



1 Statements

1.1 Class Handout, Chapter 1.3, Implications.

Let a , b , and c be integers, with a and b non-zero. If $(ab) \mid (ac)$, then $b \mid c$.

Proof 1. Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$.

Suppose $(ab) \mid (ac)$. Then $\exists k \in \mathbb{Z}$ s.t. $ac = (ab)k$.

Divide both sides of the equation by a :

$$c = bk.$$

Since $k \in \mathbb{Z}$, by definition of divides, $b \mid c$.

1.2 Class Handout, Chapter 1.4, Contrapositive and Converse

Prove that for all real numbers a and b , if $a \in \mathbb{Q}$ and $ab \notin \mathbb{Q}$, then $b \notin \mathbb{Q}$.

Proof 2. Let $a, b \in \mathbb{Q}$.

Assume for the sake of contradiction that if $a \in \mathbb{Q}$ and $ab \notin \mathbb{Q}$, we have $b \in \mathbb{Q}$.

Then, $\exists p, q, m, n \in \mathbb{Z}$ s.t. $a = \frac{m}{n}$ and $b = \frac{p}{q}$.

Hence,

$$ab = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$$

As $mp, nq \in \mathbb{Z}$, $ab \in \mathbb{Q}$.

✱ This contradicts with the fact that $ab \notin \mathbb{Q}$.

So, b must not be rational.

1.3 Chapter 1.1 # 7(c)

Prove the square of an even integer is divisible by 4.

Proof 3. Suppose $x \in \mathbb{Z}$ is even. Then $\exists k \in \mathbb{Z}$ s.t. $x = 2k$.

Then, $x^2 = (2k)^2 = 4k^2$.

Since $k^2 \in \mathbb{Z}$, we have $4 \mid 4k^2$.

Theorem 1.1 (Archimedean Principle). *For every real number x , there is an integer n , such that $n > x$.*

1.4 Chapter 1.1 # 11

For every positive real number ε , there exists a positive integer N such that $\frac{1}{n} < \varepsilon$ for all $n \geq N$.

Proof 4. Suppose $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$.

Since $\varepsilon \in \mathbb{R}$, we have $\frac{1}{\varepsilon} \in \mathbb{R}$.

Then, by Archimedean Principle, $\exists n \in \mathbb{Z}$ s.t. $n > \frac{1}{\varepsilon}$.

Hence, $n\varepsilon > 1$ or $\varepsilon > \frac{1}{n}$.

Suppose $N \in \mathbb{Z}$ s.t. $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$

$\left\lceil \frac{1}{\varepsilon} \right\rceil$ means the integer greater to $\frac{1}{\varepsilon}$ if $\frac{1}{\varepsilon} \notin \mathbb{Z}$, and the integer equals to $\frac{1}{\varepsilon}$ if $\frac{1}{\varepsilon} \in \mathbb{Z}$.

Hence, $N \geq \frac{1}{\varepsilon}$.

As $n > \frac{1}{\varepsilon}$, we have $n \geq N$

1.5 Chapter 1.1 # 12

Use the Archimedean Principle (Theorem 1.1) to prove if x is a real number, then there exists a positive integer n such that $-n < x < n$.

Proof 5. Suppose $x \in \mathbb{R}$.

Case 1 If $x > 0$, then $-x < 0$ (i.e., $-x < 0 < x$).

By the Archimedean Principle, $\exists n \in \mathbb{Z}$ s.t. $n > x$.

Multiply (-1) on both sides of the inequality:

$$-n < -x$$

As $-x < 0 < x$,

$$-n < -x < 0 < x < n,$$

which means $-n < x < n$, and n is positive.

Case 2 If $x < 0$, then $-x > 0$ (i.e., $-x > 0 > x$)

Since $x \in \mathbb{R}$, we have $-x \in \mathbb{R}$.

By the Archimedean Principle, $\exists n \in \mathbb{Z}$ s.t. $n > -x$.

Multiply (-1) on both sides of the inequality:

$$-n < x$$

As $x < 0 < -x$,

$$-n < x < 0 < -x < n,$$

which means $-n < x < n$, and n is positive.

In all cases, we have proven that $x \in \mathbb{R} \implies \exists n \in \mathbb{Z}, n > 0$ s.t. $-n < x < n$.

1.6 Chapter 1.1 # 13

Prove that if x is a positive real number, then there exists a positive integer n such that $\frac{1}{n} < x < n$.

