Emory University

MATH 212 Differential Equations Learning Notes

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Contents

1 First Order ODEs

1.1 Introduction

Definition 1.1.1 (Ordinary Differential Equations/ODEs). An *ordinary differential equation* is an equation that contains one or more derivatives of an unknown function $y = y(x)$.
Definition 1.1.2 (Order of ODEs). The *order* of an ODE is the maximum order of the derivatives appearing in the equation.
Definition 1.1.3 (Solution to ODEs). The *solution* to an ODE is a function y that satisfies the equation.

Example 1.1.4 Solve $y'' = 3x + 1$.

Solution 1.

$$\begin{aligned} y' &= \int 3x + 1 \, \mathrm{d}x = \frac{3}{2}x^2 + x + C \\ y &= \int y' \, \mathrm{d}x = \int \frac{3}{2}x^2 + x + C \, \mathrm{d}x = \frac{1}{2}x^3 + \frac{1}{x}x^2 + Cx + D. \end{aligned}$$

□

Definition 1.1.5 (Linear ODEs/Non-Linear ODEs). A first order ODE is *linear* if it can be written as

$$y' + p(x)y = f(x).$$

Otherwise, it is *non-linear*.
Definition 1.1.6 (Homogenous/Non-Homogenous Linear ODEs). If $f(x) = 0$, then the linear ODE is *homogenous*. That is,

$$y' + p(x)y = 0.$$

Otherwise, it is *non-homogenous*.
Definition 1.1.7 (Trivial/Non-Trivial Solution). $y = 0$ is a *trivial solution* to a homogenous ODE. Any other solutions are *non-trivial*.
Definition 1.1.8 (One-Parameter Family of Solutions). We call C a *parameter* and the equation, therefore solution, defines a *one-parameter family* of solutions.

Example 1.1.9 For the ODE $y' = 1$, $y_1 = x + C_1$ is a solution to it, and it is a one-parameter family of solutions. Similarly, for $y' = \frac{1}{x^2}$, the one-parameter families of solutions are defined by $y_2 = -\frac{1}{x} + C_2$ on the interval $(-\infty, 0) \cup (0, \infty)$.

Definition 1.1.10 (General Solution). Given the general form of the linear ODE $y' + p(x)y =$

$f(x)$ if p and f are continuous on some open interval (a, b) and there is a unique formula $y = y(x, c)$ and we have the following properties:

- for each fixed c, the resulting function of x is a solution of the ODE on (a, b), and
- if y is a solution of the ODE, then y can be obtained by choosing the value of c appropriately.

The function $y = y(x, c)$ is called a *general solution.*

More generally, we can write an ODE as

$$P_0(x)y' + P_1(x)y = F(x).$$

In this case, the ODE has a general solution on any open interval in which $P_0,\ P_1$, and F are continuous and $P_0 \neq 0$.

Definition 1.1.11 (Initial Value Problem (IVP)). A differential equation with an initial condition.

Example 1.1.12 Let a be a constant. Find the general solution of $y' - ay = 0$ and solve the IVP $\begin{cases} y' - ay = 0 \\ y(x_0) = y_0. \end{cases}$

Solution 2.

Classification: First order, Linear, Homogeneous.

Trivial Solution: $y = 0$.

General solution:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= ay \\ \int \frac{1}{y}\,\mathrm{d}y &= \int a\,\mathrm{d}x \\ \ln|y| &= ax + c \\ y &= e^{ax+c} = Ae^{ax}. \end{aligned}$$

This general solution includes the trivial solution.

IVP: Substitute $x = x_0$ and $y = y_0$:

$$y_0 = Ae^{ax_0} \qquad \longrightarrow \qquad A = y_0 e^{-ax_0}$$

So,

$$y^{\text{IVP}} = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

This IVP is a "generic initial condition." We need more information on x_0, y_0 *to get a more specific solution.* □

1.2 Linear First Order ODEs

Theorem 1.2.1
If p is continuous on (a, b), then the general solution of the homogeneous equation $y' + p(x)y = 0$ on (a, b) is given by

$$y = ce^{-\int p(x)\,\mathrm{d}x}.$$

Proof 1.
(a). Substitute the solution formula to show that $y = ce^{-\int p(x)\,\mathrm{d}x}$ is a solution for any choice of c.

