Linear Algebra Done Right

Jiuru Lyu

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Contents

1 Vector Spaces

1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

Definition 1.1.1 (Complex Number). A *complex number* is an ordered pair (a, b), where $a, b \in \mathbb{R}$, but we write it as $a + b\mathrm{i}$.
Notation 1.1.2. $\mathbb{C} \coloneqq \{a + b\mathrm{i} \mid a, b \in \mathbb{R}\}$
Definition 1.1.3 (Addition & Multiplication).

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)]\mathrm{i}$$

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}$$

Theorem 1.1.4 Properties of Complex Arithmetic

1. commutativity: $\alpha + \beta = \beta + \alpha; \quad \alpha\beta = \beta\alpha, \quad \forall \alpha, \beta \in \mathbb{C}.$
2. associativity: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda); \quad (\alpha\beta)\lambda = \alpha(\beta\lambda), \quad \forall \alpha, \beta, \lambda \in \mathbb{C}.$
3. identities: $\lambda + 0 = \lambda; \quad \lambda \cdot 1 = \lambda, \forall \lambda \in \mathbb{C}.$
4. additive inverse: $\forall \alpha \in \mathbb{C}, \exists$ unique $\beta \in \mathbb{C}$ *s.t.* $\alpha + \beta = 0.$
5. multiplicative inverse: $\forall \alpha \in \mathbb{C}, \alpha \neq 0, \exists$ unique $\beta \in \mathbb{C}$ *s.t.* $\alpha\beta = 1.$
6. distributivity: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta, \quad \forall \lambda, \alpha, \beta \in \mathbb{C}.$

Definition 1.1.5 (Subtraction). If $-\alpha$ is the additive inverse of α, *subtraction* on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

Definition 1.1.6 (Division). For $\alpha \neq 0$, let $\frac{1}{\alpha}$ denote the multiplicative inverse of α. Then, *division* on $\mathbb{C}$ is defined by

$$\frac{\beta}{\alpha} = \beta \cdot \left(\frac{1}{\alpha}\right)$$

Notation 1.1.7. $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
Definition 1.1.8 (List/Tuple). Suppose n is a non-negative integer. A list of length n is an ordered collection of n elements separated by commas and surrounded by parentheses: $(x_1, x_2, x_3, \cdots, x_n)$. Two lists are equal if and only if they have the same length and the same elements in the same order.

Remark. *Lists must have a FINITE length.*

Definition 1.1.9 ($\mathbb{F}^n$ and Coordinate). $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n \coloneqq \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R} \forall i = 1, \cdots, n\},$$

where x_i is the i^{th} *coordinate* of $(x_1, \cdots, x_n)$.

Example 1.1.10 $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ and $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.

Definition 1.1.11 (Addition on $\mathbb{F}^n$). *Addition* on $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n).$$

Theorem 1.1.12 Commutativity of Addition on $\mathbb{F}^n$
If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Proof 1. Suppose $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Then

$$\begin{aligned} x + y &= (x_1 + y_1, \cdots, x_n + y_n) \\ &= (y_1 + x_1, \cdots, y_n + x_n) = y + x. \end{aligned}$$

■

Definition 1.1.13 (Zero). Let 0 denote the list of length n whose coordinates are all 0: $0 := (0, \cdots, 0)$.
Definition 1.1.14 (Additive Inverse on $\mathbb{F}^n$). For $x \in \mathbb{F}^n$, the additive inverse of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ *s.t.* $x + (-x) = 0$.
Definition 1.1.15 (Scalar Multiplication in $\mathbb{F}^n$). The product of a number $\lambda \in \mathbb{F}$ and a vector $x \in \mathbb{F}^n$ is computed by multiplying each coordinate of the vector by λ :

$$\lambda x = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n),$$

where $x = (x_1, \cdots, x_n) \in \mathbb{F}^n$.

Theorem 1.1.16 Properties of Arithmetic Operations on $\mathbb{F}^n$

1. $(x + y) + z = x + (y + z) \quad \forall x, y, z \in \mathbb{F}^n$
2. $(ab)x = a(bx) \quad \forall x \in \mathbb{F}^n$ and $\forall a, b \in \mathbb{F}$.
3. $1 \cdot x = x \quad \forall x \in \mathbb{F}^n$ and $1 \in \mathbb{F}$.
4. $\lambda(x + y) = \lambda x + \lambda y \quad \forall \lambda \in \mathbb{R}$ and $\forall x, y \in \mathbb{F}^n$.
5. $(a + b)x = ax + bx \quad \forall a, b \in \mathbb{F}$ and $\forall x \in \mathbb{F}^n$.

1.2 Definition of Vector Space

Definition 1.2.1 (Addition on V). An *addition* on V is a function $(u, v) \mapsto u + v$ for all $u, v \in V$.
Definition 1.2.2 (Scalar Multiplication on V). A *scalar multiplication* on V is a function $(\lambda, v) \mapsto \lambda v$ for all $\lambda \in \mathbb{F}$ and $v \in V$.
Definition 1.2.3 (Vector Space). A *vector space* is a set V along with an addition on V and a scalar multiplication *s.t.* the following properties hold:

1. commutativity: $u + v = v + u \quad \forall u, v \in V$
2. associativity: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv) \quad \forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}$.
3. additive identity: $\exists 0 \in V$ *s.t.* $v + 0 = v \quad \forall v \in V$.
4. additive inverse: $\exists w \in V$ *s.t.* $v + w = 0 \quad \forall v \in V$.
5. multiplicative identity: $\exists 1 \in V$ *s.t.* $1 \cdot v = v \quad \forall v \in V$.
6. distributive properties: $a(u + v) = au + av$ and $(a + b)v = av + bv \quad \forall u, v \in V$ and $a, b \in \mathbb{F}$.

Definition 1.2.4 (Vector). Elements of a vector space are called *vectors* or points.
Notation 1.2.5. V is a vector space over $\mathbb{F}$.
Definition 1.2.6 (Real and Complex Vector Space). A vector space over $\mathbb{R}$ is called a *real vector space*, and a vector space over $\mathbb{C}$ is called a *complex vector space*.

Theorem 1.2.7 Unique Additive Identity of Vector Spaces
A vector space has a unique additive identity.

Proof 1. Suppose 0 and $0'$ are both additive identities for some vector space V. So,

$$\begin{aligned} 0' &= 0' + 0 && \textit{Since } 0 \textit{ is an additive identity} \\ &= 0 + 0' && \textit{commutativity} \\ &= 0. && \textit{Since } 0' \textit{ is an additive identity} \end{aligned}$$

Then, $0' = 0$. ■

Theorem 1.2.8 Unique Additive Inverse of Vector Spaces
A vector in a vector space has a unique additive inverse.

Proof 2. Let V be a vector space. Suppose w and w' are additive inverses of v for some $v \in V$. Note that

$$\begin{aligned} w &= w + 0 \\ &= w + (v + w') \\ &= (w + v) + w \\ &= 0 + w' = w'. \end{aligned}$$

■

Notation 1.2.9. Let $v, w \in V$. Then, $-v$ denotes the additive inverse of v.
Definition 1.2.10 (Subtraction). $w - v$ is defined to be $w + (-v)$.

Theorem 1.2.11
$0 \cdot v = 0 \quad \forall v \in V.$

Proof 3. Since $v \in V$, we know

$$\begin{aligned} 0 \cdot v = (0+0)v &= 0 \cdot v + 0 \cdot v \\ 0 \cdot v + (-0 \cdot v) &= 0 \cdot +0 \cdot +(-0 \cdot v) \\ 0 &= 0 \cdot v \end{aligned}$$

■

Theorem 1.2.12
$a \cdot 0 = 0 \quad \forall a \in \mathbb{F}.$

Proof 4. For $a \in \mathbb{F}$, we have

$$\begin{aligned} a \cdot 0 = a \cdot (0+0) &= a \cdot 0 + a \cdot 0 \\ a \cdot 0 + (-a \cdot 0) &= a \cdot 0 + a \cdot 0 + (-a \cdot 0) \\ 0 &= a \cdot 0. \end{aligned}$$

■

Theorem 1.2.13
$(-1)v = -v \quad \forall v \in V.$

Proof 5. For $v \in V$, we have

$$v + (-1)v = 1 \cdot v + (-1) \cdot v = (1 + (-1)) \cdot v = 0 \cdot v = 0.$$

Therefore, by definition, $(-1)v = -v$. ■

Notation 1.2.14. $\mathbb{F}^S$

1. If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.
2. For $f, g \in \mathbb{F}^S$, the <u>sum</u> $f + g \in \mathbb{F}^S$ is the function defined by $(f+g)(x) = f(x) + g(x) \quad \forall x \in S$.
3. For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the <u>product</u> $\lambda f \in \mathbb{F}^S$ is the function defined by $(\lambda f)(x) = \lambda f(x) \quad \forall x \in S$.

Theorem 1.2.15
$\mathbb{F}^S$ is a vector space.

1.3 Subspace

Definition 1.3.1 (Subspace). A subset U of V is called a *subspace* of V if U is also a vector space using the same addition and scalar multiplication as on V.

Theorem 1.3.2 Conditions for a Subspace
A subset U of V is a subspace of V if and only if U satisfies the following conditions:

1. additive identity: $0 \in U$;
2. closed under addition: $u, w \in U \implies u + w \in U$;
3. closed under scalar multiplication: $a \in \mathbb{F}$ and $u \in U \implies au \in U$.

Proof 1.
($\Rightarrow$) Suppose U is a subspace of V. By definition, U is then a vector space, and so those conditions are automatically satisfied. □
($\Leftarrow$) Suppose U satisfies the three conditions. Since U is a subset of V, U automatically has *associativity*, *commutativity*, *multiplicative identity*, and *distributivity*. So, we want to check U has additive inverse and additive identities.
For additive identity, we know $0 \in U$, by assumption.
For additive inverse, by condition #3, we know $-u = (-1)u \in U$.
Then, U is a vector space. ■

Example 1.3.3 If $b \in \mathbb{F}$, then $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.
Proof 2.
($\Rightarrow$) Suppose $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$. Then, $0 = (0, 0, 0, 0) \in U$. So, $0 = 5 \cdot 0 + b$, or $b = 0$. □
($\Leftarrow$) Suppose $b = 0$. Then, $x_3 = 5x_4$. So, $U = \{(x_1, x_2, 5x_4, x_4) \in \mathbb{F}^4\}$

1. $0 = (0, 0, 0, 0) \in U$
2. Note that
$$(x_1, x_2, 5x_4, x_4) + (y_1, y_2, 5y_4, y_4) = (x_1 + y_1, x_2 + y_2, 5(x_4 + y_4), x_4 + y_4) \in U$$
So, addition is closed under U.
3. $\forall a \in \mathbb{F}$, we have
$$a(x_1, x_2, 5x_4, x_4) = (ax_1, ax_2, 5(ax_4), ax_4) \in U$$
Then, U is a subspace of $\mathbb{F}^4$.

■

Example 1.3.4 The set of continuous real-valued functions on interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.
Proof 3.

1. 0 (zero mapping)$\in U$
2. Set f and $g \in \mathcal{C}[0,1]$, the set of continuous functions on interval $[0,1]$. Then, $f+g \in \mathcal{C}[0,1]$.
3. From Calculus, we know that $\forall a \in \mathbb{F}, \quad af \in \mathcal{C}[0,1]$.

■

Definition 1.3.5 (Sum of Subspaces). Suppose $U_1, \cdots, U_m$ are subspaces of V. The *sum* of $U_1, \cdots, U_m$, denoted as $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \cdots, U_m$:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i \quad \forall i = 1, \cdots, m\}.$$

Example 1.3.6 Suppose $U = \{(x,0,0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}$ and $W = \{(0,y,0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}$, then

$$U + W = \{(x,y,0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}.$$

Theorem 1.3.7
Suppose $U_1, \cdots, U_m$ are subspaces of V. Then, $U_1 + \cdots + U_m$ is the *smallest subspace* of V containing $U_1, \cdots, U_m$.

