1 Statements

1.1 Class Handout, Chapter 1.3, Implications.

`Let` a, b, `and` c `be integers, with` a `and` b `non-zero. If` $(ab) \mid (ac)$, `then` $b \mid c$.

Proof 1.

Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$. Suppose $(ab) \mid (ac)$. Then $\exists k \in \mathbb{Z}$ *s.t.* $ac = (ab)k$. Divide both sides of the equation by a:

$$c = bk.$$

Since $k \in \mathbb{Z}$, by definition of divides, $b \mid c$.

■

1.2 Class Handout, Chapter 1.4, Contrapositive and Converse

`Prove that for all real numbers` a `and` b, `if` $a \in \mathbb{Q}$ `and` $ab \notin \mathbb{Q}$, `then` $n \notin Q$.

Proof 2.

Let $a, b \in \mathbb{Q}$. Assume for the sake of contradiction that if $a \in \mathbb{Q}$ and $ab \notin \mathbb{Q}$, we have $b \in \mathbb{Q}$. Then, $\exists p, q, m, n \in \mathbb{Z}$ *s.t.* $a = \dfrac{m}{n}$ and $b = \dfrac{p}{q}$. Hence,

$$ab = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$$

As $mp, nq \in \mathbb{Z}$, $ab \in \mathbb{Q}$.

⋇ This contradicts with the fact that $ab \notin \mathbb{Q}$.

So, b must not be rational.

■

1.3 Chapter 1.1 # 7(c)

`Prove the square of an even integer is divisible by 4.`

Proof 3.

Suppose $x \in \mathbb{Z}$ is even. Then $\exists k \in \mathbb{Z}$ *s.t.* $x = 2k$. Then, $x^2 = (2k)^2 = 4k^2$. Since $k^2 \in \mathbb{Z}$, we have $4 \mid 4k^2$.

■

Theorem 1.1 (Archimedean Principle) *For every real number x, there is an integer n, such that $n > x$.*

> *1.4 Chapter 1.1 # 11*
>
> `For every positive real number` ε`, there exists a positive integer` N `such that` $\frac{1}{n} < \varepsilon$ `for all` $n \geq N$.

Proof 4.

Suppose $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. Since $\varepsilon \in \mathbb{R}$, we have $\frac{1}{n} \in \mathbb{R}$. Then, by Archimedean Principle, $\exists n \in \mathbb{Z}$ *s.t.* $n > \frac{1}{\varepsilon}$. Hence, $n\varepsilon > 1$ or $\varepsilon > \frac{1}{n}$.

Suppose $N \in \mathbb{Z}$ *s.t.* $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$, *where* $\left\lceil \frac{1}{\varepsilon} \right\rceil$ *means the integer greater to* $\frac{1}{\varepsilon}$ *if* $\frac{1}{\varepsilon} \notin \mathbb{Z}$, *and the integer equals to* $\frac{1}{\varepsilon}$ *if* $\frac{1}{\varepsilon} \in \mathbb{Z}$. Hence, $N \geq \frac{1}{\varepsilon}$. As $n > \frac{1}{\varepsilon}$, we have $n \geq N$

■

> *1.5 Chapter 1.1 # 12*
>
> `Use the Archimedean Principle (Theorem 1.1) to prove if` x `is a real number, then there exists a positive integer` n `such that` $-n < x < n$.

Proof 5.

Suppose $x \in \mathbb{R}$.

$\boxed{\text{Case 1}}$ If $x > 0$, then $-x < 0$ (i.e., $-x < 0 < x$). By the Archimedean Principle, $\exists n \in \mathbb{Z}$ *s.t.* $n > x$. Multiply (-1) on both sides of the inequality:

$$-n < -x$$

As $-x < 0 < x$,

$$-n < -x < 0 < x < n,$$

which means $-n < x < n$, and n is positive.

$\boxed{\text{Case 2}}$ If $x < 0$, then $-x > 0$ (i.e., $-x > 0 > x$) Since $x \in \mathbb{R}$, we have $-x \in \mathbb{R}$. By the Archimedean Principle, $\exists n \in \mathbb{Z}$ *s.t.* $n > -x$. Multiply (-1) on both sides of the inequality:

$$-n < x$$

As $x < 0 < -x$,

$$-n < x < 0 < -x < n,$$

which means $-n < x < n$, and n is positive. In all cases, we have proven that $x \in \mathbb{R} \implies \exists n \in \mathbb{Z},\ n > 0\ s.t.\ -n < x < n$.