$$y' = c\left(-\int p(x)\,\mathrm{d}x\right)' e^{-\int p(x)\,\mathrm{d}x} = -cp(x)e^{-\int p(x)\,\mathrm{d}x}.$$

Then,

$$y' + p(x)y = -cp(x)e^{-\int p(x)\,\mathrm{d}x} + cp(x)e^{-\int p(x)\,\mathrm{d}x} = 0.$$

So, $y = ce^{-\int p(x)\,\mathrm{d}x}$ is a solution for any choice of c. $\square$

(b). Want to show: any solution of $y' + p(x)y = 0$ can be written as $y = ce^{-\int p(x)\,\mathrm{d}x}$. Note that $y = 0$ is a trivial solution, so we assume $y \neq 0$.

$$\begin{aligned}
y' + p(x)y &= 0 \\
y' &= -p(x)y \\
\frac{y'}{y} &= -p(x) \\
\rightsquigarrow \int \frac{1}{y}\,\mathrm{d}y &= \int -p(x)\,\mathrm{d}x \\
\ln|y| &= -\int p(x)\,\mathrm{d}x \\
y &= ce^{-\int p(x)\,\mathrm{d}x}.
\end{aligned}$$

Note that when $c = 0$, $y = 0$ is the trivial solution. So, any solution of $y' + p(x)y = 0$ can be written as $y = ce^{-\int p(x)\,\mathrm{d}x}$. ■

Example 1.2.2 Solve the IVP

$$\begin{cases} xy' + y = 0 \\ y(1) = 3. \end{cases}$$

Solution 2.
Note that $P_0(x) = x$ and $P_1(x) = 1$, which are continuous on $\mathbb{R}$. Since we need $P_0(x) \neq 0$, $x \neq 0$. So the interval of validity is $\mathbb{R}\backslash\{0\}$.

Method 1: Separation of Variables

$$y' = -\frac{y}{x}.$$

Note that $y = 0$ is a solution. Assume $y \neq 0$.

$$\begin{aligned} \frac{y'}{y} = -\frac{1}{x} \quad \rightsquigarrow \quad \int \frac{1}{y}\,\mathrm{d}y &= -\int \frac{1}{x}\,\mathrm{d}x + k \\ \ln|y| &= -\ln|x| + k \\ |y| &= e^k \frac{1}{|x|} \\ y &= \frac{c}{x} \end{aligned}$$

$\boxed{\text{Method 2: Solution Formula}}$ By Theorem 1.2.1,

$$y = ce^{-\int p(x)\,\mathrm{d}x} = ce^{-\int \frac{1}{x}\,\mathrm{d}x} = ce^{-\ln|x|} = \frac{c}{x}.$$

$\boxed{\text{Solving the IVP}}$ Substitute $x = 1$ and $y = 3$:

$$3 = \frac{c}{1} \quad \longrightarrow \quad c = 3.$$

So, $y^{\text{IVP}} = \dfrac{3}{x}$. □

Example 1.2.3 Given the equation $(4 + x^2)y' + 2xy = 4x$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 3.

This is a first order, linear, non-homogeneous differential equation.

Note that $P_0(x) = 4 + x^2$, $P_1(x) = 2x$, $F(x) = 4x$, and $P_0 \neq 0\ \forall x \in \mathbb{R}$, so the interval of validity is $\mathbb{R}$. Also note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[4 + x^2\right] = 2x$, so the equation can be written as

$$(4 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left[4 + x^2\right]y = 4x.$$

Using the product rule to re-write the LHS as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[(4 + x^2)y\right] &= 4x \\ \int \frac{\mathrm{d}}{\mathrm{d}x}\left[(4 + x^2)y\right]\,\mathrm{d}x &= \int 4x\,\mathrm{d}x + c \\ (4 + x^2)y &= 2x^2 + c \\ y &= \frac{2x^2 + c}{4 + x^2}. \end{aligned}$$

□

Example 1.2.4 Given the equation $y' - 2y = 4 - x$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 4.

This is a first order, linear, non-homogeneous differential equation.

