Emory University

MATH 211 - Advanced Calculus (Multivariable) Learning Notes

Jiuru Lyu

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Contents

Preface

These is my personal notes for Emory University MATH 211 Advanced Calculus (Multivariable Calculus) course.

After mastering Calculus I (which covers contents concerning limits, differentiation, and basic integration) and Calculus II (which includes integration techniques and series), this course focuses on multivariable calculus, including vectors, multivariable functions, partial derivatives, optimization, multiple integrations, vector and scalar fields, Green's and Stokes' theorems, and the divergence theorem. The book used for this course is *Multivariable Calculus, 8th Edition* by James Stewart.

Throughout this personal note, I use different formats to differentiate different contents, including definitions, theorems, proofs, examples, extensions, and remarks. To be more specific:

Definition 0.0.1 (Terminology). This is a **definition**.

Theorem 0.0.1 (Theorem Name). This is a **theorem**.

Example 0.0.1. This is an **example**.

Solution.
This is the *answer* part of an **example**.

□

Remark. This is a **remark** of a definition, theorem, example, or proof.

Proof.
This is a **proof** of a theorem.

■

Extension. This is a **extension** of a theorem, proof, or example.

To better ace this course, it is recommended to do more questions than provided as examples under each section. Although each example is distinctive and representative, more questions and practice is still needed to deepen the understanding of this course.

Even though I put efforts into making as few flaws as possible when encoding these learning notes, some errors may still exist in this note. If you find any, please contact me via email: `lvjiuru@hotmail.com`.

I hope you will find my notes helpful when learning Multivariable Calculus.

Cheers,
Jiuru Lyu

1 Vectors and Geometry of Space

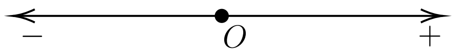
1.1 Three Dimensional Coordinate System

Definition 1.1.1 (Coordinate System). A **coordinate system** is a system that uses coordinate of a point to uniquely determine the position of the point in the space or plane.

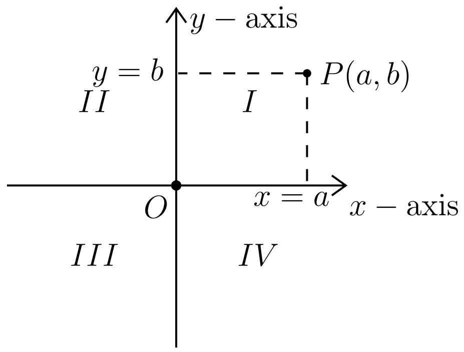
The Cartesian coordinate system is defined in different dimensions.

Definition 1.1.2 (One Dimensional Cartesian System). **One Dimensional Cartesian System** is a straight line with a fixed point as the origin and positive and negative directions.

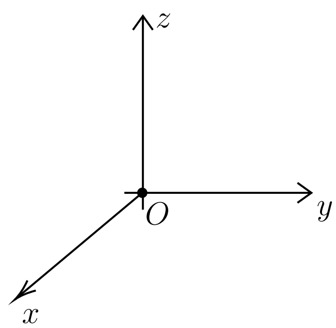
Remark. The one dimensional cartesian system is the number line:



Any point in the one dimensional Cartesian system corresponds to a number $\in \mathbb{R}$ and any number $\in \mathbb{R}$ has a location on the line. The two dimensional Cartesian system is the regular coordinate system.

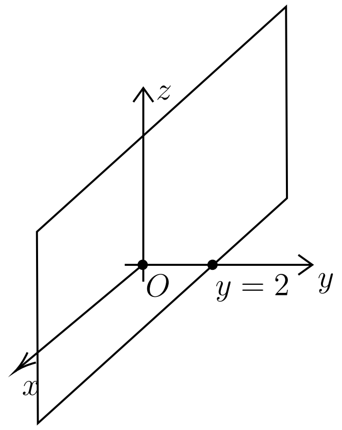


The three dimensional Cartesian system includes three perpendicular axes.

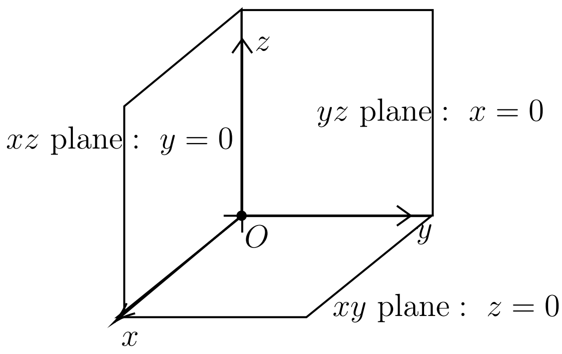


Definition 1.1.3 (Octant). A **Octant** is one of the eight divisions of the three dimensional coordinate system.

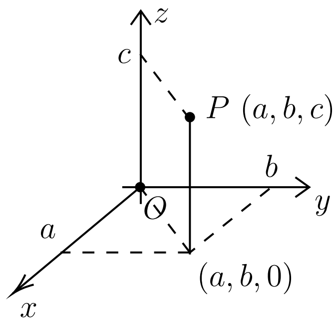
Definition 1.1.4 (Hyperplane). The hyperplane of $y = 2$ is given as below:



Specially:



Definition 1.1.5 (Points in the Three Dimensional System). $P(a, b, c)$ indicates the intersection of the three hyperplanes: $x = a$, $y = b$, and $z = c$.



For spaces in the higher dimension, we understand them via the Cartesian product.

Definition 1.1.6 (Cartesian Product).

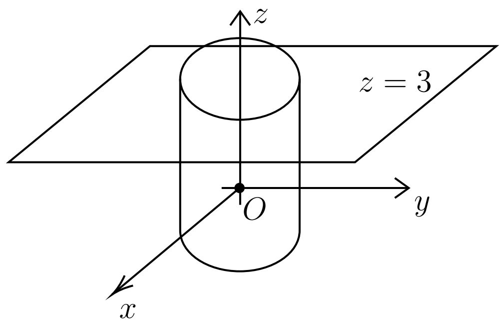
$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \cdots, x_n) | x_i \in \mathbb{R} \forall i = 1, \cdots, n\}$$

is the set of all n-tuples of real numbers and is denoted by $\mathbb{R}^n$.

Example 1.1.1. $(3, 4, 5) \in \mathbb{R}^3$ is 3 dimensional. $(3, 4, 5, 6) \in \mathbb{R}^4$ is 4 dimensional.

Example 1.1.2. Which point(s) (x, y, z) satisfies the equations

$$x^2 + y^2 = 1 \quad \text{and} \quad x = 3?$$



Those points form a circle in the hyperplane of $z = 3$ centered at the point $(0, 0, 3)$ with a radius of 1.

Theorem 1.1.1 (Distance Formula in Three Dimension). For given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the distance between them is denoted by $|P_1P_2|$ and is defined by

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

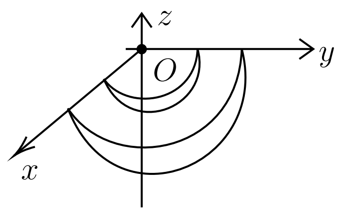
Theorem 1.1.2 (Equation of a Sphere). An equation of a sphere with a center of (a, b, c) and a radius of r is defined as

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Example 1.1.3. What is the region in $\mathbb{R}^3$ represented by the inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad \text{and} \quad z \leq 0?$$

Solution.



The region is the difference between the half spheres (the lower half of the sphere) centered at $(0, 0, 0)$ with a radius of 1 and 2.

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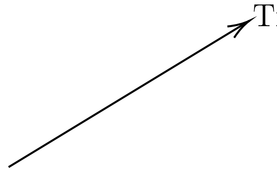
1.2 Vectors

Definition 1.2.1 (Vectors). **Vectors** are used to indicate a quantity that has both magnitude and direction.



1. Vectors are denoted as $\vec{v}$.

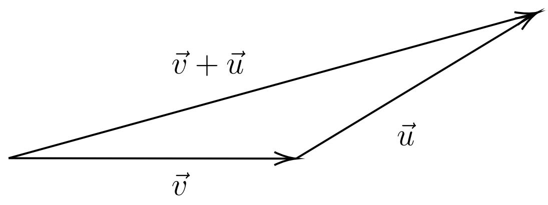
2. Magnitude

   Definition 1.2.2 (Magnitude). A vector is a line segment, of which the **magnitude** of vector denoted by $|\vec{v}|$ is the length of it and the arrow points the direction of the vector.

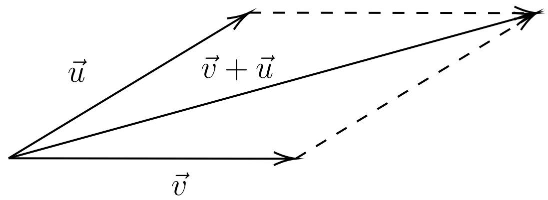
Vectors are operated in a different way:

1. Addition of Vectors:

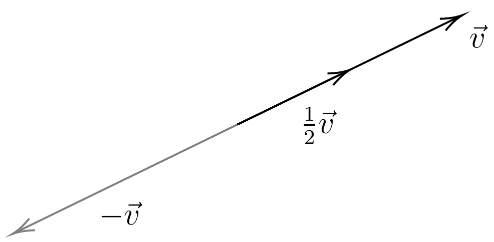
   (a) The triangle law:

   (b) The parallelogram law:

2. Scalar Multiplications:

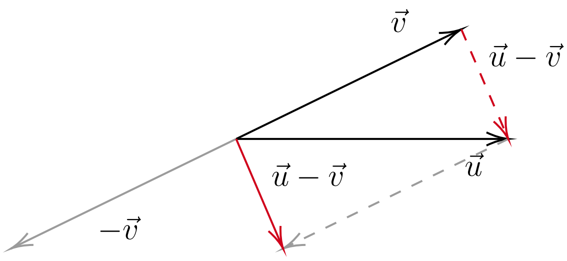
   

   Definition 1.2.3 (Scalar Multiplication). If $c \in \mathbb{R}$ and $\vec{v}$ is a vector, then $c\vec{v}$ is in the same direction of $\vec{v}$ if $c > 0$ and in the opposite direction if $c < 0$.