$\blacksquare$

1.6 Chapter 1.1 # 13

`Prove that if` x `is a positive real number, then there exists a positive integer` n `such that` $\frac{1}{n} < x < n$.

Proof 6.

Suppose $x \in \mathbb{R},\ x > 0$

$\boxed{\text{Case 1}}$ If $0 < x \leq 1$, then $\frac{1}{x} \geq 1$. Hence, $x \leq 1 \leq \frac{1}{x}$. As $x \in \mathbb{R}$, $\frac{1}{x} \in \mathbb{R}$, then by the Archimedean Principle (Theorem 1.1):

$$\exists n \in \mathbb{Z}\ s.t.\ n > \frac{1}{x}.$$

Hence, $nx > 1$ or $x > \frac{1}{n}$. As $x \leq \frac{1}{x}$, $n > \frac{1}{x}$, and $x > \frac{1}{n}$, we have

$$\frac{1}{n} < x < n.$$

$\boxed{\text{Case 2}}$ If $x > 1$, then $0 < \frac{1}{x} < 1$. Hence, $\frac{1}{x} < 1 < x$. As $x \in \mathbb{R}$, by the Archimedean Principle:

$$\exists n \in \mathbb{Z}\ s.t.\ n > x > 0$$

Hence, $\frac{1}{n} < \frac{1}{x}$. As $\frac{1}{x} < x$, $\frac{1}{n} < \frac{1}{x}$, and $n > x$, we have

$$\frac{1}{n} < x < n$$

In all cases, we proven that $x \in \mathbb{R},\ x > 0 \implies \exists n \in \mathbb{Z}, n > 0\ s.t.\ \frac{1}{n} < x < n$.

$\blacksquare$

1.7 Handout Chapter 1.4-2 More Contradictions and Equivelance

`There are no positive integer solutions to the equation` $x^2 - y^2 = 10$.

Proof 7.

Assume for the sake of contradiction that there are positive integer solutions to the equation $x^2 - y^2 = 10$. Suppose $\exists x, y \in \mathbb{Z}$ and $x > 0,\ y > 0\ s.t.\ x^2 - y^2 = 10$. Then, we have $x^2 = 10 + y^2$. Since $x > 0,\ x^2 > 0$, we have $10 + y^2 > 0$. Then, $y^2 > -10$.

$$\divideontimes \text{ This contradicts with the fact that } y^2 \geq 0 \text{ if } y \in \mathbb{Z}.$$

So, our assumption is wrong. There must be no positive integer solutions to the equation $x^2 - y^2 = 10$.

■

1.8 Handout Chapter 1.4-2 More Contradictions and Equivelance

`Show that if` $a \in \mathbb{Q}$ `and` $b \in \mathbb{Q}'$`, then` $a + b \in \mathbb{Q}'$

Remark *The notation $\mathbb{Q}$ means the set for rational numbers, and $\mathbb{Q}'$ means the set for irrational numbers.*

Proof 8.

Suppose $a \in \mathbb{Q}$ and $b \in \mathbb{Q}'$ Assume for the sake of contradiction that $a + b \in \mathbb{Q}$. Then, $\exists m, n, p, q \in \mathbb{Z}$ such that $a = \dfrac{m}{n}$ and $a + b = \dfrac{p}{q}$. Then,

$$b = \frac{p}{q} - a = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} \in \mathbb{Q}$$

Since $pn - mq \in \mathbb{Q}$ and $qn \in \mathbb{Z}$, we have $b = \dfrac{pn - mq}{qn} \in \mathbb{Q}$.

$$\divideontimes \text{ This contradicts with the fact that } b \in \mathbb{Q}'.$$

So, $a + b$ must be irrational.

■

1.9 Handout Chapter 1.4-2 More Contradictions and Equivalence

`If` $n \in \mathbb{N}$ `and` $2^n - 1$ `is prime, then` n `is prime.`

Proof 9.

