

Emory University

# MATH 212 Differential Equations Learning Notes

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# 1 First Order ODEs

## 1.1 Introduction

**Definition 1.1.1 (Ordinary Differential Equations/ODEs).** An *ordinary differential equation* is an equation that contains one or more derivatives of an unknown function  $y = y(x)$ .

**Definition 1.1.2 (Order of ODEs).** The *order* of an ODE is the maximum order of the derivatives appearing in the equation.

**Definition 1.1.3 (Solution to ODEs).** The *solution* to an ODE is a function  $y$  that satisfies the equation.

**Example 1.1.4** Solve  $y'' = 3x + 1$ .

**Solution 1.**

$$y' = \int 3x + 1 \, dx = \frac{3}{2}x^2 + x + C$$
$$y = \int y' \, dx = \int \left( \frac{3}{2}x^2 + x + C \right) dx = \frac{1}{2}x^3 + \frac{1}{2}x^2 + Cx + D.$$

□

**Definition 1.1.5 (Linear ODEs/Non-Linear ODEs).** A first order ODE is *linear* if it can be written as

$$y' + p(x)y = f(x).$$

Otherwise, it is *non-linear*.

**Definition 1.1.6 (Homogenous/Non-Homogenous Linear ODEs).** If  $f(x) = 0$ , then the linear ODE is *homogenous*. That is,

$$y' + p(x)y = 0.$$

Otherwise, it is *non-homogenous*.

**Definition 1.1.7 (Trivial/Non-Trivial Solution).**  $y = 0$  is a *trivial solution* to a homogenous ODE. Any other solutions are *non-trivial*.

**Definition 1.1.8 (One-Parameter Family of Solutions).** We call  $C$  a *parameter* and the equation, therefore solution, defines a *one-parameter family* of solutions.

**Example 1.1.9** For the ODE  $y' = 1$ ,  $y_1 = x + C_1$  is a solution to it, and it is a one-parameter family of solutions. Similarly, for  $y' = \frac{1}{x^2}$ , the one-parameter families of solutions are defined by  $y_2 = -\frac{1}{x} + C_2$  on the interval  $(-\infty, 0) \cup (0, \infty)$ .

**Definition 1.1.10 (General Solution).** Given the general form of the linear ODE  $y' + p(x)y =$

$f(x)$  if  $p$  and  $f$  are continuous on some open interval  $(a, b)$  and there is a unique formula  $y = y(x, c)$  and we have the following properties:

- for each fixed  $c$ , the resulting function of  $x$  is a solution of the ODE on  $(a, b)$ , and
- if  $y$  is a solution of the ODE, then  $y$  can be obtained by choosing the value of  $c$  appropriately.

The function  $y = y(x, c)$  is called a *general solution*.

More generally, we can write an ODE as

$$P_0(x)y' + P_1(x)y = F(x).$$

In this case, the ODE has a general solution on any open interval in which  $P_0$ ,  $P_1$ , and  $F$  are continuous and  $P_0 \neq 0$ .

**Definition 1.1.11 (Initial Value Problem (IVP)).** A differential equation with an initial condition.

**Example 1.1.12** Let  $a$  be a constant. Find the general solution of  $y' - ay = 0$  and solve

the IVP  $\begin{cases} y' - ay = 0 \\ y(x_0) = y_0. \end{cases}$

**Solution 2.**

Classification: First order, Linear, Homogeneous.

Trivial Solution:  $y = 0$ .

General solution:

$$\begin{aligned} \frac{dy}{dx} &= ay \\ \int \frac{1}{y} dy &= \int a dx \\ \ln |y| &= ax + c \\ y &= e^{ax+c} = Ae^{ax}. \end{aligned}$$

*This general solution includes the trivial solution.*

IVP: Substitute  $x = x_0$  and  $y = y_0$ :

$$y_0 = Ae^{ax_0} \longrightarrow A = y_0 e^{-ax_0}$$

So,

$$y^{\text{IVP}} = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

*This IVP is a “generic initial condition.” We need more information on  $x_0, y_0$  to get a more specific solution.* □

## 1.2 Linear First Order ODEs

### Theorem 1.2.1

If  $p$  is continuous on  $(a, b)$ , then the general solution of the homogeneous equation  $y' + p(x)y = 0$  on  $(a, b)$  is given by

$$y = ce^{-\int p(x) dx}.$$

### Proof 1.

(a). Substitute the solution formula to show that  $y = ce^{-\int p(x) dx}$  is a solution for any choice of  $c$ .