Proof 4. Suppose $U_1, \cdots, U_m$ are subspaces of U. Let $U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_j \in U_j, j = 1, \cdots m\}$. Suppose $w_j \in U_j$, then $w_1 + \cdots + w_m \in U_1 + \cdots + U_m$.

1. $U_1 + \cdots + U_m$ is a subspace of V.
   (a) Note that
   $$(u_1 + \cdots + u_m) + (w_1 + \cdots + w_m) = (u_1 + w_1) + \cdots + (u_m + w_m) \in U_1 + \cdots + U_m,$$
   so $U_1 + \cdots + U_m$ is closed under addition.
   (b) Similarly, $U_1 + \cdots + U_m$ is closed under scalar multiplication.
   (c) Note that U_j is a subspace, so $0 \in U_j$. Hence, $(0, \cdots, 0) = 0 \in U_1 + \cdots + U_m$. □
2. Now, we want to show this subspace is the smallest subspace containing $U_1, \cdots, U_m$. That is, we want to show $\forall\, W \supseteq U_1 \cup \cdots \cup U_m$, we have $W \supseteq U_1 + \cdots + U_m$.

   Note that $U_j \subseteq U_1 + \cdots + U_m$, so we have $(U_1 \cup U_2 \cup \cdots \cup U_m) \subseteq U_1 + \cdots + U_m$. This means $U_1 + \cdots + U_m$ must contain $U_1, \cdots, U_m$. Let W be some subspace containing $U_1, \cdots, U_m$. Then, for $j = 1, \cdots, m$, we have $u_j \in U_j$, which indicates $u_j \in W$. Therefore, $u_1 + \cdots + u_m \in V$ and thus $U_1 + \cdots + U_m \subseteq W$.

   Since W was arbitrary, we've shown $\forall\, W$ that contains $U_1, \cdots, U_m$, $U_1 + \cdots + U_m \subseteq W$. Therefore, $U_1 + \cdots + U_m$ is the smallest.

■

Definition 1.3.8 (Direct Sum). Suppose $U_1, \cdots, U_m$ are subspaces of V. $U_1 + \cdots + U_m$ is called a *direct sum* if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where $u_j \in U_j$.

Notation 1.3.9. If $U_1 + \cdots + U_m$ is a direct sum, then we use $U_1 \oplus \cdots \oplus U_m$ to denote it.

Example 1.3.10 Let $U = \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$ and $W = \{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\}$. Then, $\mathbb{F}^3 = U \oplus W$.

Proof 5. Note that $U + W = \{(x, y, z) \mid x, y, z \in \mathbb{F}\} = \mathbb{F}^3$. Suppose

$$(x, y, z) = (x, y, 0) + (0, 0, z), \tag{1}$$

for some $x, y, z \in \mathbb{F}$ and

$$(x, y, z) = (x', y', 0) + (0, 0, z') \tag{2}$$

for some $x', y', z' \in \mathbb{F}$. Then, (1)−(2):

$$(0, 0, 0) = (x - x', y - y', 0) + (0, 0, z - z') = (x - x', y - y', z - z').$$

Then, $x - x' = y - y' = z - z' = 0$, which indicates $x = x',\ y = y',\ z = z'$. So, by definition $U + W$ is a direct sum, or $\mathbb{F}^3 = U \oplus W$. ■

Example 1.3.11 Suppose U_j is the subspace of $\mathbb{F}^n$ *s.t.*

$$\begin{aligned} U_1 &= \{x, 0, 0, \cdots, 0 \mid x \in \mathbb{F}\} \\ U_2 &= \{0, x, 0, \cdots, 0 \mid x \in \mathbb{F}\} \\ &\vdots \\ U_n &= \{0, 0, 0, \cdots, x \mid x \in \mathbb{F}\} \end{aligned}$$

Then, $\mathbb{F}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_n$.

Proof 6. Note that $\mathbb{F}^n = U_1 + U_2 + \cdots + U_n$ is evident. Now, we'll prove that $U_1 + U_2 + \cdots + U_n$ is a direct sum. Consider $x = (x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$. Assume that

$$x = (x_1, 0, \cdots, 0) + \cdots + (0, \cdots, 0, x_n) \tag{3}$$

and

$$x = (x_1', 0, \cdots, 0) + \cdots + (0, \cdots, 0, x_n') \tag{4}$$

Then, from (3)-(4), we know that

$$0 = (x_1 - x_1', \cdots, x_n - x_n') = (0, 0, \cdots, 0).$$

Then, $\forall i = 1, \cdots, n$ we have $x_i - x_i' = 0$, or $x_i = x_i'$. Therefore, by definition, we know $U_1 + \cdots + U_n$ is a direct sum. ■

Example 1.3.12 Let
$$U_1 = \{(x, y, 0) \mid x, y \in \mathbb{F}\}$$
$$U_2 = \{(0, 0, z) \mid z \in \mathbb{F}\}$$
$$U_3 = \{(0, y, y) \mid y \in \mathbb{F}\}$$
Show that $U_1 + U_2 + U_3$ is not a direct sum.

Proof 7. Consider $(0, 0, 0) \in \mathbb{F}^3$. Note that
$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$$
and
$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1).$$
Then, $U_1 + U_2 + U_3$ is not a direct sum by definition. ■

Theorem 1.3.13
Suppose $U_1, \cdots, U_m$ are subspaces of V. Then,$U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$ is by taking each $u_j = 0$.

Proof 8.
($\Rightarrow$) Since $U_1 + \cdots + U_m$ is a direct sum, by definition, the only way to write $0 \in \mathbb{F}^n$ is to write it as
$$0 = 0 + \cdots + 0 \qquad \text{where } 0 \in U_i \forall i = 1, \cdots, m. \qquad \square$$

($\Leftarrow$) Suppose the only way to write 0 as a sum $u_1 + \cdots + u_m$ is by taking each $u_j = 0$. Assume that for some $v \in V$, we have
$$v = u_1 + \cdots + u_m, \quad u_j \in U_j \tag{5}$$
and
$$v = u_1' + \cdots + u_m', \quad u_j' \in U_j. \tag{6}$$
Then, by (5)-(6), and according to the conclusion from Example 1.3.11, we have
$$0 = (u_1 - u_1') + \cdots + (u_m - u_m') = 0 + \cdots + 0.$$
So, $\forall i \in 1, \cdots, m$, we have $u_i - u_i' = 0$. that is, $u_i = u_i'$. So, $\forall v \in V$, there is only one way to write v as a sum of $u_1 + \cdots {}_u + m$. Therefore, by definition, $U_1 + \cdots + U_m$ is a direct sum. ■

Theorem 1.3.14
Suppose U amd W are subspaces of V. Then, $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof 9.
($\Rightarrow$) Suppose $U + W$ is a direct sum. Assume $v \in U \cap W$. Then, $v \in U$ and $v \in W$. By definition of subspace, we know $-v \in W$ as well. Note that
$$0 = v + (-v) \in U \cap W.$$
Then, by Theorem 1.3.13, we know that the only representation of $0 \in U \cap W$ is $0 = 0 + 0$ since $U \cap W$

is a direct sum. Hence, it must be that $v = -v = 0$, and thus $U \cap W = \{0\}$. $\square$

($\Leftarrow$) Suppose $U \cap W = \{0\}$. Let $u \in U$ and $w \in W$ *s.t.* $u + w = 0$. Then, we have $u = -w$. Since $-w \in W$, we know $u = -w \in W$. By $u \in U$ and $u \in W$, we know that $u \in U \cap W = \{0\}$. Therefore, $0 = 0 + 0$ is the only to represent $0 \in U + W$. By Theorem 1.3.13, we know $U + W$ is a direct sum. $\blacksquare$

Remark. *When extending Theorem 1.3.14 to 3 subspaces U_1, U_2, U_3, we cannot conclude $U_1 \oplus U_2 \oplus U_3$ if we have $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$. See Example 1.3.12 as a counterexample.*

2 Finite-Dimensional Vector Spaces

2.1 Span and Linear Independence

Notation 2.1.1. We usually write list of vectors without using parentheses.

Example 2.1.2 $(4,1,6),(9,5,7)$ is a list of vectors of length 2 in $\mathbb{R}^3$.

Definition 2.1.3 (Linear Combination). A *linear combination* of a list $v_1,\cdots,v_m$ of vectors in V is a vector of the form

$$a_1v_1+\cdots+a_mv_m,$$

where $a_1,\cdots,a_m\in\mathbb{F}$.

Example 2.1.4 Since $(17,-4,2)=6(2,1,-3)+5(1,-2,4)$, we say $(17,-4,2)$ is a linear combination of $(2,1,-3),(1,-2,4)$.

Definition 2.1.5 (Span).

$$\operatorname{span}(v_1,\cdots,v_m)=\{a_1v_1+\cdots+a_mv_m\mid a_1\cdots a_m\in\mathbb{F}\}.$$

Example 2.1.6 Consider $\operatorname{span}(e_1,e_2,e_3)$:

$$\begin{aligned}\operatorname{span}(e_1,e_2,e_3)&=\{a_1e_1+a_2e_2+a_3e_3\mid a_1,a_2,a_3\in\mathbb{F}\}\\&=\{(a_1,a_2,a_3)\mid a_1,a_2,a_3\in\mathbb{F}\}=\mathbb{R}^3.\end{aligned}$$

Theorem 2.1.7
The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Proof 1. To prove this theorem, we will prove two parts: span is a subspace and span is the smallest subspace.

1. Span is a subspace of V.
   (a) By definition of span, we know $\operatorname{span}(v_1,\cdots,v_m)=\{a_1v_1+\cdots+a_mv_m\mid a_1,\cdots,a_m\in\mathbb{F}\}$. If we set $a_1,\cdots,a_m=0$, then we have $0=0v_1+\cdots+0v_m$. So, $0\in\operatorname{span}v_1,\cdots,v_m$.
   (b) Let $a_1v_1+\cdots+a_mv_m\in\operatorname{span}(v_1,\cdots,v_m)$ and $b_1v_1+\cdots+b_mv_m\in\operatorname{span}(v_1,\cdots,v_m)$. Then,

   $$(a_1v_1+\cdots+a_mv_m)+(b_1v_1+\cdots+b_mv_m)=(a_1+b_1)v_1+\cdots+(a_m+b_m)v_m.$$

   Since $(a_1+b_1),\cdots,(a_m+b_m)\in\mathbb{F}$, we know $(a_1+b_1)v_1+\cdots+(a_m+b_m)v_m\in\operatorname{span}(v_1,\cdots,v_m)$.
   (c) Let $\lambda\in\mathbb{F}$ and $a_1v_1+\cdots+a_mv_m\in\operatorname{span}(v_1,\cdots,v_m)$. Then,

   $$\lambda(a_1v_1+\cdots+a_mv_m)=\lambda a_1v_1+\cdots+\lambda a_mv_m.$$

Since $\lambda a_1, \cdots, \lambda a_m \in \mathbb{F}$, we know that $\lambda(a_1v_1 + \cdots + a_mv_m) \in \operatorname{span}(v_1, \cdots, v_m)$.

Therefore, we have proven that span is a subspace of V. $\square$

2. Now, we want to show that span is the smallest subspace.

   Let U be a subspace of V containing $v_1, \cdots, v_m$. If we can show that $\operatorname{span}(v_1, \cdots, v_m) \subseteq U$, we then know span is the smallest subspace containing $v_1, \cdots, v_m$. Since U is a subspace containing $v_1, \cdots, v_m$, it is closed under addition and scalar multiplication. So, $a_1v_1 + \cdots + a_mv_m \in \operatorname{span}(v_1, \cdots, v_m)$. Therefore, $\operatorname{span}(v_1, \cdots, v_m) \subseteq U$.

