# Linear Algebra Done Right

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## Contents

# 1 Vector Spaces

## 1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

**Definition 1.1.1 (Complex Number).** A *complex number* is an ordered pair $(a, b)$, where $a, b \in \mathbb{R}$, but we write it as $a + b\mathrm{i}$.
**Notation 1.1.2.** $\mathbb{C} \coloneqq \{a + b\mathrm{i} \mid a, b \in \mathbb{R}\}$
**Definition 1.1.3 (Addition & Multiplication).**

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)]\mathrm{i}$$

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}$$

**Theorem 1.1.4 Properties of Complex Arithmetic**

1. commutativity: $\alpha + \beta = \beta + \alpha; \quad \alpha\beta = \beta\alpha, \quad \forall \alpha, \beta \in \mathbb{C}$.
2. associativity: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda); \quad (\alpha\beta)\lambda = \alpha(\beta\lambda), \quad \forall \alpha, \beta, \lambda \in \mathbb{C}$.
3. identities: $\lambda + 0 = \lambda; \quad \lambda \cdot 1 = \lambda, \forall \lambda \in \mathbb{C}$.
4. additive inverse: $\forall \alpha \in \mathbb{C}, \exists$ unique $\beta \in \mathbb{C}$ *s.t.* $\alpha + \beta = 0$.
5. multiplicative inverse: $\forall \alpha \in \mathbb{C}, \alpha \neq 0, \exists$ unique $\beta \in \mathbb{C}$ *s.t.* $\alpha\beta = 1$.
6. distributivity: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta, \quad \forall \lambda, \alpha, \beta \in \mathbb{C}$.

**Definition 1.1.5 (Subtraction).** If $-\alpha$ is the additive inverse of $\alpha$, *subtraction* on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

**Definition 1.1.6 (Division).** For $\alpha \neq 0$, let $\dfrac{1}{\alpha}$ denote the multiplicative inverse of $\alpha$. Then, *division* on $\mathbb{C}$ is defined by

$$\frac{\beta}{\alpha} = \beta \cdot \left(\frac{1}{\alpha}\right)$$

**Notation 1.1.7.** $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
**Definition 1.1.8 (List/Tuple).** Suppose $n$ is a non-negative integer. A list of length $n$ is an ordered collection of $n$ elements separated by commas and surrounded by parentheses: $(x_1, x_2, x_3, \cdots, x_n)$. Two lists are equal if and only if they have the same length and the same elements in the same order.

**Remark.** *Lists must have a FINITE length.*

**Definition 1.1.9 ($\mathbb{F}^n$ and Coordinate).** $\mathbb{F}^n$ is the set of all lists of length $n$ of elements of $\mathbb{F}$:

$$\mathbb{F}^n \coloneqq \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R} \forall i = 1, \cdots, n\},$$

where $x_i$ is the $i^{\text{th}}$ *coordinate* of $(x_1, \cdots, x_n)$.

**Example 1.1.10** $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ and $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.

**Definition 1.1.11 (Addition on $\mathbb{F}^n$).** *Addition* on $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n).$$

**Theorem 1.1.12 Commutativity of Addition on $\mathbb{F}^n$**
If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

***Proof 1.*** Suppose $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Then

$$\begin{aligned} x + y &= (x_1 + y_1, \cdots, x_n + y_n) \\ &= (y_1 + x_1, \cdots, y_n + x_n) = y + x. \end{aligned}$$

■

**Definition 1.1.13 (Zero).** Let $0$ denote the list of length $n$ whose coordinates are all $0$: $0 := (0, \cdots, 0)$.
**Definition 1.1.14 (Additive Inverse on $\mathbb{F}^n$).** For $x \in \mathbb{F}^n$, the additive inverse of $x$, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ *s.t.* $x + (-x) = 0$.
**Definition 1.1.15 (Scalar Multiplication in $\mathbb{F}^n$).** The product of a number $\lambda \in \mathbb{F}$ and a vector $x \in \mathbb{F}^n$ is computed by multiplying each coordinate of the vector by $\lambda$ :

$$\lambda x = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n),$$

where $x = (x_1, \cdots, x_n) \in \mathbb{F}^n$.

**Theorem 1.1.16 Properties of Arithmetic Operations on $\mathbb{F}^n$**

1. $(x + y) + z = x + (y + z) \quad \forall x, y, z \in \mathbb{F}^n$
2. $(ab)x = a(bx) \quad \forall x \in \mathbb{F}^n$ and $\forall a, b \in \mathbb{F}$.
3. $1 \cdot x = x \quad \forall x \in \mathbb{F}^n$ and $1 \in \mathbb{F}$.
4. $\lambda(x + y) = \lambda x + \lambda y \quad \forall \lambda \in \mathbb{R}$ and $\forall x, y \in \mathbb{F}^n$.
5. $(a + b)x = ax + bx \quad \forall a, b \in \mathbb{F}$ and $\forall x \in \mathbb{F}^n$.

## 1.2 Definition of Vector Space

**Definition 1.2.1 (Addition on $V$).** An *addition* on $V$ is a function $(u, v) \mapsto u + v$ for all $u, v \in V$.
**Definition 1.2.2 (Scalar Multiplication on $V$).** A *scalar multiplication* on $V$ is a function $(\lambda, v) \mapsto \lambda v$ for all $\lambda \in \mathbb{F}$ and $v \in V$.
**Definition 1.2.3 (Vector Space).** A *vector space* is a set $V$ along with an addition on $V$ and a scalar multiplication *s.t.* the following properties hold:

1. commutativity: $u + v = v + u \quad \forall u, v \in V$
2. associativity: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv) \quad \forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}$.
3. additive identity: $\exists 0 \in V$ *s.t.* $v + 0 = v \quad \forall v \in V$.
4. additive inverse: $\exists w \in V$ *s.t.* $v + w = 0 \quad \forall v \in V$.
5. multiplicative identity: $\exists 1 \in V$ *s.t.* $1 \cdot v = v \quad \forall v \in V$.
6. distributive properties: $a(u + v) = au + av$ and $(a + b)v = av + bv \quad \forall u, v \in V$ and $a, b \in \mathbb{F}$.

**Definition 1.2.4 (Vector).** Elements of a vector space are called *vectors* or points.
**Notation 1.2.5.** $V$ is a vector space over $\mathbb{F}$.
**Definition 1.2.6 (Real and Complex Vector Space).** A vector space over $\mathbb{R}$ is called a *real vector space*, and a vector space over $\mathbb{C}$ is called a *complex vector space*.

**Theorem 1.2.7 Unique Additive Identity of Vector Spaces**
A vector space has a unique additive identity.

***Proof 1.*** Suppose $0$ and $0'$ are both additive identities for some vector space $V$. So,

$$\begin{aligned} 0' &= 0' + 0 && \textit{Since } 0 \textit{ is an additive identity} \\ &= 0 + 0' && \textit{commutativity} \\ &= 0. && \textit{Since } 0' \textit{ is an additive identity} \end{aligned}$$

Then, $0' = 0$. ∎

**Theorem 1.2.8 Unique Additive Inverse of Vector Spaces**
A vector in a vector space has a unique additive inverse.

***Proof 2.*** Let $V$ be a vector space. Suppose $w$ and $w'$ are additive inverses of $v$ for some $v \in V$. Note that

$$\begin{aligned} w &= w + 0 \\ &= w + (v + w') \\ &= (w + v) + w \\ &= 0 + w' = w'. \end{aligned}$$

∎

**Notation 1.2.9.** Let $v, w \in V$. Then, $-v$ denotes the additive inverse of $v$.
**Definition 1.2.10 (Subtraction).** $w - v$ is defined to be $w + (-v)$.

**Theorem 1.2.11**
$0 \cdot v = 0 \quad \forall v \in V$.

***Proof 3.*** Since $v \in V$, we know

$$\begin{aligned} 0 \cdot v = (0+0)v &= 0 \cdot v + 0 \cdot v \\ 0 \cdot v + (-0 \cdot v) &= 0 \cdot +0 \cdot +(-0 \cdot v) \\ 0 &= 0 \cdot v \end{aligned}$$

■

**Theorem 1.2.12**
$a \cdot 0 = 0 \quad \forall a \in \mathbb{F}$.

***Proof 4.*** For $a \in \mathbb{F}$, we have

$$\begin{aligned} a \cdot 0 = a \cdot (0+0) &= a \cdot 0 + a \cdot 0 \\ a \cdot 0 + (-a \cdot 0) &= a \cdot 0 + a \cdot 0 + (-a \cdot 0) \\ 0 &= a \cdot 0. \end{aligned}$$

■

**Theorem 1.2.13**
$(-1)v = -v \quad \forall v \in V$.

***Proof 5.*** For $v \in V$, we have

$$v + (-1)v = 1 \cdot v + (-1) \cdot v = (1 + (-1)) \cdot v = 0 \cdot v = 0.$$

Therefore, by definition, $(-1)v = -v$. ■

**Notation 1.2.14.** $\mathbb{F}^S$

1. If $S$ is a set, then $\mathbb{F}^S$ denotes the set of functions from $S$ to $\mathbb{F}$.
2. For $f, g \in \mathbb{F}^S$, the <u>sum</u> $f + g \in \mathbb{F}^S$ is the function defined by $(f+g)(x) = f(x) + g(x) \quad \forall x \in S$.
3. For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the <u>product</u> $\lambda f \in \mathbb{F}^S$ is the function defined by $(\lambda f)(x) = \lambda f(x) \quad \forall x \in S$.

**Theorem 1.2.15**
$\mathbb{F}^S$ is a vector space.

## 1.3 Subspace

**Definition 1.3.1 (Subspace).** A subset $U$ of $V$ is called a *subspace* of $V$ if $U$ is also a vector space using the same addition and scalar multiplication as on $V$.

**Theorem 1.3.2 Conditions for a Subspace**
A subset $U$ of $V$ is a subspace of $V$ if and only if $U$ satisfies the following conditions:

1. additive identity: $0 \in U$;
2. closed under addition: $u, w \in U \implies u + w \in U$;
3. closed under scalar multiplication: $a \in \mathbb{F}$ and $u \in U \implies au \in U$.

***Proof 1.***
($\Rightarrow$) Suppose $U$ is a subspace of $V$. By definition, $U$ is then a vector space, and so those conditions are automatically satisfied. $\square$

($\Leftarrow$) Suppose $U$ satisfies the three conditions. Since $U$ is a subset of $V$, $U$ automatically has *associativity*, *commutativity*, *multiplicative identity*, and *distributivity*. So, we want to check $U$ has additive inverse and additive identities.

For additive identity, we know $0 \in U$, by assumption.

For additive inverse, by condition #3, we know $-u = (-1)u \in U$.

Then, $U$ is a vector space. ■

**Example 1.3.3** If $b \in \mathbb{F}$, then $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

***Proof 2.***
($\Rightarrow$) Suppose $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$. Then, $0 = (0, 0, 0, 0) \in U$. So, $0 = 5 \cdot 0 + b$, or $b = 0$. $\square$

($\Leftarrow$) Suppose $b = 0$. Then, $x_3 = 5x_4$. So, $U = \{(x_1, x_2, 5x_4, x_4) \in \mathbb{F}^4\}$

1. $0 = (0, 0, 0, 0) \in U$
2. Note that
$$(x_1, x_2, 5x_4, x_4) + (y_1, y_2, 5y_4, y_4) = (x_1 + y_1, x_2 + y_2, 5(x_4 + y_4), x_4 + y_4) \in U$$
So, addition is closed under $U$.
3. $\forall a \in \mathbb{F}$, we have
$$a(x_1, x_2, 5x_4, x_4) = (ax_1, ax_2, 5(ax_4), ax_4) \in U$$
Then, $U$ is a subspace of $\mathbb{F}^4$.

■

**Example 1.3.4** The set of continuous real-valued functions on interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

***Proof 3.***

1. $0$ (zero mapping)$\in U$
2. Set $f$ and $g \in \mathcal{C}[0,1]$, the set of continuous functions on interval $[0,1]$. Then, $f+g \in \mathcal{C}[0,1]$.
3. From Calculus, we know that $\forall a \in \mathbb{F}, \quad af \in \mathcal{C}[0,1]$.

■

**Definition 1.3.5 (Sum of Subspaces).** Suppose $U_1, \cdots, U_m$ are subspaces of $V$. The *sum* of $U_1, \cdots, U_m$, denoted as $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \cdots, U_m$:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i \quad \forall i = 1, \cdots, m\}.$$

**Example 1.3.6** Suppose $U = \{(x,0,0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}$ and $W = \{(0,y,0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}$, then

$$U + W = \{(x,y,0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}.$$

**Theorem 1.3.7**
Suppose $U_1, \cdots, U_m$ are subspaces of $V$. Then, $U_1 + \cdots + U_m$ is the *smallest subspace* of $V$ containing $U_1, \cdots, U_m$.

***Proof 4.*** Suppose $U_1, \cdots, U_m$ are subspaces of $U$. Let $U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_j \in U_j, j = 1, \cdots m\}$. Suppose $w_j \in U_j$, then $w_1 + \cdots + w_m \in U_1 + \cdots + U_m$.

1. $U_1 + \cdots + U_m$ is a subspace of $V$.
   (a) Note that
   $$(u_1 + \cdots + u_m) + (w_1 + \cdots + w_m) = (u_1 + w_1) + \cdots + (u_m + w_m) \in U_1 + \cdots + U_m,$$
   so $U_1 + \cdots + U_m$ is closed under addition.
   (b) Similarly, $U_1 + \cdots + U_m$ is closed under scalar multiplication.
   (c) Note that $U_j$ is a subspace, so $0 \in U_j$. Hence, $(0, \cdots, 0) = 0 \in U_1 + \cdots + U_m$. □
2. Now, we want to show this subspace is the smallest subspace containing $U_1, \cdots, U_m$. That is, we want to show $\forall\, W \supseteq U_1 \cup \cdots \cup U_m$, we have $W \supseteq U_1 + \cdots + U_m$.

   Note that $U_j \subseteq U_1 + \cdots + U_m$, so we have $(U_1 \cup U_2 \cup \cdots \cup U_m) \subseteq U_1 + \cdots + U_m$. This means $U_1 + \cdots + U_m$ must contain $U_1, \cdots, U_m$. Let $W$ be some subspace containing $U_1, \cdots, U_m$. Then, for $j = 1, \cdots, m$, we have $u_j \in U_j$, which indicates $u_j \in W$. Therefore, $u_1 + \cdots + u_m \in V$ and thus $U_1 + \cdots + U_m \subseteq W$.

   Since $W$ was arbitrary, we've shown $\forall\, W$ that contains $U_1, \cdots, U_m$, $U_1 + \cdots + U_m \subseteq W$. Therefore, $U_1 + \cdots + U_m$ is the smallest.

■

**Definition 1.3.8 (Direct Sum).** Suppose $U_1, \cdots, U_m$ are subspaces of $V$. $U_1 + \cdots + U_m$ is called a *direct sum* if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where $u_j \in U_j$.

**Notation 1.3.9.** If $U_1 + \cdots + U_m$ is a direct sum, then we use $U_1 \oplus \cdots \oplus U_m$ to denote it.

**Example 1.3.10** Let $U = \left\{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\right\}$ and $W = \left\{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\right\}$. Then, $\mathbb{F}^3 = U \oplus W$.

***Proof 5.*** Note that $U + W = \{(x, y, z) \mid x, y, z \in \mathbb{F}\} = \mathbb{F}^3$. Suppose

$$(x, y, z) = (x, y, 0) + (0, 0, z), \tag{1}$$

for some $x, y, z \in \mathbb{F}$ and

$$(x, y, z) = (x', y', 0) + (0, 0, z') \tag{2}$$

for some $x', y', z' \in \mathbb{F}$. Then, (1)−(2):

$$(0, 0, 0) = (x - x', y - y', 0) + (0, 0, z - z') = (x - x', y - y', z - z').$$

Then, $x - x' = y - y' = z - z' = 0$, which indicates $x = x',\ y = y',\ z = z'$. So, by definition $U + W$ is a direct sum, or $\mathbb{F}^3 = U \oplus W$. ∎

**Example 1.3.11** Suppose $U_j$ is the subspace of $\mathbb{F}^n$ *s.t.*

$$\begin{aligned} U_1 &= \{x, 0, 0, \cdots, 0 \mid x \in \mathbb{F}\} \\ U_2 &= \{0, x, 0, \cdots, 0 \mid x \in \mathbb{F}\} \\ &\vdots \\ U_n &= \{0, 0, 0, \cdots, x \mid x \in \mathbb{F}\} \end{aligned}$$

Then, $\mathbb{F}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_n$.

***Proof 6.*** Note that $\mathbb{F}^n = U_1 + U_2 + \cdots + U_n$ is evident. Now, we'll prove that $U_1 + U_2 + \cdots + U_n$ is a direct sum. Consider $x = (x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$. Assume that

$$x = (x_1, 0, \cdots, 0) + \cdots + (0, \cdots, 0, x_n) \tag{3}$$

and

$$x = (x_1', 0, \cdots, 0) + \cdots + (0, \cdots, 0, x_n') \tag{4}$$

Then, from (3)-(4), we know that

$$0 = (x_1 - x_1', \cdots, x_n - x_n') = (0, 0, \cdots, 0).$$

Then, $\forall i = 1, \cdots, n$ we have $x_i - x_i' = 0$, or $x_i = x_i'$. Therefore, by definition, we know $U_1 + \cdots + U_n$ is a direct sum. ∎

**Example 1.3.12** Let

$$U_1 = \{(x, y, 0) \mid x, y \in \mathbb{F}\}$$
$$U_2 = \{(0, 0, z) \mid z \in \mathbb{F}\}$$
$$U_3 = \{(0, y, y) \mid y \in \mathbb{F}\}$$

Show that $U_1 + U_2 + U_3$ is not a direct sum.

***Proof 7.*** Consider $(0, 0, 0) \in \mathbb{F}^3$. Note that

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$$

and

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1).$$

Then, $U_1 + U_2 + U_3$ is not a direct sum by definition. ■

**Theorem 1.3.13**
Suppose $U_1, \cdots, U_m$ are subspaces of $V$. Then,$U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write $0$ as a sum $u_1 + \cdots + u_m$ is by taking each $u_j = 0$.

***Proof 8.***
($\Rightarrow$) Since $U_1 + \cdots + U_m$ is a direct sum, by definition, the only way to write $0 \in \mathbb{F}^n$ is to write it as

$$0 = 0 + \cdots + 0 \qquad \text{where } 0 \in U_i \forall i = 1, \cdots, m. \qquad \square$$

($\Leftarrow$) Suppose the only way to write $0$ as a sum $u_1 + \cdots + u_m$ is by taking each $u_j = 0$. Assume that for some $v \in V$, we have

$$v = u_1 + \cdots + u_m, \quad u_j \in U_j \tag{5}$$

and

$$v = u_1' + \cdots + u_m', \quad u_j' \in U_j. \tag{6}$$

Then, by (5)-(6), and according to the conclusion from Example 1.3.11, we have

$$0 = (u_1 - u_1') + \cdots + (u_m - u_m') = 0 + \cdots + 0.$$

So, $\forall i \in 1, \cdots, m$, we have $u_i - u_i' = 0$. that is, $u_i = u_i'$. So, $\forall v \in V$, there is only one way to write $v$ as a sum of $u_1 + \cdots_u + m$. Therefore, by definition, $U_1 + \cdots + U_m$ is a direct sum. ■

**Theorem 1.3.14**
Suppose $U$ amd $W$ are subspaces of $V$. Then, $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

***Proof 9.***
($\Rightarrow$) Suppose $U + W$ is a direct sum. Assume $v \in U \cap W$. Then, $v \in U$ and $v \in W$. By definition of subspace, we know $-v \in W$ as well. Note that

$$0 = v + (-v) \in U \cap W.$$

Then, by Theorem 1.3.13, we know that the only representation of $0 \in U \cap W$ is $0 = 0 + 0$ since $U \cap W$

is a direct sum. Hence, it must be that $v = -v = 0$, and thus $U \cap W = \{0\}$. $\square$

($\Leftarrow$) Suppose $U \cap W = \{0\}$. Let $u \in U$ and $w \in W$ *s.t.* $u + w = 0$. Then, we have $u = -w$. Since $-w \in W$, we know $u = -w \in W$. By $u \in U$ and $u \in W$, we know that $u \in U \cap W = \{0\}$. Therefore, $0 = 0 + 0$ is the only to represent $0 \in U + W$. By Theorem 1.3.13, we know $U + W$ is a direct sum. $\blacksquare$

**Remark.** *When extending Theorem 1.3.14 to 3 subspaces* $U_1, U_2, U_3$, *we cannot conclude* $U_1 \oplus U_2 \oplus U_3$ *if we have* $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$. *See Example 1.3.12 as a counterexample.*

# 2 Finite-Dimensional Vector Spaces

## 2.1 Span and Linear Independence

**Notation 2.1.1.** We usually write list of vectors without using parentheses.

**Example 2.1.2** $(4, 1, 6), (9, 5, 7)$ is a list of vectors of length $2$ in $\mathbb{R}^3$.

**Definition 2.1.3 (Linear Combination).** A *linear combination* of a list $v_1, \cdots, v_m$ of vectors in $V$ is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \cdots, a_m \in \mathbb{F}$.

**Example 2.1.4** Since $(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4)$, we say $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$.

**Definition 2.1.5 (Span).**

$$\operatorname{span}(v_1, \cdots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1 \cdots a_m \in \mathbb{F}\}.$$

**Example 2.1.6** Consider $\operatorname{span}(e_1, e_2, e_3)$ :

$$\begin{aligned} \operatorname{span}(e_1, e_2, e_3) &= \{a_1 e_1 + a_2 e_2 + a_3 e_3 \mid a_1, a_2, a_3 \in \mathbb{F}\} \\ &= \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{F}\} = \mathbb{R}^3. \end{aligned}$$

**Theorem 2.1.7**
The span of a list of vectors in $V$ is the smallest subspace of $V$ containing all the vectors in the list.

***Proof 1.*** To prove this theorem, we will prove two parts: span is a subspace and span is the smallest subspace.

1. Span is a subspace of $V$.
    (a) By definition of span, we know $\operatorname{span}(v_1, \cdots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \cdots, a_m \in \mathbb{F}\}$. If we set $a_1, \cdots, a_m = 0$, then we have $0 = 0v_1 + \cdots + 0v_m$. So, $0 \in \operatorname{span} v_1, \cdots, v_m$.
    (b) Let $a_1 v_1 + \cdots + a_m v_m \in \operatorname{span}(v_1, \cdots, v_m)$ and $b_1 v_1 + \cdots + b_m v_m \in \operatorname{span}(v_1, \cdots, v_m)$. Then,

    $$(a_1 v_1 + \cdots + a_m v_m) + (b_1 v_1 + \cdots + b_m v_m) = (a_1 + b_1) v_1 + \cdots + (a_m + b_m) v_m.$$

    Since $(a_1 + b_1), \cdots, (a_m + b_m) \in \mathbb{F}$, we know $(a_1 + b_1) v_1 + \cdots + (a_m + b_m) v_m \in \operatorname{span}(v_1, \cdots, v_m)$.
    (c) Let $\lambda \in \mathbb{F}$ and $a_1 v_1 + \cdots + a_m v_m \in \operatorname{span}(v_1, \cdots, v_m)$. Then,

    $$\lambda(a_1 v_1 + \cdots + a_m v_m) = \lambda a_1 v_1 + \cdots + \lambda a_m v_m.$$

Since $\lambda a_1, \cdots, \lambda a_m \in \mathbb{F}$, we know that $\lambda(a_1 v_1 + \cdots + a_m v_m) \in \operatorname{span}(v_1, \cdots, v_m)$.

Therefore, we have proven that span is a subspace of $V$. $\square$

2. Now, we want to show that span is the smallest subspace.

   Let $U$ be a subspace of $V$ containing $v_1, \cdots, v_m$. If we can show that $\operatorname{span}(v_1, \cdots, v_m) \subseteq U$, we then know span is the smallest subspace containing $v_1, \cdots, v_m$. Since $U$ is a subspace containing $v_1, \cdots, v_m$, it is closed under addition and scalar multiplication. So, $a_1 v_1 + \cdots + a_m v_m \in \operatorname{span}(v_1, \cdots, v_m)$. Therefore, $\operatorname{span}(v_1, \cdots, v_m) \subseteq U$.

$\blacksquare$

**Definition 2.1.8 (Span as a Verb).** If $\operatorname{span}(v_1, \cdots, v_m) = V$, we say $v_1, \cdots, v_m$ *spans* $V$.
**Definition 2.1.9 (Finite-Dimensional Vector Space).** A vector space $V$ is called *finite-dimensional* if $\exists$ a list of vectors, say $v_1, \cdots, v_m$ *s.t.* $\operatorname{span}(v_1, \cdots, v_m) = V$. In the following of this notes, we will use *f-d* as a shortcut for saying "finite-dimensional."
**Definition 2.1.10 (Infinte-Dimensional Vector Space).** A vector space $V$ is infinite-dimensional if it is not *f-d*. This is equivalent to say that $\forall$ lists of vectors in $V$, they do not span $V$.
**Definition 2.1.11 (Polynomial Functions).** A function $p : \mathbb{F} \to \mathbb{F}$ is called a *polynomial* with coefficients in $\mathbb{F}$ if $\exists\, a_0, \cdots, a_m \in \mathbb{F}$ *s.t.* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \quad \forall z \in \mathbb{F}$.
**Notation 2.1.12.** We use $\mathcal{P}(\mathbb{F})$ to denote the set of all polynomial with coefficients in $\mathbb{F}$.

**Theorem 2.1.13**
$\mathcal{P}(\mathbb{F})$ is a vector space over $\mathbb{F}$.

***Proof 2.*** Recall the definition of $\mathbb{F}^{\mathbb{F}}$. We will show $\mathcal{P}(\mathbb{F})$ is a subspace of $\mathbb{F}^{\mathbb{F}}$.

1. $0 = 0 + 0z + \cdots + 0z^m \in \mathcal{P}(\mathbb{F})$.

2. Suppose $p(z) = a_m z^m + \cdots + a_1 z + a_0$ and $q(z) = b_n z^n + \cdots + b_1 z + b_0 \in \mathcal{P}(\mathbb{F})$. WLOG, suppose $m > n$, then we have $p(z) + q(z) = a_m z^m + \cdots + (a_n + b_n) z^n + \cdots + (a_0 + b_0) \in \mathcal{P}(\mathbb{F})$.

3. Suppose $\lambda \in \mathbb{F}$. Then, $\lambda p(z) = \lambda(a_m z^m + \cdots + a_1 z + a_0) = \lambda a_m z^m + \cdots + \lambda a_0 \in \mathcal{P}(\mathbb{F})$.

Hence, we've shown $\mathcal{P}(\mathbb{F})$ is a subspace over $\mathbb{F}$. $\blacksquare$
**Definition 2.1.14 (Degree of a Polynomial).** A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have *degree* $m$ if $\exists$ scalars $a_0, \cdots, a_m \in \mathbb{F}$ with $a_m \neq 0$ *s.t.* $p(z) = a_m z^m + \cdots + a_1 z + a_0 \quad \forall z \in \mathbb{F}$. We write $\deg p = m$. Specially, $\deg 0 \coloneqq -\infty$ and $\deg a_0 \coloneqq 0$ when $a_0 \neq 0$.
**Definition 2.1.15 ($\mathcal{P}_m(\mathbb{F})$).** For $m \in \mathbb{N}^+$, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomial with coefficients in $\mathbb{F}$ and degree$\leq m$. i.e.,

$$\mathcal{P}_m(\mathbb{F}) \coloneqq \{p \in \mathcal{P}(\mathbb{F}) \mid \deg p \leq m\}.$$

**Example 2.1.16** For each $m \in \mathbb{N}$, $\mathcal{P}_m(\mathbb{F})$ is a *f-d* vector space.
***Proof 3.*** Note that $\mathcal{P}_m(\mathbb{F})$ is a vector space because it is a subspace of $\mathcal{P}(\mathbb{F})$. Suppose $p(z) \in \mathcal{P}_m(\mathbb{F})$, then $p(z) = a_0 + a_1 z + \cdots + a_m z^m \in \operatorname{span}(1, z, \cdots, z^m)$. Then, by definition, $\mathcal{P}_m(\mathbb{F})$ is *f-d*. $\blacksquare$

**Remark.** *In this proof, we are abusing notation by letting $z^k$ to denote a function.*

**Example 2.1.17** $\mathcal{P}(\mathbb{F})$ is infinite-dimensional.

***Proof 4.*** For any list of vectors in $\mathcal{P}(\mathbb{F})$, by definition of list, the length of it is finite. Suppose the highest degree in this list is $m$. Consider a polynomial with degree of $m+1:\ z^{m+1}$. Since $z^{m+1}$ cannot be written as linear combinations of the list of polynomials, we know the list does not span $\mathcal{P}(\mathbb{F})$. So, $\mathcal{P}(\mathbb{F})$ is infinite-dimensional. ∎

**Definition 2.1.18 (Linear Independence).** A list $v_1, \cdots, v_m$ of vectors in $V$ is called *linearly independent* (L.I.) if the only choice of $a_1, \cdots, a_m \in \mathbb{F}$ that makes $a_1v_1+\cdots+a_mv_m = 0$ is $a_1 = \cdots = a_m = 0$. Specially, the empty list () is declared to be L.I..

**Definition 2.1.19 (Linear Dependence).** $v_1, \cdots, v_m$ is called *linearly dependent* if it is not L.I.. Or, equivalently, $v_1, \cdots, v_m$ is *linearly dependent* if $\exists\, a_1, \cdots, a_m \in \mathbb{F}$ not all $0$ *s.t.* $\sum_{i=0}^{m} a_iv_i = 0$.

**Example 2.1.20** Let $v_1, \cdots, v_m \in V$. If $v_j$ is a linear combination of other $v$'s, then $v_1, \cdots, v_m$ is linearly dependent.

***Proof 5.*** By assumption, $v_j = a_1v_1 + \cdots + a_{j-1}v_{j-1} + a_{j+1}v_{j+a} + \cdots + a_mv_m$ for some $a_i$ not all $0$. So, $0 = a_1v_1 + \cdots + a_{j-1}v_{j-1} + a_{j+1}v_{j+1} + \cdots + a_mv_m - v_j$, a linear combination of $v_1, \cdots, v_m$. Since $-v_j$ has a coefficient of $-1 \neq 0$, by definition, $v_1, \cdots, v_m$ is not L.I.. ∎

**Lemma 2.1.21 Linear Dependence Lemma** Suppose $v_1, \cdots, v_m$ is a linearly dependent list in $V$. Then, $\exists\, j \in \{1, \cdots, m\}$ *s.t.* the following hold:

1. $v_j \in \operatorname{span}(v_1, \cdots, v_{j-1})$
2. if the $j^{\text{th}}$ term is removed from $v_1, \cdots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \cdots, v_m)$.

***Proof 6.***

1. Since $v_1, \cdots, v_m$ is linearly dependent, $a_1v_1 + \cdots + a_mv_m = 0$, for some $a_i \neq 0$. Let $j$ be the maximized index *s.t.* $a_j \neq 0$. Then, $a_{j+1} = \cdots = a_m = 0$, by this assumption. Hence,

$$\begin{aligned} a_jv_j &= -a_1v_1 - \cdots - a_{j-1}v_{j-1} - a_{j+1}v_{j+1} - \cdots - a_mv_m \\ &= -a_1v_1 - \cdots - a_{j-1}v_{j-1} \\ v_j &= -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}. \end{aligned}$$

   Since $-\frac{a_1}{a_j}, \cdots, -\frac{a_{j-1}}{a_j} \in \mathbb{F}$, we know $v_j \in \operatorname{span}(v_1, \cdots, v_{j-1})$. □

2. Consider

$$\begin{aligned} \operatorname{span}(v_1, \cdots, v_j, \cdots, v_m) &= \operatorname{span}(v_1, \cdots, -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}, \cdots, v_m) \\ &= \operatorname{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m). \end{aligned}$$

∎

**Remark.** *By using this Lemma 2.1.21, we can do lots of proofs using the "step" strategy. Namely, we start to remove vectors from a list that are linearly dependent to obtain a* L.I. *list. However, this "step" strategy can only be used when dealing with FINITE-dimensional vector spaces.*

**Theorem 2.1.22**
Let $V$ be a *f-d* vector space. Let $\operatorname{span}(w_1, \cdots, w_n) = V$. Let $u_1, \cdots, u_m$ be L.I.. Then, $m \leq n$.

***Proof 7.***
$\boxed{\text{Step } 1}$ Note that $u_1, w_1, \cdots, w_n$ is linearly dependent because $u_1 \in V = \operatorname{span}(w_1, \cdots, w_n)$. Then, by Lemma 2.1.21, we can remove one of the $w$'s, say $w_{j1}$. Then, the list becomes

$$\{u_1, w_1, \cdots, w_n\} \setminus \{w_{j1}\}.$$

$\boxed{\text{Step } 2}$ Adjoin $u_2$. Apply the same reasoning, since $\operatorname{span}(\{u_1, w_1, \cdots, w_n\} \setminus \{w_{j1}\}) = V$, we know $\{u_1, u_2, w_1, \cdots, w_n\} \setminus \{w_{j1}\}$ is linearly dependent. Since $u_2 \notin \operatorname{span}(u_1)$, Lemma 2.1.21 is not applicable to $u_2$. Now, we can remove another $w$ from the list, say $w_{j2}$. The list becomes

$$\{u_1, u_2, w_1, \cdots, w_n\} \setminus \{w_{j1}, w_{j2}\}.$$

$\vdots$

$\boxed{\text{Step } m}$ After $m$ steps, we list will become

$$\{u_1, \cdots, u_m, w_1, \cdots, w_n\} \setminus \{w_{j1}, \cdots, w_{jm}\}.$$

Since $\operatorname{span}(\{u_1, \cdots, u_m, w_1, \cdots, w_n\} \setminus \{w_{j1}, \cdots, w_{jm}\}) = V$, this list is still linearly dependent, so by Lemma 2.1.21, we know $\exists\, w$ to be removed. Therefore, $n \geq m$. ■

**Theorem 2.1.23**
Every subspace of a *f-d* vector space is *f-d*.

***Proof 8.*** Suppose $V$ to be a *f-d* vector space and $U$ to be a subspace of $V$.
$\boxed{\text{Step } 1}$ If $U = \{0\}$, then $U$ is *f-d*. If $U \neq \{0\}$, then choose $v_i \in U$ ***s.t.*** $v_1 \neq 0$.

