Emory University

MATH 211 - Advanced Calculus (Multivariable) Learning Notes

Jiuru Lyu

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Contents

Preface

These is my personal notes for Emory University MATH 211 Advanced Calculus (Multivariable Calculus) course.

After mastering Calculus I (which covers contents concerning limits, differentiation, and basic integration) and Calculus II (which includes integration techniques and series), this course focuses on multivariable calculus, including vectors, multivariable functions, partial derivatives, optimization, multiple integrations, vector and scalar fields, Green's and Stokes' theorems, and the divergence theorem. The book used for this course is *Multivariable Calculus, 8th Edition* by James Stewart.

Throughout this personal note, I use different formats to differentiate different contents, including definitions, theorems, proofs, examples, extensions, and remarks. To be more specific:

Definition 0.0.1 (Terminology). This is a **definition**.

Theorem 0.0.1 (Theorem Name). This is a **theorem**.

Example 0.0.1. This is an **example**.

Solution. This is the *answer* part of an **example**. □

Remark. This is a **remark** of a definition, theorem, example, or proof.

Proof. This is a **proof** of a theorem. ■

Extension. This is a **extension** of a theorem, proof, or example.

To better ace this course, it is recommended to do more questions than provided as examples under each section. Although each example is distinctive and representative, more questions and practice is still needed to deepen the understanding of this course.

Even though I put efforts into making as few flaws as possible when encoding these learning notes, some errors may still exist in this note. If you find any, please contact me via email: `lvjiuru@hotmail.com`.

I hope you will find my notes helpful when learning Multivariable Calculus.

Cheers,
Jiuru Lyu

1 Vectors and Geometry of Space

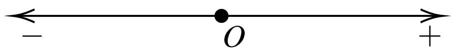
1.1 Three Dimensional Coordinate System

Definition 1.1.1 (Coordinate System). A **coordinate system** is a system that uses coordinate of a point to uniquely determine the position of the point in the space or plane.

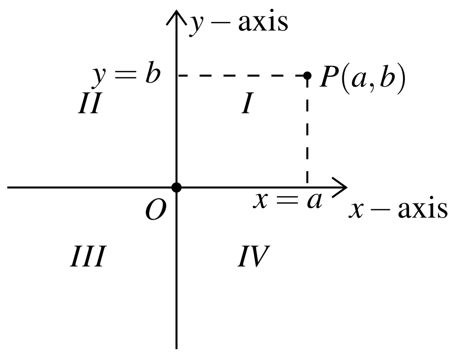
The Cartesian coordinate system is defined in different dimensions.

Definition 1.1.2 (One Dimensional Cartesian System). One Dimensional Cartesian System is a straight line with a fixed point as the origin and positive and negative directions.

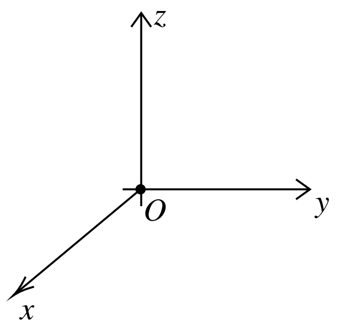
Remark. The one dimensional cartesian system is the number line:



Any point in the one dimensional Cartesian system corresponds to a number $\in \mathbb{R}$ and any number $\in \mathbb{R}$ has a location on the line. The two dimensional Cartesian system is the regular coordinate system.

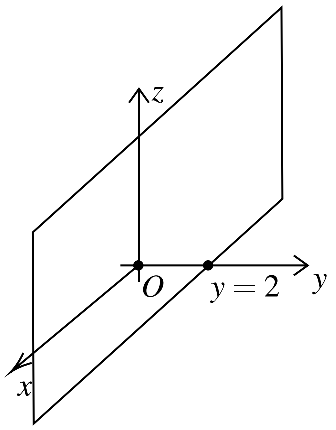


The three dimensional Cartesian system includes three perpendicular axes.

