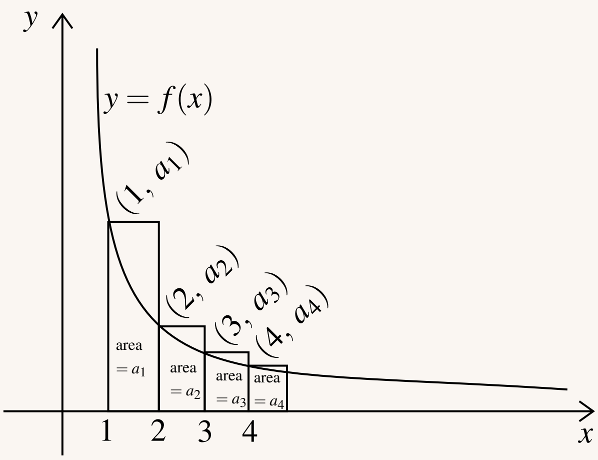


Fig. 2

In Fig. 1:

$$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)\, \mathrm{d}x$$

Thus,

- $\sum a_n$ diverges $\Longrightarrow \int_1^{\infty} f(x)\, \mathrm{d}x$ diverges.
- $\int_1^{\infty} f(x)\, \mathrm{d}x$ converges $\Longrightarrow \sum a_n$ converges.

In Fig. 2:

$$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)\, \mathrm{d}x$$

Thus,

- $\sum a_n$ converges $\Longrightarrow \int_1^{\infty} f(x)\, \mathrm{d}x$ converges.
- $\int_1^{\infty} f(x)\, \mathrm{d}x$ diverges $\Longrightarrow \sum a_n$ diverges.

Example 1

Show that the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, diverges.

Let $f(x) = \frac{1}{x}$

1. Show $f(x)$ satisfies conditions for integral test:
    - $f(x) = \frac{1}{x} \geq 0 \quad \forall x \geq 1$, obvious.
    - $f'(x) = -\frac{1}{x^2}$ is defined $\forall x \geq 1$
      $\Rightarrow f(x)$ is differentiable $\forall x \geq 1$

$\Rightarrow f(x)$ is continuous $\forall x \geq 1$

- $f'(x) = -\dfrac{1}{x^2} < 0 \quad \forall x \geq 1$
  $\Rightarrow f(x)$ is decreasing $\forall x \geq 1$

2. Use the integral test to show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges:

$$\begin{aligned}\int_1^{\infty} f(x)\,\mathrm{d}x &= \int_1^{\infty} \frac{1}{x}\,\mathrm{d}x \\ &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,\mathrm{d}x \\ &= \lim_{t\to\infty} \Big[\ln x \Big]_1^t \\ &= \lim_{t\to\infty} (\ln t - \ln 1) \\ &= \lim_{t\to\infty} \ln t = \infty\end{aligned}$$

$$\therefore \int_1^{\infty} \frac{1}{x}\,\mathrm{d}x \text{diverges}$$

By the integral test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

Example 2

Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 0$ is a constant.

Note: $p = 1 \Rightarrow$ harmonic series $\Rightarrow$ diverges. So consider $p \neq 1$:
Let $f(x) = \dfrac{1}{x^p} = x^{-p}$

1. Show that $f(x)$ satisfies conditions of integral test:

   - $f(x) = \dfrac{1}{x^p} > 0 \quad \forall x \geq 1$ and $p > 0$
   - $f'(x) = \dfrac{-p}{x^{p+1}}$ is defined $\forall x \geq 1$ and $p > 0$.
     $\therefore f(x)$ us continuous $\forall x \geq 1$ and $p > 0$.
   - $f'(x) = \frac{-p}{x^{p+1}} < 0 \forall x \geq 1$ and $p > 0$.
     $\Rightarrow f(x)$ is decreasing $\forall x \geq 1$.

2. Use the integral test to determine convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, $p \neq 1$:

   Recall from section 2.5:

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x \text{ is } \begin{cases} \text{convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{cases}$$

By the integral test:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \, dx \text{ is } \begin{cases} \text{convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{cases}$$

p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if} p > 1, \text{ and diverges if } p \leq 1.$$

Remainder Estimation

How can we estimate $S = \sum_{n=1}^{\infty} a_n$

- Choose fixed (large) value to replace ∞, and complete

$$S_n = \sum_{k=1}^{n} a_k \approx S.$$

- Question: How good is this estimation?

$$\begin{aligned} R_n &= S - S_n \quad \text{(residual/reminder)} \\ &= (a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots) \\ &\quad - (a_1 + a_2 + a_3 + \cdots + a_n) \\ &= a_{n+1} + a_{n+2} + a_{n+3} \cdots \end{aligned}$$

Our new question is: How big is R_n?

Let $f(x)$ be a function with $f(x) = a_n$, we can draw the following figures.

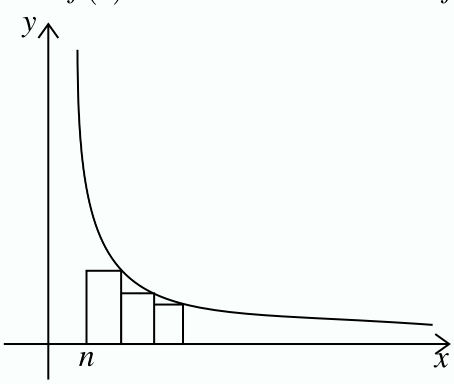


Fig.1

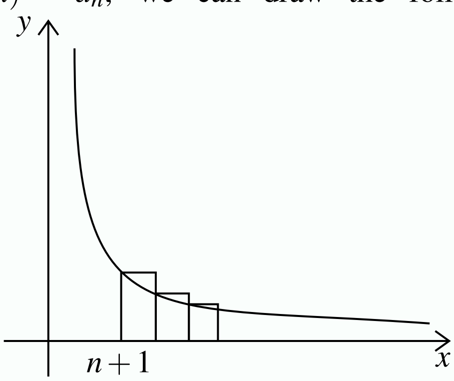


Fig. 2

From Fig. 1,

$$\text{Area} : a_{n+1} + a_{n+2} + a_{n+3} + \cdots = R_n$$

$$\int_n^{\infty} f(x) \, dx \geq R_n$$

From Fig. 2,

$$\text{Area} : a_{n+1} + a_{n+2} + a_{n+3} + \cdots = R_n$$
$$\int_{n+1}^{\infty} f(x)\, \mathrm{d}x \leq R_n$$

That is,

$$\int_{n+1}^{\infty} f(x)\, \mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\, \mathrm{d}x$$

Remainder Estimate for Partial Sums

If $S = \sum_{k=1}^{\infty} a_k$ and $S_n = \sum_{k=1}^{n} a_k = n$th partial sums, and

$$R_n = S - S_n,$$

then

$$\int_{n+1}^{\infty} f(x)\, \mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\, \mathrm{d}x$$

where $f(x)$ is a function with $f(n) = a_n$.

Example 3

Consider the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$.