We will prove the contrapositive: if n is not prime, then $2^n - 1$ is not prime. Suppose n is not prime. Then, $\exists a, b \in \mathbb{Z}$ with $1 < a, b < n\ s.t.\ n = ab$. Then, $2^n - 1 = 2^{ab} = (2^a)^b - 1$. Notice that for $x^w - 1$, by polynomial long division, have

$$x^w - 1 = (x-1)\left(x^{w-1} + x^{w-2} + \cdots + 1\right),$$

Substitute $x = 2^a$ and $w = b$, we have

$$2^n - 1 = (2^a - 1)\left[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 1\right].$$

Since $(2^a - 1) \in \mathbb{Z}$ and $\left[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 1\right] \in \mathbb{Z}$, we see that $2^n - 1$ is not prime.

■

1.10 Exam 1 Review 1-b-i

`Prove that` $[P \wedge (P \implies Q)] \implies Q$.

Proof 10.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$[P \wedge (P \Rightarrow Q)] \implies P$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

■

1.11 Exam 1 Review 1-b-ii

`Prove that` $[Q \wedge (P \implies Q)] \implies P$.

Proof 11.

P	Q	$P \Rightarrow Q$	$Q \wedge (P \Rightarrow Q)$	$[Q \wedge (P \Rightarrow Q)] \implies Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

$\blacksquare$

> *1.12 Exam 1 Review 2-a*
>
> `Given statements` P `and` Q`, prove` $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

Proof 12.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

$\blacksquare$

> *1.13 Exam 1 Review 2-b*
>
> `There is no smallest integer.`

Proof 13.

Assume for the sake of contradiction that there exists a smallest integer n. Hence, $\forall x \in \mathbb{Z}$, we have $x \geq n$. Notice that if $n > 0$, we have $0 \in \mathbb{Z}$ and $0 < n$. Hence, $n = 0$ cannot be the smallest integer ($\divideontimes$) Therefore, n most be smaller than 0. Suppose $m = -n$. Since $n \in \mathbb{Z}$, $m = -n \in \mathbb{Z} \in \mathbb{R}$ By the Archimedean Principle (Theorem 1.1), $\exists k \in \mathbb{Z}$ *s.t.* $k > m$. Hence, $k > -n$. Multiply (-1) on both sides of the inequality:

$$-k < n.$$

As $k \in \mathbb{Z}$, $-k \in \mathbb{Z}$. Then $\exists -k \in \mathbb{Z}$ *s.t.* $-k < n$.

$\divideontimes$ This contradicts with our assumption that n is the smallest integer.

Hence, our assumption must be wrong. There is no smallest integer.

$\blacksquare$

> *1.14 Exam 1 Review 2-c*
>
> `The number` $\log_2 3$ `is irrational.`

Proof 14.

Assume for the sake of contradiction that $\log_2 3$ is irrational. By definition, $\exists p, qin\mathbb{Z}$, with $q \neq 0$ $s.t.$ $\log_2 3 = \frac{p}{q}$. Observe that $\log_2 3 \neq 0$. Then $p \neq 0$ as well. By definition of logarithm,

$$2^{p/q} = 3$$
$$(2^p)^{1/q} = 3$$

Raise two sides of the equation to the power of q:

$$2^p = 3^q$$

As $p \neq 0$ and $q \neq 0$, 2^p and 3^q are not 1 $\forall p, q \in \mathbb{Z}$. Hence, 2^p is even $\forall p \in \mathbb{Z}$ and 3^q is odd $\forall q \in \mathbb{Z}$.

$\divideontimes$ This contradicts with the fact that an even number cannot equal to an odd number.

Hence, our assumption is wront. The number $\log_2 3$, then, must be irrational.

$\blacksquare$

1.15 Exam 1 Review 2-d

`There is a rational number` a `and an irrational number` b `such that` a^b `is rational.`

Proof 15.

Observe that 1 is a rational number and π is an irrational number. Suppose $a = 1$ and $b = \pi$, we have $a^b = a^\pi = 1$, which is irrational.

$\blacksquare$

Proof 16.

Recall that we have proven in the previous proof, we have proven that $\log_2 3$ is an irrational number. Recall the definition of logarithm and exponents, we have

$$2^{\log_2 3} = 3$$

Hence, we find a pair of a and b that satisfies the requirement.

$\blacksquare$

> *1.16 Exam 1 Review 2-e*
>
> `For all integers` n`, the number` $n + n^2 + n^3 + n^4$ `is even.`

Proof 17.

Suppose $n \in \mathbb{Z}$.

