Emory University

MATH 211 - Advanced Calculus (Multivariable) Learning Notes

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Contents

Preface

These is my personal notes for Emory University MATH 211 Advanced Calculus (Multivariable Calculus) course.

After mastering Calculus I (which covers contents concerning limits, differentiation, and basic integration) and Calculus II (which includes integration techniques and series), this course focuses on multivariable calculus, including vectors, multivariable functions, partial derivatives, optimization, multiple integrations, vector and scalar fields, Green's and Stokes' theorems, and the divergence theorem. The book used for this course is *Multivariable Calculus, 8th Edition* by James Stewart.

Throughout this personal note, I use different formats to differentiate different contents, including definitions, theorems, proofs, examples, extensions, and remarks. To be more specific:

Definition 0.0.1 (Terminology). This is a **definition**.

Theorem 0.0.1 (Theorem Name). This is a **theorem**.

Example 0.0.1. This is an **example**.

Solution. This is the *answer* part of an **example**. □

Remark. This is a **remark** of a definition, theorem, example, or proof.

Proof. This is a **proof** of a theorem. ■

Extension. This is a **extension** of a theorem, proof, or example.

To better ace this course, it is recommended to do more questions than provided as examples under each section. Although each example is distinctive and representative, more questions and practice is still needed to deepen the understanding of this course.

Even though I put efforts into making as few flaws as possible when encoding these learning notes, some errors may still exist in this note. If you find any, please contact me via email: `lvjiuru@hotmail.com`.

I hope you will find my notes helpful when learning Multivariable Calculus.

Cheers,
Jiuru Lyu

1 Vectors and Geometry of Space

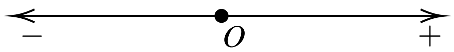
1.1 Three Dimensional Coordinate System

Definition 1.1.1 (Coordinate System). A **coordinate system** is a system that uses coordinate of a point to uniquely determine the position of the point in the space or plane.

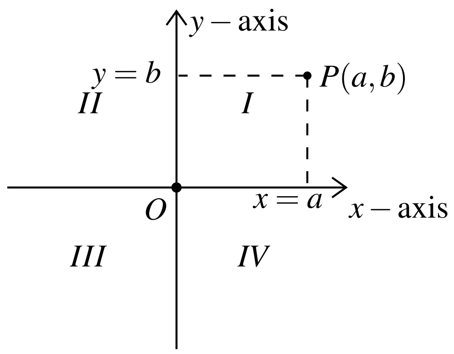
The Cartesian coordinate system is defined in different dimensions.

Definition 1.1.2 (One Dimensional Cartesian System). **One Dimensional Cartesian System** is a straight line with a fixed point as the origin and positive and negative directions.

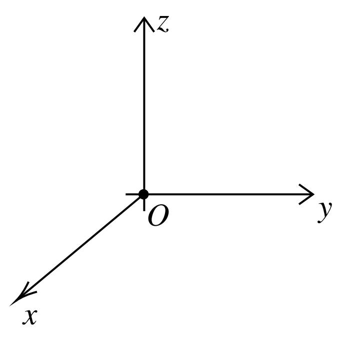
Remark. The one dimensional cartesian system is the number line:



Any point in the one dimensional Cartesian system corresponds to a number $\in \mathbb{R}$ and any number $\in \mathbb{R}$ has a location on the line. The two dimensional Cartesian system is the regular coordinate system.

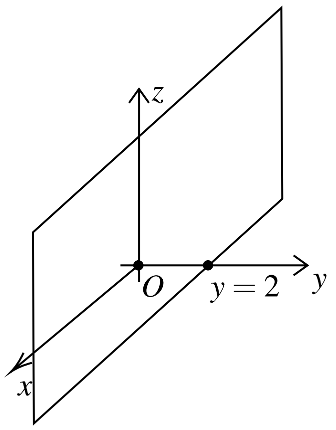


The three dimensional Cartesian system includes three perpendicular axes.

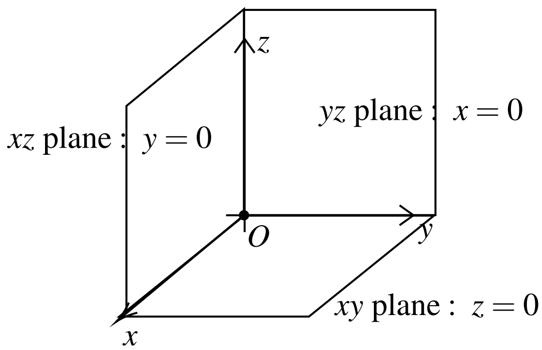


Definition 1.1.3 (Octant). A **Octant** is one of the eight divisions of the three dimensional coordinate system.

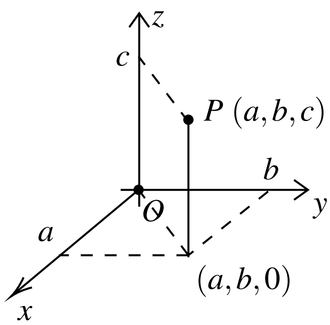
Definition 1.1.4 (Hyperplane). The hyperplane of $y = 2$ is given as below:



Specially:



Definition 1.1.5 (Points in the Three Dimensional System). $P(a,b,c)$ indicates the intersection of the three hyperplanes: $x = a$, $y = b$, and $z = c$.



For spaces in the higher dimension, we understand them via the Cartesian product.

Definition 1.1.6 (Cartesian Product).