---

1. Show that this series converges by the integral test.
2. How many terms do we need in

$$S_n = \sum_{k=2}^{n} \frac{1}{k(\ln k)^2}$$

so that S_n is accurate estimation to 0.01?
That is, we want to choose n large enough so that

$$R_n < 0.01.$$

We know

$$R_n \leq \int_{n}^{\infty} \frac{1}{x(\ln x)^2}\, \mathrm{d}x$$

$$\begin{aligned}\int_n^\infty \frac{1}{x(\ln x)^2}\,dx &= \lim_{t\to\infty}\int_n^t \frac{1}{x(\ln x)^2}\,dx\\ &= \lim_{t\to\infty}\left[-\frac{1}{\ln x}\right]_n^t\\ &= \lim_{t\to\infty}\left(-\frac{1}{\ln t}+\frac{1}{\ln n}\right)\\ &= \frac{1}{\ln n}\end{aligned}$$

$\therefore$ we need

$$\begin{aligned}\frac{1}{\ln n} &< 0.01\\ \ln n &> 100\\ n &> e^{100} \quad (\approx 2.7\times 10^{43})\end{aligned}$$

This is a really large number!

4.4 Comparison Tests

Basic Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are positive terms. Then

1. If $\sum b_n$ converges and $a_n \leq b_n\ \forall n$,
   then $\sum a_n$ also converges.
2. If $\sum b_n$ diverges and $a_n \geq b_n\ \forall n$,
   then $\sum a_n$ also diverges.

Example 1

Consider the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$

1. Compare $\dfrac{1}{\sqrt{n}-1}$ and $\dfrac{1}{\sqrt{n}}$:
$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}\ \forall n \geq 2$$
2. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}} = sum_{n=2}^{\infty}\frac{1}{n^{1/2}}$ is a divergent p-series with $p=\dfrac{1}{2}$

By the comparison test,
$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1} \text{ also diverges.}$$

Example 2

Consider the series $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

- $\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$.

$\therefore$ By comparison test,

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3+1} \text{ also converges.}$$

Example 3

Consider the series $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

- $\frac{6^n}{5^n-1} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n$
- $\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ is a divergent geometric sequence with

$$|r| = \frac{6}{5} > 1$$

$\therefore$ By comparison test,

$$\sum_{n=1}^{\infty} \frac{6^n}{5^n-1} \text{ also diverges.}$$

Example 4

Consider the series $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

- $\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} = \frac{(2k-1)(k-1)(k+1)}{(k+1)(k^2+4)^2} = \frac{(2k-1)(k-1)}{(k^2+4)^2} < \frac{(2k)(k)}{(k^2+4)^2} < \frac{2k^2}{(k^2)^2} = \frac{2}{k^2}$
- $\sum_{k=1}^{\infty} \frac{2}{k^2}$ is a convergent p-series with $p = 2 > 1$.

$\therefore$ By comparison test,

$$\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} \text{ also converges.}$$

Example 5

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$

- $\frac{1}{n^n} = \frac{1}{n \cdots n} \leq \frac{1}{n^2}$
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$

$therefore$ By comparison test,

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \text{ also converges.}$$

Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are positive term series. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C, \quad \text{where } C > 0 \text{ is finite,}$$

then either both series converge or both series diverge.

Example 6

Consider the series $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$

- Let $a_n = \frac{2}{\sqrt{n}+2}$ and $b_n = \frac{1}{\sqrt{n}}$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{\sqrt{n}+2} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{2}{1+\frac{2}{\sqrt{n}}} = 2 > 0$
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series, with $p = \frac{1}{2} < 1$

$\therefore$ By limit comparison test,

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2} \text{ also diverges.}$$

Example 7

Consider the series $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

- Let $a_n = \frac{n^2+n+1}{n^4+n^2}$ and $b_n = \frac{1}{n^2}$

- $$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2+n+1}{n^4+n^2}\cdot n^2 &= \lim_{n\to\infty}\frac{(n^2+n+1)n^2}{n^2(n^2+1)} \\ &= \lim_{n\to\infty}\frac{n^2+n+1}{n^2+1} \\ &= \lim_{n\to\infty}\frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1 > 0\end{aligned}$$

- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series, with $p = 2 > 1$.

$\therefore$ By the limit comparison test,

$$\sum_{n=1}^{\infty}\frac{n^2+n+1}{n^4+n^2} \text{ also converges.}$$

Example 8

Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$

- Let $a_n = \dfrac{1}{n^{1+1/n}} = \dfrac{1}{n\cdot n^{1/n}}$ and $b_n = \dfrac{1}{n}$

- $\displaystyle\lim_{n\to\infty}\frac{1_n}{b_n} = \lim_{n\to\infty}\frac{1}{n\cdot n^{1/n}}\cdot n = \lim_{n\to\infty}\frac{1}{n^{1/n}}$

  We need to find $\displaystyle\lim_{n\to\infty} n^{\frac{1}{n}}$:

  Let $f(x) = x^{\frac{1}{x}}$, look at $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x}}$ is in indeterminant form ∞^0.

  $$\lim_{x\to\infty} e^{\ln x^{1/x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty}\frac{\ln x}{x}}$$

  Find $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$ in indeterminant form of $\dfrac{\infty}{\infty}$ by applying L'Hopital's rule:

  $$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

  $$\therefore \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{1/x}} = e^{\lim_{x\to\infty}\frac{\ln x}{x}} = e^0 = 1$$

  $$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}} = 1 \qquad \text{(By the function method)}$$

  $$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{n^{1/n}} = \frac{1}{1} = 1 > 0$$

- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series, divergent.

$\therefore$ By the limit comparison test,

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \text{ also diverges.}$$

4.5 Alternating Series Test

Review of Important Series

Geometric: $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a\left(1+r+r^2+\cdots\right), \quad a \neq 0$

if $|r|<1$ the series converges, and $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

$p-$Series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if} p>1 \\ \text{diverges if } p \leq 1 \end{cases}$ (prove with integral test)

Harmonic: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (prove with integral test)

Alternating Harmonic: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ converges to $\ln 2$

However, the alternating harmonic series was not proven to be convergent in previous sections, which will be done in this section.

Convergence of Alternating Harmonic Series

Consider the partial sum:

$$\begin{aligned}
S_{2n} = \sum_{k=1}^{2n}(-1)^{k-1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n} \\
&= \sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{k}\right) \\
&= \sum_{k=1}^{n}\left(\frac{1}{2k-1}+\frac{1}{2k}-\frac{1}{2k}-\frac{1}{2k}\right) \\
&= \sum_{k=1}^{n}\left(\frac{1}{2k-1}+\frac{1}{2k}\right)-2\sum_{k=1}^{n}\left(\frac{1}{2k}\right) \\
&= \left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)-\underbrace{2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)}_{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}} \\
&= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n} \\
&= \sum_{k=n+1}^{2n}\frac{1}{k} \quad \Longrightarrow \text{We need to find } \lim_{n\to\infty} S_{2n} = \lim_{k\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k}.
\end{aligned}$$

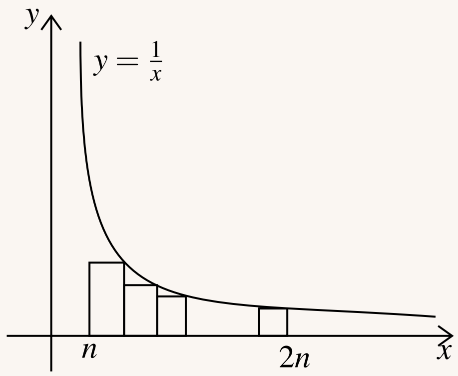


Fig.1

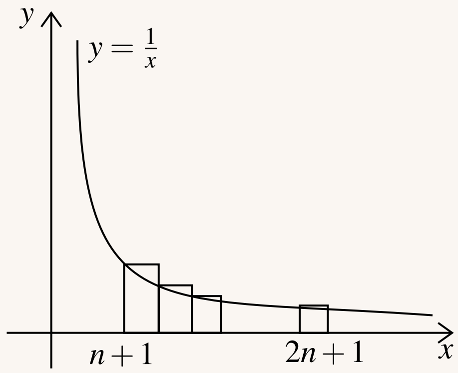


Fig. 2

From Fig.1, we know

$$\frac{1}{n+1} + \cdots + \frac{1}{2n} \leq \int_{n}^{2n} \frac{1}{x}\,\mathrm{d}x$$
$$S_{2n} \leq \int_{n}^{2n} \frac{1}{x}\,\mathrm{d}x$$