■

Definition 2.1.8 (Span as a Verb). If $\operatorname{span}(v_1, \cdots, v_m) = V$, we say $v_1, \cdots, v_m$ *spans* V.
Definition 2.1.9 (Finite-Dimensional Vector Space). A vector space V is called *finite-dimensional* if $\exists$ a list of vectors, say $v_1, \cdots, v_m$ *s.t.* $\operatorname{span}(v_1, \cdots, v_m) = V$. In the following of this notes, we will use *f-d* as a shortcut for saying "finite-dimensional."
Definition 2.1.10 (Infinte-Dimensional Vector Space). A vector space V is infinite-dimensional if it is not *f-d*. This is equivalent to say that $\forall$ lists of vectors in V, they do not span V.
Definition 2.1.11 (Polynomial Functions). A function $p : \mathbb{F} \to \mathbb{F}$ is called a *polynomial* with coefficients in $\mathbb{F}$ if $\exists\, a_0, \cdots, a_m \in \mathbb{F}$ *s.t.* $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m \quad \forall z \in \mathbb{F}$.
Notation 2.1.12. We use $\mathcal{P}(\mathbb{F})$ to denote the set of all polynomial with coefficients in $\mathbb{F}$.

Theorem 2.1.13
$\mathcal{P}(\mathbb{F})$ is a vector space over $\mathbb{F}$.

Proof 2. Recall the definition of $\mathbb{F}^{\mathbb{F}}$. We will show $\mathcal{P}(\mathbb{F})$ is a subspace of $\mathbb{F}^{\mathbb{F}}$.

1. $0 = 0 + 0z + \cdots + 0z^m \in \mathcal{P}(\mathbb{F})$.

2. Suppose $p(z) = a_mz^m + \cdots + a_1z + a_0$ and $q(z) = b_nz^n + \cdots + b_1z + b_0 \in \mathcal{P}(\mathbb{F})$. WLOG, suppose $m > n$, then we have $p(z) + q(z) = a_mz^m + \cdots + (a_n + b_n)z^n + \cdots + (a_0 + b_0) \in \mathcal{P}(\mathbb{F})$.

3. Suppose $\lambda \in \mathbb{F}$. Then, $\lambda p(z) = \lambda(a_mz^m + \cdots + a_1z + a_0) = \lambda a_mz^m + \cdots + \lambda a_0 \in \mathcal{P}(\mathbb{F})$.

Hence, we've shown $\mathcal{P}(\mathbb{F})$ is a subspace over $\mathbb{F}$. ■
Definition 2.1.14 (Degree of a Polynomial). A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have *degree* m if $\exists$ scalars $a_0, \cdots, a_m \in \mathbb{F}$ with $a_m \neq 0$ *s.t.* $p(z) = a_mz^m + \cdots + a_1z + a_0 \quad \forall z \in \mathbb{F}$. We write $\deg p = m$. Specially, $\deg 0 \coloneqq -\infty$ and $\deg a_0 \coloneqq 0$ when $a_0 \neq 0$.
Definition 2.1.15 ($\mathcal{P}_m(\mathbb{F})$). For $m \in \mathbb{N}^+$, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomial with coefficients in $\mathbb{F}$ and degree$\leq m$. i.e.,

$$\mathcal{P}_m(\mathbb{F}) \coloneqq \{p \in \mathcal{P}(\mathbb{F}) \mid \deg p \leq m\}.$$

Example 2.1.16 For each $m \in \mathbb{N}$, $\mathcal{P}_m(\mathbb{F})$ is a *f-d* vector space.
Proof 3. Note that $\mathcal{P}_m(\mathbb{F})$ is a vector space because it is a subspace of $\mathcal{P}(\mathbb{F})$. Suppose $p(z) \in \mathcal{P}_m(\mathbb{F})$, then $p(z) = a_0 + a_1z + \cdots + a_mz^m \in \operatorname{span}(1, z, \cdots, z^m)$. Then, by definition, $\mathcal{P}_m(\mathbb{F})$ is *f-d.* ■

Remark. *In this proof, we are abusing notation by letting z^k to denote a function.*

Example 2.1.17 $\mathcal{P}(\mathbb{F})$ is infinite-dimensional.

Proof 4. For any list of vectors in $\mathcal{P}(\mathbb{F})$, by definition of list, the length of it is finite. Suppose the highest degree in this list is m. Consider a polynomial with degree of $m+1$: z^{m+1}. Since z^{m+1} cannot be written as linear combinations of the list of polynomials, we know the list does not span $\mathcal{P}(\mathbb{F})$. So, $\mathcal{P}(\mathbb{F})$ is infinite-dimensional. ■

Definition 2.1.18 (Linear Independence). A list $v_1, \cdots, v_m$ of vectors in V is called *linearly independent* (L.I.) if the only choice of $a_1, \cdots, a_m \in \mathbb{F}$ that makes $a_1v_1+\cdots+a_mv_m = 0$ is $a_1 = \cdots = a_m = 0$. Specially, the empty list () is declared to be L.I..

Definition 2.1.19 (Linear Dependence). $v_1, \cdots, v_m$ is called *linearly dependent* if it is not L.I.. Or, equivalently, $v_1, \cdots, v_m$ is *linearly dependent* if $\exists\, a_1, \cdots, a_m \in \mathbb{F}$ not all 0 *s.t.* $\sum\limits_{i=0}^{m} a_iv_i = 0$.

Example 2.1.20 Let $v_1, \cdots, v_m \in V$. If v_j is a linear combination of other v's, then $v_1, \cdots, v_m$ is linearly dependent.

Proof 5. By assumption, $v_j = a_1v_1 + \cdots + a_{j-1}v_{j-1} + a_{j+1}v_{j+a} + \cdots + a_mv_m$ for some a_i not all 0. So, $0 = a_1v_1 + \cdots + a_{j-1}v_{j-1} + a_{j+1}v_{j+1} + \cdots + a_mv_m - v_j$, a linear combination of $v_1, \cdots, v_m$. Since $-v_j$ has a coefficient of $-1 \neq 0$, by definition, $v_1, \cdots, v_m$ is not L.I.. ■

Lemma 2.1.21 Linear Dependence Lemma Suppose $v_1, \cdots, v_m$ is a linearly dependent list in V. Then, $\exists\, j \in \{1, \cdots, m\}$ *s.t.* the following hold:

1. $v_j \in \operatorname{span}(v_1, \cdots, v_{j-1})$
2. if the j^{th} term is removed from $v_1, \cdots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \cdots, v_m)$.

Proof 6.

1. Since $v_1, \cdots, v_m$ is linearly dependent, $a_1v_1 + \cdots + a_mv_m = 0$, for some $a_i \neq 0$. Let j be the maximized index *s.t.* $a_j \neq 0$. Then, $a_{j+1} = \cdots = a_m = 0$, by this assumption. Hence,

$$\begin{aligned} a_jv_j &= -a_1v_1 - \cdots - a_{j-1}v_{j-1} - a_{j+1}v_{j+1} - \cdots - a_mv_m \\ &= -a_1v_1 - \cdots - a_{j-1}v_{j-1} \\ v_j &= -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}. \end{aligned}$$

   Since $-\frac{a_1}{a_j}, \cdots, -\frac{a_{j-1}}{a_j} \in \mathbb{F}$, we know $v_j \in \operatorname{span}(v_1, \cdots, v_{j-1})$. □

2. Consider

$$\begin{aligned} \operatorname{span}(v_1, \cdots, v_j, \cdots, v_m) &= \operatorname{span}(v_1, \cdots, -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}, \cdots, v_m) \\ &= \operatorname{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m). \end{aligned}$$

■

Remark. *By using this Lemma 2.1.21, we can do lots of proofs using the "step" strategy. Namely, we start to remove vectors from a list that are linearly dependent to obtain a* L.I. *list. However, this "step" strategy can only be used when dealing with FINITE-dimensional vector spaces.*

Theorem 2.1.22
Let V be a *f-d* vector space. Let $\mathrm{span}(w_1, \cdots, w_n) = V$. Let $u_1, \cdots, u_m$ be L.I.. Then, $m \leq n$.

Proof 7.
$\boxed{\text{Step } 1}$ Note that $u_1, w_1, \cdots, w_n$ is linearly dependent because $u_1 \in V = \mathrm{span}(w_1, \cdots, w_n)$. Then, by Lemma 2.1.21, we can remove one of the w's, say w_{j1}. Then, the list becomes

$$\{u_1, w_1, \cdots, w_n\} \setminus \{w_{j1}\}.$$

$\boxed{\text{Step } 2}$ Adjoin u_2. Apply the same reasoning, since $\mathrm{span}(\{u_1, w_1, \cdots, w_n\} \setminus \{w_{j1}\}) = V$, we know $\{u_1, u_2, w_1, \cdots, w_n\} \setminus \{w_{j1}\}$ is linearly dependent. Since $u_2 \notin \mathrm{span}(u_1)$, Lemma 2.1.21 is not applicable to u_2. Now, we can remove another w from the list, say w_{j2}. The list becomes

$$\{u_1, u_2, w_1, \cdots, w_n\} \setminus \{w_{j1}, w_{j2}\}.$$

$\vdots$

$\boxed{\text{Step } m}$ After m steps, we list will become

$$\{u_1, \cdots, u_m, w_1, \cdots, w_n\} \setminus \{w_{j1}, \cdots, w_{jm}\}.$$

Since $\mathrm{span}(\{u_1, \cdots, u_m, w_1, \cdots, w_n\} \setminus \{w_{j1}, \cdots, w_{jm}\}) = V$, this list is still linearly dependent, so by Lemma 2.1.21, we know $\exists\, w$ to be removed. Therefore, $n \geq m$. ■

Theorem 2.1.23
Every subspace of a *f-d* vector space is *f-d*.

Proof 8. Suppose V to be a *f-d* vector space and U to be a subspace of V.
$\boxed{\text{Step } 1}$ If $U = \{0\}$, then U is *f-d*. If $U \neq \{0\}$, then choose $v_i \in U$ ***s.t.*** $v_1 \neq 0$.

$\vdots$

$\boxed{\text{Step } j}$ If $U = \mathrm{span}(v_1, \cdots, v_{j-1})$, then U is *f-d*. If $U \neq \mathrm{span}(v_1, \cdots, v_{j-1})$, then choose $v_j \in U$ ***s.t.*** $v_j \notin \mathrm{span}(v_1, \cdots, v_{j-1})$.

By Lemma 2.1.21 and Theorem 2.1.22, we know this process will eventually terminate because the vector list that spans U cannot be longer than any spanning list of V. Therefore, U is *f-d*. ■

2.2 Bases

Definition 2.2.1 (Basis). A *basis* of V is a list of vectors in V that is L.I. and spans V.

Example 2.2.2

1. The standard basis of $\mathbb{F}^n$:

$$(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1).$$

2. $(1, 1, 0), (0, 0, 1)$ is a basis of V, where $V = \{(x, x, y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$.

   Proof 1.

   (a) Suppose $a_1(1, 1, 0) + a_2(0, 0, 1) = 0$, we have $(a_1, a_1, a_2) = 0$. So, it must be $a_1 = a_2 = 0$. Therefore, $(1, 1, 0), (0, 0, 1)$ is L.I.. □

   (b) Suppose $(x, x, y) \in V$. Note that $(x, x, y) = x(1, 1, 0) + y(0, 0, 1)$, then, $V = \operatorname{span}((1, 1, 0), (0, 0, 1))$.

   Therefore, we've proven $(1, 1, 0), (0, 0, 1)$ is a basis of V according to the definition of basis. ■

Theorem 2.2.3 Criterion for Basis
A list $v_1, \cdots, v_n \in V$ is a basis list of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \cdots + a_nv_n$, where $a_i \in \mathbb{F}$.