$$y' = c \left( -\int p(x) dx \right)' e^{-\int p(x) dx} = -cp(x)e^{-\int p(x) dx}.$$

Then,

$$y' + p(x)y = -cp(x)e^{-\int p(x) dx} + cp(x)e^{-\int p(x) dx} = 0.$$

So,  $y = ce^{-\int p(x) dx}$  is a solution for any choice of  $c$ .  $\square$

(b). Want to show: any solution of  $y' + p(x)y = 0$  can be written as  $y = ce^{-\int p(x) dx}$ . Note that  $y = 0$  is a trivial solution, so we assume  $y \neq 0$ .

$$\begin{aligned} y' + p(x)y &= 0 \\ y' &= -p(x)y \\ \frac{y'}{y} &= -p(x) \\ \rightsquigarrow \int \frac{1}{y} dy &= \int -p(x) dx \\ \ln |y| &= -\int p(x) dx \\ y &= ce^{-\int p(x) dx}. \end{aligned}$$

Note that when  $c = 0$ ,  $y = 0$  is the trivial solution. So, any solution of  $y' + p(x)y = 0$  can be written as  $y = ce^{-\int p(x) dx}$ .  $\blacksquare$

### Example 1.2.2 Solve the IVP

$$\begin{cases} xy' + y = 0 \\ y(1) = 3. \end{cases}$$

### Solution 2.

Note that  $P_0(x) = x$  and  $P_1(x) = 1$ , which are continuous on  $\mathbb{R}$ . Since we need  $P_0(x) \neq 0$ ,  $x \neq 0$ . So the interval of validity is  $\mathbb{R} \setminus \{0\}$ .

Method 1: Separation of Variables

$$y' = -\frac{y}{x}.$$

Note that  $y = 0$  is a solution. Assume  $y \neq 0$ .

$$\begin{aligned}\frac{y'}{y} = -\frac{1}{x} &\rightsquigarrow \int \frac{1}{y} dy = -\int \frac{1}{x} dx + k \\ \ln |y| &= -\ln |x| + k \\ |y| &= e^k \frac{1}{|x|} \\ y &= \frac{c}{x}\end{aligned}$$

**Method 2: Solution Formula** By Theorem 1.2.1,

$$y = ce^{-\int p(x) dx} = ce^{-\int \frac{1}{x} dx} = ce^{-\ln |x|} = \frac{c}{x}.$$

**Solving the IVP** Substitute  $x = 1$  and  $y = 3$ :

$$3 = \frac{c}{1} \longrightarrow c = 3.$$

So,  $y^{\text{IVP}} = \frac{3}{x}.$

□

**Example 1.2.3** Given the equation  $(4 + x^2)y' + 2xy = 4x$ . Classify the equation and find the general solution  $y = y(x, c)$ .

**Solution 3.**

This is a first order, linear, non-homogeneous differential equation.

Note that  $P_0(x) = 4 + x^2$ ,  $P_1(x) = 2x$ ,  $F(x) = 4x$ , and  $P_0 \neq 0 \forall x \in \mathbb{R}$ , so the interval of validity is  $\mathbb{R}$ . Also note that  $\frac{d}{dx}[4 + x^2] = 2x$ , so the equation can be written as

$$(4 + x^2)\frac{dy}{dx} + \frac{d}{dx}[4 + x^2]y = 4x.$$

Using the product rule to re-write the LHS as

$$\begin{aligned}\frac{d}{dx}[(4 + x^2)y] &= 4x \\ \int \frac{d}{dx}[(4 + x^2)y] dx &= \int 4x dx + c \\ (4 + x^2)y &= 2x^2 + c \\ y &= \frac{2x^2 + c}{4 + x^2}.\end{aligned}$$

□

**Example 1.2.4** Given the equation  $y' - 2y = 4 - x$ . Classify the equation and find the general solution  $y = y(x, c)$ .

**Solution 4.**

This is a first order, linear, non-homogeneous differential equation.