Proof 6. Suppose $x \in \mathbb{R}, x > 0$

Case 1 If $0 < x \leq 1$, then $\frac{1}{x} \geq 1$.

Hence, $x \leq 1 \leq \frac{1}{x}$.

As $x \in \mathbb{R}, \frac{1}{x} \in \mathbb{R}$, then by the Archimedean Principle (Theorem 1.1):

$$\exists n \in \mathbb{Z} \text{ s.t. } n > \frac{1}{x}.$$

Hence, $nx > 1$ or $x > \frac{1}{n}$.

As $x \leq \frac{1}{x}$, $n > \frac{1}{x}$, and $x > \frac{1}{n}$, we have

$$\frac{1}{n} < x < n.$$

Case 2 If $x > 1$, then $0 < \frac{1}{x} < 1$.

Hence, $\frac{1}{x} < 1 < x$.

As $x \in \mathbb{R}$, by the Archimedean Principle:

$$\exists n \in \mathbb{Z} \text{ s.t. } n > x > 0$$

Hence, $\frac{1}{n} < \frac{1}{x}$

As $\frac{1}{x} < x$, $\frac{1}{n} < \frac{1}{x}$, and $n > x$, we have

$$\frac{1}{n} < x < n$$

In all cases, we proven that $x \in \mathbb{R}$, $x > 0 \implies \exists n \in \mathbb{Z}, n > 0$ s.t. $\frac{1}{n} < x < n$.

1.7 Handout Chapter 1.4-2 More Contradictions and Equivalence

There are no positive integer solutions to the equation $x^2 - y^2 = 10$.

Proof 7. Assume for the sake of contradiction that there are positive integer solutions to the equation $x^2 - y^2 = 10$.

Suppose $\exists x, y \in \mathbb{Z}$ and $x > 0$, $y > 0$ s.t. $x^2 - y^2 = 10$.

Then, we have $x^2 = 10 + y^2$.

Since $x > 0$, $x^2 > 0$, we have $10 + y^2 > 0$.

Then, $y^2 > -10$.

* This contradicts with the fact that $y^2 \geq 0$ if $y \in \mathbb{Z}$.

So, our assumption is wrong. There must be no positive integer solutions to the equation $x^2 - y^2 = 10$.

1.8 Handout Chapter 1.4-2 More Contradictions and Equivalence

Show that if $a \in \mathbb{Q}$ and $b \in \mathbb{Q}'$, then $a + b \in \mathbb{Q}'$

Remark. The notation \mathbb{Q} means the set for rational numbers, and \mathbb{Q}' means the set for irrational numbers.

Proof 8. Suppose $a \in \mathbb{Q}$ and $b \in \mathbb{Q}'$

Assume for the sake of contradiction that $a + b \in \mathbb{Q}$.

Then, $\exists m, n, p, q \in \mathbb{Z}$ such that $a = \frac{m}{n}$ and $a + b = \frac{p}{q}$.

Then,

$$b = \frac{p}{q} - a = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} \in \mathbb{Q}$$

Since $pn - mq \in \mathbb{Q}$ and $qn \in \mathbb{Z}$, we have $b = \frac{pn - mq}{qn} \in \mathbb{Q}$.

* This contradicts with the fact that $b \in \mathbb{Q}'$.

So, $a + b$ must be irrational.

1.9 Handout Chapter 1.4-2 More Contradictions and Equivalence

If $n \in \mathbb{N}$ and $2^n - 1$ is prime, then n is prime.

Proof 9. We will prove the contrapositive: if n is not prime, then $2^n - 1$ is not prime.

Suppose n is not prime. Then, $\exists a, b \in \mathbb{Z}$ with $1 < a, b < n$ s.t. $n = ab$.

Then, $2^n - 1 = 2^{ab} = (2^a)^b - 1$.

Notice that for $x^w - 1$, by polynomial long division, have

$$x^w - 1 = (x - 1)(x^{w-1} + x^{w-2} + \cdots + 1),$$

Substitute $x = 2^a$ and $w = b$, we have

$$2^n - 1 = (2^a - 1) \left[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 1 \right].$$

Since $(2^a - 1) \in \mathbb{Z}$ and $\left[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 1 \right] \in \mathbb{Z}$, we see that $2^n - 1$ is not prime.

1.10 Exam 1 Review 1-b-i

Prove that $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$.

Proof 10.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

1.11 Exam 1 Review 1-b-ii

Prove that $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.

Proof 11.