Proof 2.
($\Rightarrow$) Let $v_1, \cdots, v_n$ be a basis of V. Let $v \in V$. By definition of basis, $V = \operatorname{span}(v_1, \cdots, v_n)$. So, $v \in \operatorname{span}(v_1, \cdots, v_n)$, and thus $v = a_1v_1+\cdots+a_nv_n$ for some $a_i \in \mathbb{F}$. Assume for the sake of contradiction that $v = b_1v_1 + \cdots + b_nv_n$ for some $b_i \neq a_i \in \mathbb{F}$. Then,

$$\begin{aligned} v - v &= (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n \\ 0 &= (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n. \end{aligned}$$

Since $v_1, \cdots, v_n$ is a basis, it is L.I.. So, $0 = 0v_1+\cdots+0v_n$. Therefore, we know $a_1-b_1 = \cdots = a_n-b_n = 0$. That is, $a_1 = b_1, \cdots, a_n = b_n$. ⋇ This is a contradiction with the assumption that $\exists\, a_i \neq b_i$. Hence, it must be that $v = a_1v_1 + \cdots + a_nv_n$ is unique. □

($\Leftarrow$) Suppose $v = a_1v_1 + \cdots + a_nv_n$ is the unique representation $\forall\, v \in V$. Then, $v \in \operatorname{span}(v_1, \cdots, v_n)$. Since $v \in V$, then $V \subseteq \operatorname{span}(v_1, \cdots, v_n)$. However, $v_1, \cdots, v_n \in V$, so $\operatorname{span}(v_1, \cdots, v_n) \subseteq V$. Therefore, $\operatorname{span}(v_1, \cdots, v_n) = V$. To show $v_1, \cdots, v_n$ is L.I., further consider $0 = a_1v_1 + \cdots + a_nv_n$. Since $0 \in V$, by assumption, $\exists$ a unique way to write 0 as $a_1v_1 + \cdots + a_nv_n$, and that unique way is to take every $a_i = 0$. Hence, by definition, we know $v_1, \cdots, v_n$ is L.I.. Since $v_1, \cdots, v_n$ is L.I. and $\operatorname{span}(v_1, \cdots, v_n) = V$, we know $v_1, \cdots, v_n$ is a basis list of V. ■

Theorem 2.2.4
Every spanning list can be reduced to a basis of the vector space.

Proof 3. Suppose $V = \operatorname{span}(v_1, \cdots, v_n)$. If $v_i = 0$, we just remove v_i. So, let's suppose $v_i \neq 0$.

$\boxed{\text{Step } 1}$ If $v_2 \in \text{span}(v_1)$, delete it. If $v_2 \notin \text{span}(v_2)$, keep it.

$\vdots$

$\boxed{\text{Step } j}$ If $v_j \in \text{span}(v_1, \cdots, v_{j-1})$, delete it. If $v_j \notin \text{span}(v_1, \cdots, v_{j-1})$, keep it.

$\vdots$

$\boxed{\text{Step } n}$ After n steps, we will have a "sub-list" from the original list *s.t.* it spans V and is L.I.. Therefore, the basis list is contained in the spanning list. ■

Corollary 2.2.5 Every *f-d* vector space has a basis.

Proof 4. By definition, *f-d* vector space always has a spanning list. By Theorem 2.2.4, a spanning list contain a basis. ■

Theorem 2.2.6
Every linearly independent list of vectors in a *f-d* vector space can be extended to a basis of the vector space.

Proof 5. Suppose $u_1, \cdots, u_m$ is L.I. in a *f-d* vector space of V. Let $w_1, \cdots, w_n$ be a basis of V. Then, $u_1, \cdots, u_m, w_1, \cdots, w_n$ spans V. According to Lemma 2.1.21 and Theorem 2.1.22, we can reduce $u_1, \cdots, u_m, w_1, \cdots, w_m$ to some list of $u_1, \cdots, u_m$ and some w's. ■

Theorem 2.2.7
Suppose V is *f-d* and U is a subspace of V. Then, there is a subspace W of V *s.t.* $V = U \oplus W$.

Proof 6. Since V is *f-d*, U, as V's subspace, is also *f-d*. So, $\exists$ a basis of U, say $u_1, \cdots, u_m$. Then, $u_1, \cdots, u_m$ is L.I. and $\in V$. By Theorem 2.2.6, this list can be extended to a basis

$$u_1, \cdots, u_m, w_1, \cdots, w_n \text{ of } V.$$

Let $W = \text{span}(w_1, \cdots, w_n)$. We'll show $V = U \oplus W$.

1. WTS: $V = U + W$. Suppose $v \in V$. Then,

$$v = \underbrace{a_1u_1 + \cdots + a_mu_m}_{\in U} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{\in W}.$$

   So, $v \in U + W$, or $V = U + W$. □

2. WTS: $U \cap W = \{0\}$. Suppose $v \in U \cap W$. Then, $v \in U$ and $v \in W$. So,

$$v = a_1u_1 + \cdots + a_mv_m = b_1w_1 + \cdots + b_nw_n.$$

   Hence,

$$a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0. \tag{7}$$

   Since by assumption, $u_1, \cdots, u_m, w_1, \cdots, w_n$ is a basis of V, so $u_1, \cdots, u_m, w_1, \cdots, w_n$ is L.I.. Therefore, the only way for Equation (7) to hold is when $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Hence, $v = 0u_1 + \cdots + u_m = 0$. That is, $U \cap W = \{0\}$.

Therefore, we've shown that $V = U \oplus W$. ■

2.3 Dimension

Theorem 2.3.1
Let B_1 and B_2 be two bases of V, then B_1 and B_2 have the same length.

Proof 1. Since B_1 is L.I. in V and B_2 spans V, by Theorem 2.1.22, we know $\text{len}(B_1) \leq \text{len}(B_2)$. Interchanging the roles of B_1 and B_2, we have $\text{len}(B_2) \leq \text{len}(B_1)$. So, we have $\text{len}(B_1) = \text{len}(B_2)$. ■

Definition 2.3.2 (Dimension). The *dimension* of a *f-d* vector space V is the length of any basis of V.

Notation 2.3.3. We use $\dim V$ to denote the dimension of a *f-d* vector space V.

Example 2.3.4 $\dim \mathbb{F}^n = n$ and $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$ $(1, z, z^2, \cdots, z^m)$.

Theorem 2.3.5
If V is *f-d* and U is a subspace of V, then $\dim U \leq \dim V$.

Proof 2. Let B_1 be a basis of U and B_2 be a basis of V. Then, B_1 is a L.I. list of V and B_2 spans V. Then, By Theorem 2.1.22, we know that $\text{len}(B_1) \leq \text{len}(B_2)$. So, by definition of dimension, we know $\dim U \leq \dim V$. ■

Extension. *If V is* f-d *and U is a subspace of V, given $U \subsetneq V$, then* $\dim U < \dim V$.

Proof 3. Let $u_1, \cdots, u_m$ be a basis of U. Since $U \subsetneq V$, we know $V - U \neq \varnothing$. So, choose $v \in V - U$. Then, $v \notin \text{span}(u_1, \cdots, u_m)$. Therefore, $u_1, \cdots, u_m, v$ is L.I. in V. That is

$$\begin{aligned}\dim V &\geq \dim(\text{span}(u_1, \cdots, u_m, v)) \\ &> \dim(\text{span}(u_1, \cdots, u_m)) \\ &= \dim U.\end{aligned}$$

■

Theorem 2.3.6
Let V be *f-d*, then every L.I. list of vectors in V with length $\dim V$ is a basis of V.

Proof 4. Let $v_1, \cdots, v_n \in V$ be L.I.. Let $n = \dim V$. When extending the list to basis, we get

$$\{v_1, m \cdots, v_n\} \cup \varnothing$$

as a basis of V. That is, $v_1, \cdots, v_n$ has already been a basis of V. ■

Remark. *The proof given above is not that straight-forward, so we are giving an easier-understanding proof as follows.*

Proof 5. Suppose for the sake of contradiction that $\exists v_1, \cdots, v_n \in V$ not a basis of V for $n = \dim V$. Then, $\text{span}(v_1, \cdots, v_n) \neq V$. That is, $\exists\ v_{n+1}$ *s.t.* $v_{n+1} \notin \text{span}(v_1, \cdots, v_n)$. Adding v_{n+1} to the vector list, we have $v_1, \cdots, v_n, v_{n+1}$ is L.I.. By Theorem 2.3.5, we know $\text{len}(v_1, \cdots, v_{n+1}) = n + 1 \leq \dim V$. ⋇ This contradicts with the fact that $\dim V = n < n + 1$. So, our assumption is incorrect, and it must be that $v_1, \cdots, v_n$ is a basis of V. ■

Theorem 2.3.7
Suppose V is *f-d*. Then, every spanning list of vectors in V with length $\dim V$ is a basis of V.

Example 2.3.8 Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbb{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) \mid p'(5) = 0\}.$$

Proof 6. Consider $a_1 + a_2(x-5)^2 + a_3(x-5)^3 = 0$, we will get $a_1 = a_2 = a_3 = 0$ easily from the equation. Then, $1, (x-5)^2, (x-5)^3$ is L.I.. So, by Theorem 2.3.5, we know $\dim U \geq 3$. Since $U \subsetneq \mathcal{P}_3(\mathbb{R})$, we have $\dim U < \dim \mathcal{P}_3(\mathbb{R}) = 4$. Therefore, $\dim U = 3 = \operatorname{len}(1, (x-5)^2, (x-5)^3)$. By Theorem 2.3.6, we know $1, (x-5)^2, (x-5)^3$ is a basis of U. ■

Theorem 2.3.9
If U_1 and U_2 are subspaces of a *f-d* vector space, then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Proof 7. Let $u_1, \cdots, u_m$ be a basis of $U_1 \cap U_2$, then $\dim(U_1 \cap U_2) = m$. Also, $u_1, \cdots, u_m$ is L.I. in U_1, so we can extend it to a basis of U_1 as $u_1, \cdots, u_m, v_1, \cdots, v_j$. Then, $\dim(U_1) = m + j$. Similarly, extending $u_1, \cdots, u_m$ to a basis of U_2, we will get $u_1, \cdots, u_m, w_1, \cdots, w_k$. So, $\dim(U_2) = m + k$. Now, we want to show $u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k$ is a basis of $U_1 + U_2$.

1. Since $U_1, U_2 \subseteq \operatorname{span}(u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k)$, we know that

$$\operatorname{span}(u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k) = U_1 + U_2. \qquad \square$$

2. Suppose $a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k = 0$. Then we know that

$$c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j.$$

Since $c_1w_1 + \cdots + c_kw_k \in U_2$, and $-a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j \in U_1$, we know that $c_1w_1 + \cdots + c_kw_k \in U_1 \cap U_2$. Therefore, $c_1w_1 + \cdots + c_kw_k = d_1u_1 + \cdots + d_mu_m$. Since $u_1, \cdots, u_m, w_1, \cdots, w_k$ is L.I., we know $c_1 = \cdots = c_k = 0$. So, $-a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j = 0$. Since $u_1, \cdots, u_m, v_1, \cdots, v_j$ is L.I., we have $a_1 = \cdots = a_m = b_1 = \cdots = b_j = 0$. Therefore, we've proven $u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k$ is L.I. and thus is a basis of $U_1 + U_2$. $\square$

Since $u_1, \cdots, u_m, v_1, \cdots, v_j, w_1, \cdots, w_k$ is a basis of $U_1 + U_2$, we know $\dim(U_1 + U_2) = m + j + k$. Further note that

$$\begin{aligned} \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) &= (m + j) + (m + k) - m \\ &= m + j + k \\ &= \dim(U_1 + U_2). \end{aligned}$$

■

3 Linear Maps

Notation 3.0.1. In this section, we use V and W to denote vector spaces over $\mathbb{F}$.

3.1 The Vector Space of Linear Maps

Definition 3.1.1 (Linear Map). A *linear map* from V to W is a function $T : V \to W$ with the following properties:

- additivity: $T(u+v) = Tu + Tv \qquad \forall u, v \in V$.
- homogeneity: $T(\lambda v) = \lambda(Tv) \qquad \forall \lambda \in \mathbb{F}$ and $\forall v \in V$.