Since  $P_0(x) = 1$ ,  $P_1(x) = -2y$ ,  $F(x) = 4 - x$ , and  $P_0(x) \neq 0 \forall x \in \mathbb{R}$ , the interval of validity is  $\mathbb{R}$ . Consider  $\mu = \mu(x) \neq 0$ . Multiply both sides of the equation by  $\mu(x)$ :

$$\mu(x)y' - 2\mu(x)y = \mu(x)(4 - x) \quad (1)$$

To make the LHS a product rule, we need

$$\frac{d}{dx} [\mu(x)y(x)] = \mu'(x)y(x) + \mu(x)y'(x) = \mu(x)y'(x) - 2\mu(x)y.$$

So, we have  $\mu' = -2\mu$ , or  $\mu' + 2\mu = 0$ , a first order, linear, homogeneous ODE. Solving this ODE, we get  $\mu(x) = ce^{-2x}$ . Since we only want one specific  $\mu$  that would work, take  $c = 1$ . So,  $\mu(x) = e^{-2x}$ . Substituting  $\mu(x) = e^{-2x}$  to Eq. (1):

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}(4 - x), \quad \tilde{P}_0 = e^{-2x} \neq 0, \quad \tilde{P}_1 = -2e^{-2x}.$$

Using the product rule:

$$\begin{aligned} \frac{d}{dx} [e^{-2x}y] &= 4e^{-2x} - xe^{-2x} \\ \int \frac{d}{dx} [e^{-2x}y] dx &= \int 4e^{-2x} - xe^{-2x} dx + c \\ e^{-2x}y &= \frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c \\ y &= e^{2x} \left( \frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c \right) \\ &= \frac{1}{2}x - \frac{7}{4} + ce^{2x}. \end{aligned}$$

□

**Theorem 1.2.5 Method of Integrating Factor**

Given the first order linear differential equation  $y' + p(x)y = f(x)$ , with  $p$  and  $f$  both continuous on some interval  $(a, b)$ ,

$$y(x) = \frac{1}{\mu(x)} \left[ \int \mu(x)f(x) \, dx + c \right]$$

is the general solution to the equation, with

$$\mu(x) = e^{\int p(x) \, dx}.$$

We call  $\mu(x)$  the *integrating factor*.

**Proof 5.** Consider  $\mu = \mu(x) \neq 0$ . Multiplying the both sides of  $y' + p(x)y = f(x)$  by  $\mu$ :

$$\mu y' + p\mu y = \mu f. \quad (2)$$

Impose  $\mu y' + p\mu y = \frac{d}{dx}[\mu y]$  to find  $\mu = \mu(x)$ :

$$\mu y' + p\mu y = \mu' y + \mu y'$$

$$\mu' - p\mu = 0,$$

first order, linear, homogeneous ODE

$$\mu(x) = e^{\int p(x) \, dx},$$

the integrating factor

Substitute  $\mu(x) = e^{\int p(x) \, dx}$  into Eq. (2):

$$\begin{aligned} \frac{d}{dx}[\mu y] &= \mu f \\ \int \frac{d}{dx}[\mu y] \, dx &= \int \mu f \, dx + c \\ \mu y &= \int \mu f \, dx + c \\ y(x) &= \frac{1}{\mu(x)} \left[ \int \mu(x)f(x) \, dx + c \right]. \end{aligned}$$

■

**Example 1.2.6** Give the equation  $y' + 2y = x^3 e^{-2x}$ . Classify the equation and find the general solution  $y = y(x, c)$ .

**Solution 6.**

It is a first order, linear, non-homogeneous ODE, with  $p = 2$  and  $f = x^3 e^{-2x}$ . Let  $\mu(x)$  be the integrating factor. Then,

$$\mu(x) = e^{\int 2 \, dx} = e^{2x}.$$

So, by the method of integrating factor, we know

$$\begin{aligned}\frac{d}{dx} [\mu(x)y] &= \mu(x)f(x) \\ \int \frac{d}{dx} [e^{2x}y] dx &= \int e^{2x}x^3e^{-2x} dx + c \\ e^{2x}y &= \int x^3 dx + c \\ e^{2x}y &= \frac{1}{4}x^4 + c \\ y &= \frac{1}{4}x^4e^{-2x} + ce^{-2x}.\end{aligned}$$

□

**Remark.** Re-examine the formula we derived from the method of integrating factor:

$$y(x) = \frac{1}{\mu} \int f\mu dx + \boxed{\frac{c}{\mu}}.$$

The part being boxed,  $\frac{c}{\mu}$ , is independent from  $f$  and is exactly  $ce^{-\int p dx}$  if we expand, which is the solution for a homogeneous differential equation.

**Definition 1.2.7 (Complementary Equation).** The *complementary equation* to a first order ODE  $y' + py = f$  is the homogeneous part of it. i.e.,  $y' + py = 0$ .

