# Emory University
# MATH 361 Mathematical Statistics I
# Learning Notes

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## Contents

# 1 Prerequisites

**Definition 1.0.1 (Geometric Series).** A geometric series has the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

If $|r| < 1$, then the series converges to $\dfrac{a}{1-r}$. Otherwise, it diverges.

**Example 1.0.2** Does the series $\displaystyle\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ converge or divers?

***Solution 1.***

Note that

$$2^{2n}3^{1-n} = \left(2^2\right)^n 3^{1-n} = 4^n \left(\frac{1}{3}\right)^{n-1} = 4 \cdot 4^{n-1}\left(\frac{1}{3}\right)^{n-1} = 4\left(\frac{4}{3}\right)^{n-1}.$$

So,

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

is a geometric series, with $a = 4$ and $r = \dfrac{4}{3}$.

Since $|r| = \left|\dfrac{4}{3}\right| = \dfrac{4}{3} > 1$, the series diverges. □

**Definition 1.0.3 (Taylor Series).** The Taylor series expanded about $a$ of a differentiable function $f$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots.$$

**Definition 1.0.4 (Maclaurin Series).** The Taylor series expanded about $a = 0$.

**Remark.** *The Maclurin Series of* $e^x$ *is given by* $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

**Theorem 1.0.5 Binomial Expansion**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

where $\dbinom{n}{k}$ is read as "$n$ choose $k$" and can also be written as $nCk$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

**Theorem 1.0.6 Integration by Parts**

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u.$$

**Example 1.0.7** Evaluate $\int xe^{-x} \, \mathrm{d}x$.

***Solution 2.***

Let $u = x, \ \mathrm{d}v = e^{-x} \, \mathrm{d}x$. So, $\mathrm{d}u = \mathrm{d}x$ and $v = \int e^{-x} \, \mathrm{d}x = -e^{-x}$. Then,

$$\int xe^{-x} \, \mathrm{d}x = -xe^{-x} - \int -e^{-x} \, \mathrm{d}x = -xe^{-x} - e^{-x} + C.$$

□

**Definition 1.0.8 (Type I Improper Integral).** If $\int_a^t f(x) \, \mathrm{d}x$ exists for all $t > 0$, then

$$\int_a^\infty f(x) \, \mathrm{d}x = \lim_{t \to \infty} \int_a^t f(x) \, \mathrm{d}x.$$

**Example 1.0.9** Evaluate $\int_0^\infty xe^{-x} \, \mathrm{d}x$.

***Solution 3.***

$$\begin{aligned}
\int_0^\infty xe^{-x} \, \mathrm{d}x = \lim_{t \to \infty} \int_0^t xe^{-x} \, \mathrm{d}x &= \lim_{t \to \infty} \left[ -xe^{-x} - e^{-x} \right]_0^t \\
&= \lim_{t \to \infty} \left( -te^{-t} - e^{-t} + 1 \right) \\
&= -\lim_{t \to \infty} \left( \frac{t}{e^t} \right) - \lim_{t \to \infty} e^{-t} + 1 \\
&= -\lim_{t \to \infty} \left( \frac{1}{e^t} \right) - 0 + 1 = -0 - 0 + 1 = 1.
\end{aligned}$$

□

**Example 1.0.10** Double Integrals over Irregular Domains.

Consider

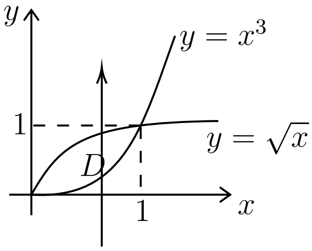
$$\iint_D 4xy - y^4 \, \mathrm{d}A,$$

where $D$ is the region bounded between $y = \sqrt{x}$ and $y = x^3$.

Evaluate this double integral over $D$.

***Solution 4.***

Firstly, we draw the diagram representing $D$ as follows:

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\begin{aligned}
\iint_D 4xy - y^3 \,\mathrm{d}A = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3 \,\mathrm{d}y\mathrm{d}x &= \int_0^1 \left[2xy^2 - \frac{1}{4}y^4\right]_{x^3}^{\sqrt{x}} \mathrm{d}x \\
&= \int_0^1 2x\bigl(x - x^6\bigr) - \frac{1}{4}\bigl(x^2 - x^{12}\bigr)\,\mathrm{d}x \\
&= \int_0^1 2x^2 - 2x^7 - \frac{1}{4}x^2 + \frac{1}{4}x^{12}\,\mathrm{d}x \\
&= \left[\frac{2}{3}x^3 - \frac{1}{4}x^8 - \frac{1}{12}x^3 + \frac{1}{52}x^{13}\right]_0^1 \\
&= \frac{2}{3} - \frac{1}{4} - \frac{1}{12} + \frac{1}{52} = \frac{55}{156}.
\end{aligned}
$$

□