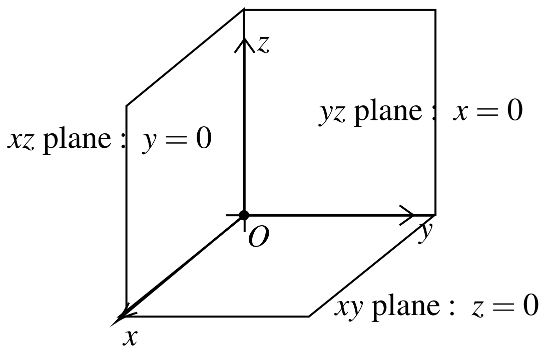


Definition 1.1.3 (Octant). A **Octant** is one of the eight divisions of the three dimensional coordinate system.

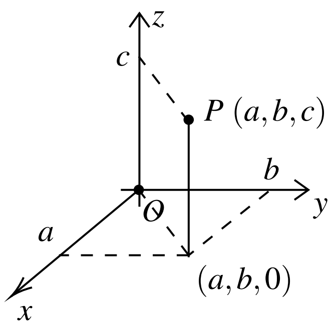
Definition 1.1.4 (Hyperplane). The hyperplane of $y = 2$ is given as below:



Specially:



Definition 1.1.5 (Points in the Three Dimensional System). $P(a,b,c)$ indicates the intersection of the three hyperplanes: $x=a$, $y=b$, and $z=c$.



For spaces in the higher dimension, we understand them via the Cartesian product.

Definition 1.1.6 (Cartesian Product).

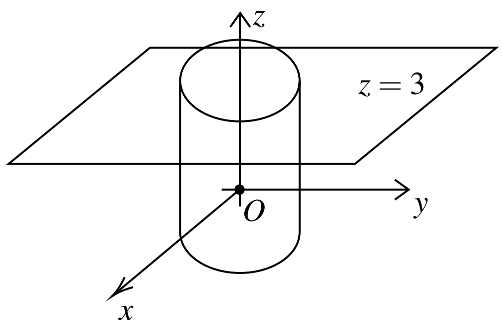
$$\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}=\{(x_1,\cdots,x_n)|x_i\in\mathbb{R}\forall i=1,\cdots,n\}$$

is the set of all n-tuples of real numbers and is denoted by $\mathbb{R}^n$.

Example 1.1.1. $(3,4,5)\in\mathbb{R}^3$ is 3 dimensional. $(3,4,5,6)\in\mathbb{R}^4$ is 4 dimensional.

Example 1.1.2. Which point(s) (x,y,z) satisfies the equations

$$x^2+y^2=1 \quad \text{and} \quad x=3?$$



Those points form a circle in the hyperplane of $z = 3$ centered at the point $(0,0,3)$ with a radius of 1.

Theorem 1.1.1 (Distance Formula in Three Dimension). For given points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, the distance between them is denoted by $|P_1P_2|$ and is defined by

$$|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}.$$

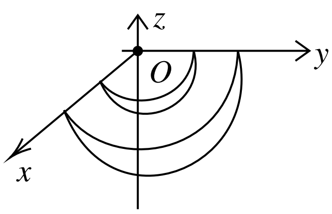
Theorem 1.1.2 (Equation of a Sphere). An equation of a sphere with a center of (a,b,c) and a radius of r is defined as

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2.$$

Example 1.1.3. What is the region in $\mathbb{R}^3$ represented by the inequalities

$$1 \le x^2+y^2+z^2 \le 4 \quad \text{and} \quad z \le 0?$$

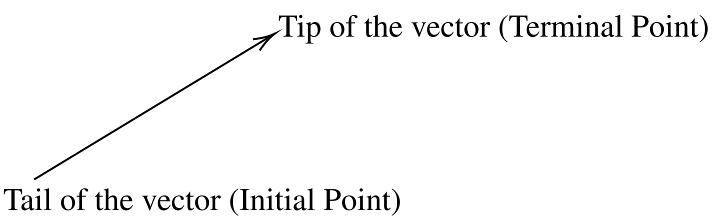
Solution.



The region is the difference between the half spheres (the lower half of the sphere) centered at $(0,0,0)$ with a radius of 1 and 2. □

1.2 Vectors

Definition 1.2.1 (Vectors). **Vectors** are used to indicate a quantity that has both magnitude and direction.