   Theorem 1.2.1. The magnitude of $c\vec{v}$:

   $$|c\vec{v}| = c|\vec{v}|.$$

3. Differences of Vectors:



The difference of vectors $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}-\vec{v}$ and is defined by

$$\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$$

4. Properties of vectors:
   Suppose $\vec{a},\ \vec{b},\ \vec{c}$ are vectors in V_n and c and d are scalars (*Those properties can be proven geometrically*):

   (a) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

   (b) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$

   (c) $\vec{a}+0=\vec{a}$

   (d) $\vec{a}+(-\vec{a})=0$

   (e) $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$

   (f) $(c+d)\vec{a}=c\vec{a}+d\vec{a}$

   (g) $(cd)\vec{a}=c(d\vec{a})$

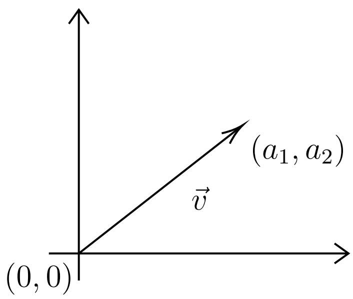
   (h) $1\cdot\vec{a}=\vec{a}$

We can link the coordinate system and vectors together:

1. **Definition 1.2.4 (Components of Vectors).** We will denote vector $\vec{v}$ as

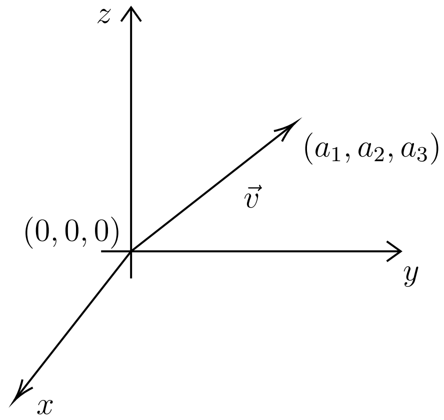
$$\vec{v}=\langle a_1,a_2\rangle,$$

   where a_1 and a_2 are called the **components** of $\vec{v}$.



2. In the three dimension:

$$\vec{v}=\langle a_1,a_2,a_3\rangle$$



3. **Definition 1.2.5.** If $A(x_1, y_1, z_1)$ as the tail of vector $\vec{v}$ and $B(x_1, y_1, z_1)$ as the tip of vector $\vec{v}$, then
$$\overrightarrow{AB} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$
$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

4. **Theorem 1.2.2.** If $\vec{v} = \langle a, b, c \rangle$ and $\vec{u} = \langle a', b', c' \rangle$, then
$$\vec{u} + \vec{v} = \langle a' + a,\ b' + b,\ c' + c \rangle$$
$$\vec{u} - \vec{v} = \langle a' - a,\ b' - b,\ c' - c \rangle$$
$$\alpha \vec{u} = \langle \alpha a',\ \alpha b',\ \alpha c' \rangle, \text{ where } \alpha \text{ is a scalar.}$$

Definition 1.2.6 (Standard Basis Vectors). In 2-D, $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$; and in 3-D, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$ are called the **standard basis vectors**.

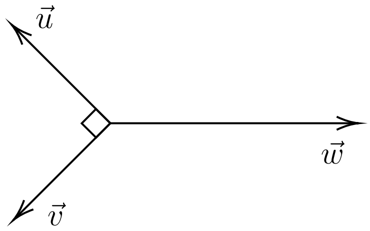
Remark. Any vectors in 2D and 3D can be written as
$$\vec{v} = \langle a, b, c \rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}.$$

Definition 1.2.7 (Unit Vector). A **unit vector**is a vector of magnitude of 1.

Example 1.2.1.
$$|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1 \text{ are unit vectors.}$$

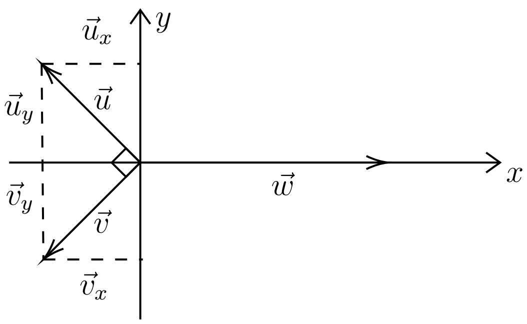
Theorem 1.2.3. To find a unit vector in the direction of any vector $\vec{v}$, we use $\dfrac{1}{|\vec{v}|}\vec{v}$. The length of vector $\dfrac{\vec{v}}{|\vec{v}|}$ is 1 and its direction is the same as $\vec{v}$.

Example 1.2.2. If the vectors in the figure satisfy $|\vec{u}| = |\vec{v}| = 1$, and $\vec{u} + \vec{v} + \vec{w} = 0$, find $|\vec{w}|$.



Solution.

Decompose the vectors:



We then have

$$\cos 45^\circ = \frac{|\vec{u}_x|}{\vec{u}} \implies |\vec{u}_x| = |\vec{u}|\cos 45^\circ;$$

$$\sin 45^\circ = \frac{|\vec{u}_y|}{\vec{u}} \implies |\vec{u}_y| = |\vec{u}|\sin 45^\circ;$$

$$\begin{aligned}\therefore \vec{u} = \langle |\vec{u}_x|,\ |\vec{u}_y\rangle &= -|\vec{u}_x|\hat{\mathbf{i}} + |\vec{u}_y|\hat{\mathbf{j}} \\ &= -\frac{\sqrt{2}}{2}|\vec{u}|\hat{\mathbf{i}} + \frac{\sqrt{2}}{2}\hat{\mathbf{j}} \\ &= \frac{\sqrt{2}}{2}|\vec{u}|(-\hat{\mathbf{i}} + \hat{\mathbf{j}})\end{aligned}$$

Similarly,

$$\vec{v} = \frac{\sqrt{2}}{2}|\vec{v}|(-\hat{\mathbf{i}} - \hat{\mathbf{j}}).$$

We know $\vec{u} + \vec{v} + \vec{w} = 0$:

$$\therefore \vec{w} + \frac{\sqrt{2}}{2}|\vec{u}|(-\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \frac{\sqrt{2}}{2}|\vec{v}|(-\hat{\mathbf{i}} - \hat{\mathbf{j}}) = 0$$

We know $|\vec{u}| = |\vec{v}| = 1$:

$$\begin{aligned}\therefore \vec{w} + \frac{\sqrt{2}}{2}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \frac{\sqrt{2}}{2}(-\hat{\mathbf{i}} - \hat{\mathbf{j}}) &= 0 \\ \vec{w} + \frac{\sqrt{2}}{2}(-\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{i}} - \hat{\mathbf{j}}) &= 0 \\ \vec{w} &= \sqrt{2}\hat{\mathbf{i}}\end{aligned}$$

$$\therefore \vec{w} = \langle \sqrt{2}, 0\rangle \implies |\vec{w}| = \sqrt{2}.$$

□

1.3 Dot Product

Definition 1.3.1 (Dot Product). If $\vec{u} = \langle x_1, y_1, z_1 \rangle$ and $\vec{v} = \langle x_2, y_2, z_2 \rangle$, then the dot product of $\vec{u}$ and $\vec{v}$ is defined as

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle \\ &= x_1 x_2 + y_1 y_1 + z_1 z_2 \end{aligned}$$

Remark. The dot product of two vectors returns a scalar.

Example 1.3.1. Let $\vec{u} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\vec{v} = 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$. Find $\vec{u} \cdot \vec{v}$.

Solution.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \langle 1, 2, -3 \rangle \cdot \langle 0, 2, -1 \rangle \\ &= (1)(0) + (2)(2) + (-3)(-1) = 7. \end{aligned}$$

□

Properties of the dot product:

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot (\vec{v} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}$
3. $m(\vec{a} \cdot \vec{b}) = (m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b}) = (\vec{a} \cdot \vec{b})m$
4. $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$
   $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$

Theorem 1.3.1.

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2.$$

Theorem 1.3.2. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta}.$$

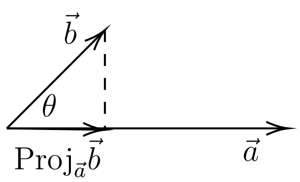
Extension.

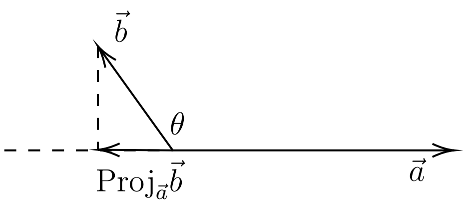
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

Extension.

$$\theta = 90^\circ \iff \vec{u} \cdot \vec{v} = 0.$$

Definition 1.3.2 (Projections). We use $\text{Proj}_{\vec{a}}\vec{b}$ to denote the **projection** of $\vec{b}$ on $\vec{a}$.





From the diagrams,

$$\cos\theta = \frac{|\text{Proj}_{\vec{a}}\vec{b}|}{|\vec{b}|} \implies |\text{Proj}_{\vec{a}}\vec{b} = \boxed{|\vec{b}|\cos\theta}.$$

We know that

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\therefore \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \boxed{|\vec{b}|\cos\theta}$$

$$\therefore |\text{Proj}_{\vec{a}}\vec{b}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}, \text{ which is a scalar.}$$

$|\text{Proj}_{\vec{a}}\vec{b}|$ is called the **scalar projection** of $\vec{b}$ on $\vec{a}$.

$$\text{Proj}_{\vec{a}}\vec{b} = |\text{Proj}_{\vec{a}}\vec{b}| \cdot \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}$$

$\text{Proj}_{\vec{a}}\vec{b}$ is called **projection** of $\vec{b}$ on $\vec{a}$ and is a vector.

Example 1.3.2. Find the scalar projection and vector projection of vector $\vec{u} = \langle 1, 1, 2\rangle$ onto $\vec{v} = \langle -2, 3, 1\rangle$.