From Fig.2, we know

$$\frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \int_{n+1}^{2n+1} \frac{1}{x}\,\mathrm{d}x$$
$$S_{2n} \geq \int_{n+1}^{2n+1} \frac{1}{x}\,\mathrm{d}x$$

$$\therefore \int_{n+1}^{2n+1} \frac{1}{x}\,\mathrm{d}x \leq S_{2n} \leq \int_{n}^{2n} \frac{1}{x}\,\mathrm{d}x$$

Do the integrals:

- $$\int_{n+1}^{2n+1} \frac{1}{x}\,\mathrm{d}x = \Big[\ln x\Big]_{n+1}^{2n+1} = \ln(2n+1) - \ln(n+1) = \ln\left(\frac{2n+1}{n+1}\right)$$
- $$\int_{n}^{2n} \frac{1}{x}\,\mathrm{d}x = \Big[\ln x\Big]_{n}^{2n} = \ln(2n) - \ln(n) = \ln\left(\frac{2n}{n}\right) = \ln 2$$

That is,

$$\ln\left(\frac{2n+1}{n+1}\right) \leq S_{2n} \leq \ln 2$$

Notice,

$$\lim_{n\to\infty} \ln\left(\frac{2n+1}{n+1}\right) = \lim_{n\to\infty} \ln\left(\frac{2+1/n}{1+1/n}\right)$$
$$= \ln 2$$
$$\lim_{n\to\infty} \ln 2 = \ln 2$$
$$\therefore \text{ When } n \to \infty: \quad \ln 2 \leq S_{2n} \leq \ln 2$$

By the squeeze theorem,

$$\lim_{n\to\infty} S_{2n} = \ln 2$$

i.e.,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\ln 2$$

However, we still need a test for other alternating series.

Alternating Series Test

If the alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

satisfies $b_n \geq b_{n+1}$ and $\lim_{n\to\infty} b_n = 0$, then the series converges.

Proof

To see why the series converges:

$$\begin{aligned} S_{2n} &= (b_1-b_2)+\cdots+(b_{2n-1}-b_{2n}) \quad \text{[Adding non-negative numbers]} \\ &= b_1-(b_2-b_3)-\cdots-(b_{2n-2}-b_{2n-1})-b_{2n} \quad \text{[Subtracting non-negative numbers]} \end{aligned}$$

$$b_1 \geq b_2 \;\Rightarrow\; b_1-b_2 \geq 0, \cdots, b_{2n-1} \geq b_{2n} \;\Rightarrow\; b_{2n-1}-b_{2n} \geq 0$$

This means:

- Even numbered partial sums are increasing non-negative numbers
- But they are bounded above by b_1
- $\lim_{n\to\infty} S_{2n}$ converges to, say, S.

Now, let's consider odd numbered partial sums:

$$\begin{aligned} S_{2n+1} &= b_1-b_2+b_3-b_4+\cdots+b_{2n-1}-b_{2n}+b_{2n+1} \\ &= S_{2n}+b_{2n+1} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} S_{2n+1} &= \lim_{n\to\infty}(S_{2n}+b_{2n+1}) \\ &= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= S+0 \\ \therefore \lim_{n\to\infty} S_{2n+1} = S &= \lim_{n\to\infty} S_{2n} \end{aligned}$$

Example 1

Consider the series $\sum_{n=1}^{\infty} n^{-1/2}$

This is a divergent p-series, with $p = \frac{1}{2} < 1$

Example 2

Consider the series $\sum_{n=1}^{\infty} (-1)^n n^{-1/2}$

Notice:

- $b_n = \frac{1}{n^{1/2}}$, $b_{n+1} = \frac{1}{(n+1)^{1/2}}$
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\therefore$ By the alternating series test,

$$\sum_{n=1}^{\infty} (-1)^n n^{-1/2} \text{ converges.}$$

Example 3

Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

This is an alternating series, with $b_n = \frac{n^2}{n^3+4}$

- Let $f(x) = \frac{x^2}{x^3+4}$, with $f(n) = b_n$

$$\begin{aligned} f'(x) = \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2} &= \frac{2x^4+8x-3x^4}{(x^3+4)^2} \\ &= \frac{8x-x^4}{(x^3+4)^2} \\ &= \frac{x(x^3-8)}{(x^3+4)^2} = \frac{x(2-x)(x^2+2x+1)}{(x^3+4)^2} \end{aligned}$$

$\therefore f'(x) < 0 \; \forall x > 2$

$\therefore f'(x)$ is decreasing $\forall x > 2$

$\therefore b_n = \frac{n^2}{n^3+4}$ is decreasing $\forall n > 2$ [by the function method]

- $\lim_{n\to\infty} \frac{n^2}{n^3+4} = \lim_{n\to\infty} \frac{1/n}{1+4/n^2} = 0$

$\therefore$ By the alternating series test,

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+4} \text{ converges.}$$

Example 4

Consider the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2n+1}$

In this case, the series diverges because

$$\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{1}{2+1/n}=\frac{1}{2}\neq 0$$

[the n-th term test]

Example 5

Consider the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\ln\left(\frac{n+1}{n}\right)$

Here, $b_n=\ln\left(\frac{n+1}{n}\right)$

- $\displaystyle\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)=\lim_{n\to\infty}\ln\left(\frac{1+1/n}{1}\right)=\ln(1)=0$
- Show that b_n is decreasing:
  Let $f(x)=\ln\left(\frac{x+1}{x}\right)$ with $f(n)=b_n$, $x>0$.

$$f(x)=\ln\left(\frac{x+1}{x}\right)=\ln(x+1)-\ln(x)$$

$$\therefore\ f'(x)=\frac{1}{x+1}-\frac{1}{x}=\frac{-1}{x(x+1)}<0\ \forall x>0$$

$f(x)$ is decreasing $\forall x>0$

$$\therefore\ b_n=\ln\left(\frac{n+1}{n}\right) \text{ is decreasing by the function method}$$

$\therefore$ By the alternating series test,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\ln\left(\frac{n+1}{n}\right) \text{ converges.}$$

4.6 Absolute Convergence and the Ratio and Root Tests

Absolute Convergence

If $\sum |a_n|$ converges, then we say that $\sum a_n$ is absolutely convergent.

Absolute Convergence Test

If $\sum a_n$ is absolutely convergent, then it is convergent.

Example 1

Consider the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$

Here $a_n = (-1)^{n-1} \frac{1}{n^2}$ and $|a_n| = \frac{1}{n^2}$

Notice:

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

$$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} \text{ is absolutely convergent.}$$

Example 2

Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

Here $a_n = (-1)^n \frac{1}{\sqrt{n}}$ and $|a_n| = \frac{1}{\sqrt{n}}$.