$\vdots$

$\boxed{\text{Step } j}$ If $U = \operatorname{span}(v_1, \cdots, v_{j-1})$, then $U$ is *f-d*. If $U \neq \operatorname{span}(v_1, \cdots, v_{j-1})$, then choose $v_j \in U$ ***s.t.*** $v_j \notin \operatorname{span}(v_1, \cdots, v_{j-1})$.

By Lemma 2.1.21 and Theorem 2.1.22, we know this process will eventually terminate because the vector list that spans $U$ cannot be longer than any spanning list of $V$. Therefore, $U$ is *f-d*. ■

## 2.2 Bases

**Definition 2.2.1 (Basis).** A *basis* of $V$ is a list of vectors in $V$ that is L.I. and spans $V$.

**Example 2.2.2**

1. The standard basis of $\mathbb{F}^n$:

$$(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1).$$

2. $(1, 1, 0), (0, 0, 1)$ is a basis of $V$, where $V = \{(x, x, y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$.

   ***Proof 1.***

   (a) Suppose $a_1(1, 1, 0) + a_2(0, 0, 1) = 0$, we have $(a_1, a_1, a_2) = 0$. So, it must be $a_1 = a_2 = 0$. Therefore, $(1, 1, 0), (0, 0, 1)$ is L.I.. □

   (b) Suppose $(x, x, y) \in V$. Note that $(x, x, y) = x(1, 1, 0) + y(0, 0, 1)$, then, $V = \text{span}((1, 1, 0), (0, 0, 1))$.

   Therefore, we've proven $(1, 1, 0), (0, 0, 1)$ is a basis of $V$ according to the definition of basis. ■

**Theorem 2.2.3 Criterion for Basis**
A list $v_1, \cdots, v_n \in V$ is a basis list of $V$ if and only if every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \cdots + a_nv_n$, where $a_i \in \mathbb{F}$.

***Proof 2.***
($\Rightarrow$) Let $v_1, \cdots, v_n$ be a basis of $V$. Let $v \in V$. By definition of basis, $V = \text{span}(v_1, \cdots, v_n)$. So, $v \in \text{span}(v_1, \cdots, v_n)$, and thus $v = a_1v_1 + \cdots + a_nv_n$ for some $a_i \in \mathbb{F}$. Assume for the sake of contradiction that $v = b_1v_1 + \cdots + b_nv_n$ for some $b_i \neq a_i \in \mathbb{F}$. Then,

$$\begin{aligned} v - v &= (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n \\ 0 &= (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n. \end{aligned}$$

Since $v_1, \cdots, v_n$ is a basis, it is L.I.. So, $0 = 0v_1 + \cdots + 0v_n$. Therefore, we know $a_1 - b_1 = \cdots = a_n - b_n = 0$. That is, $a_1 = b_1, \cdots, a_n = b_n$. ⁂ This is a contradiction with the assumption that $\exists\, a_i \neq b_i$. Hence, it must be that $v = a_1v_1 + \cdots + a_nv_n$ is unique. □

($\Leftarrow$) Suppose $v = a_1v_1 + \cdots + a_nv_n$ is the unique representation $\forall\, v \in V$. Then, $v \in \text{span}(v_1, \cdots, v_n)$. Since $v \in V$, then $V \subseteq \text{span}(v_1, \cdots, v_n)$. However, $v_1, \cdots, v_n \in V$, so $\text{span}(v_1, \cdots, v_n) \subseteq V$. Therefore, $\text{span}(v_1, \cdots, v_n) = V$. To show $v_1, \cdots, v_n$ is L.I., further consider $0 = a_1v_1 + \cdots + a_nv_n$. Since $0 \in V$, by assumption, $\exists$ a unique way to write $0$ as $a_1v_1 + \cdots + a_nv_n$, and that unique way is to take every $a_i = 0$. Hence, by definition, we know $v_1, \cdots, v_n$ is L.I.. Since $v_1, \cdots, v_n$ is L.I. and $\text{span}(v_1, \cdots, v_n) = V$, we know $v_1, \cdots, v_n$ is a basis list of $V$. ■

**Theorem 2.2.4**
Every spanning list can be reduced to a basis of the vector space.

***Proof 3.*** Suppose $V = \text{span}(v_1, \cdots, v_n)$. If $v_i = 0$, we just remove $v_i$. So, let's suppose $v_i \neq 0$.

$\boxed{\text{Step } 1}$ If $v_2 \in \operatorname{span}(v_1)$, delete it. If $v_2 \notin \operatorname{span}(v_2)$, keep it.

$\vdots$

$\boxed{\text{Step } j}$ If $v_j \in \operatorname{span}(v_1, \cdots, v_{j-1})$, delete it. If $v_j \notin \operatorname{span}(v_1, \cdots, v_{j-1})$, keep it.

$\vdots$

$\boxed{\text{Step } n}$ After $n$ steps, we will have a "sub-list" from the original list *s.t.* it spans $V$ and is L.I.. Therefore, the basis list is contained in the spanning list. ■

**Corollary 2.2.5** Every *f-d* vector space has a basis.

***Proof 4.*** By definition, *f-d* vector space always has a spanning list. By Theorem 2.2.4, a spanning list contain a basis. ■

**Theorem 2.2.6**
Every linearly independent list of vectors in a *f-d* vector space can be extended to a basis of the vector space.

***Proof 5.*** Suppose $u_1, \cdots, u_m$ is L.I. in a *f-d* vector space of $V$. Let $w_1, \cdots, w_n$ be a basis of $V$. Then, $u_1, \cdots, u_m, w_1, \cdots, w_n$ spans $V$. According to Lemma 2.1.21 and Theorem 2.1.22, we can reduce $u_1, \cdots, u_m, w_1, \cdots, w_m$ to some list of $u_1, \cdots, u_m$ and some $w$'s. ■

**Theorem 2.2.7**
Suppose $V$ is *f-d* and $U$ is a subspace of $V$. Then, there is a subspace $W$ of $V$ *s.t.* $V = U \oplus W$.

***Proof 6.*** Since $V$ is *f-d*, $U$, as $V$'s subspace, is also *f-d*. So, $\exists$ a basis of $U$, say $u_1, \cdots, u_m$. Then, $u_1, \cdots, u_m$ is L.I. and $\in V$. By Theorem 2.2.6, this list can be extended to a basis

$$u_1, \cdots, u_m, w_1, \cdots, w_n \text{ of } V.$$

Let $W = \operatorname{span}(w_1, \cdots, w_n)$. We'll show $V = U \oplus W$.

1. WTS: $V = U + W$. Suppose $v \in V$. Then,

$$v = \underbrace{a_1u_1 + \cdots + a_mu_m}_{\in U} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{\in W}.$$

   So, $v \in U + W$, or $V = U + W$. □

2. WTS: $U \cap W = \{0\}$. Suppose $v \in U \cap W$. Then, $v \in U$ and $v \in W$. So,

$$v = a_1u_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_nw_n.$$

   Hence,

$$a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0. \tag{7}$$

   Since by assumption, $u_1, \cdots, u_m, w_1, \cdots, w_n$ is a basis of $V$, so $u_1, \cdots, u_m, w_1, \cdots, w_n$ is L.I.. Therefore, the only way for Equation (7) to hold is when $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Hence, $v = 0u_1 + \cdots + u_m = 0$. That is, $U \cap W = \{0\}$.

Therefore, we've shown that $V = U \oplus W$. ■

## 2.3 Dimension

**Theorem 2.3.1**
Let $B_1$ and $B_2$ be two bases of $V$, then $B_1$ and $B_2$ have the same length.

***Proof 1.*** Since $B_1$ is L.I. in $V$ and $B_2$ spans $V$, by Theorem 2.1.22, we know $\text{len}(B_1) \leq \text{len}(B_2)$. Interchanging the roles of $B_1$ and $B_2$, we have $\text{len}(B_2) \leq \text{len}(B_1)$. So, we have $\text{len}(B_1) = \text{len}(B_2)$. ■

**Definition 2.3.2 (Dimension).** The *dimension* of a *f-d* vector space $V$ is the length of any basis of $V$.

**Notation 2.3.3.** We use $\dim V$ to denote the dimension of a *f-d* vector space $V$.

**Example 2.3.4** $\dim \mathbb{F}^n = n$ and $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$ $(1, z, z^2, \cdots, z^m)$.

**Theorem 2.3.5**
If $V$ is *f-d* and $U$ is a subspace of $V$, then $\dim U \leq \dim V$.

***Proof 2.*** Let $B_1$ be a basis of $U$ and $B_2$ be a basis of $V$. Then, $B_1$ is a L.I. list of $V$ and $B_2$ spans $V$. Then, By Theorem 2.1.22, we know that $\text{len}(B_1) \leq \text{len}(B_2)$. So, by definition of dimension, we know $\dim U \leq \dim V$. ■

**Extension.** *If $V$ is* f-d *and $U$ is a subspace of $V$, given $U \subsetneq V$, then* $\dim U < \dim V$.

***Proof 3.*** Let $u_1, \cdots, u_m$ be a basis of $U$. Since $U \subsetneq V$, we know $V - U \neq \varnothing$. So, choose $v \in V - U$. Then, $v \notin \text{span}(u_1, \cdots, u_m)$. Therefore, $u_1, \cdots, u_m, v$ is L.I. in $V$. That is

$$\begin{aligned}\dim V &\geq \dim(\text{span}(u_1, \cdots, u_m, v)) \\ &> \dim(\text{span}(u_1, \cdots, u_m)) \\ &= \dim U.\end{aligned}$$

■

**Theorem 2.3.6**
Let $V$ be *f-d*, then every L.I. list of vectors in $V$ with length $\dim V$ is a basis of $V$.

***Proof 4.*** Let $v_1, \cdots, v_n \in V$ be L.I.. Let $n = \dim V$. When extending the list to basis, we get

$$\{v_1, m \cdots, v_n\} \cup \varnothing$$

as a basis of $V$. That is, $v_1, \cdots, v_n$ has already been a basis of $V$. ■

**Remark.** *The proof given above is not that straight-forward, so we are giving an easier-understanding proof as follows.*

***Proof 5.*** Suppose for the sake of contradiction that $\exists v_1, \cdots, v_n \in V$ not a basis of $V$ for $n = \dim V$. Then, $\text{span}(v_1, \cdots, v_n) \neq V$. That is, $\exists$ $v_{n+1}$ *s.t.* $v_{n+1} \notin \text{span}(v_1, \cdots, v_n)$. Adding $v_{n+1}$ to the vector list, we have $v_1, \cdots, v_n, v_{n+1}$ is L.I.. By Theorem 2.3.5, we know $\text{len}(v_1, \cdots, v_{n+1}) = n + 1 \leq \dim V$. ⋇ This contradicts with the fact that $\dim V = n < n + 1$. So, our assumption is incorrect, and it must be that $v_1, \cdots, v_n$ is a basis of $V$. ■

**Theorem 2.3.7**
Suppose $V$ is *f-d*. Then, every spanning list of vectors in $V$ with length $\dim V$ is a basis of $V$.

**Example 2.3.8** Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace $U$ of $\mathcal{P}_3(\mathbb{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) \mid p'(5) = 0\}.$$

***Proof 6.*** Consider $a_1 + a_2(x-5)^2 + a_3(x-5)^3 = 0$, we will get $a_1 = a_2 = a_3 = 0$ easily from the equation. Then, $1, (x-5)^2, (x-5)^3$ is L.I.. So, by Theorem 2.3.5, we know $\dim U \geq 3$. Since $U \subsetneq \mathcal{P}_3(\mathbb{R})$, we have $\dim U < \dim \mathcal{P}_3(\mathbb{R}) = 4$. Therefore, $\dim U = 3 = \operatorname{len}(1, (x-5)^2, (x-5)^3)$. By Theorem 2.3.6, we know $1, (x-5)^2, (x-5)^3$ is a basis of $U$. ■

**Theorem 2.3.9**
If $U_1$ and $U_2$ are subspaces of a *f-d* vector space, then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

***Proof 7.*** Let $u_1, \cdots, u_m$ be a basis of $U_1 \cap U_2$, then $\dim(U_1 \cap U_2) = m$. Also, $u_1, \cdots, u_m$ is L.I. in $U_1$, so we can extend it to a basis of $U_1$ as $u_1, \cdots, u_m, v_1, \cdots, v_j$. Then, $\dim(U_1) = m + j$. Similarly, extending $u_1, \cdots, u_m$ to a basis of $U_2$, we will get $u_1, \cdots, u_m, w_1, \cdots, w_k$. So, $\dim(U_2) = m + k$. Now, we want to show $u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k$ is a basis of $U_1 + U_2$.

1. Since $U_1, U_2 \subseteq \operatorname{span}(u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k)$, we know that

$$\operatorname{span}(u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k) = U_1 + U_2. \qquad \square$$

2. Suppose $a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k = 0$. Then we know that

$$c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j.$$

Since $c_1w_1 + \cdots + c_kw_k \in U_2$, and $-a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j \in U_1$, we know that $c_1w_1 + \cdots + c_kw_k \in U_1 \cap U_2$. Therefore, $c_1w_1 + \cdots + c_kw_k = d_1u_1 + \cdots + d_mu_m$. Since $u_1, \cdots, u_m, w_1, \cdots, w_k$ is L.I., we know $c_1 = \cdots = c_k = 0$. So, $-a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j = 0$. Since $u_1, \cdots, u_m, v_1, \cdots, v_j$ is L.I., we have $a_1 = \cdots = a_m = b_1 = \cdots = b_j = 0$. Therefore, we've proven $u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k$ is L.I. and thus is a basis of $U_1 + U_2$. $\square$

Since $u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k$ is a basis of $U_1 + U_2$, we know $\dim(U_1 + U_2) = m + j + k$. Further note that

$$\begin{aligned}
\dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) &= (m + j) + (m + k) - m \\
&= m + j + k \\
&= \dim(U_1 + U_2).
\end{aligned}$$

■

# 3 Linear Maps

**Notation 3.0.1.** In this section, we use $V$ and $W$ to denote vector spaces over $\mathbb{F}$.

## 3.1 The Vector Space of Linear Maps

**Definition 3.1.1 (Linear Map).** A *linear map* from $V$ to $W$ is a function $T : V \to W$ with the following properties:

- additivity: $T(u+v) = Tu + Tv \qquad \forall u, v \in V$.
- homogeneity: $T(\lambda v) = \lambda(Tv) \qquad \forall \lambda \in \mathbb{F}$ and $\forall v \in V$.

**Notation 3.1.2.** The set of all linear maps from $V$ to $W$ is denoted by $\mathcal{L}(V, W)$.

**Example 3.1.3**

1. Zero-mapping: $0 \in \mathcal{L}(V, W)$ is defined by $0v = 0$.
2. Identity-mapping: $I \in \mathcal{L}(V, V)$ is defined by $Iv = v$.
3. Differentiation: $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is defined by $Dp = p'$.

   ***Proof 1.*** Note that $(f+g)' = f' + g'$ and $(\lambda f)' = \lambda f'$. ■
4. Integration: $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ is defined by $Tp = \displaystyle\int_0^1 p(x)\,\mathrm{d}x$

   ***Proof 2.*** Note that $\displaystyle\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g$ and $\displaystyle\int_0^1 \lambda f = \lambda \int_0^1 f$. ■
5. Backward shift: $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ as $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots)$.

   ***Proof 3.*** Note that

   $$\begin{aligned} T(x_1, x_2, x_3, \cdots) + T(y_1, y_2, y_3, \cdots) &= (x_2, x_3, \cdots) + (y_2, y_3, \cdots) \\ &= (x_2 + y_2, x_3 + y_3, \cdots) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots). \end{aligned}$$

   Therefore, $T$ is additive. Homogeneity of $T$ is travial and thus omitted here. ■
6. From $\mathbb{F}^n$ to $\mathbb{F}^m$, we define $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ as

   $$T(x_1, \cdots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n),$$

   where $A_{j,k} \in \mathbb{F} \quad \forall j = 1, \cdots, m$ and $k = 1, \cdots, n$.

**Theorem 3.1.4**
Suppose $v_1, \cdots, v_n$ is a basis of $V$ and $w_1, \cdots, w_n \in W$. Then, $\exists$ a unique linear map $T : V \to W$ *s.t.* $Tv_j = w_j \quad \forall j = 1, \cdots, n$.

**Remark.** *If $T$ in Theorem 3.1.1 is a linear mapping, we should have*

1. $T(v_1 + \cdots + v_n) = Tv_1 + \cdots + Tv_n = w_1 + \cdots + w_n$, *by additivity of $T$, and*
2. $T(\lambda_j v_j) = \lambda_j Tv_j$, *by homogeneity of $T$.*

*Combine the two properties, we should have*

$$T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 Tv_1 + \cdots = \lambda_n Tv_n = \lambda_1 w_1 + \cdots + \lambda_n w_n.$$

*This remark will be very helpful in our following proof of the theorem.*

***Proof 4.*** Let's define $T : V \to W$ by $T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$, where $c_1, \cdots, c_n$ are arbitrary elements of $\mathbb{F}$. Now, we want to show that $T$ is a linear mapping.

Suppose $u, v \in V$, $u = a_1 v_1 + \cdots + a_n v_n$, and $v = c_1 v_1 + \cdots + c_n v_n$. Then, we have

$$\begin{aligned} T(u+v) &= T((a_1+c_1)v_1 + \cdots + (a_n+c_n)v_n) \\ &= (a_1+c_1)w_1 + \cdots + (a_n+c_n)w_n \\ &= (a_1 w_1 + \cdots + a_n w_n) + (c_1 w_1 + \cdots + c_n w_n) \\ &= Tu + Tv. \qquad \square \end{aligned}$$

Now, we want to show $T$ has homogeneity. Suppose $\lambda \in \mathbb{F}$. Then, we know

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1 v_1 + \cdots + \lambda c_n v_n) \\ &= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\ &= \lambda(c_1 w_1 + \cdots + c_n w_n) \\ &= \lambda Tv. \qquad \square \end{aligned}$$

Also, we want to show that this $T$ satisfy the condition the theorem is asking (i.e., $Tv_j = w_j$). Note that when $c_j = 0$ and other $c$'s equal $0$, we will get $Tv_j = w_j$. $\square$

Finally, we will prove the uniqueness of this $T$. Suppose that $T' \in \mathcal{L}(V, W)$ and $T'v_j = w_j$. Let $c_1, \cdots, c_n \in \mathbb{F}$. Then, $T'(c_j v_j) = c_j w_j$. So, we know that $T'(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$. However, by definition, we know $c_1 w_1 + \cdots + c_n w_n = T(c_1 w_1 + \cdots + c_n v_n)$. So, we can conclude that $T'(c_1 v_1 + \cdots + c_n v_n) = T(c_1 w_1 + \cdots + c_n v_n)$. Thus, $T' = T$, and thus the $T$ we defined above is unique in $\mathcal{L}(V, W)$. ■

**Definition 3.1.5 (Addition and Scalar Multiplication on $\mathcal{L}(V, W)$).** Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. Then, the *addition* is defined as $(S+T)(v) := Sv + Tv$, and the *scalar multiplication* is defined as $(\lambda T)(v) := \lambda(Tv) \quad \forall v \in V$.

**Theorem 3.1.6**
$\mathcal{L}(V, W)$ is a vector space.

***Proof 5.***

1. additive identity: Note that the zero-mapping $0 \in \mathcal{L}(V, W)$ satisfies the following equation:

$$(0+T)(v) = 0v + Tv = 0 + Tv = Tv. \qquad \square$$

2. commutativity: Note that

$$(S+T)(v) = Sv + Tv = Tv + Sv = (T+S)(v). \qquad \square$$

3. associativity: Let $S, T, R \in \mathcal{L}(V, W)$. Then,

$$\begin{aligned}((S+T)+R)(v) = (S+T)(v) + Rv &= Sv + Tv + Rv \\ &= Sv + (Tv + Rv) \\ &= Sv + (T+R)(v) \\ &= (S + (T+R))(v).\end{aligned}$$

Let $a, b \in \mathbb{F}$. Then,

$$((ab)T)(v) = T(abv) = T(a(bv)) = aT(bv) = (a(bT))(v). \qquad \square$$

4. multiplicative identity: Note we have $1 \in \mathbb{F}$ *s.t.*

$$(1 \cdot T)(v) = T(1 \cdot v) = Tv. \qquad \square$$

5. additive inverse: Note that

$$(T + (-T))(v) = Tv + (-T)(v) = Tv + T(-v) = T(v - v) = T0 = 0. \qquad \square$$

6. distributivity: Note that

$$a(T+S)(v) = a(Tv + Sv) = aTv + aSv,$$

and

$$(a+b)Tv = T((a+b)v) = T(av + bv) = T(av) + T(bv) = aTv + bTv.$$

■

**Definition 3.1.7 (Product of Linear Maps).** If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by $(ST)(u) = S(Tu) \quad \forall u \in U$.

**Remark.** *Compare this definition with composite functions. $ST$ is only defined when $T$ maps into the domain of $S$.*

**Theorem 3.1.8 Algebraic Properties of Products of Linear Maps**

1. associativity: $(T_1T_2)T_3 = T_1(T_2T_3)$.
2. identity: $TI = IT = T$, where $I$ is the identity mapping
3. distributive properties: $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$.

***Proof 6.*** First, we want to show the associativity. Note that

$$[(T_1T_2)T_3](v) = (T_1T_2)(T_3v) = (T_1)(T_2(T_3v)) = (T_1)[(T_2T_3)(v)]. \qquad \square$$

Then, we want to show the identity. This proof can be done using the following diagram:

$$
\begin{tikzcd}
V \arrow[r, "T"] & W \arrow[d, "I_W"] \\
V \arrow[u, "I_V"] & W
\end{tikzcd}
\qquad \square
$$

Finally, we will show the distributive properties. Note that

$$
\begin{aligned}
[(S_1+S_2)T](v) = (S_1+S_2)(Tv) &= S_1(Tv)+S_2(Tv) \\
&= (S_1T)(v)+(S_2T)(v) \\
&= (S_1T+S_2T)(v).
\end{aligned}
$$

Similarly, we can show

$$
\begin{aligned}
[S(T_1+T_2)](v) = S[(T_1+T_2)(v)] &= S(T_1v+T_2v) \\
&= S(T_1v)+S(T_2v) \\
&= (ST_1)(v)+(ST_2)(v) \\
&= (ST_1+ST_2)(v).
\end{aligned}
$$

■

**Example 3.1.9** Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map, and $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be defined by $(Tp)(x) = x^2p(x)$. Show that $DT \neq TD$.

***Proof 7.*** Note that $(DT)p = D(Tp) = D(x^2p(x)) = 2xp(x) + x^2p'(x)$. Similarly, we can compute a general formula for $TD$: $(TD)p = T(Dp) = T(p') = x^2p'(x)$. Since $2xp(x)+x^2p'(x) \neq x^2p'(x)$, we know $DT \neq TD$. ■

**Theorem 3.1.10**
Let $T \in \mathcal{L}(V, W)$, then $T(0) = 0$.

***Proof 8.*** Since $T(0) = T(0+0) = T(0)+T(0)$, we know $0 = T(0)$, or $T(0) = 0$. ■

**Corollary 3.1.11** If $T(0) \neq 0$, then $T \notin \mathcal{L}(V, W)$.

## 3.2 Null Spaces and Ranges

**Definition 3.2.1 (Null Space/Kernel).** For $T \in \mathcal{L}(V, W)$, the *null space* of $T$, denoted $\operatorname{null} T$, is the subset of $V$ consisting of those vectors that $T$ maps to 0: $\operatorname{null} T = \{v \in V \mid Tv = 0\}$.

**Remark.** *Sometimes, null space of $T$ is also called the kernal of $T$, denoted as $\ker T$.*

**Example 3.2.2**

1. Null space of zero-mapping: Let $T$ be the zero mapping from $V$ to $W$. Since $Tv = 0 \quad \forall v \in V$, we know $\operatorname{null} T = V$.
2. $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ as $Dp = p'$: $\operatorname{null} D = \{a \mid a \in \mathbb{R}\}$.
3. $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ as $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots)$: $\operatorname{null} T = \{(a, 0, 0, \cdots) \mid a \in \mathbb{F}\}$.

**Theorem 3.2.3**
Suppose $T \in \mathcal{L}(V, W)$. Then, $\operatorname{null} T$ is a subspace of $V$.

***Proof 1.***

1. Note that $T(0) = 0$, so $0 \in \operatorname{null} T$. □
2. Suppose $u, v \in \operatorname{null} T$. Then, $Tu = Tv = 0$. So, $T(u + v) = Tu + Tv = 0 + 0 = 0$. Hence, $u + v \in \operatorname{null} T$. □
3. Suppose $u \in \operatorname{null} T$ and $\lambda \in \mathbb{F}$. Then, $Tu = 0$. So, $T(\lambda u) = \lambda Tu = \lambda \cdot 0 = 0$. Therefore, $\lambda u \in \operatorname{null} T$.

■

**Definition 3.2.4 (Injective/Injection).** A function $T : V \to W$ is called *injective* of $Tu = Tv$ implies $u = v$.

**Remark.** *Sometimes, the contrapositive will be much more helpful: $T$ is injective if $u \neq v$, then $Tu \neq v$.*

**Theorem 3.2.5**
Let $T \in \mathcal{L}(V, W)$. Then, $T$ is injective if and only if $\operatorname{null} T = \{0\}$.

***Proof 2.***
($\Rightarrow$) Suppose $T$ is an injective. We've already known that $\{0\} \subseteq \operatorname{null} T$. Then, we need to show $\operatorname{null} T \subseteq \{0\}$. Suppose $v \in \operatorname{null} T$, then $Tv = 0$. However, since $T$ is an injection, and $Tv = T0 = 0$, then we have $v = 0$. So, $\operatorname{null} T \subseteq \{0\}$. Therefore, it's sufficient to say $\operatorname{null} T = \{0\}$. □

($\Leftarrow$) Suppose $\operatorname{null} T = \{0\}$. Suppose $u, v \in V$ and $Tu = Tv$. Then, $Tu - Tv = T(u - v) = 0$. Hence, $u - v \in \operatorname{null} T$. By $\operatorname{null} T = \{0\}$, we know $u - v = 0$, so $u = v$. Then, $T$ is an injection. ■

**Definition 3.2.6 (Range/Image).** For $T \in \mathcal{L}(V, W)$, the range of $T$ is the subset of $W$ consisting of those vectors that are of the form $Tv$ for some $v \in V$: $\operatorname{range} T = \{Tv \mid v \in V\}$.

**Theorem 3.2.7**
If $T \in \mathcal{L}(V, W)$, then $\operatorname{range} T$ is a subspace of $W$.

***Proof 3.***

1. Since $T(0) = 0$, we know $0 \in \operatorname{range} T$. $\square$

2. Suppose $w_1, w_2 \in \operatorname{range} T$. Then, $\exists v_1, v_2 \in V$ *s.t.* $Tv_1 = w_1$ and $Tv_2 = w_2$. Then, $w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2)$. Since $v_1 + v_2 \in V$, we have $w_1 + w_2 = T(v_1 + v_2) \in \operatorname{range} T$. $\square$

3. Suppose $w \in \operatorname{range} T$ and $\lambda \in \mathbb{F}$. Then, $\exists v \in V$ *s.t.* $w = Tv$. So, $\lambda w = \lambda(Tv) = T(\lambda v)$. Since $\lambda v \in V$, $\lambda w = T(\lambda v) \in \operatorname{range} T$.

■

**Definition 3.2.8 (Surjective/Surjection).** A function $T : V \to W$ is called *surjective* if $\operatorname{range} T = W$.

**Remark.** *A function $T : V \to W$ is called a* bijection*, or is bijective, if it is both injective and surjective.*

**Theorem 3.2.9 Fundamental Theorem of Linear Maps**
Suppose $V$ is *f-d* and $T \in \mathcal{L}(V, W)$. Then, $\operatorname{range} T$ is *f-d* and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

***Proof 4.*** Let $u_1, \cdots, u_m$ be a basis of $\operatorname{null} T$. Then, $\dim \operatorname{null} T = m$. By Theorem 3.2.3, we know $\operatorname{null} T$ is a basis of $V$, so we can extend the basis to a basis of $V$: $u_1, \cdots, u_m, v_1, \cdots, v_n$. Thus, $\dim V = m + n$. *WTS:* $\dim \operatorname{range} T = n$. *Further WTS:* $Tv_1, \cdots, Tv_n$ *is a basis of* $\operatorname{range} T$.

Suppose $v \in V$. Then

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n.$$

Since $u_1, \cdots, u_m \in \operatorname{null} T$, we know $Tu_1, \cdots, Tu_m = 0$. Therefore,

$$Tv = a_1Tu_1 + \cdots + a_mTu_m + b_1Tv_1 + \cdots + b_nTv_n = b_1Tv_1 + \cdots + b_nTv_n.$$

Hence, $\operatorname{span}(Tv_1, \cdots, Tv_n) = \operatorname{range} T$, and thus range $T$ is *f-d. Now, WTS:* $Tv_1, \cdots, Tv_n$ *is* L.I..

Consider $c_1Tv_1 + \cdots + c_nTv_n = 0$. Then, $T(c_1v_1 + \cdots + c_nv_n) = 0$. Hence, $c_1v_1 + \cdots + c_nv_n \in \operatorname{null} T$. Since $u_1, \cdots, u_m$ is a basis of $\operatorname{null} T$, we know

$$c_1v_1 + \cdots + c_nv_n = d_1u_1 + \cdots + d_mu_m \quad \textit{f.s. } d_i \in \mathbb{F}.$$

So,

$$c_1v_1 + \cdots + c_nv_n - d_1u_1 - \cdots - d_mu_m = 0. \tag{8}$$

However, by assumption, we know $v_1, \cdots, v_n, u_1, \cdots, u_m$ is a basis of $V$, and thus it is L.I.. So, the only way to make Equation (8) hold is by taking $c_1 = \cdots = c_n = -d_1 = \cdots = -d_m = 0$. Therefore, we've shown $Tv_1, \cdots, Tv_n$ is L.I., and thus is a basis of $\operatorname{range} T$. Then, $\dim \operatorname{range} T = n$.

So, we've shown that $\dim \operatorname{null} T + \dim \operatorname{range} T = m + n = \dim V$. ■

**Theorem 3.2.10**
Suppose $V$ and $W$ are *f-d* vector spaces *s.t.* $\dim V > \dim W$. Then, no linear map from $V$ to $W$ is injective.

***Proof 5.*** Let $T \in \mathcal{L}(V, W)$. By the Fundamental Theorem of Linear Maps, we have $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Then, we know

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W > 0 \quad [\dim \operatorname{range} T \leq \dim W] \end{aligned}$$

This implies that $\operatorname{null} T \neq \{0\}$. So, $T$ is not injective by Theorem 3.2.5. ■

**Theorem 3.2.11**
Suppose $V$ and $W$ are *f-d* vector space *s.t.* $\dim V < \dim W$. Then, no linear map from $V$ to $W$ is surjective.

***Proof 6.*** We know

$$\begin{aligned}\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &\leq \dim V < \dim W\end{aligned}$$

Then, $T$ cannot be surjective by definition. ■

**Example 3.2.12** Solving Linear Systems Using Linear Maps I
For a homogenous system of linear equations,

$$\begin{cases} A_{1,1}x_1 + \cdots + A_{1,n}x_n = 0 \\ \quad\vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n = 0 \end{cases},$$

where $A_{j,k} \in \mathbb{F}$ and $(x_1, \cdots, x_n) \in \mathbb{F}^n$, we can defined a linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ as

$$T(x_1, \cdots, x_n) = \left( \sum_{k=1}^{n} A_{1,k}x_k, \cdots, \sum_{k=1}^{n} A_{m,k}x_k \right).$$

Apparently, $(x_1, \cdots, x_n) = 0$ is a solution to the system, but the question is “If there are any non-zero solutions for this linear system?”

**Theorem 3.2.13**
A homogeneous system of linear equations with more variables than equations has non-zero solutions.

***Proof 7.*** Suppose $T \in \mathcal{L}(V, W)$. Then, $\dim V = n$ and $\dim W = m$. Suppose $n > m$. So, $\dim V > \dim W$. By the Theorem 3.2.5, we know $T$ is not injective. ■

**Example 3.2.14** Solving Linear Systems Using Linear Maps II
For an inhomogeneous system of linear equations

$$\begin{cases} \sum_{k=1}^{n} A_{1,k}x_k = c_1 \\ \quad\vdots \\ \sum_{k=1}^{n} A_{m,k}x_k = c_m \end{cases},$$

where $A_{j,k} \in \mathbb{F}$ and $(c_1, \cdots, c_m) \in \mathbb{F}^m$ and $(x_1, \cdots, x_n) \in \mathbb{F}^n$, we can define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \cdots, x_m) = \left( \sum_{k=1}^{n} A_{1,k} x_k, \cdots, \sum_{k=1}^{n} A_{m,k} x_k = c_1 \right).$$

However, in this case, $(x_1, \cdots, x_n) = 0$ may not be a solution to the system.

**Theorem 3.2.15**
An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

***Proof 8.*** Suppose $T \in \mathcal{L}(V, W)$. So, $\dim V = n$ and $\dim W = m$. Suppose $n < m$. Then, $\dim V < \dim W$. By Theorem 3.2.11, we know $T$ is not surjective. ■

## 3.3 Matrices

**Definition 3.3.1 (Matrix).** Let $m, n \in \mathbb{Z}^+$. An *$m$-by-$n$ matrix* $A$ is a rectangular array of elements of $\mathbb{F}$ with $m$ rows and $n$ columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row $j$, column $k$ of $A$.

**Definition 3.3.2 (Matrix of a Linear Map).** Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_n$ is a basis of $V$ and $w_1, \cdots, w_m$ is a basis of $W$. The *matrix of $T$* with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ whose $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \cdots, v_n), (w_1, \cdots, w_m))$ is used.

**Example 3.3.3** Suppose $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ is defined by $T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$. Find the matrix of $T$ with respect to the standard bases of $\mathbb{F}^2$ and $\mathbb{F}^3$.

***Solution 1.***

Note that $T(1, 0) = (1, 2, 7)$ and $T(0, 1) = (3, 5, 9)$. Then,

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}.$$

□

**Example 3.3.4** Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find the matrix of $D$ with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$.

