

Emory University

# MATH 211 - Advanced Calculus (Multivariable) Learning Notes

Jiuru Lyu

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# 1 Vectors and Geometry of Space

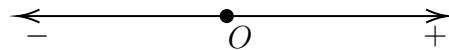
## 1.1 Three Dimensional Coordinate System

**Definition 1.1.1 (Coordinate System).** A **coordinate system** is a system that uses coordinate of a point to uniquely determine the position of the point in the space or plane.

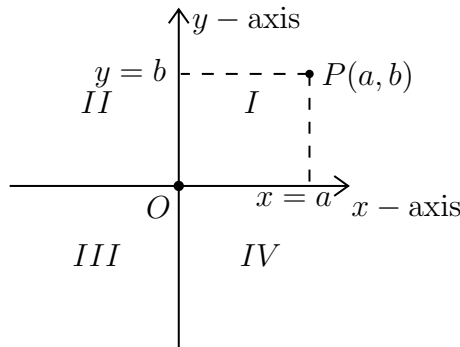
The Cartesian coordinate system is defined in different dimensions.

**Definition 1.1.2 (One Dimensional Cartesian System).** **One Dimensional Cartesian System** is a straight line with a fixed point as the origin and positive and negative directions.

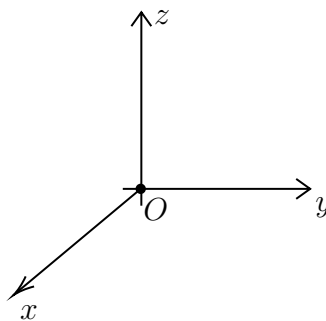
**Remark.** The one dimensional cartesian system is the number line:



Any point in the one dimensional Cartesian system corresponds to a number  $\in \mathbb{R}$  and any number  $\in \mathbb{R}$  has a location on the line. The two dimensional Cartesian system is the regular coordinate system.

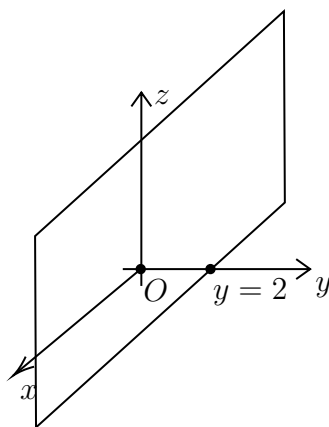


The three dimensional Cartesian system includes three perpendicular axes.

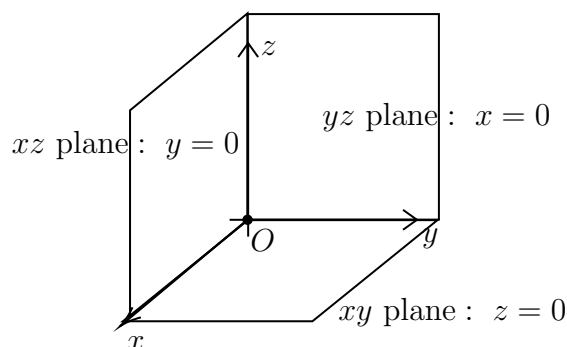


**Definition 1.1.3 (Octant).** A **Octant** is one of the eight divisions of the three dimensional coordinate system.

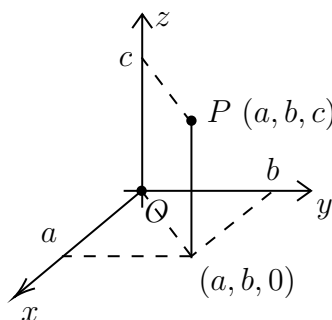
**Definition 1.1.4 (Hyperplane).** The hyperplane of  $y = 2$  is given as below:



Specially:



**Definition 1.1.5 (Points in the Three Dimensional System).**  $P(a, b, c)$  indicates the intersection of the three hyperplanes:  $x = a$ ,  $y = b$ , and  $z = c$ .



For spaces in the higher dimension, we understand them via the Cartesian product.

**Definition 1.1.6 (Cartesian Product).**

$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R} \forall i = 1, \cdots, n\}$$

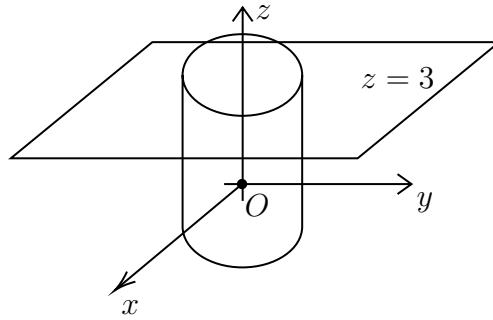
is the set of all  $n$ -tuples of real numbers and is denoted by  $\mathbb{R}^n$ .

**Example 1.1.1.**  $(3, 4, 5) \in \mathbb{R}^3$  is 3 dimensional.  $(3, 4, 5, 6) \in \mathbb{R}^4$  is 4 dimensional.

**Example 1.1.2.** Which point(s)  $(x, y, z)$  satisfies the equations

$$x^2 + y^2 = 1 \quad \text{and} \quad x = 3?$$

**Answer.**



Those points form a circle in the hyperplane of  $z = 3$  centered at the point  $(0, 0, 3)$  with a radius of 1.

□

**Theorem 1.1.1 (Distance Formula in Three Dimension).** For given points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$ , the distance between them is denoted by  $|P_1P_2|$  and is defined by

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

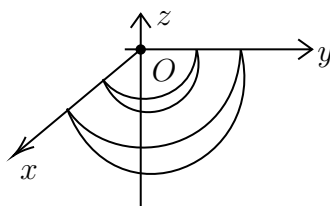
**Theorem 1.1.2 (Equation of a Sphere).** An equation of a sphere with a center of  $(a, b, c)$  and a radius of  $r$  is defined as

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

**Example 1.1.3.** What is the region in  $\mathbb{R}^3$  represented by the inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad \text{and} \quad z \leq 0?$$

**Answer.**

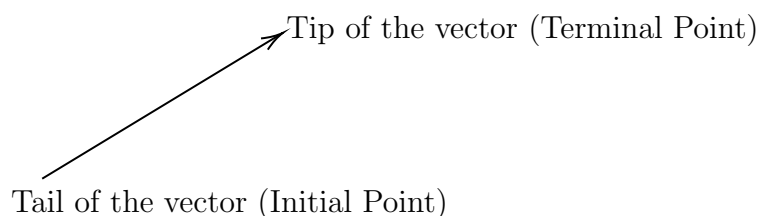


The region is the difference between the half spheres (the lower half of the sphere) centered at  $(0, 0, 0)$  with a radius of 1 and 2.

□

## 1.2 Vectors

**Definition 1.2.1 (Vectors).** **Vectors** are used to indicate a quantity that has both magnitude and direction.

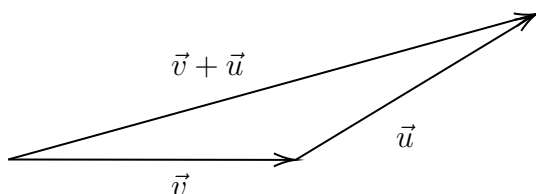


1. Vectors are denoted as  $\vec{v}$ .
2. Magnitude

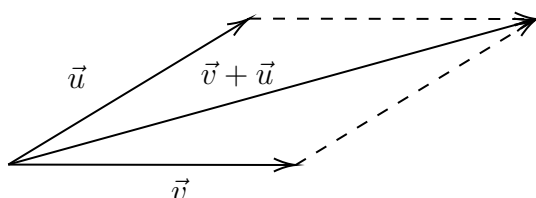
**Definition 1.2.2 (Magnitude).** A vector is a line segment, of which the **magnitude** of vector denoted by  $|\vec{v}|$  is the length of it and the arrow points the direction of the vector.

Vectors are operated in a different way:

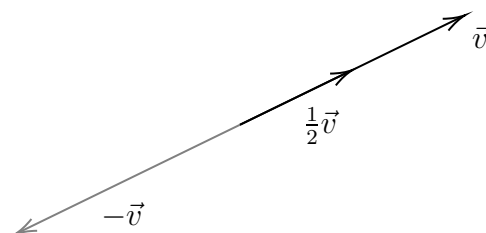
1. Addition of Vectors:
  - (a) The triangle law:



- (b) The parallelogram law:



2. Scalar Multiplications:

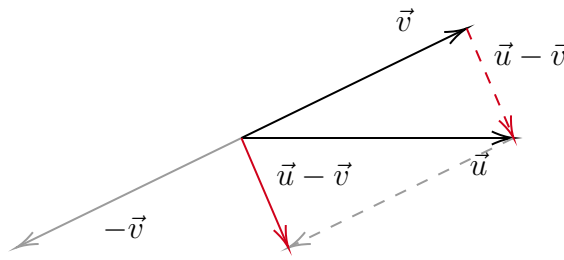


**Definition 1.2.3 (Scalar Multiplication).** If  $c \in \mathbb{R}$  and  $\vec{v}$  is a vector, then  $c\vec{v}$  is in the same direction of  $\vec{v}$  if  $c > 0$  and in the opposite direction if  $c < 0$ .

**Theorem 1.2.1.** The magnitude of  $c\vec{v}$ :

$$|c\vec{v}| = c|\vec{v}|.$$

3. Differences of Vectors:



The difference of vectors  $\vec{u}$  and  $\vec{v}$  is denoted by  $\vec{u} - \vec{v}$  and is defined by

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

4. Properties of vectors:

Suppose  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars (*Those properties can be proven geometrically*):

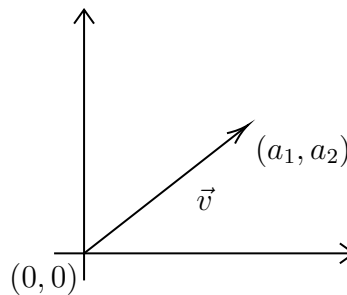
- (a)  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- (b)  $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- (c)  $\vec{a} + 0 = \vec{a}$
- (d)  $\vec{a} + (-\vec{a}) = 0$
- (e)  $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
- (f)  $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
- (g)  $(cd)\vec{a} = c(d\vec{a})$
- (h)  $1 \cdot \vec{a} = \vec{a}$

We can link the coordinate system and vectors together:

1. **Definition 1.2.4 (Components of Vectors).** We will denote vector  $\vec{v}$  as

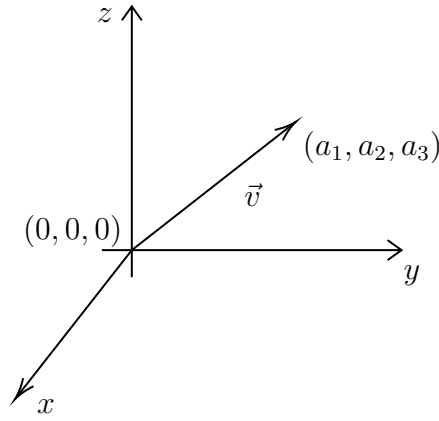
$$\vec{v} = \langle a_1, a_2 \rangle,$$

where  $a_1$  and  $a_2$  are called the **components** of  $\vec{v}$ .



2. In the three dimension:

$$\vec{v} = \langle a_1, a_2, a_3 \rangle$$



3. **Definition 1.2.5.** If  $A(x_1, y_1, z_1)$  as the tail of vector  $\vec{v}$  and  $B(x_2, y_2, z_2)$  as the tip of vector  $\vec{v}$ , then

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

4. **Theorem 1.2.2.** If  $\vec{v} = \langle a, b, c \rangle$  and  $\vec{u} = \langle a', b', c' \rangle$ , then

$$\vec{u} + \vec{v} = \langle a' + a, b' + b, c' + c \rangle$$

$$\vec{u} - \vec{v} = \langle a' - a, b' - b, c' - c \rangle$$

$$\alpha \vec{u} = \langle \alpha a', \alpha b', \alpha c' \rangle, \text{ where } \alpha \text{ is a scalar.}$$

**Definition 1.2.6 (Standard Basis Vectors).** In 2-D,  $\hat{\mathbf{i}} = \langle 1, 0 \rangle$  and  $\hat{\mathbf{j}} = \langle 0, 1 \rangle$ ; and in 3-D,  $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$ ,  $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$ , and  $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$  are called the **standard basis vectors**.

**Remark.** Any vectors in 2D and 3D can be written as

$$\vec{v} = \langle a, b, c \rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}.$$

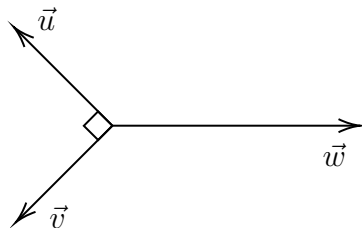
**Definition 1.2.7 (Unit Vector).** A **unit vector** is a vector of magnitude of 1.

**Example 1.2.1.**

$$|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1 \text{ are unit vectors.}$$

**Theorem 1.2.3.** To find a unit vector in the direction of any vector  $\vec{v}$ , we use  $\frac{1}{|\vec{v}|}\vec{v}$ . The length of vector  $\frac{\vec{v}}{|\vec{v}|}$  is 1 and its direction is the same as  $\vec{v}$ .

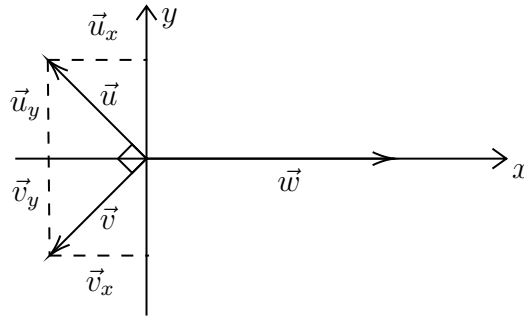
**Example 1.2.2.** If the vectors in the figure satisfy  $|\vec{u}| = |\vec{v}| = 1$ , and  $\vec{u} + \vec{v} + \vec{w} = 0$ , find  $|\vec{w}|$ .





**Answer.**

Decompose the vectors:



We then have

$$\cos 45^\circ = \frac{|\vec{u}_x|}{|\vec{u}|} \implies |\vec{u}_x| = |\vec{u}| \cos 45^\circ;$$

$$\sin 45^\circ = \frac{|\vec{u}_y|}{|\vec{u}|} \implies |\vec{u}_y| = |\vec{u}| \sin 45^\circ;$$

$$\begin{aligned} \therefore \vec{u} &= \langle |\vec{u}_x|, |\vec{u}_y| \rangle = -|\vec{u}_x|\hat{i} + |\vec{u}_y|\hat{j} \\ &= -\frac{\sqrt{2}}{2}|\vec{u}|\hat{i} + \frac{\sqrt{2}}{2}\hat{j} \\ &= \frac{\sqrt{2}}{2}|\vec{u}|(-\hat{i} + \hat{j}) \end{aligned}$$

Similarly,

$$\vec{v} = \frac{\sqrt{2}}{2}|\vec{v}|(-\hat{i} - \hat{j}).$$

We know  $\vec{u} + \vec{v} + \vec{w} = 0$ :

$$\therefore \vec{w} + \frac{\sqrt{2}}{2}|\vec{u}|(-\hat{i} + \hat{j}) + \frac{\sqrt{2}}{2}|\vec{v}|(-\hat{i} - \hat{j}) = 0$$

We know  $|\vec{u}| = |\vec{v}| = 1$ :

$$\begin{aligned} \therefore \vec{w} + \frac{\sqrt{2}}{2}(-\hat{i} + \hat{j}) + \frac{\sqrt{2}}{2}(-\hat{i} - \hat{j}) &= 0 \\ \vec{w} + \frac{\sqrt{2}}{2}(-\hat{i} + \hat{j} - \hat{i} - \hat{j}) &= 0 \\ \vec{w} &= \sqrt{2}\hat{i} \end{aligned}$$

$$\therefore \vec{w} = \langle \sqrt{2}, 0 \rangle \implies |\vec{w}| = \sqrt{2}.$$

□

### 1.3 Dot Product

**Definition 1.3.1 (Dot Product).** If  $\vec{u} = \langle x_1, y_1, z_1 \rangle$  and  $\vec{v} = \langle x_2, y_2, z_2 \rangle$ , then the dot product of  $\vec{u}$  and  $\vec{v}$  is defined as

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle \\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

**Remark.** The dot product of two vectors returns a scalar.

**Example 1.3.1.** Let  $\vec{u} = \hat{i} + 2\hat{j} - 3\hat{k}$  and  $\vec{v} = 2\hat{j} - \hat{k}$ . Find  $\vec{u} \cdot \vec{v}$ .

*Answer.*

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \langle 1, 2, -3 \rangle \cdot \langle 0, 2, -1 \rangle \\ &= (1)(0) + (2)(2) + (-3)(-1) = 7.\end{aligned}$$

□

Properties of the dot product:

1.  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2.  $\vec{a} \cdot (\vec{v} + \vec{c}) = \vec{a} \cdot \vec{v} + \vec{a} \cdot \vec{c}$
3.  $m(\vec{a} \cdot \vec{b}) = (m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b}) = (\vec{a} \cdot \vec{b})m$
4.  $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$   
 $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

**Theorem 1.3.1.**

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2.$$

**Theorem 1.3.2.** If  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$ , then

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos \theta}.$$

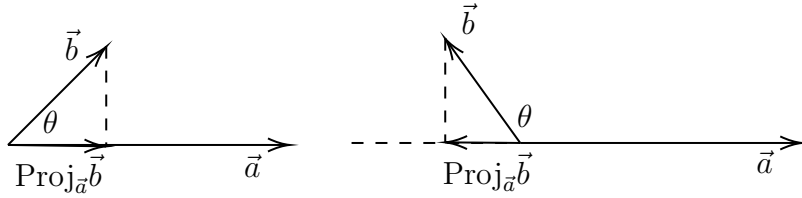
**Extension.**

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

**Extension.**

$$\theta = 90^\circ \iff \vec{u} \cdot \vec{v} = 0.$$

**Definition 1.3.2 (Projections).** We use  $\text{Proj}_{\vec{a}} \vec{b}$  to denote the **projection** of  $\vec{b}$  on  $\vec{a}$ .



From the diagrams,

$$\cos \theta = \frac{|\text{Proj}_{\vec{a}} \vec{b}|}{|\vec{b}|} \implies |\text{Proj}_{\vec{a}} \vec{b}| = \boxed{|\vec{b}| \cos \theta}.$$

We know that

$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos \theta \\ \therefore \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} &= \boxed{|\vec{b}| \cos \theta} \\ \therefore |\text{Proj}_{\vec{a}} \vec{b}| &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}, \text{ which is a scalar.} \end{aligned}$$

$|\text{Proj}_{\vec{a}} \vec{b}|$  is called the **scalar projection** of  $\vec{b}$  on  $\vec{a}$ .

$$\text{Proj}_{\vec{a}} \vec{b} = |\text{Proj}_{\vec{a}} \vec{b}| \cdot \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector}} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \cdot \vec{a}$$

$\text{Proj}_{\vec{a}} \vec{b}$  is called **projection** of  $\vec{b}$  on  $\vec{a}$  and is a vector.

**Example 1.3.2.** Find the scalar projection and vector projection of vector  $\vec{u} = \langle 1, 1, 2 \rangle$  onto  $\vec{v} = \langle -2, 3, 1 \rangle$ .

**Answer.**

$$\text{Proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v}; \quad |\text{Proj}_{\vec{v}} \vec{u}| = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

We need  $|\vec{v}| = \sqrt{4 + 9 + 1} = \sqrt{14}$  and  $\vec{u} \cdot \vec{v} = (1)(-2) + (1)(3) + (2)(1) = 3$

$$\therefore |\text{Proj}_{\vec{v}} \vec{u}| = \frac{3}{\sqrt{14}}$$

$$\text{Proj}_{\vec{v}} \vec{u} = \frac{3}{14} \cdot \vec{v} = \frac{3}{14} \cdot \langle -2, 3, 1 \rangle = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

□

## 1.4 Cross Product

**Definition 1.4.1 (Cross Product).** The **cross product** of  $\vec{u}$  and  $\vec{v}$  is denoted by  $\vec{u} \times \vec{v}$  and is a vector that is perpendicular to both  $\vec{u}$  and  $\vec{v}$ . If  $\vec{u} = \langle x_1, y_1, z_1 \rangle$  and  $\vec{v} = \langle x_2, y_2, z_2 \rangle$ , then

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = y_1 z_2 \hat{i} + x_2 z_1 \hat{j} + x_1 y_2 \hat{k} - x_2 y_1 \hat{k} - y_2 z_1 \hat{i} - x_1 z_2 \hat{j} \\ &= (y_1 z_2 - y_2 z_1) \hat{i} + (z_1 x_2 - z_2 x_1) \hat{j} + (x_1 y_2 - x_2 y_1) \hat{k}\end{aligned}$$

**Example 1.4.1.** Prove  $\vec{u} \times \vec{v}$  is perpendicular to both  $\vec{u}$  and  $\vec{v}$ .

*Proof.*

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= \langle x_1, y_1, z_1 \rangle \cdot \langle y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1 \rangle \\ &= x_1 y_1 z_2 - x_2 y_2 z_1 + x_2 y_1 z_1 - x_1 y_1 z_2 + x_1 y_2 z_1 - x_2 y_1 z_1 = 0 \\ &\therefore \vec{u} \times \vec{v} \perp \vec{u}\end{aligned}$$

Similarly,  $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0 \implies \vec{u} \times \vec{v} \perp \vec{v}$ . ■

**Theorem 1.4.1.** If  $\theta$  is the angle between vectors  $\vec{u}$  and  $\vec{v}$ , then

$$|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta.$$

*Proof.*

$$\begin{aligned}|\vec{u} \times \vec{v}|^2 &= (y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_1)^2 \\ &= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1 x_2 + y_1 y_2 + z_1 z_2)^2 \\ &= |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \\ &= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta \\ &= |\vec{u}|^2 |\vec{v}|^2 (1 - \cos^2 \theta) \\ &= |\vec{u}|^2 |\vec{v}|^2 \sin^2 \theta \\ &\therefore |\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| |\sin \theta|. \quad \blacksquare\end{aligned}$$

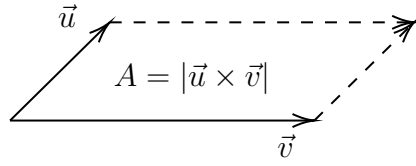
**Definition 1.4.2 (Parallel).** If two vectors,  $\vec{u}$  and  $\vec{v}$ , are parallel to each other,

$$\vec{u} = c\vec{v},$$

where  $c$  is a scalar.

**Theorem 1.4.2.** For two vectors  $\vec{u}$  and  $\vec{v}$ ,  $\vec{u} \times \vec{v} = 0$  iff  $\vec{u}$  and  $\vec{v}$  are parallel to each other.

**Theorem 1.4.3.** The length of the cross product,  $|\vec{u} \times \vec{v}|$ , is the area of the parallelogram determined by the vectors  $\vec{u}$  and  $\vec{v}$ .