Notice: $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series.

$$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \text{ does not converge absolutely.}$$

However, in section 4.5, we used the alternating series test to show that $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ does converge.

$\therefore$ In this case, we say

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \text{ is conditionally convergent.}$$

Sometimes, we may need to combine tests.

Example 3

Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^3+1}$

$|a_n| = \frac{1}{n^3+1} \therefore \sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^3+1}$

- $\frac{1}{n^3+1} < \frac{1}{n^3}$
- $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series, with $p = 3$.
- By comparison test,

$$\sum_{n=1}^{\infty}\frac{1}{n^3+1} \text{ must converge as well.}$$

$\therefore$ By absolute convergence test,

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^3+1} \text{ converges absolutely}$$

Ratio Test for Series

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum a_n$ diverges.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then we don't know – we need to use a different test.

Example 4

Use the ratio test to determine if $\sum_{n=1}^{\infty} e^{-n}n!$ converges.

$$a_n = e^{-n}n! = \frac{n!}{e^n},\ a_{n+1} = \frac{(n+1)!}{e^{n+1}} = \frac{n!(n+1)}{e^n \cdot e}$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!(n+1)}{e^n \cdot e}\cdot\frac{e^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3}\right| = \infty > 1$$

$\therefore$ By the ratio test,

$$\sum_{n=1}^{\infty} e^{-n}n! \text{ diverges.}$$

Example 5

Use the ratio test to determine if $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

$$a_n = \frac{n^2}{2^n}, a_{n+1} = \frac{(n+2)^2}{2^{n+1}} = \frac{(n+1)^2}{2^n \cdot 2}$$

$$\begin{aligned}
\therefore \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{2^n \cdot 2} \cdot \frac{2^n}{n^2} \right| &= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{2n^2} \right| \\
&= \lim_{n\to\infty} \left| \frac{1+2/n+1/n^2}{2} \right| \\
&= \frac{1}{2} < 1
\end{aligned}$$

$\therefore$ By the ratio test,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \text{ converges absolutely.}$$

Root Test for Series

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum a_n$ diverges.
- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then we don't know – we need to use a different test.

Example 6

Use the root test to determine if $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converges.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\begin{aligned}
\therefore \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} &= \lim_{n\to\infty} \left|\frac{2n+3}{3n+2}\right| \\
&= \lim_{n\to\infty} \left|\frac{2+3/n}{3+2/n}\right| \\
&= \frac{2}{3} < 1
\end{aligned}$$

$\therefore$ By the root test,

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n \text{ converges absolutely.}$$

Example 7

Consider the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)$

$$\lim_{n\to\infty}\frac{2n+3}{3n+2}=\lim_{n\to\infty}\frac{2+3/n}{3+2/n}=\frac{2}{3}\neq 0$$

$\therefore$ By the n-th term test,

$$\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right) \text{ diverges.}$$

4.7 Practice Using Various Tests for Series

Example 1

Consider the series $\sum_{n=1}^{\infty}\frac{n-1}{n^3+1}$

- $\frac{n-1}{n^3+1}<\frac{n-1}{n^3}<\frac{n}{n^3}=\frac{1}{n^2}$
- $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series, with $p=2$

$\therefore$ By comparison test, because $\frac{n-1}{n^3+1}<\frac{1}{n^2}$,

$$\sum_{n=1}^{\infty}\frac{n-1}{n^3+1} \text{ also converges.}$$

Example 2

Consider the series $\sum_{n=1}^{\infty}\frac{n+1}{n^3+1}$

- $\frac{n+1}{n^3+1}<\frac{n+1}{n^3}=\frac{1}{n^2}+\frac{1}{n^3}$
- $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series, with $p=2$;
  $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series, with $p=3$.

$$\therefore \sum_{n=1}^{\infty}\left(\frac{n+1}{n^3}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{n^3}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n^3} \text{ also converges.}$$

$\therefore$ By the comparison test, because $\dfrac{n+1}{n^3+1} < \dfrac{n+1}{n^3}$,

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3+1} \text{ also converges.}$$

Example 3

Consider the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$

$$a_n = (-1)^n \frac{n^2-1}{n^2+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{1-1/n^2}{1+1/n^2} = \text{D.N.E.} \neq 0$$

(a_n alternates between 1 and -1 when $n \to \infty$, and it never goes to 0).

$\therefore$ By the n-th term test,

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1} \text{ diverges.}$$

Example 4

Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

$$a_n = \frac{n^{2n}}{(1+n)^{3n}} = \left(\frac{n^2}{(1+n)^3}\right)^n$$

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2}{(1+n)^3}\right)\right|} &= \lim_{n\to\infty} \left|\frac{n^2}{(1+n)^3}\right| \\
&= \lim_{n\to\infty} \left|\frac{n^2}{(1+n)^3} \cdot \frac{1/n^3}{1/n^3}\right| \\
&= \lim_{n\to\infty} \left|\frac{1/n}{(1/n+1)^2}\right| \\
&= 0 < 1
\end{aligned}$$

$\therefore$ By the root test,

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}} \text{ converges absolutely.}$$

Example 5

Consider the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^4}{4^n}$

$$a_n = (-1)^n\frac{n^4}{4^n}$$

$$|a_n| = \frac{n^4}{4^n}, |a_{n+1}| = \frac{(n+1)^4}{4^{n+1}} = \frac{(n+1)^4}{4\cdot 4^n}$$

$$\begin{aligned}\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{4\cdot 4^n}\cdot\frac{4^n}{n^4}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^4}{4n^4}\right| \\ &= \lim_{n\to\infty}\left(\frac{(n+1)^4}{4n^4}\cdot\frac{1/n^4}{1/n^4}\right) \\ &= \lim_{n\to\infty}\left(\frac{(1+1/n)^4}{4}\right) \\ &= \frac{1}{4} < 1\end{aligned}$$

$\therefore$ By the ratio test,

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^4}{4^n} \text{ converges absolutely.}$$

4.8 Power Series

In this section, we consider series that include powers of a variable x, for example:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

where $c_0, c_1, \cdots$ are constants (think of this as an infinite degree polynomial).

Example 1

Use long division to find a series expression for $f(x) = \frac{1}{1-x}$

$$\begin{array}{r l}
 & 1 + x + x^2 + \cdots \\
1 - x \,\big) & 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 + 0\cdot x^3 + \cdots \\
 & \underline{1 - x} \\
 & x \\
 & \underline{x - x^2} \\
 & x^2 \\
 & \underline{x^2 - x^3} \\
 & x^3 \cdots
\end{array}$$

$\therefore f(x) = \dfrac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=1}^{\infty} x^n$

Notice: For the series to converge, we need $|x| < 1$.

In this case, x^3, x^4, $\cdots$ approach zero.

Important Question: For what values of x does $\sum_{n=0}^{\infty} c_n x^n$ converge?