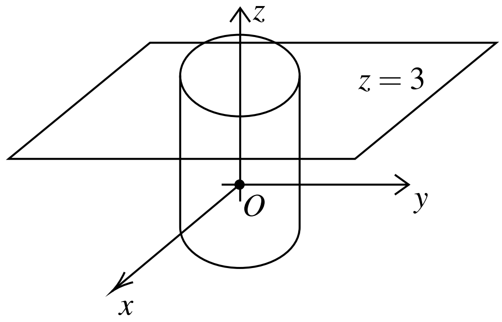
$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \cdots, x_n) | x_i \in \mathbb{R} \forall i = 1, \cdots, n\}$$

is the set of all n-tuples of real numbers and is denoted by $\mathbb{R}^n$.

Example 1.1.1. $(3,4,5) \in \mathbb{R}^3$ is 3 dimensional. $(3,4,5,6) \in \mathbb{R}^4$ is 4 dimensional.

Example 1.1.2. Which point(s) (x,y,z) satisfies the equations

$$x^2 + y^2 = 1 \quad \text{and} \quad x = 3?$$



Those points form a circle in the hyperplane of $z=3$ centered at the point $(0,0,3)$ with a radius of 1.

Theorem 1.1.1 (Distance Formula in Three Dimension). For given points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, the distance between them is denoted by $|P_1P_2|$ and is defined by

$$|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}.$$

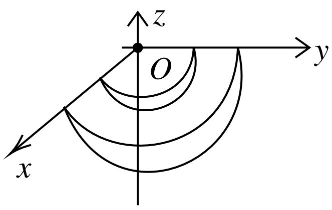
Theorem 1.1.2 (Equation of a Sphere). An equation of a sphere with a center of (a,b,c) and a radius of r is defined as

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

Example 1.1.3. What is the region in $\mathbb{R}^3$ represented by the inequalities

$$1 \le x^2+y^2+z^2 \le 4 \quad \text{and} \quad z \le 0?$$

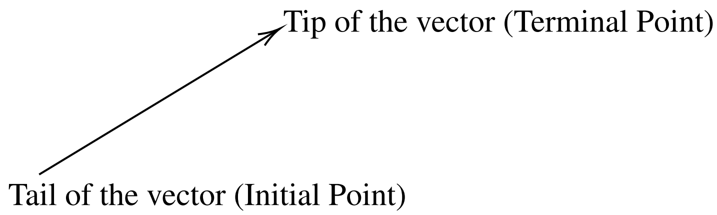
Solution.



The region is the difference between the half spheres (the lower half of the sphere) centered at $(0,0,0)$ with a radius of 1 and 2. □

1.2 Vectors

Definition 1.2.1 (Vectors). Vectors are used to indicate a quantity that has both magnitude and direction.



1. Vectors are denoted as $\vec{v}$.

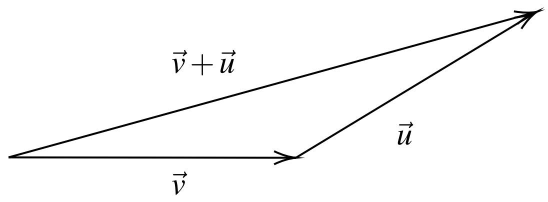
2. Magnitude

Definition 1.2.2 (Magnitude). A vector is a line segment, of which the **magnitude** of vector denoted by $|\vec{v}|$ is the length of it and the arrow points the direction of the vector.

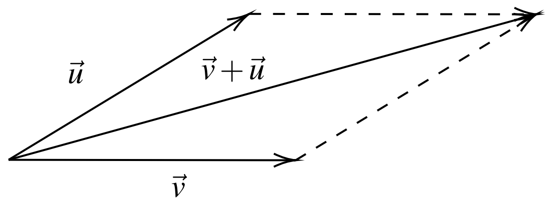
Vectors are operated in a different way:

1. Addition of Vectors:

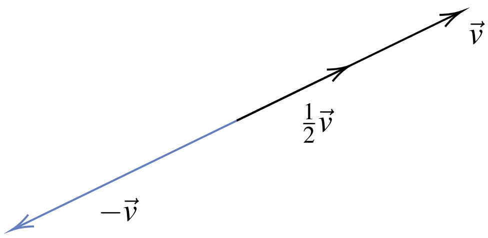
   (a) The triangle law:



   (b) The parallelogram law:



2. Scalar Multiplications:

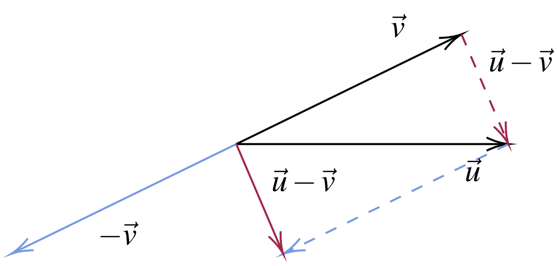


Definition 1.2.3 (Scalar Multiplication). If $c \in \mathbb{R}$ and $\vec{v}$ is a vector, then $c\vec{v}$ is in the same direction of $\vec{v}$ if $c > 0$ and in the opposite direction if $c < 0$.