1. Vectors are denoted as $\vec{v}$.

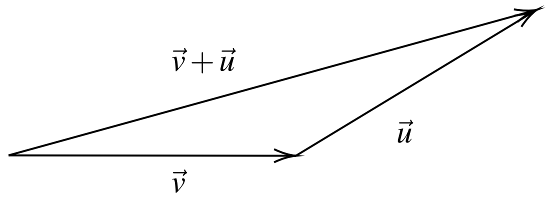
2. Magnitude

Definition 1.2.2 (Magnitude). A vector is a line segment, of which the **magnitude** of vector denoted by $|\vec{v}|$ is the length of it and the arrow points the direction of the vector.

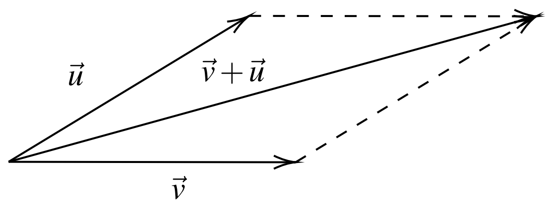
Vectors are operated in a different way:

1. Addition of Vectors:

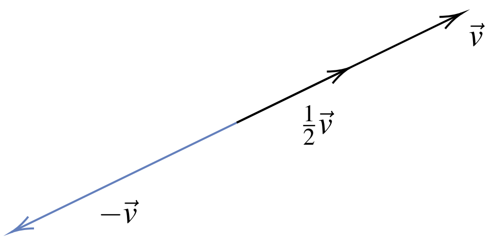
   (a) The triangle law:



   (b) The parallelogram law:



2. Scalar Multiplications:

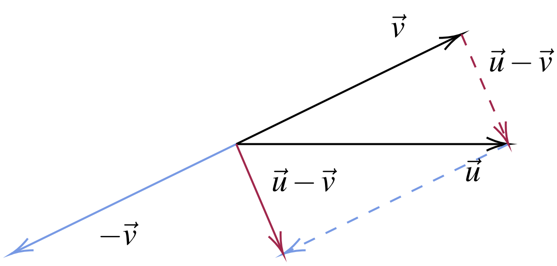


Definition 1.2.3 (Scalar Multiplication). If $c \in \mathbb{R}$ and $\vec{v}$ is a vector, then $c\vec{v}$ is in the same direction of $\vec{v}$ if $c > 0$ and in the opposite direction if $c < 0$.

Theorem 1.2.1. The magnitude of $c\vec{v}$:

$$|c\vec{v}| = c|\vec{v}|.$$

3. Differences of Vectors:



The difference of vectors $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}-\vec{v}$ and is defined by

$$\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$$

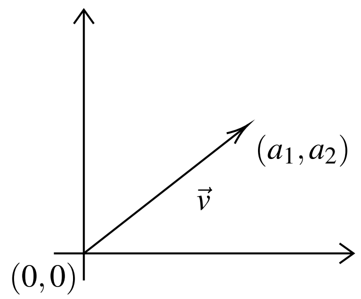
4. Properties of vectors:
   Suppose $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors in V_n and c and d are scalars (*Those properties can be proven geometrically*):
   (a) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
   (b) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
   (c) $\vec{a}+0=\vec{a}$
   (d) $\vec{a}+(-\vec{a})=0$
   (e) $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
   (f) $(c+d)\vec{a}=c\vec{a}+d\vec{a}$
   (g) $(cd)\vec{a}=c(d\vec{a})$
   (h) $1\cdot\vec{a}=\vec{a}$

We can link the coordinate system and vectors together:

1. **Definition 1.2.4 (Components of Vectors).** We will denote vector $\vec{v}$ as

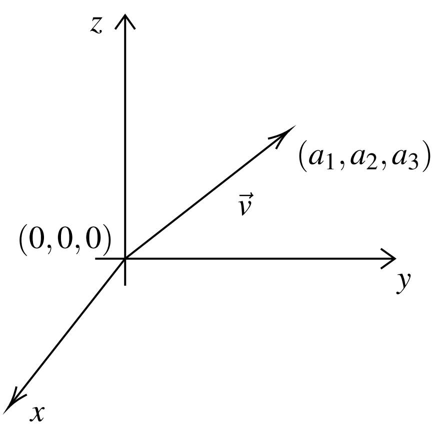
$$\vec{v}=\langle a_1,a_2\rangle,$$

   where a_1 and a_2 are called the **components** of $\vec{v}$.