**Theorem 1.4.4.**

$$\begin{aligned}\hat{i} \times \hat{j} &= \hat{k}; & \hat{j} \times \hat{k} &= \hat{i}; & \hat{k} \times \hat{i} &= \hat{j} \\ \hat{j} \times \hat{i} &= -\hat{k}; & \hat{k} \times \hat{j} &= -\hat{i}; & \hat{i} \times \hat{k} &= -\hat{j}\end{aligned}$$

Properties of cross product ( $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  are vectors, and  $c$  is a scalar):

1.  $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2.  $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3.  $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4.  $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5.  $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6.  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

**Definition 1.4.3 (Triple Product).** The **scalar triple product** is defined by

$$\vec{a} \cdot (\vec{b} \times \vec{c}).$$

**Theorem 1.4.5.**  $|\vec{a} \cdot (\vec{b} \times \vec{c})|$  denotes the volume of the parallelepiped determined by  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$ .

**Proof.**

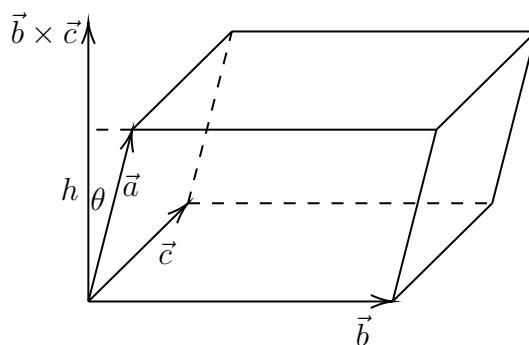
The area of the base is given by

$$A = |\vec{b} \times \vec{c}|$$

To find the volume, we need to know the height  $h$ :

$$h = |\vec{a}| |\cos \theta|$$

$$\therefore V = Ah = |\vec{b} \times \vec{c}| |\vec{a}| |\cos \theta| = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad \left[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \right]$$

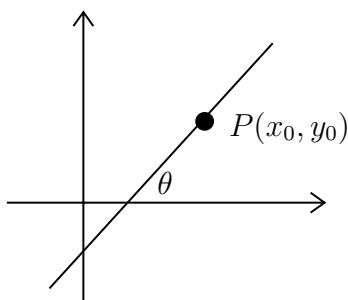


■

## 1.5 Equations of Lines and Planes

**Theorem 1.5.1 (Equation of Lines in 2D).** If we have a point  $P(x_0, y_0)$  and a direction (slope/ $\theta$ /another point on the line), we have the equation of the line:

$$\text{Given } \begin{cases} \text{slope} = m \\ P(x_0, y_0) \end{cases} \implies \text{The equation of the line: } y - y_0 = m(x - x_0).$$

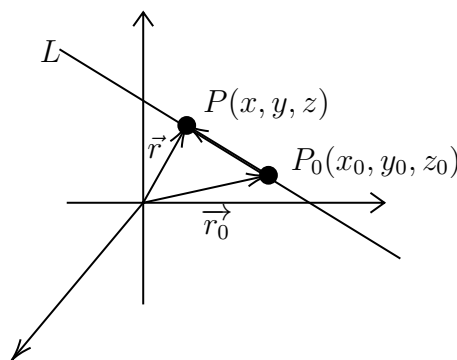


**Definition 1.5.1 (Directional Vector).** If  $\vec{v}$  is a directional vector of line  $L$ ,

$$\vec{a} = t\vec{v},$$

where  $\vec{a}$  is any vector determined by two points on the line.

**Definition 1.5.2 (Vector Equations of Lines in 3D).** Let  $\overrightarrow{P_0P} = \vec{a} \implies \vec{a} = \langle x - x_0, y - y_0, z - z_0 \rangle$



From the diagram, we also have

$$\vec{r}_0 + \vec{a} = \vec{r}.$$

As  $\vec{a} = t\vec{v}$ ,

$$\vec{r} = \vec{r}_0 + t\vec{v},$$

which is the **vector equation** of line  $L$ .

**Theorem 1.5.2.** If  $L$  is a line with point  $P(x_0, y_0, z_0)$  on it and paralleled to a direction vector  $\vec{v} = \langle a, b, c \rangle$ , we have

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle,$$

where  $t$  is a parameter and the equation is called the **vector equation** of line  $L$ .

**Extension (Parametric Equation of  $L$ ).** From  $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$ , we have

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

This system of equations is called the **parametric equation** of  $L$ .

**Extension (Symmetric Equation of  $L$ ).** From the parametric equation of  $L$ , we can derive  $t$ :

$$\begin{cases} x = x_0 + ta & \implies & t = \frac{x-x_0}{a} \\ y = y_0 + tb & \implies & t = \frac{y-y_0}{b} \\ z = z_0 + tc & \implies & t = \frac{z-z_0}{c} \end{cases}$$

As  $t$  should be equal:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

which is called the **symmetric equation** of the line with point  $P(x_0, y_0, z_0)$  and a directional vector  $\vec{v} = \langle a, b, c \rangle$ .

**Remark (Three Forms of Equation of a Line).** For line  $L$  in 3D,  $P_0(x_0, y_0, z_0)$  is on  $L$  and  $\vec{v} = \langle a, b, c \rangle$  is a directional vector of  $L$ .

1. The vector form:

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

2. The parametric form:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

3. The symmetric form:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

**Example 1.5.1.** Find the parametric and symmetric equations of the line  $L$  passing through the points  $(-8, 1, 4)$  and  $(3, -2, 4)$ .

**Answer.**

Let's set  $P_0$  to be  $(-8, 1, 4)$  and  $P_1$  to be  $(3, -2, 4)$ . So we can find the directional vector

$$\vec{v} = \overrightarrow{P_0P_1} = \langle 3 - (-8), -2 - 1, 4 - 4 \rangle = \langle 11, -3, 0 \rangle.$$

$\therefore$  The parametric equation of  $L$ :

$$\begin{cases} x = -8 + 11t \\ y = 1 - 3t \\ z = 4 + (0)t \end{cases},$$

and the symmetric equation of  $L$  is

$$\frac{x + 8}{11} = \frac{y - 1}{-3}, \quad z = 4.$$

□

Relationships of two lines in 3D:

1. Parallel: directional vectors of the two lines are parallel to each other.
2. Intersect: the two lines share one common point
3. Skewed: the two lines are neither parallel nor intersecting.

**Example 1.5.2.** Let

$$L_1 : \frac{x - 2}{1} = \frac{y - 3}{-2} = \frac{z - 1}{-3} \quad \text{and} \quad L_2 : \frac{x - 3}{1} = \frac{y + 4}{3} = \frac{z - 2}{-7}.$$

Find the relationship between  $L_1$  and  $L_2$ .

**Answer.**

$$\vec{v}_1 = \langle 1, -2, -3 \rangle; \quad \vec{v}_2 = \langle 1, 3, -7 \rangle$$

Because  $\vec{v}_1$  and  $\vec{v}_2$  are not parallel to each other,  $L_1$  and  $L_2$  are not parallel to each other.

$\therefore$   $L_1$  and  $L_2$  can only be intersecting or skewed.

To further discuss the relationship between  $L_1$  and  $L_2$ , form parametric equations:

$$L_1 : \begin{cases} x = 2 + t \\ y = 3 - 2t \\ z = 1 - 3t \end{cases} \quad L_2 : \begin{cases} x = 3 + s \\ y = -4 + 3s \\ z = 2 - 7s \end{cases}$$



If we can find a set of solutions  $t$  and  $s$  that satisfy the following system of equations, the two lines have point in common and thus is intersecting:

$$\begin{cases} 2 + t = 3 + s \\ 3 - 2t = -4 + 3s \\ 1 - 3t = 2 - 7s \end{cases} \implies \begin{cases} t - s = 1 & \textcircled{1} \\ 2t + 3s = 7 & \textcircled{2} \\ 3t - 7s = -1 & \textcircled{3} \end{cases}$$

From  $\textcircled{1}$ :

$$t = s + 1 \quad \textcircled{4}$$

Substitute  $\textcircled{2}$  with  $\textcircled{4}$ :

$$\begin{aligned} 2(s + 1) + 3s &= 7 \\ 2s + 2 + 3s &= 7 \implies 4s = 5 \implies s = 1 \\ \therefore t &= s + 1 = 1 + 1 = 2 \end{aligned}$$

Substitute  $s = 1$  and  $t = 2$  to  $\textcircled{3}$ :

$$\text{LHS} = 2(3) - 7(1) = 6 - 7 = -1 = \text{RHS}.$$

Hence,  $\begin{cases} t = 2 \\ s = 1 \end{cases}$  satisfy all three equations. Substitute  $t = 2$  to  $L_1$ :

$$x = 2 + 2 = 4, \quad y = 3 - 2(2) = -1, \quad z = 1 - 3(2) = -5.$$

$\therefore$  The two lines intersect at  $(4, -1, -5)$ .

□

**Theorem 1.5.3 (Line Segment that Connects  $\vec{\mathbf{r}}_0$  and  $\vec{\mathbf{r}}_1$ ).**

$$\vec{\mathbf{r}}(t) = (1 - t)\vec{\mathbf{r}}_0 + t\vec{\mathbf{r}}_1, \quad 1 \leq t \leq 1.$$

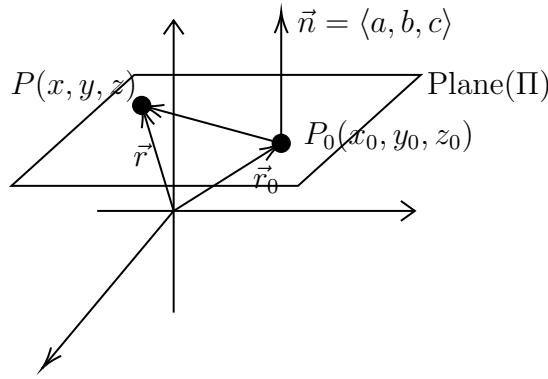
The vector equation gives a line segment the joins the tip of  $\vec{\mathbf{r}}_0$  to the tip of  $\vec{\mathbf{r}}_1$ .

**Definition 1.5.3 (Normal Vector).** A normal vector is the vector perpendicular to the plane and is often denoted as  $\vec{\mathbf{n}}$ .

**Theorem 1.5.4 (Vector Equation of a Plane).** As  $\vec{\mathbf{n}} \perp \Pi$ ,  $\vec{\mathbf{n}} \perp \overrightarrow{P_0P}$

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{\mathbf{r}} - \vec{\mathbf{r}}_0 \\ \therefore \vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) &= 0 \\ \vec{\mathbf{n}} \cdot \vec{\mathbf{r}} - \vec{\mathbf{n}} \cdot \vec{\mathbf{r}}_0 &= 0 \implies \vec{\mathbf{n}} \cdot \vec{\mathbf{r}} = \vec{\mathbf{n}} \cdot \vec{\mathbf{r}}_0, \end{aligned}$$

which is called the **vector equation** of a plane.



**Extension (Scalar Equation of a Plane).** From  $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ : As  $\vec{n} = \langle a, b, c \rangle$  and  $\vec{r} - \vec{r}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle$ , we have

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0;$$

$$\therefore a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which is the **scalar equation** of plane  $\Pi$  with point  $P_0(x_0, y_0, z_0)$  on it and a normal vector  $\vec{n} = \langle a, b, c \rangle$ .

**Extension (Linear Equation of a Plane).** From  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ :

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

Take  $d = -(ax_0 + by_0 + cz_0)$ :

$$ax + by + cz + d = 0,$$

which is called the **linear equation** of plane  $\Pi$  with point  $P_0(x_0, y_0, z_0)$  on it and a normal vector  $\vec{n} = \langle a, b, c \rangle$ .

**Remark (Equations of a Plane).** If point  $P_0(x_0, y_0, z_0)$  is on the plane  $\Pi$  and a normal vector of  $\Pi$  is  $\vec{n} = \langle a, b, c \rangle$ :

1. The vector equation:

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

2. The scalar equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

3. The linear equation:

$$ax + by + cz + d = 0,$$

$$\text{where } d = -(ax_0 + by_0 + cz_0) = -\langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$$

**Example 1.5.3.** Find an equation of the plane crossing through the points  $P(1, 3, 2)$ ,  $Q(3, -1, 6)$ , and  $R(5, 2, 0)$ .

**Answer.**

Find the normal vector using the following equation:

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

$$\overrightarrow{PQ} = \langle 3 - 1, -1 - 3, 6 - 2 \rangle = \langle 2, -4, 4 \rangle$$

$$\overrightarrow{PR} = \langle 5 - 1, 2 - 3, 0 - 2 \rangle = \langle 4, -1, -2 \rangle$$

$$\therefore \vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\hat{i} + 20\hat{j} + 14\hat{k}.$$

$$\therefore \vec{n} = \langle 12, 20, 14 \rangle, \quad P(1, 3, 2)$$

$$\therefore d = -\langle 12, 20, 14 \rangle \cdot \langle 1, 3, 2 \rangle = -(12 + 60 + 28) = -100.$$

$$\therefore \text{Linear Equation of } \Pi : 12x + 20y + 14z - 100 = 0 \implies 6x + 10y + 7z - 50 = 0.$$

□

**Theorem 1.5.5 (Relationship Between Two Planes).** If  $\vec{n}_1$  is a normal vector of plane  $\Pi_1$ , and  $\vec{n}_2$  is a normal vector of plane  $\Pi_2$ , then the angle between the two planes is given by

$$\theta = \cos^{-1} \left( \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \right).$$

i.e., the angle between the planes is the angle between the normal vectors.

**Theorem 1.5.6 (Distance from a Point to a Plane).** Distance of the point  $P(x_1, y_1, z_1)$  from the plane  $ax + by + cz + d = 0$ :

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad (1)$$

OR

$$D = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{n}|}, \quad (2)$$

where  $\vec{n}$  is the normal vector.

**Example 1.5.4.** Find the distance between the parallel planes:

$$\Pi_1 : 10x + 2y - 2z = 5 \quad \text{and} \quad \Pi_2 : 5x + y - z = 1.$$

**Answer.**

Assume point  $P(x_1, y_1, z_1)$  is on plane  $\Pi_1$ :

$$10x_1 + 2y_1 - 2z_1 = 5$$

$$\therefore 5x_1 + y_1 - z_1 = \frac{5}{2}$$

Applying formula 1:  $\vec{n} = \langle a, b, c \rangle = \langle 5, 1, -1 \rangle$ ,  $d = -1$ :

$$\therefore D = \frac{|5x_1 + y_1 - z_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|\frac{5}{2} - 1|}{\sqrt{26 + 1 + 1}} = \frac{3/2}{\sqrt{27}} = \frac{3}{2\sqrt{27}} \left( = \frac{\sqrt{3}}{6} \right).$$

□

**Extension.** Find the distance between two parallel planes:

$$\Pi_1 : ax + by + cz + d = 0 \quad \text{and} \quad \Pi_2 : ax + by + cz + d' = 0.$$

Let point  $P(x_1, y_1, z_1)$  on  $\Pi_1$ :

$$ax_1 + by_1 + cz_1 + d = 0$$

Apply formula 1:

$$D = \frac{|ax_1 + by_1 + cz_1 + d'|}{\sqrt{a^2 + b^2 + c^2}} = \frac{-d + d'}{\sqrt{a^2 + b^2 + c^2}}.$$

## 1.6 Cylinders and Quadric Surfaces

**Definition 1.6.1 (Cylinders).** A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

**Definition 1.6.2 (Quadric Surfaces).** A **quadric surface** is the graph of a second-degree equation in three variables  $x$ ,  $y$ , and  $z$ . The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where  $A, B, C, \dots, J$  are constants, but by translation and rotation it can be brought into one of the standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

**Remark.** Graphs of Quadric Surfaces (Refer to Page 877 of the Book):

1. Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses.

If  $a = b = c$ , the ellipsoid is a sphere.

2. Elliptic Paraboloid:

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses. Vertical traces are parabolas.

The variable raised to the first power indicates the axis of the paraboloid.

## 3. Hyperbolic Paraboloid:

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas. Vertical traces are parabolas.

## 4. Cone:

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.

Vertical traces in the planes  $x = k$  and  $y = k$  are hyperbolas if  $k \neq 0$  but are pairs of lines if  $k = 0$ .

## 5. Hyperboloid of One Sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses. Vertical traces are hyperbolas.

The axis of symmetry corresponds to the variable whose coefficient is negative.

## 6. Hyperboloid of Two Sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in  $z = k$  are ellipses if  $k > c$  or  $k < -c$ . Vertical traces are hyperbolas.

The two minus sign indicate two sheets.

## 2 Vector Functions

### 2.1 Vector Functions and Space Curves

**Definition 2.1.1 (Component Functions).**  $f(t)$ ,  $g(t)$ ,  $h(t)$  are real valued function and are called **component functions** of  $\vec{r}(t)$ . We write

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}.$$

**Definition 2.1.2 (Limit of Vector Functions).** To find the limit of a vector function, we check its component functions. That is

$$\lim_{t \rightarrow a} \vec{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

**Definition 2.1.3 (Continuity of Vector Functions).** A vector function  $\vec{r}(t)$  is continuous if

$$\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a).$$

**Example 2.1.1.** 1. Find the domain of

$$\vec{r}(t) = \left\langle \ln(t+1), \frac{t}{\sqrt{9-t^2}}, 2^t \right\rangle$$

**Answer.**

- Domain of  $\ln(t+1)$ :  $D_1: t+1 > 0, t > -1$
- Domain of  $\frac{t}{\sqrt{9-t^2}}$ :  $D_2: 9-t^2 > 0, -3 < t < 3$
- Domain of  $2^t$ :  $D_3: \mathbb{R}$

Find the intersection of domains of component functions:

$$D_1 \cap D_2 \cap D_3: -1 < t < 3 \quad (t \in (-1, 3))$$

□

2. Find  $\lim_{t \rightarrow 0} \vec{r}(t)$ .

**Answer.**

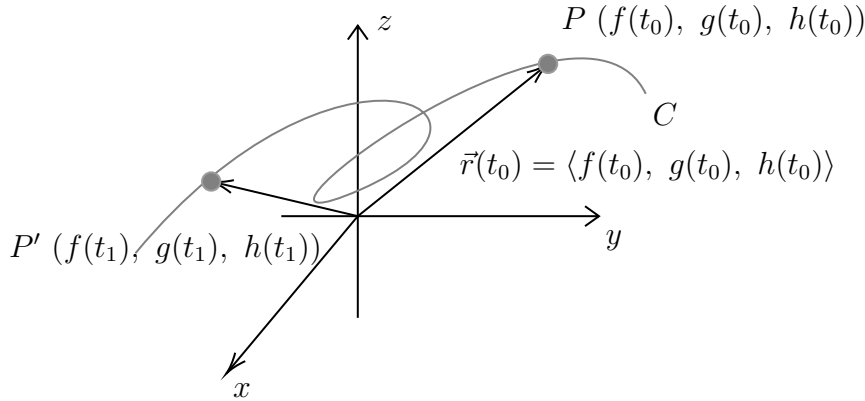
$$\begin{aligned} \lim_{t \rightarrow 0} \vec{r}(t) &= \left\langle \lim_{t \rightarrow 0} \ln(t+1), \lim_{t \rightarrow 0} \frac{t}{\sqrt{9-t^2}}, \lim_{t \rightarrow 0} 2^t \right\rangle \\ &= \left\langle \ln(1), \frac{0}{\sqrt{9}}, 2^0 \right\rangle \\ &= \langle 0, 0, 1 \rangle = \hat{\mathbf{k}} \end{aligned}$$

□

**Example 2.1.2.**

$$\begin{aligned}
 & \lim_{t \rightarrow 1} \left( \frac{t^2 - t}{t - 1} \hat{\mathbf{i}} + \sin \pi t \hat{\mathbf{j}} + \cos 2\pi t \hat{\mathbf{k}} \right) \\
 &= \lim_{t \rightarrow 1} \left( \frac{t(t - 1)}{t - 1} \hat{\mathbf{i}} + \sin \pi t \hat{\mathbf{j}} + \cos 2\pi t \hat{\mathbf{k}} \right) \\
 &= \lim_{t \rightarrow 1} t \hat{\mathbf{i}} + \lim_{t \rightarrow 1} \sin \pi t \hat{\mathbf{j}} + \lim_{t \rightarrow 1} \cos 2\pi t \hat{\mathbf{k}} \\
 &= \hat{\mathbf{i}} + \sin \pi \hat{\mathbf{j}} + \cos 2\pi \hat{\mathbf{k}} \\
 &= \hat{\mathbf{i}} + \hat{\mathbf{k}}
 \end{aligned}$$

**Definition 2.1.4 (Graphs of Vector Functions).** For a vector function  $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ , the graph of it, curve  $C$ , is defined by the moving tip of the vectors yielded from the vector function.



**Definition 2.1.5 (Space Curve).** If  $f, g, h$ , are continuous real-valued functions on an interval  $I$ , then the set  $C$  of all points  $(x, y, z)$  in space s.t.

$$x = f(t) \quad y = g(t) \quad z = h(t), \quad \text{where } t \in I$$

is called a **space curve**.

**Definition 2.1.6 (Parametric Equation).** The system of equations 
$$\begin{cases} x = f(t) \\ y = g(y) \\ z = h(t) \end{cases}$$
 is called a **parametric equation** of  $C$  and  $t$  is called the **parameter**.

## 2.2 Derivative and Integral of Vector Functions

Limits, continuity, derivative, and integrals of vector functions follow rules similar to those of scalar functions.

**Definition 2.2.1 (Derivative of Vector Functions).**

$$\frac{d\vec{\mathbf{r}}}{dt} = \lim_{h \rightarrow 0} = \frac{\vec{\mathbf{r}}(t + h) - \vec{\mathbf{r}}(t)}{h},$$

$\frac{d\vec{r}}{dt}$  or  $\vec{r}'(t)$  is the derivative of  $\vec{r}(t)$  is the limit on the right hand side exists.

**Extension.** If  $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ , then

$$\vec{r}'(t) = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}.$$

**Remark (Higher Order Derivatives).** Higher order derivatives  $\frac{d^{(n)}\vec{r}}{dt^{(n)}}$  can be defined similarly.

**Theorem 2.2.1 (Graphic Interpretation of Derivative).** When  $h \rightarrow 0$ , the vector

$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

becomes  $\vec{r}'(t)$  and therefore,  $\vec{r}'(t)$  approaches to a vector that lies on the tangent line.  $\vec{r}'(t)$  is called the **tangent vector**, and

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

is called the **unit tangent vector**.

**Example 2.2.1.** Find parametric equations of the tangent line to the vector function  $\vec{r}(t) = \langle 2 \cos t, \sin t, t \rangle$  at point  $(0, 1, \frac{\pi}{2})$ .

**Answer.**

When  $t = \frac{\pi}{2}$ ,  $2 \cos \frac{\pi}{2} = 0$ ,  $\sin \frac{\pi}{2} = 1$ .

$\therefore (0, 1, \frac{\pi}{2})$  is on the space curve of  $\vec{r}(t)$ .