Theorem 1.2.1. The magnitude of $c\vec{v}$:

$$|c\vec{v}| = c|\vec{v}|.$$

3. Differences of Vectors:



The difference of vectors $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}-\vec{v}$ and is defined by

$$\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$$

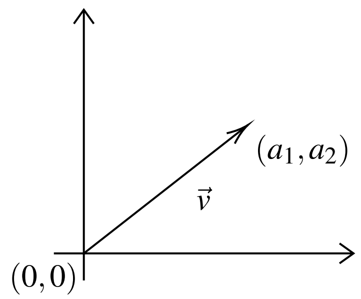
4. Properties of vectors:
   Suppose $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors in V_n and c and d are scalars (*Those properties can be proven geometrically*):
   (a) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
   (b) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
   (c) $\vec{a}+0=\vec{a}$
   (d) $\vec{a}+(-\vec{a})=0$
   (e) $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
   (f) $(c+d)\vec{a}=c\vec{a}+d\vec{a}$
   (g) $(cd)\vec{a}=c(d\vec{a})$
   (h) $1\cdot\vec{a}=\vec{a}$

We can link the coordinate system and vectors together:

1. **Definition 1.2.4 (Components of Vectors).** We will denote vector $\vec{v}$ as

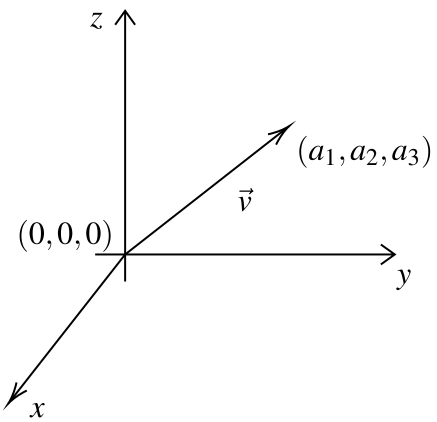
$$\vec{v}=\langle a_1,a_2\rangle,$$

   where a_1 and a_2 are called the **components** of $\vec{v}$.



2. In the three dimension:

$$\vec{v}=\langle a_1,a_2,a_3\rangle$$



3. **Definition 1.2.5.** If $A(x_1, y_1, z_1)$ as the tail of vector $\vec{v}$ and $B(x_1, y_1, z_1)$ as the tip of vector $\vec{v}$, then

$$\overrightarrow{AB} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$
$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

4. **Theorem 1.2.2.** If $\vec{v} = \langle a, b, c \rangle$ and $\vec{u} = \langle a', b', c' \rangle$, then

$$\vec{u} + \vec{v} = \langle a' + a,\ b' + b,\ c' + c \rangle$$

$$\vec{u} - \vec{v} = \langle a' - a,\ b' - b,\ c' - c \rangle$$

$$\alpha\vec{u} = \langle \alpha a',\ \alpha b',\ \alpha c' \rangle, \text{ where } \alpha \text{ is a scalar.}$$

Definition 1.2.6 (Standard Basis Vectors). In 2-D, $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$; and in 3-D, $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are called the **standard basis vectors**.

Remark. Any vectors in 2D and 3D can be written as

$$\vec{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

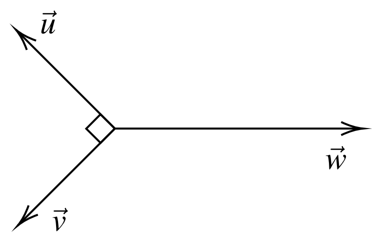
Definition 1.2.7 (Unit Vector). A **unit vector**is a vector of magnitude of 1.

Example 1.2.1.

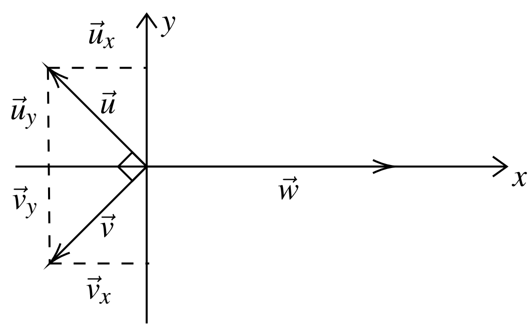
$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1 \text{ are unit vectors.}$$

Theorem 1.2.3. To find a unit vector in the direction of any vector $\vec{v}$, we use $\frac{1}{|\vec{v}|}\vec{v}$. The length of vector $\frac{\vec{v}}{|\vec{v}|}$ is 1 and its direction is the same as $\vec{v}$.

Example 1.2.2. If the vectors in the figure satisfy $|\vec{u}| = |\vec{v}| = 1$, and $\vec{u} + \vec{v} + \vec{w} = 0$, find $|\vec{w}|$.



Solution. Decompose the vectors:



We then have

$$\cos 45^\circ = \frac{|\vec{u}_x|}{\vec{u}} \Longrightarrow |\vec{u}_x| = |\vec{u}|\cos 45^\circ;$$
$$\sin 45^\circ = \frac{|\vec{u}_y|}{\vec{u}} \Longrightarrow |\vec{u}_y| = |\vec{u}|\sin 45^\circ;$$
$$\begin{aligned} \therefore \vec{u} = \langle |\vec{u}_x|,\ |\vec{u}_y\rangle &= -|\vec{u}_x|\mathbf{i} + |\vec{u}_y|\mathbf{j} \\ &= -\frac{\sqrt{2}}{2}|\vec{u}|\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} \\ &= \frac{\sqrt{2}}{2}|\vec{u}|(-\mathbf{i}+\mathbf{j}) \end{aligned}$$