Solution.

$$\text{Proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v}\ ; \quad |\text{Proj}_{\vec{v}}\vec{u}| = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

We need $|\vec{v}| = \sqrt{4+9+1} = \sqrt{14}$ and $\vec{u}\cdot\vec{v} = (1)(-2)+(1)(3)+(2)(1) = 3$

$$\therefore |\text{Proj}_{\vec{v}}\vec{u}| = \frac{3}{\sqrt{14}}$$

$$\text{Proj}_{\vec{v}}\vec{u} = \frac{3}{14}\cdot\vec{v} = \frac{3}{14}\cdot\langle -2,3,1\rangle = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle.$$

□

1.4 Cross Product

Definition 1.4.1 (Cross Product). The **cross product** of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}\times\vec{v}$ and is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$. If $\vec{u} = \langle x_1, y_1, z_1\rangle$ and $\vec{v} = \langle x_2, y_2, z_2\rangle$, then

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix} = y_1z_2\hat{\mathbf{i}} + x_2z_1\hat{\mathbf{j}} + x_1y_2\hat{\mathbf{k}} - x_2y_1\hat{\mathbf{k}} - y_2z_1\hat{\mathbf{i}} - x_1z_2\hat{\mathbf{j}}\\ &= (y_1z_2 - y_2z_1)\hat{\mathbf{i}} + (z_1x_2 - z_2x_1)\hat{\mathbf{j}} + (x_1y_2 - x_2y_1)\hat{\mathbf{k}}\end{aligned}$$

Example 1.4.1. Prove $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

Proof.

$$\begin{aligned}\vec{u}\cdot(\vec{u}\times\vec{v}) &= \langle x_1, y_1, z_1\rangle \cdot \langle y_1z_2 - y_2z_1,\ z_1x_2 - z_2x_1,\ x_1y_2 - x_2y_1\rangle \\ &= x_1y_1z_2 - x_zy_2z_1 + x_2y_1z_1 - x_1y_1z_2 + x_1y_2z_1 - x_2y_1z_1 = 0 \\ &\therefore \vec{u}\times\vec{v} \perp \vec{u}\end{aligned}$$

Similarly, $\vec{v}\cdot(\vec{u}\times\vec{v}) = 0 \Longrightarrow \vec{u}\times\vec{v}\perp\vec{v}$.

■

Theorem 1.4.1. If θ is the angle between vectors $\vec{u}$ and $\vec{v}$, then

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta.$$

Proof.

$$\begin{aligned}|\vec{u}\times\vec{v}|^2 &= (y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2 \\ &= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2 \\ &= |\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2 \\ &= |\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2\cos^2\theta \\ &= |\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta) \\ &= |\vec{u}|^2|\vec{v}|^2\sin^2\theta \\ &\therefore |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta.\end{aligned}$$

■

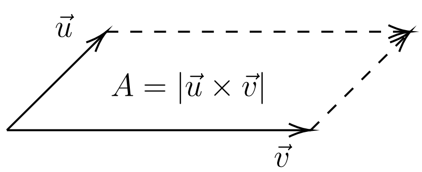
Definition 1.4.2 (Parallel). If two vectors, $\vec{u}$ and $\vec{v}$, are parallel to each other,

$$\vec{u} = c\vec{v},$$

where c is a scalar.

Theorem 1.4.2. For two vectors $\vec{u}$ and $\vec{v}$, $\vec{u}\times\vec{v} = 0$ *iff* $\vec{u}$ and $\vec{v}$ are parallel to each other.

Theorem 1.4.3. The length of the cross product, $|\vec{u}\times\vec{v}|$, is the area of the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$.



Theorem 1.4.4.

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}};\quad \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}};\quad \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}}\times\hat{\mathbf{i}} = -\hat{\mathbf{k}};\quad \hat{\mathbf{k}}\times\hat{\mathbf{j}} = -\hat{\mathbf{i}};\quad \hat{\mathbf{i}}\times\hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

Properties of cross product ($\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors, and c is a scalar):

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Definition 1.4.3 (Triple Product). The **scalar triple product** is defined by

$$\vec{a} \cdot (\vec{b} \times \vec{c}).$$

Theorem 1.4.5. $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ denotes the volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Proof.

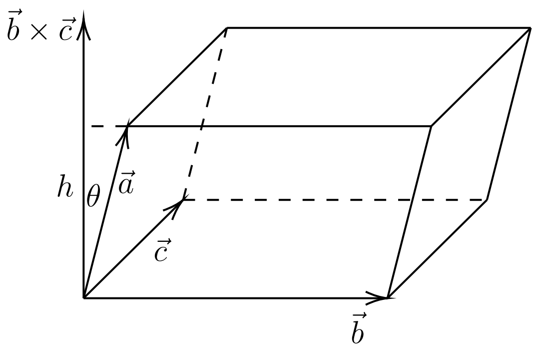
The area of the base is given by

$$A = |\vec{b} \times \vec{c}|$$

To find the volume, we need to know the height h:

$$h = |\vec{a}||\cos\theta|$$

$$\therefore V = Ah = |\vec{b} \times \vec{c}||\vec{a}||\cos\theta| = \vec{a} \cdot (\vec{b} \times \vec{c}) \qquad \left[\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta\right]$$

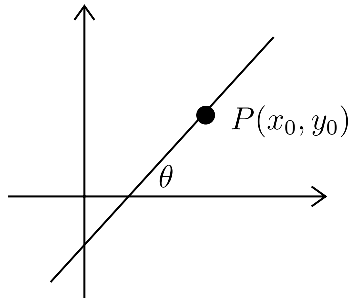


■

1.5 Equations of Lines and Planes

Theorem 1.5.1 (Equation of Lines in 2D). If we have a point $P(x_0, y_0)$ and a direction (slope/θ/another point on the line), we have the equation of the line:

$$\text{Given } \begin{cases} \text{slope} = m \\ P(x_0, y_0) \end{cases} \implies \text{The equation of the line: } y - y_0 = m(x - x_0).$$

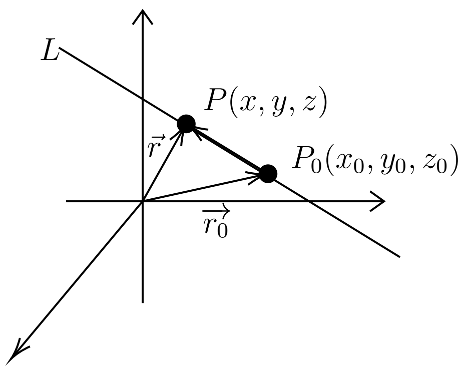


Definition 1.5.1 (Directional Vector). If $\vec{v}$ is a directional vector of line L,

$$\vec{a} = t\vec{v},$$

where $\vec{a}$ is any vector determined by two points on the line.

Definition 1.5.2 (Vector Equations of Lines in 3D). Let $\overrightarrow{P_0P} = \vec{a} \implies \vec{a} = \langle x - x_0,\ y - y_0\, z - z_0 \rangle$



From the diagram, we also have

$$\vec{\mathbf{r}}_0 + \vec{a} = \vec{\mathbf{r}}.$$

As $\vec{a} = t\vec{v}$,

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{v},$$

which is the **vector equation** of line L.

Theorem 1.5.2. If L is a line with point $P(x_0, y_0, z_0)$ on it and paralleled to a direction vector $\vec{v} = \langle a, b, c \rangle$, we have

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle,$$

where t is a parameter and the equation is called the **vector equation** of line L.

Extension (Parametric Equation of *L*). From $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$, we have

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

This system of equations is called the **parametric equation** of L.

Extension (Symmetric Equation of L). From the parametric equation of L, we can derive t:

$$\begin{cases} x = x_0 + ta \quad \implies \quad t = \frac{x-x_0}{a} \\ y = y_0 + tb \quad \implies \quad t = \frac{y-y_0}{b} \\ z = z_0 + tc \quad \implies \quad t = \frac{z-z_0}{c} \end{cases}$$

As t should be equal:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c},$$

which is called the **symmetric equation** of the line with point $P(x_0, y_0, z_0)$ and a directional vector $\vec{v} = \langle a, b, c \rangle$.

Remark (Three Forms of Equation of a Line). For line L in 3D, $P_0(x_0, y_0, z_0)$ is on L and $\vec{v} = \langle a, b, c \rangle$ is a directional vector of L.

1. The vector form:

$$\langle x, y, z \rangle = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$$

2. The parametric form:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

3. The symmetric form:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Example 1.5.1. Find the parametric and symmetric equations of the line L passing through the points $(-8, 1, 4)$ and $(3, -2, 4)$.

Solution.

Let's set P_0 to be $(-8, 1, 4)$ and P_1 to be $(3, -2, 4)$. So we can find the directional vector

$$\vec{v} = \overrightarrow{P_0P_1} = \langle 3 - (-8),\ -2 - 1,\ 4 - 4 \rangle = \langle 11, -3, 0 \rangle.$$

$\therefore$ The parametric equation of L:

$$\begin{cases} x = -8 + 11t \\ y = 1 - 3t \\ z = 4 + (0)t \end{cases},$$

and the symmetric equation of L is

$$\frac{x+8}{11} = \frac{y-1}{-3}, \quad z = 4.$$

$\square$

Relationships of two lines in 3D:

1. Parallel: directional vectors of the two lines are parallel to each other.
2. Intersect: the two lines share one common point
3. Skewed: the two lines are neither parallel nor intersecting.

Example 1.5.2. Let

$$L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} \quad \text{and} \quad L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}.$$

Find the relationship between L_1 and L_2.