$\boxed{\text{Case 1}}$ If n is even.Suppose $n = 2k$ *f.s.* $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n + n^2 + n^3 + n^4 &= (2k) + (2k)^2 + (2k)^3 + (2k)^4 \\ &= 2k + 4k^2 + 8k^3 + 16k^4 \\ &= 2(k + 2k^2 + 4k^3 + 8k^4) \end{aligned}$$

Since $(k + 2k^2 + 4k^3 + 8k^4) \in \mathbb{Z}$, we have $2(k + 2k^2 + 4k^3 + 8k^4)$ is even. Hence, $n + n^2 + n^3 + n^4$ is even when n is even.

$\boxed{\text{Case 2}}$ If n is odd. Suppose $n = 2k + 1$ *f.s.* $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n + n^2 + n^3 + n^4 &= (2k + 1) + (2k + 1)^2 + (2k + 1)^3 + (2k + 1)^4 \\ &= 2k + 1 + 4k^2 + 4k + 1 + 8k^3 + 12k^2 + 6k + 1 + 16k^4 + 32k^3 + 24k^2 + 8k + 1 \\ &= 16k^4 + 40k^3 + 40k^2 + 20k + 4 \\ &= 2(8k^4 + 20k^3 + 20k^2 + 10k + 2) \end{aligned}$$

Since $(8k^4 + 20k^3 + 20k^2 + 10k + 2) \in \mathbb{Z}$, we have $2(8k^4 + 20k^3 + 20k^2 + 10k + 2)$ is even. Hence, $n + n^2 + n^3 + n^4$ is even when n is odd.

Since integers can either be even or odd, and we have proven $n + n^2 + n^3 + n^4$ is even in either case, $n + n^2 + n^3 + n^4$ is even for all integers.

■

Definition 1.1 (Perfect Square) *A perfect square is an integer n for which there exists an integer m such that $n = m^2$.*

> *1.17 Exam 1 Review 2-f*
>
> `If` n `is a positive integer such that` n `is in the form` $4k + 2$ `or` $4k + 3$`, then` n `is not a perfect square.`

Proof 18.

We will prove the contrapositive of the statement: "If n is a perfect square, then n is a positive integer of the form $4k$ or $4k+1$ *f.s.* $k \in \mathbb{Z}$." Suppose n to be a perfect square, then $\exists m \in \mathbb{Z}$ *s.t.* $n = m^2$. $\boxed{\text{Case 1}}$ Suppose m is even, then $m = 2t$ *f.s.* $t \in \mathbb{Z}$.

$$n = m^2 = (2t)^2 = 4t^2 > 0.$$

Let $k = t^2$. Since $t^2 \in \mathbb{Z}$, we have $k \in \mathbb{Z}$.Hence, n is positive and is in the form of $4k$.

$\boxed{\text{Case 2}}$ Suppose m is odd, then $m = 2t + 1$ *f.s.* $t \in \mathbb{Z}$.

$$n = m^2 = (2t + 1)^2 = 4t^2 + 4t + 1 = 4(t^2 + t) + 1 > 1$$

Let $k = t^2 + t$. Since $(t^2 + t) \in \mathbb{Z}$, we have $k \in \mathbb{Z}$. Hence, n is in the form of $4k + 1$. Hence, we prove the contrapositive of the original statement to be true, which means our original statement is also true.

■

1.18 Exam 1 Review 2-g

`For any integer` n, $3 \mid n$ `if and only if` $3 \mid n^2$.

Proof 19.

Suppose $n \in \mathbb{Z}$.

($\Rightarrow$) Suppose $3 \mid n$. Then, $\exists k \in \mathbb{Z}$ *s.t.* $n = 3k$. Then, $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$. Since $3k^2 \in \mathbb{Z}$, by definition, $3 \mid n^2$. □

($\Leftarrow$) WTS: $3 \mid n^2 \implies 3 \mid n$. We will prove the contrapositive: If $3 \nmid n$, then $3 \nmid n^2$ Suppose $3 \nmid n$.

$\boxed{\text{Case 1}}$ Suppose $n = 3m+1$ *f.s.* $m \in \mathbb{Z}$. Then, $n^2 = (3m+1)^2 = 9m^2+6m+1$ Since $9m^2+6m+1$ cannot be written in the form of $3k$ *f.s.* $k \in \mathbb{Z}$, by definition, $3 \nmid n^2$.