**Theorem 1.2.8 Method of Variation of Parameters**

Given the first order linear differential equation  $y' + p(x)y = f(x)$ , with  $p$  and  $f$  both continuous on some interval  $(a, b)$ ,

$$y(x) = y_1(x) \left[ \int \frac{f(x)}{y_1(x)} dx + c \right]$$

is the general solution to the equation, where  $y_1$  is a solution of the complementary equation  $y' + py = 0$ .

**Proof 7.** Call  $y_1$  a solution of the complementary equation  $y' + p(x)y = 0$ . Then, we want to find  $y(x) = u(x)y_1(x)$ , the general solution of  $y' + p(x)y = f(x)$ , where  $u$  is an unknown function of  $f$ . Note that, by product rule,  $y'(x) = u'y_1 + uy_1'$ . Then, the equation becomes

$$\begin{aligned}(u'y_1 + uy_1') + p(x)(uy_1) &= f(x) \\ u'y_1 + uy_1' + puy_1 &= f \\ y_1u' + \underbrace{(y_1' + py_1)}_0 u &= f \\ y_1u' = f &\implies u(x) = \int \frac{f(x)}{y_1(x)} dx + c.\end{aligned}$$



Therefore, the formula to find  $y$  is given by

$$y = y_1 u = y_1(x) \left[ \int \frac{f(x)}{y_1(x)} dx + c \right].$$

■

**Remark.** The method of variation of parameters will be more useful when solving second or higher order differential equations.

**Example 1.2.9** Give the equation  $y' + 2y = x^3 e^{-2x}$ . Find the general solution  $y = y(x, c)$  using the method of variation of parameters.

**Solution 8.**

It is a first order, linear, non-homogeneous ODE, with  $p = 2$  and  $f = x^3 e^{-2x}$ . Let  $y_1$  be the solution of the complementary equation  $y' + 2y = 0$ . Then,  $y_1(x) = e^{-\int 2 dx} = e^{-2x}$ . By the method of variation of parameters, suppose  $y = u y_1$ , where  $u$  is an unknown function of  $x$ . Then,

$$u(x) = \int \frac{f(x)}{y_1(x)} dx + c = \int \frac{x^3 e^{-2x}}{e^{-2x}} dx + c = \int x^3 dx + c = \frac{1}{4} x^4 + c.$$

So,

$$y = u y_1 = e^{-2x} \left( \frac{1}{4} x^4 + c \right) = \frac{1}{4} x^4 e^{-2x} + c e^{-2x}.$$

□

**Theorem 1.2.10 Existence and Uniqueness Theorem**

Suppose that  $p = p(x)$  and  $f = f(x)$  are continuous on  $(a, b)$ . Then, a general solution of  $y' + p(x)y = f(x)$  on  $(a, b)$  is

$$y(x) = y_1(x) \left[ \int \frac{f(x)}{y_1(x)} dx + c \right],$$

where  $y_1(x)$  is a solution of the complementary equation (i.e.,  $y' + p(x)y = 0$ ).

If  $x_0$  is an arbitrary point in  $(a, b)$  and  $y_0$  is an arbitrary real number, then the initial value problem,

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a unique solution on  $(a, b)$ .

### 1.3 Non-Linear First Order ODEs

**Definition 1.3.1 (General Forms).** The general form of a non-linear first order ODE is given by

$$y' = f(x, y(x)).$$

If we take  $M(x, y) = -f(x, y)$  and  $N(x, y) = 1$ , we can also re-write the equation into

$$M(x, y) + N(x, y)y' = 0,$$

or

$$M(x, y) dx + N(x, y) dy = 0.$$

**Definition 1.3.2 (Separable Equations).** If  $M(x, y) = M(x)$  and  $N(x, y) = N(y)$ , then the ODE is called *separable*.

**Theorem 1.3.3 Separation of Variables (SoV)**

Consider the non-linear first order ODE  $M(x, y) + N(x, y)y' = 0$ , with  $M(x, y) = M(x)$  and  $N(x, y) = N(y)$ . Then we can find an implicit solution of the ODE in the form of

$$F(x, y) = c,$$

where  $F(x, y)$  is a function of  $x$  and  $y$  and

$$F(x, y) = \int M(x) dx + \int N(y) dy.$$

**Proof 1.** Let  $H_1'(x) = M(x)$  and  $H_2'(y) = N(y)$ . Then, the equation becomes

$$\begin{aligned} H_1'(x) + H_2'(y)y' &= 0 \\ \frac{d}{dx} [H_1(x)] + \frac{d}{dy} [H_2(y)] \frac{dy}{dx} &= 0 \end{aligned}$$