Solution.

$$\vec{v}_1 = \langle 1, -2, -3 \rangle; \quad \vec{v}_2 = \langle 1, 3, -7 \rangle$$

Because $\vec{v}_1$ and $\vec{v}_2$ are not parallel to each other, L_1 and L_2 are not parallel to each other.
$\therefore$ L_1 and L_2 can only be intersecting or skewed.
To further discuss the relationship between L_1 and L_2, form parametric equations:

$$L_1 : \begin{cases} x = 2 + t \\ y = 3 - 2t \\ z = 1 - 3t \end{cases} \qquad L_2 : \begin{cases} x = 3 + s \\ y = -4 + 3s \\ z = 2 - 7s \end{cases}$$

If we can find a set of solutions t and s that satisfy the following system of equations, the two lines have point in common and thus is intersecting:

$$\begin{cases} 2 + t = 3 + s \\ 3 - 2t = -4 + 3s \\ 1 - 3t = 2 - 7s \end{cases} \implies \begin{cases} t - s = 1 & \text{①} \\ 2t + 3s = 7 & \text{②} \\ 3t - 7s = -1 & \text{③} \end{cases}$$

From ①:

$$t = s + 1 \qquad \text{④}$$

Substitute ② with ④:

$$\begin{aligned} &2(s+1) + 3s = 7 \\ &\quad 2s + 2 + 3s = 7 \quad \Rightarrow \quad 4s = 5 \quad \Rightarrow \quad s = 1 \\ &\quad \therefore t = s + 1 = 1 + 1 = 2 \end{aligned}$$

Substitute $s = 1$ and $t = 2$ to ③:

$$\text{LHS} = 2(3) - 7(1) = 6 - 7 = -1 = \text{RHS}.$$

Hence, $\begin{cases} t = 2 \\ s = 1 \end{cases}$ satisfy all three equations. Substitute $t = 2$ to L_1:

$$x = 2 + 2 = 4, \quad y = 3 - 2(2) = -1, \quad z = 1 - 3(2) = -5.$$

$\therefore$ The two lines intersect at $(4, -1, -5)$.

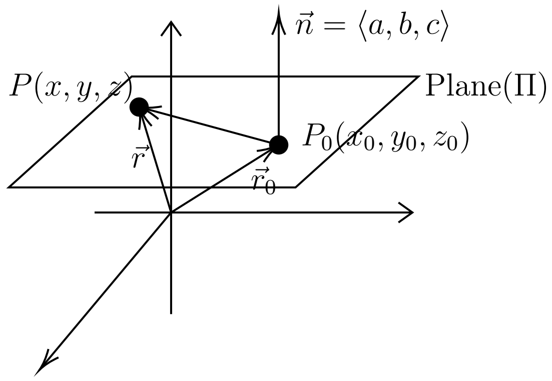
$\square$

Definition 1.5.3 (Normal Vector). A normal vector is the vector perpendicular to the plane and is often denoted as $\vec{n}$.

Theorem 1.5.3 (Vector Equation of a Plane). As $\vec{n} \perp \Pi$, $\vec{n} \perp \overrightarrow{P_0P}$

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{\mathbf{r}} - \vec{\mathbf{r}}_0 \\ \therefore \vec{n} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) &= 0 \\ \vec{n} \cdot \vec{\mathbf{r}} - \vec{n} \cdot \vec{\mathbf{r}}_0 = 0 \ &\Rightarrow \ \vec{n} \cdot \vec{\mathbf{r}} = \vec{n} \cdot \vec{\mathbf{r}}_0, \end{aligned}$$

which is called the **vector equation** of a plane.



Extension (Scalar Equation of a Plane). From $\vec{n} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$: As $\vec{n} = \langle a, b, c \rangle$ and $\vec{\mathbf{r}} - \vec{\mathbf{r}}_0 = \langle x - x_0,\ y - y_0,\ z - z_0 \rangle$, we have

$$\langle a, b, c \rangle \cdot \langle x - x_0,\ y - y_0,\ z - z_0 \rangle = 0;$$

$$\therefore a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which is the **scalar equation** of plane Π with point $P_0(x_0, y_0, z_0)$ on it and a normal vector $\vec{n} = \langle a, b, c \rangle$.

Extension (Linear Equation of a Plane). From $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$:

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

Take $d = -(ax_0 + by_0 + cz_0)$:

$$ax + by + cz + d = 0,$$

which is called the **linear equation** of plane Π with point $P_0(x_0, y_0, z_0)$ on it and a normal vector $\vec{n} = \langle a, b, c\rangle$.

Remark (Equations of a Plane). If point $P_0(x_0, y_0, z_0)$ is on the plane Π and a normal vector of Π is $\vec{n} = \langle a, b, c\rangle$:

1. The vector equation:
$$\vec{n} \cdot \vec{\mathbf{r}} = \vec{n} \cdot \vec{\mathbf{r}}_0$$

2. The scalar equation:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

3. The linear equation:
$$ax + by + cz + d = 0,$$
where $d = -(ax_0 + by_0 + cz_0) = -\langle a, b, c\rangle \cdot \langle x_0, y_0, z_0\rangle$

Example 1.5.3. Find an equation of the plane crossing through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Solution.

Find the normal vector using the following equation:

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

$$\overrightarrow{PQ} = \langle 3 - 1, -1 - 3, 6 - 2\rangle = \langle 2, -4, 4\rangle$$
$$\overrightarrow{PR} = \langle 5 - 1, 2 - 3, 0 - 2\rangle = \langle 4, -1, -2\rangle$$

$$\therefore \vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 14\hat{\mathbf{k}}.$$

$$\therefore \vec{n} = \langle 12, 20, 14\rangle, \qquad P(1, 3, 2)$$

$$\therefore d = -\langle 12, 20, 14\rangle \cdot \langle 1, 3, 2\rangle = -(12 + 60 + 28) = -100.$$

$$\therefore \text{Linear Equation of } \Pi: \; 12x + 20y + 14z - 100 = 0 \quad \Longrightarrow \quad 6x + 10y + 7z - 50 = 0.$$

□

Theorem 1.5.4 (Relationship Between Two Planes). If $\vec{n}_1$ is a normal vector of plane Π_1, and $\vec{n}_2$ is a normal vector of plane Π_2 , then the angle between the two planes is given by

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right).$$

i.e., the angle between the planes is the angle between the normal vectors.

Theorem 1.5.5 (Distance from a Point to a Plane). Distance of the point $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$:

$$D = \frac{|ax_a + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{1}$$

OR

$$D = \frac{\vec{b} \cdot \vec{n}}{|\vec{n}|}, \tag{2}$$

where $\vec{n}$ is the normal vector.

Example 1.5.4. Find the distance between the parallel planes:

$$\Pi_1 : \; 10x + 2y - 2z = 5 \quad \text{and} \quad \Pi_2 : \; 5x + y - z = 1.$$

Solution.
Assume point $P(x_1, y_!, z_1)$ is on plane Π_1:

$$\begin{aligned} 10x_1 + 2y_1 - 2z_1 &= 5 \\ \therefore \; 5x_1 + y_1 - z_1 &= \frac{5}{2} \end{aligned}$$

Applying formula 1: $\vec{n} = \langle a, b, c\rangle = \langle 5, 1, -1 rlangle, \; d = -1$:

$$\therefore \; D = \frac{|5x_1 + y_1 - z_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|\frac{5}{2} - 1|}{\sqrt{26 + 1 + 1}} = \frac{3/2}{\sqrt{27}} = \frac{3}{2\sqrt{27}} \left(= \frac{\sqrt{3}}{6} \right).$$

□

Extension. Find the distance between two parallel planes:

$$\Pi_1 : \; ax + by + cz + d = 0 \quad \text{and} \quad \Pi_2 : \; ax + by + cz + d' = 0.$$

Let point $P(x_1, y_1, z_1)$ on Π_1:

$$ax_1 + by_1 + cz_1 + d = 0$$

Apply formula 1:

$$D = \frac{|ax_1 + by_1 + cz_1 + d'|}{\sqrt{a^2 + b^2 + c^2}} = \frac{-d + d'}{\sqrt{a^2 + b^2 + c^2}}.$$

1.6 Cylinders and Quadric Surfaces

Definition 1.6.1 (Cylinders). A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

Definition 1.6.2 (Quadric Surfaces). A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gz + Hy + Iz + J = 0,$$

where $A, B, C, \cdots, J$ are constants, but by translation and rotation it can be brought into one of the standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

Remark. Graphs of Quadric Surfaces (Refer to Page 877 of the Book):

1. Ellipsoid:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
All traces are ellipses.
If $a = b = c$, the ellipsoid is a sphere.

2. Elliptic Paraboloid:
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$
Horizontal traces are ellipses. Vertical traces are parabolas.
The variable raised to the first power indicates the axis of the paraboloid.

3. Hyperbolic Paraboloid:
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$
Horizontal traces are hyperbolas. Vertical traces are parabolas.

4. Cone:
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$
Horizontal traces are ellipses.
Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

5. Hyperboloid of One Sheet:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
Horizontal traces are ellipses. Vertical traces are hyperbolas.
The axis of symmetry corresponds to the variable whose coefficient is negative.

6. Hyperboloid of Two Sheets:
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas.
The two minus sign indicate two sheets.

2 Vector Functions

2.1 Vector Functions and Space Curves

Definition 2.1.1 (Component Functions). $f(t),\ g(t),\ h(t)$ are real valued function and are called **component functions** of $\vec{\mathbf{r}}(t)$. We write

$$\vec{\mathbf{r}}(t) = \langle f(t),\ g(t),\ h(t)\rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}.$$

Definition 2.1.2 (Limit of Vector Functions). To find the limit of a vector function, we check its component functions. That is

$$\lim_{t\to a}\vec{\mathbf{r}}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t)\right\rangle$$

Definition 2.1.3 (Continuity of Vector Functions). A vector function $\vec{\mathbf{r}}(t)$ is continuous if

$$\lim_{t\to a}\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(a).$$

Example 2.1.1. 1. Find the domain of

$$\vec{\mathbf{r}}(t) = \left\langle \ln(t+1),\ \frac{t}{\sqrt{9-t^2}},\ 2^t\right\rangle$$

Solution.

- Domain of $\ln(t+1)$: $D_1 L t+1>0,\ t>-1$
- Domain of $\dfrac{t}{\sqrt{9-t^2}}$: $D_2:\ 9-t^2>0,\ -3<t<3$
- Domain of 2^t : $D_3:\ \mathbb{R}$

  Find the intersection of domains of component functions:

$$D_1\cap D_2\cap D_3:\ -1<t<3\ (t\in(-1,3))$$

□

2. Find $\lim_{t\to 0}\vec{\mathbf{r}}(t)$.

Solution.