Notation 3.1.2. The set of all linear maps from V to W is denoted by $\mathcal{L}(V, W)$.

Example 3.1.3

1. Zero-mapping: $0 \in \mathcal{L}(V, W)$ is defined by $0v = 0$.
2. Identity-mapping: $I \in \mathcal{L}(V, V)$ is defined by $Iv = v$.
3. Differentiation: $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is defined by $Dp = p'$.

   Proof 1. Note that $(f+g)' = f' + g'$ and $(\lambda f)' = \lambda f'$. ■
4. Integration: $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ is defined by $Tp = \int_0^1 p(x)\,\mathrm{d}x$

   Proof 2. Note that $\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g$ and $\int_0^1 \lambda f = \lambda \int_0^1 f$. ■
5. Backward shift: $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ as $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots)$.

   Proof 3. Note that

   $$\begin{aligned} T(x_1, x_2, x_3, \cdots) + T(y_1, y_2, y_3, \cdots) &= (x_2, x_3, \cdots) + (y_2, y_3, \cdots) \\ &= (x_2 + y_2, x_3 + y_3, \cdots) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots). \end{aligned}$$

   Therefore, T is additive. Homogeneity of T is travial and thus omitted here. ■
6. From $\mathbb{F}^n$ to $\mathbb{F}^m$, we define $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ as

   $$T(x_1, \cdots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n),$$

   where $A_{j,k} \in \mathbb{F} \quad \forall j = 1, \cdots, m$ and $k = 1, \cdots, n$.

Theorem 3.1.4
Suppose $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_n \in W$. Then, $\exists$ a unique linear map $T : V \to W$ *s.t.* $Tv_j = w_j \quad \forall j = 1, \cdots, n$.

Remark. *If T in Theorem 3.1.1 is a linear mapping, we should have*

1. $T(v_1 + \cdots + v_n) = Tv_1 + \cdots + Tv_n = w_1 + \cdots + w_n$*, by additivity of T, and*
2. $T(\lambda_j v_j) = \lambda_j T v_j$*, by homogeneity of T.*

Combine the two properties, we should have

$$T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 T v_1 + \cdots = \lambda_n T v_n = \lambda_1 w_1 + \cdots + \lambda_n w_n.$$

This remark will be very helpful in our following proof of the theorem.

Proof 4. Let's define $T : V \to W$ by $T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$, where $c_1, \cdots, c_n$ are arbitrary elements of $\mathbb{F}$. Now, we want to show that T is a linear mapping.

Suppose $u, v \in V$, $u = a_1 v_1 + \cdots + a_n v_n$, and $v = c_1 v_1 + \cdots + c_n v_n$. Then, we have

$$\begin{aligned} T(u+v) &= T((a_1+c_1)v_1 + \cdots + (a_n + c_n)v_n) \\ &= (a_1 + c_1)w_1 + \cdots + (a_n + c_n)w_n \\ &= (a_1 w_1 + \cdots + a_n w_n) + (c_1 w_1 + \cdots + c_n w_n) \\ &= Tu + Tv. \qquad \square \end{aligned}$$

Now, we want to show T has homogeneity. Suppose $\lambda \in \mathbb{F}$. Then, we know

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1 v_1 + \cdots + \lambda c_n v_n) \\ &= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\ &= \lambda(c_1 w_1 + \cdots + c_n w_n) \\ &= \lambda T v. \qquad \square \end{aligned}$$

Also, we want to show that this T satisfy the condition the theorem is asking (i.e., $Tv_j = w_j$). Note that when $c_j = 0$ and other c's equal 0, we will get $Tv_j = w_j$. $\square$

Finally, we will prove the uniqueness of this T. Suppose that $T' \in \mathcal{L}(V, W)$ and $T'v_j = w_j$. Let $c_1, \cdots, c_n \in \mathbb{F}$. Then, $T'(c_j v_j) = c_j w_j$. So, we know that $T'(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$. However, by definition, we know $c_1 w_1 + \cdots + c_n w_n = T(c_1 w_1 + \cdots + c_n v_n)$. So, we can conclude that $T'(c_1 v_1 + \cdots + c_n v_n) = T(c_1 w_1 + \cdots + c_n v_n)$. Thus, $T' = T$, and thus the T we defined above is unique in $\mathcal{L}(V, W)$. ■

Definition 3.1.5 (Addition and Scalar Multiplication on $\mathcal{L}(V, W)$). Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. Then, the *addition* is defined as $(S + T)(v) := Sv + Tv$, and the *scalar multiplication* is defined as $(\lambda T)(v) := \lambda(Tv) \quad \forall v \in V$.

Theorem 3.1.6
$\mathcal{L}(V, W)$ is a vector space.

Proof 5.

1. additive identity: Note that the zero-mapping $0 \in \mathcal{L}(V, W)$ satisfies the following equation:

$$(0 + T)(v) = 0v + Tv = 0 + Tv = Tv. \qquad \square$$

2. commutativity: Note that

$$(S+T)(v) = Sv + Tv = Tv + Sv = (T+S)(v). \qquad \square$$

3. associativity: Let $S, T, R \in \mathcal{L}(V, W)$. Then,

$$\begin{aligned}((S+T)+R)(v) = (S+T)(v) + Rv &= Sv + Tv + Rv \\ &= Sv + (Tv + Rv) \\ &= Sv + (T+R)(v) \\ &= (S+(T+R))(v).\end{aligned}$$

   Let $a, b \in \mathbb{F}$. Then,

$$((ab)T)(v) = T(abv) = T(a(bv)) = aT(bv) = (a(bT))(v). \qquad \square$$

4. multiplicative identity: Note we have $1 \in \mathbb{F}$ *s.t.*

$$(1 \cdot T)(v) = T(1 \cdot v) = Tv. \qquad \square$$

5. additive inverse: Note that

$$(T + (-T))(v) = Tv + (-T)(v) = Tv + T(-v) = T(v - v) = T0 = 0. \qquad \square$$

6. distributivity: Note that

$$a(T+S)(v) = a(Tv + Sv) = aTv + aSv,$$

   and

$$(a+b)Tv = T((a+b)v) = T(av + bv) = T(av) + T(bv) = aTv + bTv.$$

$\blacksquare$

Definition 3.1.7 (Product of Linear Maps). If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by $(ST)(u) = S(Tu) \quad \forall u \in U$.

Remark. *Compare this definition with composite functions. ST is only defined when T maps into the domain of S.*

Theorem 3.1.8 Algebraic Properties of Products of Linear Maps

1. associativity: $(T_1T_2)T_3 = T_1(T_2T_3)$.
2. identity: $TI = IT = T$, where I is the identity mapping
3. distributive properties: $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$.

Proof 6. First, we want to show the associativity. Note that

$$[(T_1T_2)T_3](v) = (T_1T_2)(T_3v) = (T_1)(T_2(T_3v)) = (T_1)[(T_2T_3)(v)]. \qquad \square$$

Then, we want to show the identity. This proof can be done using the following diagram:

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "I_W"] \\ V \arrow[u, "I_V"] & W \end{tikzcd} \qquad \square$$

Finally, we will show the distributive properties. Note that

$$\begin{aligned} [(S_1 + S_2)T](v) = (S_1 + S_2)(Tv) &= S_1(Tv) + S_2(Tv) \\ &= (S_1T)(v) + (S_2T)(v) \\ &= (S_1T + S_2T)(v). \end{aligned}$$

Similarly, we can show

$$\begin{aligned} [S(T_1 + T_2)](v) = S[(T_1 + T_2)(v)] &= S(T_1v + T_2v) \\ &= S(T_1v) + S(T_2v) \\ &= (ST_1)(v) + (ST_2)(v) \\ &= (ST_1 + ST_2)(v). \end{aligned}$$

■

Example 3.1.9 Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map, and $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be defined by $(Tp)(x) = x^2p(x)$. Show that $DT \neq TD$.

Proof 7. Note that $(DT)p = D(Tp) = D(x^2p(x)) = 2xp(x) + x^2p'(x)$. Similarly, we can compute a general formula for TD: $(TD)p = T(Dp) = T(p') = x^2p'(x)$. Since $2xp(x) + x^2p'(x) \neq x^2p'(x)$, we know $DT \neq TD$. ■

Theorem 3.1.10
Let $T \in \mathcal{L}(V, W)$, then $T(0) = 0$.

Proof 8. Since $T(0) = T(0 + 0) = T(0) + T(0)$, we know $0 = T(0)$, or $T(0) = 0$. ■

Corollary 3.1.11 If $T(0) \neq 0$, then $T \notin \mathcal{L}(V, W)$.

3.2 Null Spaces and Ranges

Definition 3.2.1 (Null Space/Kernel). For $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted $\operatorname{null} T$, is the subset of V consisting of those vectors that T maps to 0: $\operatorname{null} T = \{v \in V \mid Tv = 0\}$.

Remark. *Sometimes, null space of T is also called the kernal of T, denoted as $\ker T$.*

Example 3.2.2

1. Null space of zero-mapping: Let T be the zero mapping from V to W. Since $Tv = 0 \quad \forall v \in V$, we know $\operatorname{null} T = V$.
2. $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ as $Dp = p'$: $\operatorname{null} D = \{a \mid a \in \mathbb{R}\}$.
3. $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ as $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots)$: $\operatorname{null} T = \{(a, 0, 0, \cdots) \mid a \in \mathbb{F}\}$.

Theorem 3.2.3
Suppose $T \in \mathcal{L}(V, W)$. Then, $\operatorname{null} T$ is a subspace of V.

Proof 1.

1. Note that $T(0) = 0$, so $0 \in \operatorname{null} T$. ☐
2. Suppose $u, v \in \operatorname{null} T$. Then, $Tu = Tv = 0$. So, $T(u + v) = Tu + Tv = 0 + 0 = 0$. Hence, $u + v \in \operatorname{null} T$. ☐
3. Suppose $u \in \operatorname{null} T$ and $\lambda \in \mathbb{F}$. Then, $Tu = 0$. So, $T(\lambda u) = \lambda Tu = \lambda \cdot 0 = 0$. Therefore, $\lambda u \in \operatorname{null} T$.

■

Definition 3.2.4 (Injective/Injection). A function $T : V \to W$ is called *injective* of $Tu = Tv$ implies $u = v$.

Remark. *Sometimes, the contrapositive will be much more helpful: T is injective if $u \neq v$, then $Tu \neq v$.*

Theorem 3.2.5
Let $T \in \mathcal{L}(V, W)$. Then, T is injective if and only if $\operatorname{null} T = \{0\}$.

Proof 2.
($\Rightarrow$) Suppose T is an injective. We've already known that $\{0\} \subseteq \operatorname{null} T$. Then, we need to show $\operatorname{null} T \subseteq \{0\}$. Suppose $v \in \operatorname{null} T$, then $Tv = 0$. However, since T is an injection, and $Tv = T0 = 0$, then we have $v = 0$. So, $\operatorname{null} T \subseteq \{0\}$. Therefore, it's sufficient to say $\operatorname{null} T = \{0\}$. ☐
($\Leftarrow$) Suppose $\operatorname{null} T = \{0\}$. Suppose $u, v \in V$ and $Tu = Tv$. Then, $Tu - Tv = T(u - v) = 0$. Hence, $u - v \in \operatorname{null} T$. By $\operatorname{null} T = \{0\}$, we know $u - v = 0$, so $u = v$. Then, T is an injection. ■

Definition 3.2.6 (Range/Image). For $T \in \mathcal{L}(V, W)$, the range of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$: $\operatorname{range} T = \{Tv \mid v \in V\}$.

Theorem 3.2.7
If $T \in \mathcal{L}(V, W)$, then $\operatorname{range} T$ is a subspace of W.

Proof 3.