***Solution 2.***

Standard bases of $\mathcal{P}_3(\mathbb{R}) : 1, x, x^2, x^3$. Standard bases of $\mathcal{P}_2(\mathbb{R}) : 1, x, x^2$. Since $(x^n)' = nx^{n-1}$, so we have

$$\begin{aligned} D(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ D(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ D(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\ D(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 \end{aligned}$$

So, we have

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

□

**Definition 3.3.5 (Matrix Addition).** The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}.$$

**Theorem 3.3.6**
Suppose $S, T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

***Proof 3.*** Let $v_1, \cdots, v_n$ be a basis of V and $w_1, \cdots, w_n$ be a basis of $W$. Suppose $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = C$. Then, if $1 \leq k \leq n$, we have

$$\begin{aligned} (S+T)v_k &= Sv_k + Tv_k \\ &= (A_{1,k}w_1 + \cdots + A_{m,k}w_m) + (C_{1,k}w_1 + \cdots + C_{m,k}w_m) \\ &= (A_{1,k} + C_{1,k})w_1 + \cdots + (A_{m,k} + C_{m,k})w_m. \end{aligned}$$

Hence, we have $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$. ■

**Definition 3.3.7 (Scalar Multiplication of a Matrix).** The *product of a scalar and a matrix* is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$.

**Theorem 3.3.8**
Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

***Proof 4.*** Let $v_1, \cdots, v_n$ be a basis of V and $\mathcal{M}(T) = A$. When $1 \leq k \leq v$, note that

$$\begin{aligned} (\lambda T)v_k &= \lambda(Tv_k) \\ &= \lambda(A_{1,k}w_1 + \cdots + A_{m,k}w_m) \\ &= (\lambda A_{1,k})w_1 + \cdots + (\lambda A_{m,k})w_m. \end{aligned}$$

So, $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$. ■

**Notation 3.3.9.** $\mathbb{F}^{m,n} :=$ the set of all $m \times n$ matrices with entries in $\mathbb{F}$.

**Theorem 3.3.10**
Suppose $m, n \in \mathbb{Z}^+$. With addition and scalar multiplication defined above, $\mathbb{F}^{m,n}$ is a vector space and $\dim \mathbb{F}^{m,n} = mn$.

***Proof 5.*** It is trivial to prove $\mathbb{F}^{m,n}$ is a vector space. □

Define $A_{j,k}$ as the matrix with $1$ on its $j^{\text{th}}$ row, $k^{\text{th}}$ column and $0$ elsewhere. Then, we can see that $A_{j,k}$ for $j = 1, \cdots, m$ and $k = 1, \cdots, n$ is a basis for $\mathbb{F}^{m,n}$. So, $\dim \mathbb{F}^{m,n} = m \cdot n$. ■

**Definition 3.3.11 (Matrix Multiplication).** Suppose $A$ is an $m \times n$ matrix and $C$ is an $n \times p$ matrix. Then,

$AC$ is defined to be the $m \times p$ matrix whose entry in row $j$. column $k$ is given by

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

**Remark.** *Matrix multiplication is not commutative. i.e., $AC \neq CA$. However, it is distributive and associative.*

**Theorem 3.3.12**
If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

**Notation 3.3.13.** Suppose $A$ is an $m \times n$ matrix.

1. If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the $1 \times n$ matrix consisting of row $j$ of $A$.
2. If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the $m \times 1$ matrix consisting of column $k$ of $A$.

In other words,

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}; \qquad A_{j,\cdot} = \begin{pmatrix} A_{j,1} & \cdots & A_{j,n} \end{pmatrix} \in \mathbb{F}^{1,n}; \qquad A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix} \in \mathbb{F}^{m,1}.$$

**Theorem 3.3.14 Practical Interpretations of Matrix Multiplication**

1. Suppose $A$ is an $m \times n$ matrix and $C$ is an $n \times p$ matrix. Then, $(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$ for $1 \leq j \leq m$ and $1 \leq k \leq p$.
2. Suppose $A$ is an $m \times n$ matrix and $C$ is an $n \times p$ matrix. Then, $(AC)_{\cdot,k} = AC_{\cdot,k}$ for $1 \leq k \leq p$.
3. Suppose $A$ is an $m \times n$ matrix and $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an $n \times 1$ matrix. Then,

   $$AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}.$$

   In other words, $AC$ is a linear combination of the columns of $A$, with the scalars that multiply the columns coming from $C$.

**Example 3.3.15**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}.$$

## 3.4 Invertibility and Isomorphic Vector Spaces

**Definition 3.4.1 (Invertible).** A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if $\exists$ a linear map $S \in \mathcal{L}(W, V)$ *s.t.* $ST$ equals the identity map on $I$ and $TS$ equals the identity map on $W$.
**Definition 3.4.2 (Inverse).** A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an *inverse* of $T$.

**Theorem 3.4.3**
An invertible linear map has a unique inverse.

***Proof 1.*** Suppose $T \in \mathcal{L}(V, W)$ is invertible. Let $S_1$ and $S_2$ be inverses of $T$.Then,

$$S_1 = S_1 I = S_1(TS_2) = (S_1 T)S_2 = IS_2 = S_2.$$

Thus, $S_1 = S_2$, and so inverse is unique. ■
**Notation 3.4.4.** If $T$ is invertible, then its inverse is denoted by $T^{-1}$.

**Theorem 3.4.5**
A linear map is invertible if and only if it is injective and surjective.

***Proof 2.***
($\Rightarrow$) Let $T \in \mathcal{L}(V, W)$ be invertible. Then, $TT^{-1} = I_W$ and $T^{-1}T = T_V$. Let $Tv = 0$. Note that $(T^{-1}T)v = 0$, so $Iv = 0$ and thus $v = 0$. Therefore, $\operatorname{null} T = \{0\}$, and so $T$ is an injection.
To show $T$ is surjective, suppose $w \in W$. Note that since $T^{-1} \in \mathcal{L}(W, V)$, $T^{-1}w \in V$. So,

$$T(T^{-1}w) = (TT^{-1})w = T_W w = w \in W.$$

Therefore, $T^{-1}w$ is the $v \in V$ we intend to find. Hence, $T$ is also a surjection. □
($\Leftarrow$) Let $T$ be surjective and injective. For $w \in W$, define $Sw \in V$ *s.t.* $T(Sw) = w$. So, we know $Sw$ is unique. Since $(T \circ S)w = w$, we know $(T \circ S) = I_W$. Consider $(S \circ T)v = S(Tv)$, we have $T(S(Tv)) = Tv$, by definition of $S$. Since $T$ is injective, we know $S(Tv) = V$. So, $(S \circ T)v = v$, and thus $ST = T_V$. Therefore $T$ is invertible.
Now, we want to show $S$ is a linear map. Let $w_1, w_2 \in W$, then

$$T(S(w_1 + w_2)) = (TS)(w_1 + w_2) = I_W(w_1 + w_2) = w_1 + w_2.$$

By definition, $w_1 + w_2 = T(Sw_1) + T(Sw_3) = T(Sw_1 + Sw_2)$. So, $T(S(w_1 + w_2)) = T(Sw_1 + Sw_2)$. By $T$ is an injection, we have $S(w_1 + w_2) = Sw_1 + Sw_2$. So, $S$ is additive. Further consider

$$T(S(\lambda w)) = \lambda w = \lambda(T(Sw)) = T(\lambda Sw)$$

for some $w \in W$. Again, since $T$ is injective, $S(\lambda w) = \lambda Sw$. So, $S$ has homogeneity. Then, $S$ is a linear map. ■
**Definition 3.4.6 (Isomorphism).** An *isomorphism* is an invertible linear map.
**Definition 3.4.7 (Isomorphic).** Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.
**Notation 3.4.8.** If two vector spaces $V$ and $W$ are isomorphic, we denote them as $V \cong W$.

**Theorem 3.4.9**
Suppose $V$ and $W$ are *f-d* vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

***Proof 3.***
($\Rightarrow$) Suppose $V \cong W$. By Fundamental Theorem of Linear Maps, we know

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Since $V \cong W$, $T$ is invertible and thus is injective and surjective. So, $\dim \operatorname{null} T = 0$ and $\dim \operatorname{range} T = \dim W$. Therefore, $\dim V = 0 + \dim W = \dim W$. □

($\Leftarrow$) Suppose $\dim V = \dim W$. Suppose $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are bases of $V$ and $W$, respectively. Then, $\dim V = \dim W = n$. Here, we want to define a bijection between $V$ and $W$. Let $T$ be defined as $Tv_i = wi \quad (i = 1, \cdots, n)$.

Let $Tv = 0$. Then, $T(a_1v_1 + \cdots + a_nv_n) = 0$. So, by definition, $a_1w_1 + \cdots + a_nw_n = 0$. Since $w_1, \cdots, w_n$ is a basis, we have $a_1 = \cdots = a_n = 0$. So, $\operatorname{null} T = \{0\}$, and thus $T$ is an injection.

Let $w \in W$ be any vector. Then, we know $w = c_1w_1 + \cdots + c_nw_n$. Note that, by definition of $T$, we have $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$. Hence, $\forall w \in W, \exists v = c_1v_1 + \cdots + c_nv_n \in V$ *s.t.* $Tv = w$. Therefore, $T$ is a surjection.

Finally, it is trivial to show that $T$ is indeed a linear map, and so the proof is complete. ■

**Theorem 3.4.10**
Suppose $v_1, \cdots, v_n$ is a basis of $V$ and $w_1, \cdots, w_m$ is a basis of $W$. then, $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

***Proof 4.*** We already know $\mathcal{M}$ is linear, so we just need to show $\mathcal{M}$ is a bijection.

To prove $\mathcal{M}$ is injective, consider $\mathcal{M}(T) = 0$ for some $T \in \mathcal{L}(V, W)$. So, we get $Tv_k = 0$. Since $v_1, \cdots, v_n$ is a basis of $V$, we know $Tv = 0 \quad \forall v \in V$. Then, $T$ is the zero-mapping, or $T = 0$. Therefore, $\operatorname{null} \mathcal{M} = \{0\}$.

To show $\mathcal{M}$ is surjective, suppose $A \in \mathbb{F}^{m,n}$. Let $T$ be a linear map from $V$ to $W$ *s.t.*

$$Tv_k = \sum_{j=1}^{m} A_{j,k}w_j, \quad k = 1, \cdots, n.$$

Obviously, $\mathcal{M}(T) = A$, and thus $\operatorname{range} \mathcal{M} = \mathbb{F}^{m,n}$. So, $\mathcal{M}$ is also a surjection. ■

**Theorem 3.4.11**
Suppose $V$ and $W$ are *f-d*. Then, $\mathcal{L}(V, W)$ is *f-d* and $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

***Proof 5.*** By Theorem 3.4.10 and Theorem 3.4.9, we know $\dim \mathcal{L}(V, W) = \dim \mathbb{F}^{m,n}$. Further by Theorem 3.3.10, we know $\dim \mathbb{F}^{m,n} = (m)(n)$. As $\dim V = n$ and $\dim W = m$, so we have

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W).$$

■

**Definition 3.4.12 (Matrix of a Vector, $\mathcal{M}(v)$).** Suppose $v \in V$ and $v_1, \cdots, v_n$ is a basis of $V$. The *matrix* *of* $v$ with respect to this basis is the $n \times 1$ matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \cdots, c_n$ are scalars *s.t.* $v = c_1 v_1 + \cdots + c_n v_n$.

**Theorem 3.4.13** $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(v_k)$
Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_n$ is a basis of $V$ and $w_1, \cdots, w_m$ is a basis of $W$. Let $1 \leq k \leq n$. Then, the $k^{\text{th}}$ column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(v_k)$.

***Proof 6.*** This theorem is an immediate result by definitions of matrix of a linear mapping and a vector. ■

**Theorem 3.4.14**
Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \cdots, v_n$ is a basis of $V$ and $w_1, \cdots, w_m$ is a basis of $W$. Then, $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$.

***Proof 7.*** Note that $v = c_1 v_1 + \cdots + c_n v_n$, so we have $Tv = c_1 T v_1 + \cdots + c_n T v_n$. So, by Theorem 3.4.13, we know

$$\begin{aligned} \mathcal{M}(Tv) &= c_1 \mathcal{M}(Tv_1) + \cdots + c_n \mathcal{M}(Tv_n) \\ &= c_1 \mathcal{M}(T)_{\cdot,1} + \cdots + c_n \mathcal{M}(T)_{\cdot,n} \\ &= \mathcal{M}(T)\mathcal{M}(v). \end{aligned}$$

The final equality holds due to our interpretation of matrix multiplication as column linear combinations (Theorem 3.3.14(3)) ■

**Remark.** $\mathcal{M} : \mathbb{F}^n \to \mathbb{F}^{n,1}$ *is an isomorphism:*

$$v = c_1 v_1 + \cdots + c_n v_n \longmapsto \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

***Proof 8.*** Suppose $\mathcal{M}(v) = 0 : \ \mathcal{M}(c_1 v_1 + \cdots + c_n v_n) = 0$. So, we have $c_1 w_1 + \cdots + c_n w_n = 0$. Since $w_1, \cdots, w_n$ is a basis, $c_1 = \cdots = c_n = 0$. So, $v = 0$. Therefore, $\operatorname{null} \mathcal{M} = \{0\}$, and so $\mathcal{M}$ is injective. □

Now, prove $\mathcal{M}$ is surjective. Note that $\forall \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, we have $\mathcal{M}(c_1 v_1 + \cdots + c_n v_n) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. So, $\mathcal{M}$ is a surjection. □

Finally, its' trivial to prove $\mathcal{M}$ is a linear map. □

Since $\mathcal{M}$ is both surjective and injective, $\mathcal{M}$ is an isomorphism. ■

**Definition 3.4.15 (Operator).** A linear map from a vector space to itself is called an *operator*.

**Notation 3.4.16.** The notation $\mathcal{L}(V)$ denotes the set of all operators on $V$. So, $\mathcal{L}(v) = \mathcal{L}(V, V)$.

**Theorem 3.4.17**
Suppose $V$ is *f-d* and $T \in \mathcal{L}(V)$. Then, the following are equivalent: (a) $T$ is invertible; (b) $T$ is injective; and (c) $T$ is surjective.

***Proof 9.***

1. Clearly (a) implies (b). □
2. Suppose (b): $T$ is injective. So, $\operatorname{null} T = \{0\}$. Then, by Fundamental Theorem of Linear Maps, we know
$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = 0 + \dim \operatorname{range} T.$$
Since $\dim \operatorname{range} T = \dim V$, we know $T$ is surjective. □
3. Suppose (c): $T$ is surjective. So, $\operatorname{range} T = V$. Then, by Fundamental Theorem of Linear maps, we have
$$\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T = 0.$$
So, $\operatorname{null} T = \{0\}$, and thus $T$ is injective. Since $T$ is surjective and injective, $T$ is invertible.

■

**Example 3.4.18** Show that for each polynomial $q \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbb{F})$ such that $((x^2 + 5x + 7)p)'' = q$.

***Proof 10.*** We know that every non-zero polynomial must have a degree of $m$. So, we can think of this problem under $\mathcal{P}_m(\mathbb{R})$. Note that

$$((x^2 + 5x + 7)p)'' = 2p + (4x + 10)p' + (x^2 + 5x + 7)p'' = q.$$

Therefore, the degree of $p$ and $q$ should be the same. Define $T : \mathcal{P}_m(\mathbb{R}) \to \mathcal{P}_m(\mathbb{R})$ as

$$Tp = ((x^2 + 5x + 7)p)''.$$

Then, $T$ is an operator on $\mathcal{P}_m(\mathbb{R})$. Consider $Tp = 0$. We have $ax + b = (x^2 + 5x + 7)p$. Note that only when $p = 0$, the equation above holds. So, it must be that $p = 0$ when $Tp = 0$. That is, $\operatorname{null} T = \{0\}$, and so $T$ is injective. By Theorem 3.4.18, we know $T$ is also surjective, and so our proof is complete. ■

## 3.5 Duality

**Definition 3.5.1 (Linear Functional).** A *linear functional* on $V$ is a linear map from $V$ to $\mathbb{F}$. That is, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

**Example 3.5.2**

1. Fix $(c_1, \cdots, c_n) \in \mathbb{F}^n$. Define $\varphi : \mathbb{F}^n \to \mathbb{F}$ by $\varphi(x_1, \cdots, x_n) = c_1x_1 + \cdots + c_nx_n$. Then, $\varphi$ is a linear functional on $\mathbb{F}^n$.
2. Define $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ as $\varphi(p) = 3p''(5) + 7p(4)$.
3. Define $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ as $\varphi(p) = \displaystyle\int_0^1 p(x)\mathrm{d}x$.

**Definition 3.5.3 (Dual Space/$V'$/$V^*$).** The *dual space* of $V$, denoted as $V'$, is the vector space of all linear functionals on $V$. In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

**Theorem 3.5.4**
Suppose $V$ is *f-d*. Then, $V'$ is also *f-d* and $\dim V' = \dim V$.

***Proof 1.*** Note that for a general linear map, $\mathcal{L}(V, W) \cong \mathbb{F}^{m,n}$. So, $\mathcal{L}(V, \mathbb{F}) = V' \cong \mathbb{F}^{1,n}$. Hence,

$$\dim V' = \dim \mathbb{F}^{1,n} = 1 \cdot n = n = \dim V.$$

■

**Definition 3.5.5 (Dual Basis).** If $v_1, \cdots, v_n$ is a basis of $V$, then the *dual basis* of $v_1, \cdots, v_n$ is the list $\varphi_1, \cdots, \varphi_n$ of elements of $V'$, where each $\varphi_j$ is the linear functional on $V$ *s.t.*

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}.$$

**Example 3.5.6** Find the dual basis of $e_1, \cdots, e_n \in \mathbb{F}^n$

***Solution 2.***

$$\begin{array}{cccc} \varphi_1(e_1) = 1 & \varphi_2(e_1) = 0 & \cdots & \varphi_n(e_1) = 0 \\ \varphi_1(e_2) = 0 & \varphi_2(e_2) = 1 & \cdots & \varphi_n(e_2) = 0 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(e_n) = 1 & \varphi_2(e_n) = 0 & \cdots & \varphi_n(e_n) = 1 \end{array}$$

Define $\varphi_j$ as

$$\varphi_j(x) = \varphi_j(x_1, \cdots, x_n) = x_1\varphi_j(e_1) + \cdots + x_j\varphi_j(e_j) + \cdots + x_n\varphi_j(e_n) = x_j.$$

□

**Theorem 3.5.7**
Suppose $V$ is *f-d*. Then, the dual basis of a basis of $V$ is a basis of $V'$.

***Proof 3.*** Suppose $v_1, \cdots, v_n$ is a basis of $V$ and $\varphi_1, \cdots, \varphi_n$ denotes the dual basis. Since we've shown $\dim V = \dim V'$ in Theorem 3.5.4, we only need to show $\varphi_1, \cdots, \varphi_n$ is L.I.. Select $c_1\varphi_1 + \cdots + c_n\varphi_n = 0$. Then,

$$(c_1\varphi_1 + \cdots + c_n\varphi_n)(v) = 0 \quad \forall\, v \in V.$$

Suppose $v = v_1 + \cdots + v_n$, then

$$(c_1\varphi_1 + \cdots + c_n\varphi_n)(v_j) = c_j \quad \text{for } j = 1, \cdots, n.$$

So, $(c_1\varphi_1 + \cdots + c_n\varphi_n)(v) = c_1 + \cdots + c_n = 0$. So, it must be that $c_1 = \cdots = c_n = 0$. Therefore, $\varphi_1, \cdots, \varphi_n$ is L.I. and our proof is complete. ■

**Definition 3.5.8 (Dual Map).** If $T \in \mathcal{L}(V, W)$, then the *dual map* of $T$ is the linear map $T' \in \mathcal{L}(W', V')$ defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.

**Remark.** *The following diagram represents dual map (but not an exact representation).*

$$\begin{tikzcd}
 & V \arrow[d] \arrow[dr, "T"] & \\
T'(\varphi) = \varphi \circ T \in V' & & W \arrow[dl] \\
 & \mathbb{F} & \varphi \in \mathcal{L}(W, \mathbb{F}) = W' \arrow[ull, mapsto, bend left, "T'"]
\end{tikzcd}$$

*Also, dual map is a linear map, so it is additive and homogeneous.*

1. $T'(\varphi + \psi) = (\varphi + \psi) \circ T = \varphi \circ T + \psi \circ T = T'(\varphi) + T'(\psi)$.
2. $T'(\lambda\varphi) = (\lambda\varphi) \circ T = \lambda(\varphi \circ T) = \lambda T'(\varphi)$.

**Example 3.5.9** Suppose $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ as $Dp = p'$.

1. Define a linear functional $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ as $\varphi(p) = p(3)$. Find $D'(\varphi)$.

   ***Solution 4.***

   $$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = p'(3).$$

   □

2. Define $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, a linear functional, as $\varphi(p) = \int_0^1 p(x)\,\mathrm{d}x$. Find $D'(\varphi)$.

   ***Solution 5.***

   $$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = \int_0^1 p'(x)\,\mathrm{d}x = p(1) - p(0).$$

   □

**Theorem 3.5.10 Algebraic Properties of Dual Maps**

1. $(S+T)' = S' + T' \quad \forall\, S, T \in \mathcal{L}(V, W)$
2. $(\lambda T)' = \lambda T' \quad \forall\, T \in \mathcal{L}(V, W)$
3. $(ST)' = T'S' \quad \forall\, T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$

***Proof 6.***

1. $(S+T)' \in \mathcal{L}(W', V')$. Let $\varphi \in W'$. Then,

$$(S+T)'(\varphi) = \varphi \circ (S+T) = \varphi \circ S + \varphi \circ T = S'(\varphi) + T'(\varphi) = (S' + T')(\varphi). \qquad \square$$

2. $(\lambda T)' \in \mathcal{L}(W', V')$. Let $\varphi \in W'$. Then,

$$(\lambda T)'(\varphi) = \varphi \circ (\lambda T) = \lambda(\varphi \circ T) = \lambda T'(\varphi) = (\lambda T')(\varphi). \qquad \square$$

3. $(ST)' \in \mathcal{L}(W', U')$. Let $\varphi \in W'$. Then,

$$(ST)'(\varphi) = \varphi \circ (ST) = \varphi \circ (S \circ T) = (\varphi \circ S) \circ T = (S'(\varphi)) \circ T = T'(S'(\varphi)) = (T'S')(\varphi).$$

■

**Definition 3.5.11 (Transpose/$A^t$).** The transpose of a matrix $A$, denoted $A^t$, is the matrix obtained from $A$ by interchanging the rows and columns. i.e., $(A^t)_{k,j} = A_{j,k}$.

**Remark.** *Transpose is additive and homogeneous. That is,* $(A + C)^t = A^t + C^t$ *and* $(\lambda A)^t = \lambda A^t$.

**Theorem 3.5.12**
If $A$ is an $m \times n$ matrix and $C$ is an $n \times p$ matrix, then $(AC)^t = C^t A^t$.

***Proof 7.*** Note that

$$(AC)^t_{k,j} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k} = \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j} = (C^t A^t)_{k,j}$$

■

**Theorem 3.5.13**
Suppose $T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

***Proof 8.*** Suppose $v_1, \cdots, v_n$ is a basis of $V$, $w_1, \cdots, w_m$ is a basis of $W$, $\varphi_1, \cdots, \varphi_n$ is a basis of $V'$, and $\psi_1, \cdots, \psi_m$ is a basis of $W'$. Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Since $T'(\psi_j) = C_{1,j}\varphi_1 + \cdots + C_{n,j}\varphi_n$ and $T'(\psi_j) = \psi_j \circ T$, we have $\psi_j \circ T = C_{1,j}\varphi_1 + \cdots + C_{n,j}\varphi_n$. Consider

$$(\psi_j \circ T)(v_k) = (C_{1,j}\varphi_1 + \cdots + C_{n,j}\varphi_n)(v_k) = C_{k,j}\varphi_k(v_k) = C_{k,j}.$$

Also, we have

$$(\psi_j \circ T)(v_k) = \psi_j(Tv_k) = \psi_j(A_{1,k}w_1 + \cdots + A_{m,k}w_m) = \psi_j(A_{j,k}w_j) = A_{j,k}(\varphi_j(w_j)) = A_{j,k}.$$

Therefore, we have $A_{j,k} = C_{k,j}$, and thus $A = C^t$. So, $\mathcal{M}(T) = (\mathcal{M}(T'))^t$. ■

**Definition 3.5.14 (Annihilator/$U^0$).** For $U \subseteq V$, the *annihilator* of $U$, denoted as $U^0$, is defined by

$$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \quad \forall u \in U\}.$$

**Theorem 3.5.15**
Suppose $U \subseteq V$. Then $U^0$ is a subspace of $V'$.

***Proof 9.***

1. $0 \in U^0$: Since $0(u) = 0 \quad \forall u \in U$, then $0 \in U^0$. □

2. Let $\varphi, \psi \in U^0$. Then,

$$(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0.$$

   So, $\varphi + \psi \in U^0$. □

3. Let $\lambda \in \mathbb{F}$ and $\varphi \in U^0$. Then

$$(\lambda\varphi)(u) = \lambda\varphi(u) = \lambda \cdot 0 = 0.$$

   So, $\lambda\varphi \in U^0$.

■

**Lemma 3.5.16** Suppose $V$ is *f-d* vector space. If $U$ is a subspace of $V$ and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ *s.t.* $Tu = Su \quad \forall u \in U$.

***Proof 10.*** Suppose $u_1, \cdots, u_m$ is a basis of $U$. Then, we can extend it to a basis of $V$ as $u_1, \cdots, u_m, v_{m+1}, \cdots, v_n$. Define $T \in \mathcal{L}(V, W)$ as $Tu_i = Su_i$, $Tv_j = 0$, where $i = 1, \cdots, m$ and $j = m+1, \cdots, n$. Note that

$$\begin{aligned} Tu &= T(a_1u_1 + \cdots + a_mu_m) \\ &= a_1Tu_1 + \cdots + a_mTu_m \\ &= a_1Su_1 + \cdots + a_mSu_m \\ &= S(a_1u_1 + \cdots + a_mu_m) = Su. \end{aligned}$$

Therefore, we've found such a $T$. ■

**Theorem 3.5.17**
Let $V$ be *f-d* and $U$ be a subspace of $V$, then $\dim U + \dim U^0 = \dim V$.

***Proof 11.*** Let $i \in \mathcal{L}(U, V)$ as $i(u) = u \quad \forall u \in U$. Then, $i' \in \mathcal{L}(V', U')$. So, by Fundamental Theorem of Linear Map, we know

$$\dim V' = \dim \operatorname{null} i' + \dim \operatorname{range} i'. \tag{9}$$

By Theorem 3.5.4, we know $\dim V = \dim V'$ Note that $U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \quad \forall u \in U\}$ and

$$\begin{aligned} \text{null } i' &= \{\varphi \in V' \mid i'(\varphi) = 0\} \\ &= \{\varphi \in V' \mid \varphi \circ i = 0\} \\ &= \{\varphi \in V' \mid (\varphi \circ i)(u) = 0 \quad \forall u \in U\} \\ &= \{\varphi \in V' \mid \varphi(u) = 0 \quad \forall u \in U\} \end{aligned}$$

So, $U^0 = \text{null } i'$, and thus $\dim \text{null } i' = \dim U^0$.

Further, if $\varphi \in U'$, then $\varphi : U \to \mathbb{F}$. By Lemma 3.5.16, $\varphi$ can be extended to $\psi \in V'$ with $\psi(u) = \varphi(u) \quad \forall u \in U$. Note that $i'(\psi) = \psi \circ i$, so $(\psi \circ i)(u) = \psi(u) = \varphi(u) \quad \forall u \in U$. Then, $\exists \psi \in V'$ *s.t.* $i'(\psi) = \varphi$. So, $\varphi \in \text{range } U'$. So, $\dim \text{range } i' = \dim U' = \dim U$.

Substitute $\dim V' = \dim V$, $\dim \text{null } i' = \dim U^0$, and $\dim \text{range } i' = \dim U$ to Equation (9), we get

$$\dim V = \dim U^0 + \dim U.$$

■

**Theorem 3.5.18 The Null Space of $T'$**
Suppose $V$ and $W$ are *f-d* and $T \in \mathcal{L}(V, W)$. Then,

1. $\text{null } T' = (\text{range } T)^0$
2. $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$

***Proof 12.***

1. $(\subseteq)$ Suppose $\varphi \in \text{null } T' \subseteq W'$. Then, $T'(\varphi) = \varphi \circ T = 0 \in V'$. So, we know

   $$(\varphi \circ T)(v) = 0 \quad \forall v \in V. \quad \text{i.e., } \varphi(Tv) = 0.$$

   Note that $Tv \in \text{range } T$. By definition, we have $\varphi \in (\text{range } T)^0$ □

   $(\supseteq)$ Suppose $\varphi \in (\text{range } T)^0$. Then, $\varphi(w) = 0 \quad \forall w \in \text{range } T$. That is, $\varphi(Tv) = 0 \quad \forall v \in V$. So, $(\varphi \circ T)(v) = 0 \quad \forall v \in V$. Hence, we know $\varphi \circ T = T'(\varphi) = 0 \in V'$. Thus, $\varphi \in \text{null } T'$ ■

2. $$\begin{aligned} \dim \text{null } T' &= \dim(\text{range } T)^0 \\ &= \dim W - \dim \text{range } T \\ &= \dim W - (\dim V - \dim \text{null } T) \\ &= \dim W - \dim V + \dim \text{null } T. \end{aligned}$$

■

**Theorem 3.5.19**
Suppose $V$ and $W$ are *f-d* and $T \in \mathcal{L}(V, W)$. Then, $T$ is surjective if and only if $T'$ is injective.

***Proof 13.***
$(\Rightarrow)$ Suppose $T$ is surjective. Then, $\dim \text{range } T = W$. So, $(\text{range } T)^0 = \{0\}$. Hence,

$$\dim \text{null } T' = \dim(\text{range } T)^0 = 0.$$

Thus, $T'$ is injective. $\square$

($\Leftarrow$) Suppose $T'$ is injective. Then,

$$\dim \operatorname{null} T' = 0.$$

So, $\dim(\operatorname{range} T)^0 = \dim \operatorname{null} T' = 0$. Then, $(\operatorname{range} T)^0 = \{0\}$. So, $\dim \operatorname{range} T = W$, and thus $T$ is surjective. $\blacksquare$

> **Theorem 3.5.20 The Range of $T'$**
> Suppose $V$ and $W$ are *f-d* and $T \in \mathcal{L}(V, W)$. Then,
>
> 1. $\dim \operatorname{range} T' = \dim \operatorname{range} T$
> 2. $\operatorname{range} T' = (\operatorname{null} T)^0$

***Proof 14.***

1. By Fundamental Theorem of Linear Map, we have

$$\begin{aligned}\dim \operatorname{range} T' &= \dim W' - \dim \operatorname{null} T' \\ &= \dim W' - \dim(\operatorname{range} T)^0 \\ &= \dim W' - \dim W' + \dim \operatorname{range} T \\ &= \dim \operatorname{range} T.\end{aligned}$$

$\blacksquare$

2. Suppose $\varphi \in \operatorname{range} T' \subseteq V'$. Then, $\exists \psi \in W'$ *s.t.* $T'(\psi) = \psi \circ T = \varphi$. Let $v \in \operatorname{null} T$. Then,

$$\varphi(v) = (\psi \circ T)(v) = \psi(Tv) = \psi(0) = 0.$$

Then, $\varphi \in (\operatorname{null} T)^0$. So, $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. $\square$

Note that

$$\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V - \dim \operatorname{null} T = \dim(\operatorname{null} T)^0.$$

Then, $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$ and $\dim \operatorname{range} T' = \dim(\operatorname{null} T)^0$, so it must be that $\operatorname{range} T' = (\operatorname{null} T)^0$.

$\blacksquare$

> **Theorem 3.5.21**
> Suppose $V$ and $W$ are *f-d* and $T \in \mathcal{L}(V, W)$. Then, $T$ is injective if and only if $T'$ is surjective.

***Proof 15.***

($\Rightarrow$) If $T$ is injective, $\operatorname{null} T = \{0\}$. So,

$$\dim \operatorname{null} T = \dim V - \dim(\operatorname{null} T)^0 = \dim V - \dim \operatorname{range} T' = 0.$$

So, $\dim \operatorname{range} T' = \dim V = \dim V'$. Then, $T'$ is surjective. $\square$

($\Leftarrow$) If $T'$ is surjective, $\dim \operatorname{range} T' = \dim V' = \dim V$. So,

$$\dim \operatorname{null} T = \dim V - \dim(\operatorname{null} T)^0 = \dim V - \dim \operatorname{range} T' = 0.$$

Then, $\operatorname{null} T = \{0\}$, and so $T$ is injective. $\blacksquare$

**Definition 3.5.22 (Row Rank & Column Rank).** Suppose $A$ is an $m \times n$ matrix with entries in $\mathbb{F}$.

1. The *row rank* of $A$ is the dimension of the span of the rows of $A$ in $\mathbb{F}^{1,n}$.
2. The *column rank* of $A$ is the dimension of the span of the columns of $A$ in $\mathbb{F}^{m,1}$.

**Theorem 3.5.23**
Suppose $V$ and $W$ are *f-d* and $T \in \mathcal{L}(V, W)$. Then, $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$.

***Proof 16.*** Suppose $v_1, \cdots, v_n$ is a basis of $V$ and $w_1, \cdots, w_m$ is a basis of $W$. Then,

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

and thus

$$\mathcal{M}(Tv_k) = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix} \in \mathbb{F}^{m,1}$$

Therefore, $\mathcal{M}(T) = \begin{pmatrix} \mathcal{M}(Tv_1) & \cdots & \mathcal{M}(Tv_n) \end{pmatrix}$. Note that $\operatorname{range} T = \operatorname{span}(Tv_1, \cdots, Tv_n)$.

Define $\mathcal{M} : \operatorname{span}(Tv_1, \cdots, Tv_n) \to \operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n))$ as $w \mapsto \mathcal{M}(w)$.