Similarly,

$$\vec{v} = \frac{\sqrt{2}}{2}|\vec{v}|(-\mathbf{i}-\mathbf{j}).$$

We know $\vec{u}+\vec{v}+\vec{w}=0$:

$$\therefore \vec{w} + \frac{\sqrt{2}}{2}|\vec{u}|(-\mathbf{i}+\mathbf{j}) + \frac{\sqrt{2}}{2}|\vec{v}|(-\mathbf{i}-\mathbf{j}) = 0$$

We know $|\vec{u}| = |\vec{v}| = 1$:

$$\begin{aligned} \therefore \vec{w} + \frac{\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j}) + \frac{\sqrt{2}}{2}(-\mathbf{i}-\mathbf{j}) &= 0 \\ \vec{w} + \frac{\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j}-\mathbf{i}-\mathbf{j}) &= 0 \\ \vec{w} &= \sqrt{2}\mathbf{i} \end{aligned}$$
$$\therefore \vec{w} = \langle \sqrt{2}, 0\rangle \Longrightarrow |\vec{w}| = \sqrt{2}.$$

□

1.3 Dot Product

Definition 1.3.1 (Dot Product). If $\vec{u} = \langle x_1, y_1, z_1\rangle$ and $\vec{v} = \langle x_2, y_2, z_2\rangle$, then the dot product of $\vec{u}$ and $\vec{v}$ is defined as

$$\begin{aligned} \vec{u}\cdot\vec{v} &= \langle x_1, y_1, z_1\rangle \cdot \langle x_2, y_2, z_2\rangle \\ &= x_1x_2 + y_1y_1 + z_1z_2 \end{aligned}$$

Remark. The dot product of two vectors returns a scalar.

Example 1.3.1. Let $\vec{u} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\vec{v} = 2\mathbf{j}-\mathbf{k}$. Find $\vec{u}\cdot\vec{v}$.

Solution.

$$\begin{aligned} \vec{u}\cdot\vec{v} &= \langle 1, 2, -3\rangle \cdot \langle 0, 2, -1\rangle \\ &= (1)(0) + (2)(2) + (-3)(-1) = 7. \end{aligned}$$

□

Properties of the dot product:

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2. $\vec{a} \cdot (\vec{v} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}$

3. $m(\vec{a} \cdot \vec{b}) = (m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b}) = (\vec{a} \cdot \vec{b})m$

4. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$
   $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

Theorem 1.3.1.

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2.$$

Theorem 1.3.2. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta}.$$

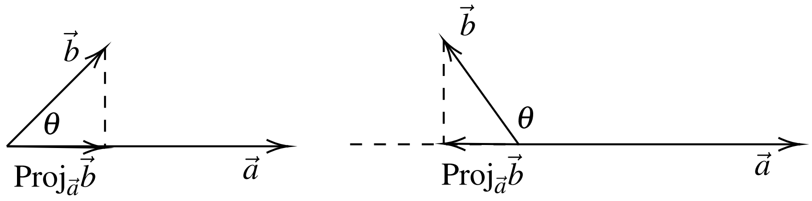
Extension.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

Extension.

$$\theta = 90^\circ \iff \vec{u} \cdot \vec{v} = 0.$$

Definition 1.3.2 (Projections). We use $\mathrm{Proj}_{\vec{a}}\vec{b}$ to denote the **projection** of $\vec{b}$ on $\vec{a}$.



From the diagrams,

$$\cos\theta = \frac{|\mathrm{Proj}_{\vec{a}}\vec{b}|}{|\vec{b}|} \implies |\mathrm{Proj}_{\vec{a}}\vec{b} = \boxed{|\vec{b}|\cos\theta}.$$

We know that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\therefore \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \boxed{|\vec{b}|\cos\theta}$$

$$\therefore |\mathrm{Proj}_{\vec{a}}\vec{b}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}, \text{ which is a scalar.}$$

$|\mathrm{Proj}_{\vec{a}}\vec{b}|$ is called the **scalar projection** of $\vec{b}$ on $\vec{a}$.

$$\mathrm{Proj}_{\vec{a}}\vec{b} = |\mathrm{Proj}_{\vec{a}}\vec{b}| \cdot \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector}} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \cdot \vec{a}$$

$\mathrm{Proj}_{\vec{a}}\vec{b}$ is called **projection** of $\vec{b}$ on $\vec{a}$ and is a vector.

Example 1.3.2. Find the scalar projection and vector projection of vector $\vec{u} = \langle 1,1,2 \rangle$ onto $\vec{v} = \langle -2,3,1 \rangle$.

Solution.

$$\text{Proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v}\,; \quad |\text{Proj}_{\vec{v}}\vec{u}| = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

We need $|\vec{v}| = \sqrt{4+9+1} = \sqrt{14}$ and $\vec{u}\cdot\vec{v} = (1)(-2)+(1)(3)+(2)(1) = 3$

$$\therefore |\text{Proj}_{\vec{v}}\vec{u}| = \frac{3}{\sqrt{14}}$$

$$\text{Proj}_{\vec{v}}\vec{u} = \frac{3}{14}\cdot\vec{v} = \frac{3}{14}\cdot\langle -2,3,1\rangle = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle.$$

□

1.4 Cross Product

Definition 1.4.1 (Cross Product). The **cross product** of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}\times\vec{v}$ and is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$. If $\vec{u} = \langle x_1, y_1, z_1\rangle$ and $\vec{v} = \langle x_2, y_2, z_2\rangle$, then

$$\begin{aligned}\vec{u}\times\vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} &= y_1z_2\mathbf{i} + x_2z_1\mathbf{j} + x_1y_2\mathbf{k} - x_2y_1\mathbf{k} - y_2z_1\mathbf{i} - x_1z_2\mathbf{j} \\ &= (y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}\end{aligned}$$