Since $P_0(x) = 1$, $P_1(x) = -2y$, $F(x) = 4 - x$, and $P_0(x) \neq 0 \, \forall x \in \mathbb{R}$, the interval of validity is $\mathbb{R}$. Consider $\mu = \mu(x) \neq 0$. Multiply both sides of the equation by $\mu(x)$:

$$\mu(x)y' - 2\mu(x)y = \mu(x)(4 - x) \tag{1}$$

To make the LHS a product rule, we need

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)y(x)\right] = \mu'(x)y(x) + \mu(x)y'(x) = \mu(x)y'(x) - 2\mu(x)y.$$

So, we have $\mu' = -2\mu$, or $\mu' + 2\mu = 0$, a first order, linear, homogeneous ODE. Solving this ODE, we get $\mu(x) = ce^{-2x}$. Since we only want one specific μ that would work, take $c = 1$. So, $\mu(x) = e^{-2x}$. Substituting $\mu(x) = e^{-2x}$ to Eq. (1):

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}(4 - x), \quad \widetilde{P}_0 = e^{-2x} \neq 0,\ \widetilde{P}_1 = -2e^{-2x}.$$

Using the product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-2x}y\right] &= 4e^{-2x} - xe^{-2x} \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-2x}y\right] \mathrm{d}x &= \int 4e^{-2x} - xe^{-2x} \, \mathrm{d}x + c \\
e^{-2x}y &= \frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c \\
y &= e^{2x}\left(\frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c\right) \\
&= \frac{1}{2}x - \frac{7}{4} + ce^{2x}.
\end{aligned}$$

□

Theorem 1.2.5 Method of Integrating Factor
Given the first order linear differential equation $y' + p(x)y = f(x)$, with p and f both continuous on some interval (a, b),

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,\mathrm{d}x + c\right]$$

is the general solution to the equation, with

$$\mu(x) = e^{\int p(x)\,\mathrm{d}x}.$$

We call $\mu(x)$ the *integrating factor.*

Proof 5. Consider $\mu = \mu(x) \neq 0$. Multiplying the both sides of $y' + p(x)y = f(x)$ by μ:

$$\mu y' + p\mu y = \mu f. \tag{2}$$

Impose $\mu y' + p\mu y = \frac{\mathrm{d}}{\mathrm{d}x}[\mu y]$ to find $\mu = \mu(x)$:

$$\begin{aligned} \mu y' + p\mu y &= \mu' y + \mu y' \\ \mu' - p\mu &= 0, && \text{first order, linear, homogeneous ODE} \\ \mu(x) &= e^{\int p(x)\,\mathrm{d}x}, && \text{the integrating factor} \end{aligned}$$

Substitute $\mu(x) = e^{\int p(x)\,\mathrm{d}x}$ into Eq. (2):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[\mu y] &= \mu f \\ \int \frac{\mathrm{d}}{\mathrm{d}x}[\mu y]\,\mathrm{d}x &= \int \mu f\,\mathrm{d}x + c \\ \mu y &= \int \mu f\,\mathrm{d}x + c \\ y(x) &= \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,\mathrm{d}x + c\right]. \end{aligned}$$

■

Example 1.2.6 Give the equation $y' + 2y = x^3e^{-2x}$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 6.

It is a first order, linear, non-homogeneous ODE, with $p = 2$ and $f = x^3e^{-2x}$. Let $\mu(x)$ be the integrating factor. Then,

$$\mu(x) = e^{\int 2\,\mathrm{d}x} = e^{2x}.$$

So, by the method of integrating factor, we know

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big[\mu(x)y\Big] &= \mu(x)f(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\Big[e^{2x}y\Big]\,\mathrm{d}x &= \int e^{2x}x^3e^{-2x}\,\mathrm{d}x + c \\
e^{2x}y &= \int x^3\,\mathrm{d}x + c \\
e^{2x}y &= \frac{1}{4}x^4 + c \\
y &= \frac{1}{4}x^4e^{-2x} + ce^{-2x}.
\end{aligned}$$

□

Remark. *Re-examine the formula we derived from the method of integrating factor:*

$$y(x) = \frac{1}{\mu}\int f\mu\,\mathrm{d}x + \boxed{\frac{c}{\mu}}.$$

The part being boxed, $\frac{c}{\mu}$, is independent from f and is exactly $ce^{-\int p\,\mathrm{d}x}$ if we expand, which is the solution for a homogeneous differential equation.

Definition 1.2.7 (Complementary Equation). The *complementary equation* to a first order ODE $y' + py = f$ is the homogeneous part of it. i.e., $y' + py = 0$.

