Emory University

MATH 212 Differential Equations Learning Notes

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Contents

1 First Order ODEs

1.1 Introduction

Definition 1.1.1 (Ordinary Differential Equations/ODEs). An *ordinary differential equation* is an equation that contains one or more derivatives of an unknown function $y = y(x)$.
Definition 1.1.2 (Order of ODEs). The *order* of an ODE is the maximum order of the derivatives appearing in the equation.
Definition 1.1.3 (Solution to ODEs). The *solution* to an ODE is a function y that satisfies the equation.

Example 1.1.4 Solve $y'' = 3x + 1$.

Solution 1.

$$\begin{aligned} y' &= \int 3x + 1 \, \mathrm{d}x = \frac{3}{2}x^2 + x + C \\ y &= \int y' \, \mathrm{d}x = \int \frac{3}{2}x^2 + x + C \, \mathrm{d}x = \frac{1}{2}x^3 + \frac{1}{x}x^2 + Cx + D. \end{aligned}$$

□

Definition 1.1.5 (Linear ODEs/Non-Linear ODEs). A first order ODE is *linear* if it can be written as

$$y' + p(x)y = f(x).$$

Otherwise, it is *non-linear*.
Definition 1.1.6 (Homogenous/Non-Homogenous Linear ODEs). If $f(x) = 0$, then the linear ODE is *homogenous*. That is,

$$y' + p(x)y = 0.$$

Otherwise, it is *non-homogenous*.
Definition 1.1.7 (Trivial/Non-Trivial Solution). $y = 0$ is a *trivial solution* to a homogenous ODE. Any other solutions are *non-trivial*.
Definition 1.1.8 (One-Parameter Family of Solutions). We call C a *parameter* and the equation, therefore solution, defines a *one-parameter family* of solutions.

Example 1.1.9 For the ODE $y' = 1$, $y_1 = x + C_1$ is a solution to it, and it is a one-parameter family of solutions. Similarly, for $y' = \dfrac{1}{x^2}$, the one-parameter families of solutions are defined by $y_2 = -\dfrac{1}{x} + C_2$ on the interval $(-\infty, 0) \cup (0, \infty)$.

Definition 1.1.10 (General Solution). Given the general form of the linear ODE $y' + p(x)y = f(x)$ if p and f are continuous on some open interval (a, b) and there is a unique formula $y = y(x, c)$ and we have the following properties:

- for each fixed c, the resulting function of x is a solution of the ODE on (a, b), and
- if y is a solution of the ODE, then y can be obtained by choosing the value of c appropriately.

The function $y = y(x, c)$ is called a *general solution.*

More generally, we can write an ODE as

$$P_0(x)y' + P_1(x)y = F(x).$$

In this case, the ODE has a general solution on any open interval in which $P_0,\ P_1,$ and F are continuous and $P_0 \neq 0$.

Definition 1.1.11 (Initial Value Problem (IVP)). A differential equation with an initial condition.

Example 1.1.12 Let a be a constant. Find the general solution of $y' - ay = 0$ and solve the IVP $\begin{cases} y' - ay = 0 \\ y(x_0) = y_0. \end{cases}$

Solution 2.

Classification: First order, Linear, Homogeneous.

Trivial Solution: $y = 0$.

General solution:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= ay \\ \int \frac{1}{y}\,\mathrm{d}y &= \int a\,\mathrm{d}x \\ \ln|y| &= ax + c \\ y &= e^{ax+c} = Ae^{ax}. \end{aligned}$$

This general solution includes the trivial solution.

IVP: Substitute $x = x_0$ and $y = y_0$:

$$y_0 = Ae^{ax_0} \quad \longrightarrow \quad A = y_0 e^{-ax_0}$$

So,

$$y^{\mathrm{IVP}} = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

This IVP is a "generic initial condition." We need more information on x_0, y_0 to get a more specific solution. $\square$

1.2 Linear First Order ODEs

Theorem 1.2.1
If p is continuous on (a, b), then the general solution of the homogeneous equation $y' + p(x)y = 0$ on (a, b) is given by

$$y = ce^{-\int p(x)\,\mathrm{d}x}.$$

Proof 1.
(a). Substitute the solution formula to show that $y = ce^{-\int p(x)\,\mathrm{d}x}$ is a solution for any choice of c.

$$y' = c\left(-\int p(x)\,\mathrm{d}x\right)' e^{-\int p(x)\,\mathrm{d}x} = -cp(x)e^{-\int p(x)\,\mathrm{d}x}.$$

Then,

$$y' + p(x)y = -cp(x)e^{-\int p(x)\,\mathrm{d}x} + cp(x)e^{-\int p(x)\,\mathrm{d}x} = 0.$$

