Emory University

MATH 211 - Advanced Calculus (Multivariable) Learning Notes

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Contents

Preface

These is my personal notes for Emory University MATH 211 Advanced Calculus (Multivariable Calculus) course.

After mastering Calculus I (which covers contents concerning limits, differentiation, and basic integration) and Calculus II (which includes integration techniques and series), this course focuses on multivariable calculus, including vectors, multivariable functions, partial derivatives, optimization, multiple integrations, vector and scalar fields, Green's and Stokes' theorems, and the divergence theorem. The book used for this course is *Multivariable Calculus, 8th Edition* by James Stewart.

Throughout this personal note, I use special "tcolorboxes" to differentiate different contents, including definitions, theorems, proofs, examples, extensions, and remarks. To be more specific:

Terminology

This is a **definition**.

Theorem Name

This is a **theorem**.

Example Number

This is the *question* part of an **example**.

This is the *answer* part of an **example**.

Remark

This is a **remark** of a definition, theorem, example, or proof.

Proof

This is a **proof** of a theorem.

Extension

This is a **extension** of a theorem, proof, or example.

To better ace this course, it is recommended to do more questions than provided as examples under each section. Although each example is distinctive and representative, more questions and practice is still needed to deepen the understanding of this course.

Even though I put efforts into making as few flaws as possible when encoding these learning notes, some errors may still exist in this note. If you find any, please contact me via email: `lvjiuru@hotmail.com`.

I hope you will find my notes helpful when learning Multivariable Calculus.

Jiuru Lyu

1 Vectors and Geometry of Space

1.1 Three Dimensional Coordinate System

Coordinate System

A **coordinate system** is a system that uses coordinate of a point to uniquely determine the position of the point in the space or plane.

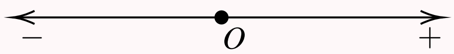
The Cartesian coordinate system is defined in different dimensions.

One Dimensional Cartesian System

One Dimensional Cartesian System is a straight line with a fixed point as the origin and positive and negative directions.

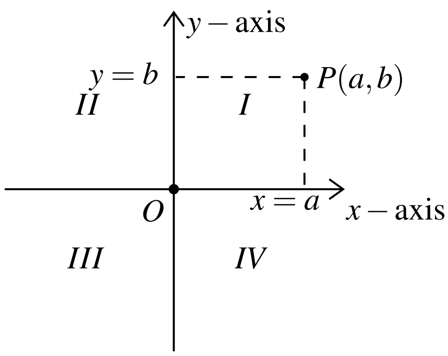
Remark

The one dimensional cartesian system is the number line:

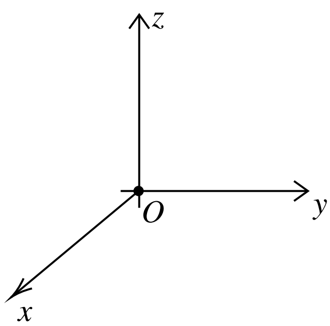


Any point in the one dimensional Cartesian system corresponds to a number $\in \mathbb{R}$ and any number $\in \mathbb{R}$ has a location on the line.

The two dimensional Cartesian system is the regular coordinate system.



The three dimensional Cartesian system includes three perpendicular axes.

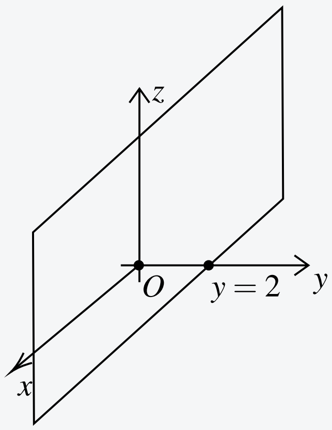


Octant

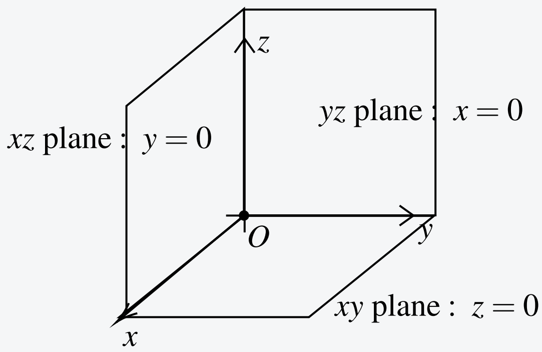
A **Octant** is one of the eight divisions of the three dimensional coordinate system.

Hyperplane

The hyperplane of $y = 2$ is given as below:

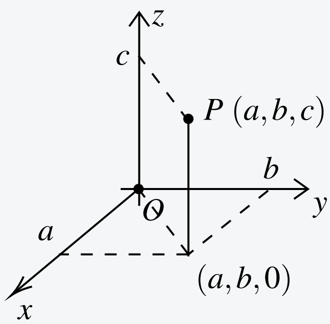


Specially:



Points in the Three Dimensional System

$P(a,b,c)$ indicates the intersection of the three hyperplanes: $x = a$, $y = b$, and $z = c$.