Note:

- If $x = 0$, then $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots = c_0$

$$\therefore \sum_{n=0}^{\infty} c_n x^{\text{converges if}}\ x = 0.$$

- If $x \neq 0$, use the ratio test:

$$a_n = c_n x^n, a_{n+1} = c_{n+1} x^{n+1} = c_{n+1} x^n \cdot x$$

$$\begin{aligned}
\therefore \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{c_{n+1} x^n \cdot x}{c_n x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \cdot x \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \cdot |x| \\
&= |x| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|
\end{aligned}$$

Suppose $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ (L can be infinity)
By the ratio test,

$$\sum_{n=1}^{\infty} c_n x^n \text{ converges absolutely}$$

if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L \cdot |x| < 1$$

$\Rightarrow |x| < \dfrac{1}{L}$ Observations:

- If $L = 0$, then $L \cdot |x| = 0 \cdot |x| = 0 < 1\ \forall x \in \mathbb{R}$

$$\therefore \sum_{n=0}^{\infty} c_n x^n \text{ converges } \forall x \in \mathbb{R}$$

- If $L = \infty$, then $L \cdot |x|$ is never less than 1 for $x \neq 0$.

$$\therefore \sum_{n=0}^{\infty} c_n x^n \text{ only converges when } x = 0$$

$$\left(\sum_{n=0}^{\infty} c_n x^n \text{ never converges } \forall x \neq 0\right)$$

- If $0 < L < \infty$, then $L \cdot |x| < 1 \ \Rightarrow\ |x| < \frac{1}{L}$

$$\therefore \sum_{n=0}^{\infty} c_n x^n \text{ converges absolutely for} |x| < \frac{1}{L}$$

and possibly for $x = \pm\dfrac{1}{L}$ (needs to be checked separately).

The Radius of Convergence

$$R = \frac{1}{L}$$

is called the radius of convergence.

Example 2

Find the radius and interval of convergence for: $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

Let $a_n = \dfrac{x^n}{n3^n}$, $a_{n+1} = \dfrac{x^{n+1}}{(n+1)3^{n+1}} = \dfrac{x^n \cdot x}{(n+1)3 \cdot 3^n}$

Ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{x^n \cdot x}{(n+1)3 \cdot 3^n} \cdot \frac{(n+1)3^{n+1}}{x^{n+1}}\right| \\
&= \lim_{n\to\infty} \left|\frac{x}{3} \cdot \frac{n}{n+1}\right| \\
&= \left|\frac{x}{3}\right| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| \\
&= \frac{|x|}{3} \lim_{n\to\infty} \left|\frac{n}{n+1}\right| \\
&= \frac{|x|}{3} \lim_{n\to\infty} \left|\frac{1}{1+1/n}\right| \\
&= \frac{|x|}{3}
\end{aligned}$$

This series converges absolutely if $\dfrac{|x|}{3} < 1 \ \Rightarrow\ |x| < 3, -3 < x < 3$

$\therefore$ Radius of convergence: $R = 3$.

To find an interval, check the end points: $x = \pm 3$

- When $x = 3$:

$$\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This series is a divergent harmonic series.

- When $x = -3$:
$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n(3^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
This series converges by the alternating series test.

$\therefore$ The interval of convergence is $\quad -3 \leq x < 3 \quad$ OR $\quad x \in [-3\,,3)$.

Example 3

Find the radius and interval of convergence for: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Let $a_n = \dfrac{2^n x^n}{n!}$, $a_{n+1} = \dfrac{2^{n+1} x^{n+1}}{(n+1)!} = \dfrac{2^n \cdot 2x^n \cdot x}{(n+1)n!}$

Ratio test:
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2^n \cdot 2x^n \cdot x}{(n+1)n!} \cdot \frac{n!}{2^n \cdot x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{2x}{n+1}\right| \\ &= |2x| \lim_{n\to\infty}\left|\frac{1}{n+1}\right| \\ &= 2|x| \cdot 0 = 0 < 1 \quad \text{for all } x\end{aligned}$$

$\therefore$ Radius of convergence: $R = \infty$

Interval of convergence: $\quad -\infty < x < \infty \quad$ OR $\quad x \in (-\infty, \infty)$

Power Series in $(x-a)$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where $c_0, c_1, \cdots$ and a are constants.

To find the radius and interval of convergence

- Again use ratio test to find:
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| &= \lim_{n\to\infty}\left|(x-a)\frac{c_{n+1}}{c_n}\right| \\ &= |x-a| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \\ &= |x-a| \cdot L \qquad \left[L = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\right]\end{aligned}$$

- Converges if $|x-a| \cdot L < 1$
  - If $L = 0$, then converges for all x:
    $\therefore\ R = \infty$, Interval: $(-\infty, \infty)$

- If $L = \infty$, then converges only if $x = a$.
  $\therefore\ R = 0$
- If $0 < L < \infty$, then converges for

$$|x-a| < \frac{1}{L} \Rightarrow -\frac{1}{L} < x-a < \frac{1}{L}$$

$$\text{Interval} \quad \begin{cases} a-\frac{1}{L} < x < a+\frac{1}{L} \\ \text{and possibly the end points} \end{cases}$$

$$\therefore R = \frac{1}{L}$$

Example 4

Find the radius and interval of convergence for: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x-2)^n}{4^n\sqrt{n}}$.

Let $|a_n| = \left|\dfrac{(x-2)^n}{4^n\sqrt{n}}\right|$, $|a_{n+1}| = \left|\dfrac{(x-2)^{n+1}}{4^{n+1}\sqrt{n+1}}\right| = \left|\dfrac{(x-2)(x-2)^n}{4\cdot 4^n\cdot\sqrt{n+1}}\right|$

$$\begin{aligned}
\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)(x-2)^n}{4\cdot 4^n\cdot\sqrt{n+1}}\cdot\frac{4^n\sqrt{n}}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x-2}{4}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\right| \\
&= \frac{|x-2|}{4}\lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right| \\
&= \frac{|x-2|}{4}\lim_{n\to\infty}\left|\sqrt{\frac{1}{1+1/n}}\right| \\
&= \frac{|x-2|}{4}
\end{aligned}$$

For convergence, we need $\dfrac{|x-2|}{4} < 1 \Rightarrow |x-2| < 4$
$\therefore$ Radius of convergence: $R = 4$
Interval: $-4 < x-2 < 4 \Rightarrow -2 < x < 6$
Check the end points:

- When $x = -2$:

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-2-2)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{(-4)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{4^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

  This is a divergent p-series, with $p = \dfrac{1}{2} < 1$

- When $x = 6$:

$$\sum_{n=1}^{\infty}(-1)^n\frac{(6-2)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{4^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$$

This series converges by the alternating series test.

$\therefore$ Interval: $-2 < x \leq 6$ OR $x \in (-2, 6]$

4.9 Representation of Functions as Power Series

Differentiating Power Series

If

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots,$$

and radius of convergence $R > 0$.
Then, $f(x)$ is differentiable on the interval $(a-R, a+R)$, and

$$\begin{aligned} f'(x) &= 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \\ &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \qquad \begin{bmatrix} \text{Only valid within the interval of convergence.} \\ \text{Not valid at the end points even they converge.} \end{bmatrix} \end{aligned}$$

Integrating Power Series

If

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots,$$

and radius of convergence $R > 0$.
Then, the indefinite integral of $f(x)$ is

$$\begin{aligned} \int f(x)\ \mathrm{d}x &= \left[c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \frac{c_3}{4}(x-a)^4 + \frac{c_4}{5}(x-a)^5 + \cdots\right] + \hat{c} \\ &= \hat{c} + \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1} \end{aligned}$$

And if $[s,t]$ is **fully contained in the interval of convergence** *(not even at the end point)*, then the definite integral is

$$\int_s^t f(x)\ \mathrm{d}x = \sum_{n=0}^{\infty}\int_s^t c_n(x-a)^n\ \mathrm{d}x$$

Example 1

Consider $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$.