So, $y = ce^{-\int p(x)\,\mathrm{d}x}$ is a solution for any choice of c. $\square$

(b). Want to show: any solution of $y' + p(x)y = 0$ can be written as $y = ce^{-\int p(x)\,\mathrm{d}x}$. Note that $y = 0$ is a trivial solution, so we assume $y \neq 0$.

$$\begin{aligned}
y' + p(x)y &= 0 \\
y' &= -p(x)y \\
\frac{y'}{y} &= -p(x) \\
\rightsquigarrow \int \frac{1}{y}\,\mathrm{d}y &= \int -p(x)\,\mathrm{d}x \\
\ln|y| &= -\int p(x)\,\mathrm{d}x \\
y &= ce^{-\int p(x)\,\mathrm{d}x}.
\end{aligned}$$

Note that when $c = 0$, $y = 0$ is the trivial solution. So, any solution of $y' + p(x)y = 0$ can be written as $y = ce^{-\int p(x)\,\mathrm{d}x}$. ■

Example 1.2.2 Solve the IVP

$$\begin{cases} xy' + y = 0 \\ y(1) = 3. \end{cases}$$

Solution 2.
Note that $P_0(x) = x$ and $P_1(x) = 1$, which are continuous on $\mathbb{R}$. Since we need $P_0(x) \neq 0$, $x \neq 0$. So the interval of validity is $\mathbb{R}\backslash\{0\}$.

$\boxed{\text{Method 1: Separation of Variables}}$

$$y' = -\frac{y}{x}.$$

Note that $y = 0$ is a solution. Assume $y \neq 0$.

$$\frac{y'}{y} = -\frac{1}{x} \quad \rightsquigarrow \quad \int \frac{1}{y}\,\mathrm{d}y = -\int \frac{1}{x}\,\mathrm{d}x + k$$

$$\ln|y| = -\ln|x| + k$$

$$|y| = e^k \frac{1}{|x|}$$

$$y = \frac{c}{x}$$

$\boxed{\text{Method 2: Solution Formula}}$ By Theorem 1.2.1,

$$y = ce^{-\int p(x)\,\mathrm{d}x} = ce^{-\int \frac{1}{x}\,\mathrm{d}x} = ce^{-\ln|x|} = \frac{c}{x}.$$

$\boxed{\text{Solving the IVP}}$ Substitute $x = 1$ and $y = 3$:

$$3 = \frac{c}{1} \quad \longrightarrow \quad c = 3.$$

So, $y^{\mathrm{IVP}} = \dfrac{3}{x}$. □

Example 1.2.3 Given the equation $(4 + x^2)y' + 2xy = 4x$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 3.

This is a first order, linear, non-homogeneous differential equation.

Note that $P_0(x) = 4 + x^2$, $P_1(x) = 2x$, $F(x) = 4x$, and $P_0 \neq 0\ \forall x \in \mathbb{R}$, so the interval of validity is $\mathbb{R}$. Also note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[4 + x^2\right] = 2x$, so the equation can be written as

$$(4 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left[4 + x^2\right]y = 4x.$$

Using the product rule to re-write the LHS as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(4 + x^2)y\right] = 4x$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left[(4 + x^2)y\right]\mathrm{d}x = \int 4x\,\mathrm{d}x + c$$

$$(4 + x^2)y = 2x^2 + c$$

$$y = \frac{2x^2 + c}{4 + x^2}.$$

□

Example 1.2.4 Given the equation $y' - 2y = 4 - x$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 4.

This is a first order, linear, non-homogeneous differential equation.

Since $P_0(x) = 1$, $P_1(x) = -2y$, $F(x) = 4 - x$, and $P_0(x) \neq 0 \; \forall x \in \mathbb{R}$, the interval of validity is $\mathbb{R}$. Consider $\mu = \mu(x) \neq 0$. Multiply both sides of the equation by $\mu(x)$:

$$\mu(x)y' - 2\mu(x)y = \mu(x)(4 - x) \tag{1}$$

To make the LHS a product rule, we need

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[\mu(x)y(x)\big] = \mu'(x)y(x) + \mu(x)y'(x) = \mu(x)y'(x) - 2\mu(x)y.$$

So, we have $\mu' = -2\mu$, or $\mu' + 2\mu = 0$, a first order, linear, homogeneous ODE. Solving this ODE, we get $\mu(x) = ce^{-2x}$. Since we only want one specific μ that would work, take $c = 1$. So, $\mu(x) = e^{-2x}$. Substituting $\mu(x) = e^{-2x}$ to Eq. (1):