Theorem 1.2.8 Method of Variation of Parameters
Given the first order linear differential equation $y' + p(x)y = f(x)$, with p and f both continuous on some interval (a, b),

$$y(x) = y_1(x)\left[\int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c\right]$$

is the general solution to the equation, where y_1 is a solution of the complementary euqation $y' + py = 0$.

Proof 7. Call y_1 a solution of the complementary equation $y' + p(x)y = 0$. Then, we want to find $y(x) = u(x)y_1(x)$, the general solution of $y' + p(x)y = f(x)$, where u is an unknown function of f. Note that, by product rule, $y'(x) = u'y_1 + uy_1'$. Then, the equation becomes

$$\begin{aligned}
(u'y_1 + uy_1') + p(x)(uy_1) &= f(x) \\
u'y_1 + uy_1' + puy_1 &= f \\
y_1u' + \underbrace{(y_1' + py_1)}_{0}\,u &= f
\end{aligned}$$

$$y_1u' = f \implies u(x) = \int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c.$$

Therefore, the formula to find y is given by

$$y = y_1 u = y_1(x)\left[\int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c\right].$$

■

Remark. *The method of variation of parameters will be more useful when solving second or higher order differential equations.*

Example 1.2.9 Give the equation $y' + 2y = x^3 e^{-2x}$. Find the general solution $y = y(x, c)$ using the method of variation of parameters.

Solution 8.

It is a first order, linear, non-homogeneous ODE, with $p = 2$ and $f = x^3 e^{-2x}$. Let y_1 be the solution of the complementary equation $y' + 2y = 0$. Then, $y_1(x) = e^{-\int 2\,\mathrm{d}x} = e^{-2x}$. By the method of variation of parameters, suppose $y = uy_1$, where u is an unknown function of x. Then,

$$u(x) = \int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c = \int \frac{x^3 e^{-2x}}{e^{-2x}}\,\mathrm{d}x + c = \int x^3\,\mathrm{d}x + c = \frac{1}{4}x^4 + c.$$

So,

$$y = uy_1 = e^{-2x}\left(\frac{1}{4}x^4 + c\right) = \frac{1}{4}x^4 e^{-2x} + ce^{-2x}.$$

□

Theorem 1.2.10 Existence and Uniqueness Theorem

Suppose that $p = p(x)$ and $f = f(x)$ are continuous on (a, b). Then, a general solution of $y' + p(x)y = f(x)$ on (a, b) is

$$y(x) = y_1(x)\left[\int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c\right],$$

where $y_1(x)$ is a solution of the complementary equation (i.e., $y' + p(x)y = 0$).

If x_0 is an arbitrary point in (a, b) and y_0 is an arbitrary real number, then the initial value problem,

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a unique solution on (a, b).

1.3 Non-Linear First Order ODEs

Definition 1.3.1 (General Forms). The general form of a non-linear first order ODE is given by

$$y' = f(x, y(x)).$$

If we take $M(x,y) = -f(x,y)$ and $N(x,y) = 1$, we can also re-write the equation into

$$M(x,y) + N(x,y)y' = 0, \quad \text{or} \quad M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0.$$

1.3.1 Separable Equations

Definition 1.3.2 (Separable Equations). If $M(x,y) = M(x)$ and $N(x,y) = N(y)$, then the ODE is called *separable*.

Theorem 1.3.3 Separation of Variables (SoV)
Consider the non-linear first order ODE $M(x,y)+N(x,y)y' = 0$, with $M(x,y) = M(x)$ and $N(x,y) = N(y)$. Then we can find an implicit solution of the ODE in the form of

$$F(x,y) = c,$$

where $F(x,y)$ is a function of x and y and

$$F(x,y) = \int M(x)\,\mathrm{d}x + \int N(y)\,\mathrm{d}y.$$

Proof 1. Let $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$. Then, the equation becomes

$$\begin{aligned} H_1'(x) + H_2'(y)y' &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_1(x)\Big] + \frac{\mathrm{d}}{\mathrm{d}y}\Big[H_2(y)\Big]\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

By using the chain rule, $\frac{\mathrm{d}}{\mathrm{d}y}\Big[H_2(y)\Big]\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_2(y(x))\Big]$. So, the equation becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_1(x)\Big] + \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_2(y(x))\Big] &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_1(x) + H_2(y)\Big] &= 0 \\ H_1(x) + H_2(y) &= c \\ \int M(x)\,\frac{\mathrm{d}}{\mathrm{d}x} + \int N(y)\,\mathrm{d}y &= c \\ F(x,y) &= c \end{aligned}$$

■

Example 1.3.4 Given the equation $y' = \dfrac{x^2}{1-y^2}$. Classify the differential equation and find the general solution.