2. In the three dimension:

$$\vec{v}=\langle a_1,a_2,a_3\rangle$$



3. **Definition 1.2.5.** If $A(x_1, y_1, z_1)$ as the tail of vector $\vec{v}$ and $B(x_1, y_1, z_1)$ as the tip of vector $\vec{v}$, then
$$\overrightarrow{AB} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$
$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

4. **Theorem 1.2.2.** If $\vec{v} = \langle a, b, c \rangle$ and $\vec{u} = \langle a', b', c' \rangle$, then
$$\vec{u} + \vec{v} = \langle a' + a,\ b' + b,\ c' + c \rangle$$
$$\vec{u} - \vec{v} = \langle a' - a,\ b' - b,\ c' - c \rangle$$
$$\alpha\vec{u} = \langle \alpha a',\ \alpha b',\ \alpha c' \rangle, \text{ where } \alpha \text{ is a scalar.}$$

Definition 1.2.6 (Standard Basis Vectors). In 2-D, $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$; and in 3-D, $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are called the **standard basis vectors**.

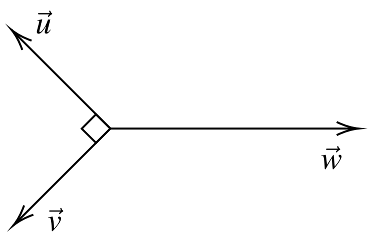
Remark. Any vectors in 2D and 3D can be written as
$$\vec{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Definition 1.2.7 (Unit Vector). A **unit vector**is a vector of magnitude of 1.

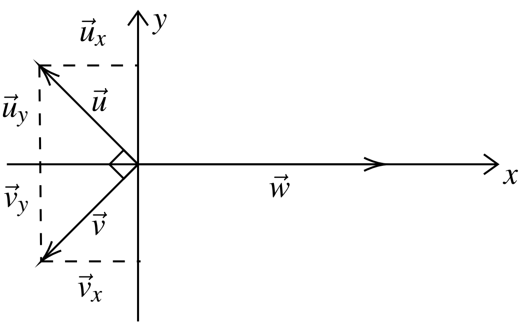
Example 1.2.1.
$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1 \text{ are unit vectors.}$$

Theorem 1.2.3. To find a unit vector in the direction of any vector $\vec{v}$, we use $\frac{1}{|\vec{v}|}\vec{v}$. The length of vector $\frac{\vec{v}}{|\vec{v}|}$ is 1 and its direction is the same as $\vec{v}$.

Example 1.2.2. If the vectors in the figure satisfy $|\vec{u}| = |\vec{v}| = 1$, and $\vec{u} + \vec{v} + \vec{w} = 0$, find $|\vec{w}|$.



Solution. Decompose the vectors:



We then have

$$\cos 45^\circ = \frac{|\vec{u}_x|}{\vec{u}} \implies |\vec{u}_x| = |\vec{u}|\cos 45^\circ;$$

$$\sin 45^\circ = \frac{|\vec{u}_y|}{\vec{u}} \implies |\vec{u}_y| = |\vec{u}|\sin 45^\circ;$$

$$\begin{aligned}\therefore \vec{u} = \langle |\vec{u}_x|,\ |\vec{u}_y\rangle &= -|\vec{u}_x|\mathbf{i} + |\vec{u}_y|\mathbf{j}\\ &= -\frac{\sqrt{2}}{2}|\vec{u}|\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\\ &= \frac{\sqrt{2}}{2}|\vec{u}|(-\mathbf{i}+\mathbf{j})\end{aligned}$$

Similarly,

$$\vec{v} = \frac{\sqrt{2}}{2}|\vec{v}|(-\mathbf{i}-\mathbf{j}).$$

We know $\vec{u}+\vec{v}+\vec{w} = 0$:

$$\therefore \vec{w} + \frac{\sqrt{2}}{2}|\vec{u}|(-\mathbf{i}+\mathbf{j}) + \frac{\sqrt{2}}{2}|\vec{v}|(-\mathbf{i}-\mathbf{j}) = 0$$