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}(4 - x), \quad \widetilde{P}_0 = e^{-2x} \neq 0, \; \widetilde{P}_1 = -2e^{-2x}.$$

Using the product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big[e^{-2x}y\big] &= 4e^{-2x} - xe^{-2x} \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\big[e^{-2x}y\big] \, \mathrm{d}x &= \int 4e^{-2x} - xe^{-2x} \, \mathrm{d}x + c \\
e^{-2x}y &= \frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c \\
y &= e^{2x}\left(\frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c\right) \\
&= \frac{1}{2}x - \frac{7}{4} + ce^{2x}.
\end{aligned}$$

□

Theorem 1.2.5 Method of Integrating Factor
Given the first order linear differential equation $y' + p(x)y = f(x)$, with p and f both continuous on some interval (a, b),

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,\mathrm{d}x + c\right]$$

is the general solution to the equation, with

$$\mu(x) = e^{\int p(x)\,\mathrm{d}x}.$$

We call $\mu(x)$ the *integrating factor*.

Proof 5. Consider $\mu = \mu(x) \neq 0$. Multiplying the both sides of $y' + p(x)y = f(x)$ by μ:

$$\mu y' + p\mu y = \mu f. \tag{2}$$

Impose $\mu y' + p\mu y = \frac{\mathrm{d}}{\mathrm{d}x}[\mu y]$ to find $\mu = \mu(x)$:

$$\begin{aligned}
\mu y' + p\mu y &= \mu' y + \mu y' \\
\mu' - p\mu &= 0, && \text{first order, linear, homogeneous ODE} \\
\mu(x) &= e^{\int p(x)\,\mathrm{d}x}, && \text{the integrating factor}
\end{aligned}$$

Substitute $\mu(x) = e^{\int p(x)\,\mathrm{d}x}$ into Eq. (2):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\mu y] &= \mu f \\
\int \frac{\mathrm{d}}{\mathrm{d}x}[\mu y]\,\mathrm{d}x &= \int \mu f\,\mathrm{d}x + c \\
\mu y &= \int \mu f\,\mathrm{d}x + c \\
y(x) &= \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,\mathrm{d}x + c\right].
\end{aligned}$$

■

Example 1.2.6 Give the equation $y' + 2y = x^3e^{-2x}$. Classify the equation and find the general solution $y = y(x, c)$.

Solution 6.

It is a first order, linear, non-homogeneous ODE, with $p = 2$ and $f = x^3e^{-2x}$. Let $\mu(x)$ be the integrating factor. Then,

$$\mu(x) = e^{\int 2\,\mathrm{d}x} = e^{2x}.$$

So, by the method of integrating factor, we know

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big[\mu(x)y\Big] &= \mu(x)f(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\Big[e^{2x}y\Big]\,\mathrm{d}x &= \int e^{2x}x^3e^{-2x}\,\mathrm{d}x + c \\
e^{2x}y &= \int x^3\,\mathrm{d}x + c \\
e^{2x}y &= \frac{1}{4}x^4 + c \\
y &= \frac{1}{4}x^4e^{-2x} + ce^{-2x}.
\end{aligned}$$

□

Remark. *Re-examine the formula we derived from the method of integrating factor:*

$$y(x) = \frac{1}{\mu}\int f\mu\,\mathrm{d}x + \boxed{\frac{c}{\mu}}.$$

The part being boxed, $\frac{c}{\mu}$*, is independent from* f *and is exactly* $ce^{-\int p\,\mathrm{d}x}$ *if we expand, which is the solution for a homogeneous differential equation.*

Definition 1.2.7 (Complementary Equation). The *complementary equation* to a first order ODE $y' + py = f$ is the homogeneous part of it. i.e., $y' + py = 0$.

Theorem 1.2.8 Method of Variation of Parameters
Given the first order linear differential equation $y' + p(x)y = f(x)$, with p and f both continuous on some interval (a, b),

$$y(x) = y_1(x)\left[\int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c\right]$$

is the general solution to the equation, where y_1 is a solution of the complementary euqation $y' + py = 0$.

Proof 7. Call y_1 a solution of the complementary equation $y' + p(x)y = 0$. Then, we want to find $y(x) = u(x)y_1(x)$, the general solution of $y' + p(x)y = f(x)$, where u is an unknown function of f. Note that, by product rule, $y'(x) = u'y_1 + uy_1'$. Then, the equation becomes

$$\begin{aligned}
(u'y_1 + uy_1') + p(x)(uy_1) &= f(x) \\
u'y_1 + uy_1' + puy_1 &= f \\
y_1u' + \underbrace{(y_1' + py_1)}_{0}u &= f
\end{aligned}$$

$$y_1u' = f \implies u(x) = \int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c.$$

Therefore, the formula to find y is given by

$$y = y_1 u = y_1(x)\left[\int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c\right].$$

■

Remark. *The method of variation of parameters will be more useful when solving second or higher order differential equations.*

Example 1.2.9 Give the equation $y' + 2y = x^3 e^{-2x}$. Find the general solution $y = y(x, c)$ using the method of variation of parameters.