Find

$$\begin{aligned}\vec{r}'(t) &= \langle (2 \cos t)', (\sin t)', t' \rangle \\ &= \langle -2 \sin t, \cos t, 1 \rangle\end{aligned}$$

When  $t = \frac{\pi}{2}$ ,

$$\vec{r}'\left(\frac{\pi}{2}\right) = \left\langle -2 \sin\left(\frac{\pi}{2}\right), \cos\left(\frac{\pi}{2}\right), 1 \right\rangle = \langle -2, 0, 1 \rangle$$

$\therefore \vec{d}$  of tangent line =  $\langle -2, 0, 1 \rangle$

$$\therefore \text{Line: } \left\langle 0, 1, \frac{\pi}{2} \right\rangle + \langle -2, 0, 1 \rangle t = \left\langle -2t, 1, \frac{\pi}{2} + t \right\rangle$$

□

**Example 2.2.2.** If  $\vec{r}(t) = (t^3 + 2t)\hat{i} - 3e^{-2t}\hat{j} + 2 \sin 5t\hat{k}$ . Find  $\frac{d\vec{r}}{dt}$ ,  $\left| \frac{d\vec{r}}{dt} \right|$ ,  $\frac{d^2\vec{r}}{dt^2}$ ,  $\left| \frac{d^2\vec{r}}{dt^2} \right|$ .

**Answer.**

$$\frac{d\vec{r}}{dt} = \langle 3t^2 + 2, -6e^{-2t}, 10 \cos 5t \rangle$$

$$\frac{d^2\vec{r}}{dt^2} = \langle 6t, -12e^{-2t}, -50 \sin 5t \rangle$$



When  $t = 0$ :

$$\vec{\mathbf{r}}'(0) = \langle 2, 6, 10 \rangle; \quad \vec{\mathbf{r}}''(0) = \langle 0, -12, 0 \rangle$$

$$\therefore |\vec{\mathbf{r}}'(0)| = \sqrt{4 + 36 + 100} = \sqrt{140} (= 2\sqrt{70}); \quad |\vec{\mathbf{r}}''(0)| = \sqrt{144} = 12.$$

□

**Theorem 2.2.2 (Properties of Differentiation).**

$$\frac{d}{dt}[\vec{\mathbf{r}}_1(t) + \vec{\mathbf{r}}_2(t)] = \frac{d}{dt}[\vec{\mathbf{r}}_1(t)] + \frac{d}{dt}[\vec{\mathbf{r}}_2(t)]$$

$$\frac{d}{dt}[\alpha \vec{\mathbf{r}}(t)] = \alpha \frac{d}{dt}[\vec{\mathbf{r}}(t)]$$

$$\frac{d}{dt}[f(t)\vec{\mathbf{r}}(t)] = f'(t)\vec{\mathbf{r}}(t) + f(t)\vec{\mathbf{r}}'(t)$$

$$\frac{d}{dt}[\vec{\mathbf{r}}_1(t) \cdot \vec{\mathbf{r}}_2(t)] = \vec{\mathbf{r}}_1'(t) \cdot \vec{\mathbf{r}}_2(t) + \vec{\mathbf{r}}_1(t) \cdot \vec{\mathbf{r}}_2'(t)$$

$$\frac{d}{dt}[\vec{\mathbf{r}}_1(t) \times \vec{\mathbf{r}}_2(t)] = \vec{\mathbf{r}}_1'(t) \times \vec{\mathbf{r}}_2(t) + \vec{\mathbf{r}}_1(t) \times \vec{\mathbf{r}}_2'(t)$$

**Example 2.2.3.** Show that if a curve lies on a sphere with center at the origin, then  $\vec{\mathbf{r}}'(t)$  is perpendicular to  $\vec{\mathbf{r}}(t)$  for any  $t$ .

**Answer.**

Let  $\vec{\mathbf{r}}(t)$  lies on a sphere, with center at the origin, and radius  $R = c$ :

$$\therefore \vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle \quad \text{and} \quad x^2(t) + y^2(t) + z^2(t) = c^2$$

$$x^2(t) + y^2(t) + z^2(t) = |\vec{\mathbf{r}}(t)|^2 = \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)$$

$$\therefore \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t) = c^2$$

Take derivative of the both sides of the euqation

$$\frac{d}{dt}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)] = \frac{d}{dt}(c^2)$$

$$\therefore \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0 \implies 2\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) = 0$$

$$\therefore \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) = 0 \implies \vec{\mathbf{r}}'(t) \perp \vec{\mathbf{r}}(t).$$

□

**Definition 2.2.2 (Definite Integral of a Vector Function).** The definite integral of a continuous vector function  $\vec{\mathbf{r}}(t)$  can be defined as

$$\int_a^b \vec{\mathbf{r}}(t) dt = \int_a^b f(t) dt \hat{\mathbf{i}} + \int_a^b g(t) dt \hat{\mathbf{j}} + \int_a^b h(t) dt \hat{\mathbf{k}},$$

if  $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ .

**Example 2.2.4.**

$$\begin{aligned}\int_0^1 \left( \frac{1}{t+1} \hat{\mathbf{i}} + \frac{1}{t^2+1} \hat{\mathbf{j}} + \frac{t}{t^2+1} \hat{\mathbf{k}} \right) dt &= \int_0^1 \frac{1}{t+1} dt \hat{\mathbf{i}} + \int_0^1 \frac{1}{t^2+1} dt \hat{\mathbf{j}} + \int_0^1 \frac{t}{t^2+1} dt \hat{\mathbf{k}} \\ &= \left[ \frac{1}{t+1} \right]_0^1 \hat{\mathbf{i}} + \left[ \frac{1}{t^2+1} \right]_0^1 \hat{\mathbf{j}} + \left[ \frac{t}{t^2+1} \right]_0^1 \hat{\mathbf{k}} \\ &= \ln(2) \hat{\mathbf{i}} + \frac{\pi}{4} \hat{\mathbf{j}} + \frac{1}{1} (\ln(2)) \hat{\mathbf{k}}\end{aligned}$$

### 3 Partial Derivative

#### 3.1 Function of Several Variables

**Definition 3.1.1 (Multivariable Functions).** A function of  $f$  of  $n$  variables is a function that takes any  $n$ -tuple  $(x_1, \dots, x_n)$  in the set  $D$  to a number in  $\mathbb{R}$ , where

$$D = \left\{ (x_1, \dots, x_n) \mid x_i \in \mathbb{R} \text{ and } f \text{ is defined in } (x_1, \dots, x_n) \right\}$$

**Example 3.1.1.**  $f(x, y) = \sqrt{x^2 + y^2 - 4}$ :  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$   
 $(x, y) \mapsto \text{a number like } r$

Domain of  $f$ : all  $(x, y) \in \mathbb{R}$  s.t.  $x^2 + y^2 - 4 \geq 0$ . (i.e., Everything exclude the circle centered at the origin with a radius of 2.)

**Definition 3.1.2 (Graphs of a Two-Variable Function).** The graph of a two-variable function with domain  $D$  is the set of all points  $(x, y, z) \in \mathbb{R}^3$  s.t.  $z = f(x, y)$  and  $(x, y) \in D$ .

**Definition 3.1.3 (Vector Functions).**

$$\vec{r}: \mathbb{R} \rightarrow V_n$$

$$t \mapsto \langle f(t), g(t), h(t), \dots \rangle,$$

where  $V_n$  is a set of all vectors with  $n$  components, and  $t$  is a parameter.

**Remark.** We will only work with  $V_3$ , i.e.,  $\vec{r}: \mathbb{R} \rightarrow V_3$   
 $t \mapsto \langle f(t), g(t), h(t) \rangle$ .

**Theorem 3.1.1.** A multivariable function creates a surface in the space. if two surfaces intersect each other, then the intersection identifies a curve.

**Example 3.1.2.** Find a vector function  $\vec{r}(t)$  that represents the curve of intersection of two surfaces

$$z = \sqrt{x^2 + y^2} \quad \text{and} \quad z = 3 + y.$$

**Answer.**

Solve the system of equation  $\begin{cases} x = \sqrt{x^2 + y^2} \\ z = 3 + y \end{cases}$ .

Hence,

$$\begin{aligned} \sqrt{x^2 + y^2} &= 3 + y \\ x^2 + y^2 &= (3 + y)^2 = y^2 + 6y + 9 \\ x^2 &= 6y + 9 \\ y &= \frac{x^2 - 9}{6} \\ \therefore z &= 3 + y = \frac{x^2 + 0}{6} \end{aligned}$$

Let  $x = t$ :

$$\vec{\mathbf{r}}(t) = \langle x, t, z \rangle = \left\langle t, \frac{t^2 - 9}{6}, \frac{t^2 + 9}{6} \right\rangle$$

□

**Example 3.1.3.** Do the same for surfaces

$$z = 3x^2 + y^2 \quad \text{and} \quad y = 5x^2$$

**Answer.**

Solve the system of equations  $\begin{cases} z = 3x^2 + y^2 \\ y = 5x^2 \end{cases}$ .

$$\therefore 5x^2 = 3x^2 + y^2 \implies z = 3x^2 + (5x^2)^2 = 3x^2 + 25x^4$$

Let  $x = t$ :

$$\vec{\mathbf{r}}(t) = \langle x, t, z \rangle = \left\langle t, 5t^2, 3t^2 + 25t^4 \right\rangle$$

□

**Definition 3.1.4 (Level Curves).** The level curve of a two variable function  $z = f(x, y)$  is a curve  $f(x, y) = k$  (in the  $xy$ -plane). That means all values of  $x$  and  $y$  that have the same value  $z = k$ .

**Theorem 3.1.2 (Application of Level Curve).** Given that a point  $(a, b)$  is on the level curve of  $f(x, y)$  for  $k = c$ , then we know  $f(a, b) = c$ .

## 3.2 Limit and Continuity

**Definition 3.2.1 (Limit).** For two variable function  $z = f(x, y)$ , we check limit when  $(x, y) \rightarrow (a, b)$ . Therefore, we can make  $(x, y)$  closer to  $a(b)$  from infinitely many directions. Therefore,

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if in all directions that  $(x, y)$  approaches to  $(a, b)$ , we have  $f(x, y) \rightarrow L$ .

**Definition 3.2.2 (Precise Definition of Limit).**  $\forall$  given  $\varepsilon > 0$ ,  $\exists$  associated  $\delta > 0$  s.t. if  $(x, y) \in D$  and  $d((x, y), (a, b)) < \delta \implies d(f(x, y), L) < \varepsilon$ , where  $d((x, y), (a, b))$  is the distance between  $(x, y)$  and  $(a, b)$  and is calculated by  $\sqrt{(x - a)^2 + (y - b)^2}$ .

**Example 3.2.1.** Consider function  $f(x, y) = \frac{xy}{x^2 + y^2}$ , and identify if it has a limit at  $(0, 0)$  or not.

**Answer.**

In the direction of  $x$ -axis ( $y = 0$ ), we have  $f(x, y) = \frac{x \cdot 0}{x^2 + 0^2} = 0$  and  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$  along the  $x$ -axis.

In the direction of  $y$ -axis ( $x = 0$ ), we have  $f(x, y) = 0$ , and  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$  along the  $y$ -axis.

If  $y = x$ ,  $f(x, y) = f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$ , and  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \frac{1}{2}$  along the line  $y = x$ . □

**Example 3.2.2.** Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$ .

**Answer.**

By looking at the graph of the function, we think it has a limit at  $(0, 0)$ . This is not enough, and later we will be able to say that limit exists by converting it to polar coordinate.

Let  $y = mx$ :

$$f(x, y) = f(x, mx) = \frac{x^2 \cdot mx}{x^2 + (mx)^2} = \frac{x^3 m}{x^2(1 + m^2)} = \frac{m}{1 + m^2} x$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0 \text{ along the line of } y = mx.$$

□

**Example 3.2.3.**

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^4} = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^3 y}{x^4 + y^4} \text{ D.N.E. } \left( \text{check } \begin{cases} x = 0 \\ y = x \end{cases} \right)$$

**Definition 3.2.3 (Continuity).** Functions of two-variables is continues at  $(a, b)$  if

$$\lim_{(x,y) \rightarrow (a,b)} = f(a, b).$$

**Example 3.2.4.** Find  $\lim_{(x,y) \rightarrow (1,2)} (x^2 y^3 - x^3 y^2 + 3x + 2y)$ .

**Answer.**

As  $x^2 y^3 - x^3 y^2 + 3x + 2y$  is a polynomial and continuous everywhere, so

$$\lim_{(x,y) \rightarrow (1,2)} (x^2 y^3 - x^3 y^2 + 3x + 2y) = (1)^2 (2)^3 - (1)^3 (2)^2 + 3(1) + 2(2) = 1.$$

□

**Example 3.2.5.**  $f(x, y) = \frac{x^2 y}{x^2 + y^2}$  is not continuous at  $(0, 0)$ , but

$$g(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \text{ is continuous at } (0, 0).$$

### 3.3 Partial Derivatives

In two-variable functions, we will have partial derivatives  $f_x$  (derivative with respect to  $x$ ) and  $f_y$  (derivative with respect to  $y$ ).

**Definition 3.3.1 (Partial Derivative).** If  $f(x, y)$  is a two variable function, then its partial derivatives are  $f_x$  and  $f_y$  and is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

**Example 3.3.1.** Let  $f(x, y) = x^3 + x^2y^3 - 2y$  and find  $f_x(2, 1)$  and  $f_y(2, 1)$

**Answer.**

Find  $f_x(x, y)$ : keep  $y$  constant.

$$f_x(x, y) = 3x^2 + 2xy^3$$

$$\therefore f_x(2, 1) = 3(2)^2 + 2(2)(1)^3 = 16$$

Find  $f_y(x, y)$ : keep  $x$  constant.

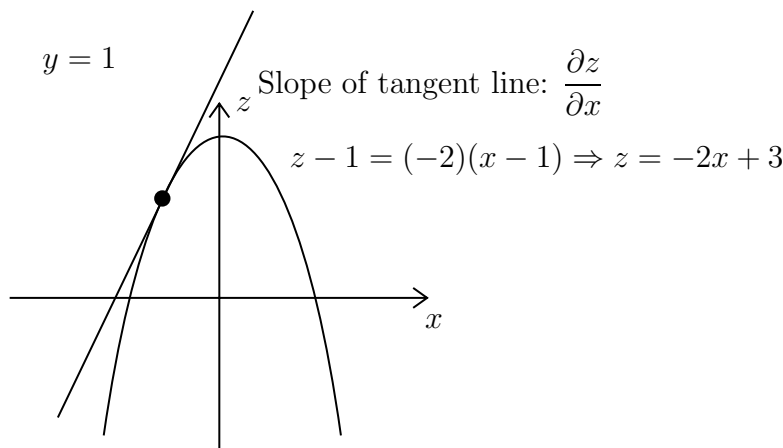
$$f_y(x, y) = 3x^2y^2 - 2$$

$$\therefore f_y(2, 1) = 3(2)^2(1)^2 - 2 = 10$$

□

**Example 3.3.2.** Let  $f(x, y) = 4 - x^2 - 2y^2$ . Find  $f_x(1, 1)$  and interpret the values.

**Answer.**



$$f(1, 1) = 4 - 1 - 2 = 1 \implies A(1, 1, 1) \text{ lies on } f(x, y).$$

$$\frac{\partial f}{\partial x} = -2x \implies \frac{\partial f}{\partial x}(1, 1) = -2$$

Let's consider  $y = 1$ :

The plane  $y = 1$  will intersect with  $f(x, y)$  at a line  $\vec{r}(t)$ .

$$\text{Solve } \vec{\mathbf{r}}(t) : \begin{cases} z = 4 - x^2 - 2y^2 \\ y = 1 \end{cases}$$

$$\Rightarrow z = 4 - x^2 - 2 = 2 - x^2$$

$$\therefore \vec{\mathbf{r}}(t) = \langle t, 1, 2 - t^2 \rangle, \quad \vec{\mathbf{r}}'(t) = \langle 1, 0, -2t \rangle$$

At point  $A(1, 1, 1)$ ,  $t = 1$ .

$\therefore \vec{\mathbf{r}}'(1) = \langle 1, 0, -2 \rangle$ , which is a directional vector of the tangent line.

$\therefore$  Tangent line:

$$L : x = 1 + t, \quad y = 1, \quad z = 1 - 2t$$

□

**Definition 3.3.2 (Higher Order Partial Derivative).**

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$

$$\frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$

**Theorem 3.3.1 (Clairaut's Theorem).** If  $f$  is continuous on a disk  $D$ , then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

**Definition 3.3.3 (Functions With More Than Two Variables).** If  $U = f(x_1, \dots, x_n)$ , its partial derivative with respect to  $x_i$  is

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} \\ &= \frac{\partial U}{\partial x_i} \end{aligned}$$

### 3.4 Tangent Plane and Linear Approximation

**Theorem 3.4.1 (Tangent Plane).** If  $f$  has continuous partial derivatives, an equation of the tangent plane to the surface  $z = f(x, y)$  at the point  $(x_0, y_0, z_0)$  is

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

**Example 3.4.1.** Find the tangent plane of  $f(x, y) = 2x^2 + y^2$  at  $(1, 1, 3)$ .

**Answer.**

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4x & \frac{\partial f}{\partial y} &= 2y \\ \therefore \frac{\partial f}{\partial x}(1, 1) &= 4 & \frac{\partial f}{\partial y}(1, 1) &= 2\end{aligned}$$

$\therefore$  Tangent plane at  $(1, 1, 3)$ :

$$\Pi : z - 3 = 4(x - 1) + 2(y - 1).$$

□

**Definition 3.4.1 (Linearization and Linear Approximation).** Similar to single variable calculus, we can approximate the value of a function at a point using the tangent line:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is the **linearization** of  $f(x, y)$  at point  $(a, b)$ :

$$f(x, y) \approx L(x, y)$$

is called the **linear approximation** or the **tangent plane approximation** of  $f$  at  $(a, b)$ .

**Definition 3.4.2 (Differentiable Functions).** A **differentiable function** is a function that the linear approximation is a good approximation when  $(x, y)$  are very close to  $(a, b)$ .

**Theorem 3.4.2 (A sufficient condition for differentiability).** If partial derivative  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exists near  $(a, b)$  and are continuous at  $(a, b)$ , then  $f$  is differentiable at  $(a, b)$ .

**Example 3.4.2.** Show that function  $f(x, y) = \frac{\sqrt{x}}{y}$  is differentiable at  $(16, 5)$  and use it to approximate  $\frac{\sqrt{16.02}}{4.96}$ .

**Answer.**

$$\begin{aligned}f(16, 5) &= \frac{\sqrt{16}}{5} = \frac{4}{5}; & \frac{\partial f}{\partial x} &= \frac{1}{2y\sqrt{x}}; & \frac{\partial f}{\partial y} &= -\frac{\sqrt{x}}{y^2}. \\ \therefore \frac{\partial f}{\partial x} \Big|_{(16, 5)} &= \frac{1}{2(5)\sqrt{16}} = \frac{1}{40}; & \frac{\partial f}{\partial y} \Big|_{(16, 5)} &= -\frac{\sqrt{16}}{25} = -\frac{4}{25}.\end{aligned}$$

As  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exists and is continuous at  $(x, y) = (16, 5)$ ,  $f(x, y)$  is differentiable at  $(16, 5)$ .

Then, the approximation is

$$f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$



At  $a = 16$  and  $b = 5$ :

$$\begin{aligned}\frac{\sqrt{x}}{y} &\approx \frac{4}{5} + \frac{1}{40}(x - 16) + \left(-\frac{4}{25}\right)(y - 5) \\ &= \frac{4}{5} + \frac{1}{40}x - \frac{2}{5} - \frac{4}{25}y + \frac{4}{5} \\ &= \frac{1}{40} - \frac{4}{25}y + \frac{6}{5}.\end{aligned}$$

Therefore,  $\frac{\sqrt{16.02}}{4.96} \approx \frac{1}{40}(16.02) - \frac{4}{25}(4.96) + \frac{6}{5} \approx 0.807$ .

□

**Definition 3.4.3 (Differentials).**

$$\Delta z = \Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$$

$$dz = f_x(a, b)dx + f_y(a, b)dy$$

**Extension (Differentials in Higher Dimensions).** Let  $U = f(x_1, x_2, \dots, x_n)$ , we have

$$dU = f_{x_1}(a_1, \dots, a_n)dx_1 + f_{x_2}(a_1, \dots, a_n)dx_2 + \dots + f_{x_n}(a_1, \dots, a_n)dx_n$$

$$\Delta U = \Delta f = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) - f(x_1, \dots, x_n)$$

### 3.5 The Chain Rule

**Theorem 3.5.1 (The Multivariable Chain Rule).** Let  $U$  be a differentiable function of  $n$  variables  $x_1, \dots, x_n$ , and each  $x_i$  for  $i = 1, \dots, n$  is a differentiable function of  $t_1, \dots, t_m$ . Then, we have

$$\frac{\partial U}{\partial t_i} = \frac{\partial U}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial U}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial U}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

**Example 3.5.1.** Let  $U = x^4y + y^2z^3$  and  $x = rse^t$ ,  $y = rs^2e^{-t}$ , and  $z = r^2s \sin(t)$ . Find the value of  $\frac{\partial U}{\partial s}$  when  $r = 2$ ,  $s = 1$ ,  $t = 0$ .

**Answer.**

From the multivariable chain rule, we know

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial s}$$

$$\frac{\partial U}{\partial x} = 4x^3y; \quad \frac{\partial U}{\partial x} = x^4 + 2yz^3; \quad \frac{\partial U}{\partial x} = 3y^2z^2;$$

$$\frac{\partial x}{\partial s} = re^t; \quad \frac{\partial y}{\partial s} = 2rse^{-t}; \quad \frac{\partial z}{\partial s} = r^2 \sin t.$$

$$\therefore \frac{\partial U}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2 \sin t)$$

When  $r = 2$ ,  $s = 1$ ,  $t = 0$ , we have

$$x = 2, \quad y = 2, \quad z = 0.$$

$$\therefore \frac{\partial U}{\partial s} \bigg|_{(r,s,t)=(2,1,0)} = (4(2)^3(2))(2) + (2^4)(2 \cdot 2) + 0 = 128 + 64 = 192.$$

□

**Example 3.5.2.** If  $z = f(x, y)$  has continuous second order partial derivatives and  $x = r^2 + s^2$  and  $y = 2rs$ . Find  $\frac{\partial z}{\partial r}$  and  $\frac{\partial^2 z}{\partial r^2}$ .