P	Q	$P \Rightarrow Q$	$Q \wedge (P \Rightarrow Q)$	$[Q \wedge (P \Rightarrow Q)] \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

1.12 Exam 1 Review 2-a

Given statements P and Q , prove $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

Proof 12.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

1.13 Exam 1 Review 2-b

There is no smallest integer.

Proof 13. Assume for the sake of contradiction that there exists a smallest integer n . Hence, $\forall x \in \mathbb{Z}$, we have $x \geq n$.

Notice that if $n > 0$, we have $0 \in \mathbb{Z}$ and $0 < n$.

Hence, $n = 0$ cannot be the smallest integer (\ast)

Therefore, n must be smaller than 0.

Suppose $m = -n$.

Since $n \in \mathbb{Z}$, $m = -n \in \mathbb{Z} \in \mathbb{R}$

By the Archimedean Principle (Theorem 1.1), $\exists k \in \mathbb{Z}$ s.t. $k > m$.

Hence, $k > -n$.

Multiply (-1) on both sides of the inequality:

$$-k < n.$$

As $k \in \mathbb{Z}$, $-k \in \mathbb{Z}$. Then $\exists -k \in \mathbb{Z}$ s.t. $-k < n$.

\ast This contradicts with our assumption that n is the smallest integer.

Hence, our assumption must be wrong. There is no smallest integer.

1.14 Exam 1 Review 2-c

The number $\log_2 3$ is irrational.

Proof 14. Assume for the sake of contradiction that $\log_2 3$ is irrational.

By definition, $\exists p, q \in \mathbb{Z}$, with $q \neq 0$ s.t. $\log_2 3 = \frac{p}{q}$.

Observe that $\log_2 3 \neq 0$. Then $p \neq 0$ as well.

By definition of logarithm,

$$2^{p/q} = 3$$

$$(2^p)^{1/q} = 3$$

Raise two sides of the equation to the power of q :

$$2^p = 3^q$$

As $p \neq 0$ and $q \neq 0$, 2^p and 3^q are not 1 $\forall p, q \in \mathbb{Z}$.

Hence, 2^p is even $\forall p \in \mathbb{Z}$ and 3^q is odd $\forall q \in \mathbb{Z}$.

* This contradicts with the fact that an even number cannot equal to an odd number.

Hence, our assumption is wrong. The number $\log_2 3$, then, must be irrational.

1.15 Exam 1 Review 2-d

There is a rational number a and an irrational number b such that a^b is rational.

Proof 15. Observe that 1 is a rational number and π is an irrational number.

Suppose $a = 1$ and $b = \pi$, we have $a^b = a^\pi = 1$, which is irrational.

Proof 16. Recall that we have proven in the previous proof, we have proven that $\log_2 3$ is an irrational number.

Recall the definition of logarithm and exponents, we have

$$2^{\log_2 3} = 3$$

Hence, we find a pair of a and b that satisfies the requirement.

1.16 Exam 1 Review 2-e

For all integers n , the number $n + n^2 + n^3 + n^4$ is even.

Proof 17. Suppose $n \in \mathbb{Z}$.

Case 1 If n is even.

Suppose $n = 2k$ f.s. $k \in \mathbb{Z}$.

Then,

$$\begin{aligned} n + n^2 + n^3 + n^4 &= (2k) + (2k)^2 + (2k)^3 + (2k)^4 \\ &= 2k + 4k^2 + 8k^3 + 16k^4 \\ &= 2(k + 2k^2 + 4k^3 + 8k^4) \end{aligned}$$

Since $(k + 2k^2 + 4k^3 + 8k^4) \in \mathbb{Z}$, we have $2(k + 2k^2 + 4k^3 + 8k^4)$ is even.

Hence, $n + n^2 + n^3 + n^4$ is even when n is even.

Case 2 If n is odd.

Suppose $n = 2k + 1$ f.s. $k \in \mathbb{Z}$.

Then,

$$\begin{aligned} n + n^2 + n^3 + n^4 &= (2k + 1) + (2k + 1)^2 + (2k + 1)^3 + (2k + 1)^4 \\ &= 2k + 1 + 4k^2 + 4k + 1 + 8k^3 + 12k^2 + 6k + 1 + 16k^4 + 32k^3 + 24k^2 + 8k + 1 \\ &= 16k^4 + 40k^3 + 40k^2 + 20k + 4 \\ &= 2(8k^4 + 20k^3 + 20k^2 + 10k + 2) \end{aligned}$$

Since $(8k^4 + 20k^3 + 20k^2 + 10k + 2) \in \mathbb{Z}$, we have $2(8k^4 + 20k^3 + 20k^2 + 10k + 2)$ is even.