Solution 8.

It is a first order, linear, non-homogeneous ODE, with $p = 2$ and $f = x^3 e^{-2x}$. Let y_1 be the solution of the complementary equation $y' + 2y = 0$. Then, $y_1(x) = e^{-\int 2\,\mathrm{d}x} = e^{-2x}$. By the method of variation of parameters, suppose $y = uy_1$, where u is an unknown function of x. Then,

$$u(x) = \int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c = \int \frac{x^3 e^{-2x}}{e^{-2x}}\,\mathrm{d}x + c = \int x^3\,\mathrm{d}x + c = \frac{1}{4}x^4 + c.$$

So,

$$y = uy_1 = e^{-2x}\left(\frac{1}{4}x^4 + c\right) = \frac{1}{4}x^4 e^{-2x} + ce^{-2x}.$$

□

Theorem 1.2.10 Existence and Uniqueness Theorem

Suppose that $p = p(x)$ and $f = f(x)$ are continuous on (a, b). Then, a general solution of $y' + p(x)y = f(x)$ on (a, b) is

$$y(x) = y_1(x)\left[\int \frac{f(x)}{y_1(x)}\,\mathrm{d}x + c\right],$$

where $y_1(x)$ is a solution of the complementary equation (i.e., $y' + p(x)y = 0$).

If x_0 is an arbitrary point in (a, b) and y_0 is an arbitrary real number, then the initial value problem,

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a unique solution on (a, b).

1.3 Separable Equations

Definition 1.3.1 (General Forms). The general form of a non-linear first order ODE is given by

$$y' = f(x, y(x)).$$

If we take $M(x,y) = -f(x,y)$ and $N(x,y) = 1$, we can also re-write the equation into

$$M(x,y) + N(x,y)y' = 0, \quad \text{or} \quad M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0.$$

Definition 1.3.2 (Separable Equations). If $M(x,y) = M(x)$ and $N(x,y) = N(y)$, then the ODE is called *separable*.

Theorem 1.3.3 Separation of Variables (SoV)
Consider the non-linear first order ODE $M(x,y) + N(x,y)y' = 0$, with $M(x,y) = M(x)$ and $N(x,y) = N(y)$. Then we can find an implicit solution of the ODE in the form of

$$F(x,y) = c,$$

where $F(x,y)$ is a function of x and y and

$$F(x,y) = \int M(x)\,\mathrm{d}x + \int N(y)\,\mathrm{d}y.$$

Proof 1. Let $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$. Then, the equation becomes

$$\begin{aligned} H_1'(x) + H_2'(y)y' &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_1(x)\Big] + \frac{\mathrm{d}}{\mathrm{d}y}\Big[H_2(y)\Big]\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

By using the chain rule, $\frac{\mathrm{d}}{\mathrm{d}y}\Big[H_2(y)\Big]\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_2(y(x))\Big]$. So, the equation becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_1(x)\Big] + \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_2(y(x))\Big] &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\Big[H_1(x) + H_2(y)\Big] &= 0 \\ H_1(x) + H_2(y) &= c \\ \int M(x)\,\frac{\mathrm{d}}{\mathrm{d}x} + \int N(y)\,\mathrm{d}y &= c \\ F(x,y) &= c \end{aligned}$$

■

Example 1.3.4 Given the equation $y' = \dfrac{x^2}{1-y^2}$. Classify the differential equation and find the general solution.

Solution 2.
It is a first order, non-linear ODE. Since $y' = \dfrac{x^2}{1-y^2}$, so we have $1 - y^2 \neq 0$. That is,

$y^2 \neq 1$, or $y \neq \pm 1$. Using the separation of variables (SoV), we have

$$\begin{aligned}(1-y^2)y' &= x^2\\ \int 1-y^2 \, \mathrm{d}y &= \int x^2 \, \mathrm{d}x\\ y-\frac{1}{3}y^3 &= \frac{1}{3}x^3+c\\ y-\frac{1}{3}y^3-\frac{1}{3}x^3 &= c\\ 3y-y^3-x^3 &= c\end{aligned}$$

□

Example 1.3.5 Given the equation $y' = \dfrac{(y-3)\cos x}{1+2y^2}$. Classify the equation and find the general solution.