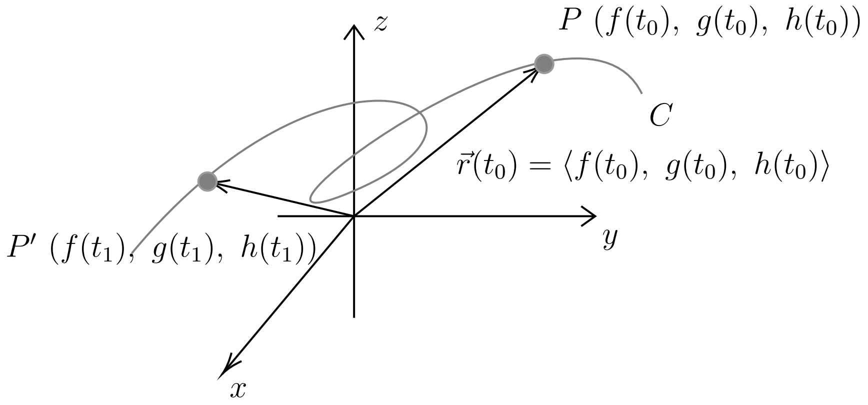
$$\begin{aligned}\lim_{t\to 0}\vec{\mathbf{r}}(t) &= \left\langle \lim_{t\to 0}\ln(t+1),\ \lim_{t\to 0}\frac{t}{\sqrt{9-t^2}},\ \lim_{t\to 0}2^t\right\rangle\\ &= \left\langle \ln(1),\ \frac{0}{\sqrt{9}},\ 2^0\right\rangle\\ &= \langle 0,0,1\rangle = \hat{\mathbf{k}}\end{aligned}$$

□

Example 2.1.2.

$$\begin{aligned}
&\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\hat{\mathbf{i}}+\sin\pi t\hat{\mathbf{j}}+\cos 2\pi t\hat{\mathbf{k}}\right)\\
=&\lim_{t\to 1}\left(\frac{t(t-1)}{t-1}\hat{\mathbf{i}}+\sin\pi t\hat{\mathbf{j}}+\cos 2\pi t\hat{\mathbf{k}}\right)\\
=&\lim_{t\to 1} t\hat{\mathbf{i}}+\lim_{t\to 1}\sin\pi t\hat{\mathbf{j}}+\lim_{t\to 1}\cos 2\pi t\hat{\mathbf{k}}\\
=&\hat{\mathbf{i}}+\sin\pi\hat{\mathbf{j}}+\cos 2\pi\hat{\mathbf{k}}\\
=&\hat{\mathbf{i}}+\hat{\mathbf{k}}
\end{aligned}$$

Definition 2.1.4 (Graphs of Vector Functions). For a vector function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$, the graph of it, curve C, is defined by the moving tip of the vectors yielded from the vector function.



Definition 2.1.5 (Space Curve). If f, g, h, are continuous real-valued functions on an interval I, then the set C of all points (x,y,z) in space s.t.

$$x=f(t)\quad y=g(t)\quad z=h(t),\quad \text{where} t\in I$$

is called a **space curve**.

Definition 2.1.6 (Parametric Equation). The system of equations $\begin{cases}x=f(t)\\ y=g(y)\\ z=h(t)\end{cases}$ is called a **parametric equation** of C and t is called the **parameter**.

2.2 Derivative and Intergral of Vector Functions

Limits, continuity, derivative, and integrals of vector functions follow rules similar to those of scalar functions.

Definition 2.2.1 (Derivative of Vector Functions).

$$\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}=\lim_{h\to 0}=\frac{\vec{\mathbf{r}}(t+h)-\vec{\mathbf{r}}(t)}{h},$$

$\dfrac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}$ or $\vec{\mathbf{r}}'(t)$ is the derivative of $\vec{\mathbf{r}}(t)$ is the limit on the right hand side exists.

Extension. If $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, then

$$\vec{\mathbf{r}}'(t) = f'(t)\hat{\mathbf{i}} + g'(t)\hat{\mathbf{j}} + h'(t)\hat{\mathbf{k}}.$$

Remark (Higher Order Derivatives). Higher order derivatives $\dfrac{\mathrm{d}^{(n)}\vec{\mathbf{r}}}{dt^{(n)}}$ can be defined similarly.

Theorem 2.2.1 (Graphic Interpretation of Derivative). When $h \to 0$, the vector

$$\frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h}$$

becomes $\vec{\mathbf{r}}(t)$ and therefore, $\vec{\mathbf{r}}'(t)$ approaches to a vector that lies on the tangent line. $\vec{\mathbf{r}}'(t)$ is called the **tangent vector**, and

$$\vec{T} = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}$$

is called the **unit tangent vector**.

Example 2.2.1. Find parametric equations of the tangent line to the vector function $\vec{\mathbf{r}}(t) = \langle 2\cos t,\ \sin t,\ t\rangle$ at point $(0, 1, \dfrac{\pi}{2})$.

Solution.

When $t = \frac{\pi}{2}$, $2\cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1.$

$\therefore (0, 1, \frac{\pi}{2})$ is on the space curve of $\vec{\mathbf{r}}(t)$.

Find

$$\begin{aligned}\vec{\mathbf{r}}'(t) &= \langle (2\cos t)',\ (\sin t)',\ t'\rangle \\ &= \langle -2\sin t,\ \cos t, 1\rangle\end{aligned}$$

When $t = \frac{\pi}{2}$,

$$\vec{\mathbf{r}}'\left(\frac{\pi}{2}\right) = \langle -2\sin\left(\frac{\pi}{2}\right),\ \cos\left(\frac{\pi}{2}\right), 1\rangle = \langle -2, 0, 1\rangle$$

$\therefore \vec{d}$ of tangent line $= \langle -2, 0, 1\rangle$

$$\therefore \text{Line: } \langle 0, 1, \frac{\pi}{2}\rangle + \langle -2, 0, 1\rangle t = \langle -2t, 1, \frac{\pi}{2} + t\rangle$$

□

Example 2.2.2. If $\vec{\mathbf{r}}(t) = (t^3 + 2t)\hat{\mathbf{i}} - 3e^{-2t}\hat{\mathbf{j}} + 2\sin 5t\hat{\mathbf{k}}$. Find $\dfrac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}$, $\left|\dfrac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}\right|$, $\dfrac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2}$, $\left|\dfrac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2}\right|$.

Solution.

$$\begin{aligned}\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} &= \langle 3t^2 + 2,\ 6e^{-2t},\ 10\cos 5t\rangle \\ \frac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2} &= \langle 6t,\ -12e^{-2t},\ -50\sin 5t\rangle\end{aligned}$$

When $t = 0$:

$$\vec{\mathbf{r}}'(0) = \langle 2,\ 6,\ 10\rangle; \qquad \vec{\mathbf{r}}''(0) = \langle 0,\ -12,\ 0\rangle$$

$$\therefore |\vec{\mathbf{r}}'(0)| = \sqrt{4+36+100} = \sqrt{140}(= 2\sqrt{70}); \quad |\vec{\mathbf{r}}''(0)| = \sqrt{144} = 12.$$

□

Theorem 2.2.2 (Properties of Differentiation).

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t) + \vec{\mathbf{r}}_2(t)] = \frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t)] + \frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_2(t)]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha\vec{\mathbf{r}}(t)] = \alpha\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}(t)]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)\vec{\mathbf{r}}(t)] = f'(t)\vec{\mathbf{r}}(t) + f(t)\vec{\mathbf{r}}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t)\cdot\vec{\mathbf{r}}_2(t)] = \vec{\mathbf{r}}_1'(t)\cdot\vec{\mathbf{r}}_2(t) + \vec{\mathbf{r}}_1(t)\cdot\vec{\mathbf{r}}_2'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t)\times\vec{\mathbf{r}}_2(t)] = \vec{\mathbf{r}}_1'(t)\times\vec{\mathbf{r}}_2(t) + \vec{\mathbf{r}}_1(t)\times\vec{\mathbf{r}}_2'(t)$$

Example 2.2.3. Show that if a curve lies on a sphere with center at the origin, then $\vec{\mathbf{r}}'(t)$ is perpendicular to $\vec{\mathbf{r}}(t)$ for any t.

Solution.

Let $\vec{\mathbf{r}}(t)$ lies on a sphere, with center at the origin, and radius $R = c$:

$$\therefore \vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle \quad \text{and} \quad x^2(t) + y^2(t) + z^2(t) = c^2$$

$$x^2(t) + y^2(t) + z^2(t) = |\vec{\mathbf{r}}(t)|^2 = \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)$$

$$\therefore \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t) = c^2$$

Take derivative of the both sides of the euqation

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)] = \frac{\mathrm{d}}{\mathrm{d}t}(c^2)$$

$$\therefore \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) = 0 \implies 2\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t) = 0$$

$$\therefore \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t) = 0 \implies \vec{\mathbf{r}}'(t) \perp \vec{\mathbf{r}}(t).$$

□

Definition 2.2.2 (Definite Integral of a Vector Function). The definite integral of a continuous vector function $\vec{\mathbf{r}}(t)$ can be defined as

$$\int_a^b \vec{\mathbf{r}}(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t\hat{\mathbf{i}} + \int_a^b g(t)\mathrm{d}t\hat{\mathbf{j}} + \int_a^b h(t)\mathrm{d}t\hat{\mathbf{k}},$$

if $\vec{\mathbf{r}}(t) = \Big\langle f(t), g(t), h(t)\Big\rangle$.