**Answer.**

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}$$

Since

$$\frac{\partial}{\partial r} = 2r; \quad \frac{\partial}{\partial r} = 2s$$

$$\therefore \frac{\partial z}{\partial r} = 2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y}.$$

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial r} \right) = \frac{\partial}{\partial r} \left( 2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y} \right) \\ &= 2 \frac{\partial}{\partial r} \left( r \frac{\partial z}{\partial x} \right) + 2 \frac{\partial}{\partial r} \left( s \frac{\partial z}{\partial y} \right) \\ &= 2 \left[ \frac{\partial}{\partial r}(r) \cdot \frac{\partial z}{\partial x} + r \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) \right] + 2 \left[ \frac{\partial}{\partial r}(s) \cdot \frac{\partial z}{\partial y} + s \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right) \right] \end{aligned}$$

Notice that  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  are functions dependent on  $x$  and  $y$ , so to find their partial derivatives with respect to  $r$ , we need to apply multivariable chain rule again:

$$\begin{aligned} \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \cdot \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial r} \\ \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \cdot \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} \\ \therefore \frac{\partial^2 z}{\partial r^2} &= 2 \frac{\partial z}{\partial x} + 2r \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial r} \right) + 2s \left( \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} \right) \end{aligned}$$

□

**Theorem 3.5.2 (Implicit Differentiation).** If we have two-variable function like  $F(x, y) = 0$ , where  $y$  depends on  $x$ , we use the multivariable chain rule to differential the both sides of  $F(x, y)$ :

$$\begin{aligned} \frac{\partial F}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_1 + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} &= 0 \\ \frac{\partial F}{\partial x} &= - \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} \\ \therefore \frac{dy}{dx} &= - \frac{\partial F / \partial x}{\partial F / \partial y} = - \frac{F_x}{F_y} \end{aligned}$$

**Example 3.5.3.** Find  $y'$  if  $x^3 + y^3 = 6xy$

**Answer.**

Method1 Applying the formula:

$$\begin{aligned} F_x &= 3x^2 - 6y \\ F_y &= 3y^2 - 6x \\ \therefore \frac{dy}{dx} &= -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} \end{aligned}$$

Method2 Find derivatives of the both sides:

$$\begin{aligned} x^3 + y^3 - 6xy &= 0 \\ 3x^2 + 3y^2 \frac{dy}{dx} - 6y - 6x \frac{dy}{dx} &= 0 \\ (3y^2 - 6x) \frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

□

**Theorem 3.5.3 (Multivariable Implicit Differentiation).** If  $z = f(x, y)$ , consider a function

$$F(x, y, z) = F(x, y, f(x, y))$$

Then, by the multivariable chain rule, we differentiate both sides of  $F(x, y, f(x, y)) = 0$ :

$$\frac{\partial F}{\partial x} \underbrace{\frac{dx}{dx}}_1 + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{\partial F / \partial x}{\partial F / \partial z}$$

Similarly, we have

$$\frac{\partial F}{\partial y} \underbrace{\frac{dy}{dy}}_1 + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{\partial F / \partial y}{\partial F / \partial z}$$

**Example 3.5.4.** Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $x^3 + y^3 + z^3 + 6xyz = 1$ .

**Answer.**

In order to find  $\frac{\partial z}{\partial x}$ , differentiate both sides with respect to  $x$ :

$$\begin{aligned} 3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} &= 0 \\ (3x^2 + 6yz) \frac{\partial z}{\partial x} &= -(3x^2 + 6yz) \\ \frac{\partial z}{\partial x} &= -\frac{3x^2 + 6yz}{3z^2 + 6xy} \left( = -\frac{x^2 + 2yz}{z^2 + 2xy} \right) \end{aligned}$$

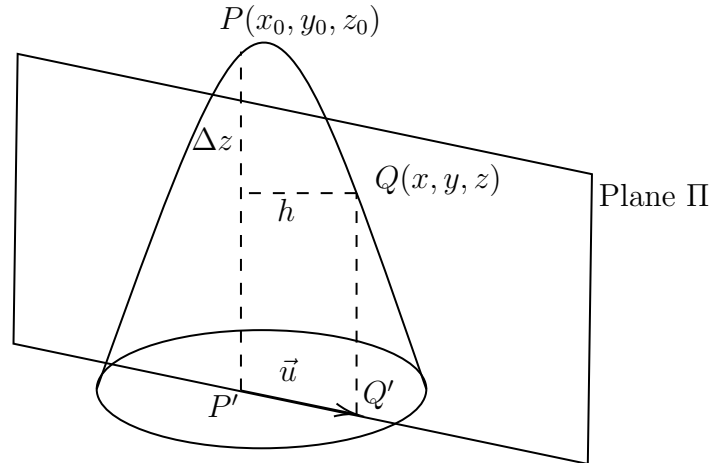
In order to find  $\frac{\partial z}{\partial y}$ , differentiate both sides with respect to  $y$ :

$$\begin{aligned} 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6xz + 6xy \frac{\partial z}{\partial y} &= 0 \\ (3z^2 + 6xy) \frac{\partial z}{\partial y} &= -(3y^2 + 6xz) \\ \frac{\partial z}{\partial y} &= -\frac{3y^2 + 6xz}{3z^2 + 6xy} \left( = -\frac{y^2 + 2xz}{z^2 + 2xy} \right) \end{aligned}$$

□

### 3.6 Directional Derivatives and Gradient

To formally study directional derivatives, we start from the ideas of it. We want to study the change of  $z = f(x, y)$  in the direction of the unit vector  $\vec{u} = \langle a, b \rangle = a\hat{i} + b\hat{j}$ . ( $\sqrt{a^2 + b^2} = 1$ ). We intersect surface  $z = f(x, y)$  with plane  $\Pi$  that passes through the point  $P(x_0, y_0, z_0)$  vertically and in the direction of vector  $\vec{u} = \langle a, b \rangle$ .



So, we have

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + h, y_0 + h) - f(x_0, y_0)}{h}$$

**Definition 3.6.1 (Directional Derivative).** The directional derivative of  $f$  at  $(x_0, y_0)$  in the direction of a vector  $\vec{u} = \langle a, b \rangle$  is defined as

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Now, let  $g(h) = f(x_0 + ha, y_0 + hb)$ , then we have

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = g'(0)$$

To find  $g'(h)$ , we use the multivariable chain rule:

$$g'(h) = \frac{\partial g}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial g}{\partial y} \cdot \frac{dy}{dh} \quad \text{where} \quad \begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}.$$

From  $\begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}$ , we have  $\frac{\partial x}{\partial h} = a$  and  $\frac{\partial y}{\partial h} = b$ .

$$\begin{aligned} \therefore g'(h) &= \frac{\partial g}{\partial x} \cdot a + \frac{\partial g}{\partial y} \cdot b \\ &= a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} \quad \left[ g(h) \text{ is in fact } f(x, y) \right] \end{aligned}$$

When  $h \rightarrow 0$ ,

$$\begin{aligned} g'(0) &= a \cdot f_x(x_0, y_0) + b \cdot f_y(x_0, y_0) \\ \therefore D_{\vec{u}}f(x_0, y_0) &= a \cdot f_x(x_0, y_0) + b \cdot f_y(x_0, y_0) \\ &= \langle a, b \rangle \cdot \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \end{aligned}$$

**Theorem 3.6.1 (Directional Derivative in Dot Product).**

$$D_{\vec{u}}f(x_0, y_0) = \vec{u} \cdot \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle = \vec{u} \cdot \nabla f(x_0, y_0)$$

**Definition 3.6.2 (Gradient Vector).** A gradient vector of  $f$  is a vector function defined as

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}}.$$

The notation “ $\nabla$ ” is called nabla.

**Extension.** If  $f$  is a function as  $f(x_1, \dots, x_n)$ , then

$$\nabla f = \langle f_{x_1}, f_{x_2}, f_{x_3} \dots, f_{x_n} \rangle.$$

**Theorem 3.6.2 (Properties of Gradient).** From the dot product definition of directional vector, we know that

$$D_{\vec{u}}f = \nabla f \cdot \vec{u}.$$

Then, if  $\theta$  is the angle between  $\nabla f$  and  $\vec{u}$ , we have

$$D_{\vec{u}}f = |\nabla f| |\vec{u}| \cos \theta.$$

Thus,

$$\max D_{\vec{u}}f = |\nabla f| |\vec{u}| \text{ when } \theta = 0$$

(or, the vector  $\vec{\mathbf{u}}$  is in the direction of  $\nabla f$ .) Since  $\vec{\mathbf{u}}$  is a unit vector,  $|\vec{\mathbf{u}}| = 1$ . So when  $\vec{\mathbf{u}}$  is in the same direction of  $\nabla f$ , we have

$$\max D_{\vec{\mathbf{u}}}f = |\nabla f|.$$

On the other hand, if  $\vec{\mathbf{u}}$  and  $\nabla f$  are in the opposite direction, we have  $\theta = \pi$  and  $\cos \theta = \cos(\pi) = -1$ .

$$\therefore \min D_{\vec{\mathbf{u}}}f = |\nabla f||\vec{\mathbf{u}}| \cos \theta = -|\nabla f|$$

**Extension.** If  $\vec{\mathbf{u}}$  is a unit vector and  $\vec{\mathbf{u}} = \langle a, b \rangle$  and  $f$  has continuous second partial derivatives, then

$$D_{\vec{\mathbf{u}}}^2 f = f_{xx}a + 2f_{xy}ab + f_{yy}b.$$

**Example 3.6.1.** If  $f(x, y) = xe^y$ , then

1. Find the rate of change of  $f$  at the point  $P(2, 0)$  in the direction from  $P$  to  $Q\left(\frac{1}{2}, 2\right)$ .

**Answer.**

$$\frac{\partial f}{\partial x} = e^y; \quad \frac{\partial f}{\partial y} = xe^y; \quad \overrightarrow{PQ} = \left\langle \frac{1}{2} - 2, 2 - 0 \right\rangle = \left\langle -\frac{3}{2}, 2 \right\rangle; \quad |\overrightarrow{PQ}| = \sqrt{\frac{9}{4} + 4} = \frac{5}{2}$$

$$\therefore \vec{\mathbf{u}} = \left\langle -\frac{3}{2} \cdot \frac{2}{5}, 2 \cdot \frac{2}{5} \right\rangle = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle; \quad \nabla f = \langle e^y, xe^y \rangle.$$

Therefore,

$$D_{\vec{\mathbf{u}}}f = \nabla f \cdot \vec{\mathbf{u}} = \langle e^y, xe^y \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{3}{5}e^y + \frac{4}{5}xe^y.$$

At point  $P(2, 0)$ ,

$$D_{\vec{\mathbf{u}}}f(2, 0) = -\frac{3}{5}e^0 + \frac{4}{5} \cdot 2 \cdot e^0 = -\frac{3}{5} + \frac{8}{5} = 1.$$

□

2. In what direction does  $f$  have the maximum rate of change? What is this maximum rate of change?

**Answer.**

$$\nabla f(2, 0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$$

Hence, in direction  $\nabla f = \langle 1, 2 \rangle$ ,  $f$  has the maximum rate of change. The maximum rate of change is  $|\nabla f(2, 0)| = \sqrt{5}$ .

□

**Theorem 3.6.3 (Gradient and Tangent Plane).** The equation of the tangent plane for the surface  $z = f(x, y)$  at the point  $P(x_0, y_0, z_0)$  is given by:

$$\nabla f \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or (for implicit functions)

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0.$$

The normal line of the plane is given by

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}.$$

**Remark (Gradient and Multivariable Chain Rule).** If  $F(x, y, z) = k$  and  $x, y, z$  are dependent of  $t$ , then we differentiate both sides with respect to  $t$  to get:

$$\begin{aligned} \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} &= 0 \\ \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle &= 0 \\ \nabla F \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle &= 0 \end{aligned}$$

**Theorem 3.6.4 (Graphical Interpretation of Gradient Vector).** In general, the gradient vector at  $P$ ,  $\nabla F(x_0, y_0, z_0)$  is perpendicular to the tangent vector  $\mathbf{r}'(t_0)$  to any curve  $C$  that passes through the point  $P$  on the surface  $S$ . Similar properties hold on level curves.

### 3.7 Maximum and Minimum Values

**Definition 3.7.1 (Local Maximum and Local Minimum).** A function  $f(x, y)$  has a **local maximum** at point  $(a, b)$  if  $\forall (x, y)$  near point  $(a, b)$ , we have  $f(x, y) \leq f(a, b)$ . The function  $f(x, y)$  has a **local minimum** at point  $(a, b)$  if  $\forall (x, y)$  near point  $(x, y)$ , we have  $f(x, y) \geq f(a, b)$ .

**Remark.** “near point  $(a, b)$ ” refers to a disk centered at  $(a, b)$ .

**Definition 3.7.2 (Absolute Maximum and Absolute Minimum).** If the equalities  $f(x, y) \leq f(a, b)$  and  $f(x, y) \geq f(a, b)$  holds for any  $(x, y)$  in the domain of  $f(x, y)$ , then we call them **absolute maximum** or **absolute minimum**.

**Theorem 3.7.1.** If  $f$  has local maximum or minimum at  $(a, b)$ , and the first order partial derivatives of  $f$  exist at  $(a, b)$ , then  $f_x(a, b)$  and  $f_y(a, b)$  are equal to 0. In other words,

$$\nabla f(a, b) = 0.$$

**Corollary 3.1.** As a result of Theorem 3.7.1, the equation of the tangent plane at  $(a, b)$  is

$$\begin{aligned} z - \overbrace{f(a, b)}^{z_0} &= \overbrace{f_x(a, b)}^0(x - a) + \overbrace{f_y(a, b)}^0(y - b) \\ z - z_0 &= 0. \end{aligned}$$

In other words, the tangent plane is horizontal.

**Definition 3.7.3 (Critical Points).** A point  $(a, b)$  is called the **critical point** if  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$  or if one of the partial derivatives does not exist.

**Remark.** At a critical point, we may have maximum or minimum or neither (saddle point).

**Definition 3.7.4 (Determinant).** The determinant ( $\Delta$  or  $D$ ) is defined as

$$\begin{aligned} D &= \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\ &= f_{xx}f_{yy} - f_{xy}f_{yx} \\ &= f_{xx}f_{yy} - (f_{xy})^2. \end{aligned}$$

**Theorem 3.7.2 (Second Derivative Test).** Let  $(a, b)$  be a critical point and second partial derivatives of  $f$  (i.e.,  $f_{xx}$ ,  $f_{xy}$ ,  $f_{yx}$ ,  $f_{yy}$ ) are continuous on a disk centered at  $(a, b)$ . Then

1. If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local minimum.
2. If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local maximum.
3. If  $D < 0$ , then  $f(a, b)$  is not a local maximum or local minimum, and it is called a **saddle point**.

**Remark.** At saddle points, the tangent plane will intersect with the surface of  $f$ .

**Example 3.7.1.** For function  $f(x, y) = 4 + x^3 + y^3 - 3xy$ . Check if  $f(x, y)$  has local maximum, local minimum, and saddle points.

**Answer.**

$$\frac{\partial f}{\partial x} = 3x^2 - 3y; \quad \frac{\partial f}{\partial y} = 3y^2 - 3x$$

$$\text{Solve } \begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 3y = 0 & \textcircled{1} \\ 3y^2 - 3x = 0 & \textcircled{2} \end{cases}.$$

From  $\textcircled{1}$ :  $y = x^2$ .

Substitute  $y = x^2$  to  $\textcircled{2}$ :

$$3(x^2)^2 - 3x = 0$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0 \implies x = 0 \text{ or } x = 1$$



$$\therefore y = 0^2 = 0 \quad \text{or} \quad y = 1^2 = 1$$

$$\therefore \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

i.e., Critical points are at  $(0, 0)$  and  $(1, 1)$ .

Find  $D$ :

$$D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} = 36xy - 9.$$

Apply the second derivative test:

1.  $D(0, 0) = -9 < 0 \implies (0, 0)$  is a saddle point.
2.  $D(1, 1) = 36 - 9 = 27 > 0$  and  $\frac{\partial^2 f}{\partial x^2} = 6(1) > 0 \implies (1, 1)$  is a local minimum.

□

**Theorem 3.7.3 (Extreme Value Theorem, EVT).** We are expanding the Extreme Value Theorem from a single variable version to a multivariable version:

1. Single Variable Version: any continuous function on a closed interval  $I$  has a maximum or minimum value in that interval  $I$ .
2. Multivariable Version: For a multivariable function  $f(x_1, \dots, x_n)$  on a **closed and bounded** region  $D$  in  $\mathbb{R}^n$ .  $f$  has both maximum and minimum values in that region.

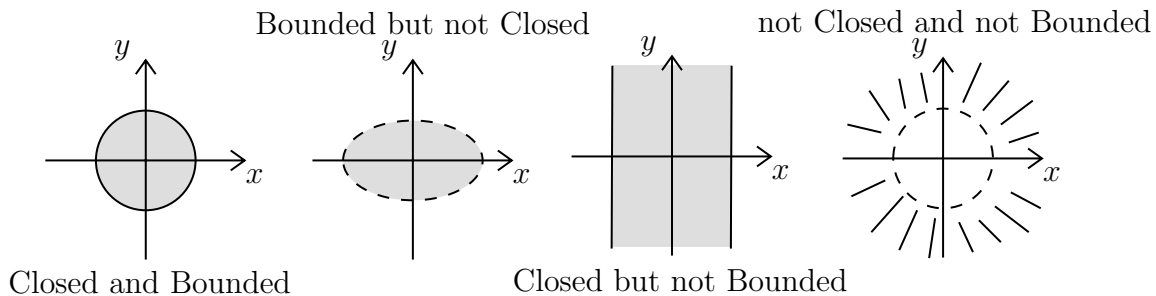
**Definition 3.7.5 (Bounded Region).**  $D$  is bounded if there exists some ball

$$x_1^2 + x_2^2 + \dots + x_n^2 \leq R^2$$

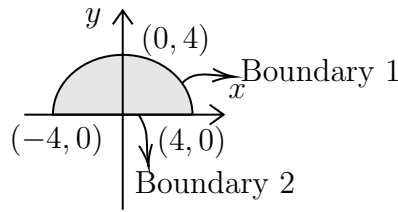
that contains  $D$ .

**Definition 3.7.6 (Closed Region).** Closed region  $D$  is a region that includes the boundaries.

**Example 3.7.2 (Bounded and Closed Region).** The following are examples of closed and bounded regions.



**Example 3.7.3.** Find the extreme values of the function  $f(x, y) = x^2 + 2y^2 - x^2y$  on the following region:



**Answer.**

We can write the region  $D$  as the following set:

$$D = \{(x, y) \mid x^2 + y^2 \leq 6, y \geq 0\}.$$

**Step 1** Find the critical points of the function that are inside the boundary (interior to the boundary).

$$f(x, y) = x^2 + 2y^2 - x^2y \Rightarrow \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x - 2xy, 4y - x^2 \rangle.$$

$$\text{Set } \nabla f(x, y) = 0 : \begin{cases} 2x - 2xy = 0 & \textcircled{1} \\ 4y - x^2 = 0 & \textcircled{2}. \end{cases}$$

From  $\textcircled{2}$ :  $y = \frac{x^2}{4}$ . Substitute this result into  $\textcircled{1}$ :

$$2x - 2x \cdot \frac{x^2}{4} = 0$$

$$2x - \frac{1}{2}x^3 = 0 \Rightarrow x \left( 2 - \frac{1}{2}x^2 \right) = 0$$

$$\therefore \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x^2 = 4 \\ y = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 2 \\ y = 1 \end{cases} \quad \text{or} \quad \begin{cases} x = -2 \\ y = 1 \end{cases}$$

All the points  $(0, 0)$ ,  $(2, 1)$ , and  $(-2, 1)$  are inside the boundary.

**Step 2** Check the boundaries for maximum and minimum.

**Check Boundary 1:**  $x^2 + y^2 = 16, \quad 0 \leq y \leq 4$ .

$$\begin{aligned} f(x, y) &= x^2 + 2y^2 - x^2y = 16 + y^2 - (16 - y^2)y \\ &= 16 + y^2 - 16y + y^3 \end{aligned}$$

$$f(y) = y^3 + y^2 - 16y + 16 \rightarrow \text{one variable function}$$

$$f'(y) = 3y^2 + 2y - 16 = 0 \quad y = -\frac{8}{3}, \quad y = 2.$$

Since  $0 \leq y \leq 4$ ,  $y = 2$ .

When  $y = 2$ ,  $x = \pm\sqrt{16-4} = \pm 2\sqrt{3}$ .

$$f(y) = 2^3 + 2^2 - 16(2) + 16 = 8 + 4 - 32 + 16 = -4$$

When  $y = 4$ ,  $x = 0$ .

$$f(x, y) = 16 + 16 - 64 + 64 = 32$$

When  $y = 0$ ,  $x = \pm 4$ .

$$f(x, y) = 16 \rightarrow (\text{not a extreme value})$$

Hence, we have  $-4 \leq f(x, y) \leq 32$  on Boundary 1.

**Check boundary 2:**  $-4 \leq x \leq 4$ ,  $y = 0$ .

$$f(x, y) = x^2 + 2y^2 - x^2y = x^2$$

Since  $0 \leq x^2 \leq 16$ ,  $0 \leq f(x, y) \leq 16$ .

**Step 3** List all the points and values:

Point	Value
$(0, 0)$	$f(0, 0) = 0$
$(2, 1)$	$f(2, 1) = 2$
$(-2, 1)$	$f(-2, 1) = 2$
$(2\sqrt{3}, 2)$	$f(2\sqrt{3}, 2) = -4$
$(-2\sqrt{3}, 2)$	$f(-2\sqrt{3}, 2) = -4$
$(0, 4)$	$f(0, 4) = 32$

Hence, minimum occurs at  $(2\sqrt{3}, 2)$  and  $(-2\sqrt{3}, 2)$ , and the function value is  $-4$  at minimum. The maximum occurs at  $(0, 4)$ , and the function value is  $32$  at maximum.

□

### 3.8 Lagrange Multiplier

**Definition 3.8.1 (Optimization).** Find minimum or maximum values of a function subject to constraints.

**Remark.** The constraints can be an equality or an inequality.

**Definition 3.8.2 (Objective Function).** The function  $f$  we are working with is called the **objective function** or **cost function**.

**Definition 3.8.3 (Linear and Non-Linear Optimization).** If the objective function is linear, the process is called **linear programming** or **linear optimization**. If the objective function is not linear, the process is called **non-linear optimization**.

**Theorem 3.8.1 (Lagrange Multiplier).** The minimum or maximum value of  $f(x_1, \dots, x_n)$  subject to the condition  $g(x_1, \dots, x_n) = k$ , where  $f$  and  $g$  are differentiable, occur when the gradient vectors,  $\nabla f$  and  $\nabla g$ , are parallel. That is,

$$\nabla f(x_1, \dots, x_n) = \lambda \nabla g(x_1, \dots, x_n)$$

for some  $\lambda$ .

**Extension (Lagrange Multiplier with Multiple Constrains).** If we have two constrains  $g(x_1, \dots, x_n) = k$  and  $h(x_1, \dots, x_n) = m$ , then the minimum or maximum value of  $f(x_1, \dots, x_n)$  occurs at

$$\nabla f(x_1, \dots, x_n) = \lambda \nabla g(x_1, \dots, x_n) + \mu \nabla h(x_1, \dots, x_n)$$

for some  $\lambda$  and  $\mu$ .

**Example 3.8.1.** Maximize  $f(x, y) = xy$  on the curve  $x^2 + y^2 = 4$ .

**Answer.**

In this example,  $f(x, y) = xy$ ,  $g(x, y) = x^2 + y^2$ , and  $k = 4$ . Then,

$$\nabla f(x, y) = \langle y, x \rangle \quad \nabla g(x, y) = \langle 2x, 2y \rangle.$$

Attempt to solve  $\nabla f(x, y) = \lambda \nabla g(x, y)$ :

$$\langle y, x \rangle = \lambda \langle 2x, 2y \rangle.$$

$$\text{So, we have } \begin{cases} y = 2\lambda x & \textcircled{1} \\ x = 2\lambda y & \textcircled{2} \end{cases}$$

Substitute  $\textcircled{1}$  into  $\textcircled{2}$  we have  $x = 2\lambda(2\lambda x)$ , or  $x = 4\lambda^2 x$ .