1. $\mathcal{M}$ is surjective: Note that

$$c_1\mathcal{M}(Tv_1) + \cdots + c_n\mathcal{M}(Tv_n) = \mathcal{M}(c_1Tv_1 + \cdots + c_nTv_n).$$

   Since $c_1Tv_1 + \cdots + c_nTv_n \in \operatorname{range} T$, we know $\mathcal{M}$ is surjective. □

2. $\mathcal{M}$ is injective: Let

$$\mathcal{M}(c_1Tv_1 + \cdots + c_nTv_n) = 0. \tag{10}$$

   We can reduce $c_1Tv_1 + \cdots + c_nTv_n$ to a basis $Tv_{j_1}, \cdots, Tv_{j_m}$. Then, Equation (10) becomes $\mathcal{M}(a_1Tv_{j_1} + \cdots + a_mTv_{j_m}) = 0$. By definition of matrix, we know $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = 0$. So, $a_1 = \cdots = a_m = 0$ and $a_1Tv_{j_1} + \cdots + a_mTv_{j_m} = 0$. So, $\mathcal{M}$ is injective. □

Since $\mathcal{M}$ is both surjective and injective, $\mathcal{M}$ is a bijection. Thus, $\mathcal{M}$ is an isomorphism between $\operatorname{span}(Tv_1, \cdots, Tv_n)$ and $\operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n))$. In other words,

$$\operatorname{span}(Tv_1, \cdots, Tv_n) \cong \operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n)).$$

Then, $\dim \operatorname{span}(Tv_1, \cdots, Tv_n) = \dim \operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n))$. That is,

$$\dim \operatorname{range} T = \text{column rank of } T.$$

■

**Theorem 3.5.24 Row Rank Equals Column Rank**
Suppose $A \in \mathbb{F}^{m,n}$. Then, the row rank of $A$ equals the column rank of $A$.

***Proof 17.*** Define $T : \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$. Then, $\mathcal{M}(T) = A$, where $\mathcal{M}(T)$ is computed with respect to the standard basis of $\mathbb{F}^{n,1}$ and $\mathbb{F}^{m,1}$. Note that

$$\begin{aligned}
\text{column rank of } A &= \text{column rank of } \mathcal{M}(T) \\
&= \dim \operatorname{range} T && \text{Theorem 3.5.23} \\
&= \dim \operatorname{range} T' && \text{Theorem 3.5.20(1)} \\
&= \text{column rank of } \mathcal{M}(T') \\
&= \text{column rank of } A^t && \text{Theorem 3.5.13} \\
&= \text{row rank of } A
\end{aligned}$$

■

**Definition 3.5.25 (Rank).** The *rank* of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of $A$, denoted as $\operatorname{rank} A$.

## 3.6 Quotients of Vector Spaces

**Definition 3.6.1 ($v + U$/Affine Subset).** Suppose $v \in V$ and $U$ is a subspace of $V$. Then

$$v + U \coloneqq \{v + u \mid u \in U\}.$$

An *affine subset* of $V$ is a subset of $V$ of the form $v + U$ for some $v \in V$ and some subspace $U$ of $V$. The affine subset is said to be *parallel* to $U$.

**Definition 3.6.2 (Quotient Space, $V/U$).** Suppose $U$ is a subspace of $V$. Then the quotient space $V/U$ is the set of all affine subsets of $V$ parallel to $U$. In other words,

$$V/U \coloneqq \{v + U \mid v \in V\}.$$

**Example 3.6.3** If $U = \left\{(x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ with slope of $2$.

**Theorem 3.6.4**
Suppose $U$ is a subspace of $V$ and $v, w \in V$. Then, the following are equivalent:

1. $v - w \in U$
2. $v + U = w + U$
3. $(v + U) \cap (w + U) \neq \varnothing$

***Proof 1.***

1. We want to show (1) $\implies$ $(2)$. Suppose $v - w \in U$. Note that $v + u = w + ((v - w) + u)$. Since $v - u$ and $u \in U$, we have $(v - w) + u \in U$. So, $v + u \in w + U$. Similarly, we can show that $w + u \in v + U$. Then, we have $v + U = w + U$. $\square$

2. Now, we want to show (2) $\implies$ (3): Suppose $v + U = w + U$. Then, we have $(v + U) \cap (w + U) \neq \varnothing$, which is evident from the assumption. $\square$

3. Finally, we will show (3) $\implies$ $(1)$. Suppose $(v + U) \cap (w + U) \neq \varnothing$. Then, $\exists u_1, u_2 \in U$ *s.t.* $v + u_1 = w + u_2$. So we have $v - w = u_2 - u_1 \in U$.

$\blacksquare$

**Definition 3.6.5 (Addition & Scalar Multiplication on $V/U$).** Suppose $U$ is a subspace of $V$. Then, *addition* and *scalar multiplication* is defined on $V/U$ by

$$(v + U) + (w + U) = (v + w) + U$$

and

$$\lambda(v + U) = (\lambda v) + U$$

for $v, w \in U$ and $\lambda \in \mathbb{F}$.

**Theorem 3.6.6**
Suppose $U$ is a subspace of $V$. Then, $V/U$, with the operations of addition and scalar multiplication defined above, is a vector space.

***Proof 2.***

1. Addition on $V/U$ makes sense.

   Note the addition can be written in the language of mapping as $+ : V/U \times V/U \to V/U$. So, we have $(v+U, w+U) \mapsto (v+w)+U$. Suppose $\exists\, \hat{v}.\hat{w} \in V$ *s.t.* $v+U = \hat{v}+U$ and $w+U = \hat{w}+U$. Note that $v-\hat{v} \in U$ and $w-\hat{w} \in U$ by Theorem 3.6.4. Then, $(v-\hat{v})+(w-\hat{w}) \in U$. So, we have $(v+w)-(\hat{v}+\hat{w}) in U$. Further, by Theorem 3.6.4, we have
   $$(v+w)+U = (\hat{v}+\hat{w})+U. \qquad \square$$

2. Scalar multiplication on $V/U$ makes sense.

   We can write the scalar multiplication on $V/U$ as a mapping: $\cdot : \mathbb{F} \times V/U \to V/U$ defined as $(\lambda, v+U) \mapsto \lambda v + U$. Suppose $\exists\, \hat{v} \in V$ *s.t.* $v+U = \hat{v}+U$. So we know $v-\hat{v} \in U$, and thus $\lambda(v-\hat{v}) = \lambda v - \lambda\hat{v} \in U$. By Theorem 3.6.4, we then have $(\lambda v)+U = (\lambda\hat{v})+U$. Thus, the scalar multiplication makes sense. $\square$

3. additive identity: $0+U = U. \quad \square$

4. additive inverse: $(-v)+U. \quad \square$

5. commutativity:
   $$\begin{aligned}(v+U)+(w+U) &= (v+w)+U = (w+v)+U \\ &= (w+U)+(v+U). \qquad \square\end{aligned}$$

6. associativity:
   $$\begin{aligned}[(v+U)+(w+U)]+(x+U) &= [(v+w)+U]+(x+U) \\ &= [(v+w)+x]+U \\ &= [v+(w+x)]+U \\ &= (v+U)+[(w+x)+U] \\ &= (v+U)+[(x+U)+(x+U)]. \qquad \square\end{aligned}$$

7. multiplicative identity: $1\cdot(v+U) = (1\cdot v)+U = v+U. \quad \square$

8. distributivity:
   $$\begin{aligned}a[(v+U)+(w+U)] &= a[(v+w)+U] \\ &= a(v+w)+U \\ &= (av+aw)+U \\ &= (av+U)+(aw+U) \\ &= a(v+U)+a(w+U).\end{aligned}$$

$$
\begin{aligned}
(a+b)(v+U) &= (a+b)v+U \\
&= (av+bv)+U \\
&= (av+U)+(bv+U) \\
&= a(v+U)+b(v+U)
\end{aligned}
$$

■

**Definition 3.6.7 (Quotient Map).** Suppose $U$ is a subspace of $V$. The *quotient map* $\pi$ is the linear map $\pi : V \to V/U$ defined by $\pi(v) := v + U \quad \forall v \in V$.

**Remark.** *Here are some properties of the quotient map:*

1. $\pi(v)$ *is defined* $\forall v \in V$. *Thus,* $\pi$ *is surjective.*
2. $\operatorname{null} \pi = \{v \in V \mid \pi(v) = 0\}$. *If* $\pi(v) = 0$, *then* $v + U = U = 0 + U$. *So,* $v - 0 \in U$ *by Theorem 3.6.4. Then,* $v \in U$. *So,* $\operatorname{null} \pi \subseteq U$. *Further,* $\forall v \in U$, *if* $\pi(v) = 0$, *then* $v \in \operatorname{null} \pi$, *then* $U \subseteq \operatorname{null} \pi$. *So,* $U = \operatorname{null} \pi$.
3. $\pi(v+w) = (v+w)+U = (v+U)+(w+U) = \pi(v)+\pi(w)$.
4. $\pi(\lambda v) = (\lambda v)+U = \lambda(v+U) = \lambda \pi(v)$.

**Theorem 3.6.8**
Suppose $V$ is *f-d* and $U$ is a subspace of $V$. Then

$$\dim V/U = \dim V - \dim U.$$

***Proof 3.*** By Fundamental Theorem of Linear Map, we have

$$\dim V = \dim \operatorname{null} \pi + \dim \operatorname{range} \pi. \tag{11}$$

Since $\operatorname{null} \pi = U$ from the Remark, we have $\dim \operatorname{null} \pi = \dim U$. Further, since $\pi$ is surjective as mentioned in the Remark, $\operatorname{range} \pi = V/U$. Hence, $\dim \operatorname{range} \pi = \dim V/U$. Therefore, Equation (11) becomes

$$\dim V = \dim U + \dim V/U,$$

or we have

$$\dim V/U = \dim V - \dim U$$

■

**Definition 3.6.9 ($\tilde{T}$).** Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T} : V/(\operatorname{null} T) \to W$ by $\tilde{T}(v + \operatorname{null} T) - Tv$.

***Proof 4.***

1. This definition makes sense

   Suppose $u, v \in V$ *s.t.* $u + \operatorname{null} T = v + \operatorname{null} T$. By Theorem 3.6.4, we know $u - v \in \operatorname{null} T$. Then, $T(u-v) = 0$, or $Tu = Tv$. □

2. $\tilde{T}$ is a linear map.

$$
\begin{aligned}
\tilde{T}[(u+\operatorname{null} T)+(v+\operatorname{null} T)] &= \tilde{T}[(u+v)+\operatorname{null} T] \\
&= T(u+v) \\
&= Tu+Tv = \tilde{T}(u+\operatorname{null} T)+\tilde{T}(v+\operatorname{null} T). \qquad \square
\end{aligned}
$$

$$\begin{aligned}\tilde{T}[\lambda(u + \operatorname{null} T)] &= \tilde{T}(\lambda u + \operatorname{null} T) \\ &= T(\lambda u) \\ &= \lambda T u \\ &= \lambda T(u + \operatorname{null} T).\end{aligned}$$

■

**Theorem 3.6.10**
Suppose $T \in \mathcal{L}(V, W)$. Then,

1. $\tilde{T}$ is injective.
2. $\operatorname{range} \tilde{T} = \operatorname{range} T$.
3. $V/(\operatorname{null} T) \cong \operatorname{range} T$.

***Proof 5.***

1. Suppose $v \in V$ and $\tilde{T}(v + \operatorname{null} T) = 0$. Then, $Tv = 0$. So, $v \in \operatorname{null} T$, or $v - 0 \in \operatorname{null} T$. By Theorem 3.6.4, we then have $v + \operatorname{null} T = 0 + \operatorname{null} T$. Then, it implies $\operatorname{null} \tilde{T} = 0$. So, $\tilde{T}$ is injective. □
2. By definition of $\tilde{T}$, it must be $\operatorname{range} \tilde{T} = \operatorname{range} T$. □
3. Note that $\dim V/(\operatorname{null} T) = \dim \operatorname{null} \tilde{T} + \dim \operatorname{range} \tilde{T} = 0 + \dim \operatorname{range} T$. Then, by Theorem 3.4.9, we know two vector spaces are isomorphic if and only if their dimensions are equal. Then,

$$V(\operatorname{null} T) \cong \operatorname{range} T.$$

■

# 4 Eigenvectors and Invariant Subspaces

## 4.1 Invariant Subspaces

**Theorem 4.1.1**
Suppose $V$ is *f-d* with $\dim V = n \geq 1$. Then, $\exists\, 1-$dimensional subspaces $U_1, \cdots, U_n$ of $V$ *s.t.*

$$V = U_1 \oplus \cdots \oplus U_n.$$

***Proof 1.*** Choose a basis $v_1, \cdots, v_n$ of $V$. Then, we know $V = \operatorname{span}(v_1) + \cdots + \operatorname{span}(v_n)$. Also, $\forall v \in V$, we have $v = a_1 v_1 + \cdots + a_n v_n$ with $a_j v_j \in \operatorname{span}(v_j)$. Set $a_1 v_1 + \cdots + a_n v_n = 0$. Since $v_1, \cdots, v_n$ is a basis, it must be $a_1 = \cdots = a_n = 0$. Then,

$$V = \operatorname{span}(v_1) \oplus \cdots \oplus \operatorname{span}(v_n).$$

■

**Theorem 4.1.2**
Suppose $U_1, \cdots, U_m$ are *f-d* subspaces of $V$ *s.t.* $U_1 + \cdots + U_m$ is a direct sum. Then, $U_1 \oplus \cdots \oplus U_m$ is *f-d* and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

***Proof 2.*** Suppose $u_{k,1}, \cdots, u_{k.j_k}$ is a basis of the subspace $U_k$. Then, any vector in $\bigoplus_{i=1}^{m} U_i$ is in the form of $u_1 + \cdots + u_m, \quad u_j \in U_j$. Also,

$$u_i = \sum_{k=1}^{j_i} a_{i,k} u_{i,k}.$$

So,

$$u_1 + \cdots + u_m = \sum_{k=1}^{j_1} a_{1,k} u_{1,k} + \cdots + \sum_{k=1}^{j_m} a_{m,k} u_{m,k}.$$

Then, $u_1 + \cdots + u_m$ is a linear combination of $u_{1,1}, \cdots, u_{j,m}$. So, the direct sum is *f-d.* □

Further, suppose

$$\sum_{k=1}^{j_1} a_{1,k} u_{1,k} + \cdots + \sum_{k=1}^{j_m} a_{m,k} u_{m,k} = 0.$$

Since $U_1 + \cdots + U_m$ is a direct sum, it must be

$$\sum_{k=1}^{j_1} a_{1,k} u_{1,k} = \cdots = \sum_{k=1}^{j_m} a_{a,k} u_{m,k} = 0.$$

Since we selected bases, $a_{1,k} = \cdots = a_{m,k} = 0$. So, $u_{1,1}, \cdots, u_{m,j_m}$ is a basis of $U_1 \oplus \cdots \oplus U_m$. Then,

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

■

**Definition 4.1.3 (Invariant Subspace).** Suppose $T \in \mathcal{L}(V)$. A subspace $U$ of $V$ is called *invariant* under $T$ if $u \in U$ implies $Tu \in U$.

**Example 4.1.4** Suppose $T \in \mathcal{L}(V)$. Show that each of the following subspaces of $V$ is invariant under $T$:

1. $\{0\}$

   ***Proof 3.*** $T0 = 0 \in \{0\}$ ■

2. $V$

   ***Proof 4.*** $u \in V \implies Tu \in V$ ■

3. $\operatorname{null} T$

   ***Proof 5.*** $u \in \operatorname{null} T \implies Tu = 0 \in \operatorname{range} T$ ■

4. $\operatorname{range} T$

   ***Proof 6.*** $u \in \operatorname{range} T \implies Tu \in \operatorname{range} T$ ■

**Example 4.1.5** Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is defined by $Tp = p'$. Then, $\mathcal{P}_4(\mathbb{R})$ is invariant under $T$.
***Proof 7.*** Note that $Tp_4) \in \mathcal{P}_4(\mathbb{R})$. Then, $\mathcal{P}_4(\mathbb{R})$ is invariant under $T$. ■

**Definition 4.1.6 (Eigenvalue).** Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an *eigenvalue* of $T$ if $\exists\, v \in V$ *s.t.* $v \neq 0$ and $Tv = \lambda v$.

**Corollary 4.1.7** $T$ has a $1-$dimensional invariant subspace if and only if $T$ has an eigenvalue.

***Proof 8.***

($\Rightarrow$) Suppose $\operatorname{span}(v)$ is invariant under $T$. Let $U$ be defined as $U = \{\lambda v \mid \lambda \in \mathbb{F}\} = \operatorname{span}(v)$. Then. $U$ is the invariant subspace under $T$ and $\dim U = 1$. Then, $\forall v \in V$, we have $Tv \in U$. Hence, $\exists \lambda \in \mathbb{F}$ *s.t.* $Tv = \lambda v$. Then, $\lambda$ is an eigenvalue. □

($\Leftarrow$) Suppose $\lambda \in \mathbb{F}$ is an eigenvalue. Then, $Tv = \lambda v$. Hence, $\operatorname{span}(v)$ is a $1$ =dimensional invariant subspace under $T$. ■

**Theorem 4.1.8 Equivalent Conditions to be an Eigenvalue**
Suppose $V$ is *f-d*, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then, the following are equivalent:

1. $\lambda$ is an eigenvalue of $T$.
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

***Proof 9.***

1. (1) $\implies$ (2): Suppose $\lambda$ is an eigenvalue of $T$. Then, $\exists v \in V$ *s.t.* $v \neq 0$ and $Tv - \lambda v$. So, $Tv - \lambda v = (T - \lambda I)v = 0$. Since $v \neq 0$, $\operatorname{null}(T - \lambda I) \neq \{0\}$, and thus $T$ is not injective. □
2. Note that $T - \lambda I$ is an operator by itself. By Theorem 3.4.17, we know (2), (3), and (4) are equivalent.

3. (4) $\implies$ (1): Suppose $T-\lambda I$ is not invertible. Then, it is not injective. So, $\exists v \neq 0$ *s.t.* $(T-\lambda I)v = 0$. That is, $Tv - \lambda Iv = Tv - \lambda v = 0$. So, $Tv = \lambda v$. Then, $\lambda$ is an eigenvalue of $T$.

■

**Definition 4.1.9 (Eigenvector).** Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of $T$. A vector $v \in V$ is called an *eigenvector* of $T$ corresponding to $\lambda$ if $v \neq 0$ and $Tv = \lambda v$.

**Corollary 4.1.10** A vector $v \in V$ with $v \neq 0$ is an eigenvector of $T$ with respect to $\lambda$ if and only if $v \in \operatorname{null}(T - \lambda I)$.

***Proof 10.*** Note that $Tv = \lambda v$ if and only if $(T-\lambda I)v = 0$. ■

**Example 4.1.11** Suppose $T \in \mathcal{L}(\mathbb{F}^2)$ is defined by $T(w,z) = (-z, w)$.

1. Find the eigenvalues and eigenvectors of $T$ if $\mathbb{F} = \mathbb{R}$.

   ***Solution 11.***

   Let $T(2,z) = \lambda(w,z)$. So, $(-z,w) = (\lambda w, \lambda z)$. Then, solve $\begin{cases} -z = \lambda w \\ w = \lambda z \end{cases}$.

   Then, we have $\lambda^2 z + z = 0$. If $z \neq 0$, $\lambda^2 + 1 = 0$. This equation has no solutions on $\mathbb{R}$. So $T$ has no eigenvalues. If $w = 0, z = 0$, then $T(w,z) = T(0.0) = T0$. By definition, $T$ has no eigenvalues. □

2. Find the eigenvalues and eigenvectors of $T$ if $\mathbb{F} = \mathbb{C}$.

   ***Solution 12.***

   Applying similar rational, $z \neq 0$ and solve $\lambda^2 + 1 = 0$. Then, we have $\lambda = \pm \mathrm{i}$. If $\lambda = \mathrm{i}$, then $-z = \mathrm{i}w$. So, $v = (w,z) = (w, -\mathrm{i}w)$. If $\lambda - \mathrm{i}$, then $-z = -\mathrm{i}w$, or $z = \mathrm{i}w$. So, $v = (w, \mathrm{i}w)$. □

**Theorem 4.1.12**
Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of $T$ and $v_1, \cdots, v_m$ are corresponding eigenvectors. Then, $v_1, \cdots, v_m$ is L.I..

***Proof 13.*** Suppose for the sake of contradiction that $v_1, \cdots, v_m$ is linearly dependent. Let $k$ be the smallest positive integer *s.t.* $v_k \in \operatorname{span}(v_1, \cdots, v_{k-1})$. Then, $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$. Applying $T$, we have

$$\lambda_k v_k = a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}. \tag{12}$$

Since $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$, we also have

$$\lambda_k v_k = a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}. \tag{13}$$

So, by Equation (13)-(12), we have

$$0 = a_1(\lambda_k - \lambda_1) v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1}) v_{k-1}.$$

By assumption, $v_1, \cdots, v_{k-1}$ is L.I.. Then, it must be that $a_1 = \cdots = a_{k-1} = 0$ since $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues. Therefore, $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1} = 0$. ⋇ This contradicts with the fact that $v_k$ is an eigenvector, which cannot be 0. So,it must be that $v_1, \cdots, v_m$ are L.I. ■

**Theorem 4.1.13**
Suppose $V$ is *f-d*. Then, each operator on $V$ has at most $\dim V$ distinct eigenvalues.

***Proof 14.*** Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of $T$. Let $v_1, \cdots, v_m$ be corresponding eigenvectors. By Theorem 4.1.12, we know $v_1, \cdots, v_m$ is $\mathrm{L.I.}$. Further by Theorem 2.3.5, we know $\dim \operatorname{span}(v_1, \cdots, v_m) \leq \dim V$. That is, $m \leq \dim V$ as desired. ■

## 4.2 Eigenvectors and Upper-Triangular Matrices

**Definition 4.2.1 ($T^m$).** Suppose $T \in \mathcal{L}(V)$ and $m$ is a positive integer. Then, $T^m$ is defined by

$$T^m := \underbrace{T \cdots T}_{m \text{ times}}.$$

Specially, $T^0$ is defined to be the identity operator $I$ on $V$. Further, if $T$ is invertible with inverse $T^{-1}$, then $T^{-m}$ is defined by $T^{-m} := (T^{-1})^m$.

**Theorem 4.2.2**

$$T^m T^n = T^{m+n}; \qquad (T^m)^n = T^{mn}.$$

**Definition 4.2.3 ($p(T)$).** Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m, \quad z \in \mathbb{F}.$$

Then, $p(T)$ is the operator defined by

$$p(T) := a_o I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

**Example 4.2.4** Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation operator defined by $Dq = q'$ and $p$ is the polynomuial defined by $p(x) = 7 - 3x + 5x^2$. Find $p(D)$ and $(p(D))q$.

***Solution 1.***

$$\begin{aligned} p(D) &= 7I - 3D + 5D^2 \\ (p(D))q &= (7I - 3D + 5D^2)q \\ &= 7Iq - 3Dq + 5D^2 q \\ &= 7q - 3q' + 5q''. \end{aligned}$$

□

**Theorem 4.2.5**
If we fix an operator $T \in \mathcal{L}(V)$, then the function from $\mathcal{P}(\mathbb{F})$ to $\mathcal{L}(V)$ given by $p \mapsto p(T)$ is linear.

***Proof 2.*** Suppose $f : \mathcal{P}(\mathbb{F}) \to \mathcal{L}(V)$ is defined by $p \mapsto p(T)$. Suppose

$$p = a_0 + a_1 z + \cdots + a_m z^m \mapsto a_0 I + a_1 T + \cdots + a_m T^m$$

and

$$q = b_0 + b_1 z + \cdots + b_m z^m \mapsto b_0 I + b_1 T + \cdots + b_m T^m.$$

Then,

$$\begin{aligned} f(p+q) &= (a_0 + b_0)I + (a_1 + b_1)T + \cdots + (a_m + b_m)T^m \\ &= (a_0 I + a_1 T + \cdots + a_m T^m) + (b_0 I + b_1 T + \cdots + b_m T^m) \\ &= f(p) + f(q). \end{aligned}$$

Further, suppose $\lambda \in \mathbb{F}$, then

$$\begin{aligned} f(\lambda p) &= \lambda a_0 I + \lambda a_1 T + \cdots + \lambda a_m T^m \\ &= \lambda(a_0 I + a_1 T + \cdots + a_m T^m) \\ &= \lambda f(p). \end{aligned}$$

■

**Definition 4.2.6 (Product of Polynomials).** If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by $(pq)(z) := p(z)q(z)$ for $z \in \mathbb{F}$.

**Remark.** $(pq)(z) = p(z)q(z) = q(z)p(z) = (qp)(z)$ *for* $z \in \mathbb{F}$.

**Theorem 4.2.7 Multiplicative Properties**
Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$. Then

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

***Proof 3.***

1. Suppose $p(z) = \sum_{j=0}^{m} a_j z^j$ and $q(z) = \sum_{k=0}^{n} b_k z^k$. Then

$$(pq)(z) = p(z)q(z) = \sum_{j=0}^{m} a_j z^j \sum_{k=0}^{n} b_k z^k = \sum_{j=0}^{m}\sum_{k=0}^{n} a_j b_k z^{j+k}$$

   So, by definition, we have

$$p(T)q(T) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_j b_k T^{j+k} = \left(\sum_{j=0}^{m} a_j T^j\right) \cdot \left(\sum_{k=0}^{n} b_k T^k\right) = p(T)q(T). \qquad \square$$

2. Similar to the Remark,

$$p(T)q(T) = (pq)(T) = (qp)(T) = q(T)p(T).$$

■

**Theorem 4.2.8 Fundamental Theorem of Algebra**
Every non-constant polynomial with complex coefficients has a zero.

**Theorem 4.2.9 Existence of Eigenvalues**
Every operator on a *f-d*, non-zero, complex vector space has an eigenvalue.

***Proof 4.*** Let $V$ be a complex vector space with dimension $n > 0$. Suppose $T \in \mathcal{L}(V)$. Choose $v \in V$ *s.t.* $v \neq 0$. Then, $v, Tv, T^2v, \cdots, T^n v$ is linearly dependent because $\dim V = n$ but the length of the list is $n + 1 > n$. Hence, $\exists\, a_0, a_1, \cdots, a_n$ not all $0 \in \mathbb{C}$ *s.t.*

$$0 = a_0 v + a_1 T v + \cdots + a_n T^n v \tag{14}$$

By Fundamental Theorem of Algebra (Theorem 4.2.8), we have

$$a_0 + a_1 z + \cdots + a_n z^n = c(z - \lambda_1) \cdots (z - \lambda_m)$$

with $c \in \mathbb{C},\ c \neq 0$, and $\lambda_j \in \mathbb{C}$. Then, Equation (14) becomes

$$\begin{aligned} 0 &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= (a_0 I + a_1 T + \cdots + a_n T^n) v \\ &= c(T - \lambda_1 I) \cdots (T - \lambda_m I) v \end{aligned}$$

Since $v \neq 0$ and $c \neq 0$, it must be some $T - \lambda_i I = 0$. Thus, $T = \lambda_i I$, and $\lambda_i$ is an eigenvalue of $T$. ■

**Definition 4.2.10 (Diagonal of a Matrix).** The *diagonal of a square matrix* consists of the entires along the line from the upper left corner to the bottom right corner.

**Definition 4.2.11 (Upper-Triangular Matrix).** A matrix is called *upper-triangular* if all the entires below the diagonal equal $0$. Typically, we present an upper triangular matrix in the form

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

**Theorem 4.2.12 Conditions for Upper-Triangular Matrix**
Suppose $T \in \mathcal{L}(V)$ and $v_1, \cdots, v_n$ is a basis of $V$. Then, the following are equivalent:

1. the matrix of $T$ with respect to $v_1, \cdots, v_n$ is upper triangular.
2. $Tv_j \in \operatorname{span}(v_1, \cdots, v_j)$ for each $j = 1, \cdots, n$
3. $\operatorname{span}(1, \cdots, v_j)$ is invariant under $T$ for each $j = 1, \cdots, n$.

***Proof 5.***

1. First, we will show (1) $\iff$ (2).

   Suppose $\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ & \ddots & \vdots \\ 0 & & A_{n,n} \end{pmatrix}$. Then,

   $$\begin{aligned} Tv_1 &= A_{1,1} v_1 \\ Tv_2 &= A_{1,2} v_1 + a_{2,2} v_2 \\ &\vdots \\ Tv_j &= A_{1,j} v_1 + \cdots + A_{j,j} v_j. \end{aligned}$$

   So, $Tv_j \in \operatorname{span}(v_1, \cdots, v_j)$. The reverse implication is trivial to prove. □

2. (3) $\implies$ (2) is obvious and trivial to prove.

3. Lastly, we want to show (2) $\implies$ (3).

Note that for each fixed $j = 1, \cdots, n$, we have

$$\begin{aligned} Tv_1 &\in \operatorname{span}(v_1) \subseteq \operatorname{span}(v_1, \cdots, v_j) \\ Tv_2 &\in \operatorname{span}(v_1, v_2) \subseteq \operatorname{span}(v_1, \cdots, v_j) \\ &\vdots \\ Tv_j &\in \operatorname{span}(v_1, \cdots, v_j) \end{aligned}$$

Let $v \in \operatorname{span}(v_1, \cdots, v_j)$. Then, $v$ is a linear combination of $v_1, \cdots, v_j$, then

$$Tv \in \operatorname{span}(v_1, \cdots, v_j).$$

That is, $\operatorname{span}(v_1, \cdots, v_j)$ is invariant under $T$.

■

**Definition 4.2.13 (Quotient Operator).** Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$ invariant under $T$. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by $(T/U)(v+U) \coloneqq Tv + U$.

***Proof 6.*** The definition makes sense, and here is the proof. If $v + U = w + U$, then $v - w \in U$. So, $T(v-w) \in U$ since $U$ is invariant. That is, $Tv - Tw \in U$. Then, $Tv + U = Tw + U$. ■

**Theorem 4.2.14**
Suppose $U$ is a subspace of $V$. Let $v_1 + U, \cdots, v_m + U$ be a basis of $V/U$ and $u_1, \cdots, u_n$ be a basis of $U$. Then, $v_1, \cdots, v_m, u_1, \cdots, u_n$ is a basis of $V$.

***Proof 7.*** Let $v \in V$. Then $v + U \in V/U$. So, $v + U = a_1v_1 + \cdots + a_mv_m + U$, uniquely. Then, by Theorem 3.6.4, we have $v-(a_1v_1+\cdots+a_mv_m) \in U$. Therefore, $v-(a_1v_1+\cdots+a_mv_m) = b_1u_1+\cdots+b_nu_n$, uniquely. So, $v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n$. uniquely. By definition, $v_1, \cdots, v_m, u_1, \cdots, u_n$ is a basis of $V$. ■

**Theorem 4.2.15**
Suppose $V$ is a *f-d* complex vector space and $T \in \mathcal{L}(V)$. Then, $T$ has an upper-triangular matrix with respect to some basis of $V$.

***Proof 8.***

$\boxed{\text{Base Case}}$ When $\dim V = 1$, the implication holds.

$\boxed{\text{Inductive Steps}}$ Suppose the implication is true for some complex vector space with dimension of $n-1$. Let $\dim V = n$ and $v_1$ be any eigenvector of $T$. Suppose $U = \operatorname{span}(v_1)$. Then, $U$ is invariant under $T$. Note that $\dim V/U = \dim V - \dim U = n-1$, so we can use the inductive hypothesis on the quotient operator $T/U \in \mathcal{L}(V/U)$. Then, $\exists$ a basis $v_2 + U, \cdots, v_n + U \in V/U$ *s.t.* $T/U$ has an upper-triangular matrix. By Theorem 4.2.12, we have

$$(T/U)(v_j + U) \in \operatorname{span}(v_2 + U, \cdots, c = v_j + U) \quad \text{for } j \in \{2, \cdots, n\}.$$

So, $Tv_j + U = (c_2v_2 + \cdots + c_jv_j) + U$. Then,

$$Tv_j - (c_2v_2 + \cdots + c_jv_j) \in U = \operatorname{span}(v_1).$$

So, $Tv_j - (c_2v_2 + \cdots + c_jv_j) = c_1v_1$ for some $c_1 \in \mathbb{F}$. Then, $Tv_j = c_1v_1 + c_2v_2 + \cdots + c_jv_j$. So, $Tv_j \in \operatorname{span}(v_1, \cdots, v_j)$ for $j \in \{1, \cdots, n\}$. Since by Theorem 4.2.14, $v_1, \cdots, v_n$ is a basis of $V$, further by Theorem 4.2.12, $T$ has an upper-triangular matrix with respect to $v_1, \cdots, v_n$. So, the implication is true for $\dim V = n$.

Since the implication is true for $\dim V = 1$ and is true for $\dim V = n$ whenever it is hold for $\dim V = n - 1$, by the Principle of Mathematical Induction, the implication is true for all positive integers $n$. Hence, the proof is complete. ■

## 4.3 Eigenspaces and Diagonal Matrices

**Definition 4.3.1 (Diagonal Matrix).** A *diagonal matrix* is a square matrix that is $0$ everywhere except possibly along the diagonal.

**Definition 4.3.2 (Eigenspace, $E(\lambda, T)$).** Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The *eigenspace* of $T$ corresponding to $\lambda$, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \text{null } (T - \lambda I).$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of $T$ corresponding to $\lambda$, along with the $0$ vector.

**Theorem 4.3.3 Sum of Eigenspaces is a Direct Sum**
Suppose $V$ is *f-d* and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of $T$. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Further

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

***Proof 1.*** Suppose $u_1 + \cdots + u_m = 0$, where $u_j \in E(\lambda_j, T)$. If some $u_i \neq 0$, then $u_1 + \cdots + u_m$ can never be $0$ because $u_1, \cdots, u_m$, as eigenvectors corresponding to distinct eigenvalues, is L.I.. Hence, the only way for $u_1 + \cdots + u_m$ to be $0$ is by taking $u_1 = \cdots = u_m = 0$. Hence, we know $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum. □

By Theorem 4.1.2, we know

$$\begin{aligned} \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) &= \dim E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T) \\ &\leq \dim V. \end{aligned}$$

■

**Definition 4.3.4 (Diagonalizable).** An operator $T \in \mathcal{L}(V)$ is called *diagonalizable* if the operator has a diagonal matrix with respect to some basis of $V$.