We know $|\vec{u}| = |\vec{v}| = 1$:

$$\begin{aligned}\therefore \vec{w} + \frac{\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j}) + \frac{\sqrt{2}}{2}(-\mathbf{i}-\mathbf{j}) &= 0\\ \vec{w} + \frac{\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j}-\mathbf{i}-\mathbf{j}) &= 0\\ \vec{w} &= \sqrt{2}\mathbf{i}\end{aligned}$$

$$\therefore \vec{w} = \langle\sqrt{2},0\rangle \implies |\vec{w}| = \sqrt{2}.$$

□

1.3 Dot Product

Definition 1.3.1 (Dot Product). If $\vec{u} = \langle x_1, y_1, z_1\rangle$ and $\vec{v} = \langle x_2, y_2, z_2\rangle$, then the dot product of $\vec{u}$ and $\vec{v}$ is defined as

$$\begin{aligned}\vec{u}\cdot\vec{v} &= \langle x_1, y_1, z_1\rangle \cdot \langle x_2, y_2, z_2\rangle\\ &= x_1x_2 + y_1y_1 + z_1z_2\end{aligned}$$

Remark. The dot product of two vectors returns a scalar.

Example 1.3.1. Let $\vec{u} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\vec{v} = 2\mathbf{j}-\mathbf{k}$. Find $\vec{u}\cdot\vec{v}$.

Solution.

$$\begin{aligned}\vec{u}\cdot\vec{v} &= \langle 1,2,-3\rangle \cdot \langle 0,2,-1\rangle\\ &= (1)(0)+(2)(2)+(-3)(-1) = 7.\end{aligned}$$

□

Properties of the dot product:

1. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
2. $\vec{a}\cdot(\vec{v}+\vec{c})=\vec{a}\vec{b}+\vec{a}\vec{c}$
3. $m(\vec{a}\cdot\vec{b})=(m\vec{a})\cdot\vec{b}=\vec{a}\cdot(m\vec{b})=(\vec{a}\cdot\vec{b})m$
4. $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$
   $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$

Theorem 1.3.1.

$$\vec{u}\cdot\vec{u}=|\vec{u}|^2.$$

Theorem 1.3.2. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

$$\boxed{\vec{u}\cdot\vec{v}=|\vec{u}|\cdot|\vec{v}|\cos\theta}.$$

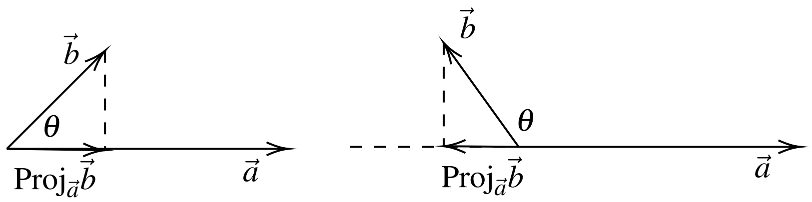
Extension.

$$\cos\theta=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

Extension.

$$\theta=90^\circ\iff\vec{u}\cdot\vec{v}=0.$$

Definition 1.3.2 (Projections). We use $\mathrm{Proj}_{\vec{a}}\vec{b}$ to denote the **projection** of $\vec{b}$ on $\vec{a}$.



From the diagrams,

$$\cos\theta=\frac{|\mathrm{Proj}_{\vec{a}}\vec{b}|}{|\vec{b}|}\Longrightarrow|\mathrm{Proj}_{\vec{a}}\vec{b}=\boxed{|\vec{b}|\cos\theta}.$$

We know that

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

$$\therefore\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}=\boxed{|\vec{b}|\cos\theta}$$

$$\therefore|\mathrm{Proj}_{\vec{a}}\vec{b}|=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|},\text{ which is a scalar.}$$

$|\mathrm{Proj}_{\vec{a}}\vec{b}|$ is called the **scalar projection** of $\vec{b}$ on $\vec{a}$.

$$\mathrm{Proj}_{\vec{a}}\vec{b}=|\mathrm{Proj}_{\vec{a}}\vec{b}|\cdot\underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector}}=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}$$

$\mathrm{Proj}_{\vec{a}}\vec{b}$ is called **projection** of $\vec{b}$ on $\vec{a}$ and is a vector.