Solution 2.

It is a first order, non-linear ODE. Since $y' = \dfrac{x^2}{1-y^2}$, so we have $1 - y^2 \neq 0$. That is, $y^2 \neq 1$, or $y \neq \pm 1$. Using the separation of variables (SoV), we have

$$\begin{aligned}
(1-y^2)y' &= x^2 \\
\int 1 - y^2 \, \mathrm{d}y &= \int x^2 \, \mathrm{d}x \\
y - \frac{1}{3}y^3 &= \frac{1}{3}x^3 + c \\
y - \frac{1}{3}y^3 - \frac{1}{3}x^3 &= c \\
3y - y^3 - x^3 &= c
\end{aligned}$$

□

Example 1.3.5 Given the equation $y' = \dfrac{(y-3)\cos x}{1+2y^2}$. Classify the equation and find the general solution.

Solution 3.

It is a first order, non-linear ODE. Since $1 + 2y^2 \neq 0 \quad \forall y \in \mathbb{R}$. Note that if we take $y - 3 = 0$, we get $y = 3$, a constant solution to the differential equation. Now, assume $y \neq 3$. Then, use SoV:

$$\int \frac{1+2y^2}{y-3} \, \mathrm{d}y = \int \cos x \, \mathrm{d}x + c = \sin x + c.$$

Set $t = y - 3$, $\mathrm{d}t = \mathrm{d}y$. So, $y = t + 3$ and $y^2 = (t+3)^2$. Then,

$$\begin{aligned}
\int \frac{1+2y^2}{y-3} \, \mathrm{d}y = \int \frac{1+2(t+3)^2}{t} \, \mathrm{d}t &= \int \frac{1+2t^2+12t+18}{t} \, \mathrm{d}t \\
&= \int \frac{19}{t} + 12 + 2t \, \mathrm{d}t \\
&= 19 \ln|t| + 12t + t^2 \\
&= 19 \ln|y-3| + 12(y-3) + (y-3)^2 \\
&= 19 \ln|y-3| + 6y + y^2 - 27.
\end{aligned}$$

So,

$$19 \ln|y-3| + y^2 + 6y - 27 - \sin x = c$$

□

Example 1.3.6 Give the equation $y' = \frac{1}{2}x(1-y^2)$. Classify the equation and find the general solution.

Solution 4.

It is a first order, non-linear ODE. Notice that we have the constant solutions when we take $1-y^2=0$, or $y=\pm 1$. Now, assume $y \neq \pm 1$. Using SoV:

$$\int \frac{2}{1-y^2}\,\mathrm{d}y = \int x\,\mathrm{d}x + c = \frac{1}{2}x^2 + c.$$

Note that $\frac{2}{1-y^2} = \frac{2}{(1-y)(1+y)}$. Use partial fractions. Assume

$$\frac{2}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}.$$

Then, we get $A(1+y)+B(1-y)=2$. That is, $(A+B)+(A-B)y=2$. Attempting to solve the system of equations $\begin{cases} A-B=0 \\ A+B=2 \end{cases}$, then we get $A=B=1$.Therefore,

$$\frac{2}{1-y^2} = \frac{1}{1-y} + \frac{1}{1+y}.$$

Then,

$$\begin{aligned}
\int \frac{1}{1-y} + \frac{1}{1+y}\,\mathrm{d}y &= \frac{1}{2}x^2 + c \\
-\ln|1-y| + \ln|1+y| &= \frac{1}{2}x^2 + c \\
\ln|1-y| - \ln|1+y| &= -\frac{1}{2}x^2 + c \\
\ln\left|\frac{1-y}{1+y}\right| &= -\frac{1}{2}x^2 + c \\
\left|\frac{y-1}{y+1}\right| &= e^{-\frac{1}{2}x^2+c} = e^{-\frac{1}{2}x^2}e^c \\
\frac{y-1}{y+1} &= c_2 e^{-\frac{1}{2}x^2} \\
y-1 &= (y+1)c_2 e^{-\frac{1}{2}x^2} \\
(1-c_2 e^{-\frac{1}{2}x^2})y &= 1 + c_2 e^{-\frac{1}{2}x^2} \\
y &= \frac{1+c_2 e^{-\frac{1}{2}x^2}}{1-c_2 e^{-\frac{1}{2}x^2}}
\end{aligned}$$

The value of c_2 si chosen according to the sign of $\frac{y-1}{y+1}$. □

Example 1.3.7 Given the equation $y' = \dfrac{3x^2+4x+2}{2(y-1)}$, with $y(0) = 1$. Classify the equation, find the general solution, and solve the IVP.