1. Since $T(0) = 0$, we know $0 \in \operatorname{range} T$. $\square$
2. Suppose $w_1, w_2 \in \operatorname{range} T$. Then, $\exists v_1, v_2 \in V$ *s.t.* $Tv_1 = w_1$ and $Tv_2 = w_2$. Then, $w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2)$. Since $v_1 + v_2 \in V$, we have $w_1 + w_2 = T(v_1 + v_2) \in \operatorname{range} T$. $\square$
3. Suppose $w \in \operatorname{range} T$ and $\lambda \in \mathbb{F}$. Then, $\exists v \in V$ *s.t.* $w = Tv$. So, $\lambda w = \lambda(Tv) = T(\lambda v)$. Since $\lambda v \in V$, $\lambda w = T(\lambda v) \in \operatorname{range} T$.

$\blacksquare$

Definition 3.2.8 (Surjective/Surjection). A function $T : V \to W$ is called *surjective* if $\operatorname{range} T = W$.

Remark. *A function $T : V \to W$ is called a* bijection*, or is bijective, if it is both injective and surjective.*

Theorem 3.2.9 Fundamental Theorem of Linear Maps
Suppose V is *f-d* and $T \in \mathcal{L}(V, W)$. Then, $\operatorname{range} T$ is *f-d* and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof 4. Let $u_1, \cdots, u_m$ be a basis of $\operatorname{null} T$. Then, $\dim \operatorname{null} T = m$. By Theorem 3.2.3, we know $\operatorname{null} T$ is a basis of V, so we can extend the basis to a basis of V: $u_1, \cdots, u_m, v_1, \cdots, v_n$. Thus, $\dim V = m + n$. *WTS:* $\dim \operatorname{range} T = n$. *Further WTS:* $Tv_1, \cdots, Tv_n$ *is a basis of* $\operatorname{range} T$.

Suppose $v \in V$. Then

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n.$$

Since $u_1, \cdots, u_m \in \operatorname{null} T$, we know $Tu_1, \cdots, Tu_m = 0$. Therefore,

$$Tv = a_1 Tu_1 + \cdots + a_m Tu_m + b_1 Tv_1 + \cdots + b_n Tv_n = b_1 Tv_1 + \cdots + b_n Tv_n.$$

Hence, $\operatorname{span}(Tv_1, \cdots, Tv_n) = \operatorname{range} T$, and thus range T is *f-d. Now, WTS:* $Tv_1, \cdots, Tv_n$ *is* L.I..

Consider $c_1 Tv_1 + \cdots + c_n Tv_n = 0$. Then, $T(c_1 v_1 + \cdots + c_n v_n) = 0$. Hence, $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$. Since $u_1, \cdots, u_m$ is a basis of $\operatorname{null} T$, we know

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m \quad \textit{f.s. } d_i \in \mathbb{F}.$$

So,

$$c_1 v_1 + \cdots + c_n v_n - d_1 u_1 - \cdots - d_m u_m = 0. \tag{8}$$

However, by assumption, we know $v_1, \cdots, v_n, u_1, \cdots, u_m$ is a basis of V, and thus it is L.I.. So, the only way to make Equation (8) hold is by taking $c_1 = \cdots = c_n = -d_1 = \cdots = -d_m = 0$. Therefore, we've shown $Tv_1, \cdots, Tv_n$ is L.I., and thus is a basis of $\operatorname{range} T$. Then, $\dim \operatorname{range} T = n$.

So, we've shown that $\dim \operatorname{null} T + \dim \operatorname{range} T = m + n = \dim V$. $\blacksquare$

Theorem 3.2.10
Suppose V and W are *f-d* vector spaces *s.t.* $\dim V > \dim W$. Then, no linear map from V to W is injective.

Proof 5. Let $T \in \mathcal{L}(V, W)$. By the Fundamental Theorem of Linear Maps, we have $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Then, we know

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W > 0 \quad [\dim \operatorname{range} T \leq \dim W] \end{aligned}$$

This implies that $\operatorname{null} T \neq \{0\}$. So, T is not injective by Theorem 3.2.5. ■

Theorem 3.2.11
Suppose V and W are *f-d* vector space *s.t.* $\dim V < \dim W$. Then, no linear map from V to W is surjective.

Proof 6. We know

$$\begin{aligned}\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &\leq \dim V < \dim W\end{aligned}$$

Then, T cannot be surjective by definition. ■

Example 3.2.12 Solving Linear Systems Using Linear Maps I
For a homogenous system of linear equations,

$$\begin{cases} A_{1,1}x_1 + \cdots + A_{1,n}x_n = 0 \\ \qquad\vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n = 0 \end{cases},$$

where $A_{j,k} \in \mathbb{F}$ and $(x_1, \cdots, x_n) \in \mathbb{F}^n$, we can defined a linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ as

$$T(x_1, \cdots, x_n) = \left(\sum_{k=1}^{n} A_{1,k}x_k, \cdots, \sum_{k=1}^{n} A_{m,k}x_k \right).$$

Apparently, $(x_1, \cdots, x_n) = 0$ is a solution to the system, but the question is "If there are any non-zero solutions for this linear system?"

Theorem 3.2.13
A homogeneous system of linear equations with more variables than equations has non-zero solutions.

Proof 7. Suppose $T \in \mathcal{L}(V, W)$. Then, $\dim V = n$ and $\dim W = m$. Suppose $n > m$. So, $\dim V > \dim W$. By the Theorem 3.2.5, we know T is not injective. ■

Example 3.2.14 Solving Linear Systems Using Linear Maps II
For an inhomogeneous system of linear equations

$$\begin{cases} \displaystyle\sum_{k=1}^{n} A_{1,k}x_k = c_1 \\ \qquad\vdots \\ \displaystyle\sum_{k=1}^{n} A_{m,k}x_k = c_m \end{cases},$$

where $A_{j,k} \in \mathbb{F}$ and $(c_1, \cdots, c_m) \in \mathbb{F}^m$ and $(x_1, \cdots, x_n) \in \mathbb{F}^n$, we can define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \cdots, x_m) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \cdots, \sum_{k=1}^{n} A_{m,k} x_k = c_1 \right).$$

However, in this case, $(x_1, \cdots, x_n) = 0$ may not be a solution to the system.

Theorem 3.2.15
An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof 8. Suppose $T \in \mathcal{L}(V, W)$. So, $\dim V = n$ and $\dim W = m$. Suppose $n < m$. Then, $\dim V < \dim W$. By Theorem 3.2.11, we know T is not surjective. ■

3.3 Matrices

Definition 3.3.1 (Matrix). Let $m, n \in \mathbb{Z}^+$. An *m-by-n matrix* A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

Definition 3.3.2 (Matrix of a Linear Map). Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W. The *matrix of* T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ whose $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \cdots, v_n), (w_1, \cdots, w_m))$ is used.

Example 3.3.3 Suppose $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ is defined by $T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$. Find the matrix of T with respect to the standard bases of $\mathbb{F}^2$ and $\mathbb{F}^3$.

Answer 1.

Note that $T(1, 0) = (1, 2, 7)$ and $T(0, 1) = (3, 5, 9)$. Then,

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}.$$

□

Example 3.3.4 Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find the matrix of D with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$.

Answer 2.

Standard bases of $\mathcal{P}_3(\mathbb{R}) : 1, x, x^2, x^3$. Standard bases of $\mathcal{P}_2(\mathbb{R}) : 1, x, x^2$. Since $(x^n)' = nx^{n-1}$, so we have

$$\begin{aligned} D(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ D(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ D(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\ D(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 \end{aligned}$$

So, we have

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

□

Definition 3.3.5 (Matrix Addition). The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}.$$

Theorem 3.3.6
Suppose $S, T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Proof 3. Let $v_1, \cdots, v_n$ be a basis of V and $w_1, \cdots, w_n$ be a basis of W. Suppose $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = C$. Then, if $1 \leq k \leq n$, we have

$$\begin{aligned} (S+T)v_k &= Sv_k + Tv_k \\ &= (A_{1,k}w_1 + \cdots + A_{m,k}w_m) + (C_{1,k}w_1 + \cdots + C_{m,k}w_m) \\ &= (A_{1,k} + C_{1,k})w_1 + \cdots + (A_{m,k} + C_{m,k})w_m. \end{aligned}$$

Hence, we have $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$. ■

Definition 3.3.7 (Scalar Multiplication of a Matrix). The *product of a scalar and a matrix* is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$.

Theorem 3.3.8
Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

Proof 4. Let $v_1, \cdots, v_n$ be a basis of V and $\mathcal{M}(T) = A$. When $1 \leq k \leq v$, note that

$$\begin{aligned} (\lambda T)v_k &= \lambda(Tv_k) \\ &= \lambda(A_{1,k}w_1 + \cdots + A_{m,k}w_m) \\ &= (\lambda A_{1,k})w_1 + \cdots + (\lambda A_{m,k})w_m. \end{aligned}$$

So, $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$. ■

Notation 3.3.9. $\mathbb{F}^{m,n} :=$ the set of all $m \times n$ matrices with entries in $\mathbb{F}$.

Theorem 3.3.10
Suppose $m, n \in \mathbb{Z}^+$. With addition and scalar multiplication defined above, $\mathbb{F}^{m,n}$ is a vector space and $\dim \mathbb{F}^{m,n} = mn$.

Proof 5. It is trivial to prove $\mathbb{F}^{m,n}$ is a vector space. □

Define $A_{j,k}$ as the matrix with 1 on its j^{th} row, k^{th} column and 0 elsewhere. Then, we can see that $A_{j,k}$ for $j = 1, \cdots, m$ and $k = 1, \cdots, n$ is a basis for $\mathbb{F}^{m,n}$. So, $\dim \mathbb{F}^{m,n} = m \cdot n$. ■

Definition 3.3.11 (Matrix Multiplication). Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then,

AC is defined to be the $m \times p$ matrix whose entry in row j. column k is given by

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

Remark. *Matrix multiplication is not commutative. i.e., $AC \neq CA$. However, it is distributive and associative.*

Theorem 3.3.12
If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Notation 3.3.13. Suppose A is an $m \times n$ matrix.

1. If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the $1 \times n$ matrix consisting of row j of A.
2. If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the $m \times 1$ matrix consisting of column k of A.

In other words,

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}; \qquad A_{j,\cdot} = \begin{pmatrix} A_{j,1} & \cdots & A_{j,n} \end{pmatrix} \in \mathbb{F}^{1,n}; \qquad A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix} \in \mathbb{F}^{m,1}.$$

Theorem 3.3.14 Practical Interpretations of Matrix Multiplication

1. Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then, $(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$ for $1 \leq j \leq m$ and $1 \leq k \leq p$.
2. Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then, $(AC)_{\cdot,k} = AC_{\cdot,k}$ for $1 \leq k \leq p$.
3. Suppose A is an $m \times n$ matrix and $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an $n \times 1$ matrix. Then,

$$AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}.$$

In other words, AC is a linear combination of the columns of A, with the scalars that multiply the columns coming from C.

Example 3.3.15

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}.$$

3.4 Invertibility and Isomorphic Vector Spaces

Definition 3.4.1 (Invertible). A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if $\exists$ a linear map $S \in \mathcal{L}(W, V)$ *s.t.* ST equals the identity map on I and TS equals the identity map on W.
Definition 3.4.2 (Inverse). A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an *inverse* of T.

Theorem 3.4.3
An invertible linear map has a unique inverse.

Proof 1. Suppose $T \in \mathcal{L}(V, W)$ is invertible. Let S_1 and S_2 be inverses of T.Then,

$$S_1 = S_1 I = S_1(TS_2) = (S_1 T)S_2 = IS_2 = S_2.$$

Thus, $S_1 = S_2$, and so inverse is unique. ■
Notation 3.4.4. If T is invertible, then its inverse is denoted by T^{-1}.

Theorem 3.4.5
A linear map is invertible if and only if it is injective and surjective.