$\boxed{\text{Case 2}}$ Suppose $n = 3m + 2$ *f.s.* $m \in \mathbb{Z}$Then, $n^2 = (3m + 2)^2 = 9m^2 + 12m + 4$Since $9m^2 + 12m + 4$ cannot be written in the form of $3k$ for some $k \in \mathbb{Z}$, by definition, $3 \nmid n^2$. Hence, we proved the contrapositive, and thus the original statement is true.

Therefore, $n \mid n \iff 3 \mid n^2$.

■

1.19 Exam 1 Review 2-h

`There exists an integer` n `such that` $12 \mid n^2$ `but` $12 \nmid n$.

Proof 20.

Observe that if we take $n = 6$, we have $n^2 = 36$. Since $n^2 = 36 = 3 \times 12$, we know $12 \mid n^2$. However, $12 \nmid 6$ since 6 cannot be written as $12k$ for all $k \in \mathbb{Z}$. Hence, there exists an integer $n = 6$ *s.t.* $12 \mid n^2$ but $12 \nmid n$.

■

1.20 Exam 1 Review 2-i

`For every integer` a`, the numbers` a `and` $(a+1)(a-1)$ `have opposite parity.`

Proof 21.

Suppose $a \in \mathbb{Z}$.

$\boxed{\text{Case 1}}$ Suppose a is even. Then $a = 2k$ *f.s.* $k \in \mathbb{Z}$. Then,

$$(a+1)(a-1) = a^2 - 1 = (2k)^2 - 1 = 4k^2 - 1 = 2(2k^2) - 1.$$

Since $2k^2 \in \mathbb{Z}$, we have $(a+1)(a-1)$ is odd. That is, a and $(a+1)(a-1)$ have opposite parity.

$\boxed{\text{Case 2}}$ Suppose a is odd. Then $a = 2k+1$ *f.s.* $k \in \mathbb{Z}$. Hence,

$$(a+1)(a-1) = a^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 2(2k^2 + 2k).$$

Since $2k^2 + 2k \in \mathbb{Z}$, we have $(a+1)(a-1)$ is even. As a result, a and $(a+1)(a-1)$ have opposite parity.

In both cases, we've shown that a and $(a+1)(a-1)$ have opposite parity.

■

1.21 Exam 1 Review 2-j

`Suppose` $x \in \mathbb{R}$`. If` x^2 `is irrational, then` x `is irrational.`

Proof 22.

We will prove the contrapositive: "If x is rational, then x^2 is rational." Suppose $x \in \mathbb{Q}$, then $x = \dfrac{p}{q}$ *f.s.* $p, q \in \mathbb{Z}$, assuming p and 1 have no common factors and $q \neq 0$. Then,

$$x^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}.$$

As $p, q \in \mathbb{Z}$, we have $p^2, q^2 \in \mathbb{Z}$. Hence, $x^2 = \dfrac{p^2}{q^2} \in \mathbb{Q}$. Therefore, if x is rational, so is x^2.

■

1.22 Exam 1 Review 2-k

`For any integers` a `and` b`, if` ab `is even, then` a `is even or` b `is even.`

Proof 23.

We will prove the contrapositive: "If a is odd and b is odd, then ab is odd." Suppose $a, b \in \mathbb{Z}$ and a and b are both odd. Then, $\exists k, l \in \mathbb{Z}$ *s.t.* $a = 2k + 1 \quad$ and $\quad b = 2l + 1$. Then,

$$ab = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1.$$

Since $2kl + k + l \in \mathbb{Z}$, we have ab is odd.

$\blacksquare$

1.23 Exam 1 Review 2-l

`For` $n \in \mathbb{N}$`,` n`,` $n + 2$`, and` $n + 4$ `are all prime if and only if` $n = 3$`.`

Proof 24.

($\Rightarrow$) WTS: $n, n + 2$, and $n + 4$ are all prime $\implies n = 3$. We will prove the contrapositive: $n \neq 3 \implies n, n + 2$, or $n + 4$ is not prime.

$\boxed{\text{Case 1}}$ Suppose $0 < n < 3$.

① If $n = 1$, then $n = 1$ is not a prime.

② If $n = 2$, then $n = 2$ is a prime number, but $n + 2 = 2 + 2 = 4$ is not a prime.

Hence, if $0 < n < 3$, $n, n + 2$, or $n + 4$ is not a prime.

$\boxed{\text{Case 2}}$ Suppose $n > 3$.

① If $n = 3k$ *f.s.* $k \in \mathbb{Z}$, then n is not a prime because $3 \mid n$.