Divide  $x$  on both sides of the equation, we have  $4\lambda^2 = 1$  or  $\lambda^2 = \frac{1}{4}$ . Hence,  $\lambda = \pm \frac{1}{2}$ .

$$\boxed{\lambda = \frac{1}{2}}: y = 2\left(\frac{1}{2}\right)x = x$$

Substitute  $y = x$  into  $x^2 + y^2 = 4$ :  $2x^2 = 4$ , or  $x^2 = 2$ . So  $x = \pm\sqrt{2}$ .

Hence, critical points when  $\lambda = \frac{1}{2}$ :  $(\sqrt{2}, \sqrt{2})$  or  $(-\sqrt{2}, -\sqrt{2})$ .

The values of function are  $f(\sqrt{2}, \sqrt{2}) = \sqrt{2} \cdot \sqrt{2} = 2$  and  $f(-\sqrt{2}, -\sqrt{2}) = (-\sqrt{2})(-\sqrt{2})$ .

$$\boxed{\lambda = -\frac{1}{2}}: y = 2\left(-\frac{1}{2}\right)x = -x.$$

Substitute  $y = -x$  into  $x^2 + y^2 = 4$ :  $2x^2 = 4$  and  $x = \pm\sqrt{2}$ .

Hence, critical points are  $(\sqrt{2}, -\sqrt{2})$  and  $(-\sqrt{2}, \sqrt{2})$ .

The respective values of the function are  $f(\sqrt{2}, -\sqrt{2}) = -2$  and  $f(-\sqrt{2}, \sqrt{2}) = -2$ .

Hence, the maximum occurs at  $(\sqrt{2}, \sqrt{2})$  and  $(-\sqrt{2}, -\sqrt{2})$ , with the maximum value of 2. and the minimum occurs at  $(\sqrt{2}, -\sqrt{2})$  and  $(-\sqrt{2}, \sqrt{2})$ , with the minimum value of -2.

□

**Extension (Lagrange Multiplier with an Inequality Constrain).** If we are having an inequality constrain, we need to check if any critical points of  $\nabla f = 0$  satisfies the inequality, if so, the critical points from  $\nabla f = 0$  will be the maximum or minimum point for this optimization. If we do not have any critical points of  $\nabla f = 0$ , critical points calculated from the Lagrange Multiplier will be the maximum or minimum point for the optimization.

## 4 Multiple Integrals

### 4.1 Double Integral Over Rectangles

**Definition 4.1.1 (Double Integral).** Suppose  $f(x, y)$  is a two-variable function, then the double integral of it over rectangles is defined by

$$\iint_R f(x, y) \, dA = \lim_{m, n \rightarrow \infty} \sum_{j=1}^m \sum_{i=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

if the limit exists.

**Theorem 4.1.1.** If  $f(x, y) \geq 0$ , then the volume  $V$  of the solid that lies above the rectangle  $R$  and below the surface  $z = f(x, y)$  is

$$V = \iint_R f(x, y) \, dA$$

**Example 4.1.1.** Approximate the volume of  $f(x, y) = x^2y$  when  $R = [0, 2] \times [0, 1]$ . Use midpoint approximation and  $m = n = 2$ .

**Answer.**

We can compute the following  $(x, y)$  points that are used for the approximation:

$$(x_{11}, y_{11}) = \left(\frac{1}{2}, \frac{1}{4}\right) \quad (x_{12}, y_{12}) = \left(\frac{1}{2}, \frac{3}{4}\right) \quad (x_{21}, y_{21}) = \left(\frac{3}{2}, \frac{1}{4}\right) \quad (x_{22}, y_{22}) = \left(\frac{3}{2}, \frac{3}{4}\right)$$

We can also compute the value of  $\Delta A$  :

$$\Delta A = \Delta x \cdot \Delta y = 1 \times \frac{1}{2} = \frac{1}{2}.$$

Hence, we can approximate the volume:

$$\begin{aligned} V &\approx \Delta A \left[ f(x_{11}, y_{11}) + f(x_{12}, y_{12}) + f(x_{21}, y_{21}) + f(x_{22}, y_{22}) \right] \\ &= \frac{1}{2} \left[ f\left(\frac{1}{2}, \frac{1}{4}\right) + f\left(\frac{1}{2}, \frac{3}{4}\right) + f\left(\frac{3}{2}, \frac{1}{4}\right) + f\left(\frac{3}{2}, \frac{3}{4}\right) \right] \\ &= \frac{1}{2} \left[ \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)^2 \cdot \left(\frac{3}{4}\right) + \left(\frac{3}{2}\right)^2 \cdot \left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2 \cdot \left(\frac{3}{4}\right) \right] \\ &= \frac{1}{2} \left( \frac{1}{4} + \frac{9}{4} \right) \\ &= \frac{10}{8} = \frac{5}{4}. \end{aligned}$$

□

**Theorem 4.1.2 (Calculating Double Integrals).** In order to compute the double integral on  $R = [a, b] \times [c, d]$ :

$$\iint_R f(x, y) \, dA$$

1. First, we hold  $x$  fixed and find the integral

$$A(x) = \int_c^d f(x, y) \, dy$$

The result is an expression on  $x$  is called the integration with respect to  $y$ .

2. Then, we find the integral

$$\begin{aligned} V &= \int_a^b A(x) \, dx = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] dx \\ &= \int_a^b \int_c^d f(x, y) \, dy dx \end{aligned}$$

**Theorem 4.1.3 (Fubini's Theorem).** Suppose  $f$  is a continuous function of  $x$  and  $y$  on the rectangle  $R = \{(a, y) \mid a \leq x \leq b, c \leq y \leq d\}$ . Then,

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy dx = \int_c^d \int_a^b f(x, y) \, dx dy.$$

**Example 4.1.2.** Evaluate  $\int_0^3 \int_1^2 x^2 y \, dy dx$ .

**Answer.**

$$\begin{aligned} \int_0^3 \int_1^2 x^2 y \, dy dx &= \int_0^3 \left[ \frac{1}{2} x^2 y^2 \right]_1^2 dx \\ &= \int_0^3 \left( \frac{1}{2} (4) x^2 - \frac{1}{2} x^2 \right) dx \\ &= \int_0^3 \frac{3}{2} x^2 \, dx \\ &= \left[ \frac{1}{3} \cdot \frac{3}{2} x^3 \right]_0^3 = \frac{1}{2} (27) = \frac{27}{2} \end{aligned}$$

□

**Example 4.1.3.** Evaluate the double integral

$$\iint_R y \sin(xy) \, dA, \quad \text{where } R = [1, 2] \times [0, \pi].$$

**Answer.**

From the Fubini's Theorem,

$$\iint_R y \sin(xy) \, dA = \int_1^2 \int_0^\pi y \sin(xy) \, dy dx = \int_0^\pi \int_1^2 y \sin(xy) \, dx dy$$

Let  $u = xy$ , then  $\frac{du}{dx} = y$ , which is  $du = ydx$ .

$$\begin{aligned}
 \therefore \int_0^\pi \int_1^2 y \sin xy \, dx dy &= \int_0^\pi \int_y^{2y} \sin(u) \, du dy \\
 &= \int_0^\pi [-\cos(u)]_y^{2y} dy \\
 &= - \int_0^\pi \cos(2y) - \sin(y) \, dy \\
 &= - \left[ \frac{1}{2} \sin(2y) - \sin(y) \right]_0^\pi \\
 &= - \left( \frac{1}{2} (\sin(2\pi) - \sin(0)) - (\sin(\pi) - \sin(0)) \right) \\
 &= 0
 \end{aligned}$$

□

**Theorem 4.1.4.** For a double integral  $f(x, y) = g(x) \cdot h(x)$  on the rectangle  $R = [a, b] \times [c, d]$ ,

$$\iint_R g(x) \cdot h(x) \, dA = \int_a^b g(x) \, dx \cdot \int_c^d h(x) \, dy$$

**Example 4.1.4.** Evaluate the double integral

$$\iint_R \sin(x) \cos(y) \, dA, \quad \text{where } R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$$

**Answer.**

By the Fubini's Theorem,

$$\begin{aligned}
 \iint_R \sin(x) \cos(y) \, dA &= \int_0^{\pi/2} \sin(x) \, dx \cdot \int_0^{\pi/2} \cos(y) \, dy \\
 &= [-\cos x]_0^{\pi/2} \cdot [\sin(y)]_0^{\pi/2} \\
 &= \left[-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right] \cdot \left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] \\
 &= (1)(1) = 1
 \end{aligned}$$

□

**Definition 4.1.2 (Average Value).** In two-variable functions, then the average value of  $f$  on the rectangle  $R = [a, b] \times [c, d]$ ,  $f_{\text{ave}}$  is given by

$$f_{\text{ave}} = \frac{\iint_R f(x, y) \, dA}{A(R)} \quad \text{or} \quad \iint_R f(x, y) \, dA = A(R) \cdot f_{\text{ave}}.$$

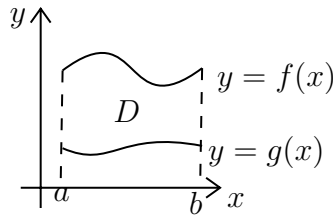


## 4.2 Double Integral Over General Region

**Definition 4.2.1 (Double Integral Over a General Region).** Furthering the definition of double integral over a rectangle, we use the notation  $\iint_D f(x, y) \, dA$  to represent a double integral of  $f(x, y)$  over a general region  $D$ .

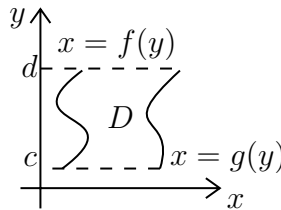
**Theorem 4.2.1 (Two Fundamental Types of Region  $D$ ).** Here, we discuss two fundamental types of region  $D$ , which includes one variable to be dependent on the other.

1.  $D = \{(x, y) \mid a < x < b, g(x) \leq y \leq f(x)\}$



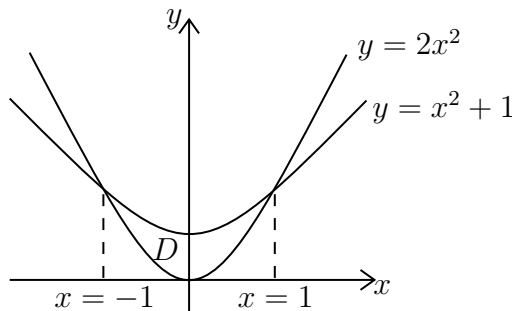
$$\iint_D f(x, y) \, dA = \int_{g(x)}^{f(x)} \int_a^b f(x, y) \, dx \, dy$$

2.  $D = \{(x, y) \mid f(y) \leq x \leq g(y), c < y < d\}$



$$\iint_D f(x, y) \, dA = \int_{f(y)}^{g(y)} \int_c^d f(x, y) \, dy \, dx$$

**Example 4.2.1.** Find  $\iint_D x+2y \, dA$ , where  $D$  is the region bounded by  $y = 2x^2$  and  $y = x^2+1$ .



**Answer.**

$$\begin{aligned} \iint_D f(x, y) \, dA &= \int_{-1}^1 \int_{2x^2}^{x^2+1} x+2y \, dy \, dx = \int_{-1}^1 \left[ xy + y^2 \right]_{2x^2}^{x^2+1} dx \\ &= \int_{-1}^1 x(x^2+1) + (x^2+1)^2 - x(2x^2) - (2x^2)^2 \, dx \end{aligned}$$

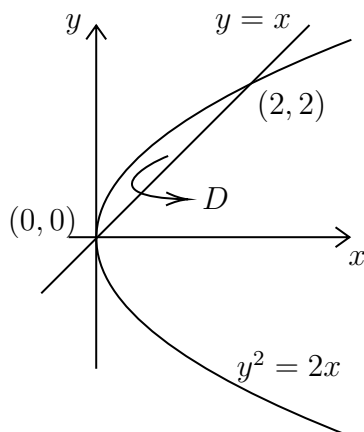
$$\begin{aligned}
\therefore \iint_D x + 2y \, dA &= \int_{-1}^1 x(x^2 + 1) + (x^2 + 1)^2 - x(2x^2) - (2x^2)^2 \, dx \\
&= \int_{-1}^1 -3x^4 - x^3 + 2x^2 + x + 1 \, dx \\
&= \left[ -\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \right]_{-1}^1 \\
&= -\frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1 - \left( \frac{3}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 1 \right) \\
&= -\frac{6}{5} + \frac{4}{3} + 2 \\
&= \frac{32}{15}
\end{aligned}$$

□

**Theorem 4.2.2.**

$$\iint_D 1 \, dA = A(D) = \text{Area of } D.$$

**Example 4.2.2.** Sketch the region  $D$  in the  $xy$ -plane bounded by  $y^2 = 2x$  and  $y = x$ . Find the area of  $D$ .

**Answer.**

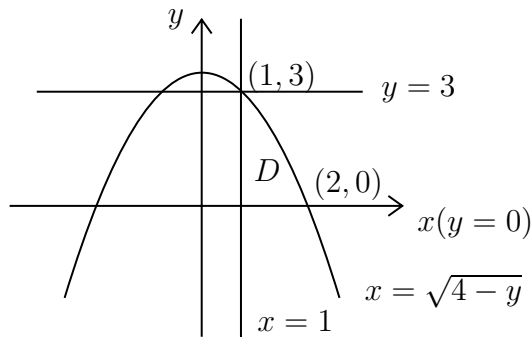
$$\begin{aligned}
\text{Area of } D &= \iint_D 1 \, dA = \iint_D 1 \, dydx \\
&= \int_0^2 \int_x^{\sqrt{2x}} 1 \, dydx \\
&= \int_0^2 (\sqrt{2x} - x) \, dx \\
&= \left[ \sqrt{2} \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^2 \right]_0^2 \\
&= \left( \frac{2\sqrt{2}}{3} (\sqrt{2})^3 - \frac{1}{2}(4) - 0 \right) \\
&= \frac{8}{3} - 2 = \frac{2}{3}
\end{aligned}$$

□

**Example 4.2.3.** Given  $\int_0^3 \int_1^{\sqrt{4-y}} x + y \, dx \, dy$ .

(a) Sketch the region.

**Answer.**



□

(b) Interchange the order.

**Answer.**

$$\int_0^3 \int_1^{\sqrt{4-y}} x + y \, dx \, dy = \int_1^2 \int_0^{4-x^2} x + y \, dy \, dx$$

□

(c) Evaluate the integral.

**Answer.**

$$\begin{aligned} \int_1^2 \int_0^{4-x^2} x + y \, dy \, dx &= \int_1^2 \left[ xy + \frac{1}{2}y^2 \right]_0^{4-x^2} dx \\ &= \int_1^2 \left[ x(4-x^2) + \frac{1}{2}(4-x^2)^2 \right] dx \\ &= \int_1^2 \left( 4x - x^3 + \frac{1}{2}(16 + x^4 - 8x^2) \right) dx \\ &= \int_1^2 \left( \frac{1}{2}x^4 - x^3 - 4x^2 + 4x + 8 \right) dx \\ &= \left[ \frac{1}{2} \cdot \frac{1}{5}x^5 - \frac{1}{4}x^4 - 4 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 + 8x \right]_1^2 \\ &= \frac{1}{10}(2^5 - 1) - \frac{1}{4}(2^4 - 1) - \frac{4}{3}(2^3 - 1) + 2(2^2 - 1) + 8(2 - 1) \\ &= \frac{31}{10} - \frac{15}{4} - \frac{28}{3} + 6 + 8 \\ &= \frac{241}{60} \end{aligned}$$

□

**Theorem 4.2.3.** Properties of Double Integral:

1.

$$\iint_D [f(x, y) + g(x, y)] \, dA = \iint_D f(x, y) \, dA + \iint_D g(x, y) \, dA$$

2.

$$\iint_D cf(x, y) \, dA = c \iint_D f(x, y) \, dA$$

3. If  $D = D_1 + D_2$ , then

$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA$$

4. If  $f(x, y) \geq g(x, y)$ , then

$$\iint_D f(x, y) \, dA \geq \iint_D g(x, y) \, dA$$

5. If  $m \leq f(x, y) \leq M$  and  $A(D)$  is the area of the region  $D$ , then

$$m \cdot A(D) \leq \iint_D f(x, y) \, dA \leq M \cdot A(D).$$

**Example 4.2.4.** Estimate the integral  $\iint_D e^{\sin x \cos y} \, dA$ , where  $D$  is a disk centered at origin with a radius of 2.**Answer.**Since  $-1 \leq \sin x \leq 1$  and  $-1 \leq \cos y \leq 1$ , we have

$$-1 \leq \sin x \cos y \leq 1.$$

Therefore,

$$e^{-1} \leq e^{\sin x \cos y} \leq e^1.$$

$$\iint_D e^{-1} \, dA \leq \iint_D e^{\sin x \cos y} \, dA \leq \iint_D e^1 \, dA.$$

Recall that

$$\iint_D 1 \, dA = \text{Area of the disk} = 2^2\pi = 4\pi.$$

$$\iint_D e^{-1} \, dA = e^{-1} \iint_D 1 \, dA = \frac{4\pi}{e} \quad \text{and} \quad \iint_D e^1 \, dA = 4e\pi.$$

$$\frac{4\pi}{e} \leq \iint_D e^{\sin x \cos y} \, dA \leq 4e\pi.$$

□

### 4.3 Changing Variables in Double Integrals

**Theorem 4.3.1 (Transformation of Double Integral).**

$$\iint_R F(x, y) \, dx dy = \iint_{R'} F(f(u, v), g(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where  $x = f(u, v)$  and  $y = g(u, v)$ .  $R'$  is the region in  $uv$ -plane which  $R$  is mapped under the transformation  $T = \begin{cases} x = f(u, v) \\ y = g(u, v) \end{cases}$ .

**Definition 4.3.1 (Jacobian).** The Jacobian of transformation  $T = \begin{cases} x = f(u, v) \\ y = g(u, v) \end{cases}$  is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v}.$$

**Example 4.3.1.** If  $u = x^2 - y^2$  and  $v = 2xy$ . Find  $\frac{\partial(x, y)}{\partial(u, v)}$  in terms of  $u$  and  $v$ .

*Answer.*

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2x & 2y \\ -2y & 2x \end{vmatrix}} = \frac{1}{4x^2 + 4y^2}$$

$$u = x^2 - y^2, \quad v = 2xy$$

Note that:

$$\begin{aligned} (x^2 - y^2)^2 &= (x^2 + y^2)^2 - (2xy)^2 \\ u^2 &= (x^2 + y^2)^2 - v^2 \\ (x^2 + y^2)^2 &= u^2 + v^2 \\ x^2 + y^2 &= \pm \sqrt{u^2 + v^2} \end{aligned}$$

Therefore,

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\pm 4\sqrt{u^2 + v^2}}$$

□

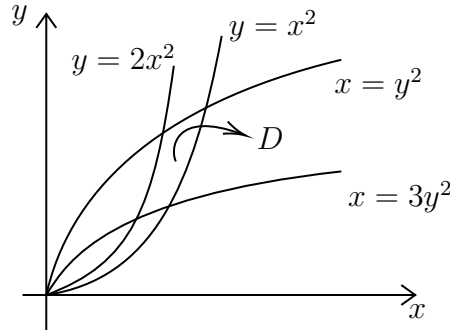
**Theorem 4.3.2 (Absolute Value of Jacobian).** In fact, the absolute value of Jacobian  $\frac{\partial(x, y)}{\partial(u, v)}$  is the ratio between corresponding area elements in the  $xy$ -plane and the  $uv$ -plane.

$$dA = dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

**Example 4.3.2.** Find the area of the finite plane region bounded by the four parabolas:

$$y = x^2, \quad y = 2x^2, \quad x = y^2, \quad x = 3y^2$$

**Answer.**



From  $\begin{cases} y = x^2 \\ y = 2x^2 \end{cases}$ , we know  $\begin{cases} \frac{y}{x^2} = 1 \\ \frac{y}{x^2} = 2 \end{cases}$ . Let  $u = \frac{y}{x^2}$ :  $\begin{cases} u = 1 \\ u = 2 \end{cases}$ .

Similarly, let  $v = \frac{x}{y^2}$ , then  $\begin{cases} v = 1 \\ v = 3 \end{cases}$ .

So, the region  $D$  is transformed to a rectangle in the  $uv$ -plane.

Let  $u = \frac{y}{x^2}$  and  $v = \frac{x}{y^2}$ , where  $1 \leq u \leq 2$  and  $1 \leq v \leq 3$ .

$$\iint_D dA = \iint_R \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -\frac{2x}{y^3} \end{vmatrix}} = \frac{1}{\frac{4}{x^2 y^2} - \frac{1}{x^2 y^2}} = \frac{x^2 y^2}{3}. \end{aligned}$$

Note that  $uv = \frac{y}{x^2} \cdot \frac{x}{y^2} = \frac{1}{xy}$ , so  $u^2 v^2 = \frac{1}{x^2 y^2}$ . Hence,  $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{3u^2 v^2}$ .

Therefore,

$$\begin{aligned} \iint_D dA &= \int_1^3 \int_1^2 \frac{1}{3u^2 v^2} du dv = \frac{1}{3} \int_1^3 \int_1^2 \frac{1}{u^2 v^2} du dv \\ &= \frac{1}{3} \int_1^3 \left[ -\frac{1}{uv^2} \right]_1^2 dv \\ &= \frac{1}{3} \int_1^3 \left( -\frac{1}{2v^2} \right) dv \\ &= -\frac{1}{6} \int_1^3 \frac{1}{v^2} dv = -\frac{1}{6} \left[ -\frac{1}{v} \right]_1^3 = -\frac{1}{6} \left( -1 + \frac{1}{3} \right) = \frac{1}{9} \end{aligned}$$

□

## 4.4 Double Integral in Polar Coordinates

**Theorem 4.4.1 (Double Integral in Polar Coordinates).** In polar coordinates,  $x^2 + y^2 = r$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Therefore,

$$\iint_R F(x, y) \, dA = \iint_R F(x, y) \, dx dy = \iint_{R'} F(r \cos \theta, r \sin \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr d\theta.$$

Since

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r(\cos^2 \theta + \sin^2 \theta) = r,$$

we have

$$\iint_R F(x, y) \, dx dy = \iint_{R'} F(r \cos \theta, r \sin \theta) r \, dr d\theta.$$

**Example 4.4.1.** Evaluate  $\iint_D \frac{y^2}{x^2} \, dA$ , where  $D$  is the region limited to

$$0 \leq a \leq x^2 + y^2 \leq b \quad y = 0, \quad x = y, \quad x, y > 0.$$

**Answer.**

$$\begin{aligned} I &= \iint_D \frac{y^2}{x^2} \, dA = \int_0^{\pi/4} \int_a^b \tan^2 \theta \cdot r \, dr d\theta \\ &= \int_0^{\pi/4} \left[ \tan^2 \theta \frac{r^2}{2} \right]_a^b \, d\theta \\ &= \int_0^{\pi/4} \tan^2 \theta \frac{b^2 - a^2}{2} \, d\theta \\ &= \frac{b^2 - a^2}{2} \left[ \tan \theta - \theta \right]_0^{\pi/4} \\ &= \frac{b^2 - a^2}{2} \left( 1 - \frac{\pi}{4} \right). \end{aligned}$$

**Remark.** To evaluate  $\int \tan^2 \theta \, d\theta = \int \frac{\sin^2 \theta}{\cos^2 \theta} \, d\theta$ , we apply  $\sin^2 \theta = 1 - \cos^2 \theta$  :

$$\int \frac{\sin^2 \theta}{\cos^2 \theta} \, d\theta = \int \frac{1 - \cos^2 \theta}{\cos^2 \theta} \, d\theta = \int \frac{1}{\cos^2 \theta} d\theta - \int d\theta = \tan \theta - \theta + C.$$

□

**Example 4.4.2.** Show  $I = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$ .

**Answer.**

We try to find  $I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-x^2} \, dx$  Further, we have

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-y^2} \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} \, dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx dy$$

Then, we change it to the polar coordinate:

$$\begin{aligned}
 I^2 &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr d\theta = 2\pi \int_0^\infty e^{-r^2} r \, dr \\
 &= 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_0^\infty \\
 &= -\pi \left( \lim_{t \rightarrow \infty} \frac{1}{e^{t^2}} - e^0 \right) \\
 &= \pi(0 - 1) = \pi.
 \end{aligned}$$

□

## 4.5 Triple Integrals

**Definition 4.5.1 (Triple Integral).** Find a bounded function  $f(x, y, z)$  defined on a rectangular box,  $B : \begin{cases} x_1 \leq x \leq x_2 \\ y_1 \leq y \leq y_2 \\ z_1 \leq z \leq z_2 \end{cases}$ , then, the triple integral on that box is defined as

$$\iiint_B f(x, y, z) \, dV = \lim_{n, m, l \rightarrow \infty} \sum_{k=1}^n \sum_{j=1}^m \sum_{i=1}^l f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \underbrace{\Delta x \Delta y \Delta z}_{\Delta V}$$

if the limit exists.