Example 1.4.1. Prove $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

Proof.

$$\begin{aligned}\vec{u}\cdot(\vec{u}\times\vec{v}) &= \langle x_1, y_1, z_1\rangle \cdot \langle y_1z_2 - y_2z_1,\ z_1x_2 - z_2x_1,\ x_1y_2 - x_2y_1\rangle \\ &= x_1y_1z_2 - x_zy_2z_1 + x_2y_1z_1 - x_1y_1z_2 + x_1y_2z_1 - x_2y_1z_1 = 0 \\ &\therefore \vec{u}\times\vec{v} \perp \vec{u}\end{aligned}$$

Similarly, $\vec{v}\cdot(\vec{u}\times\vec{v}) = 0 \Longrightarrow \vec{u}\times\vec{v}\perp\vec{v}$. ■

Theorem 1.4.1. If θ is the angle between vectors $\vec{u}$ and $\vec{v}$, then

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta.$$

Proof.

$$\begin{aligned}|\vec{u}\times\vec{v}|^2 &= (y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2 \\ &= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2 \\ &= |\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2 \\ &= |\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2\cos^2\theta \\ &= |\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta) \\ &= |\vec{u}|^2|\vec{v}|^2\sin^2\theta\end{aligned}$$

$$\therefore |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta.$$

■

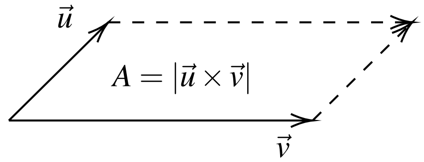
Definition 1.4.2 (Parallel). If two vectors, $\vec{u}$ and $\vec{v}$, are parallel to each other,

$$\vec{u} = c\vec{v},$$

where c is a scalar.

Theorem 1.4.2. For two vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \times \vec{v} = 0$ *iff* $\vec{u}$ and $\vec{v}$ are parallel to each other.

Theorem 1.4.3. The length of the cross product, $|\vec{u} \times \vec{v}|$, is the area of the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$.



Theorem 1.4.4.

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}; \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}; \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}; \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}; \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Properties of cross product ($\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors, and c is a scalar):

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Definition 1.4.3 (Triple Product). The **scalar triple product** is defined by

$$\vec{a} \cdot (\vec{b} \times \vec{c}).$$

Theorem 1.4.5. $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ denotes the volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

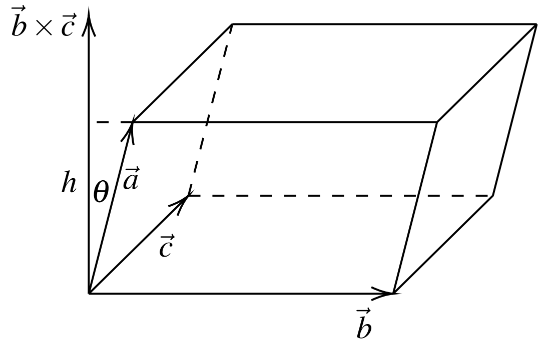
Proof. The area of the base is given by

$$A = |\vec{b} \times \vec{c}|$$

To find the volume, we need to know the height h:

$$h = |\vec{a}||\cos\theta|$$

$$\therefore V = Ah = |\vec{b}\times\vec{c}||\vec{a}||\cos\theta| = \vec{a}\cdot(\vec{b}\times\vec{c}) \qquad \left[\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta\right]$$

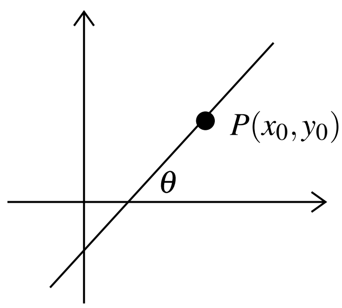


■

1.5 Equations of Lines and Planes

Theorem 1.5.1 (Equation of Lines in 2D). If we have a point $P(x_0, y_0)$ and a direction (slope/θ/another point on the line), we have the equation of the line:

$$\text{Given } \begin{cases} \text{slope} = m \\ P(x_0, y_0) \end{cases} \implies \text{The equation of the line: } y - y_0 = m(x - x_0).$$

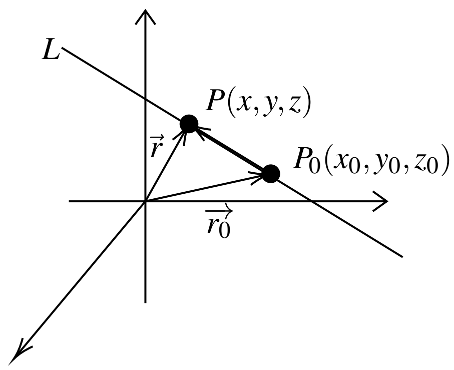


Definition 1.5.1 (Directional Vector). If $\vec{v}$ is a directional vector of line L,

$$\vec{a} = t\vec{v},$$

where $\vec{a}$ is any vector determined by two points on the line.

Definition 1.5.2 (Vector Equations of Lines in 3D). Let $\overrightarrow{P_0P} = \vec{a} \implies \vec{a} = \langle x - x_0,\ y - y_0\, z - z_0\rangle$



From the diagram, we also have

$$\vec{r}_0 + \vec{a} = \vec{r}.$$

As $\vec{a} = t\vec{v}$,

$$\vec{r} = \vec{r}_0 + t\vec{v},$$

which is the **vector equation** of line L.