Example 2.2.4.

$$\begin{aligned}
\int_0^1 \left(\frac{1}{t+1}\hat{\mathbf{i}} + \frac{1}{t^2+1}\hat{\mathbf{j}} + \frac{t}{t^2+1}\hat{\mathbf{k}}\right) \mathrm{d}t &= \int_0^1 \frac{1}{t+1}\mathrm{d}t\hat{\mathbf{i}} + \int_0^1 \frac{1}{t^2+1}\mathrm{d}t\hat{\mathbf{j}} + \int_0^1 \frac{t}{t^2+1}\mathrm{d}t\hat{\mathbf{k}} \\
&= \left[\frac{1}{t+1}\right]_0^1 \hat{\mathbf{i}} + \left[\frac{1}{t^2+1}\right]_0^1 \hat{\mathbf{j}} + \left[\frac{t}{t^2+1}\right]_0^1 \hat{\mathbf{k}} \\
&= \ln(2)\hat{\mathbf{i}} + \frac{\pi}{4}\hat{\mathbf{j}} + \frac{1}{1}(\ln(2))\hat{\mathbf{k}}
\end{aligned}$$

3 Partial Derivative

3.1 Function of Several Variables

Definition 3.1.1 (Multivariable Functions). A function of f of n variables is a function that takes any n-tuple $(x_1, \cdots, x_n)$ in the set D to a number in $\mathbb{R}$, where

$$D = \left\{(x_1, \cdots, x_n) | x_i \in \mathbb{R} \text{ and } f \text{ is defined in } (x_1, \cdots, x_n)\right\}$$

Example 3.1.1. $f(x,y) = \sqrt{x^2+y^2-4}$: $f: \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R} \\ (x,y) \longmapsto \text{ a number like } r \end{array}$

Domain of f: all $(x,y) \in \mathbb{R}$ $s.t.$ $x^2+y^2-4 \geq 0$. (i.e., Everything exclude the circle centered at the origin with a radius of 2.)

Definition 3.1.2 (Graphs of a Two-Variable Function). The graph of a two-variable function with domain D is the set of all points $(x,y,z) \in \mathbb{R}^3$ $s.t.$ $z = f(x,y)$ and $(x,y) \in D$.

Definition 3.1.3 (Vector Functions).

$$\vec{\mathbf{r}} : \begin{array}{l} \mathbb{R} \longrightarrow V_n \\ t \longmapsto \langle f(t),\ g(t),\ h(t), \cdots \rangle \end{array},$$

where V_n is a set of all vectors with n components, and t is a parameter.

Remark. We will only work with V_3, i.e., $\vec{\mathbf{r}} : \begin{array}{l} \mathbb{R} \longrightarrow V_3 \\ t \longmapsto \langle f(t),\ g(t),\ h(t) \rangle \end{array}$.

Theorem 3.1.1. A multivariable function creates a surface in the space. if two surfaces intersect each other, then the intersection identifies a curve.

Example 3.1.2. Find a vector function $\vec{\mathbf{r}}(t)$ that represents the curve of intersection of two surfaces

$$z = \sqrt{x^2+y^2} \qquad \text{and} \qquad z = 3+y.$$

Solution.

Solve the system of equation $\begin{cases} x = \sqrt{x^2+y^2} \\ z = 3+y \end{cases}$.

Hence,

$$\begin{aligned} \sqrt{x^2+y^2} &= 3+y \\ x^2+y^2 &= (3+y)^2 = y^2+6y+9 \\ x^2 &= 6y+9 \\ y &= \frac{x^2-9}{6} \\ \therefore z =& 3+y = \frac{x^2+0}{6} \end{aligned}$$

Let $x = t$:

$$\vec{\mathbf{r}}(t) = \langle x, t, z\rangle = \left\langle t,\ \frac{t^2-9}{6},\ \frac{t^2+9}{6}\right\rangle$$

□

Example 3.1.3. Do the same for surfaces

$$z = 3x^2 + y^2 \qquad \text{and} \qquad y = 5x^2$$

Solution.

Solve the system of equations $\begin{cases} z = 3x^2 + y^2 \\ y = 5x^2 \end{cases}$.

$$\therefore 5x^2 = 3x^2 + y^2 \implies z = 3x^2 + (5x^2)^2 = 3x^2 + 25x^4$$

Let $x = t$:

$$\vec{\mathbf{r}}(t) = \langle x, t, z\rangle = \left\langle t,\ 5t^2,\ 3t^2 + 25t^4\right\rangle$$

□

Definition 3.1.4 (Level Curves). The level curve of a two variable function $z = f(x, y)$ is a curve $f(x, y) = k$ (in the xy-plane). That means all values of x and y that have the same value $z = k$.

Theorem 3.1.2 (Application of Level Curve). Given that a point (a, b) is on the level curve of $f(x, y)$ for $k = c$, then we know $f(a, b) = c$.

3.2 Limit and Continuity

Definition 3.2.1 (Limit). For two variable function $z = f(x, y)$, we check limit when $(x, y) \to (a, b)$. Therefore, we can make (x, y) closer to $a(b)$ from infinitely many directions. Therefore,

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if in all directions that (x, y) approaches to (a, b), we have $f(x, y) \to L$.

Definition 3.2.2 (Precise Definition of Limit). $\forall$ given $\varepsilon > 0$, $\exists$ associated $\delta > 0$s.t. if $(x, y) \in D$ and $d((x, y), (a, b)) < \delta \implies d(f(x, y), L) < \varepsilon$, where $d((x, y), (a, b))$ is the distance between (x, y) and (a, b) and is calculated by $\sqrt{(x-a)^2 + (y-b)^2}$.

Example 3.2.1. Consider function $f(x, y) = \dfrac{xy}{x^2 + y^2}$, and identify if it is has a limit at $(0, 0)$ or not.

Solution.

In the direction of x-axis ($y = 0$), we have $f(x, y) = \dfrac{x \cdot 0}{x^2 + 0^2} = 0$ and $\lim_{(x,y)\to(0,0)} f(, y) = 0$ along the x-axis.

In the direction of y-axis ($x = 0$), we have $f(x, y) = 0.$, and $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ along the y-axis.

If $y = x$, $f(x, y) = f(x, x) = \dfrac{x^2}{x^2 + x^2} = \dfrac{1}{2}$, and $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \dfrac{1}{2}$ along the line $y = x$.

□

Example 3.2.2. Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2 + y^2}$.

Solution.

By looking at the graph of the function, we think it has a limit at $(0, 0)$. This is not enough, and later we will be able to say that limit exists by converting it to polar coordinate.

Let $y = mx$:

$$f(x, y) = f(x, mx) = \frac{x^2 \cdot mx}{x^2 + (mx)^2} = \frac{x^3 m}{x^2(1 + m^2)} = \frac{m}{1 + m^2}x$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x, y) = 0 \text{ along the line of } y = mx.$$

□

Example 3.2.3.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^4} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^3 y}{x^4 + y^4} \text{ D.N.E.} \left(\text{check } \begin{cases} x = 0 \\ y = x \end{cases}\right)$$

Definition 3.2.3 (Continuity). Functions of two-variables is continues at (a, b) if

$$\lim_{(x,y)\to(a,b)} = f(a, b).$$

Example 3.2.4. Find $\lim\limits_{(x,y)\to(1,2)} (x^2 y^3 - x^3 y^2 + 3x + 2y)$.

Solution.

As $x^2 y^3 - x^3 y^2 + 3x + 2y$ is a polynomial and continuous everywhere, so

$$\lim_{(x,y)\to(1,2)} (x^2 y^3 - x^3 y^2 + 3x + 2y) = (1)^2(2)^3 - (1)^3(2)^2 + 3(1) + 2(2) = 1.$$

□

Example 3.2.5. $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ is not continuous at $(0, 0)$, but

$$g(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \text{ is continuous at } (0, 0).$$

3.3 Partial Derivatives

In two-variable functions, we will have partial derivatives f_x (derivative with respect to x) and f_y (derivative with respect to y).

Definition 3.3.1 (Partial Derivative). If $f(x, y)$ is a two variable function, then its partial derivatives are f_x and f_y and is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Example 3.3.1. Let $f(x, y) = x^3 + x^2y^3 - 2y$ and find $f_x(2, 1)$ and $f_y(2, 1)$

Solution.

Find $f_x(x, y)$: keep y constant.

$f_x(x, y) = 3x^2 + 2xy^3$

$$\therefore f_x(2, 1) = 3(2)^2 + 2(2)(1)^3 = 16$$

Find $f_y(x, y)$: keep x constant.

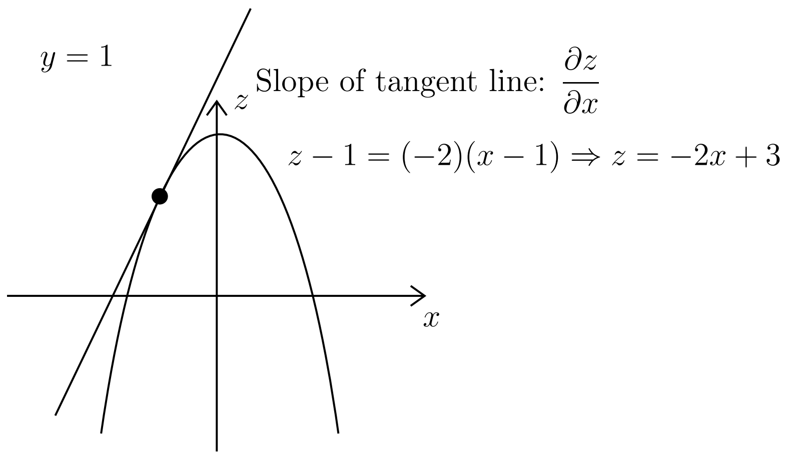
$f_y(x, y) = 3x^2y^2 - 2$

$$\therefore f_y(2, 1) = 3(2)^2(1)^2 - 2 = 10$$

□

Example 3.3.2. Let $f(x, y) = 4 - x^2 - 2y^2$. Find $f_x(1, 1)$ and interpret the values.

Solution.



$f(1, 1) = 4 - 1 - 2 = 1 \implies A(1, 1, 1)$ lies on $f(x, y)$.

$\frac{\partial f}{\partial x} = -2x \implies \frac{\partial f}{\partial x}(1, 1) = -2$

Let's consider $y = 1$:

The plane $y = 1$ will intersect with $f(x, y)$ at a line $\vec{\mathbf{r}}(t)$.