Solution 3.

It is a first order, non-linear ODE. Since $1+2y^2 \neq 0 \quad \forall y \in \mathbb{R}$. Note that if we take $y-3=0$, we get $y=3$, a constant solution to the differential equation. Now, assume $y \neq 3$. Then, use SoV:

$$\int \frac{1+2y^2}{y-3} \, \mathrm{d}y = \int \cos x \, \mathrm{d}x + c = \sin x + c.$$

Set $t = y-3$, $\mathrm{d}t = \mathrm{d}y$. So, $y = t+3$ and $y^2 = (t+3)^2$. Then,

$$\begin{aligned}\int \frac{1+2y^2}{y-3} \, \mathrm{d}y = \int \frac{1+2(t+3)^2}{t} \, \mathrm{d}t &= \int \frac{1+2t^2+12t+18}{t} \, \mathrm{d}t\\ &= \int \frac{19}{t} + 12 + 2t \, \mathrm{d}t\\ &= 19\ln|t| + 12t + t^2\\ &= 19\ln|y-3| + 12(y-3) + (y-3)^2\\ &= 19\ln|y-3| + 6y + y^2 - 27.\end{aligned}$$

So,

$$19\ln|y-3| + y^2 + 6y - 27 - \sin x = c$$

□

Example 1.3.6 Give the equation $y' = \dfrac{1}{2}x(1-y^2)$. Classify the equation and find the general solution.

Solution 4.

It is a first order, non-linear ODE. Notice that we have the constant solutions when we take $1 - y^2 = 0$, or $y = \pm 1$. Now, assume $y \neq \pm 1$. Using SoV:

$$\int \frac{2}{1-y^2}\,\mathrm{d}y = \int x\,\mathrm{d}x + c = \frac{1}{2}x^2 + c.$$

Note that $\dfrac{2}{1-y^2} = \dfrac{2}{(1-y)(1+y)}$. Use partial fractions. Assume

$$\frac{2}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}.$$

Then, we get $A(1+y) + B(1-y) = 2$. That is, $(A+B) + (A-B)y = 2$. Attempting to solve the system of equations $\begin{cases} A - B = 0 \\ A + B = 2 \end{cases}$, then we get $A = B = 1$.Therefore,

$$\frac{2}{1-y^2} = \frac{1}{1-y} + \frac{1}{1+y}.$$

Then,

$$\begin{aligned}
\int \frac{1}{1-y} + \frac{1}{1+y}\,\mathrm{d}y &= \frac{1}{2}x^2 + c \\
-\ln|1-y| + \ln|1+y| &= \frac{1}{2}x^2 + c \\
\ln|1-y| - \ln|1+y| &= -\frac{1}{2}x^2 + c \\
\ln\left|\frac{1-y}{1+y}\right| &= -\frac{1}{2}x^2 + c \\
\left|\frac{y-1}{y+1}\right| &= e^{-\frac{1}{2}x^2 + c} = e^{-\frac{1}{2}x^2}e^c \\
\frac{y-1}{y+1} &= c_2 e^{-\frac{1}{2}x^2} \\
y - 1 &= (y+1)c_2 e^{-\frac{1}{2}x^2} \\
(1 - c_2 e^{-\frac{1}{2}x^2})y &= 1 + c_2 e^{-\frac{1}{2}x^2} \\
y &= \frac{1 + c_2 e^{-\frac{1}{2}x^2}}{1 - c_2 e^{-\frac{1}{2}x^2}}
\end{aligned}$$

The value of c_2 si chosen according to the sign of $\dfrac{y-1}{y+1}$. □

Example 1.3.7 Given the equation $y' = \dfrac{3x^2 + 4x + 2}{2(y-1)}$, with $y(0) = 1$. Classify the equation, find the general solution, and solve the IVP.

Solution 5.

It is a first order, nonlinear, separable ODE. Note that $y - 1 \neq 0$, so $y \neq 1$. Assume

$y \neq 1$, use SoV:

$$\int 2(y-1)\,\mathrm{d}y = \int 3x^2 + 4x + 2\,\mathrm{d}x + c$$
$$(y-1)^2 = x^3 + 2x^2 + 2x + c$$
$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + c$$
$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

Substitute $y = -1$ when $x = 0$:

$$1 + 2 = c \implies c = 3.$$

So,

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3, \quad y \neq 1$$
$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$$
$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$
$$y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4}$$
$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

If $y = -1$ and $x = 0$: $-1 = 1 \pm \sqrt{4} = 1 \pm 2$. So, it must be that $-1 = 1 - 2$. So,

$$y^{\text{IVP}} = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

Note that now we have another condition for x:

$$x^3 + 2x^2 + 2x + 4 \geq 0$$
$$(x+2)(x^2+2) \geq 0$$
$$x + 2 \geq 0$$
$$x \geq -2$$

So,

$$y^{\text{IVP}} = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}, \text{ with } y \neq 1 \text{ and } x \geq -2.$$

□

Example 1.3.8 Solve the IVP $\begin{cases} y' = \sqrt[3]{y} = y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$.