Solution 5.

It is a first order, nonlinear, separable ODE. Note that $y - 1 \neq 0$, so $y \neq 1$. Assume $y \neq 1$, use SoV:

$$\begin{aligned}
\int 2(y-1)\,\mathrm{d}y &= \int 3x^2 + 4x + 2\,\mathrm{d}x + c \\
(y-1)^2 &= x^3 + 2x^2 + 2x + c \\
y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + c \\
y^2 - 2y &= x^3 + 2x^2 + 2x + c.
\end{aligned}$$

Substitute $y = -1$ when $x = 0$:

$$1 + 2 = c \implies c = 3.$$

So,

$$\begin{aligned}
y^2 - 2y &= x^3 + 2x^2 + 2x + 3, \quad y \neq 1 \\
y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + 4 \\
(y-1)^2 &= x^3 + 2x^2 + 2x + 4 \\
y - 1 &= \pm\sqrt{x^3 + 2x^2 + 2x + 4} \\
y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.
\end{aligned}$$

If $y = -1$ and $x = 0$: $-1 = 1 \pm \sqrt{4} = 1 \pm 2$. So, it must be that $-1 = 1 - 2$. So,

$$y^{\text{IVP}} = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

Note that now we have another condition for x:

$$\begin{aligned}
x^3 + 2x^2 + 2x + 4 &\geq 0 \\
(x+2)(x^2+2) &\geq 0 \\
x + 2 &\geq 0 \\
x &\geq -2
\end{aligned}$$

So,

$$y^{\text{IVP}} = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}, \text{ with } y \neq 1 \text{ and } x \geq -2.$$

□

Example 1.3.8 Solve the IVP $\begin{cases} y' = \sqrt[3]{y} = y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$.

Solution 6.

It is a first order, nonlinear, separable ODE. The initial interval of validity: $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Note that if $y = 0$, there is a constant solution. Assume $y \neq 0$, use SoV:

$$\begin{aligned} \int y^{-\frac{1}{3}}\,\mathrm{d}y &= \int \mathrm{d}x + c \\ \frac{3}{2}y^{\frac{2}{3}} &= x + c \\ y^{\frac{2}{3}} &= \frac{2}{3}x + c \\ y &= \pm\left(\frac{2}{3}x + c\right)^{\frac{3}{2}} \end{aligned}$$

Substitute $y(0) = 0$:

$$0 = 0 + c \implies c = 0.$$

So,

$$y^{\text{IVP}} = \pm\left(\frac{2}{3}x\right)^{\frac{3}{2}}.$$

□

Theorem 1.3.9 Existence and Uniqueness of Solutions to Nonlinear ODEs
Consider the IVP

$$y' = f(x, y(x)) \quad \text{with } y(x_0) = y_0.$$

- If f is continuous on an open rectangle $R\{a < x < b, c < y < d\}$ that contains (x_0, y_0), then the IVP has *at least* one solution on some open subinterval of (a, b) that contains x_0.
- If both f and $\dfrac{\partial f}{\partial y}$ are continuous on R, then the IVP has a *unique* solution on some open subinterval of (a, b) that contains x_0.

Example 1.3.10 In the IVP above (Example 1.3.8), $f(x, y) = y^{\frac{1}{3}}$, and so $\dfrac{\partial f}{\partial y} = \dfrac{1}{3}y^{-\frac{2}{3}}$, which is not continuous at $y = 0$. So, the IVP $\nexists$ a unique solution on the interval given: $R = \{x \in \mathbb{R}, y \in \mathbb{R}\}$.