② If $n = 3k + 1$ *f.s.* $k \in \mathbb{Z}$, then $n + 2 = 3k + 1 + 2 = 3k + 3 = 3(k + 1)$. Since $k + 1 \in \mathbb{Z}$, we have $3 \mid n + 2$. Then, $n + 2$ is not a prime.

③ If $n = 3k + 2$ *f.s.* $k \in \mathbb{Z}$, then $n + 4 = 3k + 2 + 4 = 3k + 6 = 3(k + 2)$. Since $k + 2 \in \mathbb{Z}$, we know that $3 \mid n + 4$. Therefore, $n + 4$ is not a prime.

Hence, if $n > 3$, we also have $n, n + 2$, or $n + 4$ is not a prime.

In both cases, we have proven that if $n \neq 3$, then $n, n + 2$, or $n + 4$ is not a prime. $\square$

($\Leftarrow$) Note that when $n = 3$, we have $n + 2 = 3 + 2 = 5$ and $n + 4 = 3 + 4 = 7$. Since $3, 5$, and 7 are all primes, we have shown that when $n = 3$, n, $n + 2$, and $n + 4$ are all primes.

■

1.24 Exam 1 Review 3-a

`Prove or disprove: Every real number is less than or equal to its square.`

Disproof 25.

Prove.

■

1.25 Exam 1 Review 3-b

`Prove or disprove: The sum of two integers is never equal to their product.`

Disproof 26.

Prove.

■

1.26 Exam 1 Review 3-c

`Prove or disprove: There exists a non-zero integer whose cube equals its negative.`

Disproof 27.

Prove.

■

1.27 Exam 1 Review 3-d

Prove or disprove: Fall all $x \in \mathbb{R}$, $x \leq x^2$ or $0 \leq x < 1$.

Proof 28.

Prove.

■

1.28 Chapter 1.4 # 20-a

Let n be an integer. Prove that n is even if and only if n^3 is even.

Proof 29.

Prove.

■

1.29 Chapter 1.4 # 20-b

Let n be an integer. Prove that n is odd if and only if n^3 is odd.

Proof 30.

Prove.

■

1.30 Chapter 1.4 # 21

Prove that $\sqrt[3]{2}$ is irrational.

Proof 31.

Prove.

■

2 Sets

2.1 Handout Chapter 2.1 - Sets and Subsets

`Prove that` $\{12a + 4b \mid a, b \in \mathbb{Z}\} = \{4c \mid c \in \mathbb{Z}\}$.

Proof 1.

($\subseteq$) Suppose $x \in \{12a + 4b \mid a, b \in \mathbb{Z}\}$. Then, $x = 12a + 4b$ *f.s.* $a, b \in \mathbb{Z}$. So, $x = 12a + 4b = 4(3a + b)$. As $3a + b \in \mathbb{Z}$, we have $x \in \{4c \mid c \in \mathbb{Z}\}$. By definition, $\{12a + 4b \mid a, b \in \mathbb{Z}\} \subseteq \{4c \mid c \in \mathbb{Z}\}$.

($\supseteq$) Suppose $x \in \{4c \mid c \in \mathbb{Z}\}$. Then, $x = 4c$ *f.s.* $c \in \mathbb{Z}$. Suppose $c = 3a + b$ *f.s.* $a, b \in \mathbb{Z}$. Then, $x = 4c = 4(3a + b) = 12a + 4b$. By definition, $\{4c \mid c \in \mathbb{Z}\} \subseteq \{12a + 4b \mid a, b \in \mathbb{Z}\}$

Hence, we have proven $\{12a + 4b \mid a, b \in \mathbb{Z}\} = \{4c \mid c \in \mathbb{Z}\}$.

■

2.2 Exam 1 Review 2-m

`If` $A = \{x \mid x = n^4 - 1,\ n \in \mathbb{Z}\}$ `and` $B = \{x \mid x = m^2 - 1,\ m \in \mathbb{Z}\}$, `then` $A \subseteq B$.

Proof 2.

Prove.

■

2.3 Exam 1 Review 2-n

`If` A, B, `and` C `are sets, then` $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof 3.

Prove.

■

2.4 Exam 1 Review 2-o

`For subsets` A `and` B `of a universal set` U, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof 4.

Prove.