**Theorem 4.5.1 (Fubini's Theorem for Triple Integral).** If  $f(x, y, z)$  is continuous over a box  $B$ , where  $B$  is defined by  $B = \{(x, y, z) \mid x_1 \leq x \leq x_2, y_1 \leq y \leq y_2, z_1 \leq z \leq z_2\}$ , then

$$\iiint_B f(x, y, z) \, dV = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y, z) \, dx dy dz.$$

**Theorem 4.5.2.**

$$\iiint_B dV = V(B) = \text{Volume of the box } B$$

In more general cases,

$$\iiint_E dV = V(E) = \text{Volume of a more general bounded region } E,$$

where  $E$  is a general bounded region.

**Theorem 4.5.3 (Volume of a Sphere).**

$$V(\text{Sphere}) = \iiint_E dV = \frac{4}{3} \pi a^3, \text{ where } E \text{ is bounded by } x^2 + y^2 + z^2 \leq a$$

**Example 4.5.1.** Evaluate  $\iiint_E 2 + x - \sin z \, dV$ , where  $E$  is bounded by  $x^2 + y^2 + z^2 \leq a$



**Answer.**

$x$  and  $\sin z$  are odd functions, so integrals of them are 0 on a symmetric region.

Note that  $E$ , by definition, is sphere centered at origin, with a radius of  $a$ , which is a symmetric region, so we have

$$\iiint_E x \, dV = \iiint_E \sin z \, dV = 0.$$

Plugging into the integral, we will have

$$\iiint_E 2 + x - \sin z \, dV = \iiint_E 2 \, dV + \iiint_E x \, dV + \iiint_E \sin z \, dV = \iiint_E 2 \, dV = \frac{8}{3}\pi a^3.$$

□

**Example 4.5.2.** Evaluate  $\iiint_B xyz^2 \, dV$ , where  $B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$ .

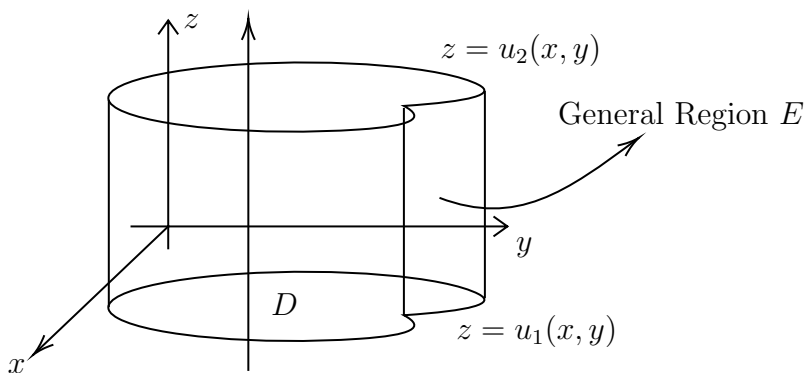
**Answer.**

$$\begin{aligned} \iiint_B xyz^2 \, dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx dy dz \\ &= \int_0^3 \int_{-1}^2 \left[ \frac{1}{2} x^2 y z^2 \right]_0^1 dy dz \\ &= \int_0^3 \int_{-1}^2 \frac{1}{2} y z^2 \, dy dz \\ &= \int_0^3 \left[ \frac{1}{4} y^2 z^2 \right]_{-1}^2 dz \\ &= \int_0^3 \frac{1}{4} (4) z^2 - \frac{1}{4} z^2 \, dz \\ &= \left[ \frac{1}{3} z^3 - \frac{1}{12} z^3 \right]_0^3 \\ &= \frac{1}{3} (27) - \frac{1}{12} (27) = 9 - \frac{9}{4} = \frac{27}{4} \end{aligned}$$

□

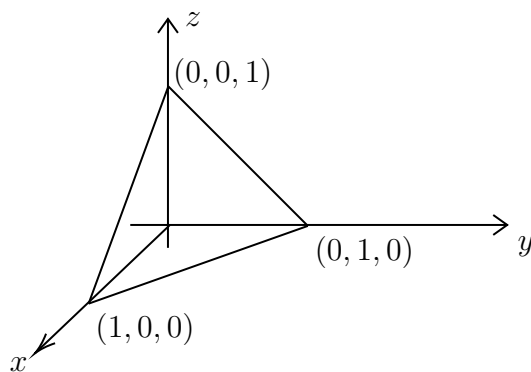
**Theorem 4.5.4 (Triple Integral Over a General Region).** If we can write  $z = u(x, y)$  as function of  $x$  and  $y$ , then we can change the triple integral into double integral. The following diagram shows this case.

$$\begin{aligned} \iiint_E f(x, y, z) \, dV &= \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right] dA \\ &= \int_a^b \int_{g_1}^{g_2} \int_{u_1}^{u_2} f(x, y, z) \, dz dx dy, \quad g(y) = x \\ \text{OR} \quad &= \int_c^d \int_{h_1}^{h_2} \int_{u_1}^{u_2} f(x, y, z) \, dz dy dx, \quad h(x) = y \end{aligned}$$



**Example 4.5.3.** Evaluate  $\iiint_E z \, dV$ , where  $E$  is the solid tetrahedron bounded by the following planes:

$$x = 0; \quad y = 0; \quad z = 0; \quad x + y + z = 1.$$



**Answer.**

$$\begin{aligned}
 \iiint_E z \, dV &= \iint_D \left[ \int_0^{1-x-y} z \, dz \right] dA \\
 &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx \\
 &= \int_0^1 \int_0^{1-x} \left[ d \frac{1}{2} z^2 \right]_0^{1-x-y} dy \, dx \\
 &= \int_0^1 \int_0^{1-x} \frac{1}{2} (1-x-y)^2 dy \, dx \\
 &= \frac{1}{2} \int_0^1 \int_0^{1-x} 1 + x^2 + y^2 - 2x - 2y + 2xy \, dy \, dx \\
 &= \frac{1}{2} \int_0^1 \left[ y + x^2 y + \frac{1}{3} y^3 - 2xy - \frac{2}{2} y^2 + \frac{2}{2} xy^2 \right]_0^{1-x} dx \\
 &= \frac{1}{2} \int_0^1 (1-x) + x^2(1-x) + \frac{1}{3}(1-x)^3 - 2x(1-x) - (1-x)^2 + x(1-x)^2 dx \\
 &= \frac{1}{2} \int_0^1 \left( 1-x+x^2-x^3 + \frac{1}{3}(1-x)^3 - 2x+2x^2 - 1+2x-x^2+x-2x^2+x^3 \right) dx \\
 &= \frac{1}{2} \int_0^1 \frac{1}{3} (1-x^3+3x^2-3x) dx \\
 &= \frac{1}{6} \left[ x - \frac{1}{4} x^4 + \frac{3}{3} x^3 - \frac{3}{2} x^2 \right]_0^1 = \frac{1}{6} \left( 1 - \frac{1}{4} + 1 - \frac{3}{2} \right) = \frac{1}{6} \left( \frac{1}{4} \right) = \frac{1}{24}.
 \end{aligned}$$

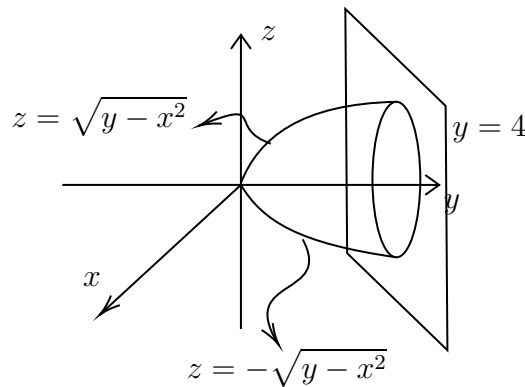
□

**Extension.** Similarly, we can have other types of triple integrals over the general region:

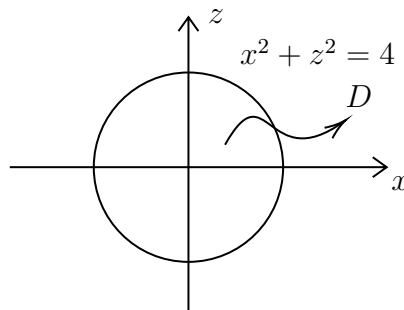
$$\iiint_E f(x, y, z) \, dV = \iint_D \left[ \int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) \, dx \right] dA$$

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[ \int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) \, dy \right] dA$$

**Example 4.5.4.** Evaluate  $\iiint_E \sqrt{x^2 + z^2} \, dV$ , where  $E$  is the region bounded by  $y = x^2 + z^2$  and  $y = 4$ .



**Answer.**



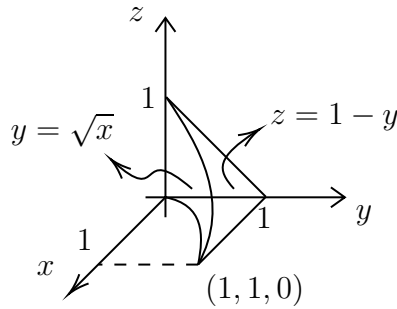
$$\begin{aligned} \iiint_E \sqrt{x^2 + z^2} \, dV &= \iint_D \left[ \int_{x^2+z^2}^4 \sqrt{x^2 + z^2} \, dy \right] dA \\ &= \iint_D \left[ (4 - x^2 - z^2) \sqrt{x^2 + z^2} \right] dA \end{aligned}$$

Now, change to polar coordinate:  $r^2 = x^2 + z^2$ ,  $0 \leq r \leq 2$ ,  $0 \leq \theta \leq 2\pi$ . So,

$$\begin{aligned} \iiint_E \sqrt{x^2 + z^2} \, dV &= \iint_{D'} (4 - r^2) \sqrt{r^2} \cdot r \, dr d\theta = \int_0^{2\pi} \int_0^2 4r^2 - r^4 \, dr d\theta \\ &= 2\pi \left[ \frac{4}{3} r^3 - \frac{1}{5} r^5 \right]_0^2 \\ &= 2\pi \left( \frac{4}{3} (8) - \frac{32}{5} \right) = \frac{128}{15} \pi \end{aligned}$$

□

**Example 4.5.5.** Given  $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) \, dz dy dx$ . Rewrite the triple integral using other five orders.



**Answer.**

① Change to  $dz dx dy$ :

$$\iiint_E f(x, y, z) \, dV = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z) \, dz dx dy$$

② Change to  $dx dy dz$ :

$$\begin{aligned} \iiint_E f(x, y, z) \, dV &= \iint_D \left[ \int_0^{y^2} f(x, y, z) \, dx \right] dA \\ &= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z) \, dx dz dy \end{aligned}$$

③ Change to  $dx dy dz$ : From  $z = 1 - y$ , we have  $y = 1 - z$ . So,

$$\iiint_E f(x, y, z) \, dV = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z) \, dx dy dz$$

④ Change to  $dy dz dx$ :

$$\begin{aligned} \iiint_E f(x, y, z) \, dV &= \iint_D \left[ \int_{\sqrt{x}}^{1-z} f(x, y, z) \, dy \right] dA \\ &= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z) \, dy dz dx \end{aligned}$$

⑤ Change to  $dy dx dz$ : Since  $z = 1 - \sqrt{x}$ , we have  $\sqrt{x} = 1 - z$ , or  $x = (1 - z)^2$ :

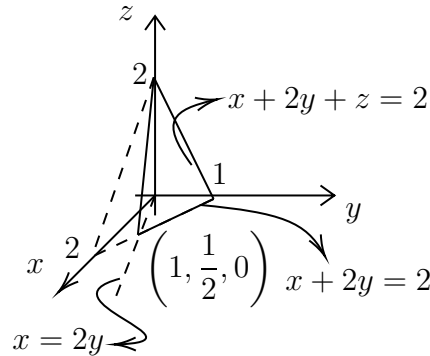
$$\iiint_E f(x, y, z) \, dV = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z) \, dy dx dz$$

□

**Remark.** One application of triple integral is to find volume of a region.

**Example 4.5.6.** Find the volume of the region bounded by the following planes:

$$x + 2y + z = 2, \quad x = 2y, \quad x = 0, \quad z = 0.$$



**Answer.**

From  $x + 2y + z = 2$ , we know that  $z = 2 - x - 2y$ . So we have

$$\begin{aligned} V &= \iiint_E 1 \, dV = \iint_D \left[ \int_0^{2-x-2y} 1 \, dz \right] dA \\ &= \int_0^1 \int_{x/2}^{(2-x)/2} \int_0^{2-x-2y} 1 \, dz \, dy \, dx \\ &= \int_0^1 \int_{x/2}^{(2-x)/2} (2-x-2y) \, dy \, dx \\ &= \int_0^1 \left[ (2-x)y - y^2 \right]_{x/2}^{(2-x)/2} dx \\ &= \int_0^1 \left( (2-x)(1-x) - \frac{1}{4}x^2 - 1 + x + \frac{1}{4}x^2 \right) dx \\ &= \int_0^1 (x^2 - 2x + 1) \, dx \\ &= \left[ \frac{1}{3}x^3 - x^2 + x \right]_0^1 = \frac{1}{3} - 1 + 1 = \frac{1}{3} \end{aligned}$$

□

## 4.6 Changing Variables in Triple Integrals

**Theorem 4.6.1 (Change of Variables in Triple Integrals).** Consider the transformation

$$T = \begin{cases} x = f(u, v, w) \\ y = g(u, v, w) \\ z = h(u, v, w) \end{cases} . \text{ We have } dV = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw, \text{ where}$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Then, we have

$$\iiint_E f(x, y, z) dx dy dz = \iiint_{E'} g(u, v, w) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw.$$

**Remark.** The determinant of triangular and diagonal matrices is the product of the elements on the main diagonal. Suppose matrix **A** and **B** are defined as follows:

$$\mathbf{A} = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Then  $\det(\mathbf{A}) = \det(\mathbf{B}) = abc$ .

**Example 4.6.1.** Find the volume of ellipsoid is given by  $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

**Answer.**

Consider the transformation:  $x = au, \quad y = bv, \quad z = cw$ .

Then,

$$E' : \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \leq 1$$

$$u^2 + v^2 + w^2 \leq 1$$

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

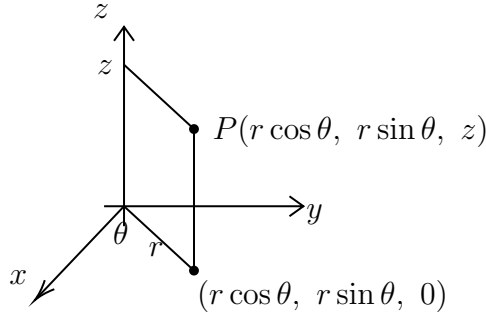
So,

$$\iiint_E 1 \, dV = \iiint_{E'} abc \, dV = abc \times V(\text{ball with radius} = 1) = abc \left( \frac{4}{3} \pi \right).$$

□

**Remark.** In 3D, there are two alternatives to Cartesian coordinate system: Cylindrical coordinate system and spherical coordinate system.

**Definition 4.6.1 (Cylindrical Coordinate System).** Uses polar coordinate in the  $xy$ -plane while retaining the Cartesian  $z$  coordinate for measuring vertical distance.



In Cylindrical Coordinate system,  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and  $z = z$ . So,

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

So,

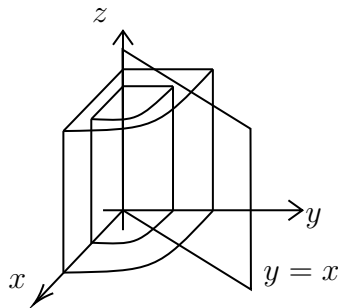
$$dV = r dr d\theta dz.$$

**Theorem 4.6.2 (Change Triple Integrals to Cylindrical Coordinate System).**

$$\iiint_E f(x, y, z) dV = \int_{z=u_1(x,y)}^{z=u_2(x,y)} \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r \cos \theta, r \sin \theta, z) r dr d\theta dz.$$

**Example 4.6.2.** Evaluate  $I = \iiint_E x^2 + y^2 dV$  over the first octant region bounded by the cylinders  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$  and planes  $z = 0$ ,  $z = 1$ ,  $x = 0$ , and  $y = x$ .

**Answer.**



Change to Cylindrical Coordinate System:  $r^2 = x^2 + y^2$ , where  $1 \leq r \leq 2$ ,  $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ ,  $0 \leq z \leq 1$ . Then,

$$\begin{aligned} I &= \int_0^1 \int_{\pi/4}^{\pi/2} \int_1^2 r^2 \cdot r dr d\theta dz \\ &= (1 - 0) \left( \frac{\pi}{2} - \frac{\pi}{4} \right) \left( \frac{2^4}{4} - \frac{1^4}{4} \right) = \frac{15}{16} \pi \end{aligned}$$

□

**Example 4.6.3.** Evaluate  $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz dy dx$ .

**Answer.**

Change to Cylindrical Coordinate system:  $r^2 = x^2 + y^2$ . So,  $r \leq z \leq 2$ .

Since  $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ , so  $0 \leq y^2 \leq 4-x^2$

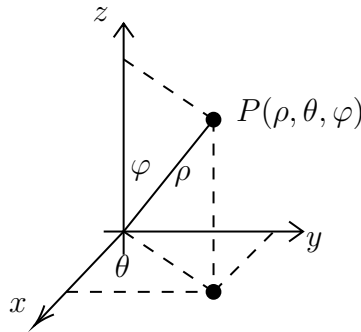
That is,  $0 \leq y^2 + x^2 \leq 4$ , or  $0 \leq r^2 \leq 4$ .

So,  $0 \leq r \leq 2$ .

$$\begin{aligned}
 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz dy dx &= \int_0^\pi \int_0^2 \int_r^2 r^2 \cdot r \, dz dr d\theta \\
 &= (2\pi) \int_0^2 r^3(2-r) \, dr \\
 &= (2\pi) \int_0^2 (2r^3 - r^4) \, dr \\
 &= (2\pi) \left[ \frac{1}{2}r^4 - \frac{1}{5}r^5 \right]_0^2 \\
 &= (2\pi) \left( 8 - \frac{32}{5} \right) = \frac{16}{5}\pi
 \end{aligned}$$

□

**Definition 4.6.2 (Spherical Coordinate System).** Here we define  $\rho$  is the distance from the origin to  $P$ ,  $\varphi$  is the angle between the line  $OP$  and the positive  $z$ -axis ( $0 \leq \varphi \leq \pi$ ), and  $\theta$  is the angle between  $OP'$  (the projection of  $OP$  onto the  $xy$ -plane) and the positive  $x$ -axis ( $0 \leq \theta \leq 2\pi$ ). So a point  $P(\rho, \theta, \varphi)$  is represented in the following graph.



Using trigonometric identities, we know  $z = \rho \cos(\varphi)$  and  $OP' = \rho \sin(\varphi)$ . Then,  $x = \rho \sin(\varphi) \cos(\theta)$  and  $y = \rho \sin(\varphi) \sin(\theta)$ . Also, applying the formula, we know  $\frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} = \rho^2 \sin(\varphi)$ . Therefore,

$$\iiint_E f(x, y, z) \, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin(\varphi) \cos(\theta), \rho \sin(\varphi) \sin(\theta), \rho \cos(\varphi)) \rho^2 \sin(\varphi) d\rho d\theta d\varphi,$$

where  $a \leq \rho \leq b$ ,  $\alpha \leq \theta \leq \beta$ ,  $c \leq \varphi \leq d$ .



**Example 4.6.4.** Evaluate  $\iiint_E e^{(x^2+y^2+z^2)^{3/2}} dV$ , where  $E = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ .

**Answer.**

Change to spherical coordinate:  $\rho^2 = x^2 + y^2 + z^2$ .

$$\begin{aligned} \iiint_E e^{(x^2+y^2+z^2)^{3/2}} dV &= \iiint_{E'} e^{(\rho^2)^{3/2}} \rho^2 \sin(\varphi) d\rho d\theta d\varphi \\ &= \iiint_{E'} e^{\rho^3} \rho^2 \sin(\varphi) d\rho d\theta d\varphi \\ &= \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 e^{\rho^3} \sin(\varphi) d\rho d\theta d\varphi \\ &= \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3} d\rho. \end{aligned}$$

Let  $u = \rho^3$ , then  $du = 3\rho^2 d\rho$ . So,  $\int \rho^2 e^{\rho^3} d\rho = \frac{1}{3} \int e^u du = \frac{1}{3} e^u = \frac{1}{3} e^{\rho^3}$ .

So,

$$\begin{aligned} \iiint_E e^{(x^2+y^2+z^2)^{3/2}} dV &= \left[ -\cos(\varphi) \right]_0^\pi (2\pi) \left[ \frac{1}{3} e^{\rho^3} \right]_0^1 \\ &= (1 + 1)(2\pi) \left( \frac{1}{3} e - \frac{1}{3} \right) \\ &= \frac{4}{3} \pi (e - 1). \end{aligned}$$

□

## 4.7 Applications of Multiple Integrals

**Theorem 4.7.1 (Surface Area).** The key idea is to use the tangent plane at any point like  $P_{ij}(x_i, y_j, z_k)$  to approximate the surface near the point  $P_{ij}$ .

Divide region  $D$  into small rectangles,  $R_{ij}$ . So,

$$\Delta A = A(R_{ij}) = \Delta x \Delta y$$

Let  $(x_i, y_j)$  be a point on  $R_{ij}$ , and its corresponding point on the surface is given by

$$P_{ij}(x_i, y_j, f(x_i, y_j))$$

The tangent plane to the surface  $S$  at point  $P_{ij}$  is an approximation of the surface around  $P_{ij}$ . Therefore,  $\Delta S_{ij} \approx \Delta T_{ij}$ . So,

$$A(S) \approx \sum_{i=1}^n \sum_{j=1}^m \Delta T_{ij}$$

and

$$A(S) = \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \Delta T_{ij}$$

To find  $\Delta T_{ij}$ , we use cross product:  $A(\Delta T_{ij}) = |\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$ .