Theorem 1.5.2. If L is a line with point $P(x_0, y_0, z_0)$ on it and paralleled to a direction vector $\vec{v} = \langle a, b, c\rangle$, we have

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle,$$

where t is a parameter and the equation is called the **vector equation** of line L.

Extension (Parametric Equation of L). From $\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$, we have

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

This system of equations is called the **parametric equation** of L.

Extension (Symmetric Equation of L). From the parametric equation of L, we can derive t:

$$\begin{cases} x = x_0 + ta & \implies & t = \frac{x - x_0}{a} \\ y = y_0 + tb & \implies & t = \frac{y - y_0}{b} \\ z = z_0 + tc & \implies & t = \frac{z - z_0}{c} \end{cases}$$

As t should be equal:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

which is called the **symmetric equation** of the line with point $P(x_0, y_0, z_0)$ and a directional vector $\vec{v} = \langle a, b, c\rangle$.

Remark (Three Forms of Equation of a Line). For line L in 3D, $P_0(x_0, y_0, z_0)$ is on L and $\vec{v} = \langle a, b, c\rangle$ is a directional vector of L.

1. The vector form:

$$\langle x, y, z\rangle = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc\rangle$$

2. The parametric form:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

3. The symmetric form:
$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Example 1.5.1. Find the parametric and symmetric equations of the line L passing through the points $(-8,1,4)$ and $(3,-2,4)$.

Solution. Let's set P_0 to be $(-8,1,4)$ and P_1 to be $(3,-2,4)$. So we can find the directional vector
$$\vec{v}=\overrightarrow{P_0P_1}=\langle 3-(-8),\ -2-1,\ 4-4\rangle=\langle 11,-3,0\rangle.$$

$\therefore$ The parametric equation of L:
$$\begin{cases} x=-8+11t \\ y=1-3t \\ z=4+(0)t \end{cases},$$

and the symmetric equation of L is
$$\frac{x+8}{11}=\frac{y-1}{-3},\quad z=4.$$

$\square$

Relationships of two lines in 3D:

1. Parallel: directional vectors of the two lines are parallel to each other.
2. Intersect: the two lines share one common point
3. Skewed: the two lines are neither parallel nor intersecting.

Example 1.5.2. Let
$$L_1:\frac{x-2}{1}=\frac{y-3}{-2}=\frac{z-1}{-3}\quad\text{and}\quad L_2:\frac{x-3}{1}=\frac{y+4}{3}=\frac{z-2}{-7}.$$

Find the relationship between L_1 and L_2.

Solution.
$$\vec{v}_1=\langle 1,-2,-3\rangle;\quad \vec{v}_2=\langle 1,3,-7\rangle$$

Because $\vec{v}_1$ and $\vec{v}_2$ are not parallel to each other, L_1 and L_2 are not parallel to each other.
$\therefore L_1$ and L_2 can only be intersecting or skewed.
To further discuss the relationship between L_1 and L_2, form parametric equations:

$$L_1:\begin{cases} x=2+t \\ y=3-2t \\ z=1-3t \end{cases}\qquad L_2:\begin{cases} x=3+s \\ y=-4+3s \\ z=2-7s \end{cases}$$

If we can find a set of solutions t and s that satisfy the following system of equations, the two lines have point in common and thus is intersecting:

$$\begin{cases} 2+t=3+s \\ 3-2t=-4+3s \\ 1-3t=2-7s \end{cases} \Longrightarrow \begin{cases} t-s=1 & ① \\ 2t+3s=7 & ② \\ 3t-7s=-1 & ③ \end{cases}$$

From ①:

$$t=s+1 \qquad ④$$

Substitute ② with ④:

$$\begin{aligned} &2(s+1)+3s=7 \\ &\quad 2s+2+3s=7 \quad \Rightarrow \quad 4s=5 \quad \Rightarrow \quad s=1 \\ &\quad \therefore t=s+1=1+1=2 \end{aligned}$$

Substitute $s=1$ and $t=2$ to ③:

$$\text{LHS} = 2(3)-7(1)=6-7=-1=\text{RHS}.$$

Hence, $\begin{cases} t=2 \\ s=1 \end{cases}$ satisfy all three equations. Substitute $t=2$ to L_1:

$$x=2+2=4, \quad y=3-2(2)=-1, \quad z=1-3(2)=-5.$$

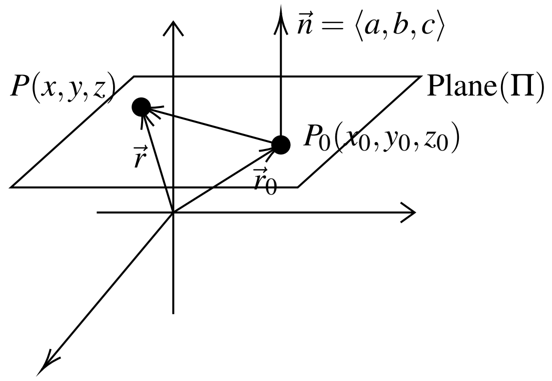
$\therefore$ The two lines intersect at $(4,-1,-5)$. □

Definition 1.5.3 (Normal Vector). A normal vector is the vector perpendicular to the plane and is often denoted as $\vec{n}$.

