Linear Algebra Done Right

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Contents

1 Vector Spaces

1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

Definition 1.1.1 (Complex Number). A *complex number* is an ordered pair (a, b), where $a, b \in \mathbb{R}$, but we write it as $a + b\mathrm{i}$.

Notation 1.1 $\mathbb{C} := \{a + b\mathrm{i} \mid a, b \in \mathbb{R}\}$

Definition 1.1.2 (Addition & Multiplication).

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)]\mathrm{i}$$

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}$$

Theorem 1.1.1 (Properties of Complex Arithmetic).

1. commutativity: $\alpha + \beta = \beta + \alpha; \quad \alpha\beta = \beta\alpha, \quad \forall \alpha, \beta \in \mathbb{C}$.
2. associativity: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda); \quad (\alpha\beta)\lambda = \alpha(\beta\lambda), \quad \forall \alpha, \beta, \lambda \in \mathbb{C}$.
3. identities: $\lambda + 0 = \lambda; \quad \lambda \cdot 1 = \lambda, \forall \lambda \in \mathbb{C}$.
4. additive inverse: $\forall \alpha \in \mathbb{C}, \exists$ unique $\beta \in \mathbb{C}$ *s.t.* $\alpha + \beta = 0$.
5. multiplicative inverse: $\forall \alpha \in \mathbb{C}, \alpha \neq 0, \exists$ unique $\beta \in \mathbb{C}$ *s.t.* $\alpha\beta = 1$.
6. distributivity: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta, \quad \forall \lambda, \alpha, \beta \in \mathbb{C}$.

Definition 1.1.3 (Subtraction). If $-\alpha$ is the additive inverse of α, *subtraction* on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

Definition 1.1.4 (Division). For $\alpha \neq 0$, let $\dfrac{1}{\alpha}$ denote the multiplicative inverse of α. Then, *division* on $\mathbb{C}$ is defined by

$$\frac{\beta}{\alpha} = \beta \cdot \left(\frac{1}{\alpha}\right)$$

Notation 1.2 $\mathbb{F}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$.

Definition 1.1.5 (List/Tuple). Suppose n is a non-negative integer. A list of length n is an ordered collection of n elements separated by commas and surrounded by parentheses: $(x_1, x_2, x_3, \cdots, x_n)$. Two lists are equal if and only if they have the same length and the same elements in the same order.

Remark *Lists must have a FINITE length.*

Definition 1.1.6 ($\mathbb{F}^n$ and Coordinate). $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n := \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R} \forall i = 1, \cdots, n\},$$

where x_i is the i^{th} *coordinate* of $(x_1, \cdots, x_n)$.

Example 1.1.1 $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ and $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.
Definition 1.1.7 (Addition on $\mathbb{F}^n$). *Addition* on $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n).$$

Theorem 1.1.2 (Commutativity of Addition on $\mathbb{F}^n$).
If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.
Proof 1. Suppose $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Then

$$\begin{aligned} x + y &= (x_1 + y_1, \cdots, x_n + y_n) \\ &= (y_1 + x_1, \cdots, y_n + x_n) = y + x. \end{aligned}$$

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Definition 1.1.8 (Zero). Let 0 denote the list of length n whose coordinates are all 0: $0 \coloneqq (0, \cdots, 0)$.
Definition 1.1.9 (Additive Inverse on $\mathbb{F}^n$). For $x \in \mathbb{F}^n$, the additive inverse of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ *s.t.* $x + (-x) = 0$.
Definition 1.1.10 (Scalar Multiplication in $\mathbb{F}^n$). The product of a number $\lambda \in \mathbb{F}$ and a vector $x \in \mathbb{F}^n$ is computed by multiplying each coordinate of the vector by λ :

$$\lambda x = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n),$$

where $x = (x_1, \cdots, x_n) \in \mathbb{F}^n$.
Theorem 1.1.3 (Properties of Arithmetic Operations on $\mathbb{F}^n$).

1. $(x + y) + z = x + (y + z) \quad \forall x, y, z \in \mathbb{F}^n$
2. $(ab)x = a(bx) \quad \forall x \in \mathbb{F}^n$ and $\forall a, b \in \mathbb{F}$.
3. $1 \cdot x = x \quad \forall x \in \mathbb{F}^n$ and $1 \in \mathbb{F}$.
4. $\lambda(x + y) = \lambda x + \lambda y \quad \forall \lambda \in \mathbb{R}$ and $\forall x, y \in \mathbb{F}^n$.
5. $(a + b)x = ax + bx \quad \forall a, b \in \mathbb{F}$ and $\forall x \in \mathbb{F}^n$.

1.2 Definition of Vector Space

Definition 1.2.1 (Addition on V). An *addition* on V is a function $(u, v) \mapsto u + v$ for all $u, v \in V$.
Definition 1.2.2 (Scalar Multiplication on V). A *scalar multiplication* on V is a function $(\lambda, v) \mapsto \lambda v$ for all $\lambda \in \mathbb{F}$ and $v \in V$.
Definition 1.2.3 (Vector Space). A *vector space* is a set V along with an addition on V and a scalar multiplication *s.t.* the following properties hold:

1. commutativity: $u + v = v + u \quad \forall u, v \in V$
2. associativity: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv) \quad \forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}$.
3. additive identity: $\exists 0 \in V$ *s.t.* $v + 0 = v \quad \forall v \in V$.
4. additive inverse: $\exists w \in V$ *s.t.* $v + w = 0 \quad \forall v \in V$.

5. multiplicative identity: $\exists 1 \in V \ s.t. \ 1 \cdot v = v \quad \forall v \in V$.
6. distributive properties: $a(u+v) = au + av$ and $(a+b)v = av + bv \quad \forall u, v \in V$ and $a, b \in \mathbb{F}$.

Definition 1.2.4 (Vector). Elements of a vector space are called *vectors* or points.

Notation 1.3 *V is a vector space over $\mathbb{F}$.*

Definition 1.2.5 (Real and Complex Vector Space). A vector space over $\mathbb{R}$ is called a *real vector space*, and a vector space over $\mathbb{C}$ is called a *complex vector space*.

Theorem 1.2.1 (Properties of Vector Spaces).

1. A vector space has a unique additive identity.

   Proof 1.

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2. A vector in a vector space has a unique additive inverse.

   Proof 2.

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1.3 Subspace

2 Finite-Dimensional Vector Spaces

2.1 Span and Linear Independence

2.2 Bases

2.3 Dimension