Emory University

MATH 212 Differential Equations Learning Notes

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Contents

1 First Order ODEs

1.1 Introduction

Definition 1.1.1 (Ordinary Differential Equations/ODEs). An *ordinary differential equation* is an equation that contains one or more derivatives of an unknown function $y = y(x)$.
Definition 1.1.2 (Order of ODEs). The *order* of an ODE is the maximum order of the derivatives appearing in the equation.
Definition 1.1.3 (Solution to ODEs). The *solution* to an ODE is a function y that satisfies the equation.

Example 1.1.4 Solve $y'' = 3x + 1$.

Solution 1.

$$\begin{aligned} y' &= \int 3x + 1 \,\mathrm{d}x = \frac{3}{2}x^2 + x + C \\ y &= \int y' \,\mathrm{d}x = \int \frac{3}{2}x^2 + x + C \,\mathrm{d}x = \frac{1}{2}x^3 + \frac{1}{x}x^2 + Cx + D. \end{aligned}$$

□

Definition 1.1.5 (Linear ODEs/Non-Linear ODEs). A first order ODE is *linear* if it can be written as

$$y' + p(x)y = f(x).$$

Otherwise, it is *non-linear.*
Definition 1.1.6 (Homogenous/Non-Homogenous Linear ODEs). If $f(x) = 0$, then the linear ODE is *homogenous.* That is,

$$y' + p(x)y = 0.$$

Otherwise, it is *non-homogenous.*
Definition 1.1.7 (Trivial/Non-Trivial Solution). $y = 0$ is a *trivial solution* to a homogenous ODE. Any other solutions are *non-trivial.*
Definition 1.1.8 (One-Parameter Family of Solutions). We call C a *parameter* and the equation, therefore solution, defines a *one-parameter family* of solutions.

Example 1.1.9 For the ODE $y' = 1$, $y_1 = x + C_1$ is a solution to it, and it is a one-parameter family of solutions. Similarly, for $y' = \dfrac{1}{x^2}$, the one-parameter families of solutions are defined by $y_2 = -\dfrac{1}{x} + C_2$ on the interval $(-\infty, 0) \cup (0, \infty)$.

Definition 1.1.10 (General Solution). Given the general form of the linear ODE $y' + p(x)y =$

$f(x)$ if p and f are continuous on some open interval (a, b) and there is a unique formula $y = y(x, c)$ and we have the following properties:

- for each fixed c, the resulting function of x is a solution of the ODE on (a, b), and
- if y is a solution of the ODE, then y can be obtained by choosing the value of c appropriately.

The function $y = y(x, c)$ is called a *general solution.*

More generally, we can write an ODE as

$$P_0(x)y' + P_1(x)y = F(x).$$

In this case, the ODE has a general solution on any open interval in which $P_0,\ P_1$, and F are continuous and $P_0 \neq 0$.

Definition 1.1.11 (Initial Value Problem (IVP)). A differential equation with an initial condition.

Example 1.1.12 Let a be a constant. Find the general solution of $y' - ay = 0$ and solve the IVP $\begin{cases} y' - ay = 0 \\ y(x_0) = y_0. \end{cases}$

Solution 2.

Classification: First order, Linear, Homogeneous.

Trivial Solution: $y = 0$.

General solution:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= ay \\ \int \frac{1}{y}\,\mathrm{d}y &= \int a\,\mathrm{d}x \\ \ln|y| &= ax + c \\ y &= e^{ax+c} = Ae^{ax}. \end{aligned}$$

This general solution includes the trivial solution.

IVP: Substitute $x = x_0$ and $y = y_0$:

$$y_0 = Ae^{ax_0} \quad \longrightarrow \quad A = y_0 e^{-ax_0}$$

So,

$$y^{\text{IVP}} = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

This IVP is a "generic initial condition." We need more information on x_0, y_0 to get a more specific solution. □

1.2 The Method of Integrating Factors

Theorem 1.2.1
If p is continuous on (a,b), then the general solution of the homogeneous equation $y' + p(x)y = 0$ on (a,b) is given by

$$y = ce^{-\int p(x)\,\mathrm{d}x}.$$

Proof 1.
(a). Substitute the solution formula to show that $y = ce^{-\int p(x)\,\mathrm{d}x}$ is a solution for any choice of c.

$$y' = c\left(-\int p(x)\,\mathrm{d}x\right)' e^{-\int p(x)\,\mathrm{d}x} = -cp(x)e^{-\int p(x)\,\mathrm{d}x}.$$

Then,

$$y' + p(x)y = -cp(x)e^{-\int p(x)\,\mathrm{d}x} + cp(x)e^{-\int p(x)\,\mathrm{d}x} = 0.$$

So, $y = ce^{-\int p(x)\,\mathrm{d}x}$ is a solution for any choice of c. □

(b). Want to show: any solution of $y' + p(x)y = 0$ can be written as $y = ce^{-\int p(x)\,\mathrm{d}x}$. Note that $y = 0$ is a trivial solution, so we assume $y \neq 0$.

$$\begin{aligned} y' + p(x)y &= 0 \\ y' &= -p(x)y \\ \frac{y'}{y} &= -p(x) \\ \rightsquigarrow \int \frac{1}{y}\,\mathrm{d}y &= \int -p(x)\,\mathrm{d}x \\ \ln|y| &= -\int p(x)\,\mathrm{d}x \\ y &= ce^{-\int p(x)\,\mathrm{d}x}. \end{aligned}$$

Note that when $c = 0$, $y = 0$ is the trivial solution. So, any solution of $y' + p(x)y = 0$ can be written as $y = ce^{-\int p(x)\,\mathrm{d}x}$. ■

Example 1.2.2 Solve the IVP

$$\begin{cases} xy' + y = 0 \\ y(1) = 3. \end{cases}$$

Solution 2.
Note that $P_0(x) = x$ and $P_1(x) = 1$, which are continuous on $\mathbb{R}$. Since we need $P_0(x) \neq 0$, $x \neq 0$. So the interval of validity is $\mathbb{R}\backslash\{0\}$.