Theorem 1.5.3 (Vector Equation of a Plane). As $\vec{n} \perp \Pi$, $\vec{n} \perp \overrightarrow{P_0P}$

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{r}-\vec{r}_0 \\ \therefore \vec{n}\cdot(\vec{r}-\vec{r}_0) &= 0 \\ \vec{n}\cdot\vec{r}-\vec{n}\cdot\vec{r}_0 &= 0 \ \Rightarrow \ \vec{n}\cdot\vec{r}=\vec{n}\cdot\vec{r}_0, \end{aligned}$$

which is called the **vector equation** of a plane.



Extension (Scalar Equation of a Plane). From $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$: As $\vec{n}=\langle a,b,c\rangle$ and $\vec{r}-\vec{r}_0=\langle x-x_0,\ y-y_0,\ z-z_0\rangle$, we have

$$\langle a,b,c\rangle\cdot\langle x-x_0,\ y-y_0,\ z-z_0\rangle=0;$$

$$\therefore a(x-x_0)+b(y-y_0)+c(z-z_0)=0,$$

which is the **scalar equation** of plane Π with point $P_0(x_0,y_0,z_0)$ on it and a normal vector $\vec{n}=\langle a,b,c\rangle$.

Extension (Linear Equation of a Plane). From $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$:

$$ax+by+cz-(ax_0+by_0+cz_0)=0$$

Take $d=-(ax_0+by_0+cz_0)$:

$$ax+by+cz+d=0,$$

which is called the **linear equation** of plane Π with point $P_0(x_0,y_0,z_0)$ on it and a normal vector $\vec{n}=\langle a,b,c\rangle$.

Remark (Equations of a Plane). If point $P_0(x_0,y_0,z_0)$ is on the plane Π and a normal vector of Π is $\vec{n}=\langle a,b,c\rangle$:

1. The vector equation:
$$\vec{n}\cdot\vec{r}=\vec{n}\cdot\vec{r}_0$$
2. The scalar equation:
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$
3. The linear equation:
$$ax+by+cz+d=0,$$
where $d=-(ax_0+by_0+cz_0)=-\langle a,b,c\rangle\cdot\langle x_0,y_0,z_0\rangle$

Example 1.5.3. Find an equation of the plane crossing through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

Solution. Find the normal vector using the following equation:

$$\vec{n}=\overrightarrow{PQ}\times\overrightarrow{PR}$$

$$\overrightarrow{PQ}=\langle 3-1,-1-3,6-2\rangle=\langle 2,-4,4\rangle$$
$$\overrightarrow{PR}=\langle 5-1,2-3,0-2\rangle=\langle 4,-1,-2\rangle$$

$$\therefore \vec{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-4&4\\4&-1&-2\end{vmatrix}=12\mathbf{i}+20\mathbf{j}+14\mathbf{k}.$$

$$\therefore \vec{n}=\langle 12,20,14\rangle,\qquad P(1,3,2)$$

$$\therefore d = -\langle 12, 20, 14\rangle \cdot \langle 1, 3, 2\rangle = -(12+60+28) = -100.$$

$$\therefore \text{Linear Equation of } \Pi: \ 12x+20y+14z-100 = 0 \quad \Longrightarrow \quad 6x+10y+7z-50 = 0.$$

□

Theorem 1.5.4 (Relationship Between Two Planes). If $\vec{n}_1$ is a normal vector of plane Π_1, and $\vec{n}_2$ is a normal vector of plane Π_2 , then the angle between the two planes is given by

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right).$$

i.e., the angle between the planes is the angle between the normal vectors.

Theorem 1.5.5 (Distance from a Point to a Plane). Distance of the point $P(x_1, y_1, z_1)$ from the plane $ax+by+cz+d = 0$:

$$D = \frac{|ax_a+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \tag{1}$$

OR

$$D = \frac{\vec{b}\cdot\vec{n}}{|\vec{n}|}, \tag{2}$$

where $\vec{n}$ is the normal vector.

Example 1.5.4. Find the distance between the parallel planes:

$$\Pi_1: \ 10x+2y-2z = 5 \quad \text{and} \quad \Pi_2: \ 5x+y-z = 1.$$

Solution. Assume point $P(x_1, y_!, z_1)$ is on plane Π_1:

$$10x_1+2y_1-2z_1 = 5$$
$$\therefore\ 5x_1+y_1-z_1 = \frac{5}{2}$$

Applying formula 1: $\vec{n} = \langle a, b, c\rangle = \langle 5, 1, -1 rlangle,\ d = -1$:

$$\therefore\ D = \frac{|5x_1+y_1-z_1+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|\frac{5}{2}-1|}{\sqrt{26+1+1}} = \frac{3/2}{\sqrt{27}} = \frac{3}{2\sqrt{27}}\left(=\frac{\sqrt{3}}{6}\right).$$

□

Extension. Find the distance between two parallel planes:

$$\Pi_1: \ ax+by+cz+d = 0 \quad \text{and} \quad \Pi_2: \ ax+by+cz+d' = 0.$$

Let point $P(x_1, y_1, z_1)$ on Π_1:

$$ax_1+by_1+cz_1+d = 0$$

Apply formula 1:

$$D = \frac{|ax_1+by_1+cz_1+d'|}{\sqrt{a^2+b^2+c^2}} = \frac{-d+d'}{\sqrt{a^2+b^2+c^2}}.$$

1.6 Cylinders and Quadric Surfaces

Definition 1.6.1 (Cylinders). A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

Definition 1.6.2 (Quadric Surfaces). A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gz+Hy+Iz+J=0,$$

where $A,B,C,\cdots,J$ are constants, but by translation and rotation it can be brought into one of the standard forms:

$$Ax^2+By^2+Cz^2+J=0 \quad \text{or} \quad Ax^2+By^2+Iz=0.$$

Remark. Graphs of Quadric Surfaces (Refer to Page 877 of the Book):

1. Ellipsoid:
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$
All traces are ellipses.
If $a=b=c$, the ellipsoid is a sphere.