Solution 6.

It is a first order, nonlinear, separable ODE. The initial interval of validity: $x \in \mathbb{R}$ and

$y \in \mathbb{R}$. Note that if $y = 0$, there is a constant solution. Assume $y \neq 0$, use SoV:

$$\begin{aligned} \int y^{-\frac{1}{3}}\,\mathrm{d}y &= \int \,\mathrm{d}x + c \\ \frac{3}{2}y^{\frac{2}{3}} &= x + c \\ y^{\frac{2}{3}} &= \frac{2}{3}x + c \\ y &= \pm\left(\frac{2}{3}x + c\right)^{\frac{3}{2}} \end{aligned}$$

Substitute $y(0) = 0$:

$$0 = 0 + c \implies c = 0.$$

So,

$$y^{\text{IVP}} = \pm\left(\frac{2}{3}x\right)^{\frac{3}{2}}.$$

□

Theorem 1.3.9 Existence and Uniqueness of Solutions to Nonlinear ODEs
Consider the IVP

$$y' = f(x, y(x)) \quad \text{with } y(x_0) = y_0.$$

- If f is continuous on an open rectangle $R\{a < x < b, c < y < d\}$ that contains (x_0, y_0), then the IVP has *at least* one solution on some open subinterval of (a, b) that contains x_0.
- If both f and $\dfrac{\partial f}{\partial y}$ are continuous on R, then the IVP has a *unique* solution on some open subinterval of (a, b) that contains x_0.

Example 1.3.10 In the IVP above (Example 1.3.8), $f(x, y) = y^{\frac{1}{3}}$, and so $\dfrac{\partial f}{\partial y} = \dfrac{1}{3}y^{-\frac{2}{3}}$, which is not continuous at $y = 0$. So, the IVP $\nexists$ a unique solution on the interval given: $R = \{x \in \mathbb{R}, y \in \mathbb{R}\}$.

1.4 Exact Equations

Theorem 1.4.1 Multivariable Chain Rule
Given $F(x, y) = c$, where $y = y(x)$. Then, the total derivative with respect to x is

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Example 1.4.2 Exact ODEs

$$\begin{aligned} M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y &= 0 \\ M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \quad \text{if } y = y(x) \\ M(x,y)\frac{\mathrm{d}x}{\mathrm{d}y} + N(x,y) &= 0 \quad \text{if } x = x(y). \end{aligned}$$

Example 1.4.3 Show that $x^4y^3 + x^2y^5 + 2xy = c$ is an implicit solution of

$$\left(4x^3y^3 + 2xy^5 + 2y\right)\,\mathrm{d}x + \left(3x^4y^2 + 5x^2y^4 + 2x\right)\,\mathrm{d}y = 0.$$

Solution 1.

$$\frac{\partial F}{\partial x} = 4x^3y^3 + 2xy^5 + 2y; \quad \frac{\partial F}{\partial y} = 3x^4y^2 + 5x^2y^4 + 2x$$

If $y = y(x)$:

$$\left(4x^3y^3 + 2xy^5 + 2y\right) + \left(3x^4y^2 + 5x^2y^4 + 2x\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

If $x = x(y)$:

$$\left(4x^3y^3 + 2xy^5 + 2y\right)\frac{\mathrm{d}x}{\mathrm{d}y} + \left(3x^4y^2 + 5x^2y^4 + 2x\right) = 0.$$

So the implicit function is a solution to the differential equation. □

Theorem 1.4.4
Given an implicit function $F(x,y) = c$. It is a solution to the differential equation if

$$F_x\,\mathrm{d}x + F_y\,\mathrm{d}y = 0.$$

Definition 1.4.5 (Exact ODEs). We say that an ODE of the form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is *exact* if $\exists\, F(x,y) = c$, with F_x and F_y continuous, such that

$$M(x,y) = F_x(x,y) \quad \text{and} \quad N(x,y) = F_y(x,y).$$

Theorem 1.4.6 Characterization of Exact ODEs
Suppose that M and N are continuous on R and have continuous partial derivatives $M_y,\ N_x$ on R. Then,

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

is exact if and only if

$$M_y(x,y) = N_x(x,y)$$

on R.