Method 1: Separation of Variables

$$y' = -\frac{y}{x}.$$

Note that $y = 0$ is a solution. Assume $y \neq 0$.

$$\begin{aligned} \frac{y'}{y} = -\frac{1}{x} \quad \rightsquigarrow \quad \int \frac{1}{y}\,\mathrm{d}y &= -\int \frac{1}{x}\,\mathrm{d}x + k \\ \ln|y| &= -\ln|x| + k \\ |y| &= e^k \frac{1}{|x|} \\ y &= \frac{c}{x} \end{aligned}$$

$\boxed{\text{Method 2: Solution Formula}}$ By Theorem 1.2.1,

$$y = ce^{-\int p(x)\,\mathrm{d}x} = ce^{-\int \frac{1}{x}\,\mathrm{d}x} = ce^{-\ln|x|} = \frac{c}{x}.$$

$\boxed{\text{Solving the IVP}}$ Substitute $x = 1$ and $y = 3$:

$$3 = \frac{c}{1} \quad \longrightarrow \quad c = 3.$$

So, $y^{\mathrm{IVP}} = \dfrac{3}{x}$. □

Example 1.2.3 Given the equation $(4 + x^2)y' + 2xy = 4x$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 3.

This is a first order, linear, non-homogeneous differential equation.

Note that $P_0(x) = 4 + x^2$, $P_1(x) = 2x$, $F(x) = 4x$, and $P_0 \neq 0\ \forall x \in \mathbb{R}$, so the interval of validity is $\mathbb{R}$. Also note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[4 + x^2\right] = 2x$, so the equation can be written as

$$(4 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left[4 + x^2\right]y = 4x.$$

Using the product rule to re-write the LHS as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[(4 + x^2)y\right] &= 4x \\ \int \frac{\mathrm{d}}{\mathrm{d}x}\left[(4 + x^2)y\right]\mathrm{d}x &= \int 4x\,\mathrm{d}x + c \\ (4 + x^2)y &= 2x^2 + c \\ y &= \frac{2x^2 + c}{4 + x^2}. \end{aligned}$$

□

Example 1.2.4 Given the equation $y' - 2y = 4 - x$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 4.

This is a first order, linear, non-homogeneous differential equation.

Since $P_0(x) = 1$, $P_1(x) = -2y$, $F(x) = 4 - x$, and $P_0(x) \neq 0\ \forall x \in \mathbb{R}$, the interval of validity is $\mathbb{R}$. Consider $\mu = \mu(x) \neq 0$. Multiply both sides of the equation by $\mu(x)$:

$$\mu(x)y' - 2\mu(x)y = \mu(x)(4 - x) \tag{1}$$

To make the LHS a product rule, we need

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)y(x)\right] = \mu'(x)y(x) + \mu(x)y'(x) = \mu(x)y'(x) - 2\mu(x)y.$$

So, we have $\mu' = -2\mu$, or $\mu' + 2\mu = 0$, a first order, linear, homogeneous ODE. Solving this ODE, we get $\mu(x) = ce^{-2x}$. Since we only want one specific μ that would work, take $c = 1$. So, $\mu(x) = e^{-2x}$. Substituting $\mu(x) = e^{-2x}$ to Eq. (1):

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}(4 - x), \quad \widetilde{P}_0 = e^{-2x} \neq 0,\ \widetilde{P}_1 = -2e^{-2x}.$$

Using the product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-2x}y\right] &= 4e^{-2x} - xe^{-2x} \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-2x}y\right] \mathrm{d}x &= \int 4e^{-2x} - xe^{-2x}\, \mathrm{d}x + c \\
e^{-2x}y &= \frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c \\
y &= e^{2x}\left(\frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c\right) \\
&= \frac{1}{2}x - \frac{7}{4} + ce^{2x}.
\end{aligned}$$

□

Theorem 1.2.5 Method of Integrating Factor
Given the first order linear differential equation $y' + p(x)y = f(x)$, with p and f both continuous on some interval (a, b),

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,\mathrm{d}x + c\right]$$

is the general solution to the equation, with

$$\mu(x) = e^{\int p(x)\,\mathrm{d}x}.$$

We call $\mu(x)$ the *integrating factor*.

Proof 5. Consider $\mu = \mu(x) \neq 0$. Multiplying the both sides of $y' + p(x)y = f(x)$ by μ:

$$\mu y' + p\mu y = \mu f. \tag{2}$$

Impose $\mu y' + p\mu y = \frac{\mathrm{d}}{\mathrm{d}x}[\mu y]$ to find $\mu = \mu(x)$:

$$\begin{aligned} \mu y' + p\mu y &= \mu' y + \mu y' \\ \mu' - p\mu &= 0, && \text{first order, linear, homogeneous ODE} \\ \mu(x) &= e^{\int p(x)\,\mathrm{d}x}, && \text{the integrating factor} \end{aligned}$$

Substitute $\mu(x) = e^{\int p(x)\,\mathrm{d}x}$ into Eq. (2):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[\mu y] &= \mu f \\ \int \frac{\mathrm{d}}{\mathrm{d}x}[\mu y]\,\mathrm{d}x &= \int \mu f\,\mathrm{d}x + c \\ \mu y &= \int \mu f\,\mathrm{d}x + c \\ y(x) &= \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,\mathrm{d}x + c\right]. \end{aligned}$$

■

2 Second Order ODEs

3 System of ODEs