Hence, $n + n^2 + n^3 + n^4$ is even when n is odd.

Since integers can either be even or odd, and we have proven $n + n^2 + n^3 + n^4$ is even in either case, $n + n^2 + n^3 + n^4$ is even for all integers.

Definition 1.1 (Perfect Square). A perfect square is an integer n for which there exists an integer m such that $n = m^2$.

1.17 Exam 1 Review 2-f

If n is a positive integer such that n is in the form $4k + 2$ or $4k + 3$, then n is not a perfect square.

Proof 18. We will prove the contrapositive of the statement: “If n is a perfect square, then n is a positive integer of the form $4k$ or $4k + 1$ f.s. $k \in \mathbb{Z}$.”

Suppose n to be a perfect square, then $\exists m \in \mathbb{Z}$ s.t. $n = m^2$.

Case 1 Suppose m is even, then $m = 2t$ f.s. $t \in \mathbb{Z}$.

$$n = m^2 = (2t)^2 = 4t^2 > 0.$$

Let $k = t^2$. Since $t^2 \in \mathbb{Z}$, we have $k \in \mathbb{Z}$.

Hence, n is positive and is in the form of $4k$.

Case 2 Suppose m is odd, then $m = 2t + 1$ f.s. $t \in \mathbb{Z}$.

$$n = m^2 = (2t + 1)^2 = 4t^2 + 4t + 1 = 4(t^2 + t) + 1 > 1$$

Let $k = t^2 + t$. Since $(t^2 + t) \in \mathbb{Z}$, we have $k \in \mathbb{Z}$.

Hence, n is in the form of $4k + 1$.

Hence, we prove the contrapositive of the original statement to be true, which means our original statement is also true.

1.18 Exam 1 Review 2-g

For any integer n , $3 \mid n$ if and only if $3 \mid n^2$.

Proof 19. Suppose $n \in \mathbb{Z}$.

(\Rightarrow) Suppose $3 \mid n$. Then, $\exists k \in \mathbb{Z}$ s.t. $n = 3k$.

$$\text{Then, } n^2 = (3k)^2 = 9k^2 = 3(3k^2).$$

Since $3k^2 \in \mathbb{Z}$, by definition, $3 \mid n^2$.

(\Leftarrow) WTS: $3 \mid n^2 \implies 3 \mid n$

We will prove the contrapositive: If $3 \nmid n$, then $3 \nmid n^2$

Suppose $3 \nmid n$.

Case 1 Suppose $n = 3m + 1$ f.s. $m \in \mathbb{Z}$.

$$\text{Then, } n^2 = (3m + 1)^2 = 9m^2 + 6m + 1$$

Since $9m^2 + 6m + 1$ cannot be written in the form of $3k$ f.s. $k \in \mathbb{Z}$, by definition, $3 \nmid n^2$.

Case 2 Suppose $n = 3m + 2$ f.s. $m \in \mathbb{Z}$

$$\text{Then, } n^2 = (3m + 2)^2 = 9m^2 + 12m + 4$$

Since $9m^2 + 12m + 4$ cannot be written in the form of $3k$ for some $k \in \mathbb{Z}$, by definition, $3 \nmid n^2$.

Hence, we proved the contrapositive, and thus the original statement is true.

Therefore, $n \mid n \iff 3 \mid n^2$.

1.19 Exam 1 Review 2-h

There exists an integer n such that $12 \mid n^2$ but $12 \nmid n$.

Proof 20. Prove.

1.20 Exam 1 Review 2-i

For every integer a , the numbers a and $(a+1)(a-1)$ have opposite parity.

Proof 21. Prove.

1.21 Exam 1 Review 2-j

Suppose $x \in \mathbb{R}$. If x^2 is irrational, then x is irrational.

Proof 22. Prove.

1.22 Exam 1 Review 2-k

For any integers a and b , if ab is even, then a is even or b is even.

Proof 23. Prove.

1.23 Exam 1 Review 2-l

For $n \in \mathbb{N}$, n , $n+2$, and $n+4$ are all prime if and only if $n=3$.

Proof 24. Prove.

1.24 Exam 1 Review 3-a

Prove or disprove: Every real number is less than or equal to its square.