$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$ is a geometric series, with $r = -x$

(a) Find the radius and interval of convergence.

For a geometric series to converge,

$$|r| = |-x| = |x| < 1$$

$\therefore$ Radius: $R = 1$

Interval: $-1 < x < 1$ (It does not converge at end points)

(b) Find $f'(x)$ [only valid for $x \in (-1, 1)$]

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \\ \therefore f'(x) &= 0 - 1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n n x^{n-1} \end{aligned}$$

(c) Find $\int f(x)\,\mathrm{d}x$ [only valid for $x \in (-1, 1)$]

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \\ \therefore \int f(x)\,\mathrm{d}x &= \left[x - \frac{1}{2}x + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots \right] + C \\ &= C + \sum_{n=0}^{\infty} (-1) \frac{1}{n+1} x^{n+1} \end{aligned}$$

(d) Find $\int_0^2 f(x)\,\mathrm{d}x$

Not possible because $[0, 2]$ is not fully contained in the interval of convergence $(-1, 1)$.

One approach to find power series representation of $f(x)$

Recall: Geometric series converges if $|r| < 1$, and

$$\begin{aligned}\sum_{n=0}^{\infty} ar^n &= a + ar + ar^2 + ar^3 + \cdots \\ &= a(1 + r + r^2 + r^3 + \cdots) \\ &= \frac{a}{1-r}\end{aligned}$$

In a previous example in section 4.8, we looked at $\dfrac{1}{1-x} = \dfrac{a}{1-r} \Rightarrow a = 1,\ x = r$

$$\therefore \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Example 2

Find a power series representation for $f(x) = \dfrac{1}{2+x}$

Make $f(x) = \dfrac{1}{2+x}$ look like $\dfrac{a}{1-r}$ Note:

$$\frac{1}{2+x} = \frac{1}{2(1+x/2)} = \frac{1/2}{1+x/2} = \frac{1/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$\Rightarrow a = \frac{1}{2},\ r = -\frac{x}{2}$$

$\therefore$ If $|r| = \left|-\dfrac{x}{2}\right| = \dfrac{|x|}{2} < 1 \Rightarrow |x| < 2$ Then, $\dfrac{1}{2+x}$ is the sum of a geometric series with $a = \dfrac{1}{2},\ r = -\dfrac{x}{2}$

$$\begin{aligned}\frac{a}{1-r} &= a(1 + r + r^2 + r^3 + \cdots) \\ \therefore \frac{1}{2+x} &= \frac{1}{2}\left(1 + \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 + \left(-\frac{x}{2}\right)^3 + \cdots\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{2}\left(\frac{x}{2}\right)^n\end{aligned}$$

Remark

We will do many examples that use geometric series, but we may also need to be creative.

Basic approach:

- Start with a basic series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ OR $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$, $|u| < 1$
- Try to relate a given function to $\frac{1}{1-x}$ or $\frac{1}{1-u}$ using multiplication, substitution, differentiation, or integration.

Example 3

Find a power series representation for $f(x) = \frac{1}{1-x^3}$

Here we can use substitution, with $u = x^3$.
Start with

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \ |u| < 1$$

Using $u = x^3$:

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n, \ |x^3| < 1$$

$$\therefore f(x) = \frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}, \ |x| < 1$$

Example 4

Find a power series representation for $f(x) = \frac{x}{1+9x^2}$

- First, notice:

$$\frac{x}{1+9x^2} = x \cdot \left[\frac{1}{1+9x^2}\right] = x\left[\frac{1}{1-(-9x^2)}\right]$$

- Using $u = -9x^2$, we get:

$$\begin{aligned}\frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} &= \sum_{n=0}^{\infty} (-9x^2)^n, \ |-9x^2| < 1 \\ &= \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}, \ |3x| < 1 \ \left[R = \frac{1}{3}\right]\end{aligned}$$

$$\begin{aligned}\therefore f(x) = \frac{x}{1+9x^2} = x\left[\frac{1}{1+9x^2}\right] &= x\sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}, \ |x| < \frac{1}{3} \\ &= \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n+1}, \ |x| < \frac{1}{3}\end{aligned}$$

Example 5

Find a power series representation for $f(x) = \dfrac{5x}{(1-x)^2}$

- First, notice
$$f(x) = 5x \cdot \left[\frac{1}{(1-x)^2}\right]$$
- Now, how to write $\dfrac{1}{(1-x)^2}$ in terms of $\dfrac{1}{1-u}$?
  Differentiation:
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x)^{-1}\right] = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$
$$\begin{aligned}\therefore f(x) = \frac{5x}{(1-x)^2} = 5x \cdot \left[\frac{1}{(1-x)^2}\right] &= 5x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(1-x)^2}\right] \\ &= 5x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} x^n\right], \ |x| < 1 \\ &= 5x \cdot \sum_{n=0}^{\infty} nx^{n-1}, \ |x| < 1 \\ &= \sum_{n=0}^{\infty} 5nx^n, \ |x| < 1\end{aligned}$$

Example 6

Find a power series representation for $f(x) = \tan^{-1} x$

- We need something like $\dfrac{1}{1-u} = \sum_{n=0}^{\infty} u^n$, $|u| < 1$
- Recall:
$$f(x) = \tan^{-1} x = \int \frac{1}{1+x^2}\,\mathrm{d}x$$
  Now look at
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}, \ |x| < 1$$
$$\begin{aligned}\therefore \tan^{-1} x = \int \frac{1}{1+x^2}\,\mathrm{d}x = \int \left[\sum_{n=0}^{\infty}(-1)^n x^{2n}\right] \mathrm{d}x, \ |x| < 1 \\ = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,\mathrm{d}x \\ = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C, \ |x| < 1\end{aligned}$$

Since C can be any constant, we can take $C=0$.

$$\therefore f(x)=\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},\ |x|<1$$

Example 7

Evaluate the integral $\displaystyle\int\frac{x-\tan^{-1}x}{x^3}\,\mathrm{d}x$

- From the previous example:

$$\begin{aligned}\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},\ |x|<1\\ &=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots\end{aligned}$$

- Now:

$$\begin{aligned}\therefore x-\tan^{-1}x&=x-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots\right)\\ &=\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}-\frac{x^9}{9}+\cdots\\ \therefore\ \frac{x-\tan^{-1}x}{x^3}&=\frac{1}{x^3}\cdot\left(\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}-\frac{x^9}{9}+\cdots\right)\\ &=\frac{1}{3}-\frac{x^2}{5}+\frac{x^4}{7}-\frac{x^6}{9}+\cdots\\ \therefore\int\frac{x-\tan^{-1}x}{x^3}\,\mathrm{d}x&=\int\left[\frac{1}{3}-\frac{x^2}{5}+\frac{x^4}{7}-\frac{x^6}{9}+\cdots\right]\mathrm{d}x\\ &=\frac{1}{3}x-\frac{1}{3}\frac{x^3}{5}+\frac{1}{5}\frac{x^5}{7}-\frac{1}{7}\frac{x^7}{9}+\cdots\\ &=\frac{x}{3}-\frac{x^3}{3\cdot5}+\frac{x^5}{5\cdot7}-\frac{x^7}{7\cdot9}+\cdots\\ &=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)(2n+1)},\ |x|<1\end{aligned}$$