Solve $\vec{\mathbf{r}}(t)$: $\begin{cases} z = 4 - x^2 - 2y^2 \\ y = 1 \end{cases}$

$$\Rightarrow z = 4 - x^2 - 2 = 2 - x^2$$

$$\therefore \vec{\mathbf{r}}(t) = \langle t, 1, 2 - t^2 \rangle,\ \vec{\mathbf{r}}'(t) = \langle 1, 0, -2t \rangle$$

At point $A(1, 1, 1)$, $t = 1$.

$\therefore \vec{\mathbf{r}}'(1) = \langle 1, 0, -2 \rangle$, which is a directional vector of the tangent line.

$\therefore$ Tangent line:

$$L:\ x = 1 + t,\ y = 1,\ z = 1 - 2t$$

□

Definition 3.3.2 (Higher Order Partial Derivative).

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Theorem 3.3.1 (Clairaut's Theorem). If f is continuous on a disk D, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Definition 3.3.3 (Functions With More Than Two Variables). If $U = f(x_1, \cdots, x_n)$, its partial derivative with respect to x_i is

$$\begin{aligned} \frac{\partial f}{\partial y_i} &= \lim_{h \to 0} \frac{f(x_a, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h} \\ &= \frac{\partial U}{\partial x_i} \end{aligned}$$

3.4 Tangent Plane and Linear Approximation

Theorem 3.4.1 (Tangent Plane). If f has continuous partial derivatives, an equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Example 3.4.1. Find the tangent plane of $f(x,y) = 2x^2 + y^2$ at $(1,1,3)$.

Solution.

$$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = 2y$$

$$\therefore \frac{\partial f}{\partial x}(1,1) = 4 \qquad \frac{\partial f}{\partial y}(1,1) = 2$$

$\therefore$ Tangent plane at $(1,1,3)$:

$$\Pi : z - 3 = 4(x-1) + 2(y-3).$$

$\square$

Definition 3.4.1 (Linearization and Linear Approximation). Similar to single variable calculus, we can approximate the value of a function at a point using the tangent line:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is the **linearization** of $f(x,y)$ at point (a,b):

$$f(x,y) \approx L(x,y)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a,b).

Definition 3.4.2 (Differentiable Functions). A **differentiable function** is a function that the linear approximation is a good approximation when (x,y) are very close to (a,b).

Theorem 3.4.2 (A sufficient condition for differentiability). If partial derivative $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exists near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Example 3.4.2. Show that function $f(x,y) = \dfrac{\sqrt{x}}{y}$ is differentiable at $(16,5)$ and use it to approximate $\dfrac{\sqrt{16.02}}{4.96}$.

Solution.

$$f(16,5) = \frac{\sqrt{16}}{5} = \frac{4}{5}; \quad \frac{\partial f}{\partial x} = \frac{1}{2y\sqrt{x}}; \quad \frac{\partial f}{\partial y} = -\frac{\sqrt{x}}{y^2}.$$

$$\therefore \left.\frac{\partial f}{\partial x}\right|_{(16,5)} = \frac{1}{2(5)\sqrt{16}} = \frac{1}{40}; \quad \left.\frac{\partial f}{\partial y}\right|_{(16,5)} = -\frac{\sqrt{16}}{25} = -\frac{4}{25}.$$

As $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exists and is continuous at $(x,y) = (16,5)$, $f(x,y)$ is differentiable at $(16,5)$.

Then, the approximation is

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

At $a = 16$ and $b = 5$:

$$\begin{aligned}\frac{\sqrt{x}}{y} &\approx \frac{4}{5} + \frac{1}{40}(x-16) + \left(-\frac{4}{25}\right)(y-5)\\ &= \frac{4}{5} + \frac{1}{40}x - \frac{2}{5} - \frac{4}{25}y + \frac{4}{5}\\ &= \frac{1}{40} - \frac{4}{25}y + \frac{6}{5}.\end{aligned}$$

Therefore, $\dfrac{\sqrt{16.02}}{4.96} \approx \dfrac{1}{40}(16.02) - \dfrac{4}{25}(4.96) + \dfrac{6}{5} \approx 0.807.$

□

Definition 3.4.3 (Differentials).

$$\Delta z = \Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$$

$$\mathrm{d}z = f_x(a,b)\mathrm{d}x + f_y(a,b)\mathrm{d}y$$

Extension (Differentials in Higher Dimensions). Let $U = f(x_1, x_2, \cdots, x_n)$, we have

$$\mathrm{d}U = f_{x_1}(a_1, \cdots, a_n)\mathrm{d}x_1 + f_{x_2}(a_1, \cdots, a_n)\mathrm{d}x_2 + \cdots + f_{x_n}(a_1, \cdots, a_n)\mathrm{d}x_n$$

$$\Delta U = \Delta f = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_n + \Delta x_n) - f(x_a, \cdots, x_n)$$

3.5 The Chain Rule

Theorem 3.5.1 (The Multivariable Chain Rule). Let U be a differentiabel function of n variables $x_1, \cdots, x_n$, and each x_i for $i = 1, \cdots, n$ is a differentiable function of $t_1, \cdots, t_m$. Then, we have

$$\frac{\partial U}{\partial t_i} = \frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial U}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Example 3.5.1. Let $U = x^4y + y^2z^3$ and $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin(t)$. Find thee value of $\dfrac{\partial U}{\partial s}$ when $r = 2,\ s = 1,\ t = 0$.

Solution.

From the multivariable china rule, we know

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial U}{\partial x} = 4x^3y; \quad \frac{\partial U}{\partial x} = x^4 + 2yz^3; \quad \frac{\partial U}{\partial x} = 3y^2z^2;$$

$$\frac{\partial x}{\partial s} = re^t; \quad \frac{\partial y}{\partial s} = 2rse^{-t}; \quad \frac{\partial x}{\partial s} = r^2\sin t.$$

$$\therefore \frac{\partial U}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When $r = 2,\ s = 1,\ t = 0$, we have

$$x = 2,\ y = 2,\ z = 0.$$

$$\therefore \left.\frac{\partial U}{\partial s}\right|_{(r,s,t)=(2,1,0)} = (4(2)^3(2))(2) + (2^4)(2\cdot 2) + 0 = 128 + 64 = 192.$$

□

Example 3.5.2. If $z = f(x, y)$ has continuous second order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$.

Solution.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Since

$$\frac{\partial}{\partial r} = 2r; \qquad \frac{\partial y}{\partial r} = 2s$$

$$\therefore \frac{\partial z}{\partial r} = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}.$$

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) &= \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)\\ &= 2\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial x}\right) + 2\frac{\partial}{\partial r}\left(s\frac{\partial z}{\partial y}\right)\\ &= 2\left[\frac{\partial}{\partial r}(r)\cdot\frac{\partial z}{\partial x} + r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\right] + 2\left[\frac{\partial}{\partial r}(s)\cdot\frac{\partial z}{\partial y} + s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\right]\end{aligned}$$

Notice that $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are functions dependent on x and y, so to find their partial derivatives with respect to r, we need to apply multivariable chain rule again:

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}$$

$$\therefore \frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 2r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}\right) + 2s\left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)$$

□

Theorem 3.5.2 (Implicit Differentiation). If we have two-variable function like $F(x, y) = 0$, where y depends on x, we use the multivariable chain rule to differential the both sides of $F(x, y)$:

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\mathrm{d}x}{\mathrm{d}x}}_{1} + \frac{\partial F}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

Example 3.5.3. Find y' if $x^3 + y^3 = 6xy$

Solution.

$\boxed{\text{Method1}}$ Applying the formula:

$$\begin{aligned} F_x &= 3x^2 - 6y \\ F_y &= 3y^2 - 6x \\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} \end{aligned}$$

$\boxed{\text{Medthod2}}$ Find derivatives of the both sides:

$$\begin{aligned} x^3 + y^3 - 6xy &= 0 \\ 3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 6y - 6x\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (3y^2 - 6x)\frac{\mathrm{d}y}{\mathrm{d}x} &= 6y - 3x^2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

□

Theorem 3.5.3 (Multivariable Implicit Differentiation). If $z = f(x, y)$, consider a function

$$F(x, y, z) = F(x, y, f(x, y))$$

Then, by the multivariable chain rule, we differentiate both sides of $F(x, y, f(x, y)) = 0$:

$$\frac{\partial F}{\partial x}\underbrace{\frac{\mathrm{d}x}{\mathrm{d}x}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

Similarly, we have

$$\frac{\partial F}{\partial y}\underbrace{\frac{\mathrm{d}y}{\mathrm{d}y}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

Example 3.5.4. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution.

In order to find $\dfrac{\partial z}{\partial x}$, differentiate both sides with respect to x:

$$\begin{aligned} 3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} &= 0 \\ (3z^2 + 6xy)\frac{\partial z}{\partial x} &= -(3x^2 + 6yz) \\ \frac{\partial z}{\partial x} &= -\frac{3x^2 + 6yz}{3z^2 + 6xy}\left(= -\frac{x^2 + 2yz}{z^2 + 2xy}\right) \end{aligned}$$

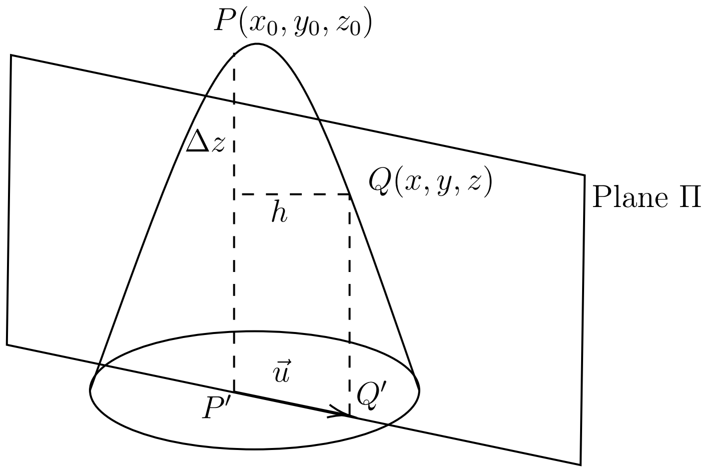
In order to find $\dfrac{\partial z}{\partial y}$, differentiate both sides with respect to y:

$$\begin{aligned} 3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} &= 0 \\ (3z^2 + 6xy)\frac{\partial z}{\partial y} &= -(3y^2 + 6xz) \\ \frac{\partial z}{\partial y} &= -\frac{3y^2 + 6xz}{3z^2 + 6xy}\left(= -\frac{y^2 + 2xz}{z^2 + 2xy}\right) \end{aligned}$$

□

3.6 Directional Derivatives and Gradient

To formally study directional derivatives, we start from the ideas of it. We want to study the change of $z = f(x, y)$ in the direction of the unit vector $\vec{u} = \langle a, b\rangle = a\hat{\mathbf{i}} + \hat{\mathbf{j}}$. ($\sqrt{a^2 + b^2} = 1$). We intersect surface $z = f(x, y)$ with plane Π that passes through the point $P(x_0, y_0, z_0)$ vertically and in the direction of vector $\vec{u} = \langle a, b\rangle$.