- Slope of  $\vec{\mathbf{a}} = f_x(x_i, y_j) = \frac{\Delta z}{\Delta x}$

$$\implies \Delta z = \Delta x f_x(x_i, y_j), \quad \vec{\mathbf{a}} = \Delta x \hat{\mathbf{i}} + \Delta x f_x(x_i, y_j) \hat{\mathbf{k}}.$$

- Slope of  $\vec{\mathbf{b}} = f_y(x_i, y_j) = \frac{\Delta z}{\Delta y}$

$$\implies \Delta z = \Delta y f_y(x_i, y_j), \quad \vec{\mathbf{b}} = \Delta y \hat{\mathbf{j}} + \Delta y f_y(x_i, y_j) \hat{\mathbf{k}}.$$

So,

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Delta x & 0 & \Delta x f_x(x_i, y_j) \\ 0 & \Delta y & \Delta y f_y(x_i, y_j) \end{vmatrix} = (-f_x(x_i, y_j) \hat{\mathbf{i}} - f_y(x_i, y_j) \hat{\mathbf{j}} + \hat{\mathbf{k}}) \Delta x \Delta y \\ &= (-f_x(x_i, y_j) \hat{\mathbf{i}} - f_y(x_i, y_j) \hat{\mathbf{j}} + \hat{\mathbf{k}}) \Delta A \end{aligned}$$

So,

$$\begin{aligned} A(\Delta T_{ij}) &= |\vec{\mathbf{a}} \times \vec{\mathbf{b}}| \\ &= \sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1} \Delta A \end{aligned}$$

Therefore,

$$\begin{aligned} S &= \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \Delta T_{ij} \\ &= \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1} \Delta A \\ &= \boxed{\iint_D \sqrt{f_x^2(x, y) + f_y^2(x, y) + 1} \, dA} \\ &= \boxed{\iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dA} \end{aligned}$$

**Example 4.7.1.** Find the surface area of the paraboloid  $z = x^2 + y^2$  that lies under  $z = 9$ .

**Answer.**

$$\begin{aligned} S &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2} \, dA \\ &= \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA \end{aligned}$$

Change to polar coordinate:  $0 \leq r \leq 3$  and  $0 \leq \theta \leq 2\pi$ :

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^3 \sqrt{1+4r^2} r dr d\theta \\ &= 2\pi \int_0^3 r \cdot \sqrt{1+4r^2} dr \end{aligned}$$

Let  $u = 1 + 4r^2$ , so  $du = 8r dr$ . So,

$$\begin{aligned} \int r \sqrt{1+4r^2} dr &= \frac{1}{8} \int \sqrt{u} du \\ &= \frac{1}{8} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{12} u^{3/2} \end{aligned}$$

Therefore,

$$\begin{aligned} S &= 2\pi \int_0^3 r \cdot \sqrt{1+4r^2} dr \\ &= 2\pi \left[ \frac{1}{12} (1+4r^2)^{3/2} \right]_0^3 \\ &= \frac{\pi}{6} (37\sqrt{37} - 1). \end{aligned}$$

□

**Example 4.7.2.** Find the area of the part of the plane  $z = ax + by + c$  that projects onto a region in the  $xy$ -plane with an area of  $A$ .

**Answer.**

$$\text{Area} = \iint_D \sqrt{a^2 + b^2 + 1} dA = \sqrt{a^2 + b^2 + 1} \iint_D dA$$

Since  $\iint_D dA = A$  is given,

$$\text{Area} = \sqrt{a^2 + b^2 + 1}(A) = A\sqrt{a^2 + b^2 + 1}.$$

□

**Definition 4.7.1 (Mass from Density Function).** Let  $D$  be a lamina (a thin plate) made of materials whose density varies across  $D$ . Let  $\rho(x, y)$  be the density of  $D$  at point  $(x, y)$ , we define

$$m(D) = \iint_D \rho(x, y) dA$$

as the total mass of  $D$  with density function  $\rho$ .

**Remark.** If we change  $\rho(x, y)$  to be probability functions,  $m(D)$  can be regarded as the cumulative probability.

**Definition 4.7.2 (Center of Mass).** The center of mass is denoted by the point  $(\bar{x}, \bar{y})$  on  $D$  such that if we place a support at that point, the lamina  $D$  will have a perfect balance.

**Definition 4.7.3 (Moment).** We define the moment of the lamina  $D$  over the  $y$ -axis as

$$\iint_D x\rho(x, y) \, dA$$

and the moment of the lamina  $D$  over the  $x$ -axis as

$$\iint_D y\rho(x, y) \, dA.$$

**Theorem 4.7.2 (Calculate Center of Mass).** We use moment of the lamina to calculate the center of mass:

$$\bar{x} = \frac{\iint_D x\rho(x, y) \, dA}{m(D)}; \quad \bar{y} = \frac{\iint_D y\rho(x, y) \, dA}{m(D)}.$$

**Example 4.7.3.** The geometric model of a material body is a plane region  $R$  bounded by  $y = x^2$  and  $y = \sqrt{2 - x^2}$  on the interval  $[0, 1]$ . The density function is  $\rho(x, y) = xy$ . Find the center of mass of  $R$ .

**Answer.**

We know

$$m(D) = \iint_D xy \, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} xy \, dy dx = \frac{7}{24}.$$

Applying the formula to calculate the center of mass, we get

$$\bar{x} = \frac{\iint_D x\rho(x, y) \, dA}{m(D)} = \frac{\frac{17}{105}}{\frac{7}{24}}$$

and

$$\bar{y} = \frac{\iint_D y\rho(x, y) \, dA}{m(D)} = \frac{\frac{13}{120} + \frac{4\sqrt{2}}{15}}{\frac{7}{24}}.$$

□

## 4.8 Multiple Integral – Practice

**Example 4.8.1.** If  $D$  is the triangle with vertices  $(-2, 0)$ ,  $(0, 4)$ , and  $(8, 0)$ , calculate  $\iint_D xy^2 \, dA$ .

**Answer.**

- Using the order  $dydx$ , we have

$$\int_{-2}^0 \int_0^{2x+4} xy^2 \, dy dx + \int_0^8 \int_0^{-x/2+4} xy^2 \, dy dx$$

It is not easy to calculate the integral as two parts.

- Using the order  $dx dy$ , we have

$$\begin{aligned}
 \int_0^4 \int_{-2+y/2}^{8-2y} xy^2 \, dx dy &= \int_0^4 \left[ \frac{1}{2} x^2 y^2 \right]_{-2+y/2}^{8-2y} dy \\
 &= \int_0^4 30y^2 - 15y^3 + \frac{15}{8}y^4 \, dy \\
 &= \left[ 30y^2 - 15y^3 + \frac{15}{8}y^4 \right]_0^4 \\
 &= 640 - 960 + 384 = 64.
 \end{aligned}$$

□

**Example 4.8.2.** If  $D$  is the region bounded by  $y = x^2$  and  $y = 8 - x^2$ , calculate  $\iint_D x^3 \, dA$ .

**Answer.**

$D$  is a symmetric region about  $x = 0$  and function  $f(x, y) = x^3$  is an odd function with respect to  $x$ . Therefore,

$$\iint_D x^3 \, dA = 0.$$

□

**Example 4.8.3.** Calculate the area of the region bounded by two parabolas  $y = x^2$  and  $x = y^2$ .

**Answer.**

$$\begin{aligned}
 A(D) &= \iint_D 1 \, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy dx \\
 &= \int_0^1 \sqrt{x} - x^2 \, dx \\
 &= \left[ \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1 \\
 &= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
 \end{aligned}$$

□

**Example 4.8.4.** Let  $D$  be the unit disk:  $x^2 + y^2 \leq 1$ . Calculate  $\iint_D (2 - x)(3 + y) \, dA$ .

**Answer.**

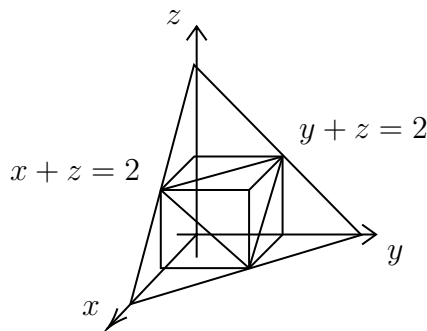
$D$  is a symmetric region in  $x$  and  $y$ . So,

$$\begin{aligned}
 \iint_D (2 - x)(3 + y) \, dA &= \iint_D 6 - 3x + 2y - xy \, dA \\
 &= \iint_D 6 \, dA - \underbrace{\iint_D 3x \, dA}_{=0 \text{ (symmetric in } x)}} + \underbrace{\iint_D -xy + 2y \, dA}_{=0 \text{ (symmetric in } y)}} \\
 &= 6 \times A(D) = 6\pi.
 \end{aligned}$$

□

**Example 4.8.5.** Find  $\iiint_E x \, dV$ , where  $E$  is the tetrahedron bounded by the plane

$$x = 1, \quad y = 1, \quad z = 1, \quad x + y + z = 2.$$



**Answer.**

$$\begin{aligned} \iiint_E x \, dV &= \iint_D \left[ \int_{2-x-y}^1 x \, dz \right] dA \\ &= \int_0^1 \int_{1-x}^1 \int_{2-x-y}^1 x \, dz dy dx \\ &= \int_0^1 \int_{1-x}^1 x(1 - 2 + x + y) \, dy dx \\ &= \int_0^1 \int_{1-x}^1 x^2 + xy - x \, dy dx \\ &= \int_0^1 x^3 + x^2 - \frac{1}{2}x^3 - x^2 \, dx \\ &= \int_0^1 \frac{1}{2}x^3 \, dx = \left[ \frac{1}{2}x^3 \right]_0^1 = \frac{1}{8}. \end{aligned}$$

□

**Example 4.8.6.** Plot the cylindrical coordinate of  $\left(4, \frac{\pi}{3}, -3\right)$  and find its rectangular coordinates.

**Answer.**

$$r = 4, \quad \theta = \frac{\pi}{3}, \quad z = -3.$$

$$x = r \cos \theta = 4 \cdot \cos \left( \frac{\pi}{3} \right) = 4 \cdot \frac{1}{2} = 2$$

$$y = r \sin \theta = 4 \cdot \sin \left( \frac{\pi}{3} \right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

$$\text{Rectangular coordinate: } (2, 2\sqrt{3}, -3).$$

□

**Example 4.8.7.** Find the volume enclosed by the cone  $z = \sqrt{x^2 + y^2}$  and  $x^2 + y^2 + z^2 = 2$ .

**Answer.**

Change to cylindrical coordinate:  $x^2 + y^2 = r^2$  and  $z = z$  :

$$0 \leq r \leq \sqrt{2}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1.$$

So,

$$\iiint_E dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^1 dz dr d\theta = 2\pi(\sqrt{2})(1) = 2\sqrt{2}\pi.$$

□

## 5 Vector Calculus

### 5.1 Vector Fields

**Definition 5.1.1 (Vector Field).** Let  $D$  be a region (or a set) in  $\mathbb{R}^n$ . A vector field on  $\mathbb{R}^n$  is a function  $\vec{\mathbf{F}}$  that assigns to each point  $(x_1, \dots, x_n)$  a  $n$ -dimensional vector  $\vec{\mathbf{F}}(x_1, \dots, x_n)$ .

**Example 5.1.1.**

$$\vec{\mathbf{F}}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}},$$

where  $P$  and  $Q$  are scalar functions. Sometimes,  $P$  and  $Q$  are called scalar fields.

$$\vec{\mathbf{F}}(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}},$$

where  $P$ ,  $Q$ , and  $R$  are scalar functions or scalar fields.

**Remark.** In fact, vector fields can model velocity, magnetic force, fluid motion, and gradient.

**Definition 5.1.2 (Gradient Fields).** let  $f$  be a scalar function of two (or three) variables on  $\mathbb{R}^2$  (or  $\mathbb{R}^3$ ). Its gradient is a vector field on  $\mathbb{R}^2$  (or  $\mathbb{R}^3$ ) given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

or

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}.$$

**Example 5.1.2.** Find the gradient vector field of  $f(x, y) = x^2y - y^3$ .

**Answer.**

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} = 2xy\hat{\mathbf{i}} + (x^2 - 3y^2)\hat{\mathbf{j}}$$

□

**Remark.** Properties of Gradient Fields

- Gradient vectors are perpendicular to the level curves
- Gradient vectors point in the direction of maximum change in value of the function at a given point.
- The magnitudes of gradient vectors are a measure of local intensity change at a given point.



## 5.2 Line Integrals

In this section, we define line integral similar to a single integral, but instead of interval, we integrate over a curve.

**Definition 5.2.1 (Line Integral).** Let  $f$  be defined on a differentiable curve  $C$ , where

$$C = \begin{cases} x(t) \\ y(t) \end{cases}, \quad a \leq t \leq b.$$

We choose  $(x_i^*, y_i^*)$  on sub-arc correspond to  $t_i^*$ . We calculate

$$\sum_{i=1}^n f(x_i^*, y_i^*) \Delta S_i.$$

When  $n \rightarrow \infty$ , we define the line integral of  $f$  along curve  $C$  as

$$\int_C f(x, y) \, ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta S_i$$

if the limit exists.

**Theorem 5.2.1 (Length of a Curve).** The length of a curve  $C$  defined by  $\begin{cases} x(t) \\ y(t) \end{cases}$  is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

**Theorem 5.2.2 (Calculating Line Integrals).** Applying Theorem 5.2.1, we have

$$\int_C f(x, y) \, ds = \int_a^b f(x, y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

**Example 5.2.1.** Evaluate  $\int_C 2 + x^2 y \, ds$  over the upper half of the unit circle  $x^2 + y^2 = 1$ .

**Answer.**

We know  $C : \begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}, \quad 0 \leq t \leq \pi$ . So,  $x'(t) = -\sin t$  and  $y'(t) = \cos t$ .

$$\begin{aligned} \int_C 2 + x^2 y \, ds &= \int_0^\pi (2 + x^2 y) \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) \, dt \\ &= \left[ 2t \right]_0^\pi - \frac{1}{3} \left[ \cos^3 t \right]_0^\pi = 2\pi - \frac{1}{3}(-2) = 2\pi + \frac{2}{3}. \end{aligned}$$

□

**Theorem 5.2.3 (Price-wise Smooth Line Integrals).** If  $C$  is a piece-wise smooth curve defined by  $C_1 + C_2 + \cdots + C_n$ . Then, the line integral over  $C$  is

$$\int_C f(x, y) \, dx = \int_{C_1} f(x, y) \, ds + \int_{C_2} f(x, y) \, ds + \cdots + \int_{C_n} f(x, y) \, ds$$

**Theorem 5.2.4 (Vector Representation of a Line Segment).** The vector representation of a line segment starts at  $\vec{r}_0$  and ends at  $\vec{r}_1$  is given by

$$\vec{r}(t) = (1 - t)\vec{r}_0 + t\vec{r}_1 \quad 0 \leq t \leq 1.$$

**Definition 5.2.2 (Line Integrals with Respect to  $x$  and  $y$ ).**

$$\int_C f(x, y) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i = \int_a^b f(x(t), y(t)) x'(t) \, dt$$

$$\int_C f(x, y) \, dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i = \int_a^b f(x(t), y(t)) y'(t) \, dt$$

**Theorem 5.2.5.**

$$\int_C P(x, y) \, dx + \int_C Q(x, y) \, dy = \int_C P(x, y) dx + Q(x, y) dy$$

**Example 5.2.2.** Evaluate  $\int_C y^2 dx + x dy$ , where  $C$  is

1. A line segment from  $(-5, -3)$  to  $(0, 2)$

**Answer.**

The equation of the line is  $y + 3 = x + 5$ .

Set  $y + 3 = x + 5 = t$ . We get  $y(t) = t - 3$  and  $x(t) = t - 5$ .

So,  $dy = dt$  and  $dx = dt$ .

From  $(-5, -3)$  to  $(0, 2) : 0 \leq t \leq 5$ .

$$\begin{aligned} \int_C y^2 dx + x dy &= \int_0^5 (t - 3)^2 dx + (t - 5) dy \\ &= \int_0^5 (t - 3)^2 dt + (t - 5) dt \\ &= \int_0^5 (t^2 + 9 - 6t + t - 5) \, dt \\ &= \int_0^5 t^2 - 5t + 4 \, dt \\ &= \left[ \frac{1}{3} t^3 - \frac{5}{2} t^2 + 4t \right]_0^5 = -\frac{5}{6} \end{aligned}$$

□

2. The parabola of  $x = 4 - y^2$  from  $(-5, -3)$  to  $(0, 2)$

**Answer.**

Let  $y = t$ , so  $x(t) = 4 - t^2$ .

So,  $dy = dt$  and  $dx = -2t dt$ .

Since  $-3 \leq y \leq 2$ , we know  $-3 \leq t \leq 2$ . So,

$$\begin{aligned} \int_C y^2 dx + x dy &= \int_{-3}^2 t^2(-2t) dt + (4 - t^2) dt \\ &= \int_{-3}^2 -2t^3 + 4t - t^2 dt \\ &= \left[ -\frac{1}{2}t^4 - \frac{1}{3}t^3 + 4t \right]_{-3}^2 = \frac{245}{6}. \end{aligned}$$

□

**Theorem 5.2.6.** The line integral depends on the path in general. Line integral depends on the orientation of the path.

$$\int_{-C} f(x, y) ds = - \int_C f(x, y) ds.$$

**Definition 5.2.3 (Vector Representation of Line Integrals).** Let  $\vec{r}(t) = \langle x(t), y(t) \rangle = x(t)\hat{i} + y(t)\hat{j}$ . Then,  $\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$ . So,

$$\int_C f(x, y) ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

**Definition 5.2.4 (Line Integrals in Spaces).**

$$\begin{aligned} \int_C f(x, y, z) ds &= \int_a^b f(x, y, z) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt, \end{aligned}$$

where  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ .

**Theorem 5.2.7.** Specially, if  $f(x, y, z) = 1$ , we have

$$L = \text{length of the curve } C = \int_C ds = \int_a^b |\vec{r}'(t)| dt.$$

**Example 5.2.3.** Evaluate  $\int_C y \sin z ds$ , where  $C = \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}, 0 \leq t \leq 2\pi$  (the circular helix).

**Answer.**

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t, \quad 0 \leq t \leq 2\pi$$

$$x'(t) = -\sin t, \quad y'(t) = \cos t, \quad z'(t) = 1.$$

So,

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1 + 1} = \sqrt{2}.$$

$$\begin{aligned} \int_C y \sin z \, ds &= \int_0^{2\pi} \sin t \cdot \sin t (\sqrt{2}) \, dt \\ &= \sqrt{2} \int_0^{2\pi} \sin^2 t \, dt \\ &= \sqrt{2} \int_0^{2\pi} \pi \frac{1}{2} (1 - \cos 2t) \, dt \\ &= \frac{\sqrt{2}}{2} \left[ t - \frac{1}{2} \sin 2t \right]_0^{2\pi} \\ &= \frac{2}{2} (2\pi) = \sqrt{2}\pi. \end{aligned}$$

□

**Example 5.2.4.** 1. Find the vector representation of the line segment starting at  $(2, 0, 0)$  and ending at  $(3, 4, 5)$ .

**Answer.**

$$\begin{aligned} \vec{r}(t) &= (1-t)\langle 2, 0, 0 \rangle + t\langle 3, 4, 5 \rangle, \quad 0 \leq t \leq 1 \\ &= \langle 2 - 2t + 3t, 4t, 5t \rangle \\ &= \langle 2 + t, 4t, 5t \rangle, \quad 0 \leq t \leq 1. \end{aligned}$$

□

2. Evaluating  $\int_C y \, dx + z \, dy + x \, dz$ , where  $C$  is the line segment from the previous question.

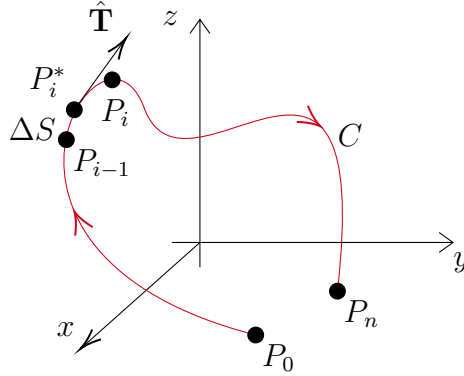
**Answer.**

$$x(t) = 2 + t, \quad dx = dt, \quad y(t) = 4t, \quad dy = 4dt, \quad z(t) = 5t, \quad dz = 5dt.$$

$$\begin{aligned} \int_C y \, dx + z \, dy + x \, dz &= \int_0^1 4t \, dt + 5t(4) \, dt + (2+t)(5) \, dt \\ &= \int_0^1 29t + 10 \, dt \\ &= \left[ \frac{29}{2} t^2 + 10t \right]_0^1 \\ &= \frac{29}{2} + 10 = \frac{49}{2}. \end{aligned}$$

□

**Definition 5.2.5 (Line Integrals of Vector Fields).** Let  $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$  be a continuous force field on  $\mathbb{R}^3$ . We want to compute the work done by this force in moving a particle along a smooth curve  $C$ .



So, we divide  $C$  into  $n$  sub-arc with length  $\Delta S$ . Particles moves along curve  $C$  from  $P_{i-1}$  to  $P_i$  in the direction of the unit tangent vector  $\hat{\mathbf{T}}(t_i^*)$  at  $P_i^*$ . The work done by the force  $\vec{F}$  in moving from  $P_{i-1}$  to  $P_i$  is

$$W \approx \vec{F} \cdot \vec{D} = \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \hat{\mathbf{T}}(t_i^*) \Delta S.$$

So,

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \hat{\mathbf{T}}(t_i^*) \right] \Delta S \\ &= \int_C \vec{F}(x, y, z) \cdot \hat{\mathbf{T}}(x, y, z) ds \\ &= \int_C \vec{F} \cdot \hat{\mathbf{T}} \, ds \end{aligned}$$

where  $\hat{\mathbf{T}}$  is the unit tangent vector at the point  $(x, y, z)$ .

Since  $ds = |\vec{r}'(t)| dt$  and  $\hat{\mathbf{T}}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ , we have

$$\begin{aligned} W &= \int_a^b \left( \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \right) \cdot |\vec{r}'(t)| \, dt \\ &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \end{aligned}$$

Therefore, for a continuous vector field  $\vec{F}$  defined on a smooth curve  $C$  given by a vector function  $\vec{r}(t)$ ,  $a \leq t \leq b$ , the line integral on  $\vec{F}$  along  $C$  is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_C \vec{F} \cdot \hat{\mathbf{T}} \, ds.$$

**Theorem 5.2.8.** If  $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$  is a vector field and  $\vec{\mathbf{r}} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ , then

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) dt \\ &= \int_a^b \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\ &= \int_a^b \left( P(x, y, z) \frac{dx}{dt} + Q(x, y, z) \frac{dy}{dt} + R(x, y, z) \frac{dz}{dt} \right) dt \\ &= \boxed{\int_a^b Pdx + Qdy + Rdz}\end{aligned}$$

**Example 5.2.5.** Evaluate  $\int_C \vec{\mathbf{F}} d\vec{\mathbf{r}}$ , where  $\vec{\mathbf{F}} = xy\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$  and  $C = \begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$ ,

where  $0 \leq t \leq 1$ .