4.10 Maclaurin and Taylor Series

Question: Given a function $f(x$ that is continuously differentiable, can we find constants $c_0,c_1,c_2,\cdots$ such that

$$f(x)=\sum_{n=0}^{\infty}c_nx^n?$$

$$\begin{aligned}f(x)&=\sum_{n=0}^{\infty}c_nx^n\\ &=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots\end{aligned}$$

Then,

- $f(0) = c_0 + c_1 \cdot 0 + c_2 \cdot 0 + \cdots = c_0$

$$\Rightarrow c_0 = f(0) = \frac{f(0)}{0!}$$

- $f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots$

$$\Rightarrow f'(0) = c_1 \;\Rightarrow\; c_1 = f'(0) = \frac{f'(0)}{1!}$$

- $f''(x) = 2c_2 + 2 \cdot 3c_3x + 3 \cdot 4c_4x^2 + \cdots$

$$\Rightarrow f''(0) = 2c_2 \;\Rightarrow\; c_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$

- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4x + \cdots$

$$\Rightarrow f'''(0) = 2 \cdot 3c_3 \;\Rightarrow\; c_3 = \frac{f'''(0)}{2 \cdot 3} = \frac{f'''(0)}{3!}$$

In general, if $f(x)$ has n continuous derivatives, then

$$c_n = \frac{f^{(n)}(0)}{n!}$$

Note:

- $f^{(n)}$ is the nth derivative of $f(x)$
- $f^{(0)}(x) = f(x)$
- $0! = 1$

Taylor series and Maclaurin series

A **Taylor series about** $x = a$ is defined as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

A **Maclaurin series** is a special case of a Taylor series with $a = 0$, i.e.,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

Example 1

Find a Maclaurin series for $f(x) = \ln(x+1)$.

	$f^{(n)}(x)$	$f^{(n)}(0)$
$n=0$	$f(x) = \ln(1+x)$	$f(0) = \ln(1) = 0$
$n=1$	$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$	$f'(0) = 1$
$n=2$	$f''(x) = -(1+x)^{-2}$	$f''(0) = -1$
$n=3$	$f'''(x) = 2(1+x)^{-3}$	$f'''(0) = 2$
$n=4$	$f^{(4)}(x) = -2\cdot 3(1+x)^{-4}$	$f^{(4)}(0) = -2\cdot 3 = -3!$
$n=5$	$f^{(5)}(x) = 2\cdot 3\cdot 4(1+x)^{-5}$	$f^{(5)}(0) = 2\cdot 3\cdot 4 = 4!$

$$\begin{aligned}
\therefore f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\
&= 0 + x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3!}{4!}x^4 + \frac{4!}{5!}x^5 + \cdots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \\
&= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}
\end{aligned}$$

Example 2

Find the radius and interval of convergence for the previous example.

Apply ratio test: $|a_n| = \dfrac{x^n}{n}$, $|a_{n+1}| = \dfrac{x^{n+1}}{n+1} = \dfrac{x^n\cdot x}{n+1}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^n\cdot x}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x\cdot n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x| < 1$$

$\therefore$ Radius of convergence: $R = 1$.

Check the end points:

- When $x = -1$:

  $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^{n-1}(-1)}{n} = \sum_{n=1}^{\infty}(-1)\frac{1}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$$

  Harmonic series – divergent.

- When $x = 1$:

  $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(1)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

  Alternating harmonic series – convergent

$\therefore$ Interval of convergence: $-1 < x \le 1$ OR $x \in (-1, 1]$

Example 3

Use the Taylor series

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, |x| < 1$$

to find a Taylor series (about $x = 0$) for $\ln(1+7x)$.

$$\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{u^n}{n}$$

Let $u = 7x$:

$$\begin{aligned}
\ln(1+7x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(7x)^n}{n}, \quad |7x| < 1 \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} (7)^n \frac{x^n}{n}, \quad |x| < \frac{1}{7}
\end{aligned}$$

Example 4

Find a Taylor series for $f(x) = \ln(x)$ about $x = 1$

	$f^{(n)}(x)$	$f^{(n)}(1)$
$n=0$	$f(x) = \ln(x)$	$f(1) = 0$
$n=1$	$f'(x) = \frac{1}{x} = x^{-1}$	$f'(1) = 1$
$n=2$	$f''(x) = -x^{-2}$	$f''(1) = -1$
$n=3$	$f'''(x) = 2x^{-3}$	$f'''(1) = 2$
$n=4$	$f^{(4)}(x) = -2 \cdot 3x^{-4}$	$f^{(4)}(1) = -2 \cdot 3 = -3!$
$n=5$	$f^{(5)}(x) = 2 \cdot 3 \cdot 4x^{-5}$	$f^{(5)}(1) = 2 \cdot 3 \cdot 4 = 4!$

$$\begin{aligned}
\therefore f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\
&= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\
&= 0 + (x-1) + \frac{-1}{21}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \frac{-3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5 + \cdots \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}
\end{aligned}$$

<u>Exercise</u>: Show that radius of converges is $R = 1$.

Example 5

Find a Taylor series for $f(x) = e^x$ about $x = 0$ and the radius of convergence of it.

	$f^{(n)}(x)$	$f^{(n)}(0)$
$n=0$	$f(x)=e^x$	$f(0)=1$
$n=1$	$f'(x)=e^x$	$f'(0)=1$
$n=2$	$f''(x)=e^x$	$f''(0)=1$
$n=3$	$f'''(x)=e^x$	$f'''(0)=1$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Apply the ratio test: $a_n = \dfrac{x^n}{n!}$, $a_{n+1} = \dfrac{x^{n+1}}{(n+1)!} = \dfrac{x^n \cdot x}{n!(n+1}$

$$\begin{aligned}\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{x^n \cdot x}{n!(n+1)} \cdot \frac{n!}{x^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{x}{n+1}\right| \\ &= |x| \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0 < 1 \quad \forall x\end{aligned}$$

$\therefore$ Radius of convergence: $R = \infty$.

Example 6

Find a Taylor series for $f(x) = \cos x$ about $x = 0$ and the radius of convergence of it.

	$f^{(n)}(x)$	$f^{(n)}(0)$
$n=0$	$f(x)=\cos x$	$f(0)=1$
$n=1$	$f'(x)=-\sin x$	$f'(0)=0$
$n=2$	$f''(x)=-\cos x$	$f''(0)=-1$
$n=3$	$f'''(x)=\sin x$	$f'''(0)=0$
$n=4$	$f^{(4)}(x)=\cos x$	$f^{(4)}(0)=1$
$n=5$	$f^{(5)}(x)=-\sin x$	$f^{(5)}(0)=0$
$n=6$	$f^{(6)}(x)=-\cos x$	$f^{(6)}(0)=-1$

$$\begin{aligned}\therefore f(x) &= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= 1 + 0 \cdot x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

Apply the ratio test: $|a_n| = \frac{x^{2n}}{(2n)!}$, $|a_{n+1}| = \frac{x^{2(n+1)}}{(2(n+1))!} = \frac{x^{2n+2}}{(2n+1)!} = \frac{x^{2n}\cdot x^2}{(2n)!(2n+1)(2n+2)}$.

$$\begin{aligned}\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2n}\cdot x^2}{(2n)!(2n+1)(2n+2)}\cdot\frac{(2n)!}{x^{2n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n+2)}\right| \\ &= |x^2|\lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n+2)}\right| = 0 < 1 \quad \forall x\end{aligned}$$

$\therefore$ Radius of convergence: $R = \infty$.