Example 1.4.7 Show this ODE is exact:

$$\left(4x^3y^3 + 2xy^5 + 2y\right)\,\mathrm{d}x + \left(3x^4y^2 + 5x^2y^4 + 2x\right)\,\mathrm{d}y = 0.$$

Solution 2.

$$\begin{aligned} M(x,y) &= 4x^3y^3 + 2xy^5 + 2y; \quad & M_y(x,y) &= 12x^3y^2 + 10xy^4 + 2 \\ N(x,y) &= 3x^4y^2 + 5x^2y^4 + 2x; \quad & N_x(x,y) &= 12x^3y^2 + 10xy^4 + 2 \end{aligned}$$

Since $M_y(x,y) = N_x(x,y)$, it is exact. $\square$

Example 1.4.8 Find the general solution of

$$\left(4x^3y^3 + 3x^2\right)\,\mathrm{d}x + \left(3x^4y^2 + 6y^2\right)\,\mathrm{d}y = 0$$

Solution 3.
Note that $M(x,y) = 4x^3y^3 + 3x^2$ and $N(x,y) = 3x^4y^2 + 6y^2$, so

$$M_y(x,y) = 12x^3y^2; \quad N_x(x,y) = 12x^3y^2.$$

Since $M_y(x,y) = N_x(x,y)$, the ODE is exact. Then,

$$\begin{aligned} F(x,y) &= \int M(x,y)\,\mathrm{d}x + \varphi(y) \\ &= \int 4x^3y^3 + 3x^2\,\mathrm{d}x + \varphi(y) \\ &= x^4y^4 + x^4 + \varphi(y). \end{aligned}$$

Since $F_y(x,y) = 3x^4y^2 + \varphi'(y) = 3x^4y^2 + 6y^2$, we have $\varphi'(y) = 6y^2$. That is,

$$\varphi(y) = \int 6y^2\,\mathrm{d}y + c = 2y^3 + c.$$

So, $F(x,y) = x^4y^3 + x^3 + 2y^3 + c$. Then, the implicit solution is

$$x^4y^3 + x^3 + 2y^3 = c.$$

Alternatively, we can use $F(x,y) = \displaystyle\int N(x,y)\,\mathrm{d}y + \psi(x)$ to get the same result. $\square$

Theorem 1.4.9 Method of Integrating Factors for Exact ODEs
Given $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is not exact. Consider $\mu = \mu(x,y) \neq 0$. Multiply

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,\mathrm{d}x} \quad \text{or} \quad \mu(y) = e^{\int \frac{N_x - M_y}{M}\,\mathrm{d}y},$$

we could make the ODE exact and thus solvable.

Proof 4. Given $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is not exact. Consider $\mu = \mu(x,y) \neq 0$. Multiply both sides by μ:

$$\mu M(x,y)\,\mathrm{d}x + \mu N(x,y)\,\mathrm{d}y = 0 \tag{3}$$

Then, we have

$$\widetilde{M}(x,y) = \mu M; \quad \text{and} \quad \widetilde{N}(x,y) = \mu N.$$

Thus, the condition for Eq. (3) to be exact is $\widetilde{M}_y = \widetilde{N}_x$, or $(\mu M)_y = (\mu N)_x$. By product rule,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x \tag{4}$$

Remark. *Eq. (4) is a PDE, which we cannot solve. So, make the assumption that μ is a function of only x or only y.*

- If $\mu = \mu(x)$, then $\mu_y = 0$. So, we have

$$\begin{aligned} \mu M_y &= \mu_x N + \mu N_x \\ N\mu_x + (N_x - M_y)\mu &= 0 \\ \mu(x) &= e^{\int \frac{M_y - N_x}{N}\,\mathrm{d}x}. \end{aligned}$$

  If $\mu = \mu(y)$, then $\mu_x = 0$. So, we have

$$\begin{aligned} \mu_y M + \mu M_y &= \mu N_x \\ M\mu_y + (M_y - N_x)\mu &= 0 \\ \mu(x) &= e^{\int \frac{N_x - M_y}{M}\,\mathrm{d}y}. \end{aligned}$$

$\blacksquare$

Remark. *To decide on if we should use $\mu(x)$ or $\mu(y)$, test if $\dfrac{M_y - N_x}{N}$ is a function of x only and inf $\dfrac{N_x - M_y}{M}$ is a function of y only.*

Example 1.4.10 Given the equation $(3x + 2y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0$. Find the integrating factor and solve the ODE.

Solution 5.