2. Elliptic Paraboloid:
$$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$
Horizontal traces are ellipses. Vertical traces are parabolas.
The variable raised to the first power indicates the axis of the paraboloid.

3. Hyperbolic Paraboloid:
$$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$
Horizontal traces are hyperbolas. Vertical traces are parabolas.

4. Cone:
$$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$
Horizontal traces are ellipses.
Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.

5. Hyperboloid of One Sheet:
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$
Horizontal traces are ellipses. Vertical traces are hyperbolas.
The axis of symmetry corresponds to the variable whose coefficient is negative.

6. Hyperboloid of Two Sheets:

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus sign indicate two sheets.

2 Vector Functions

2.1 Vector Functions and Space Curves

Definition 2.1.1 (Multivariable Functions). A function of f of n variables is a function that takes any n-tuple $(x_1, \cdots, x_n)$ in the set D to a number in $\mathbb{R}$, where

$$D = \left\{ (x_1, \cdots, x_n) | x_i \in \mathbb{R} \text{ and } f \text{ is defined in } (x_1, \cdots, x_n) \right\}$$

Example 2.1.1. $f(x,y) = \sqrt{x^2 + y^2 - 4}$: $f : \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R} \\ (x,y) \longmapsto \text{ a number like } r \end{array}$

Domain of f: all $(x,y) \in \mathbb{R}$ *s.t.* $x^2 + y^2 - 4 \geq 0$. (i.e., Everything exclude the circle centered at the origin with a radius of 2.)

Definition 2.1.2 (Graphs of a Two-Variable Function). The graph of a two-variable function with domain D is the set of all points $(x,y,z) \in \mathbb{R}^3$ *s.t.* $z = f(x,y)$ and $(x,y) \in D$.

Definition 2.1.3 (Vector Functions).

$$\vec{r} : \begin{array}{l} \mathbb{R} \longrightarrow V_n \\ t \longmapsto \langle f(t),\ g(t),\ h(t), \cdots \rangle \end{array},$$

where V_n is a set of all vectors with n components, and t is a parameter.

Remark. We will only work with V_3, i.e., $\vec{r} : \begin{array}{l} \mathbb{R} \longrightarrow V_3 \\ t \longmapsto \langle f(t),\ g(t),\ h(t) \rangle \end{array}$.

Definition 2.1.4 (Component Functions). $f(t),\ g(t),\ h(t)$ are real valued function and are called **component functions** of $\vec{r}(t)$. We write

$$\vec{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

Definition 2.1.5 (Limit of Vector Functions). To find the limit of a vector function, we check its component functions. That is

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle$$

Definition 2.1.6 (Continuity of Vector Functions). A vector function $\vec{r}(t)$ is continuous if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

Example 2.1.2. 1. Find the domain of

$$\vec{r}(t) = \left\langle \ln(t+1),\ \frac{t}{\sqrt{9 - t^2}},\ 2^t \right\rangle$$

Solution.

- Domain of $\ln(t+1)$: $D_1 L t+1>0,\ t>-1$
- Domain of $\dfrac{t}{\sqrt{9-t^2}}$: $D_2 :\ 9-t^2>0,\ -3<t<3$
- Domain of 2^t : $D_3 :\ \mathbb{R}$

  Find the intersection of domains of component functions:

$$D_1 \cap D_2 \cap D_3 :\ -1<t<3\ (t \in (-1,3))$$

□

2. Find $\lim\limits_{t\to 0} \vec{r}(t)$.

   Solution.

$$\begin{aligned}\lim_{t\to 0} \vec{r}(t) &= \left\langle \lim_{t\to 0} \ln(t+1),\ \lim_{t\to 0} \frac{t}{\sqrt{9-t^2}},\ \lim_{t\to 0} 2^t \right\rangle \\ &= \left\langle \ln(1),\ \frac{0}{\sqrt{9}},\ 2^0 \right\rangle \\ &= \langle 0,0,1\rangle = \mathbf{k}\end{aligned}$$

□

Example 2.1.3.

$$\begin{aligned}&\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\mathbf{i}+\sin \pi t\mathbf{j}+\cos 2\pi t\mathbf{k}\right)\\ =&\lim_{t\to 1}\left(\frac{t(t-1)}{t-1}\mathbf{i}+\sin \pi t\mathbf{j}+\cos 2\pi t\mathbf{k}\right)\\ =&\lim_{t\to 1} t\mathbf{i}+\lim_{t\to 1}\sin \pi t\mathbf{j}+\lim_{t\to 1}\cos 2\pi t\mathbf{k}\\ =&\mathbf{i}+\sin \pi\mathbf{j}+\cos 2\pi\mathbf{k}\\ =&\mathbf{i}+\mathbf{k}\end{aligned}$$

Definition 2.1.7 (Graphs of Vector Functions). For a vector function $\vec{r}(t) = f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, the graph of it, curve C, is defined by the moving tip of the vectors yielded from the vector function.