**Theorem 4.3.5 Conditions Equivalent to Diagonalizability**
Suppose $V$ is *f-d* and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ denote the distinct eigenvalues of $T$. Then, the following are equivalent:

1. $T$ is diagonalizable.
2. $V$ has a basis consisting of eigenvectors of $T$.
3. $\exists\, 1-$dimensional subspaces $U_1, \cdots, U_n$ of $V$, each invariant under $T$, *s.t.* $V = U_1 \oplus \cdots \oplus U_n$.
4. $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.
5. $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

**Remark.** *To prove this theorem, we will prove (1) $\iff$ (2), (2) $\iff$ (3), (2) $\implies$ (4), (4) $\implies$ (5), and (5) $\implies$ (2).*

***Proof 2.***

1. (1) $\iff$ (2): By definition, we know $T$ is diagonalizable if and only if $\exists$ a basis $v_1, \cdots, v_n$ of $T$ *s.t.*

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

which holds if and only if we have $Tv_1 = \lambda_1 v_1, \cdots, Tv_n = \lambda_n v_n$. i.e., $v_1, \cdots, v_n$ are eigenvectors of $T$. $\square$

2. (2) $\implies$ (3): Suppose $v_1, \cdots, v_n$ is a basis of $V$. Then, for some $v \in V$, we have $v = a_1v_1 + \cdots + a_nv_n$. So, we know $V = \operatorname{span}(v_1) + \cdots + \operatorname{span}(v_n)$. Further, let $a_1v_1 + \cdots + a_mv_m = 0$. Since $v_1, \cdots, v_n$ is a basis, it must be $a_1 = \cdots = a_m = 0$. So, there is only one way to represent $0$. So,

$$V = \operatorname{span}(v_1) \oplus \cdots \oplus \operatorname{span}(v_n).$$

Now, we want to show each $\operatorname{span}(v_j)$ is invariant. Consider $T(c_jv_j) = c_jTv_j = c_j\lambda_jv_j \in \operatorname{span}(v_j)$. So, $\operatorname{span}(v_j)$ is invariant. $\square$

3. (3) $\implies$ (2): Suppose $\exists$ $1-$dimensional subspaces $U_1, \cdots, U_n$ of $V$, each invariant under $T$, *s.t.* $V = U_1 \oplus \cdots \oplus U_n$. Then, $\forall v \in V$, we have $v = a_1u_1 + \cdots + a_nu_n$ uniquely. Then, $u_1, \cdots, u_n$ is a basis of $V$. Since $U_1, \cdots, U_n$ are $1-$dimensional invariant subspaces, $u_1, \cdots, u_n$ are the eigenvalues. $\square$

4. (2) $\implies$ (4): Suppose $V$ has a basis consisting of eigenvectors of $T$. Then, $v = a_1v_1 + \cdots + a_nv_n$ is a linear combination of eigenvectors of $T$. By definition, $E(\lambda_j, T)$ contains the eigenvectors corresponding to $\lambda_j$. Further since $\lambda_1, \cdots, \lambda_m$ is distinct, corresponding eigenvectors are L.I.. Then, $E(\lambda_j, T) \cap E(\lambda_i, T) = \{0\}$ if $i \neq j$. Then, we have

$$v = a_1v_1 + \cdots + a_nv_n \in E(\lambda_1, T) + \cdots + E(\lambda_m, T).$$

Hence, $V = \mathbb{E}(\lambda_1, T) + \cdots + E(\lambda_m, T)$. Further by Theorem 4.3.3, we have

$$V = \mathbb{E}(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T). \qquad \square$$

5. (4) $\implies$ (5): This conclusion can be deduced from Theorem 4.3.3 and its proof.

6. (5) $\implies$ (2): Suppose $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$. Select $B_j$, the basis of $E(\lambda_j, T)$ for $j = 1, \cdots, m$. Denote $\dim V = n$. Then, if we put these bases together as $B_1, \cdots, B_m$, we can write the collection as $v_1, \cdots, v_n$. Suppose $a_1v_1 + \cdots + a_nv_n = 0$. Let $u_j$ denote the sum of all the terms $a_kv_k$ *s.t.* $v_k \in E(\lambda_j, T)$. Then, the equation becomes $u_1 + \cdots + u_m = 0$ and each $u_j \in E(\lambda_j, T)$. Since eigenvectors corresponding to distinct eigenvalues are L.I., it must be that $u_1 = \cdots = u_m = 0$. Further, by definition of $u_j$, and since $u_k's$ are bases of $E(\lambda_j, T)$, it must be $a_1 = \cdots = a_n = 0$. So, we know $v_1, \cdots, v_n$ is L.I.. Further, since $\operatorname{len}(v_1, \cdots, v_n) = n = \dim V$, we know that $v_1, \cdots, v_n$ is a basis of $V$.

$\blacksquare$

**Theorem 4.3.6**
If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then $T$ is diagonalizable.

***Proof 3.*** Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues: $\lambda_1, \cdots, \lambda_{\dim V}$. Then, it has $v_1, \cdots, v_{\dim V}$ as corresponding eigenvectors and is L.I.. Note that $\operatorname{len}(v_1, \cdots, v_{\dim V}) = \dim V$. So, $v_1, \cdots, v_{\dim V}$ is a basis of $V$. By Theorem 4.3.5, with respect to this basis consisting of eigenvectors, $T$ has a diagonal matrix. ■

**Example 4.3.7** The *Fibonacci Sequence* $F_1, F_2, \cdots$ is defined by

$$F_1 = F_2 = 3 \qquad \text{and} \qquad F_n = F_{n-2} + F_{n-1} \quad \text{for } n \geq 3.$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

1. Show that $T^n(0,1) = (F_n, F_{n+1})$ for each $n \in \mathbb{Z}^+$.

   ***Proof 4.***

   - Base Case: Note that $T(0,1) = (1,1) = (F_1, F_2)$.
   - Inductive Process: Suppose $T^{n-1}(0,1) = (F_{n-1}, F_n)$. Then,

   $$\begin{aligned} T^n = [T(T^{n-1})](0,1) &= T[T^{n-1}(0,1)] \\ &= T(F_{n-1}, F_n) \\ &= (F_n, F_{n-1} + F_n) \\ &= (F_n, F_{n+1}). \end{aligned}$$

   So, $T^n(0,1) = (F_n, F_{n+1}) \quad \forall\, n \in \mathbb{Z}^+$ by Principle of Mathematical Induction.

   ■

2. Find the eigenvalues of $T$.

   ***Solution 5.***

   Suppose $T(x,y) = \lambda(x,y)$. So, $(y, x+y) = (\lambda x, \lambda y)$. Solve $\begin{cases} y = \lambda x \\ x + y = \lambda y \end{cases}$. That is, $x + \lambda x = \lambda^2 x$, or $\lambda^2 x - \lambda x - x = 0$. It follows $x \neq 0$, so solving $\lambda^2 - \lambda - 1 = 0$, we get

   $$\lambda_1 = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1 - \sqrt{5}}{2}.$$

   □

3. Since $T$ has two eigenvalues, $T$ should have a basis of $\mathbb{R}^2$ consisting of eigenvectors. Find the basis.

   ***Solution 6.***

When $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$, we have $y = \lambda x = \dfrac{1+\sqrt{5}}{2}x$. So, $v_1 = \left(x, \dfrac{1+\sqrt{5}}{2}x\right) = x\left(1, \dfrac{1+\sqrt{5}}{2}\right)$. That is,

$$v_1 = \left(1, \frac{1+\sqrt{5}}{2}\right).$$

Similarly, we have

$$v_2 = \left(1, \frac{1-\sqrt{5}}{2}\right).$$

Further, it follows that

$$\mathcal{M}(T, v_1, v_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

□

4. Find $F_n$ using an expression of $n$ only.

***Solution 7.***

Note that $(0,1) = \dfrac{1}{\sqrt{5}}(v_1 - v_2)$. So, we have

$$\begin{aligned}
T^n(0,1) &= T^n\left(\frac{1}{\sqrt{5}}(v_1 - v_2)\right) \\
&= \frac{1}{\sqrt{5}}T^n(v_1 - v_2) \\
&= \frac{1}{\sqrt{5}}(T^n v_1 - T^n v_2) \\
&= \frac{1}{\sqrt{5}}(\lambda_1^n v_1 - \lambda_2^n v_2) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n \left(1, \frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n \left(1, \frac{1-\sqrt{5}}{2}\right)\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n, \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \\
&= (F_n, F_{n+1}).
\end{aligned}$$

So, we have

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

□

# 5 Inner Product Spaces

## 5.1 Inner Products and Norms

**Definition 5.1.1 (Dot Product).** For $x, y \in \mathbb{R}^n$, the *dot product* of $x$ and $y$, denoted $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.

**Theorem 5.1.2 Properties of dot Product**

1. $x \cdot x = x_1^2 + \cdots + x_n^2 \geq 0 \quad \forall x \in \mathbb{R}^n$.
2. $x \cdot x = 0$ if and only if $x = 0$.
3. For $y \in \mathbb{R}^n$, define $f : \mathbb{R}^n \to \mathbb{R}$ as $x \mapsto x \cdot y$. Then, $f$ is linear.
4. $\forall x, y \in \mathbb{R}^n,\ x \cdot y = y \cdot x$.

***Proof 1.*** Properties #1, #2, and #4 are trivial to prove, so the proof is omitted. Here we complete a proof for property #3, linearity of dot product. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $x \mapsto x \cdot y$ for a fixed $y \in \mathbb{R}^n$. Note that

$$\begin{aligned} f(a+b) = (a+b) \cdot y &= (a_1 + b_1)y_1 + \cdots + (a_n + b_n)y_n) \\ &= (a_1 y_1 + \cdots + a_n y_n) + (b_1 y_1 + \cdots + b_n y_n) \\ &= f(a) + f(b). \end{aligned}$$

Further, notice that

$$\begin{aligned} f(\lambda x) = (\lambda x) \cdot y &= (\lambda x_1)y_1 + \cdots + (\lambda x_n)y_n \\ &= \lambda(x_1 y_x + \cdots + x_n y_n = \lambda f(x). \end{aligned}$$

■

**Remark.** *For $w, z \in \mathbb{C}^n$, we define the dot product of $w$ and $z$, denoted as $\langle w, z \rangle$, as*

$$\langle w, z \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}.$$

**Definition 5.1.3 (Inner Product).** An *inner product* on $V$ is a function that takes each ordered pair $(u, v)$ of elements of $V$ to a number $\langle u, v \rangle \in \mathbb{F}$ and has the following properties:

1. positivity: $\langle v, v \rangle \geq 0 \quad \forall v \in V$.
2. definiteness: $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. additivity in first slot: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \quad \forall u, v, w \in V$.
4. homogeneity in first slot: $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \quad \forall \lambda \in \mathbb{F}$ and $\forall u, v \in V$.
5. conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle} \quad \forall u, v \in V$.

**Example 5.1.4** Here, we offer some examples of inner product. Note that there might be multiple inner products over a vector space, as long as the following the definition and properties given in Definition 5.1.3.

1. Consider $\mathbb{C}[-1,1]$, the set of continuous real-valued functions on the interval $[-1,1]$. An inner product can be defined as $\langle f, g\rangle = \displaystyle\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$.

   ***Proof 2.***

   (a) $\langle f, f\rangle = \displaystyle\int_{-1}^{1} f^2(x)\,\mathrm{d}x \geq 0$.

   (b) $\langle f, f\rangle = 0$ if and only if $f(x) = 0$.

   (c) Note that

   $$\begin{aligned}\langle f+g, h\rangle &= \int_{-1}^{1}[f(x)+g(x)]h(x)\,\mathrm{d}x \\ &= \int_{-1}^{1} f(x)h(x)+g(x)h(x)\,\mathrm{d}x \\ &= \int_{-1}^{1} f(x)h(x)\,\mathrm{d}x + \int_{-1}^{1} g(x)h(x)\,\mathrm{d}x \\ &= \langle f, h\rangle + \langle g, h\rangle.\end{aligned}$$

   (d) $\langle \lambda f, g\rangle = \displaystyle\int_{-1}^{1} \lambda f(x)g(x)\,\mathrm{d}x = \lambda\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x = \lambda\langle f, g\rangle$.

   (e) $\langle g, f\rangle = \displaystyle\int_{-1}^{1} g(x)f(x)\,\mathrm{d}x = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x = \langle f, g\rangle = \overline{\langle f, g\rangle}$.

   ■

2. An inner product on $\mathcal{P}(\mathbb{R})$ can be defined as $\langle p, q\rangle = \displaystyle\int_{0}^{\infty} p(x)q(x)e^{-x}\,\mathrm{d}x$

   ***Proof 3.*** The definition makes sense. Consider the inner product as $\langle\ \rangle : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$. Note that $\infty \notin \mathbb{R}$. So we need to show the improper integral converges to a finite number under any circumstances. Consider

   $$\frac{x^2 p(x)q(x)}{e^x} = \frac{p(x)q(x)e^{-x}}{\frac{1}{x^2}}.$$

   Note that

   $$\lim_{x\to\infty} \frac{p(x)q(x)e^{-x}}{\frac{1}{x^2}} = 0$$

   Further since $\displaystyle\int_0^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges as it is a $p$-series with $p = 2 > 1$, we know it must be $\displaystyle\int_0^\infty p(x)q(x)e^{-x}\,\mathrm{d}x$ converges as well, by the comparison test. The remaining job is to show this definition of $\langle\ \rangle$ indeed retain the five properties as required in Definition 5.1.3, which is trivial and so is omitted. ■

**Definition 5.1.5 (Inner Product Space).** An *inner product space* is a vector space $V$ along with an inner product on $V$.

**Example 5.1.6** Euclidean Inner Product on $\mathbb{F}^n$ is defined as

$$\langle (w_1, \cdots, w_n), (z_1, \cdots, z_n) \rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n},$$

where $(w_1, \cdots, w_n), (z_1, \cdots, z_n) \in \mathbb{F}^n$.

**Notation 5.1.7.** For the rest of this Chapter, without otherwise specification, $V$ denotes an inner product space over $\mathbb{F}$.

**Remark.** *If not explicitly defined, the inner product is the Euclidean inner product as defined in Example 5.1.6.*

**Theorem 5.1.8 Basic Properties of an Inner Product**

1. For each fixed $u \in V$, the function that takes $v$ to $\langle v, u \rangle$ is a linear map from $V$ to $\mathbb{F}$.
2. $\langle 0, u \rangle = 0$ for every $u \in V$.
3. $\langle u, 0 \rangle = 0$ for ever $u \in V$.
4. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.
5. $\langle u, \lambda v \rangle = \overline{\lambda}\langle u, v \rangle \quad \forall \lambda \in \mathbb{F}$ and $u, v \in V$.

***Proof 4.***

1. Define $f : V \to \mathbb{F}$ as $v \mapsto \langle v, u \rangle$ for some fixed $u \in V$. Then,
$$f(v + w) = \langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = f(v) + f(w).$$
$$f(\lambda v) = \langle \lambda v, u \rangle = \lambda \langle v, u \rangle = \lambda f(v). \qquad \square$$
2. Since $f$ is a linear map, then $f(0) = \langle 0, u \rangle = 0. \qquad \square$
3. Note that $\langle u, 0 \rangle = \overline{\langle 0, u \rangle} = \overline{0} = 0. \qquad \square$
4. Notice
$$\begin{aligned} \langle u, v + w \rangle = \overline{\langle v + w, u \rangle} &= \overline{\langle v, u \rangle + \langle w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle. \qquad \square \end{aligned}$$
5. Observe that
$$\begin{aligned} \langle u, \lambda v \rangle = \overline{\langle \lambda v, u \rangle} &= \overline{\lambda \langle v, u \rangle} \\ &= \overline{\lambda} \cdot \overline{\langle v, u \rangle} = \overline{\lambda}\langle u, v \rangle. \end{aligned}$$

■

**Definition 5.1.9 (Norm).** Suppose $V$ is a vector space. Then, the *norm* of $v$ is a real-valued function $\| \, \| : V \to \mathbb{R}$ satisfying the following properties:

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
2. $\|\alpha v\| = |\alpha|\|v\| \quad \forall \alpha \in \mathbb{F}$ and $v \in V$.
3. triangle inequality: $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in \mathbb{F}$.

**Definition 5.1.10 (Norm Induced by An Inner Product).** For $v \in V$, $\|v\| = \sqrt{\langle v, v\rangle}$ is a *norm* on $V$.

**Remark.** *We will prove Definition 5.1.10 is indeed a definition of norm that satisfies the conditions indicated by Definition 5.1.9 throughout the rest of this section.*

**Theorem 5.1.11 Basic Properties of Norms**
Let $v \in V$. Then,

1. $\|v\| = 0$ if and only if $v = 0$.
2. $\|\lambda v\| = |\lambda|\|v\| \quad \forall \lambda \in \mathbb{F}$.

***Proof 5.***

1. $\|v\| = 0$ if and only if $\sqrt{\langle v, v\rangle} = 0$, which is equivalent to $\langle v, v\rangle = 0$. By properties of an inner product, $\langle v, v\rangle = 0$ if and only if $v = 0$. So, the proof is complete. □
2. Consider
$$\|\lambda v\|^2 = \langle \lambda v, \lambda v\rangle = \lambda \cdot \overline{\lambda}\langle v, v\rangle = |\lambda|^2\langle v, v\rangle.$$
So, $\|\lambda v\| = \sqrt{|\lambda|^2\langle v, v\rangle} = |\lambda|\|v\|$.

■

**Definition 5.1.12 (Orthogonal).** Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v\rangle = 0$.

**Theorem 5.1.13 Orthogonality and** $0$

1. $0$ is orthogonal to every vector in $V$.
2. $0$ is the only vector in $V$ that is orthogonal to itself.

***Proof 6.***

1. As $\langle 0, u\rangle = 0 \quad \forall u \in V$, the proof is complete. □
2. Note that $\langle v, v\rangle = 0$ if and only if $v = 0$, so we complete the proof. □

■

**Theorem 5.1.14 Pythagorean Theorem**
Suppose $u$ and $v$ are orthogonal vectors in $V$, then
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

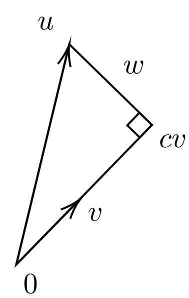
***Proof 7.*** Note that

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle.\end{aligned}$$

Since $u$ and $v$ are orthogonal, $\langle u,v\rangle = \langle v,u\rangle = 0$. So, $\|u+v\|^2 = \langle u,u\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2$. ∎

**Theorem 5.1.15 An Orthogonal Decomposition**
Suppose $u, v \in V$, with $v \neq 0$. Set $c = \dfrac{\langle u,v\rangle}{\|v\|^2}$ and $w = u - \dfrac{\langle u,v\rangle}{\|v\|^2}v$. Then, $\rangle w, v\rangle = 0$ and $u = cv + w$.

***Proof 8.***




The idea is the find $c, w$ *s.t.* $\langle v, w\rangle = 0$ and $w = u - cv$. That is, $u = w + cv$. Since $\langle v, w\rangle = 0$, then we have

$$\langle v, u-cv\rangle = 0 = \langle u-cv, v\rangle = \langle u,v\rangle - c\|v\|^2.$$

So,

$$c = \frac{\langle u,v\rangle}{\|v\|^2}$$

and

$$w = u - cv = u - \frac{\langle u,v\rangle}{\|v\|^2}v.$$

∎

**Theorem 5.1.16 Cauchy-Schwarz Inequality**
Suppose $u, v \in V$. Then,

$$|\langle u,v\rangle| \leq \|u\|\|v\|.$$

This inequality is an equality if and only if one of $u, v$ is a scalar multiples of the other.

***Proof 9.*** If $v = 0$, then $|\langle u,v\rangle| = 0 = \|u\|\|v\|$. So, we can assume $v \neq 0$. Consider the orthogonal decomposition,

$$u = \frac{\langle u,v\rangle}{\|v\|^2}\cdot v + w.$$

Then, by the Pythagorean Theorem, we have

$$\begin{aligned}\|u\|^2 = \left\|\frac{\langle u,v\rangle}{\|v\|^2}\cdot v\right\|^2 + \|w\|^2 &= \frac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2 + \|w\|^2 \\ &= \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \|w\|^2 \geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}\end{aligned}$$

As $\|v\|^2 > 0$, we have $\|u\|^2\|v\|^2 \geq |\langle u, v\rangle|^2$. Further since $\|u\| \geq 0,\ \|v\| \geq 0$, and $|\langle u, v\rangle| \geq 0$, then

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

The equality holds if and only if $\|w\|^2 = 0$. That is, $w = 0$ from the orthogonal decomposition. In other words, $u$ and $v$ are linearly dependent. ■

**Theorem 5.1.17 Triangle Inequality**
Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of $u, v$ is a non-negative multiple of the other.

***Proof 10.*** Note that

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\
&= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \overline{\langle u, v\rangle} \\
&= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\,(\langle u, v\rangle) \\
&\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v\rangle| \\
&\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| && \text{Cauchy-Schwarz Inequality} \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

Since $\|u + v\| \geq 0,\ \|u\| \geq 0$, and $\|v\| \geq 0$, we have

$$\|u + v\| \leq \|u\| + \|v\|.$$

The equality holds if and only if $\langle u, v\rangle = \|u\|\|v\|$. That is, when $u$ and $v$ are linearly dependent to each other. ■

**Remark.** *After proving this triangle inequality, we finally, and officially, complete our proof to show the norm induced by an inner product as stated in Definition 5.1.10 is indeed a norm satisfying the formal definition of norms as stated in Definition 5.1.9.*

**Theorem 5.1.18 Parallelogram Equality**
Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2\big(\|u\|^2 + \|v\|^2\big).$$

***Proof 11.*** Note that

$$\begin{aligned}
\|u + v\|^2 + \|u - v\|^2 &= \langle u + v, u + v\rangle + \langle u - v, u - v\rangle \\
&= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle u, u\rangle + \langle v, v\rangle - \langle u, v\rangle - \langle v, u\rangle \\
&= \|u\|^2 + \|u\|^2 + \|v\|^2 + \|v\|^2 \\
&= 2\big(\|u\|^2 + \|v\|^2\big).
\end{aligned}$$

■

**Theorem 5.1.19**
Suppose $V$ is a real inner product space. Then,

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

***Proof 12.*** Note that

$$\begin{aligned}\|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\ &= \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle - \big(\|u\|^2 + \|v\|^2 - 2\langle u, v\rangle\big) \\ &= 4\langle u, v\rangle.\end{aligned}$$

So, we have

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

■

**Theorem 5.1.20**
Suppose $V$ is a complex inner product space. Then,

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+\mathrm{i}v\|^2\mathrm{i} - \|u-\mathrm{i}v\|^2\mathrm{i}}{4}.$$

***Proof 13.*** Note that

$$\begin{aligned}&\langle u+v, u+v\rangle - \langle u-v, u-v\rangle + \langle u+\mathrm{i}v, u+\mathrm{i}v\rangle\mathrm{i} - \langle u-\mathrm{i}v, u-\mathrm{i}v\rangle\mathrm{i} \\ &= 2\langle u, v\rangle + 2\langle v, u\rangle + (2\langle u, \mathrm{i}v\rangle + 2\langle \mathrm{i}v, u\rangle)\mathrm{i} \\ &= 2\langle u, v\rangle + 2\langle v, u\rangle + (-2\mathrm{i}\langle u, v\rangle + 2\mathrm{i}\langle v, u\rangle)\mathrm{i} \\ &= 2\langle u, v\rangle + 2\langle v, u\rangle + 2\langle u, v\rangle - 2\langle v, u\rangle \\ &= 4\langle u, v\rangle.\end{aligned}$$

so, we have

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+\mathrm{i}v\|^2\mathrm{i} - \|u-\mathrm{i}v\|^2\mathrm{i}}{4}.$$

■

**Theorem 5.1.21**
Let $U$ be a vector space. If $\|\ \|$ is a norm on $U$ satisfying the parallelogram equality, then there is an inner product $\langle\ \rangle$ on $U$ *s.t.* $\|u\| = \sqrt{\langle u, u\rangle} \quad \forall u \in U$.

## 5.2 Orthonormal Bases

**Definition 5.2.1 (Orthonormal).** A list of vectors is called *orthonormal* if each vector in the list has norm $1$ and is orthogonal to all the other vectors in the list. In other words, a list $e_1, \cdots, e_m$ of vectors in $V$ is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

**Theorem 5.2.2**
If $e_1, \cdots, e_m$ is an orthonormal list of vectors in $V$, then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2 \quad \forall a_1, \cdots, a_m \in \mathbb{F}.$$

***Proof 1.*** Note that

$$\langle a_1 e_1, a_2 e_2 + \cdots + a_m e_m \rangle = \langle a_1 e_1, a_2 e_2 \rangle + \cdots + \langle a_1 e_1, a_m e_m \rangle = 0.$$

So, by the Pythagorean Theorem, we have

$$\begin{aligned} \|a_1 e_1 + \cdots + a_m e_m\|^2 &= \|a_1 e_1\|^2 + \|a_2 e_2 + \cdots + a_m e_m\|^2 \\ &= \|a_1 e_1\|^2 + \|a_2 e_2\|^2 + \cdots + \|a_m e_m\|^2 \\ &= |a_1|^2 + |a_2|^2 + \cdots + |a_m|^2. \end{aligned}$$

■

**Theorem 5.2.3**
Every orthonormal list of vectors is L.I..

***Proof 2.*** Suppose $e_1, \cdots, e_m$ is an orthonormal list of vectors in $V$. Then, $\|a_1 e_1 + \cdots + a_m e_m\|^2 = 0$. By Theorem 5.2.2, it is equivalent to $|a_1|^2 + \cdots + |a_m|^2 = 0$. Since each $|a_j| \geq 0$, it must be $a_j = 0$ for all $j = 1, \cdots, m$. Therefore, the orthonormal list is L.I.. ■

**Definition 5.2.4 (Orthonormal Basis).** An *orthonormal basis* of $V$c is an orthonormal list of vectors in $V$ that is also a basis of $V$.

**Theorem 5.2.5**
Every orthonormal list of vectors in $V$ with length $\dim V$c is an orthonormal basis of $V$.

***Proof 3.*** By Theorem 5.2.3, any orthonormal list of vectors must be L.I.. Further since it has length $\dim V$, it is a basis of $V$. So, by definition, it is an orthonormal basis of $V$. ■

**Theorem 5.2.6**
Suppose $e_1, \cdots, e_n$ is an orthonormal basis of $V$ and $v \in V$. Then, $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$, and $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$.

***Proof 4.*** Suppose $v \in V$ and $v = a_1 e_1 + \cdots + a_n e_n$. Then,

$$\langle v, e_j \rangle = \langle a_1 e_1 + \cdots + a_n e_n, e_j \rangle = \langle a_j e_j, e_j \rangle = a_j.$$

So, we have

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n.$$

Further, by Theorem 5.2.2, we have

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

■

**Theorem 5.2.7 Gram-Schmidt Procedure**
Suppose $v_1, \cdots, v_m$ is L.I. list of vectors in $V$. Let $e_1 = \frac{v_1}{\|v_1\|}$. For $j = 2, \cdots, m$, define $e_j$ inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\|}. \tag{15}$$

Then, $e_1, \cdots, e_m$ is an orthonormal list of vectors in $V$ *s.t.* $\operatorname{span}(v_1, \cdots, v_j) = \operatorname{span}(e_1, \cdots, e_j)$ for $j = 1, \cdots, m$.

***Proof 5.*** To prove that Gram-Schmidt Procedure indeed produces an orthonormal list of vectors in $V$, we will use prove by mathematical induction.

$\boxed{\text{Base Case}}$ Suppose $j = 1$, then $\operatorname{span}(v_1) = \operatorname{span}(e_1)$ since $v_1$ is a positive multiple of $e_1$. So, the conclusion holds when $j = 1$.

$\boxed{\text{Inductive Steps}}$ Suppose for some $1 < j < m$, we have $\operatorname{span}(v_1, \cdots, v_{j-1}) = \operatorname{span}(e_1, \cdots, e_{j-1})$. Since$v_1, \cdots, v_m$ is L.I., we know $v_j \notin \operatorname{span}(v_1, \cdots, v_{j-1})$. That is, $v_j \notin \operatorname{span}(e_1, \cdots, e_{j-1})$. (*If* $v_j \in \operatorname{span}(e_1, \cdots, e_{j-1})$, *then* $v_j = \langle v_j, e_1 \rangle e_1 + \cdots + \langle v_j, e_{j-1} \rangle e_{j-1}$.*) Then, we are dividing by* $0$ *in Equation (15).* So, we are not dividing by $0$ in Equation (15). Dividing a vector by its norm produces a new vector with norm $1$, so $\|e_j\| = 1$. Now, we want to verify $e_j$ is orthogonal to $e_1, \cdots, e_{j-1}$. Pick some $k$ *s.t.* $1 \leq k < j$. Then

$$\begin{aligned}
\langle e_j, e_k \rangle &= \left\langle \frac{v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\|}, e_k \right\rangle \\
&= \frac{\langle v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}, e_k \rangle}{\|\langle v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\|} \\
&= \frac{\langle v_j, e_k \rangle - \langle v_j, e_k \rangle}{\|\langle v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\|} \\
&= 0
\end{aligned}$$

Then, $e_1, \cdots, e_j$ is an orthonormal basis, and $v_j \in \operatorname{span}(e_1, \cdots, e_j)$ since $v_j$ is a linear combination of $e_1, \cdots, e_j$ by Equation (15). Further, we have

$$\dim \operatorname{span}(v_1, \cdots, v_j) = \dim \operatorname{span}(e_1, \cdots, e_j)$$

and

$$\operatorname{span}(v_1, \cdots, v_j) \subseteq \operatorname{span}(e_1, \cdots, e_j).$$

That is, exactly, $\operatorname{span}(v_1, \cdots, v_j) = \operatorname{span}(e_1, \cdots, e_j)$. ■

**Theorem 5.2.8**
Every *f-d* inner product space has an orthonormal basis.

***Proof 6.*** Suppose $V$ is *f-d*, and select a basis of $V$. Apply Gram-Schmidt Procedure (Theorem 5.2.7) to this basis, we then have an orthonormal basis of $V$. ■

**Theorem 5.2.9**
Suppose $V$ is *f-d*. Then, every orthonormal list of vectors in $V$ can be extended to an orthonormal basis of $V$.

***Proof 7.*** Suppose $e_1, \cdots, e_m$ is an orthonormal list of vectors in $V$. Then, $e_1, \cdots, e_m$ is L.I. and can be extended to a basis $e_1, \cdots, e_m, v_1, \cdots, v_n$ of $V$. Apply Gram-Schmidt Procedure to this basis, we get an orthonormal list $e_1, \cdots, e_m, f_1, \cdots, f_n$. Here, $e_1, \cdots, e_m$ is unchanged since they are already orthonormal. Then, $e_1, \cdots, e_m, f_1, \cdots, f_n$ is an orthonormal basis of $V$. ■

**Theorem 5.2.10**
Suppose $T \in \mathcal{L}(V)$. If $T$ has an upper-triangular matrix with respect to some basis of $V$, then $T$ has an upper-triangular matrix with respect to some orthonormal basis of $V$.

***Proof 8.*** Suppose $\mathcal{M}(T)$ is upper-triangular with respect to a basis $v_1, \cdots, v_n$ of $V$. Then, we know $\text{span}(v_1, \cdots, v_j)$ is invariant under $T$ for $j = 1, \cdots, n$. Apply Gram-Schmidt Procedure to $v_1, \cdots, v_n$, we will get an orthonormal basis $e_1, \cdots, e_n$ of $V$. Further, since $\text{span}(e_1, \cdots, e_j) = \text{span}(v_1, \cdots, v_j)$ for $j = 1, \cdots, n$, we know $\text{span}(e_1, \cdots, e_j)$ is invariant under $T$. Therefore, $T$ has an upper-triangular matrix with respect to the orthonormal basis $e_1, \cdots, e_n$. ■

**Theorem 5.2.11 Schur's Theorem**
Suppose $V$ is a *f-d* complex vector space and $T \in \mathcal{L}(V)$. Then, $T$ has an upper-triangular matrix with respect to some orthonormal basis of $V$.

***Proof 9.*** Since $V$ is a *f-d* complex vector space, $T$ must have an upper-triangular matrix with respect to some basis of $V$. Further, by Theorem 5.2.10, $T$ must have an upper-triangular matrix with respect to an orthonormal basis of $V$. ■

**Example 5.2.12** The function $\varphi : \mathbb{F}^3 \to \mathbb{F}$ defined by

$$\varphi(z_1, z_2, z_3) = 2z_1 - 5z_2 + z_3$$

is a linear functional on $\mathbb{F}^3$. We could write this linear functional in the form $\varphi(z) = \langle z, u \rangle$ for every $z \in \mathbb{F}^3$, where $u = \langle 2, -5, 1 \rangle$.

**Theorem 5.2.13 Riesz Representation Theorem**
Suppose $V$ is *f-d* and $\varphi$ is a linear functional on $V$. Then, there is a unique vector $u \in V$ *s.t.* $\varphi(v) = \langle v, u \rangle$ for every $v \in V$.

***Proof 10.*** Let $e_1, \cdots, e_n$ be an orthonormal basis of $V$. Then, for all $v \in V$, we have

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n.$$

So,

$$\begin{aligned}\varphi(v) &= \varphi(\langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n) \\ &= \langle v, e_1\rangle \varphi(e_1) + \cdots + \langle v, e_n\rangle \varphi(e_n) \\ &= \langle v, \overline{\varphi(e_1)} e_1\rangle + \cdots + \langle v, \overline{\varphi(e_n)} e_n\rangle \\ &= \langle v, \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n\rangle.\end{aligned}$$

Suppose $\exists u_1, u_2 \in V$ *s.t.* $\varphi(v) = \langle v, u_1\rangle = \langle v, u_2\rangle$. Then, $\langle v, u_1\rangle - \langle v, u_2\rangle = \langle v, u_1 - u_2\rangle = 0$. Let $v = u_1 - u_2$, then we have $\langle u_1 - u_2, u_1 - u_2\rangle = 0$. So, it must be $u_1 = u_2$. Therefore, $\exists$ a unique $u \in V$ and

$$u = \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n \textit{ s.t. } \varphi(v) = \langle v, u\rangle \quad \forall v \in V.$$

■

**Example 5.2.14** Find $u \in \mathcal{P}_2(\mathbb{R})$ *s.t.* $\int_{-1}^{1} p(t)(\cos(\pi t))\, \mathrm{d}t = \int_{-1}^{1} p(t)u(t)\, \mathrm{d}t$ for every $p \in \mathcal{P}_2(\mathbb{R})$.

