# Emory University
# **MATH 347 Non Linear Optimization**
# Learning Notes

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January 18, 2024

## Contents

# 1 Math Preliminaries

## 1.1 Introduction to Optimization

**Definition 1.1 (Optimization Problem).** The main optimization problem can be stated as follows

$$\min_{x \in S} f(x), \tag{1}$$

where

- $x$ is the *optimization variable*,
- $S$ is the *feasible set*, and
- $f$ is the *objective function*.

**Remark 1.1** $\max\limits_{x \in S} f(x) = -\min\limits_{x \in S} -f(x)$. *Hence, we will only study minimization problems.*

**Theorem 1.2 Solving an Optimization Problem**

- Theoretical Analysis: analytic solution
- Numerical solution/optimization

**Definition 1.3 (Solution Methods depend on the type of $x$, $S$, and $f$).**

- When $x$ is continuous (e.g., $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, $\dots$), then the optimization problem stated in Eq. (1) is a *continuous optimization problem.* *It will also be the focus of this class.*

  Opposite to continuous optimization problems, we have *discrete optimization problem* if $x$ is discrete.

  If $x$ has both types of components, then we call the problem *mixed.*

- Depending on $S$, we can have
  - *Unconstrained problems*: where $S = \mathbb{R}^n$, $S = \mathbb{R}^{m\times n}$, $\dots$ $(m, n$ are fixed).
  - *Constrained problems*: where $S \subsetneq \mathbb{R}^n$, $S \subsetneq \mathbb{R}^{m\times n}$, $\dots$.

    *Both types of problems will be studied.*

- Depending on $f$, we have
  - *Smooth optimization problems*: $f$ has first and/or second order derivatives.

    *Only smooth optimization problems will be studied.*
  - *Non-smooth optimization problems*: $f$ is not differentiable.

**Definition 1.4 (Linear Optimization/Program).** If $f$ is linear and $S$ consists of linear constrains, then the optimization problem is called a *linear problem/program.*

**Example 1.5 Classification of Optimization Problems**

1. Consider the following problem

$$\min_{x_1,x_2,x_3} x_1^2 - 4x_1x_2 + 3x_2x_3 + \sin x_3$$

***Solution 1.***

- Optimization variable: $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. $\longrightarrow$ continuous.
- Feasible set: $S = \mathbb{R}^3$. $\longrightarrow$ unconstrained.
- Objective function: $f(x_1, x_2, x_3) = x_1^2 - 4x_1x_2 + 3x_2x_3 + \sin x_3$. $\longrightarrow$ smooth but non-linear.

$\square$

2. Consider the following problem

$$\max_{\substack{4x_1+7x_2+3x_3\leq 1\\ x_1,x_2,x_3\geq 0}} x_1 + 2x_2 + 3x_3$$

***Solution 2.***

- Optimization variable: $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. $\longrightarrow$ continuous.
- Feasible set: $S = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0, 4x_1 + 7x_2 + 3x_3 \leq 1\} \subsetneq \mathbb{R}^3$. $\longrightarrow$ constrained.
- Objective function: $f(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$. $\longrightarrow$ smooth and linear.

$\square$

**Remark 1.2** *This problem can be considered as the budget constrained optimization problem in Economics.*

3. Consider the following problem

$$\min_{x_1,x_2\geq 0} 4x_1 - 3|x_2| + \sin\big(x_1^2 - 2x_2\big)$$

***Solution 3.***

- Optimization variable: $x = (x_1, x_2) \in \mathbb{R}^2$. $\longrightarrow$ continuous.
- Feasible set: $S = \{(x_1, x_2) : x_1, x_2 \geq 0\} \subsetneq \mathbb{R}62$. $\longrightarrow$ constrained.
- Objective function: $f(x_1, x_2) = 4x_1 - 3|x_2| + \sin(x_1^2 - 2x_2)$. $\longrightarrow$ non-smooth and non-linear.

$\square$

**Remark 1.3** *In this particular problem,* $x_2 \geq 0$*, and so* $f(x_1, x_2) = 4x_1 - 3x_2 + \sin(x_1^2 - 2x_2)$ *on the feasible set. Hence, this problem can be equivalently written as*

$$\min_{x_1, x_2 \geq 0} 4x_1 - 3x_2 + \sin\left(x_1^2 - 2x_2\right),$$

*which is a smooth optimization problem.*

# 2 Unconstrained Optimization

# 3 Least Square

# 4 Constrained Optimization