Proof 2.
($\Rightarrow$) Let $T \in \mathcal{L}(V, W)$ be invertible. Then, $TT^{-1} = I_W$ and $T^{-1}T = T_V$. Let $Tv = 0$. Note that $(T^{-1}T)v = 0$, so $Iv = 0$ and thus $v = 0$. Therefore, $\operatorname{null} T = \{0\}$, and so T is an injection.
To show T is surjective, suppose $w \in W$. Note that since $T^{-1} \in \mathcal{L}(W, V)$, $T^{-1}w \in V$. So,

$$T(T^{-1}w) = (TT^{-1})w = T_W w = w \in W.$$

Therefore, $T^{-1}w$ is the $v \in V$ we intend to find. Hence, T is also a surjection. □
($\Leftarrow$) Let T be surjective and injective. For $w \in W$, define $Sw \in V$ *s.t.* $T(Sw) = w$. So, we know Sw is unique. Since $(T \circ S)w = w$, we know $(T \circ S) = I_W$. Consider $(S \circ T)v = S(Tv)$, we have $T(S(Tv)) = Tv$, by definition of S. Since T is injective, we know $S(Tv) = V$. So, $(S \circ T)v = v$, and thus $ST = T_V$. Therefore T is invertible.
Now, we want to show S is a linear map. Let $w_1, w_2 \in W$, then

$$T(S(w_1 + w_2)) = (TS)(w_1 + w_2) = I_W(w_1 + w_2) = w_1 + w_2.$$

By definition, $w_1 + w_2 = T(Sw_1) + T(Sw_3) = T(Sw_1 + Sw_2)$. So, $T(S(w_1 + w_2)) = T(Sw_1 + Sw_2)$. By T is an injection, we have $S(w_1 + w_2) = Sw_1 + Sw_2$. So, S is additive. Further consider

$$T(S(\lambda w)) = \lambda w = \lambda(T(Sw)) = T(\lambda Sw)$$

for some $w \in W$. Again, since T is injective, $S(\lambda w) = \lambda Sw$. So, S has homogeneity. Then, S is a linear map. ■
Definition 3.4.6 (Isomorphism). An *isomorphism* is an invertible linear map.
Definition 3.4.7 (Isomorphic). Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.
Notation 3.4.8. If two vector spaces V and W are isomorphic, we denote them as $V \cong W$.

Theorem 3.4.9
Suppose V and W are *f-d* vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Proof 3.
($\Rightarrow$) Suppose $V \cong W$. By Fundamental Theorem of Linear Maps, we know

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Since $V \cong W$, T is invertible and thus is injective and surjective. So, $\dim \operatorname{null} T = 0$ and $\dim \operatorname{range} T = \dim W$. Therefore, $\dim V = 0 + \dim W = \dim W$. $\square$

($\Leftarrow$) Suppose $\dim V = \dim W$. Suppose $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are bases of V and W, respectively. Then, $\dim V = \dim W = n$. Here, we want to define a bijection between V and W. Let T be defined as $Tv_i = wi \quad (i = 1, \cdots, n)$.

Let $Tv = 0$. Then, $T(a_1v_1 + \cdots + a_nv_n) = 0$. So, by definition, $a_1w_1 + \cdots + a_nw_n = 0$. Since $w_1, \cdots, w_n$ is a basis, we have $a_1 = \cdots = a_n = 0$. So, $\operatorname{null} T = \{0\}$, and thus T is an injection.

Let $w \in W$ be any vector. Then, we know $w = c_1w_1 + \cdots + c_nw_n$. Note that, by definition of T, we have $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$. Hence, $\forall w \in W, \exists v = c_1v_1 + \cdots + c_nv_n \in V$ *s.t.* $Tv = w$. Therefore, T is a surjection.

Finally, it is trivial to show that T is indeed a linear map, and so the proof is complete. ■

Theorem 3.4.10
Suppose $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W. then, $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

Proof 4. We already know $\mathcal{M}$ is linear, so we just need to show $\mathcal{M}$ is a bijection.

To prove $\mathcal{M}$ is injective, consider $\mathcal{M}(T) = 0$ for some $T \in \mathcal{L}(V, W)$. So, we get $Tv_k = 0$. Since $v_1, \cdots, v_n$ is a basis of V, we know $Tv = 0 \quad \forall v \in V$. Then, T is the zero-mapping, or $T = 0$. Therefore, $\operatorname{null} \mathcal{M} = \{0\}$.

To show $\mathcal{M}$ is surjective, suppose $A \in \mathbb{F}^{m,n}$. Let T be a linear map from V to W *s.t.*

$$Tv_k = \sum_{j=1}^{m} A_{j,k}w_j, \quad k = 1, \cdots, n.$$

Obviously, $\mathcal{M}(T) = A$, and thus $\operatorname{range} \mathcal{M} = \mathbb{F}^{m,n}$. So, $\mathcal{M}$ is also a surjection. ■

Theorem 3.4.11
Suppose V and W are *f-d*. Then, $\mathcal{L}(V, W)$ is *f-d* and $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

Proof 5. By Theorem 3.4.10 and Theorem 3.4.9, we know $\dim \mathcal{L}(V, W) = \dim \mathbb{F}^{m,n}$. Further by Theorem 3.3.10, we know $\dim \mathbb{F}^{m,n} = (m)(n)$. As $\dim V = n$ and $\dim W = m$, so we have

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W).$$

■

Definition 3.4.12 (Matrix of a Vector, $\mathcal{M}(v)$). Suppose $v \in V$ and $v_1, \cdots, v_n$ is a basis of V. The *matrix* *of* v with respect to this basis is the $n \times 1$ matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \cdots, c_n$ are scalars *s.t.* $v = c_1 v_1 + \cdots + c_n v_n$.

Theorem 3.4.13 $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(v_k)$
Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W. Let $1 \leq k \leq n$. Then, the k^{th} column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(v_k)$.

Proof 6. This theorem is an immediate result by definitions of matrix of a linear mapping and a vector. ■

Theorem 3.4.14
Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W. Then, $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$.

Proof 7. Note that $v = c_1 v_1 + \cdots + c_n v_n$, so we have $Tv = c_1 T v_1 + \cdots + c_n T v_n$. So, by Theorem 3.4.13, we know

$$\begin{aligned} \mathcal{M}(Tv) &= c_1 \mathcal{M}(Tv_1) + \cdots + c_n \mathcal{M}(Tv_n) \\ &= c_1 \mathcal{M}(T)_{\cdot,1} + \cdots + c_n \mathcal{M}(T)_{\cdot,n} \\ &= \mathcal{M}(T)\mathcal{M}(v). \end{aligned}$$

The final equality holds due to our interpretation of matrix multiplication as column linear combinations (Theorem 3.3.14(3)) ■

Remark. $\mathcal{M} : \mathbb{F}^n \to \mathbb{F}^{n,1}$ *is an isomorphism:*

$$v = c_1 v_1 + \cdots + c_n v_n \longmapsto \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Proof 8. Suppose $\mathcal{M}(v) = 0 : \ \mathcal{M}(c_1 v_1 + \cdots + c_n v_n) = 0$. So, we have $c_1 w_1 + \cdots + c_n w_n = 0$. Since $w_1, \cdots, w_n$ is a basis, $c_1 = \cdots = c_n = 0$. So, $v = 0$. Therefore, $\operatorname{null} \mathcal{M} = \{0\}$, and so $\mathcal{M}$ is injective. □

Now, prove $\mathcal{M}$ is surjective. Note that $\forall \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, we have $\mathcal{M}(c_1 v_1 + \cdots + c_n v_n) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. So, $\mathcal{M}$ is a surjection. □

Finally, its' trivial to prove $\mathcal{M}$ is a linear map. □

Since $\mathcal{M}$ is both surjective and injective, $\mathcal{M}$ is an isomorphism. ■

Definition 3.4.15 (Operator). A linear map from a vector space to itself is called an *operator*.

Notation 3.4.16. The notation $\mathcal{L}(V)$ denotes the set of all operators on V. So, $\mathcal{L}(v) = \mathcal{L}(V, V)$.

Theorem 3.4.17
Suppose V is *f-d* and $T \in \mathcal{L}(V)$. Then, the following are equivalent: (a) T is invertible; (b) T is injective; and (c) T is surjective.

Proof 9.

1. Clearly (a) implies (b). □

2. Suppose (b): T is injective. So, $\operatorname{null} T = \{0\}$. Then, by Fundamental Theorem of Linear Maps, we know
$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = 0 + \dim \operatorname{range} T.$$
Since $\dim \operatorname{range} T = \dim V$, we know T is surjective. □

3. Suppose (c): T is surjective. So, $\operatorname{range} T = V$. Then, by Fundamental Theorem of Linear maps, we have
$$\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T = 0.$$
So, $\operatorname{null} T = \{0\}$, and thus T is injective. Since T is surjective and injective, T is invertible.

■

Example 3.4.18 Show that for each polynomial $q \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbb{F})$ such that $((x^2 + 5x + 7)p)'' = q$.

Proof 10. We know that every non-zero polynomial must have a degree of m. So, we can think of this problem under $\mathcal{P}_m(\mathbb{R})$. Note that

$$((x^2 + 5x + 7)p)'' = 2p + (4x + 10)p' + (x^2 + 5x + 7)p'' = q.$$

Therefore, the degree of p and q should be the same. Define $T : \mathcal{P}_m(\mathbb{R}) \to \mathcal{P}_m(\mathbb{R})$ as

$$Tp = ((x^2 + 5x + 7)p)''.$$

Then, T is an operator on $\mathcal{P}_m(\mathbb{R})$. Consider $Tp = 0$. We have $ax + b = (x^2 + 5x + 7)p$. Note that only when $p = 0$, the equation above holds. So, it must be that $p = 0$ when $Tp = 0$. That is, $\operatorname{null} T = \{0\}$, and so T is injective. By Theorem 3.4.18, we know T is also surjective, and so our proof is complete. ■

3.5 Duality

Definition 3.5.1 (Linear Functional). A *linear functional* on V is a linear map from V to $\mathbb{F}$. That is, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

Example 3.5.2

1. Fix $(c_1, \cdots, c_n) \in \mathbb{F}^n$. Define $\varphi : \mathbb{F}^n \to \mathbb{F}$ by $\varphi(x_1, \cdots, x_n) = c_1 x_1 + \cdots + c_n x_n$. Then, φ is a linear functional on $\mathbb{F}^n$.
2. Define $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ as $\varphi(p) = 3p''(5) + 7p(4)$.
3. Define $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ as $\varphi(p) = \displaystyle\int_0^1 p(x)\mathrm{d}x$.

Definition 3.5.3 (Dual Space/V'/V^*). The *dual space* of V, denoted as V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

Theorem 3.5.4
Suppose V is *f-d*. Then, V' is also *f-d* and $\dim V' = \dim V$.

Proof 1. Note that for a general linear map, $\mathcal{L}(V, W) \cong \mathbb{F}^{m,n}$. So, $\mathcal{L}(V, \mathbb{F}) = V' \cong \mathbb{F}^{1,n}$. Hence,

$$\dim V' = \dim \mathbb{F}^{1,n} = 1 \cdot n = n = \dim V.$$

■

Definition 3.5.5 (Dual Basis). If $v_1, \cdots, v_n$ is a basis of V, then the *dual basis* of $v_1, \cdots, v_n$ is the list $\varphi_1, \cdots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V *s.t.*

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}.$$

Example 3.5.6 Find the dual basis of $e_1, \cdots, e_n \in \mathbb{F}^n$

Answer 2.

$$\begin{array}{cccc} \varphi_1(e_1) = 1 & \varphi_2(e_1) = 0 & \cdots & \varphi_n(e_1) = 0 \\ \varphi_1(e_2) = 0 & \varphi_2(e_2) = 1 & \cdots & \varphi_n(e_2) = 0 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(e_n) = 1 & \varphi_2(e_n) = 0 & \cdots & \varphi_n(e_n) = 1 \end{array}$$

Define φ_j as

$$\varphi_j(x) = \varphi_j(x_1, \cdots, x_n) = x_1\varphi_j(e_1) + \cdots + x_j\varphi_j(e_j) + \cdots + x_n\varphi_j(e_n) = x_j.$$

□

Theorem 3.5.7
Suppose V is *f-d*. Then, the dual basis of a basis of V is a basis of V'.