For spaces in the higher dimension, we understand them via the Cartesian product.

Cartesian Product

$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \cdots, x_n) | x_i \in \mathbb{R} \forall i = 1, \cdots, n\}$$

is the set of all n-tuples of real numbers and is denoted by $\mathbb{R}^n$.

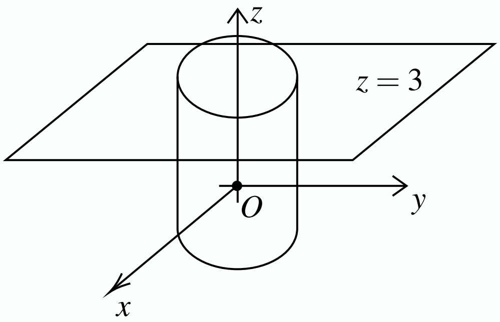
Example 1

$(3,4,5) \in \mathbb{R}^3$ is 3 dimensional.
$(3,4,5,6) \in \mathbb{R}^4$ is 4 dimensional.

Example 2

Which point(s) (x,y,z) satisfies the equations

$$x^2+y^2=1 \quad \text{and} \quad x=3?$$



Those points form a circle in the hyperplane of $z=3$ centered at the point $(0,0,3)$ with a radius of 1.

Distance Formula in Three Dimension

For given points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, the distance between them is denoted by $|P_1P_2|$ and is defined by

$$|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}.$$

Equation of a Sphere

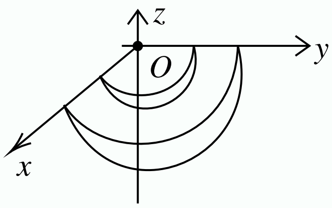
An equation of a sphere with a center of (a,b,c) and a radius of r is defined as

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

Example 3

What is the region in $\mathbb{R}^3$ represented by the inequalities

$$1 \leq x^2+y^2+z^2 \leq 4 \quad \text{and} \quad z \leq 0?$$

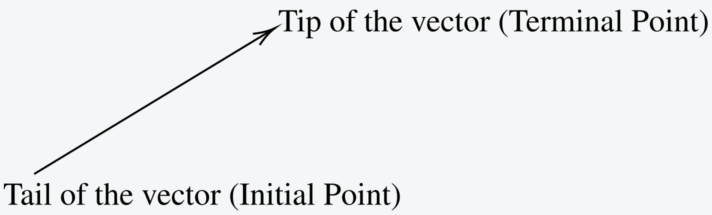


The region is the difference between the half spheres (the lower half of the sphere) centered at $(0,0,0)$ with a radius of 1 and 2.

1.2 Vectors

Vectors

Vectors are used to indicate a quantity that has both magnitude and direction.



1. Vectors are denoted as $\vec{v}$.
2. Magnitude

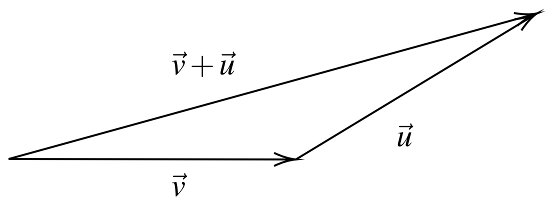
   Magnitude

   A vector is a line segment, of which the magnitude of vector denoted by $|\vec{v}|$ is the length of it and the arrow points the direction of the vector.

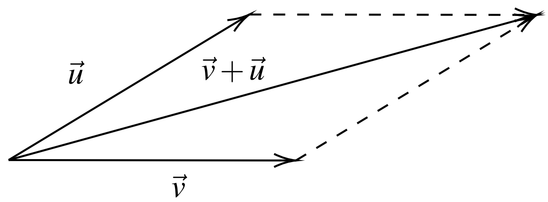
Vectors are operated in a different way.

1. Addition of Vectors:

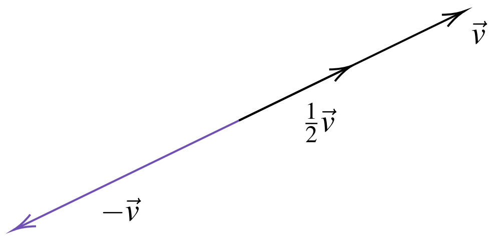
   (a) The triangle law:



   (b) The parallelogram law:



2. Scalar Multiplications:



Scalar Multiplication

If $c \in \mathbb{R}$ and $\vec{v}$ is a vector, then $c\vec{v}$ is in the same direction of $\vec{v}$ if $c > 0$ and in the opposite direction if $c < 0$.

Magnitude After Scalar Multiplication

The magnitude of $c\vec{v}$: $|c\vec{v}| = c|\vec{v}|$.