Important Taylor/Maclaurin Series

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty}(-1)^n\frac{X^{2n+1}}{(2n+1)!}\end{aligned}$$

Note: They converges for all x

Example 7

Find a Taylor series about $x = 0$ for $f(x) = xe^{x^2}$.

$$e^u = \sum_{n=0}^{\infty}\frac{u^n}{n!} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots$$

Let $u = x^2$:

$$e^{x^2} = \sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

$$\therefore f(x) = xe^{x^2} = x\sum_{n=0}^{\infty}\frac{x^{2n}}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!}$$

Example 8

Find the sum of the series $\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{6^{2n}(2n)!}$.

$$\sum_{n=1}^{\infty}(-1)^n\frac{\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$$

Recall: $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\therefore \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

4.11 Application of Taylor Polynomials

nth degree Taylor Polynomial Approximation

If a Taylor series for $f(x)$ about $x = a$ is given as

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \end{aligned}$$

then the **nth degree Taylor polynomial approximation of** $f(x)$ **about** $x = a$ is

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Note: $T_n(x)$ is a polynomial approximation of $f(x)$:

$$T_n(x) \approx f(x)$$

Example 1

(a) State the Taylor series for $f(x) = e^x$ about $x = 0$.

$$f(x) = e^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(b) Find the degree 1 Taylor polynomial approximation of $f(x)$ about $x = 0$.

$$T_1(x) = 1 + x$$

(c) Find the degree 3 Taylor polynomial approximation of $f(x)$ about $x = 0$.

$$\begin{aligned} T_3(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \end{aligned}$$

Question: How good are Taylor polynomial approximations? (Consider $T_1(x)$)

- Taylor series about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- Consider $T_1(x)$:

$$\boxed{T_1(x) = f(a) + f'(a)(x-a)}$$

- Assume $|f''(x)| \leq M$ for $|x-a| \leq d$

$$\therefore -M \leq f''(x) \leq M$$

Illustration

$$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x$$

$$\therefore |f''(x)| = |-\sin x| \leq 1$$

Take $M = 1$

- Because $f''(x) \leq M$, the inequality still holds if we integrate:

$$\begin{aligned} \int_a^x f''(t)\,\mathrm{d}t &\leq \int_a^x M\,\mathrm{d}t \\ \Big[f(t)\Big]_a^x &\leq \left[f'(a)t + \frac{M}{2}(t-a)^2\right]_a^x \\ f(x) - f(a) &\leq f'(a)(x-a) + \frac{M}{2}(x-a)^2 - \frac{M}{2}(a-a)^2 \\ f(x) &\leq \underbrace{\boxed{f(a) + f'(a)(x-a)}}_{T_1(x)} + \frac{M}{2}(x-a)^2 \\ \therefore f(x) &\leq T_1(x) + \frac{M}{2}(x-a)^2 \\ f(x) - T_1(x) &\leq \frac{M}{2}(x-a)^2 \end{aligned}$$

$\therefore$ The remainder $|R_1(x)|$:

$$|f(x) - T_1(x)| \leq \frac{M}{2}|x-a|^2$$

Taylor Remainder Inequality

Suppose

- the Taylor series for $f(x)$ is given by:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

- and the nth degree Taylor polynomial approximate is given by:

$$f(x)\approx T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $|f^{(n+1)}(x)|\le M$ for $|x-a|\le d$, then

$$|R_n(x)|=|f(x)-T_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1},\quad |x-a|\le d$$

Example 2

Consider $f(a)=\cos x$

(a) Find a 4th degree Taylor polynomial approximation to $f(x)$ about $x=\dfrac{\pi}{3}$ (or equivalently, at $a=\dfrac{\pi}{3}$).

$$\begin{aligned}T_4(x)&=\sum_{k=0}^{4}\frac{f^{(k)}(a)}{k!}(x-a)^k\\&=f\left(\frac{\pi}{3}\right)+f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right)+\frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2+\frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2\\&\quad+\frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3+\frac{f^{(4)}\left(\frac{\pi}{3}\right)}{4!}\left(x-\frac{\pi}{3}\right)^4\end{aligned}$$

	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
$n=0$	$f(x)=\cos x$	$f\left(\frac{\pi}{3}\right)=\frac{1}{2}$
$n=1$	$f'(x)=-\sin x$	$f'\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$
$n=2$	$f''(x)=-\cos x$	$f''\left(\frac{\pi}{3}\right)=-\frac{1}{2}$
$n=3$	$f'''(x)=\sin x$	$f'''\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$
$n=4$	$f^{(4)}(x)=\cos x$	$f^{(4)}\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$$\begin{aligned}\therefore T_4(x)&=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{2}\cdot\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{2}\cdot\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{2}\cdot\frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4\\&\boxed{=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4}\end{aligned}$$

(b) How good is the approximation when $0 \le x \le \frac{2\pi}{3}$
That is, use the Taylor inequality to find an upper bond.
The remainder:
$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}, \; |f^{(n+1)}(x)| \le M$$
Here $n = 4, \; n+1 = 5, \; a = \frac{\pi}{3}$:
$$|R_4(x)| \le \frac{M}{5!}\left|x - \frac{\pi}{3}\right|^5, \text{ where } |f^{(5)}(x)| \le M$$
To find M:
$$f(x) = \cos x, \; f'(x) = -\sin x, \; \cdots, \; f^{(4)}(x) = \cos x, \; f^{(5)}(x) = -\sin x.$$
$$\therefore |f^{(5)}(x)| = |-\sin x| \le 1, \text{ for } 0 \le x \le \frac{2\pi}{3} \;\Rightarrow\; M = 1$$
So far, we have
$$|R_4(x)| \le \frac{1}{5!}\left|x - \frac{\pi}{3}\right|^5, \; 0 \le x \le \frac{2\pi}{3}$$
Now, we need to ask: How large can $\left|x - \frac{\pi}{3}\right|^5$ be on $0 \le x \le \frac{2\pi}{3}$?
This is a maximum problem on a closed interval.
$\therefore$ The maximum occurs at a critical point of $g(x) = \left|x - \frac{\pi}{3}\right|^5$ or at one of the end points.
Recall: Try $g(x) = |u| = \sqrt{u^2} = (u^2)^{1/2}$
$$\frac{dg}{dx} = \frac{d}{dx}\left[\left|x - \frac{\pi}{3}\right|\right] = \left|x - \frac{\pi}{3}\right|^4 \frac{d}{dx}\left[\left|x - \frac{\pi}{3}\right|\right] = 5\left|x - \frac{\pi}{3}\right|^4 \frac{(x - \frac{\pi}{3})}{|x - \frac{\pi}{3}|} = 5\left|x - \frac{\pi}{3}\right|^3\left(x - \frac{\pi}{3}\right)$$
$\therefore$ Critical points: $x = \frac{\pi}{3}$
$$g\left(\frac{\pi}{3}\right) = 0$$
$$g(0) = \left|0 - \frac{\pi}{3}\right|^5 = \left(\frac{\pi}{3}\right)^5$$
$$g\left(\frac{2\pi}{3}\right) = \left|\frac{2\pi}{3} - \frac{\pi}{3}\right|^5 = \left(\frac{\pi}{3}\right)^5$$
$$\therefore |R_4(x)| \le \frac{1}{5!}\left(\frac{\pi}{3}\right)^5$$