Disproof 25.

Prove.

■

1.25 Exam 1 Review 3-b

Prove or disprove: The sum of two integers is never equal to their product.

Disproof 26.

Prove. ■

1.26 Exam 1 Review 3-c

Prove or disprove: There exists a non-zero integer whose cube equals its negative.

Disproof 27.

Prove. ■

1.27 Exam 1 Review 3-d

Prove or disprove: For all $x \in \mathbb{R}$, $x \leq x^2$ or $0 \leq x < 1$.

Proof 28. Prove.

1.28 Chapter 1.4 # 20-a

Let n be an integer. Prove that n is even if and only if n^3 is even.

Proof 29. Prove.

1.29 Chapter 1.4 # 20-b

Let n be an integer. Prove that n is odd if and only if n^3 is odd.

Proof 30. Prove.

1.30 Chapter 1.4 # 21

Prove that $\sqrt[3]{2}$ is irrational.

Proof 31. Prove.

2 Sets

2.1 Handout Chapter 2.1 - Sets and Subsets

Prove that $\{12a + 4b \mid a, b \in \mathbb{Z}\} = \{4c \mid c \in \mathbb{Z}\}$.

Proof 1. (\subseteq) Suppose $x \in \{12a + 4b \mid a, b \in \mathbb{Z}\}$

Then, $x = 12a + 4b$ f.s. $a, b \in \mathbb{Z}$.

$$x = 12a + 4b = 4(3a + b)$$

As $3a + b \in \mathbb{Z}$, we have $x \in \{4c \mid c \in \mathbb{Z}\}$

By definition, $\{12a + 4b \mid a, b \in \mathbb{Z}\} \subseteq \{4c \mid c \in \mathbb{Z}\}$

(\supseteq) Suppose $x \in \{4c \mid c \in \mathbb{Z}\}$

Then, $x = 4c$ f.s. $c \in \mathbb{Z}$.

Suppose $c = 3a + b$ f.s. $a, b \in \mathbb{Z}$.

Then, $x = 4c = 4(3a + b) = 12a + 4b$.

By definition, $\{4c \mid c \in \mathbb{Z}\} \subseteq \{12a + 4b \mid a, b \in \mathbb{Z}\}$

Hence, we have proven $\{12a + 4b \mid a, b \in \mathbb{Z}\} = \{4c \mid c \in \mathbb{Z}\}$.

2.2 Exam 1 Review 2-m

If $A = \{x \mid x = n^4 - 1, n \in \mathbb{Z}\}$ and $B = \{x \mid x = m^2 - 1, m \in \mathbb{Z}\}$, then $A \subseteq B$.

Proof 2. Prove.

2.3 Exam 1 Review 2-n

If A , B , and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof 3. Prove.

2.4 Exam 1 Review 2-o

For subsets A and B of a universal set U , $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof 4. Prove.

2.5 Exam 1 Review 2-p

Suppose that A , B , and C are subsets of a universal set U . Let P and Q be the following statements:

P : $A \subseteq B$ or $A \subseteq C$; and

Q : $A \subseteq B \cap C$.

Write the statement $P \Rightarrow Q$, its converse and contrapositive. Prove the truth statements among these or give counterexamples.

Proof 5. Prove.

2.6 Handout Chapter 2.2-Combining Sets

Let $A = \{6a + 4 \mid x \in \mathbb{Z}\}$ and $B = \{18b - a \mid b \in \mathbb{Z}\}$. Prove or disprove: $A \subseteq B$.

Disproof 6.

Prove.

■

2.7 Handout Chapter 2.2-Combining Sets

Let $A = \{6a + 4 \mid x \in \mathbb{Z}\}$ and $B = \{18b - a \mid b \in \mathbb{Z}\}$. Prove or disprove: $B \subseteq A$.

Proof 7. Prove.

2.8 Handout Chapter 2.2-Combining Sets

If A and B are sets, then $\mathcal{P}(A) - \mathcal{P}(B) = \mathcal{P}(A - B)$.

Proof 8. Prove.

2.9 Handout Chapter 2.2-Combining Sets

If A , B , and C are sets, and $A \times B = B \times C$, then $A = B$.

Proof 9. Prove.

2.10 Chapter 2.1 # 6

Let $n \in \mathbb{Z}$ and let $A = n\mathbb{Z}$. Prove that if $x, y \in A$, then $x + y \in A$ and $xy \in A$.

Proof 10. Prove.