1.3.2 Exact Equations

Theorem 1.3.11 Multivariable Chain Rule
Given $F(x, y) = c$, where $y = y(x)$. Then, the total derivative with respect to x is

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Example 1.3.12 Exact ODEs

$$\begin{aligned} M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y &= 0 \\ M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \quad \text{if } y = y(x) \\ M(x, y)\frac{\mathrm{d}x}{\mathrm{d}y} + N(x, y) &= 0 \quad \text{if } x = x(y). \end{aligned}$$

Example 1.3.13 Show that $x^4y^3 + x^2y^5 + 2xy = c$ is an implicit solution of

$$\left(4x^3y^3 + 2xy^5 + 2y\right)\,\mathrm{d}x + \left(3x^4y^2 + 5x^2y^4 + 2x\right)\,\mathrm{d}y = 0.$$

Solution 7.

$$\frac{\partial F}{\partial x} = 4x^3y^3 + 2xy^5 + 2y; \quad \frac{\partial F}{\partial y} = 3x^4y^2 + 5x^2y^4 + 2x$$

If $y = y(x)$:

$$\left(4x^3y^3 + 2xy^5 + 2y\right) + \left(3x^4y^2 + 5x^2y^4 + 2x\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

If $x = x(y)$:

$$\left(4x^3y^3 + 2xy^5 + 2y\right)\frac{\mathrm{d}x}{\mathrm{d}y} + \left(3x^4y^2 + 5x^2y^4 + 2x\right) = 0.$$

So the implicit function is a solution to the differential equation. □

Theorem 1.3.14
Given an implicit function $F(x, y) = c$. It is a solution to the differential equation if

$$F_x\,\mathrm{d}x + F_y\,\mathrm{d}y = 0.$$

Definition 1.3.15 (Exact ODEs). We say that an ODE of the form $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$ is *exact* if $\exists\ F(x, y) = c$, with F_x and F_y continuous, such that

$$M(x, y) = F_x(x, y) \quad \text{and} \quad N(x, y) = F_y(x, y).$$

Theorem 1.3.16 Characterization of Exact ODEs
Suppose that M and N are continuous on R and have continuous partial derivatives $M_y,\ N_x$ on R. Then,

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

is exact if and only if

$$M_y(x,y) = N_x(x,y)$$

on R.

Example 1.3.17 Show this ODE is exact:

$$\left(4x^3y^3 + 2xy^5 + 2y\right)\mathrm{d}x + \left(3x^4y^2 + 5x^2y^4 + 2x\right)\mathrm{d}y = 0.$$

Solution 8.

$$M(x,y) = 4x^3y^3 + 2xy^5 + 2y; \quad M_y(x,y) = 12x^3y^2 + 10xy^4 + 2$$

$$N(x,y) = 3x^4y^2 + 5x^2y^4 + 2x; \quad N_x(x,y) = 12x^3y^2 + 10xy^4 + 2$$

Since $M_y(x,y) = N_x(x,y)$, it is exact. □

Example 1.3.18 Find the general solution of

$$\left(4x^3y^3 + 3x^2\right)\mathrm{d}x + \left(3x^4y^2 + 6y^2\right)\mathrm{d}y = 0$$

Solution 9.
Note that $M(x,y) = 4x^3y^3 + 3x^2$ and $N(x,y) = 3x^4y^2 + 6y^2$, so

$$M_y(x,y) = 12x^3y^2; \quad N_x(x,y) = 12x^3y^2.$$

Since $M_y(x,y) = N_x(x,y)$, the ODE is exact. Then,

$$\begin{aligned} F(x,y) &= \int M(x,y)\,\mathrm{d}x + \varphi(y) \\ &= \int 4x^3y^3 + 3x^2\,\mathrm{d}x + \varphi(y) \\ &= x^4y^4 + x^4 + \varphi(y). \end{aligned}$$

Since $F_y(x,y) = 3x^4y^2 + \varphi'(y) = 3x^4y^2 + 6y^2$, we have $\varphi'(y) = 6y^2$. That is,

$$\varphi(y) = \int 6y^2\,\mathrm{d}y + c = 2y^3 + c.$$

So, $F(x,y) = x^4y^3 + x^3 + 2y^3 + c$. Then, the implicit solution is

$$x^4y^3 + x^3 + 2y^3 = c.$$

Alternatively, we can use $F(x,y) = \int N(x,y)\,\mathrm{d}y + \psi(x)$ to get the same result. $\square$

2 Second Order ODEs

3 System of ODEs