■

2.5 Exam 1 Review 2-p

Suppose that A, B, and C are subsets of a universal set U. Let P and Q be the following statements:

P: $A \subseteq B$ or $A \subseteq C$; and

Q: $A \subseteq B \cap C$.

Write the statement $P \Rightarrow Q$, its converse and contrapositive. Prove the truth statements among these or give counterexamples.

Proof 5.

Prove.

■

2.6 Handout Chapter 2.2-Combining Sets

Let $A = \{6a + 4 \mid x \in \mathbb{Z}\}$ and $B = \{18b - a \mid b \in \mathbb{Z}\}$. Prove or disprove: $A \subseteq B$.

Disproof 6.

Prove.

■

2.7 Handout Chapter 2.2-Combining Sets

Let $A = \{6a + 4 \mid x \in \mathbb{Z}\}$ and $B = \{18b - a \mid b \in \mathbb{Z}\}$. Prove or disprove: $B \subseteq A$.

Proof 7.

Prove.

■

2.8 Handout Chapter 2.2-Combining Sets

If A and B are sets, then $\mathcal{P}(A) - \mathcal{P}(B) = \mathcal{P}(A - B)$.

Proof 8.

Prove.

■

2.9 Handout Chapter 2.2-Combining Sets

If A, B, and C are sets, and $A \times B = B \times C$, then $A = B$.

Proof 9.

Prove. ■

2.10 Chapter 2.1 # 6

Let $n \in \mathbb{Z}$ and let $A = n\mathbb{Z}$. Prove that if $x, y \in A$, then $x + y \in Z$ and $xy \in A$.

Proof 10.

Prove. ■

2.11 Chapter 2.1 # 10

Let n and m be integers. Let $A = n\mathbb{Z}$ and $B = m\mathbb{Z}$. Prove that if n is a multiplier of m, then $A \subseteq B$.

Proof 11.

Prove. ■

2.12 Chapter 2.1 # 12

Let $A = \{n \in \mathbb{Z} \mid n \text{ is a multiple of } 4\}$ and $B = \left\{n \in \mathbb{Z} \mid n^2 \text{ is a multiple of } 4\right\}$. Prove that $A \subseteq B$ and $B \nsubseteq A$.

Proof 12.

Prove. ■

Proof 13.

Prove. ■

2.13 Chapter 2.1 # 13

If $A = \{n \in \mathbb{Z} \mid n + 3 \text{ is odd}\}$, then A is equal to the set of all even integers.

Proof 14.

Prove. ■

2.14 Chapter 2.1 # 15

Let $A = \{n \in \mathbb{Z} \mid n = 4t + 1 \text{ for some } t \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} \mid n = 4t + 9 \text{ for some } t \in \mathbb{Z}\}$. Prove that $A = B$.

Proof 15.

Prove. ■

2.15 Chapter 2.1 # 16

Let $A = \{n \in \mathbb{Z} \mid n = 3t + 1 \text{ for some } t \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} \mid n = 3t + 2 \text{ for some } t \in \mathbb{Z}\}$. Prove that A and B have no elements in common.

Proof 16.

Prove. ■

2.16 Chapter 2.3 # 8

Let $A_i = (-i, i) = \{x \in \mathbb{R} \mid -i < x < i\}$. Prove that $\bigcup_{i=1}^{\infty}(-i.i) = \mathbb{R}$ and $\bigcap_{i=1}^{\infty}(-i, i) = (-1, 1)$.

Proof 17.

Prove. ■

Proof 18.

Prove. ■

> *2.17 Chapter 2.3 # 10*
>
> `Let` $A_i = \{1, 2, 3, \cdots, i\}$ `for` $i \in \mathbb{Z}^+$. `Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$.

Proof 19.

Prove.

■

Claim. $\bigcap_{i=1}^{\infty} A_i = \{1\}$.

Proof 20.

Prove.

■

> *2.18 Chapter 2.3 # 10*
>
> `Let` $A_i = [i, i+1) = \{x \in \mathbb{R} \mid i \leq x < i+1\}$ `for` $i \in \mathbb{Z}^+$. `Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = \{x \in \mathbb{R} \mid x \geq 1\}$.

Proof 21.

Prove.

■

Claim. $\bigcap_{i=1}^{\infty} A_i = \varnothing$.

Proof 22.

Prove.