**Remark.** *Define an inner product on* $\mathcal{P}_2(\mathbb{R})$ *as* $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\, \mathrm{d}x$ *to solve this problem.*

***Solution 11.***

Let $\varphi \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}), \mathbb{R})$ be defined as $\varphi(t) = \int_{-1}^{1} p(t)(\cos(\pi t))\, \mathrm{d}t$. Note that $1, x, x^2$ is a basis of $\mathcal{P}_2(\mathbb{R})$. To find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, apply Gram-Schmidt Procedure, we have

$$e_1 = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} 1 \cdot 1\, \mathrm{d}t}} = \sqrt{\frac{1}{2}}.$$

Since $x - \langle x, e_1\rangle e_1 = x - \int_{-1}^{1} x\sqrt{\frac{1}{2}}\, \mathrm{d}x \cdot \sqrt{\frac{1}{2}} = x$, and $\|x\| = \sqrt{\int_{-1}^{1} x^2\, \mathrm{d}x} = \sqrt{\frac{2}{3}}$, we have

$$e_2 = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x.$$

Further, consider

$$\begin{aligned}x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2 &= x^2 - \int_{-1}^{1} x^2\sqrt{\frac{1}{2}}\, \mathrm{d}x \cdot \sqrt{\frac{1}{2}} - \int_{-1}^{1} x^2\sqrt{\frac{3}{2}}x\, \mathrm{d}x \cdot \sqrt{\frac{3}{2}}x \\ &= x^2 - \frac{1}{3},\end{aligned}$$

and note that

$$\left\|x^2-\frac{1}{3}\right\|=\sqrt{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2\,\mathrm{d}x}=\sqrt{\int_{-1}^{1}x^4-\frac{2}{3}x^2+\frac{1}{9}\,\mathrm{d}x}=\sqrt{\frac{8}{45}}.$$

So, we have

$$e_3=\frac{x^2-\dfrac{1}{3}}{\sqrt{\dfrac{8}{45}}}=\sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right).$$

That is, $e_1=\sqrt{\frac{1}{2}},\ e_2=\sqrt{\frac{3}{2}}x,\ e_3=\sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)$ is an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$. Then, we have

$$\varphi(e_1)=\int_{-1}^{1}\sqrt{\frac{1}{2}}\cos(\pi t)\,\mathrm{d}t=\sqrt{\frac{1}{2}}\int_{-1}^{1}\cos(\pi t)\,\mathrm{d}t=0$$

$$\varphi(e_2)=\int_{-1}^{1}\sqrt{\frac{3}{2}}t\cos(\pi t)\,\mathrm{d}t=\sqrt{\frac{3}{2}}\int_{-1}^{1}t\cos(\pi t)\,\mathrm{d}t=0$$

$$\begin{aligned}\varphi(e_3)&=\int_{-1}^{1}\sqrt{\frac{45}{8}}\left(t^2-\frac{1}{3}\right)\cos(\pi t)\,\mathrm{d}t\\&=\sqrt{\frac{45}{8}}\int_{-1}^{1}t^2\cos(\pi t)\,\mathrm{d}t-\sqrt{\frac{45}{8}}\cdot\frac{1}{3}\underbrace{\int_{-1}^{1}\cos(\pi t)\,\mathrm{d}t}_{0}\\&=\sqrt{\frac{45}{8}}\int_{-1}^{1}t^2\cos(\pi t)\,\mathrm{d}t\\&=\sqrt{\frac{45}{8}}\left(-\frac{4}{\pi^2}\right).\end{aligned}$$

So, by Theorem 5.2.15 and its proof, we know

$$\begin{aligned}u=\varphi(e_1)e_1+\varphi(e_2)e_2+\varphi(e_3)e_3&=0+0+\sqrt{\frac{45}{8}}\left(-\frac{4}{\pi^2}\right)\sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)\\&=\frac{45}{8}\left(-\frac{4}{\pi^2}\right)\left(x^2-\frac{1}{3}\right)\\&=-\frac{45}{2\pi^2}\left(x^2-\frac{1}{3}\right).\end{aligned}$$

$\square$

## 5.3 Orthogonal Complements and Minimization Problems

**Definition 5.3.1 (Orthogonal Complement, $U^\perp$).** If $U$ is a subset of $V$, then the *orthogonal complement* of $U$, denoted $U^\perp$, is the set of all vectors in $V$ that are orthogonal to every vector in $U$:

$$U^\perp = \{v \in v \mid \langle v, u\rangle = 0 \quad \forall u \in U\}.$$

**Theorem 5.3.2 Basic Properties of Orthogonal Complements**

1. If $U$ is a subset of $V$, then $U^\perp$ is a subspace of $V$.
2. $\{0\}^\perp = V$.
3. $V^\perp = \{0\}$.
4. If $U$ is a subset of $V$, then $U \cap U^\perp \subseteq \{0\}$.
5. If $U$ and $W$ are subsets of $V$ and $U \subseteq W$, then $W^\perp \subseteq U^\perp$.

***Proof 1.***

1. Let $v, w \in U^\perp$. Then $\langle v + w, u\rangle = \langle v, w\rangle + \langle w, u\rangle = 0 + 0 = 0$. So, $v + w \in U^\perp$. Further, suppose $\lambda \in \mathbb{F}$. Then $\langle \lambda v, u\rangle = \lambda\langle v, u\rangle = \lambda \cdot 0 = 0$. So, $\lambda v \in U^\perp$. Finally since $\langle 0, u\rangle = 0$, we know $0 \in U^\perp$. Then, $U^\perp$ is a subspace of $V$. □
2. Since $\langle v, 0\rangle = 0 \quad \forall v \in V$, we know $\{0\}^\perp = V$. □
3. Suppose $v \in V^\perp$. Then, $\langle v, v\rangle = 0$. By property of an inner product, it must be that $v = 0$. So, $V^\perp = \{0\}$. □
4. Suppose $U$ is a subset of $V$. Let $v \in U \cap U^\perp$. Then, $v \in U$ and $v \in U^\perp$. So, $\langle v, v\rangle = 0$. Then, it must be that $v = 0$. So, $U \cap U^\perp \subseteq \{0\}$. □
5. Suppose $U$ and $W$ are subsets of $V$ with $U \subseteq W$. Suppose $v \in W^\perp$. Then, $\langle v, u\rangle = 0 \quad \forall u \in W$. Since $U \subseteq W$, we have $\langle v, w\rangle = 0 \quad \forall u \in U$. That is, $v \in U^\perp$. Then, we have $W^\perp \subseteq U^\perp$.

■

**Theorem 5.3.3**
Suppose $U$ is a *f-d* subspace of $V$. Then, $V = U \oplus U^\perp$.

***Proof 2.*** Suppose $u \in U$ and $w \in U^\perp$. Then, $\forall v \in V$, we have $v = cu + w$ for some $c \in \mathbb{F}$ and $\langle u, w\rangle = 0$. Then, we have $V = U + U^\perp$. Further, by Theorem 5.3.2(4), $U \cap U^\perp = \{0\}$ since $U$ and $U^\perp$ are all subspaces of $V$. Hence, $V = U \oplus U^\perp$. ■

**Corollary 5.3.4** Suppose $V$ is *f-d* and $U$ is a subspace of $V$. Then, $\dim U^\perp = \dim V - \dim U$.

**Theorem 5.3.5**
Suppose $U$ is a *f-d* subspace of $V$. Then, $U = (U^\perp)^\perp$.

***Proof 3.***
($\subseteq$). Suppose $u \in U$. Then, $\langle u, v\rangle = 0 \quad \forall v \in U^\perp$. Then, $u \in (U^\perp)^\perp$. That is, $U \subseteq (U^\perp)^\perp$. □

($\supseteq$). Suppose $v \in (U^\perp)^\perp$. Then, $v = u + w$ for some $u \in U$ and $w \in U^\perp$. Then, $w = v - u \in (U^\perp)^\perp$. Since $U \subseteq (U^\perp)^\perp$, we know $u \in U^\perp$. Then, $v - u \in (U^\perp)^\perp$. Hence, $v - u \in U^\perp \cap (U^\perp)^\perp$. That is, $v - u$ is orthogonal to itself. So, it must be that $v - u = 0$ or $v = u$. Since $u \in U$ and $v \in U$, we have shown that $(U^\perp)^\perp \subseteq U$. ■

**Definition 5.3.6 (Orthogonal Projection, $P_U$).** Suppose $U$ is a *f-d* subspace of $V$. Then orthogonal projection of $V$ onto $U$ is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then, $P_U v = u$.

**Remark.** *By Theorem 5.3.3, $V = U \oplus U^\perp$, which ensures each $v \in V$ can be uniquely represented in the form of $u + w$ with $u \in U$ and $w \in U^\perp$, and thus $P_U$ is well-defined.*

**Example 5.3.7** Suppose $x \in V$ with $x \neq 0$ and $U = \operatorname{span}(x)$. Show that

$$P_U v = \frac{\langle v, x\rangle}{\|x^2\|} x \qquad \forall v \in V.$$

***Proof 4.*** Suppose $v \in V$. Then,

$$v = \frac{\langle v, x\rangle}{\|x^2\|} x + \left(v - \frac{\langle v, x\rangle}{\|x^2\|} x\right),$$

where $\frac{\langle v, x\rangle}{\|x^2\|} x \in \operatorname{span}(x)$ and $v - \frac{\langle v, x\rangle}{\|x^2\|} x \in U^\perp$. Thus, $P_U v = \frac{\langle v, x\rangle}{\|x^2\|} x$. ■

**Theorem 5.3.8 Properties of Orthogonal Projections**
Suppose $U$ is a *f-d* subspace of $V$ and $v \in V$. Then,

1. $P_U \in \mathcal{L}(V)$.
2. $P_U u = u \quad \forall u \in U$.
3. $P_U w = 0 \quad \forall w \in U^\perp$.
4. $\operatorname{range} P_U = U$.
5. $\operatorname{null} P_U = U^\perp$.
6. $v - P_U v \in U^\perp$.
7. $P_U^2 = P_U$.
8. $\|P_U v\| \leq \|v\|$.
9. for every orthonormal basis $e_1, \cdots, e_m$ of $U$,

$$P_U v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m.$$

***Proof 5.***

1. Suppose $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$, where $v_1, v_2 \in V$, $u_1, u_2 \in U$, and $w_1, w_2 \in U^\perp$. Then,

$v_1 + v_2 = (u_1 + u_2) + (w_1 + w_2)$, where $u_1 + u_2 \in U$ and $w_1 + w_2 \in U^\perp$. So,

$$P_U(v_1 + v_2) = u_1 + u_2 = P_U v_1 + P_U v_2.$$

Additionally, suppose $\lambda \in \mathbb{F}$. Then, $\lambda v_1 = \lambda u_1 + \lambda w_1$, where $\lambda u_1 \in U$ and $\lambda w_1 \in U^\perp$. Then,

$$P_U(\lambda v_1) = \lambda u_1 = \lambda P_U(v_1). \qquad \square$$

2. Suppose $u \in U$. Then, $u = u + 0$, where $u \in U$ and $0 \in U^\perp$. So, $P_U u = u$. $\square$

3. Suppose $w \in U^\perp$. Then, $w = 0 + w$, where $0 \in U$ and $w \in U^\perp$. So, $P_U w = 0$. $\square$

4. By definition of $P_U$, we have $\operatorname{range} P_U \subseteq U$. By Theorem 5.3.8(2), we know $U \subseteq \operatorname{range} P_U$. So, $\operatorname{range} P_U = U$. $\square$

5. By Theorem 5.3.8(3), we have $U^\perp \subseteq \operatorname{null} P_U$. Further note if $v \in \operatorname{null} P_U$, then $v = 0 + v$ with $0 + u$ and $v \in U^\perp$. So, $\operatorname{null} P_U \subseteq U^\perp$. That is, $\operatorname{null} P_U = U$. $\square$

6. If $v = u + w$ with $u \in U$ and $w \in U^\perp$, then

$$v - P_U v = v - u = w \in U^\perp. \qquad \square$$

7. If $v = u + w$ with $u \in U$ and $w \in U^\perp$, then

$$(P_U^2)v = P_U(P_U v) = P_U u = u = P_U v.$$

So, $P_U^2 = P_U$. $\square$

8. If $v = u + w$ with $u \in U$ and $w \in U^\perp$, then we have

$$\|P_U v\|^2 = \|u\|^2 \leq \|u\|^2 + \|w\|^2 = \|v\|^2$$

by the Pythagorean Theorem. $\square$

9. If $v = u + w$ with $u \in U$ and $w \in U^\perp$, then

$$v = u + w = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m + (v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m).$$

Since $e_1, \cdots, e_m$ is an orthonormal basis of $U$, we have $\langle ve_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m \in U$. Now, consider

$$\begin{aligned}
\big\langle \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m, v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m \big\rangle &= \big\langle \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m, v \big\rangle - \|u\|^2 \\
&= \langle v, e_1\rangle\langle e_1, v\rangle + \cdots + \langle v, e_m\rangle\langle e_m, v\rangle - \|u\|^2 \\
&= \langle v, e_1\rangle\overline{\langle v, e_1\rangle} + \cdots + \langle v, e_m\rangle\overline{\langle v, e_m\rangle} - \|u\|^2 \\
&= |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 - \|u\|^2 \\
&= \|u\|^2 - \|u\|^2 = 0 \quad (By\ Theorem\ 5.2.2)
\end{aligned}$$

Then, $v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m \in U^\perp$. So, we have $P_U v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$.

$\blacksquare$

**Theorem 5.3.9 Minimizing the Distance to a Subspace**
Suppose $U$ is a *f-d* subspace of $V$, $v \in V$, and $u \in U$. Then, $\|v - P_U v\| \leq \|v - u\|$. The inequality is an equality if and only if $u = P_U v$.

***Proof 6.*** Note that $\|v - P_U v\|^2 \leq \|v - P_U v\|^2 + \|P_U v - u\|^2$ since $\|P_U v - u\|^2 \geq 0$. Further, since $v - P_U v \in U^\perp$ by Theorem 5.3.8(6) and $P_U v - u \in U$ by the Pythagorean Theorem, we have

$$\|v - P_U v\|^2 + \|P_U v - u\|^2 = \|v = P_U v + P_U v - u\|^2 = \|v - u\|^2.$$

Then, $\|u - P_U v\|^2 \leq \|v = P_U v\|^2 + \|P_U v - u\|^2 = \|v - u\|^2$. Since $\|v - P_U v\|^2 \geq 0$ and $\|v - u\|^2 \geq 0$, we have $\|v - P_U v\| \leq \|v - u\|$. The equality holds if and only if $\|P_U v - u\|^2 = 0$. That is, $\|P_U v - u\| = 0$, $P_U v - u = 0$, or $P_U v = u$. ■

**Example 5.3.10** In $\mathbb{R}^4$, set $U = \operatorname{span}((1,1,0,0),(1,1,1,2))$. Find $u \in U$ ***s.t.*** $\|u - (1,2,3,4)\|$ is as small as possible.

***Solution 7.***

By Theorem 5.3.9, we need to find $P_U v = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2$. Thus, we need to use Gram-Schmidt Procedure to find $e_1$ and $e_2$:

$$e_1 = \frac{1}{\sqrt{2}}(1,1,0,0) \qquad \text{and} \qquad e_2 = \frac{1}{\sqrt{5}}(0,0,1,2).$$

Set $v = (1,2,3,4)$, we have

$$\begin{aligned} P_U v &= \langle (1,2,3,4), \frac{1}{\sqrt{2}}(1,1,0,0) \rangle \frac{1}{\sqrt{2}}(1,1,0,0) + \langle (1,2,3,4), \frac{1}{\sqrt{5}}(0,0,1,2) \rangle \frac{1}{\sqrt{5}}(0,0,1,2) \\ &= \left( \frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5} \right). \end{aligned}$$

□

# 6 Operators on Inner Product Spaces

## 6.1 Self-Adjoint and Normal Operators

**Definition 6.1.1 (Adjoint, $T^*$).** Suppose $T \in \mathcal{L}(V, W)$. The *adjoint* of $T$ is the function $T^* : W \to V$ *s.t.*

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and every $w \in W$.

**Theorem 6.1.2**
If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

***Proof 1.***

1. The definition of adjoint makes sense.

   Suppose $T \in \mathcal{L}(V, W)$. Fix $w \in W$. Let $f : V \to \mathbb{F}$ be defined as $v \mapsto \langle Tv, w \rangle$. Then, $f$ is a linear functional on $V$. Note that

   $$\begin{aligned} f(au + bv) = \langle T(au + bv), w \rangle &= \langle aTu + bTv, w \rangle \\ &= a\langle Tu, w \rangle + b\langle Tv, w \rangle \\ &= af(u) + b(fv). \end{aligned}$$

   By Riesz Representation Theorem, we know $f(v) = \langle v, \Delta \rangle$ for some $\Delta \in V$. We call this unique $\Delta$ as $T^* w$. That is, for each $w \in W$, $\exists$ unique $T^* w \in V$. So, $T^*$ is well-defined as a function from $W$ to $V$. $\square$

2. Adjoint is a linear map.

   Suppose $w_1, w_2 \in W$. If $v \in V$, then

   $$\begin{aligned} \langle v, T^*(w_1 + w_2) \rangle = \langle Tv, w_1 + w_2 \rangle &= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle \\ &= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle \\ &= \langle v, T^* w_1 + T^* w_2 \rangle. \end{aligned}$$

   So, $T^*(w_1 + w_2) = T^* w_1 + T^* w_2$. $\square$

   Now fix $w \in W$ and $\lambda \in \mathbb{F}$. If $v \in V$, then

   $$\begin{aligned} \langle v, T^*(\lambda w) \rangle = \langle Tv, \lambda w \rangle &= \overline{\lambda} \langle Tv, w \rangle \\ &= \overline{\lambda} \langle v, T^* w \rangle \\ &= \langle v, \lambda T^* w \rangle. \end{aligned}$$

   So, we know $T^*(\lambda w) = \lambda T^* w$. $\square$

   Thus, we've shown $T^*$ is a linear map as desired.

$\blacksquare$

**Example 6.1.3** Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$. Find a formula for $T^*$.

***Solution 2.***

Define $T^* : \mathbb{R}^2 \to \mathbb{R}^3$. Let $y = (y_1, y_2) \in \mathbb{R}^2$. Then,

$$\begin{aligned}\langle x, T^* y\rangle = \langle Tx, y\rangle &= y_1 x_2 + 3y_1 x_3 + 2x_1 y_2 \\ &= \langle (x_1, x_2, x_3), (2y_2, y_1, 3y_1)\rangle.\end{aligned}$$

Thus, $T^* : \mathbb{R}^2 \to \mathbb{R}^3$ is defined as $T^*(y_1, y_2) = (2y_2, y_1, 3y_1)$. □

**Example 6.1.4** Fix $u \in V$ and $x \in W$. Define $T \in \mathcal{L}(V, W)$ by $Tv = \langle v, u\rangle x$ for every $v \in V$. Find a formula for $T^*$.

***Solution 3.***

Define $T^* \in \mathcal{L}(W, V)$. Consider

$$\begin{aligned}\langle v, T^* w\rangle = \langle Tv, w\rangle &= \big\langle \langle v, u\rangle x, w\big\rangle \\ &= \langle v, u\rangle\langle x, w\rangle \\ &= \big\langle v, \langle w, x\rangle u\big\rangle.\end{aligned}$$

So, we have $T^* w = \langle w, x\rangle u$. □

**Theorem 6.1.5 Properties of the Adjoint**

1. $(S + T)^* = S^* + T^* \quad \forall S, T \in \mathcal{L}(V, W)$.
2. $(\lambda T)^* = \overline{\lambda} T^* \quad \forall \lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$.
3. $(T^*)^* = T \quad \forall T \in \mathcal{L}(V, W)$.
4. $I^* = I$, where $I$ is the identity operator on $V$.
5. $(ST)^* = T^* S^* \quad \forall T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$.

***Proof 4.***

1. Consider
$$\begin{aligned}\langle v, (S+T)^* w\rangle = \langle (S+T)v, w\rangle &= \langle Sv, w\rangle + \langle Tv, w\rangle \\ &= \langle v, S^* w\rangle + \langle v, T^* w\rangle \\ &= \langle v, S^* w + T^* w\rangle \\ &= \langle v, (S^* + T^*)w\rangle.\end{aligned}$$
So, we have $(S + T)^* w = (S^* + T^*)w \quad \forall w \in W$. □

2. Note that
$$\begin{aligned}\langle v, (\lambda T)^* w\rangle = \langle (\lambda T)v, w\rangle &= \lambda\langle Tv, w\rangle \\ &= \lambda\langle v, T^* w\rangle \\ &= \langle v, \overline{\lambda} T^* w\rangle.\end{aligned}$$

So, we get $(\lambda T)^* w = \overline{\lambda} T^* w \quad \forall w \in W.$ $\square$

3. Consider
$$\begin{aligned}\langle v, (T^*)^* w\rangle = \langle T^* v, w\rangle &= \overline{\langle w, T^* v\rangle} \\ &= \overline{\langle Tw, v\rangle} \\ &= \langle v, Tw\rangle.\end{aligned}$$
So, it is $(T^*)^* w = Tw \quad \forall w \in W.$ $\square$

4. Note we have
$$\langle v, I^* w\rangle = \langle Iv, w\rangle = \langle v, w\rangle.$$
So, $I^* w = w \quad \forall w \in W$. That is, $I^* = I$. $\square$

5. We have
$$\begin{aligned}\langle v, (ST)^* w\rangle = \langle (ST)v, w\rangle &= \langle S(Tv), w\rangle \\ &= \langle Tv, S^* w\rangle \\ &= \langle v, T^*(S^* w)\rangle.\end{aligned}$$
So, $(ST)^* w = T^*(S^* w) = (T^* S^*) w \quad \forall w \in W.$

■

**Theorem 6.1.6 Null Space and Range of $T^*$**
Suppose $T \in \mathcal{L}(V, W)$. Then,

1. $\text{null}\, T^* = (\text{range}\, T)^\perp$.
2. $\text{range}\, T = (T^*)^\perp$.
3. $\text{null}\, T = (\text{range}\, T^*)^\perp$.
4. $\text{range}\, T^* = (\text{null}\, T)^\perp$.

***Proof 5.***

1. Suppose $w \in \text{null}\, T^*$. Then, $T^* w = 0$. So, $\langle v, T^* w\rangle = 0$. That is, $\langle Tv, w\rangle = 0 \quad \forall v \in 0$. Then, $w$ is orthogonal to any $Tv$. That is, $w \in (\text{range}\, T)^\perp$. Conversely, if $w \in (\text{range}\, T)^\perp$, we have $\langle Tv, w\rangle = 0$, and thus $\langle v, T^* w\rangle = 0$, or $T^* w = 0$. That is, $w \in \text{null}\, T^*$. Hence, $\text{null}\, T^* = (\text{range}\, T)^\perp$. $\square$

2. Note that $(\text{null}\, T^*)^\perp = \left((\text{range}\, T)^\perp\right)^\perp = \text{range}\, T.$ $\square$

3. Suppose $v \in \text{null}\, T$. Then, $Tv = 0$, and $\langle Tv, w\rangle = 0$. So, $\langle v, T^* w\rangle = 0 \quad \forall w \in W$. Then, $v$ is orthogonal to every vectors in $T^* w$. So, $v \in (\text{range}\, T^*)^\perp$. In the other way around, if we assume $v \in (\text{range}\, T^*)^\perp$, then $\langle v, T^* w\rangle = \langle Tv, w\rangle = 0$. So, $Tv = 0$, and thus $v \in \text{null}\, T$. Hence, we have $\text{null}\, T = (\text{range}\, T^*)^\perp$. $\square$

4. Consider $(\text{null}\, T)^\perp = \left((\text{range}\, T^*)^\perp\right)^\perp = \text{range}\, T^*.$

■

**Definition 6.1.7 (Conjugate Transpose).** The *conjugate transpose* of an $m \times n$ matrix is the $n \times m$ matrix obtained by interchanging the rows and columns and then taking the conjugate of each entry.

**Theorem 6.1.8**
Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \cdots, e_n$ is an orthonormal basis of $V$ and $f_1, \cdots, f_m$ is an orthonormal basis of $W$. Then, $\mathcal{M}(T^*, (f_1, \cdots, f_m), (e_1, \cdots, e_m))$ is the conjugate transpose of $\mathcal{M}(T, (e_1, \cdots, e_n), (f_1, \cdots, f_m))$.

***Proof 6.*** Suppose $\mathcal{M}(T)$ denote the matrix $\mathcal{M}(T, (e_1, \cdots, e_n), (f_1, \cdots, f_m))$ and let $\mathcal{M}(T^*)$ denote the matrix $\mathcal{M}(T^*, (f_1, \cdots, f_m), (e_1, \cdots, e_m))$. Then, note that $Te_k = \langle Te_k, f_1\rangle f_1 + \cdots + \langle Te_k, f_m\rangle f_m$. So,

$$(\mathcal{M}(T))_{j,k} = \langle Te_k, f_j\rangle.$$

Further, consider $T^* f_k = \langle T^* f_k, e_1\rangle e_1 + \cdots + \langle T^* f_k, e_n\rangle e_n$. That is,

$$\begin{aligned}(\mathcal{M}(T^*))_{j,k} = \langle T^* f_k, e_j\rangle &= \overline{\langle e_j, T^* f_k\rangle} \\ &= \overline{\langle Te_j, f_k\rangle} \\ &= \overline{(\mathcal{M}(T))_{k,j}}\end{aligned}$$

So, we've shown that $\mathcal{M}(T^*)$ is the conjugate transpose of $\mathcal{M}(T)$. ■

**Definition 6.1.9 (Self-Adjoint).** An operator $T \in \mathcal{L}(V)$ is called *self-adjoint* if $T = T^*$. In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if $\langle Tv, w\rangle = \langle v, Tw\rangle \quad \forall v, w \in V$.

**Theorem 6.1.10**
The sum of two self-adjoint operators is self-adjoint, and the product of a real scalar and a self-adjoint operator is self-adjoint.

***Proof 7.***

1. Suppose $T, S \in \mathcal{L}(V)$ are self-adjoint. Then,

$$(S + T)^* = S^* + T^* = S + T.$$

   So, $S + T$ is self-adjoint. □

2. Let $\lambda \in \mathbb{R}$. Then,

$$(\lambda T)^* = \lambda T^* = \lambda T.$$

   So, $\lambda T$ is self-adjoint.

■

**Theorem 6.1.11**
Every eigenvalue of a self-adjoint operator is real.

***Proof 8.*** Suppose $T$ is a self-adjoint operator on $V$. Let $\lambda$ be an eigenvalue of $T$, and let $v$ be a non-zero vector in $V$ *s.t.* $Tv = \lambda v$. Then,

$$\lambda\|v\|^2 = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\|v\|^2.$$

So, it must be $\lambda = \overline{\lambda}$, which means $\lambda$ is real. ■

**Theorem 6.1.12**
Suppose $V$ is a complex inner product space and $T \in \mathcal{L}(V)$. Suppose $\langle Tv, v\rangle = 0 \quad \forall v \in V$. Then, $T = 0$.

***Proof 9.*** Note that

$$\begin{aligned}\langle Tu, w\rangle &= \frac{1}{4}\Big[\big\langle T(u+w), u+w\big\rangle - \big\langle T(u-w), u-w\big\rangle\Big] \\ &\quad + \frac{\mathrm{i}}{4}\Big[\big\langle T(u+\mathrm{i}w), u+\mathrm{i}w\big\rangle - \big\langle T(u-\mathrm{i}w), (u-\mathrm{i}w)\big\rangle\Big] \\ &= 0 \quad \forall u, w \in V.\end{aligned}$$

Let $w = Tu \in V$. Then, $\langle Tu, Tu\rangle = 0$. That is, $Tu = 0 \quad \forall u \in V$. So, $T = 0$. ■

**Theorem 6.1.13**
Suppose $V$ is a complex inner product space and $T \in \mathcal{L}(V)$. Then, $T$ is self-adjoint if and only if $\langle Tv, v\rangle \in \mathbb{R} \quad \forall v \in V$.

***Proof 10.***
($\Rightarrow$) Let $v \in V$. Then,

$$\begin{aligned}\langle Tv, v\rangle - \overline{\langle Tv, v\rangle} &= \langle Tv, v\rangle - \langle v, Tv\rangle \\ &= \langle Tv, v\rangle - \langle T^*v, v\rangle \\ &= \langle (T - T^*)v, v\rangle\end{aligned} \tag{16}$$

If $\langle Tv, v\rangle \in \mathbb{R} \quad \forall v \in V$, then Equation (16)$= 0$. That is, $\langle (T - T^*)v, v\rangle = 0 \quad \forall v \in V$. So, $T - T^* = 0$, or $T = T^*$. That is, $T$ is self-adjoint. □

($\Leftarrow$) Conversely, if $T$ is self-adjoint, then Equation (16)$= 0$. That is, $\langle Tv, v\rangle = \overline{\langle Tv, v\rangle} = 0$, or we have $\langle Tv, v\rangle = \overline{\langle Tv, v\rangle}$. This is equivalent to the conclusion $\langle Tv, v\rangle \in \mathbb{R}$. ■

**Theorem 6.1.14**
Suppose $T$ is a self-adjoint operator on $V$ *s.t.* $\langle Tv, v\rangle = 0 \quad \forall v = V$. Then, $T = 0$.

***Proof 11.*** We've already shown this to be true under a complex inner product space. Thus, we can assume $V$ is a real inner product space. If $u, w \in V$, then

$$\begin{aligned}\langle Tu, w\rangle &= \frac{1}{4}\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle \\ &= 0 \quad \forall u, w \in V.\end{aligned}$$

Let $w = Tu$. Then, $\langle Tu, Tu\rangle = 0$, or $Tu = 0 \quad \forall u \in V$. So, $T = 0$. ■

**Definition 6.1.15 (Normal Operator).** An operator on an inner product space is called *normal* if it commutes with its adjoint. In other words, $T \in \mathcal{L}(V)$ is normal if $TT^* = T^*T$.

**Example 6.1.16** Let $T$ be the operator on $\mathbb{F}^2$ whose matrix with respect to the standard basis is

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

Show that $T$ is not self-adjoint but is still normal.

***Proof 12.*** Since $\mathcal{M}(T) = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ and $\mathcal{M}(T^*) = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$, then $\mathcal{M}(T) \neq \mathcal{M}(T^*)$, and thus it is not self-adjoint. However, note that

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}.$$

So, by definition, $T$ is normal. ■

**Theorem 6.1.17**
An operator $T \in \mathcal{L}(V)$ is normal if and only if $\|Tv\| = \|T^*v\| \quad \forall v \in V$.

***Proof 13.*** Note that

$$\begin{aligned} T \text{ is normal} &\iff T^*T - TT^* = 0 \\ &\iff \big\langle (T^*T - TT^*)v, v \big\rangle = 0 \quad \forall v \in V \\ &\iff \langle T^*Tv, v \rangle = \langle TT^*v, v \rangle \quad \forall v \in V \\ &\iff \langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle \quad \forall v \in V \\ &\iff \|Tv\|^2 = \|T^*v\|^2 \quad \forall v \in V. \end{aligned}$$

Since $\|Tv\| \geq 0$ and $\|T^*v\| \geq 0$, it is equivalent to

$$\|Tv\| = \|T^*v\| \quad \forall v \in V.$$

■

**Theorem 6.1.18**
Suppose $T \in \mathcal{L}(V)$ is normal and $v \in V$ is an eigenvector of $T$ with eigenvalue $\lambda$. Then, $v$ is also an eigenvector of $T^*$ with eigenvalue $\overline{\lambda}$.

***Proof 14.*** Note that $(T - \lambda I)^* = T^* - \overline{\lambda} I$. Consider $(T - \lambda I)(T - \lambda I)^* = TT^* - \overline{\lambda}T - \lambda T^* + \lambda\overline{\lambda}$ and $(T - \lambda I)^*(T - \lambda I) = T^*T - \overline{\lambda}T - \lambda T^* + \lambda\overline{\lambda}$. Since, $T$ is normal, $TT^* = T^*T$. So.

$$(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)^*(T - \lambda I).$$

That is, $T - \lambda I$ is also normal. So, by Theorem 6.1.17, we have

$$\|(T - \lambda I)v\| = \|(T^* - \overline{\lambda} I)v\| = 0.$$

That is, $T^*v = \overline{\lambda}v$, or $v$ is an eigenvector of $T^*$ with eigenvalue $\overline{\lambda}$. ■

**Theorem 6.1.19**
Suppose $T \in \mathcal{L}(V)$ is normal. Then, eigenvectors of $T$ corresponding to distinct eigenvalues are orthogonal.

***Proof 15.*** Suppose $\alpha, \beta$ are distinct eigenvalues of $T$, with corresponding eigenvectors $u, v$. Then, $Tu = \alpha u$ and $Tv = \beta v$. By Theorem 6.1.18, we have $T^*v = \overline{\beta}v$. So, we have

$$\begin{aligned}(\alpha - \beta)\langle u, v\rangle &= \langle \alpha u, v\rangle - \langle u, \overline{\beta}v\rangle \\ &= \langle Tu, v\rangle - \langle U, T^*v\rangle \\ &= \langle Tu, v\rangle - \langle Tu, v\rangle \\ &= 0.\end{aligned}$$

Since $\alpha \neq \beta$, it must be $\langle u, v\rangle = 0$. So, $u$ and $v$ are orthogonal. ■

## 6.2 The Spectral Theorem

**Theorem 6.2.1 Complex Spectral Theorem**
Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then, the following are equivalent:

1. $T$ is normal.
2. $V$ has an orthonormal basis consisting of eigenvectors of $T$.
3. $T$ has a diagonal matrix with respect to some orthonormal basis of $V$.

***Proof 1.*** Note that (2) $\iff$ (3) is obvious by Theorem 4.3.5. No we need to show (3) $\iff$ (1) to complete the proof. $\square$

Suppose (3). Then, $\mathcal{M}(T)$ is diagonal. That is, $\mathcal{M}(T^*)$ is also diagonal. Then, $\mathcal{M}(T)\mathcal{M}(T^*) = \mathcal{M}(T^*)\mathcal{M}(T)$. That is $\mathcal{M}(TT^*) = \mathcal{M}(T^*T)$, or $TT^* = T^*T$. So, $T$ is normal. $\square$

Suppose (1). That is, $T$ is normal. Then, by Schur's Theorem, $\exists$ an orthonormal basis $e_1, \cdots, e_n$ of $V$ *s.t.* $\mathcal{M}(T, (e_1, e_n))$ is an upper triangular matrix. Suppose

$$\mathcal{M}(T, (e_1, \cdots, e_n)) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}.$$

Then,

$$\mathcal{M}(T^*, (e_1, \cdots, e_n)) = \begin{pmatrix} \overline{a_{1,1}} & & 0 \\ \vdots & \ddots & \\ \overline{a_{1,n}} & \cdots & \overline{a_{n,n}} \end{pmatrix}.$$

Then, $Te_1 = a_{1,1}e_1$ and $T^*e_1 = \overline{a_{1,1}}e_1 + \cdots + \overline{a_{1,n}}e_n$. Further, note that $\|Te_1\|^2 = |a_{1,1}|^2$ and $\|T^*e_1\|^2 = |a_{1,1}|^2 + \cdots + |a_{1,n}|^2$. Since $\|Te_1\|^2 = \|T^*e_1\|^2$, we have $|a_{1,1}|^2 = |a_{1,1}|^2 + \cdots + |a_{1,n}|^2$. Then, it must be that $|a_{1,2}|^2 + \cdots + |a_{1,n}|^2 = 0$. Applying this procedure to $\|Te_2\|^2 = \|T^*e_2\|^2, \cdots, \|Te_n\|^2 = \|T^*e_n\|^2$, we have $|a_{j,k}| = 0$ when $j \neq k$. So, $\mathcal{M}(T)$ is a diagonal matrix. $\blacksquare$

**Lemma 6.2.2 Invertible Quadratic Expressions** Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are *s.t.* $b^2 < 4c$. Then, $T^2 + bT + cI$ is invertible.