Note that $M(x, y) = 3x + 2y^2$ and $N(x, y) = 2xy$. So, $M_y = 4y$ and $N_x = 2y$. So, the ODE is not exact, and we should find an integrating factor. It is easier to divide by N since there is only 1 term:

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x},$$

is a function of x only. So, by the Method of Integrating Factors, we know

$$\mu(x) = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\ln x} = x \quad (x \neq 0).$$

Now, multiply both decide of the original ODE by x:

$$(3x^2 + 2xy^2)\,\mathrm{d}x + 2x^2y\,\mathrm{d}y = 0,$$

where $\widetilde{M}(x, y) = 3x^2 + 2xy^2$ and $\widetilde{N}(x, y) = 2x^2y$. So,

$$\begin{aligned} F(x, y) &= \int \widetilde{N}(x, y)\,\mathrm{d}y + \psi(x) \\ &= \int 2x^2y\,\mathrm{d}y + \psi(x) \\ &= x^2y^2 + \psi(x) \end{aligned}$$

Compute the partial deriviate of F with respect to x, we get

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= 2xy^2 + \psi'(x) = 3x^2 + 2xy^2 \\ \psi'(x) &= 3x^2 \\ \psi(x) &= \int 3x^2\,\mathrm{d}x + c = x^3 + c \end{aligned}$$

So, the solution is

$$x^2y^2 + x^3 = c.$$

□

Example 1.4.11 Solve the equation $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,\mathrm{d}x + (3x^2y^2 + 4y)\,\mathrm{d}y = 0$.

Solution 6.

- Test if the equation is exact:

$$M(x,y) = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x; \quad M_y = 6xy^2 - 6x^3y^2 - 8xy$$

$$N(x,y) = 3x^2y^2 + 4y; \quad N_x = 6xy^2$$

Since $M_y \neq N_x$, the equation is no exact.

- Find an integrating factor. Try:

$$\frac{M_y - N_x}{N} = \frac{6xy^2 - 6x^3y^2 - 8xy - 6xy^2}{3x^2y^2 + 4y} = \frac{-2x(3x^2y^2 + 4y)}{3x^2y^2 + 4y} = -2x,$$

which is a function of x only. So, the integrating factor $\mu = \mu(x) \neq 0$ is

$$\mu = e^{\int -2x \, \mathrm{d}x} = e^{-x^2}.$$

- Multiply the equation by μ:

$$e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right) \mathrm{d}x + e^{-x^2}\left(3x^2y^2 + 4y\right) \mathrm{d}y = 0.$$

We can easily show that the equation is exact.

$$\begin{aligned} F(x,y) &= \int e^{-x^2}\left(3x^2y^2 + 4y\right) \mathrm{d}y + \psi(x) \\ &= e^{-x^2}\left(x^2y^3 + 2y^2\right) + \psi(x). \end{aligned}$$

So,

$$\begin{aligned} F_x(x,y) &= -(2x)e^{-x^2}\left(x^2y^3 + 2y^2\right) + e^{-x^2}\left(2xy^3\right) + \psi'(x) \\ &= e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2\right) + \psi'(x). \end{aligned}$$

Note that

$$\begin{aligned} e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2\right) + \psi'(x) &= e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right) \\ \psi'(x) &= e^{-x^2}(2x) \\ \psi(x) &= \int (2x)e^{-x^2} \, \mathrm{d}x + c = -e^{-x^2} + c. \end{aligned}$$

- So, the solution is

$$e^{-x^2}\left(x^2y^3 + 2y^2 - 1\right) = c.$$

$\square$

1.5 Autonomous ODEs

Definition 1.5.1 (Autonomous ODEs). ODEs in the form $y' = f(y)$ is called *autonomous ODEs*. In other words, the dependent variable x does not appear explicitly in the equation.

Remark. *Autonomous ODEs are separable.*

Example 1.5.2 Exponential Growth

Let $y = \varphi(t)$ be the population of the given species at time t. Assume that the rate of change of the population is proportional to the current value of y, and the constant of proportionality is given by r, called the rate of growth $(r > 0)$ or decline $(r < 0)$.

A differential equation to model this situation is given by

$$\begin{cases} y' = ry \\ y(0) = y_0 \end{cases}$$

and the solution is

$$y = y_0 e^{rt}.$$

Example 1.5.3 Logistic Growth

The *logistic growth* is modeled by the differential equation

$$y' = r\Big(1 - \frac{y}{K}\Big)y,$$

where $r > 0$ represents the intrinsic growth rate in absence of limiting factors, and $K > 0$ is the environmental carrying capacity.

2 Second Order ODEs

3 System of ODEs