■

> *2.19 Chapter 2.3 # 12*
>
> `Let` $A_i = \left(\frac{1}{i}, i\right] = \left\{x \in \mathbb{R} \mid \frac{1}{i} < x \leq i\right\}$ `for` $i \geq 2$. `Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = (0, \infty)$.

Proof 23.

Prove.

■

Claim. $\displaystyle\bigcap_{i=1}^{\infty} A_i = \left(\frac{1}{2}, 2\right]$.

Proof 24.

Prove.

■

> *2.20 Chapter 2.3 # 13*
>
> `Let` $A_i = \left[i, 1 + \dfrac{1}{i}\right]$ `for` $i \in \mathbb{Z}^+$. `Compute` $\displaystyle\bigcup_{i=1}^{\infty} A_i$ `and` $\displaystyle\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\displaystyle\bigcup_{i=1}^{\infty} A_i = [1, 2]$.

Proof 25.

Prove.

■

Claim. $\displaystyle\bigcap_{i=1}^{\infty} A_i = \{1\}$.

Proof 26.

Prove.

■

> *2.21 Chapter 2.3 # 14*
>
> `Let` $A_i = \left(i, 1 + \dfrac{1}{i}\right)$ `for` $i \in \mathbb{Z}^+$. `Compute` $\displaystyle\bigcup_{i=1}^{\infty} A_i$ `and` $\displaystyle\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\displaystyle\bigcup_{i=1}^{\infty} A_i = (1, 2)$, and $\displaystyle\bigcap_{i=1}^{\infty} A_i = \varnothing$.

Proof 27.

Similar proofs as done in the previous exercise.

■

> *2.22 Exam 2 Review 2*
>
> `For sets` A, B, C, D`, prove that` $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof 28.

Prove.

■

2.23 Exam 2 Review 3

Given the indexed sets, compute the unions and intersections. Give full and careful proofs of each: $A_i = [i-1, i]$ for $i = 1, \cdots, n$. Compute $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$.

Claim. $\bigcap_{i=1}^{n} A_i = \begin{cases} A_1, & n = 1 \\ A_1 \cap A_2 = \{1\}, & n = 2 \\ \varnothing, & n \geq 3 \end{cases}$.

Proof 29.

Prove. ∎

Claim. $\bigcup_{i=1}^{n} A_i = [0, n]$.

Proof 30.

∎

2.24 Exam 2 Review 4

Here's a mathematical statement:

(s): for all sets A and B, $A \subseteq B$ implies that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

State the converse (s_1) of (s), the contrapositive (s_2) of (s), the negation $(\neg s)$ of (s). Which of the statements (s), (s_1), (s_2), $(\neg s)$ are true?

Claim. (s) is true.

Proof 31.

Prove. ∎

Claim. (s_1) is true.

Proof 32.

Prove. ∎

Claim. (s_2) is true.

Proof 33.

Prove.

■

Claim. $(\neg s)$ is false.

Proof 34.

Prove.

■

2.25 Exam 2 Review 5

`For all sets` A `and` B`, if` $\mathcal{P}(A) = \mathcal{P}(B)$`, then` $A = B$.

Proof 35.

Prove.

■

3 Integers

3.1 Handout Chapter 5.1-5.2-Axioms of Integers

`Claim: Let` $a, b \in \mathbb{Z}$. `Then` $(-a)(-b) = ab$.

Proof 1.

Prove.

■

3.2 Chapter 5.1 # 1-a

$-(-a) = a$ `for all` $a \in \mathbb{Z}$.

Proof 2.

Prove.

■

3.3 Chapter 5.1 # 1-c

$a(b - c) = ab - ac$ `for all` $a, b, c \in \mathbb{Z}$.

Proof 3.

Prove.

■

3.4 Chapter 5.1 # 2

`Let` $a, b \in \mathbb{Z}$. `Prove that` $-(a+b) = -a - b$.

Proof 4.

Prove.

■

3.5 Chapter 5.1 # 3

`Let` $a, b \in \mathbb{Z}$. `Suppose that` $a < b$. `Prove that` $(-a) > (-b)$.

Proof 5.

Prove.

■

3.6 Exam 2 Review 1

`State the Well Ordering Principle for` $\mathbb{N}$.

Claim. Prove.

3.7 Exam 2 Review 6-a

`Every non-empty subset of the rational numbers` $\mathbb{Q}$ `contains a minimum element.`

Counterexample. Prove.

Counterexample. Prove

Proof 6.

■