By using the chain rule,  $\frac{d}{dy} [H_2(y)] \frac{dy}{dx} = \frac{d}{dx} [H_2(y(x))]$ . So, the equation becomes

$$\begin{aligned} \frac{d}{dx} [H_1(x)] + \frac{d}{dx} [H_2(y(x))] &= 0 \\ \frac{d}{dx} [H_1(x) + H_2(y)] &= 0 \\ H_1(x) + H_2(y) &= c \\ \int M(x) \frac{d}{dx} + \int N(y) dy &= c \\ F(x, y) &= c \end{aligned}$$

■

**Example 1.3.4** Given the equation  $y' = \frac{x^2}{1 - y^2}$ . Classify the differential equation and find the general solution.

**Solution 2.**

It is a first order, non-linear ODE. Since  $y' = \frac{x^2}{1 - y^2}$ , so we have  $1 - y^2 \neq 0$ . That is,  $y^2 \neq 1$ , or  $y \neq \pm 1$ . Using the separation of variables (SoV), we have

$$\begin{aligned}(1 - y^2)y' &= x^2 \\ \int (1 - y^2) dy &= \int x^2 dx \\ y - \frac{1}{3}y^3 &= \frac{1}{3}x^3 + c \\ y - \frac{1}{3}y^3 - \frac{1}{3}x^3 &= c \\ 3y - y^3 - x^3 &= c\end{aligned}$$

□

**Example 1.3.5** Given the equation  $y' = \frac{(y - 3) \cos x}{1 + 2y^2}$ . Classify the equation and find the general solution.

**Solution 3.**

It is a first order, non-linear ODE. Since  $1 + 2y^2 \neq 0 \quad \forall y \in \mathbb{R}$ . Note that if we take  $y - 3 = 0$ , we get  $y = 3$ , a constant solution to the differential equation. Now, assume  $y \neq 3$ . Then, use SoV:

$$\int \frac{1 + 2y^2}{y - 3} dy = \int \cos x dx + c = \sin x + c.$$

Set  $t = y - 3$ ,  $dt = dy$ . So,  $y = t + 3$  and  $y^2 = (t + 3)^2$ . Then,

$$\begin{aligned}\int \frac{1 + 2y^2}{y - 3} dy &= \int \frac{1 + 2(t + 3)^2}{t} dt = \int \frac{1 + 2t^2 + 12t + 18}{t} dt \\ &= \int \frac{19}{t} + 12 + 2t dt \\ &= 19 \ln |t| + 12t + t^2 \\ &= 19 \ln |y - 3| + 12(y - 3) + (y - 3)^2 \\ &= 19 \ln |y - 3| + 6y + y^2 - 27.\end{aligned}$$

So,

$$19 \ln |y - 3| + y^2 + 6y - 27 - \sin x = c$$

□

**Example 1.3.6** Give the equation  $y' = \frac{1}{2}x(1 - y^2)$ . Classify the equation and find the general solution.

**Solution 4.**

It is a first order, non-linear ODE. Notice that we have the constant solutions when we take  $1 - y^2 = 0$ , or  $y = \pm 1$ . Now, assume  $y \neq \pm 1$ . Using SoV:

$$\int \frac{2}{1 - y^2} dy = \int x dx + c = \frac{1}{2}x^2 + c.$$

Note that  $\frac{2}{1 - y^2} = \frac{2}{(1 - y)(1 + y)}$ . Use partial fractions. Assume

$$\frac{2}{(1 - y)(1 + y)} = \frac{A}{1 - y} + \frac{B}{1 + y}.$$

Then, we get  $A(1 + y) + B(1 - y) = 2$ . That is,  $(A + B) + (A - B)y = 2$ . Attempting to solve the system of equations  $\begin{cases} A - B = 0 \\ A + B = 2 \end{cases}$ , then we get  $A = B = 1$ . Therefore,

$$\frac{2}{1 - y^2} = \frac{1}{1 - y} + \frac{1}{1 + y}.$$

Then,

$$\begin{aligned} \int \frac{1}{1 - y} + \frac{1}{1 + y} dy &= \frac{1}{2}x^2 + c \\ -\ln |1 - y| + \ln |1 + y| &= \frac{1}{2}x^2 + c \\ \ln |1 + y| - \ln |1 - y| - \frac{1}{2}x^2 &= c. \end{aligned}$$

□

## 2 Second Order ODEs

## 3 System of ODEs