***Proof 2.*** Let $v \in V$ *s.t.* $v \neq 0$. Note that

$$\begin{aligned} \langle (T^2 + bT + cI)v, v \rangle &= \langle T^2 v, v \rangle + b\langle Tv, v \rangle + c\langle v, v \rangle \\ &= \langle Tv, Tv \rangle + b\langle Tv, v \rangle + c\|v\|^2 && T \ is \ self-adjoint \\ &\geq \|Tv\|^2 - |b|\|Tv\|\|v\| + c\|v\|^2 && Cauchy-Schwarz \\ &= \left(\|Tv\| - \frac{|b|\|v\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2 \\ &> 0 && b^2 < 4c \end{aligned}$$

Then, $\forall v \neq 0$, $\langle (T^2 + bT + cI)v, v \rangle > 0$. So, it must be that $(T^2 + bT + cI)v = 0$ if and only if $v = 0$. Then, $\operatorname{null}(T^2 + bT + cI) = \{0\}$. Thus, $T^2 + bT + cI$ is injective, and thus it is invertible. $\blacksquare$

**Lemma 6.2.3** Suppose $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then, $T$ has an eigenvalue.

***Proof 3.*** Let $m = \dim V$ and choose $v \in V$. Then, $v, Tv, \cdots, T^n v$ cannot be L.I. because we have $n + 1 > \dim V$ vectors in the list. So, $\exists a_0, \cdots, a_n \in \mathbb{R}$ *s.t.* $a_0 v + a_1 Tv + \cdots + a_n T^n v = 0$. Make the $a$'s the coefficient of a polynomial then

$$a_0 + a_1 x + \cdots + a_n x^n = c(x^2 + b_1 x + c_1)\cdots(x^2 + b_M x + c_M)(x - \lambda_1)\cdots(x - \lambda_m),$$

where $c$ is a non-zero real number, each $b_j, c_j, \lambda_j \in \mathbb{R}$, each $b_j < 4c_j$, and $m + M \geq 1$. Then, we have

$$\begin{aligned} 0 &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= (a_0 I + a_1 T + \cdots + a_n T^n) v \\ &= c(T^2 + b_1 T + c_1 I)\cdots(T^2 + b_M T + c_M I)(T - \lambda_1 I)\cdots(T - \lambda_m I). \end{aligned}$$

By Lemma 6.2.2, $T^2 + b_j T + c_j I$ is invertible. Since $c \neq 0$, it must be that $0 = (T - \lambda_1 I)\cdots(T - \lambda_m I)$. Hence, $T - \lambda_j I$ is not injective for at least one $j$. So, $T$ has at least one eigenvalue. ■

**Definition 6.2.4 (Restriction Operator, $T|_U$).** Suppose $T \in \mathcal{L}(V)$ and $U$ is an invariant subspace of $V$ under $T$. Then, the *restriction operator*, $T|_U \in \mathcal{L}(V)$, is defined as $T|_U(u) = Tu$ for $u \in U$.

**Theorem 6.2.5**
Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $U$ is a subspace of $V$ that is invariant under $T$. Then,

1. $U^\perp$ is invariant under $T$;
2. $T|_U \in \mathcal{L}(U)$ is self-adjoint;
3. $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

***Proof 4.***

1. Suppose $v \in U^\perp$ and $u \in U$. Then, $\langle v, Tu \rangle = \langle Tv, u \rangle = 0$ since $U$ is invariant under $T$ (and hence $Tu \in U$) and $v \in U^\perp$. Then, we have $Tv \in U^\perp \quad \forall v \in U^\perp$, proving $U^\perp$ is an invariant subspace under $T$. □
2. Note that if $u, v \in U$, then

   $$\langle (T|_U)u, v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle = \langle u, (T|_U)v \rangle.$$

   Therefore, $T|_U$ is self-adjoint. □
3. Replace $U$ with $U^\perp$ in (2) and apply the conclusion from (1), and we complete the proof.

■

**Theorem 6.2.6 Real Spectral Theorem**
Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then, the following are equivalent:

1. $T$ is self-adjoint;
2. $V$ has an orthonormal basis consisting of eigenvectors of $T$.
3. $T$ has a diagonal matrix with respect to some orthonormal basis of $V$.

***Proof 5.*** Similar to the complex case, (2) $\iff$ (3) is obvious. So, we will show (3) $\implies$ (1) and (1) $\implies$ (2) to complete the proof. □

Suppose (3) holds. Then, $\mathcal{M}(T)$ is diagonal. So, we have $\mathcal{M}(T)^t = \mathcal{M}(T)$. That is, $T = T^*$, and thus $T$ is self-adjoint. $\square$

Suppose (1) holds. We will use mathematical induction on $\dim V$. $\boxed{\text{Base Case}}$ When $\dim V = 1$. Clearly, (1) $\implies$ (2). $\boxed{\text{Inductive Steps}}$ Assume $\dim V > 1$ and (1) $\implies$ (2) holds for all cases with dimension $\dim V - 1$. Let $u$ be an eigenvector of $T$ with $\|u\| = 1$. Let $U = \operatorname{span}(u)$. Then, $\dim U = 1$. Since $V = U \oplus U^\perp$, we know $\dim U^\perp = \dim V - \dim U = \dim V - 1$. So, (1) $\implies$ (2) holds on $U^\perp$. That is, $\exists$ an orthonormal basis of $U^\perp$ consisting of eigenvectors of $T|_{U^\perp}$. Now, add $u$ to this orthonormal basis, we get a basis of $V$. Further since $u \in U$, this basis is an orthonormal basis of $V$ consisting of eigenvectors of $T$. $\blacksquare$

## 6.3 Positive Operators and Isometries

**Definition 6.3.1 (Positive Operator).** An operator $T \in \mathcal{L}(V)$ is called *positive* if $T$ is self-adjoint and $\langle Tv, v\rangle \geq 0 \quad \forall v \in V$.

**Definition 6.3.2 (Square Root).** An operator $R$ is called a *square root* of an operator $T$ if $R^2 = T$.

**Example 6.3.3** Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and $R \in \mathcal{L}(\mathbb{R}^3)$ be defined as $T(z_1, z_2, z_3) = (z_3, 0, 0)$ and $R(z_1, z_2, z_3) = (z_2, z_3, 0)$. Then, $R$ is a square root of $T$.

***Proof 1.*** Since $R^2(z_1, z_2, z_3) = R(z_2, z_3, 0) = (z_3, 0, 0) = T(z_1, z_2, z_3)$, $R$ is a square root of $T$. ■

**Theorem 6.3.4 Characterization of Positive Operators**
Let $T \in \mathcal{L}(V)$. Then, the following are equivalent:

1. $T$ is positive;
2. $T$ is self-adjoint and all the eigenvalues of $T$ are non-negative;
3. $T$ has a positive square root;
4. $T$ has a self-adjoint square root;
5. $\exists$ an operator $R \in \mathcal{L}(V)$ *s.t.* $T = R^*R$.

***Proof 2.***

(1) $\implies$ (2): Since $T$ is positive, then $T$ is self-adjoint. Let $\lambda$ be an eigenvalue of $T$. Then, $Tv = \lambda v$ for some $v \in V$. Then, $\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle = \lambda\|v\|^2$. Since $T$ is positive, $\langle Tv, v\rangle \geq 0$. Further since $\|v\|^2 \geq 0$, it must also be $\lambda \geq 0$. So, we complete the proof. □

(2) $\implies$ (3): Suppose $T$ is self-adjoint and all the eigenvalues of $T$ are non-negative. By the Spectrum Theorem, $\exists$ an orthonormal basis $e_1, \cdots, e_n$, where $e_1, \cdots, e_n$ are eigenvectors of $T$. Let $\lambda_1, \cdots, \lambda_n$ be the corresponding eigenvalues, where $\lambda_j \geq 0$. Let $R \in \mathcal{L}(V)$ *s.t.* $Re_j = \sqrt{\lambda_j}e_j$. Then

$$\begin{aligned}\langle Rv, v\rangle &= \left\langle a_1\sqrt{\lambda_1}e_1 + \cdots + a_n\sqrt{\lambda_n}e_n, a_1e_1 + \cdots + a_ne_n\right\rangle \\ &= |a_1|^2\sqrt{\lambda_1} + \cdots + |a_n|^2\sqrt{\lambda_n} \geq 0.\end{aligned}$$

Further, we can verity $R$ is self-adjoint (proof omitted). So, $R$ is positive by definition. Note that

$$R^2e_j = R\left(\sqrt{\lambda_j}e_j\right) = \sqrt{\lambda_j}\sqrt{\lambda_j}e_j = \lambda_je_j = Te_j.$$

So, $R$ is a square root of $T$. □

(3) $\implies$ (4): Suppose $T$ has a positive square root. By definition, positive operators are self-adjoint. □

(4) $\implies$ (5): Suppose $T$ has a self-adjoint square root. Then, we have $R \in \mathcal{L}(V)$ *s.t.* $R^* = R$ and $R^2 = T$. That is, $T = R^2 = RR = R^*R$. □

(5) $\implies$ (1): Suppose $\exists$ an operator $R \in \mathcal{L}(V)$ *s.t.* $T = T^*T$. Then,

$$T^* = (R^*R)^* = R^*(R^*)^* = R^*R = T.$$

So, $T$ is self-adjoint. Now, since

$$\langle Tv, v\rangle = \langle R^*Rv, v\rangle = \langle Rv, Rv\rangle = \|Rv\|^2 \geq 0,$$

we have $T$ is a positive operator. ■

**Theorem 6.3.5**
Each positive operator on $V$ has a unique positive square root.

***Proof 3.*** Let $T$ be a positive operator on $V$. Select $v$ to be an eigenvector of $T$ with corresponding eigenvalue of $\lambda$. Then, we have $Tv = \lambda v$. Let $R$ be a positive square root of $T$. Apply Spectrum Theorem to $R$, then $\exists$ an orthonormal basis $e_1, \cdots, e_n$, where $e_1, \cdots, e_n$ are eigenvectors of $R$. Then, $\exists \lambda_1, \cdots, \lambda_n \geq 0$ *s.t.* $Re_j = \sqrt{\lambda_j} e_j$. Suppose $v \in V$ and $v = a_1e_1 + \cdots + a_ne_n$. Then,

$$Rv = a_1\sqrt{\lambda_1}e_1 + \cdots + a_n\sqrt{\lambda_n}e_n \quad \text{and} \quad R^2v = a_1\lambda_1e_1 + \cdots + a_n\lambda_ne_n.$$

Further, $Tv = \lambda v = \lambda a_1e_1 + \cdots + \lambda a_ne_n$. Since $R^2v = Tv$, we know

$$a_1(\lambda_1 - \lambda)e_1 + \cdots + a_n(\lambda_n - \lambda)e_n = 0.$$

Since $e_1, \cdots, e_n$ is an orthonormal basis, for each $j = 1, \cdots, n$, we have $a_j(\lambda_j - \lambda) = 0$. So, it must be $a_j = 0$ or $\lambda_j = \lambda$. If $a_j = 0$, then we can delet it from the representation of $v$. So,

$$v = \sum_{\{j|\lambda_j = \lambda\}} a_je_j$$

Hence,

$$Rv = \sum_{\{j|\lambda_j = \lambda\}} a_j\sqrt{\lambda}e_j = \sqrt{\lambda}v.$$

■

**Definition 6.3.6 (Isometry).** An operator $S \in \mathcal{L}(V)$ is called an *isometry* if $\|Sv\| = \|v\| \quad \forall v \in V$. In other words, an operator is an isometry if ti preserves norms.

**Example 6.3.7** Let $\lambda_1, \cdots, \lambda_n \in \mathbb{F}$ with $|\lambda_j|$ and $S \in \mathcal{L}(V)$ *s.t.* $Se_j = \lambda_je_j$ for some orthonormal bases $e_1, \cdots, e_n$ of $V$. Then, $S$ is an isometry.

***Proof 4.*** Let $v \in V$. Then, $v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$. So, $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$. Further, $Sv = \lambda_1\langle v, e_1\rangle e_1 + \cdots + \lambda_n\langle v, e_n\rangle e_n$, and thus $\|Sv\|^2 = |\lambda_1|^2|\langle v, e_1\rangle|^2 + \cdots + |\lambda_n|^2|\langle v, e_n\rangle|^2$. Since $|\lambda_j| = 1$, we know

$$\|Sv\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 = \|v\|^2.$$

So, $\|Sv\| = \|v\|$ since $\|Sv\| \geq 0$ and $\|v\| \geq 0$. That is, by definition, $S$ is an isometry. ■

**Theorem 6.3.8 Characterization of Isometries**
Suppose $S \in \mathcal{L}(V)$. Then, the following are equivalent:

1. $S$ is an isometry.
2. $\langle Su, Sv \rangle = \langle u, v \rangle \quad \forall u, v \in V$;
3. $Se_1, \cdots, Se_n$ is orthonormal for every orthonormal list of vectors $e_1, \cdots, e_n$ in $V$;
4. $\exists$ an orthonormal basis $e_1, \cdots, e_n$ of $V$ *s.t.* $Se_1, \cdots, Se_n$ is orthonormal;
5. $S^*S = I$;
6. $SS^* = I$;
7. $S^*$ is an isometry;
8. $S$ is invertible and $S^{-1} = S^*$.

***Proof 5.***
(1) $\implies$ (2): Note that

$$\begin{aligned}\langle Su, Sv \rangle = \frac{\|Su + Sv\|^2 - \|Su - Sv\|^2}{4} &= \frac{\|S(u+v)\|^2 - \|S(u-v)\|^2}{4} \\ &= \frac{\|u+v\|^2 - \|u-v\|^2}{4} \\ &= \langle u, v \rangle \qquad \square\end{aligned}$$

(2) $\implies$ (3): We have

$$\langle Se_i, Se_j \rangle = \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So, $Se_1, \cdots, Se_n$ are orthonormal. $\square$

(3) $\implies$ (4): Suppose $e_1, \cdots, e_m$ is orthonormal. We can extend it to a basis of $V$: $e_1, \cdots, e_m, v_{m+1}, \cdots, v_n$. Then, apply the Gram-Schmidt Procedure, we get an orthonormal basis, $e_1, \cdots, e_m, e_{m+1}, \cdots, e_n$ of $V$. $\square$

(4) $\implies$ (5): Suppose $e_1, \cdots, e_n$ is an orthonormal basis of $V$. Then,

$$\langle S^*Se_j, e_k \rangle = \langle Se_j, Se_k \rangle = \langle e_j, e_j \rangle.$$

Suppose $u, v \in V$ *s.t.* $u = a_1e_1 + \cdots + a_ne_n$ and $v = b_1e_1 + \cdots + b_ne_n$. Then,

$$\begin{aligned}\langle S^*Su, v \rangle = \langle Su, Sv \rangle &= \langle S(a_1e_1 + \cdots + a_ne_n), S(b_1e_1 + \cdots + b_ne_n) \rangle \\ &= \langle a_1Se_1 + \cdots + a_nSe_n, b_1Se_1 + \cdots + b_nSe_n \rangle \\ &= \langle a_1Se_1, b_1Se_1 \rangle + \cdots + \langle a_nSe_n, b_nSe_n \rangle \\ &= a_1\overline{b_1}\|Se_1\|^2 + \cdots + a_n\overline{b_n}\|Se_n\|^2 \\ &= a_1\overline{b_1} + \cdots + a_n\overline{b_n} \\ &= \langle u, v \rangle.\end{aligned}$$

So, $S^*Su = u$, or $S^*S = I$. $\square$

(5) $\implies$ (6): Suppose $S^*S = I$. Then, $S = S^*$. So, $SS^* = I$. $\square$

(6) $\implies$ (7): Suppose $S^*S = I$. Then,

$$\|S^*v\|^2 = \langle S^*v, S^*v\rangle = \langle SS^*v, v\rangle = \langle v, v\rangle = \|v\|^2. \qquad \square$$

(7) $\implies$ (8): Suppose $S^*$ is an isometry. Then, we know $S^*S = I$ and $SS^* = I$ by the proofs done above. So, $S$ is invertible, and $S^{-1} = S^*$. $\square$

(8) $\implies$ (1): Finally, suppose $S$ is invertible and $S^{-1} = S^*$. Then, $S^*S = I$. Note that

$$\|Sv\|^2 = \langle Sv, Sv\rangle = \langle S^*Sv, v\rangle = \langle v, v\rangle = \|v\|^2.$$

$\blacksquare$

**Theorem 6.3.9**
Suppose $V$ is a complex inner product space and $S \in \mathcal{L}(V)$. Then, $S$ is an isometry if and only if $\exists$ an orthonormal basis of $V$ consisting of eigenvectors of $S$ whose corresponding eigenvalues all have absolute value of $1$.

***Proof 6.***

($\Rightarrow$): By the Spectrum Theorem, $\exists$ an orthonormal basis $e_1, \cdots, e_n$, where $e_1, \cdots, e_n$ are eigenvectors of $S$. Suppose $\lambda_1, \cdots, \lambda_n$ are the corresponding eigenvalues. Then, we have

$$\|Se_j\| = \|\lambda_j e_j\| = |\lambda_j|.$$

Since $S$ is an isometry, $\|Se_j\| = \|e_j\| = 1$. So, $|\lambda_j| = \|Se_j\| = 1$. $\square$

($\Leftarrow$): This direction is proven in Example 6.3.7. $\blacksquare$

## 6.4 Polar Decomposition and SVD

**Notation 6.4.1.** If $T$ is a positive operator, then $\sqrt{T}$ denotes the unique positive square root of $T$.

**Remark.** *We want to verify that the definition of* $\sqrt{T^*T}$ *is reasonable:* $\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \geq 0$. *Also,* $(T^*T)^* = T^*T$. *So,* $T^*T$ *is a positive operator, and thus* $\sqrt{T^*T}$ *is well-defined.*

> **Theorem 6.4.2 Polar Decomposition**
> Suppose $T \in \mathcal{L}(V)$. Then, $\exists$ an isometry $S \in \mathcal{L}(V)$ *s.t.* $T = S\sqrt{T^*T}$.

***Proof 1.***
$\boxed{\text{Step } 1}$ Characteristics of $\mathrm{range}\ \sqrt{T^*T}$: Note that

$$\begin{aligned} \|Tv\|^2 = \langle Tv, Tv\rangle &= \langle T^*Tv, v\rangle \\ &= \langle \sqrt{T^*T}\sqrt{T^*T}v, v\rangle \\ &= \langle \sqrt{T^*T}v, \sqrt{T^*T}v\rangle \\ &= \|\sqrt{T^*T}v\|^2. \end{aligned}$$

So, $\forall v \in V$, we have $\|Tv\| = \|\sqrt{T^*T}v\|$. Define $S_1 : \mathrm{range}\ \sqrt{T^*T} \to \mathrm{range}\ T$ as $S_1\left(\sqrt{T^*T}v\right) = Tv$. Then, we have $\|S_1\sqrt{T^*T}v\| = \|Tv\|$.

1. Now, we want to verify that $S_1$ is well-defined. Suppose $v_1, v_2 \in V$ *s.t.* $\sqrt{T^*T}v_1 = \sqrt{T^*T}v_2$. Then,

$$\begin{aligned} \|Tv_1 - Tv_2\| = \|T(v_1 - v_2)\| &= \|\sqrt{T^*T}(v_1 - v_2)\| \\ &= \|\sqrt{T^*T}v_1 - \sqrt{T^*T}v_2\| \\ &= 0. \end{aligned}$$

   So, $S_1$ is well-defined.

2. Further, we want to show $S_1$ is linear. By using the linearity of $T$, we can easily prove that $S_1$ is also linear.

3. Additionally, $S_1$ is surjective by definition of $S_1$.

4. Also, $S_1$ is isometry. Note that $\forall u \in \mathrm{range}\ \sqrt{T^*T}$, we have $\|S_1 u\| = \|u\|$ since $\|\sqrt{T^*T}v\| = \|Tv\|$.

5. Hence, $S_1$ is injective: Note that $\|S_1 v\| = 0$ if and only if $\|v\| = 0$, which is equivalent to $v = 0$. So, $\mathrm{null}\ S_1 = \{0\}$. $\square$

$\boxed{\text{Step } 2}$ Extend $S_1$ to an isometry on $V$. Note that we have $\dim\mathrm{range}\ \sqrt{T^*T} = \dim\mathrm{range}\ T$. So, we know $\dim\left(\mathrm{range}\ \sqrt{T^*T}\right)^\perp = \dim\left(\mathrm{range}\ T\right)^\perp$. Select an orthonormal basis $e_1, \cdots, e_m$ of $\left(\mathrm{range}\ \sqrt{T^*T}\right)^\perp$ and an orthonormal basis $f_1, \cdots, f_m$ of $(\mathrm{range}\ T)^\perp$. Now, let's define $S_2 : \left(\mathrm{range}\ \sqrt{T^*T}\right)^\perp \to (\mathrm{range}\ T)^\perp$ as $S_1(a_1e_1 + \cdots + a_me_m) = a_1f_1 + \cdots + a_mf_m$. We can then not only show $S_2$ is well-defined but also $S_2$

is linear. Moreover, $\forall w \in \left(\text{range}\sqrt{T^*T}\right)^\perp$, if $w = a_1e_1 + \cdots + a_me_m$, we have

$$\begin{aligned}\|S_2w\|^2 = \|S_2(a_1e_1 + \cdots + a_me_m)\|^2 &= \|a_1f_1 + \cdots + a_mf_m\|^2 \\ &= |a_1|^2 + \cdots + |a_m|^2 \\ &= \|a_1e_1 + \cdots + a_me_m\|^2 \\ &= \|w\|^2.\end{aligned}$$

So, $\|S_2w\| = \|w\|$. Now, we define

$$Sv = \begin{cases} S_1v, & v \in \text{range}\sqrt{T^*T} \\ S_2v, & v \in \left(\text{range}\sqrt{T^*T}\right)^\perp \end{cases}$$

Note that since $V = \text{range}\sqrt{T^*T} \oplus \left(\text{range}\sqrt{T^*T}\right)^\perp$, we can uniquely represent $v \in V$ as $v = u + w$ for some $u \in \text{range}\sqrt{T^*T}$ and $w \in \left(\text{range}\sqrt{T^*T}\right)^\perp$. Hence, we can also write the definition of $S$ as $Sv = S_1u + S_2w$. If we select $\sqrt{T^*T}v \in \text{range}\sqrt{T^*T}$, then we have $S\left(\sqrt{T^*T}v\right) = S_1\left(\sqrt{T^*T}v\right) = Tv$. Therefore, $T = S\sqrt{T^*T} \quad \forall v \in V$. □

Finally, we will show $S$ is an isometry. Note that $v = u + w$. So, by Pythagorean Theorem,

$$\begin{aligned}\|Sv\|^2 = \|S_1u + S_2w\|^2\|S_1u\|^2 + \|S_2w\|^2 \\ &= \|u\|^2 + \|w\|^2 \\ &= \|v\|^2.\end{aligned}$$

■

**Definition 6.4.3 (Singular Values).** Suppose $T \in \mathcal{L}(V)$. The *singular values* of $T$ are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue $\lambda$ repeated $\dim E\left(\lambda, \sqrt{T^*T}\right)$ times.

**Example 6.4.4** Define $T \in \mathcal{L}(\mathbb{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

Find the singular values of $T$.

***Solution 2.***

Suppose $v = (z_1, z_2, z_3, z_4) \in \mathbb{F}^4$ and $w = (y_1, y_2, y_3, y_4) \in \mathbb{F}^4$. Consider

$$\begin{aligned}\langle v, T^*w\rangle &= \langle Tv, w\rangle \\ &= \langle (0, 3z_1, 2z_2, -3z_4), (y_1, y_2, y_3, y_4)\rangle \\ &= 3z_1\overline{y_2} + 2z_2\overline{y_3} - 3z_4\overline{y_4} \\ &= \langle (z_1, z_2, z_3, z_3), (3y_2, 2y_3, 0, -3y_4)\rangle.\end{aligned}$$

So, $T^*w = T^*(y_1, y_2, y_3, y_4) = (3y_2, 2y_3, 0, -3y_4)$. Then, $T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4)$. Then, $\sqrt{T^*T}(z_2, z_2, z_3, z_4) = (3_z1, 2z_2, 0, 3z_4)$. So, the eigenvalues of $\sqrt{T^*T}$ are 3, 2, and $0$. Also,

$$\dim E\left(3, \sqrt{T^*T}\right) = 2, \quad \dim E\left(2, \sqrt{T^*T}\right) = \dim E\left(0, \sqrt{T^*T}\right) = 1.$$

So, the singular values are $3, 3, 2, 0$. □

**Theorem 6.4.5 Singular Value Decomposition (SVD)**
Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \cdots, s_n$. Then, $\exists$ orthonormal bases $e_1, \cdots, e_n$ and $f_1, \cdots, f_n$ of $V$ *s.t.*

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$.

**Remark.** *Relevant Theorem used in proving SVD: Spectrum Theorem, Characterization and Properties of Isometry, and Polar Decomposition.*

***Proof 3.*** Apply the Spectrum Theorem to $\sqrt{T^*T}$, we know $\exists$ an orthonormal basis $e_1, \cdots, e_n$ of $V$ *s.t.*

$$\sqrt{T^*T}e_j = s_j e_j \quad \forall j = 1, \cdots, n.$$

Note that $\forall v \in V$, we have

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n \tag{17}$$

Apply $\sqrt{T^*T}$ to Equation (17) we have

$$\sqrt{T^*T}v = s_1\langle v, e_1\rangle e_1 + \cdots + s_n\langle v, e_n\rangle e_n. \tag{18}$$

By Polar Decomposition, $\exists$ an isometry $S \in \mathcal{L}(V)$ *s.t.* $T = S\sqrt{T^*T}$. Apply $S$ to Equation (18), we get

$$S\left(\sqrt{T^*T}v\right) = s_1\langle v, e_1\rangle Se_1 + \cdots + s_n\langle v, e_n\rangle Se_n.$$

By the characteristics of isometry, since $e_1, \cdots, e_n$ is an orthonormal basis, $Se_1, \cdots, Se_n$ is also an orthonormal basis. Let $f_j = Se_j$. Then,

$$Tv = S\sqrt{T^*T}v = s_1\langle v, e_e\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n.$$

■

**Theorem 6.4.6**
Suppose $T \in \mathcal{L}(V)$. Then, the singular values of $T$ are the non-negative square roots of the eigenvalues of $T^*T$, with each eigenvalue $\lambda$ repeated $\dim E(\lambda, T^*T)$ times.

***Proof 4.*** By the Spectrum Theorem, $\exists$ an orthonormal basis $e_1, \cdots, e_n$ and non-negative number $\lambda_1, \cdots, \lambda_n$ *s.t.* $T^*Te_j = \lambda_j e_j \quad \forall j = 1, \cdots, n$. Then, we have $\sqrt{T^*T}e_j = \sqrt{\lambda_j}e_j \quad \forall j = 1, \cdots, n$, which completes the proof. ■

# 7 Operators on Complex Vector Spaces

## 7.1 Generalized Eigenvectors, Nilpotent Operators

**Theorem 7.1.1**
Suppose $T \in \mathcal{L}(V)$. Then,

$$\{0\} \subseteq \operatorname{null} T^0 \subseteq \operatorname{null} T^1 \subseteq \cdots \subseteq \operatorname{null} T^k \subseteq \operatorname{null} T^{k+1} \subseteq \cdots$$

***Proof 1.*** Let $k \in \mathbb{N}^+$. Let $v \in \operatorname{null} T^k$. Then, $T^k v = 0$. Then, we know $T(T^k v) = T^{k+1} v = 0$. So, $v \in \operatorname{null} T^{k+1}$. That is, $\operatorname{null} T^k \subseteq \operatorname{null} T^{k+1}$ as desired. ■

**Theorem 7.1.2**
Suppose $T \in \mathcal{L}(V)$. Suppose $m$ is a non-negative integer *s.t.* $\operatorname{null} T^m = \operatorname{null} T^{m+1}$. Then,

$$\operatorname{null} T^m = \operatorname{null} T^{m+1} = \operatorname{null} T^{m+2} = \operatorname{null} T^{m+3} = \cdots$$

***Proof 2.*** Let $k \in \mathbb{N}$. We've already shown $\operatorname{null} T^{m+k} \subseteq \operatorname{null} T^{m+k+1}$ in Theorem 7.1.1. Now, let $v \in \operatorname{null} T^{m+k+1}$. So, $T^{m+k+1}(v) = 0$. That is, $T^{m+1}\left(T^k v\right) = 0$. So, $T^k v \in \operatorname{null} T^{m+1} = \operatorname{null} T^m$. In other words, $T^m\left(T^k v\right) = T^{m+k}(v) = 0$. So, $v \in \operatorname{null} T^{m+k}$. Then, $\operatorname{null} T^{m+k+1} \subseteq \operatorname{null} T^{m+k}$. Hence,

$$\operatorname{null} T^{m+k} = \operatorname{null} T^{m+k+1}.$$

■

**Theorem 7.1.3**
Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then,

$$\operatorname{null} T^n = \operatorname{null} T^{n+1} = \operatorname{null} T^{n+2} = \cdots$$

***Proof 3.*** Suppose for the sake of contradiction that $\operatorname{null} T^n \neq \operatorname{null} T^{n+1}$. Then,

$$\operatorname{null} T^0 \subsetneq \operatorname{null} T \subsetneq T^2 \subsetneq \cdots \subsetneq \operatorname{null} T^n \subsetneq T^{n+1}.$$

As the symbol $\subsetneq$ means "contained in but not equal to," at each of the strict inclusions in the chain above, the dimension increases by at least $1$. That is, $\dim \operatorname{null} T^{n+1} \geq n+1$. ⋇ This is a contradiction because a subspace of $V$ $(\operatorname{null} T^{n+1})$ cannot be a dimension larger than $\dim V = n$. So, it must be that our assumption is wrong, and $\operatorname{null} T^n = \operatorname{null} T^{n+1}$. ■

**Theorem 7.1.4**
Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then,

$$V = \operatorname{null} T^n \oplus \operatorname{range} T^n.$$

***Proof 4.*** Note that $\dim V = \dim \operatorname{null} T^n + \dim \operatorname{range} T^n$ by the Fundamental Theorem of Linear Maps. So, we only need to prove $(\operatorname{null} T^n) \cap (\operatorname{range} T^n) = \{0\}$. Let $v \in (\operatorname{null} T^n) \cap (\operatorname{range} T^n)$. Then, $\exists u \in V$ ***s.t.*** $v = T^n u$. Since $v \in \operatorname{null} T^n$, $T^N v = T^n(T^n u) = 0$. That is, $T^{2n} u = T^n v = 0$. Therefore, $u \in \operatorname{null} T^{2n} = \operatorname{null} T^n$. So, we now have $T^n u = 0$. Hence, $v = T^n u = 0$. Then, i $(\operatorname{null} T^n) \cap (\operatorname{range} T^n) = \{0\}$, and thus $V = \operatorname{null} T^n \oplus \operatorname{range} T^n$. ■

**Definition 7.1.5 (Generalized Eigenvector).** Suppose $T \in \mathcal{L}(V)$ and $\lambda$ is an eigenvalue of $T$. A vector $v \in V$ is called a *generalized eigenvector* of $T$ corresponding to $\lambda$ if $v \neq 0$ and $(T - \lambda I)^j v = 0$ for some positive integer $j$.

**Definition 7.1.6 (Generalized Eigenspace, $G(\lambda, T)$).** Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The *generalized eigenspace* of $T$ corresponding to $\lambda$, denoted $G(\lambda, T)$, is defined to be the set of all generalized eigenvectors of $T$ corresponding to $\lambda$, along with the $0$ vector.

**Theorem 7.1.7**
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then,

$$G(\lambda, T) = \text{null}\,(T - \lambda I)^{\dim V}.$$

***Proof 5.***
($\subseteq$): Let $v \in G(\lambda, T)$. Then, $\exists\, j \in \mathbb{N}^+$ *s.t.*

$$v \in \text{null}\,(T - \lambda I)^j.$$

Since $\text{null}\,(T - \lambda I)^j \subseteq \text{null}\,(T - \lambda)^{j+1} \subseteq \cdots \subseteq \text{null}\,(T - \lambda I)^{\dim V}$, we have $v \in \text{null}\,(T - \lambda I)^{\dim V}$. So, $G(\lambda, T) \subseteq \text{null}\,(T - \lambda I)^{\dim V}$.

($\supseteq$): Conversely, suppose $v \in \text{null}\,(T - \lambda I)^{\dim V}$. Then,

$$(T - \lambda I)^{\dim V} v = 0.$$

By definition, $v$ is a generalized eigenvector, and so $v \in G(\lambda, T)$. Then, $\text{null}\,(T - \lambda I)^{\dim V} \subseteq G(\lambda, T)$. ■

**Theorem 7.1.8**
Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T$ and $v_1, \ldots, v_m$ are corresponding generalized eigenvectors. Then, $v_1, \ldots, v_m$ is L.I..