So, we have

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + h, y_0 + h) - f(x_0, y_0)}{h}$$

Definition 3.6.1 (Directional Derivative). The directional derivative of f at (x_0, y_0) in the direction of a vector $\vec{u} = \langle a, b\rangle$ is defined as

$$D_{\vec{u}}f(x_0y_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Now, let $g(h) = f(x_0 + ha, y_0 + hb)$, then we have

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h\to 0}\frac{g(h) - g(0)}{h} = g'(0)$$

To find $g'(h)$, we use the multivariable chain rule:

$$g'(h) = \frac{\partial g}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}h} + \frac{\partial g}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}h} \quad \text{where } \begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}.$$

From $\begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}$, we have $\dfrac{\partial x}{\partial h} = a$ and $\dfrac{\partial y}{\partial h} = b$.

$$\begin{aligned} \therefore g'(h) &= \frac{\partial g}{\partial x} \cdot a + \frac{\partial g}{\partial y} \cdot b \\ &= a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} \qquad \Big[g(h) \text{ is in fact } f(x,y)\Big] \end{aligned}$$

When $h \to 0$,

$$g'(0) = a \cdot f_x(x_0, y_0) + b \cdot f_y(x_0, y_0)$$

$$\begin{aligned} \therefore D_{\vec{u}} f(x_0, y_0) &= a \cdot f_x(a_0, y_0) + b \cdot f_y(x_0, y_0) \\ &= \langle a, b \rangle \cdot \langle f_x(a_0, y_0), f_y(x_0, y_0) \rangle \end{aligned}$$

Theorem 3.6.1 (Directional Derivative in Dot Product).

$$D_{\vec{u}} f(x_0, y_0) = \vec{u} \cdot \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle = \vec{u} \cdot \nabla f(x_0, y_0)$$

Definition 3.6.2 (Gradient Vector). A gradient vector of f is a vector function defined as

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}}.$$

The notation "∇" is called nabla.

Extension. If f is a function as $f(x_1, \cdots, x_n)$, then

$$\nabla f = \langle f_{x_1}, f_{x_2}, f_{x_3} \cdots, f_{x_n} \rangle.$$

Theorem 3.6.2 (Properties of Gradient). From the dot product definition of directional vector, we know that

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

Then, if θ is the angle between ∇f and $\vec{u}$, we have

$$D_{\vec{u}} f = |\nabla f||\vec{u}| \cos \theta.$$

Thus,

$$\max D_{\vec{u}} f = |\nabla f||\vec{u}| \text{ when } \theta = 0$$

(or, the vector $\vec{u}$ is in the direction of ∇f.) Since $\vec{u}$ is a unit vector, $|\vec{u}| = 1$. So when $\vec{u}$ is in the same direction of ∇f, we have

$$\max D_{\vec{u}} f = |\nabla f|.$$

On the other hand, if $\vec{u}$ and ∇f are in the opposite direction, we have $\theta = \pi$ and $\cos\theta = \cos(\pi) = -1$.

$$\therefore \min D_{\vec{u}} f = |\nabla f||\vec{u}|\cos\theta = -|\nabla f|$$

Extension. If $\vec{u}$ is a unit vector and $\vec{u} = \langle a, b\rangle$ and f has continuous second partial derivatives, then

$$D^2_{\vec{u}} f = f_{xx} a + 2 f_{xy} ab + f_{yy} b.$$

Example 3.6.1. If $f(x, y) = xe^y$, then

1. Find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

   Solution.

$$\frac{\partial f}{\partial x} = e^y; \quad \frac{\partial f}{\partial y} = xe^y; \quad \overrightarrow{PQ} = \left\langle \frac{1}{2} - 2, 2 - 0 \right\rangle = \left\langle -\frac{3}{2}, 2 \right\rangle; \quad \left|\overrightarrow{PQ}\right| = \sqrt{\frac{9}{4} + 4} = \frac{5}{2}$$

$$\therefore \vec{u} = \left\langle -\frac{3}{2} \cdot \frac{2}{5}, 2 \cdot \frac{2}{5} \right\rangle = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle; \quad \nabla f = \langle e^y, xe^y \rangle.$$

   Therefore,

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \langle e^y, xe^y \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{3}{5} e^y + \frac{4}{5} xe^y.$$

   At point $P(2, 0)$,

$$D_{\vec{u}} f(2, 0) = -\frac{3}{5} e^0 + \frac{4}{5} \cdot 2 \cdot e^0 = -\frac{3}{5} + \frac{8}{5} = 1.$$

   □

2. In what direction does f have the maximum rate of change? What is this maximum rate of change?

   Solution.

$$\nabla f(2, 0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$$

   Hence, in direction $\nabla f = \langle 1, 2 \rangle$, f has the maximum rate of change. The maximum rate of change is $|\nabla f(2, 0)| = \sqrt{5}$.

   □

Theorem 3.6.3 (Gradient and Tangent Plane). The equation of the tangent plane for the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0) =$ is given by:

$$\nabla f \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or (for implicit functions)

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0.$$

The normal line of the plane is given by

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}.$$

Remark (Gradient and Multivariable Chain Rule). If $F(x, y, z) = k$ and x, y, z are dependent of t, then we differentiate both sides with respect to t to get:

$$\frac{\partial F}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle = 0$$

$$\nabla F \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle = 0$$

Theorem 3.6.4 (Graphical Interpretation of Gradient Vector). In general, the gradient vector at P, $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{\mathbf{r}}'(t_0)$ to any curve C that passes through the point P on the surface S. Similar properties hold on level curves.

3.7 Maximum and Minimum Values

Definition 3.7.1 (Local Maximum and Local Minimum). A function $f(x, y)$ has a **local maximum** at point (a, b) if $\forall (x, y)$ near point (a, b), we have $f(x, y) \le f(a, b)$. The function $f(x, y)$ has a **local minimum** at point (a, b) if $\forall (x, y)$ near point (x, y), we have $f(x, y) \ge f(a, b)$.

Remark. "near point (a, b)" refers to a disk centered at (a, b).

Definition 3.7.2 (Absolute Maximum and Absolute Minimum). If the equalities $f(x, y) \le f(a, b)$ and $f(x, y) \ge f(a, b)$ holds for any (x, y) in the domain of $f(x, y)$, then we call them **absolute maximum** or **absolute minimum**.

Theorem 3.7.1. If f has local maximum or minimum at (a, b), and the first order partial derivatives of f exist at (a, b), then $f_x(a, b)$ and $f_y(a, b)$ are equal to 0. In other words,

$$\nabla f(a, b) = 0.$$

Corollary 3.1. As a result of Theorem 3.7.1, the equation of the tangent plane at (a,b) is

$$\begin{aligned} z - \overbrace{f(a,b)}^{z_0} &= \overbrace{f_x(a,b)}^{0}(x-a) + \overbrace{f_y(a,b)}^{0}(y-b) \\ z - z_0 &= 0. \end{aligned}$$

In other words, the tangent plane is horizontal.

Definition 3.7.3 (Critical Points). A point (a,b) is called the **critical point** if $f_x(a,b)=0$ and $f_y(a,b)=0$ or if one of the partial derivatives does not exist.

Remark. At a critical point, we may have maximum or minimum or neither (saddle point).

Definition 3.7.4 (Determinant). The determinant (Δ or D) is defined as

$$\begin{aligned} D &= \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\ &= f_{xx}f_{yy} - f_{xy}f_{yx} \\ &= f_{xx}f_{yy} - (f_{xy})^2. \end{aligned}$$

Theorem 3.7.2 (Second Derivative Test). Let (a,b) be a critical point and second partial derivatives of f (i.e., f_{xx}, f_{xy}, f_{yx}, f_{yy}) are continuous on a disk centered at (a,b). Then

1. If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.
2. If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.
3. If $D<0$, then $f(a,b)$ is not a local maximum or local minimum, and it is called a **saddle point**.

Remark. At saddle points, the tangent plane will intersect with the surface of f.

Example 3.7.1. For function $f(x,y)=4+x^3+y^3-3xy$. Check it $f(x,y)$ has local maximum, local minimum, and saddle points.

Solution.

$$\frac{\partial f}{\partial x} = 3x^3 - 3y; \qquad \frac{\partial f}{\partial y} = 3y^2 - 3x$$

Solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 3y = 0 & ① \\ 3y^2 - 3x = 0 & ② \end{cases}.$

From ①: $y = x^2$.

Substitute $y = x^2$ to ②:

$$\begin{aligned} 3\left(x^2\right)^2 - 3x &= 0 \\ x^4 - x &= 0 \\ x(x^3-1) &= 0 \Longrightarrow x = 0 \text{ or } x = 1 \end{aligned}$$

$$\therefore y = 0^2 = 0 \qquad \text{or} \qquad y = 1^2 = 1$$

$$\therefore \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

i.e., Critical points are at $(0, 0)$ and $(1, 1)$.

Find D:

$$D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} = 36xy - 9.$$

Apply the second derivative test:

1. $D(0, 0) = -9 < 0 \Longrightarrow (0, 0)$ is a saddle point.

2. $D(1, 1) = 36 - 9 = 27 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} = 6(1) > 0 \Longrightarrow (1, 1)$ is a local minimum.

□