**Answer.**

$$x(t) = t, \quad dx = dt; \quad y(t) = t^2, \quad dy = 2t dt; \quad z(t) = t^3, \quad dz = 3t^2 dt$$

So,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b Pdx + Qdy + Rdz \\ &= \int_0^1 xydt + yz(2t)dt + zx(3t^2)dt \\ &= \int_0^1 t^3 + 5t^6 dt \\ &= \left[ \frac{1}{4}t^4 + \frac{5}{7}t^7 \right]_0^1 = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}.\end{aligned}$$

□

## 5.3 The Fundamental Theorem of Line Integral

**Theorem 5.3.1 (The Fundamental Theorem of Line Integral).**

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)),$$

where  $C$  is a smooth curve with vector function  $\vec{\mathbf{r}}(t)$ , with  $a \leq t \leq b$  and  $f$  is a differentiable function of two or three variables whose gradient vector,  $\nabla f$ , is continuous on  $C$

**Proof.**

Let  $I$  be the line integral defined by

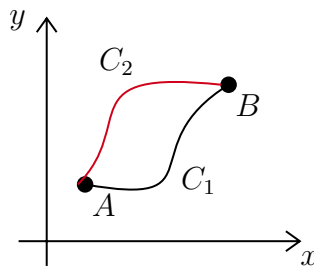
$$I = \int_C \nabla f \cdot d\vec{\mathbf{r}}.$$

Then,

$$\begin{aligned}
 I &= \int_a^b \langle f_x(\vec{r}(t)), f_y(\vec{r}(t)), f_z(\vec{r}(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt \\
 &= \int_a^b (f_x(\vec{r}(t))x'(t) + f_y(\vec{r}(t))y'(t) + f_z(\vec{r}(t))z'(t)) dt \\
 &= \int_a^b \left( \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\
 &= \int_a^b \frac{d}{dt} (F(\vec{r}(t))) dt = f(\vec{r}(b)) - f(\vec{r}(a)).
 \end{aligned}$$

■

**Remark (Independence of Path).** Let  $C_1$  and  $C_2$  be two paths that have the same initial and terminal points.



We know that, in general,

$$\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$$

But we can show

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$$

The key difference here is that we may not be able to find a function  $f$  whose gradient  $\nabla f = \vec{F}$ , the vector field.

**Definition 5.3.1 (Conservative Vector Function).** We say that vector function  $\vec{F}$  is conservative if there exists a function  $f(x, y, z)$  such that  $\nabla f = \vec{F}$ .

**Theorem 5.3.2 (Testing Conservative).** A vector field  $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$  is conservative and  $P, Q, R$  have continuous first order partial derivatives if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

**Theorem 5.3.3 (Independence of Path).** The line integral of a conservative vector field depends only on initial and terminal points and is independent of path.

**Definition 5.3.2 (Independence of Path).** Let  $\vec{F}$  be a continuous vector field with domain  $D$ . We say that  $\int_C \vec{F} \cdot d\vec{r}$  is independent of path if

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

for any two paths  $C_1$  and  $C_2$  in  $D$  that have the same initial and terminal points.

**Lemma 5.1.** Let  $\int_C \vec{F} \cdot d\vec{r}$  be independent of path where  $C$  is a closed path, then  $\int_C \vec{F} \cdot d\vec{r} = 0$ .

**Proof.**

Divide  $C$  into two paths,  $C_1$  and  $C_2$ .

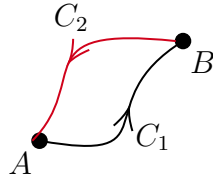
Then,

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} \\ &= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r}. \end{aligned}$$

Since  $\vec{F}$  is independent of path, we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r}.$$

So,  $\int_C \vec{F} \cdot d\vec{r} = 0$ .



■

**Lemma 5.2.** If  $\int_C \vec{F} \cdot d\vec{r} = 0$  for every closed path in  $D$ , then  $\int_C \vec{F} \cdot d\vec{r}$  is independent of path in  $D$ .

**Proof.**

We have  $\int_C \vec{F} \cdot d\vec{r} = 0$  for any closed  $C$  in  $D$ .

$$\begin{aligned} 0 &= \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} \\ &= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} \end{aligned}$$

So,  $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r}$ .

Therefore,  $\vec{F}$  is independent of path.

■

**Theorem 5.3.4.** From Lemma 5.1 and Lemma 5.2, we have  $\int_C \vec{F} \cdot d\vec{r}$  is independent of path in  $D$  if and only if  $\int_C \vec{F} \cdot d\vec{r} = 0$  for every closed  $C$  in  $D$ .



**Theorem 5.3.5 (Test for Conservation).** If the vector field  $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$  is conservative and  $P, Q, R$  have continuous first order partial derivatives, then the following is true:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}; \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

**Proof.**

Since  $\vec{\mathbf{F}}$  is conservative, there exists a function  $f$  such that

$$\vec{\mathbf{F}} = \nabla f = f_x \hat{\mathbf{i}} + f_y \hat{\mathbf{j}} + f_z \hat{\mathbf{k}}.$$

So,

$$P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}} = f_x \hat{\mathbf{i}} + f_y \hat{\mathbf{j}} + f_z \hat{\mathbf{k}}.$$

That is,

$$\begin{cases} P = f_x \\ Q = f_y \\ R = f_z \end{cases} \implies \begin{cases} \frac{\partial P}{\partial y} = f_{yx} = f_{xy} = \frac{\partial f_y}{\partial x} = \frac{\partial Q}{\partial x} \\ \frac{\partial P}{\partial z} = f_{zx} = f_{xz} = \frac{\partial f_z}{\partial x} = \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial z} = f_{zy} = f_{yz} = \frac{\partial f_z}{\partial y} = \frac{\partial R}{\partial y} \end{cases}$$

**Example 5.3.1.** Consider the vector field

$$\vec{\mathbf{F}} = Ax \sin(\pi y) \hat{\mathbf{i}} + (x^2 \cos(\pi y) + B y e^{-z}) \hat{\mathbf{j}} + y^2 e^{-z} \hat{\mathbf{k}}.$$

1. For what values of  $A$  and  $B$  is the vector field  $\vec{\mathbf{F}}$  conservative?

**Answer.**

We know:  $P = Ax \sin(\pi y)$ ,  $Q = (x^2 \cos(\pi y) + B y e^{-z})$ ,  $R = y^2 e^{-z}$ .

Then, by Theorem 5.3.5, we should have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}; \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

From  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ , we know  $Ax\pi \sin(\pi y) = 2x \cos(\pi y)$ , so  $A = \frac{2}{\pi}$ .

From  $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$ , we know  $0 = 0$ .

From  $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ , we know  $-B y e^{-z} = 2y e^{-z}$ , and thus  $B = -2$ . Therefore,

$$\boxed{\vec{\mathbf{F}} = \frac{2x}{\pi} \sin(\pi y) \hat{\mathbf{i}} + (x^2 \cos(\pi y) - 2y e^{-z}) \hat{\mathbf{j}} + y^2 e^{-z} \hat{\mathbf{k}}}$$

Now, since  $\vec{\mathbf{F}}$  is conservative, we can find an  $f$  such that  $\nabla f = \vec{\mathbf{F}}$ .

So, we have  $\frac{\partial f}{\partial x} = \frac{2x}{\pi} \sin(\pi y)$ .

$$f = \int \frac{2x}{\pi} \sin(\pi y) \, dx + g(y, z) = \frac{x^2}{\pi} \sin(\pi y) + g(y, z).$$

Hence,  $\frac{\partial f}{\partial y} = x^2 \cos(\pi y) + \frac{\partial g}{\partial y} = x^2 \cos(\pi y) - 2ye^{-z}$ .

$$\begin{aligned} \frac{\partial g}{\partial y} &= -2ye^{-z} \\ g(y, z) &= \int -2ye^{-z} \, dy + h(z) \\ g(y, z) &= -y^2 e^{-z} + h(z). \end{aligned}$$

So,

$$f = \frac{x^2}{\pi} \sin(\pi y) - y^2 e^{-z} + h(z)$$

So,  $\frac{\partial f}{\partial z} = -(-y^2 e^{-z}) + \frac{dh}{dz} = y^2 e^{-z}$ . Then, we would have  $\frac{dh}{dz} = 0$ , and thus  $h(z) = 0$ .

Therefore,

$$f = \frac{x^2}{\pi} \sin(\pi y) - y^2 e^{-z}$$

□

2. Using your answer in the previous question to evaluate  $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ , where  $C$  is

(a) The curve  $\vec{\mathbf{r}} = \cos(t)\hat{\mathbf{i}} + \sin(2t)\hat{\mathbf{j}} + \sin^2(t)\hat{\mathbf{k}}$ .

**Answer.**

Since we have  $\vec{\mathbf{r}}(0) = \langle 1, 0, 0 \rangle = \hat{\mathbf{i}}$  and  $\vec{\mathbf{r}}(2\pi) = \langle 1, 0, 0 \rangle = \hat{\mathbf{i}}$ , we know that  $\vec{\mathbf{r}}(t)$  is a closed curve. Therefore, by Theorem 5.3.4, since  $\vec{\mathbf{F}}$  is conservative, we have

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0.$$

□

(b) Curve of intersection of the paraboloid  $z = x^2 + 4y^2$  and the plane  $z = 3x - 2y$  from  $(0, 0, 0)$  to  $\left(1, \frac{1}{2}, 2\right)$

**Answer.**

By Theorem 5.3.1, the Fundamental Theorem of Line Integral, we know

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

So,

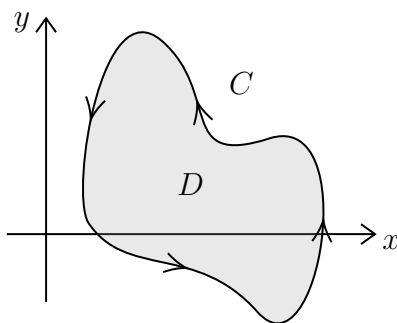
$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \left[ \frac{x^2}{\pi} \sin(\pi y) - y^2 e^{-z} \right]_{(0,0,0)}^{(1,1/2,2)} \\ &= \frac{1}{\pi} - \frac{1}{4e^2}.\end{aligned}$$

□

## 5.4 Green's Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve  $C$  and a double integral over the plane  $D$  bounded by  $C$ .

**Definition 5.4.1 (Simply Connected Regions).** Simply connected regions are regions that every simple closed curves in  $D$  enclosed only points that are in  $D$ .

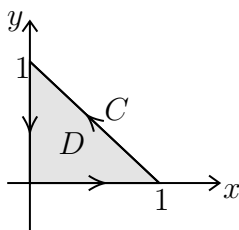


**Theorem 5.4.1 (Green's Theorem).** Let  $C$  be positively oriented piecewise-smooth simple closed curve in the plane, and let  $D$  be the region bounded by  $C$ . If  $P$  and  $Q$  have continuous partial derivatives on an open region that contains  $D$ , then

$$\oint_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

**Remark.** “Positively oriented” means the direction is counter-clockwise.

**Example 5.4.1.** Evaluate  $I = \oint_C x^4 dx + xy dy$ , where  $C$  is the following oriented triangle:



**Answer.**

By Green's Theorem, we have

$$I = \oint_C x^4 dx + xy dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Since  $P = x^4$  and  $Q = xy$ , we know  $\frac{\partial Q}{\partial x} = y$  and  $\frac{\partial P}{\partial y} = 0$ . Therefore,

$$\begin{aligned} I &= \iint_D (y - 0) dA = \int_0^1 \int_0^{1-x} y dy dx = \int_0^1 \left[ \frac{1}{2} y^2 \right]_0^{1-x} dx = \frac{1}{2} \left[ \frac{1}{3} (1-x)^3 \right]_0^1 \\ &= \frac{1}{6} ((1-1)^3 - (0-1)^3) = \frac{1}{6}. \end{aligned}$$

□

**Example 5.4.2.** Evaluate  $\oint_C (3y - e^{\sin x}) dx + (7 + \sqrt{y^4 + 1}) dy$  over  $C$  as  $x^2 + y^2 = 9$ .

**Answer.**

By Green's Theorem,

$$\begin{aligned} \oint_C (3y - e^{\sin x}) dx + (7 + \sqrt{y^4 + 1}) dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D (7 - 3) dA \\ &= 4 \iint_D dA \\ &= 4A(D) = 4(9\pi) = 36\pi. \end{aligned}$$

□

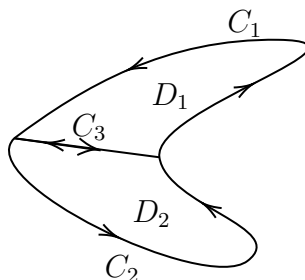
**Remark (A Special Case).** We can see that if  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ , we have

$$\oint_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D dA = A(D).$$

Also,

$$A(D) = \oint_C x dy = - \oint_C y dx = \frac{1}{2} \oint_C x dy - y dx.$$

**Theorem 5.4.2 (Extension of Green's Theorem 1).** We can extend Green's Theorem to finite union of simply connected regions:



$$\int_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

**Proof.**

Let  $I = \int_C Pdx + Qdy$ . Then,

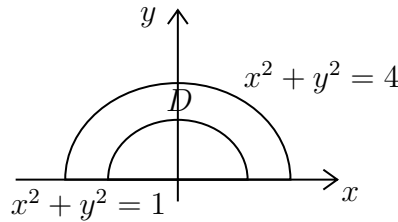
$$\begin{aligned} I &= \int_{C_1 \cup C_3} Pdx + Qdy + \int_{C_2 \cup (-C_3)} Pdx + Qdy \\ &= \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA + \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \\ &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA. \end{aligned}$$

■

**Theorem 5.4.3 (Extension of Green's Theorem 2).** Green's Theorem can be applied to regions with holes (regions that are not simply connected):

$$\int_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

**Example 5.4.3.** Evaluate  $\oint_C y^2 dx + 3xy dy$  along  $C$  as the following:



**Answer.**

Use the extension of the Green's Theorem:

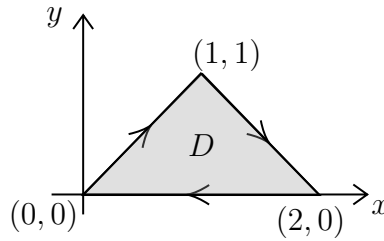
$$I = \oint_C y^2 dx + 3xy dy = \iint_D (3y - 2y) dA = \iint_D y dA.$$

Change to polar coordinates:  $1 \leq r \leq 2$ ,  $0 \leq \theta \leq \pi$ ,  $y = r \sin \theta$ .

$$\begin{aligned} I &= \int_0^\pi \int_1^2 r \sin \theta \cdot r dr d\theta = \int_0^\pi \sin \theta d\theta \int_1^2 r^2 dr \\ &= \left[ -\cos \theta \right]_0^\pi \left[ \frac{1}{3} r^3 \right]_1^2 \\ &= (-(-1) - (-1)) \left( \frac{8}{3} - \frac{1}{3} \right) \\ &= 2 \left( \frac{7}{3} \right) = \frac{14}{3}. \end{aligned}$$

□

**Example 5.4.4.** Evaluate  $\oint_C (x^2 - xy)dx + (xy - x^2)dy$ , where  $C$  is given by the following triangle.



**Answer.**

This question is left as an exercise so the steps are omitted, but the answer should be

$$I = -\frac{4}{3}.$$

□

## 5.5 Curl and Divergence

**Definition 5.5.1 (Divergence and Curl).** For a vector field  $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ , we define divergence and curl as

$$\begin{aligned}\operatorname{div} \vec{\mathbf{F}} &= \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\end{aligned}$$

$$\begin{aligned}\operatorname{curl} \vec{\mathbf{F}} &= \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{\mathbf{i}} - \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{\mathbf{j}} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{\mathbf{i}} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{\mathbf{j}} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}}\end{aligned}$$

**Example 5.5.1.** Find the divergence and curl of the vector field

$$\vec{\mathbf{F}} = xy\hat{\mathbf{i}} + (y^2 - z^2)\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

**Answer.**

$$\begin{aligned}\operatorname{div} \vec{\mathbf{F}} &= \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xy, (y^2 - z^2), yz \rangle \\ &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) \\ &= y + 2y + y = 4y.\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 - z^2 & yz \end{vmatrix} \\
&= \left( \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - z^2) \right) \hat{\mathbf{i}} + (0 - 0) \hat{\mathbf{j}} + \left( \frac{\partial}{\partial x}(y^2 - z^2) - \frac{\partial}{\partial y}(xy) \right) \hat{\mathbf{k}} \\
&= (z + 2z) \hat{\mathbf{i}} - 0 + (0 - x) \hat{\mathbf{k}} \\
&= 3z \hat{\mathbf{i}} - x \hat{\mathbf{k}}.
\end{aligned}$$

□

**Theorem 5.5.1 (Properties of Curl, Divergence, and Gradient).** Let  $f$  be a scalar field and  $\vec{\mathbf{F}}$  be a vector field. Suppose  $f$  and  $\vec{\mathbf{F}}$  are all smooth and have all partial derivatives continuous, then

$$1. \nabla \cdot (\nabla \times \vec{\mathbf{F}}) = 0 \text{ or in words, } \operatorname{div} (\operatorname{curl} \vec{\mathbf{F}}) = 0$$

**Proof.**

$$\begin{aligned}
\nabla \cdot (\nabla \times \vec{\mathbf{F}}) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \\
&= \frac{\partial}{\partial x} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= 0
\end{aligned}$$

■

$$2. \nabla \times (\nabla f) = 0 \text{ or in words, } \nabla \times (\operatorname{gradient} f) = 0$$

**Proof.**

$$\begin{aligned}
\nabla \times (\nabla f) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \hat{\mathbf{i}} - \left( \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \hat{\mathbf{j}} + \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \hat{\mathbf{k}} \\
&= 0
\end{aligned}$$

■

**Remark.** If  $\vec{\mathbf{F}}$  is conservative, then  $\vec{\mathbf{F}} = \nabla f$  and

$$\operatorname{curl} \vec{\mathbf{F}} = \operatorname{curl} (\nabla f) = 0.$$

**Theorem 5.5.2.** If  $\vec{\mathbf{F}}$  is a vector field on  $\mathbb{R}^3$  and its component functions,  $P$ ,  $Q$ , and  $R$ , have continuous partial derivatives and  $\operatorname{curl} \vec{\mathbf{F}} = 0$ , then  $\vec{\mathbf{F}}$  is conservative.

**Example 5.5.2.** Show that

$$\vec{\mathbf{F}}(x, y, z) = y^2 z^3 \hat{\mathbf{i}} + 2xyz^3 \hat{\mathbf{j}} + 3xy^2 z^2 \hat{\mathbf{k}}$$

is a conservative field and find a function  $f$  such that  $\vec{\mathbf{F}} = \nabla f$ .

**Answer.**

Note that

$$\operatorname{curl} \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} = 0$$

Also,  $y^2 z^3$ ,  $2xyz^3$ , and  $3xy^2 z^2$  are in  $\mathbb{R}^3$  and have continuous partial derivatives.

Therefore, by Theorem 5.5.2,  $\vec{\mathbf{F}}$  is conservative.

Now, we can find the  $f$  such that  $\nabla f = \vec{\mathbf{F}}$ .

So,

$$\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}} = y^2 z^3 \hat{\mathbf{i}} + 2xyz^3 \hat{\mathbf{j}} + 3xy^2 z^2 \hat{\mathbf{k}}$$

That is,

$$\frac{\partial f}{\partial x} = y^2 z^3; \quad \frac{\partial f}{\partial y} = 2xyz^3; \quad \frac{\partial f}{\partial z} = 3xy^2 z^2.$$

From  $\frac{\partial f}{\partial x} = y^2 z^3$ , we have  $f = xy^2 z^3 + g(y, z)$

So,

$$\frac{\partial f}{\partial y} = 2xyz^3 + \frac{\partial g}{\partial y} = 2xyz^3.$$

We have  $\frac{\partial g}{\partial y} = 0$ , which means  $g(y, z) = h(z)$ .

So,

$$\frac{\partial f}{\partial z} = 3xy^2 z^2 + \frac{dh}{dz} = 3xy^2 z^2$$

Similarly,  $\frac{dh}{dz} = 0$ , so  $h(z)$  is a constant function.

Hence,

$$\boxed{f = xy^2 z^3 + C}$$

□



**Definition 5.5.2 (Laplace Operator/Laplacian).** The Laplace operator (or laplacian) is denoted as  $\nabla \cdot \nabla$  or  $\nabla^2$  and is defined by

$$\nabla^2 = \left\langle \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \right\rangle$$

**Theorem 5.5.3 (More Properties).** Let  $f$  and  $g$  be scalar fields and  $\vec{\mathbf{F}}$  and  $\vec{\mathbf{G}}$  be vector fields. Define

$$\begin{aligned} (f\vec{\mathbf{F}})(x, y, z) &= f(x, y, z)\vec{\mathbf{F}}(x, y, z) \\ (\vec{\mathbf{F}} \cdot \vec{\mathbf{G}})(x, y, z) &= \vec{\mathbf{F}}(x, y, z) \cdot \vec{\mathbf{G}}(x, y, z) \\ (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) &= \vec{\mathbf{F}}(x, y, z) \times \vec{\mathbf{G}}(x, y, z) \end{aligned}$$

Suppose  $f, g, \vec{\mathbf{F}}$  and  $\vec{\mathbf{G}}$  are all smooth and have all partial derivatives continuous, then

1.  $\nabla \cdot (\vec{\mathbf{F}} + \vec{\mathbf{G}}) = \nabla \cdot \vec{\mathbf{F}} + \nabla \cdot \vec{\mathbf{G}}$
2.  $\nabla \times (\vec{\mathbf{F}} + \vec{\mathbf{G}}) = \nabla \times \vec{\mathbf{F}} + \nabla \times \vec{\mathbf{G}}$
3.  $\nabla \cdot (f\vec{\mathbf{F}}) = f\nabla \cdot \vec{\mathbf{F}} + \vec{\mathbf{F}} \cdot \nabla f$
4.  $\nabla \times (f\vec{\mathbf{F}}) = f\nabla \times \vec{\mathbf{F}} + (\nabla f) \times \vec{\mathbf{F}}$
5.  $\nabla \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = \vec{\mathbf{G}} \cdot \nabla \times \vec{\mathbf{F}} - \vec{\mathbf{F}} \cdot \nabla \times \vec{\mathbf{G}}$
6.  $\nabla \cdot (\nabla f \times \nabla g) = 0$
7.  $\boxed{\nabla \times (\nabla \times \vec{\mathbf{F}}) = \nabla(\nabla \cdot \vec{\mathbf{F}}) - \nabla^2 \vec{\mathbf{F}}}$

**Theorem 5.5.4 (Stoke's Theorem).** Let  $S$  be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve  $C$  with positive orientation. Let  $\vec{\mathbf{F}}$  be a vector field whose components have continuous partial derivatives on an open region in  $\mathbb{R}^3$  that contains  $S$ , then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \nabla \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$