***Proof 6.*** Let $a_1, \ldots, a_m \in \mathbb{C}$ *s.t.*

$$0 = a_1 v_1 + \cdots + a_m v_m. \tag{19}$$

Let $k$ be the largest non-negative integer such that $(T - \lambda_1 I)^k v_1 \neq 0$. Let $w = (T - \lambda_1)^k v_1$, then

$$\begin{aligned}(T - \lambda_1 I)w &= (T = \lambda_1 I)(T - \lambda_1 I)^k v = 0 \\ &= (T - \lambda_1 I)^{k+1} v = 0\end{aligned}$$

So, $w$ is an eigenvector, and

$$Tw = \lambda_1 w. \tag{20}$$

Minus $\lambda w$ from both sides of Equation (20), we have

$$(T - \lambda I)w = (\lambda_1 - \lambda)w \qquad \forall v \in \mathbb{F}$$

Then, $(T - \lambda I)^n w = (\lambda_1 - \lambda)^n w,\ \lambda \in \mathbb{F},\ n = \dim V$. Apply the operator $(T - \lambda_1 I)^k (T - \lambda_2 I)^n \cdots (T - \lambda_m I)^m$

to both sides of Equation (19), we have

$$\begin{aligned}
0 &= (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n (a_1 v_1 + \cdots + a_m v_m) \\
&= (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n (a_m v_m) + \cdots + (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n (a_1 v_1) \\
&= (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n (a_1 v_1) \\
&= a_1 (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n w \\
&= a_1 \underbrace{(T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n}_{\neq 0} \underbrace{w}_{\neq 0}
\end{aligned}$$

So, it must be $a_1 = 0$. Apply the same rationale, we can show $a_1 = \cdots = a_m = 0$. Therefore, $v_1, \ldots, v_m$ is L.I. by definition. ■

**Definition 7.1.9 (Nilpotent).** An operator is called *nilpotent* if some power of it equals $0$.

**Theorem 7.1.10**
Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then, $N^{\dim V} = 0$.

***Proof 7.*** Note that $\text{null}\,(N - 0I)^{\dim V} = G(0, N) = V$. So, we have proven $N^{\dim V} = 0$. ■

**Lemma 7.1.11** Suppose $N \in \mathcal{L}(V)$ has a basis such that $\mathcal{M}(N)$ is an upper-triangular matrix with its diagonal all $0$. Then, $N$ is nilpotent.

***Proof 8.*** Suppose the basis is $v_1, \ldots, v_n$ and

$$A = \mathcal{M}(N) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix}.$$

Then,

$$\begin{aligned}
Nv_1 &= 0 \\
Nv_2 &= A_{1,2} v_1 + 0, \quad N^2 v_2 = A_{1,2} N v_1 = 0 \\
&\vdots \\
Nv_n &= A_{1,n} v_1 + \cdots + A_{n-1,n} v_{n-1} + 0.
\end{aligned}$$

So, $N^n v_n = A_{1,n} N^{n-1} v_1 + A_{2,n} N^{n-1} v_2 + \cdots + A_{n-1,n} N^{n-1} v_{n-1} = 0$. That is, $N^n = 0$. So, we've shown that $N$ is nilpotent. ■

**Theorem 7.1.12 Matrix of a Nilpotent Operator**
Suppose $N$ is a nilpotent operator on $V$. Then, $\exists$ a basis of $V$ with respect to which the matrix of $N$ has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix};$$

where all entries on and below the diagonal are $0$'s.

***Proof 9.*** Let $k \in \mathbb{N} \cup \{0\}$ be the smallest such that $N^k = 0$. So, we have $\text{null}\, N^k = V$ and $k \leq n$. So,

$N^j \neq 0 \quad \forall j < k$. So, we have

$$\{0\} = \text{null } N^0 \subsetneq \text{null } N^1 \subsetneq \text{null } N^2 \subsetneq \cdots \subsetneq \text{null } N^k.$$

Select $v_1^1, \ldots, v_n^1, v_1^2, \ldots, v_{n_2}^2, \ldots, v_1^k, \ldots, v_{n_k}^k$ as a basis of $N$. It can be also written as $v_1, \ldots, v_n$.

1. Let $j$ be an index such that $v_j \in \text{null } N$. Then, $Nv_j = 0$.
2. Let $j$ be an index such that $v_j \in \text{null } N^2$. Then, $N^2(v_j) = N(Nv_j) = 0$. So, $Nv_j \in \text{null } N$.

So, $Nv_j = \sum_{\{i | v_i \in \text{null } N\}} A_{i,j} v_j, \quad i < j.$ ■

**Theorem 7.1.13**
Let $T \in \mathcal{L}(V)$ *s.t.* $T$ is no nilpotent. Suppose $\dim V = n$. Then, $V = \text{null } T^{n-1} \oplus \text{range } T^{n-1}$.

***Proof 10.*** Since $T$ is not nilpotent, $N^n \neq 0$. So, $\text{null } N^n \subsetneq V$. That is,

$$0 \subseteq \text{null } T \subseteq \text{null } T^2 \subseteq \cdots \subseteq \text{null } T^{n-1} \subseteq \text{null } T^n \subsetneq V.$$

So, it must be the case that $\text{null } T^{n-1} = \text{null } T^n$.

Suppose $v \in (\text{null } T^{n-1}) \cap (\text{range } T^{n-1})$. Then, $\exists u \in V$ *s.t.* $T^{n-1}u = v$. Note that

$$T^{n-1}v = T^{n-1}(T^{n-1}u) = T^{2n-2}u = T^n u = 0.$$

So, $u \in \text{null } T^n = \text{null } T^{n-1}$. That is, $T^{n-1}u = 0$. So, $v = 0$. Then, $(\text{null } T^{n-1}) \cap (\text{range } T^{n-1}) = \{0\}$, and thus $V = \text{null } T^{n-1} \oplus \text{range } T^{n-1}$. ■

**Theorem 7.1.14**
Suppose $T \in \mathcal{L}(V)$, $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq \beta$. Then,

$$G(\alpha, T) \cap G(\beta, T) = \{0\}.$$

***Proof 11.*** Let $v \in G(\alpha, T) \cap G(\beta, T)$ with $v \neq 0$. Then, we know $v$ is a generalized eigenvector of $\alpha$ and $\beta$ at the same time. However, given $\alpha \neq \beta$, their corresponding generalized eigenvectors should be L.I.. ⋇ This contradicts with the fact that $v$ cannot be L.I. with $v$. Then, our assumption is wrong, and $G(\alpha, T) \cap G(\beta, T) = \{0\}$. ■

## 7.2 Decomposition of an Operator

**Theorem 7.2.1**
Suppose $T \in \mathcal{L}(V)$ and $p = \mathcal{P}(\mathbb{F})$. Then, $\operatorname{null} p(T)$ and $\operatorname{range} p(T)$ are invariant under $T$.

***Proof 1.*** Let $v \in \operatorname{null} p(T)$. Then, $p(T)(Tv) = T(p(T)v) = T(0) = 0$. So, $\operatorname{null} p(T)$ is invariant under $T$. Suppose $v \in \operatorname{range} p(T)$, then $\exists u \in V$ *s.t.* $p(T)u = v$. Then, $Tv = T(p(T)u) = p(T)(Tu) \in \operatorname{range} p(T)$. So, $\operatorname{range} p(T)$ is also invariant under $T$. ■

**Theorem 7.2.2**
Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T$. Then,

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$.
2. each $G(\lambda_j, T)$ is invariant under $T$.
3. each $(T - \lambda_j I) |_{G(\lambda_j,T)}$ is nilpotent.

***Proof 2.***

1. We will prove it by induction. Obviously, the conclusion follows when $n = 1$. Now, consider $n > 1$. Suppose the conclusion holds for all spaces with dimension $\leq n - 1$.

   *WTS: the conclusion is true for* $\dim V = n$.

   Consider $V = \operatorname{null}(T - \lambda_1 I)^n \oplus \operatorname{range}(T - \lambda_1 I)^n = G(\lambda_1, T) \oplus U$ if we fix $U = \operatorname{range}(T - \lambda_1 I)^n$. Obviously ,$G(\lambda_1, T) \neq \{0\}$. So, $\dim U < n$, and so our inductive hypothesis is applicable to $U$. Note that $G(\lambda_i, T) \cap G(\lambda_j, T) = \{0\}$ if $i \neq j$. Then, $\lambda_2, \ldots, \lambda_m$ are eigenvalues of $T |_U$. So, $U = G(\lambda_2, T |_U) \oplus \cdots \oplus G(\lambda_m, T |_U)$. Then, $V = G(\lambda_1, T) \oplus G(\lambda_2, T |_U) \oplus \cdots \oplus G(\lambda_m, T |_U)$.

   *WTS:* $G(\lambda_j, T |_U) = G(\lambda_j, T)$

   Note that $G(\lambda_j, T |_U) \subseteq G(\lambda_j, T)$ is evident. Conversely, suppose $v \in G(\lambda_k, T) \subseteq V$. Then, $v = v_1 + u$ for some $v_1 \in G(\lambda_1, T)$ and $u \in U$. Further, by our inductive hypothesis, we have

   $$u = v_2 + \cdots + v_m \quad \text{for some} \quad v_j \in G(\lambda_j, T |_U) \subseteq G(\lambda_j, T).$$

   Then, $v = v_1 + u = v_1 + v_2 + \cdots + v_m \in G(\lambda_k, T)$. That is, $v_1 + \cdots + (v_k - v) + \cdots + v_m = 0$. Then, $v_1 \in G(\lambda_1, T), \ldots, v_k - v \in G(\lambda_k, T), \ldots, v_m \in G(\lambda_m, T)$. Therefore, $v_1, \ldots, v_k - v, \ldots, v_m$ are L.I.. So, it must be that $v_1 = \cdots = v_k - 2 = \cdots = v_m = 0$. So, $v = v_1 + u = 0 + u = u$. Then, $v \in U$. So, $v \in G(\lambda_k, T) \cap U = G(\lambda_k, T |_U)$. As $k$ was arbitrary, we've shown $G(\lambda_k, U) \subseteq G(\lambda_k, T |_U)$. So, $G(\lambda_j, T |_U) = G(\lambda_j, T)$. We complete our proof.

2. Note that $G(\lambda_j, T) = \operatorname{null}(T - \lambda_j I)^n = \operatorname{null} p(T)$ if $p(z) = (z - \lambda_j)^n$. By Theorem 7.2.1, $\operatorname{null} p(T)$ is invariant under $T$. So, it follows that $G(\lambda_j, T)$ is also invariant under $T$. □

3. By definition, we have $G(\lambda_j, T) = \operatorname{null}(T - \lambda_j I)^n$. Then, $\left[(T - \lambda_j I) |_{G(\lambda_j,T)}\right]^n = 0$. So, by definition, $(T - \lambda_j I) |_{G(\lambda_j,T)}$ is nilpotent.

■

**Corollary 7.2.3** Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Then, $\exists$ a basis of $V$ consisting of generalized eigenvectors of $T$.

**Definition 7.2.4 (Multiplicity).** Suppose $T \in \mathcal{L}(V)$. The *(algebraic) multiplicity* of an eigenvalue $\lambda$ of $T$ is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$. In other words, the multiplicity of an eigenvalue $\lambda$ of $T$ equals $\dim \operatorname{null} (T - \lambda I)^{\dim V}$. The *geometric multiplicity* of an eigenvalue $\lambda$ of $T$ is $\dim E(\lambda, T)$.

**Theorem 7.2.5**
Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Then, the sum of the multiplicities of all eigenvalues of $T$ equals $\dim V$.

***Proof 3.*** By Theorem 7.2.2 (1), we know $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$. So, we have

$$\dim V = \dim G(\lambda_1, T) + \cdots + \dim G(\lambda_m, T).$$

■

**Definition 7.2.6 (Block Diagonal Matrix).** A *block diagonal matrix* is a square matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where $A_1, \ldots, A_m$ are square matrices lying along the diagonal and all the other entries of the matrix equal $0$.

**Theorem 7.2.7**
Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T$, with multiplicities $d_1, \ldots, d_m$. Then, $\exists$ a basis of $V$ with respect to which $T$ has a black diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where each $a_j$ is $d_j$-by-$d_j$ upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}.$$

***Proof 4.*** Note that $Tv_k = A_{1,k}v_1 + \cdots + A_{k,k}v_k + \cdots + A_{n,k}v_n$. Also, $(T - \lambda_j I) \mid_{G(\lambda_j, T)}$ is nilpotent. For each $G(\lambda_j, T)$, choose a basis of $G(\lambda_j, T)$ and $\dim G(\lambda_j, T) = d_j$. Then,

$$\mathcal{M}\Big((T - \lambda_j I) \mid_{G(\lambda_j, T)}\Big) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

Since $\mathcal{M}\Big((T-\lambda_j I)\mid_{G(\lambda_j,T)}\Big)=\mathcal{M}\Big(T\mid_{G(\lambda_j,T)}\Big)-\mathcal{M}(\lambda_j I)$, we have

$$\begin{aligned}\mathcal{M}\Big(T\mid_{G(\lambda_j,T)}\Big) &= \begin{pmatrix}0 & & *\\ & \ddots & \\ 0 & & 0\end{pmatrix}+\mathcal{M}(\lambda_j I)\\ &= \begin{pmatrix}0 & & *\\ & \ddots & \\ 0 & & 0\end{pmatrix}+\begin{pmatrix}\lambda_j & & *\\ & \ddots & \\ 0 & & \lambda_j\end{pmatrix}\\ &= \begin{pmatrix}\lambda_j & & *\\ & \ddots & \\ 0 & & \lambda_j\end{pmatrix}.\end{aligned}$$

Put all the bases of $G(\lambda_j,T)$ together, we have completed the proof. ■

## 7.3 Characteristic and Minimal Polynomials

**Definition 7.3.1 (Characteristic Polynomial).** Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of $T$, with multiplicities $d_1, \dots, d_m$. The polynomial

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

is called the *characteristic polynomial* of $T$.

**Theorem 7.3.2**
Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Then,

1. the characteristic polynomial of $T$ has degree $\dim V$;
2. the zeros of the characteristic polynomial of $T$ are eigenvalues of $T$.

***Proof 1.***

1. Note that $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$. So, $\dim V = d_1 + \cdots + d_m$. That is, the characteristic polynomial of $T$ has degree $\dim V$. $\square$
2. By the definition of characteristic polynomial, it is evidently true.

$\blacksquare$

**Theorem 7.3.3 Cayley-Hamilton Theorem**
Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Let $q$ denote the characteristic polynomial of $T$. Then, $q(T) = 0$.

***Proof 2.*** Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of $T$ and $d_1, \dots, d_m$ are their corresponding multiplicities. For each $j = 1, \dots, m$, we have $(T - \lambda_j I) \mid_{G(\lambda_j, T)}$ is nilpotent. Then, $(T - \lambda_j I)^{d_j} \mid_{G(\lambda_j, T)} = 0$. Since $q(z) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$, we know $q(T) = (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m}$. Consider $v \in V$. Since $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$, then $v = a_1 v_1 + \cdots + a_m v_m$, where $v_j \in G(\lambda_j, T)$. Then,

$$\begin{aligned} q(T)v &= q(T)(a_1 v_1 + \cdots + a_m v_m) \\ &= a_1 q(T) v_1 + \cdots + a_m q(T) v_m. \end{aligned}$$

For simplicity, consider

$$\begin{aligned} q(T)v_j &= (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m} v_j \\ &= (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m} (T - \lambda_j I)^{d_j} v_j. \end{aligned}$$

Since $v_j \in G(\lambda_j, T)$, we know $(T - \lambda_j I)^{d_j} v_j = 0$. Then, $q(T)v_j = 0$ for each $j = 1, \dots, m$. So, $q(T)v = 0$. That is, $q(T) = 0$. $\blacksquare$

**Definition 7.3.4 (Monic Polynomial).** A *monic polynomial* is a polynomial whose highest-degree coefficient equals $1$.

**Theorem 7.3.5**
Suppose $T \in \mathcal{L}(V)$. Then, $\exists$ a unique monic polynomial $p$ of smallest degree such that $p(T) = 0$.

***Proof 3.*** Let $\dim V = n$. Then, the list $I, T, T^2, \ldots, T^{n^2}$ is not L.I. in $\mathcal{L}(V)$ because $\mathcal{L}(V)$ has dimension $n^2$ and we have a list of length $n^2 + 1$. Let $m$ be the smallest positive integer such that the list $I, T, T^2, \ldots, T^m$ is linearly dependent. Then, by the Linear Dependence Lemma, $T^m$ is a linear combination of $I, T, \ldots, T^{m-1}$. So, we have

$$a_0 I + a_1 T + a_2 T^2 + \cdots + a_{m-1} T^{m-1} + T^m = 0 \tag{21}$$

Define a monic $p \in \mathcal{P}(\mathbb{F})$ as $p(z) = a_0 + z_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + z^m$. Then, Equation (21) implies $p(T) = 0$. Now, we will prove the uniqueness. Suppose $\exists$ a monic $q \in \mathcal{P}(\mathbb{F})$ with $\deg q = m$ *s.t.* $q(T) = 0$. Then, $(p-q)(T) = p(T) - q(T) = 0$ and $\deg(p-q) < m$. Hence, $p = q$. ■

**Definition 7.3.6 (Minimal Polynomial).** Suppose $T \in \mathcal{L}(V)$. Then, the *minimal polynomial* of $T$ is the unique monic polynomial $p$ of smallest degree such that $p(T) = 0$.

**Corollary 7.3.7** By the Cayley-Hamilton Theorem, the minimal polynomial of each $T \in \mathcal{L}(V)$ has degree $\leq \dim V$.

**Theorem 7.3.8 Division Algorithm of Polynomials**
Suppose $p, s \in \mathcal{P}(\mathbb{F})$ with $s \neq 0$. Then, $\exists$ unique $q, r \in \mathcal{P}(\mathbb{F})$ *s.t.* $p = sq + r$ and $\deg r < \deg s$.

***Proof 4.*** Let $\deg p = n$ and $\deg s = m$. If $n < m$, then $q = 0$ and $r = p$. Now, we assume $n \geq m$. Define $T : \mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F}) \to \mathcal{P}_n(\mathbb{F})$ as $T(q, r) = sq + r$. It is easy to verify that $T$ is a linear map. If $(q, r) \in \operatorname{null} T$, then $sq + r = 0$. So, $q = r = 0$. That is, $\dim \operatorname{null} T = 0$ and $T$ is injective. Further, note that $\dim(\mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F})) = (n-m+1) + (m-1+1) = n+1$ and $\dim \operatorname{range} T = n+1 = \dim \mathcal{P}_n(\mathbb{F})$. Since $\operatorname{range} T \subseteq \mathcal{P}_n(\mathbb{F})$ and $\dim \operatorname{range} T = \dim \mathcal{P}_n(\mathbb{F})$, we have $\operatorname{range} T = \mathcal{P}_n(\mathbb{F})$. Therefore, $T$ is surjective. ■

**Theorem 7.3.9**
Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{F})$. Then, $q(T) = 0$ if and only if $q$ is a polynomial multiple of the minimal polynomial of $T$.

***Proof 5.*** Let $p$ be the minimal polynomial of $T$.

($\Leftarrow$): Suppose $q = sp$. Then, $q(T) = s(T)p(T) = 0$. □

($\Rightarrow$): Suppose $q(T) = 0$. By division algorithm of polynomials, $q = sp + r$ with $\deg r < \deg p$. Then, $q(T) = s(T)p(T) + r(T) = 0$. Note that $p(T) = 0$, so $r(T) = 0$. Then, $r = 0$. It must be $q = sp$. ■

**Theorem 7.3.10 Characteristic Polynomial and Minimal Polynomial**
Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then, the characteristic polynomial of $T$ is a polynomial multiple of the minimal polynomial of $T$.

***Proof 6.*** Suppose $q$ is a characteristic polynomial of $T$. Then, by Cayley-Hamilton Theorem, $q(T) = 0$. Further by Theorem 7.3.9, $q$ is a polynomial multiple of the minimal polynomial of $T$. ■

**Theorem 7.3.11**
Let $T \in \mathcal{L}(V)$. Then, the zeros of the minimal polynomial of $T$ are precisely the eigenvalues of $T$.

**Remark.** *"Precisely" means "is and only is." So, we need to prove the theorem from two directions.*

***Proof 7.*** Suppose $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + z^m$ is the minimal polynomial of $T$.

($\Rightarrow$): Suppose $p(\lambda) = 0$. *WTS: $\lambda$ is the eigenvalue.* Since $p(\lambda) = 0$, we have $p(z) = (z - \lambda)q(z)$. Then, $p(T) = (T - \lambda I)q(T) = 0$. Then, $\deg q < \deg p$ and $p(T)v = (T - \lambda I)q(T)v = 0 \quad \forall v \in V$. So, $\exists v \in V$ *s.t.* $q(T)v \neq 0$. So, it must be that $T - \lambda I$ is not injective, and thus $\lambda$ is an eigenvalue of $T$. □

($\Leftarrow$): Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of $T$. Then, $\exists v \in V$ *s.t.* $Tv = \lambda v$ with $v \neq 0$. Consider $T^j v = \lambda^j v$. Then,

$$\begin{aligned} p(T)V &= (a_0 I + a_1 T + \cdots + a_{m-1}T^{m-1} + T^m)v \\ &= (a_0 + a_1\lambda + \cdots + a_{m-1}\lambda^{m-1} + \lambda^m)v \\ &= p(\lambda)v = 0 \end{aligned}$$

Since $v \neq 0$, it must be $p(\lambda) = 0$. ■

**Example 7.3.12** Suppose $T \in \mathcal{L}(\mathbb{C}^3)$ be defined as

$$T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3, 6z_2 + 2z_3, 7z_3).$$

Then,

$$\mathcal{M}(T) = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}.$$

Find the minimal polynomial of $T$.

***Solution 8.***

Since $\mathcal{M}(T) = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}$, the eigenvalues of $T$ are $6, 6, 7$. The multiplicity of $6$ is $2$ and that of $7$ is $1$. So, the characteristic polynomial of $T$ is $q(z) = (z-6)^2(z-7)$. Then, the minimal polynomial is polynomial multiple of $(z-6)(z-7)$. So, the minimal polynomial of $T$ should be $(z-6)(z-7)$ or $(z-6)^2(z-7)$. Note that

$$\begin{aligned} \mathcal{M}\big[(T-6I)^2(T-7I)\big] &= (\mathcal{M}(T-6I))^2\mathcal{M}(T-7I) \\ &= \begin{pmatrix} 0 & 010 & \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \end{aligned}$$

and

$$\mathcal{M}[(T-6I)(T-7I)] = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

So, $(z-6)^2(z-7)$ is the minimal polynomial of $T$. □

**Example 7.3.13** Find the minimal polynomial of operator $T \in \mathcal{L}(\mathbb{C}^3)$ defined by $T(z_1, z_2, z_3) = (6z_1, 6z_2, 7z_3)$.

***Solution 9.***

Note that

$$\mathcal{M}(T) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

Then, the characteristic polynomial is $q(z) = (z-6)^2(z-7)$. The minimal polynomial could be $(z-6)^2(z-7)$ or $(z-6)(z-7)$. Since

$$\begin{aligned} \mathcal{M}[(T-6I)(T-7I)] &= \mathcal{M}(T-6I)\mathcal{M}(T-7I) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \end{aligned}$$

the minimal polynomial of $T$ is $(z-6)(z-7)$. □

**Theorem 7.3.14**
Suppose $T \in \mathcal{L}(V)$. $T$ is invertible if and only if the constant term in the minimal polynomial of $T$ is non-zero.

***Proof 10.*** Let $p(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} + z^m$ be the minimal polynomial of $T$.
($\Rightarrow$) We will prove the contrapositive: "If $a_0 = 0$, then $T$ is not invertible." Suppose $a_0 = 0$. Then,

$$p(z) = a_1 z + \cdots + a_{m-1}z^{m-1} + z^m.$$

Then, $p(0) = 0$. So, $0$ is an eigenvalue of $T$. That is, $Tv = 0$ for some $v \neq 0$. Then, $T$ is not injective, and thus is not invertible. □

($\Leftarrow$) We will prove the contrapositive: "If $T$ is not invertible, then $a_0 = 0$." Suppose $T$ is not invertible. Then, $T$ is not injective. So, $\exists v \neq 0$ *s.t.* $Tv = 0$. That is, $Tv = 0 \cdot v$ or $0$ is an eigenvalue of $T$. So, $p(z) = zq(z)$, and thus $a_0 = 0$. ■

**Theorem 7.3.15**
Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. $V$ has a basis consisting of eigenvectors of $T$ if and only if the minimal polynomial of $T$ has no repeated roots.

## 7.4 Jordan Form

**Example 7.4.1** Let $N \in \mathcal{L}(\mathbb{F}^4)$ be the nilpotent operator $N(z_1, z_2, z_3, z_4) = (0, z_1, z_2, z_3)$. Let $v = (1, 0, 0, 0)$. Then, $Nv = (0, 1, 0, 0)$, $N^2 v = (0, 0, 1, 0)$, and $N^3 v = (0, 0, 0, 1)$. Note that $v, Nv, N^2v, N^3v$ is a basis of $\mathbb{F}^4$, and the matrix of $N$ with respect to this basis is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Example 7.4.2** Let $N \in \mathcal{L}(\mathbb{F}^6)$ be the nilpotent operator defined by

$$N(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0).$$

Let $v_1 = (1, 0, 0, 0, 0, 0)$, $v_2 = (0, 0, 0, 1, 0, 0)$, and $v_3 = (0, 0, 0, 0, 0, 1)$. Then, we have $N^2 v_1$, $Nv_1$, $Nv_2$, $v_2$, $v_3$ to be a basis of $\mathbb{F}^6$. The matrix of $N$ with respect to this basis is

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix}$$

**Theorem 7.4.3**
Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then, $\exists v_1, \ldots, v_n \in V$ and $m_1, \ldots, m_n \in \mathbb{N}^+$ such that

1. $N^{m_1} v_1, \ldots, Nv_1, v_1, \ldots, N^{m_n} v_n, \ldots, Nv_n, v_n$ is a basis of $V$;
2. $N^{m_1+1} v_1 = \cdots = N^{m_n+1} v_n = 0$.

***Proof 1.*** We will prove by induction on $\dim V$.

Base Case When $\dim V = 1$, the conclusions obviously hold.

Inductive Steps Assume $\dim V > 1$ and the conclusions hold for all spaces with dimension smaller than $\dim V$. Since $N$ is nilpotent, it is not injective and thus is not surjective. So, $\operatorname{range} N \subsetneq V$. That is, $\dim \operatorname{range} N < \dim V$. Since $N$ is nilpotent, it is not injective and thus is not surjective. So, $\operatorname{range} N \subsetneq V$. that is, $\dim \operatorname{range} N < \dim V$. Apply the inductive hypothesis on $\operatorname{range} N$. Consider $N|_{\operatorname{range} N} \in \mathcal{L}(\operatorname{range} N)$, then $\exists v_1, \ldots, v_n \in \operatorname{range} N$ and $m_1, \ldots, m_n \in \mathbb{N}^+$ such that

$$N^{m_1} v_1, \ldots, Nv_1, v_1, \ldots, N^{m_n} v_n, \ldots, Nv_n, v_n. \tag{22}$$

is a basis of $\operatorname{range} N$, and $N^{m_1+1} v_1 = \cdots = N^{m_n+1} v_n = 0$. For each $j$, $v_j \in \operatorname{range} N$. Then, $\exists u_j \in$

$V$ *s.t.* $v_j = Nu_j$. So, $N^{k+1}u_j = N^k v_j \quad \forall k \in \mathbb{N}^+$. We now claim the following list of vectors is L.I.:

$$N^{m_1+1}u_1, \dots, Nu_1, u_1, \dots, N^{m_n+1}u_n, \dots, Nu_n, u_n \tag{23}$$

Let $a_1^{m_1+1}N^{m_1+1}u_1 + \cdots + a_1^1 Nu_1 + a_1^0 u_1 + \cdots + a_n^{m_n+1}N^{m_n+1}u_n + \cdots + a_n^1 Nu_n + a_n^0 u_n = 0$. Then,

$$a_1^{m_1+1}N^{m_1}v_1 + \cdots + a_1^1 v_1 + a_1^0 u_1 + \cdots + a_n^{m_n+1}N^{m_n}v_n + \cdots + a_n^1 v_n + a_n^0 u_n = 0. \tag{24}$$

Apply $N$ to both sides of the Equation (24),

$$\underbrace{a_1^{m_1+1}N^{m_1+1}v_1}_{0} + \cdots + a_1^1 Nv_1 + a_1^0 \underbrace{Nu_1}_{v_1} + \cdots + \underbrace{a_n^{m_n+1}N^{m_n+1}v_n}_{0} + \cdots + a_n^1 Nv_n + a_n^0 \underbrace{Nu_n}_{v_n} = 0.$$

So,

$$a_1^{m_1}N^{m_1}v_1 + \cdots + a_1^1 Nv_1 + a_1^0 v_1 + \cdots + a_n^{m_n}N^{m_n}v_n + \cdots + a_n^1 Nv_n + a_n^0 v_n = 0.$$

Since Equation (22) is a basis, it must be all the coefficients equal to $0$. Meanwhile, reconsider Equation (24). It becomes

$$a_1^{m_1+1}N^{m_1}v_1 + \cdots + a_n^{m_n+1}N^{m_n}v_n = 0.$$

As $N^{m_1}, \dots, N^{m_n}$ is included in the list of vector stated in Equation (22), they must also be L.I.. Thus, we have $a_1^{m_1+1} = \cdots = a_n^{m_n+1} = 0$. So, we have proven the claim by showing Equation (23) is indeed a list of L.I. vectors. Now, extend Equation (23) into a bassi of $V$:

$$N^{m_1+1}u_1, \dots, Nu_1, u_1, \dots, N^{m_n+1}u_n, \dots, Nu_n, u_n, w_1, \dots, w_p \tag{25}$$

Then, each $Nw_j \in \operatorname{range} N = \operatorname{span}($Equation (22)$)$ *s.t.* $Nw_j = Nx_j$. Now, suppose $u_{n+j} = w_j - x_j$, and we have $Nu_{n+j} = 0$. Hence,

$$N^{m_1+1}u_1, \dots, Nu_1, u_1, \dots, N^{m_n+1}u_n, \dots, Nu_n, u_n, u_{n+1}, \dots, u_{n+p} \tag{26}$$

spans $V$ because it contains each $x_j$ and $u_{n+j}$ and thus $w_j$. Since Equation (25) and Equation (26) have the same length, Equation (26) is a basis of $V$ satisfying the desired condition. ■

**Definition 7.4.4 (Jordan Basis).** Suppose $T \in \mathcal{L}(V)$. A basis of $V$ is called a *Jordan basis* of $T$ if $\mathcal{M}(T)$ with respect to this basis has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix},$$

where each $A_j$ is an upper-triangular matrix of the form

$$\begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

**Theorem 7.4.5 Jordan Form**
Suppose $V$ is a complex vector space. If $T \in \mathcal{L}(V)$, then $\exists$ a basis of $V$ that is a Jordan basis for $T$.

***Proof 2.*** First consider a nilpotent operator $N \in \mathcal{L}(V)$. Suppose $v_1, \ldots, v_n \in \mathcal{L}(V)$ satisfy the condition in Theorem 7.4.3. For each $j$, note that the list of vectors $N^{m_j}v_j, N^{m_{j-1}}v_j, \ldots, Nv_j, v_j$ correspond to a matrix of $N$ as

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0. \end{pmatrix}$$

Hence, the conclusion holds for a nilpotent operator. Assume $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T$. Then, we have the generalized eigenspace decomposition:

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T),$$

where each $(T - \lambda_j I) \mid_{G(\lambda_j, T)}$ is nilpotent. Thus, some basis of each $G(\lambda_j, T)$ is a Jordan basis of $T - \lambda_j I$. So,

$$\mathcal{M}\Big((T - \lambda_j I) \mid_{G(\lambda_j, T)}\Big) = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

and

$$\mathcal{M}\Big(T \mid_{G(\lambda_j, T)}\Big) = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

Also, the dimension of the matrix is $\dim G(\lambda_j, T)$. ■

# 8 Operators on Real Vectors Spaces

## 8.1 Complexification

**Definition 8.1.1 (Complexification of $V$/$V_{\mathbb{C}}$).** Suppose $V$ is a real vector space. The *complexification* of $V$, denoted $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair $(u, v)$, where $u, v \in V$, but we will write this as $u + \mathrm{i}v$.

**Definition 8.1.2 (Addition & Multiplication on $V_{\mathbb{C}}$).**

1. *Addition* on $V_{\mathbb{C}}$ is defined by

$$(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2) = (u_1 + u_2) + \mathrm{i}(v_1 + v_2).$$

   for $u_1, u_2, v_1, v_2 \in V$.

2. *Complex Scalar Multiplication* on $V_{\mathbb{C}}$ is defined by

$$(a + b\mathrm{i})(u + \mathrm{i}v) = (au - bv) + \mathrm{i}(av + bu)$$

   for $a, b \in \mathbb{R}$ and $u, v \in V$.

**Theorem 8.1.3**
Suppose $V$ is a real vector space. Then, with the definition of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space.

***Proof 1.***

1. Addition. Let $u_j + \mathrm{i}v_j \in \mathbb{C}$.

   (a) commutativity:

$$\begin{aligned}(u_1 + \mathrm{i}v_1) + (u_2 + \mathrm{i}v_2) &= (u_1 + u_2) + \mathrm{i}(v_1 + v_2) \\ &= (u_2 + u_1) + \mathrm{i}(v_2 + v_1) \\ &= (u_2 + \mathrm{i}v_2) + (u_1 + \mathrm{i}v_1). \qquad \square\end{aligned}$$

   (b) associativity:

$$\begin{aligned}((u_1, v_1) + (u_2, v_2)) + (u_3, v_3) &= (u_1 + u_2, v_1 + v_2) + (u_3, v_3) \\ &= (u_1 + u_2 + u_3, v_1 + v_2 + v_3) \\ &= (u_1 + (u_2 + u_3), v_1 + (v_2 + v_3)) \\ &= (u_1, v_1) + ((u_2, v_2) + (u_3, v_3)). \qquad \square\end{aligned}$$

   (c) identity:

$$\begin{aligned}(0, 0) + (u, v) = (0 + u, 0 + v) &= (u + 0, v + 0) \\ &= (u, v) + (0, 0) \\ &= (u, v). \qquad \square\end{aligned}$$

   (d) inverse:

$$(-u, -v) + (u, v) = (-u + u, -v + v) = (0, 0). \qquad \square$$

2. Scalar Multiplication: Let $(u, v) \in V_{\mathbb{C}}$, $a + b\mathrm{i}$ and $c + d\mathrm{i} \in \mathbb{C}$.

   (a) identity:
   $$(1 + 0\mathrm{i})(u + \mathrm{i}v) = u + \mathrm{i}v + 0\mathrm{i}u - 0v = u + \mathrm{i}v. \qquad \square$$

   (b) associativity: can be easily verified. omitted.

   (c) distributivity: can be easily verified. omitted.

■

**Theorem 8.1.4**
Suppose $V$ is a real vector space.

1. If $v_1 \ldots, v_n$ is a basis of $V$ (as a real vector space), then $v_1, \ldots, v_n$ is a basis of $V_{\mathbb{C}}$ (as a complex vector space).
2. The dimension of $V_{\mathbb{C}}$ (as a complex vector space) equals the dimension of $V$ (as a real vector space).

***Proof 2.***

■

## 8.2 Operators on Real Inner Product Spaces

# 9 Trace and Determinant

## 9.1 Trace

## 9.2 Determinant