Proof 3. Suppose $v_1, \cdots, v_n$ is a basis of V and $\varphi_1, \cdots, \varphi_n$ denotes the dual basis. Since we've shown $\dim V = \dim V'$ in Theorem 3.5.4, we only need to show $\varphi_1, \cdots, \varphi_n$ is L.I.. Select $c_1\varphi_1 + \cdots + c_n\varphi_n = 0$. Then,

$$(c_1\varphi_1 + \cdots + c_n\varphi_n)(v) = 0 \quad \forall\, v \in V.$$

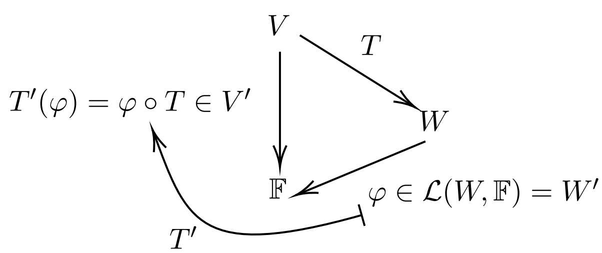
Suppose $v = v_1 + \cdots + v_n$, then

$$(c_1\varphi_1 + \cdots + c_n\varphi_n)(v_j) = c_j \quad \text{for } j = 1, \cdots, n.$$

So, $(c_1\varphi_1 + \cdots + c_n\varphi_n)(v) = c_1 + \cdots + c_n = 0$. So, it must be that $c_1 = \cdots = c_n = 0$. Therefore, $\varphi_1, \cdots, \varphi_n$ is L.I. and our proof is complete. ■

Definition 3.5.8 (Dual Map). If $T \in \mathcal{L}(V, W)$, then the *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.

Remark. *The following diagram represents dual map (but not an exact representation).*



Also, dual map is a linear map, so it is additive and homogeneous.

1. $T'(\varphi + \psi) = (\varphi + \psi) \circ T = \varphi \circ T + \psi \circ T = T'(\varphi) + T'(\psi)$.
2. $T'(\lambda\varphi) = (\lambda\varphi) \circ T = \lambda(\varphi \circ T) = \lambda T'(\varphi)$.

Example 3.5.9 Suppose $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ as $Dp = p'$.

1. Define a linear functional $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ as $\varphi(p) = p(3)$. Find $D'(\varphi)$.

   Answer 4.

   $$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = p'(3).$$

   □

2. Define $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, a linear functional, as $\varphi(p) = \displaystyle\int_0^1 p(x)\,\mathrm{d}x$. Find $D'(\varphi)$.

   Answer 5.

   $$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = \int_0^1 p'(x)\,\mathrm{d}x = p(1) - p(0).$$

   □

Theorem 3.5.10 Algebraic Properties of Dual Maps

1. $(S+T)' = S' + T' \quad \forall\, S, T \in \mathcal{L}(V, W)$
2. $(\lambda T)' = \lambda T' \quad \forall\, T \in \mathcal{L}(V, W)$
3. $(ST)' = T'S' \quad \forall\, T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$

Proof 6.

1. $(S+T)' \in \mathcal{L}(W', V')$. Let $\varphi \in W'$. Then,
$$(S+T)'(\varphi) = \varphi \circ (S+T) = \varphi \circ S + \varphi \circ T = S'(\varphi) + T'(\varphi) = (S' + T')(\varphi). \qquad \square$$
2. $(\lambda T)' \in \mathcal{L}(W', V')$. Let $\varphi \in W'$. Then,
$$(\lambda T)'(\varphi) = \varphi \circ (\lambda T) = \lambda(\varphi \circ T) = \lambda T'(\varphi) = (\lambda T')(\varphi). \qquad \square$$
3. $(ST)' \in \mathcal{L}(W', U')$. Let $\varphi \in W'$. Then,
$$(ST)'(\varphi) = \varphi \circ (ST) = \varphi \circ (S \circ T) = (\varphi \circ S) \circ T = (S'(\varphi)) \circ T = T'(S'(\varphi)) = (T'S')(\varphi).$$

■

Definition 3.5.11 (Transpose/A^t). The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows and columns. i.e., $(A^t)_{k,j} = A_{j,k}$.

Remark. *Transpose is additive and homogeneous. That is,* $(A+C)^t = A^t + C^t$ *and* $(\lambda A)^t = \lambda A^t$.

Theorem 3.5.12
If A is an $m \times n$ matrix and C is an $n \times p$ matrix, then $(AC)^t = C^t A^t$.

Proof 7. Note that

$$(AC)^t_{k,j} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k} = \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j} = (C^t A^t)_{k,j}$$

■

Theorem 3.5.13
Suppose $T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof 8. Suppose $v_1, \cdots, v_n$ is a basis of V, $w_1, \cdots, w_m$ is a basis of W, $\varphi_1, \cdots, \varphi_n$ is a basis of V', and $\psi_1, \cdots, \psi_m$ is a basis of W'. Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Since $T'(\psi_j) = C_{1,j}\varphi_1 + \cdots + C_{n,j}\varphi_n$ and $T'(\psi_j) = \psi_j \circ T$, we have $\psi_j \circ T = C_{1,j}\varphi_1 + \cdots + C_{n,j}\varphi_n$. Consider

$$(\psi_j \circ T)(v_k) = (C_{1,j}\varphi_1 + \cdots + C_{n,j}\varphi_n)(v_k) = C_{k,j}\varphi_k(v_k) = C_{k,j}.$$

Also, we have

$$(\psi_j \circ T)(v_k) = \psi_j(Tv_k) = \psi_j(A_{1,k}w_1 + \cdots + A_{m,k}w_m) = \psi_j(A_{j,k}w_j) = A_{j,k}(\varphi_j(w_j)) = A_{j,k}.$$

Therefore, we have $A_{j,k} = C_{k,j}$, and thus $A = C^t$. So, $\mathcal{M}(T) = (\mathcal{M}(T'))^t$. ■

Definition 3.5.14 (Annihilator/U^0). For $U \subseteq V$, the *annihilator* of U, denoted as U^0, is defined by

$$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \quad \forall u \in U\}.$$

Theorem 3.5.15
Suppose $U \subseteq V$. Then U^0 is a subspace of V'.

Proof 9.

1. $0 \in U^0$: Since $0(u) = 0 \quad \forall u \in U$, then $0 \in U^0$. □

2. Let $\varphi, \psi \in U^0$. Then,
$$(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0.$$
So, $\varphi + \psi \in U^0$. □

3. Let $\lambda \in \mathbb{F}$ and $\varphi \in U^0$. Then
$$(\lambda\varphi)(u) = \lambda\varphi(u) = \lambda \cdot 0 = 0.$$
So, $\lambda\varphi \in U^0$.

■

Lemma 3.5.16 Suppose V is *f-d* vector space. If U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ *s.t.* $Tu = Su \quad \forall u \in U$.

Proof 10. Suppose $u_1, \cdots, u_m$ is a basis of U. Then, we can extend it to a basis of V as $u_1, \cdots, u_m, v_{m+1}, \cdots, v_n$. Define $T \in \mathcal{L}(V, W)$ as $Tu_i = Su_i$, $Tv_j = 0$, where $i = 1, \cdots, m$ and $j = m+1, \cdots, n$. Note that

$$\begin{aligned} Tu &= T(a_1u_1 + \cdots + a_mu_m) \\ &= a_1Tu_1 + \cdots + a_mTu_m \\ &= a_1Su_1 + \cdots + a_mSu_m \\ &= S(a_1u_1 + \cdots + a_mu_m) = Su. \end{aligned}$$

Therefore, we've found such a T. ■

Theorem 3.5.17
Let V be *f-d* and U be a subspace of V, then $\dim U + \dim U^0 = \dim V$.

Proof 11. Let $i \in \mathcal{L}(U, V)$ as $i(u) = u \quad \forall u \in U$. Then, $i' \in \mathcal{L}(V', U')$. So, by Fundamental Theorem of Linear Map, we know

$$\dim V' = \dim \operatorname{null} i' + \dim \operatorname{range} i'. \tag{9}$$

By Theorem 3.5.4, we know $\dim V = \dim V'$ Note that $U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \quad \forall u \in U\}$ and

$$\begin{aligned} \operatorname{null} i' &= \{\varphi \in V' \mid i'(\varphi) = 0\} \\ &= \{\varphi \in V' \mid \varphi \circ i = 0\} \\ &= \{\varphi \in V' \mid (\varphi \circ i)(u) = 0 \quad \forall u \in U\} \\ &= \{\varphi \in V' \mid \varphi(u) = 0 \quad \forall u \in U\} \end{aligned}$$

So, $U^0 = \operatorname{null} i'$, and thus $\dim \operatorname{null} i' = \dim U^0$.

Further, if $\varphi \in U'$, then $\varphi : U \to \mathbb{F}$. By Lemma 3.5.16, φ can be extended to $\psi \in V'$ with $\psi(u) = \varphi(u) \quad \forall u \in U$. Note that $i'(\psi) = \psi \circ i$, so $(\psi \circ i)(u) = \psi(u) = \varphi(u) \quad \forall u \in U$. Then, $\exists \psi \in V'$ *s.t.* $i'(\psi) = \varphi$. So, $\varphi \in \operatorname{range} U'$. So, $\dim \operatorname{range} i' = \dim U' = \dim U$.

Substitute $\dim V' = \dim V$, $\dim \operatorname{null} i' = \dim U^0$, and $\dim \operatorname{range} i' = \dim U$ to Equation (9), we get

$$\dim V = \dim U^0 + \dim U.$$

■

Theorem 3.5.18 The Null Space of T'
Suppose V and W are *f-d* and $T \in \mathcal{L}(V, W)$. Then,

1. $\operatorname{null} T' = (\operatorname{range} T)^0$
2. $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

Proof 12.

1. ($\subseteq$) Suppose $\varphi \in \operatorname{null} T' \subseteq W'$. Then, $T'(\varphi) = \varphi \circ T = 0 \in V'$. So, we know

   $$(\varphi \circ T)(v) = 0 \quad \forall v \in V. \quad \text{i.e., } \varphi(Tv) = 0.$$

   Note that $Tv \in \operatorname{range} T$. By definition, we have $\varphi \in (\operatorname{range} T)^0$ □

   ($\supseteq$) Suppose $\varphi \in (\operatorname{range} T)^0$. Then, $\varphi(w) = 0 \quad \forall w \in \operatorname{range} T$. That is, $\varphi(Tv) = 0 \quad \forall v \in V$. So, $(\varphi \circ T)(v) = 0 \quad \forall v \in V$. Hence, we know $\varphi \circ T = T'(\varphi) = 0 \in V'$. Thus, $\varphi \in \operatorname{null} T'$ ■

2. $$\begin{aligned} \dim \operatorname{null} T' &= \dim (\operatorname{range} T)^0 \\ &= \dim W - \dim \operatorname{range} T \\ &= \dim W - (\dim V - \dim \operatorname{null} T) \\ &= \dim W - \dim V + \dim \operatorname{null} T. \end{aligned}$$

■

Theorem 3.5.19
Suppose V and W are *f-d* and $T \in \mathcal{L}(V, W)$. Then, T is surjective if and only if T' is injective.

Proof 13.
($\Rightarrow$) Suppose T is surjective. Then, $\dim \operatorname{range} T = W$. So, $(\operatorname{range} T)^0 = \{0\}$. Hence,

$$\dim \operatorname{null} T' = \dim (\operatorname{range} T)^0 = 0.$$

Thus, T' is injective. $\square$

($\Leftarrow$) Suppose T' is injective. Then,

$$\dim \operatorname{null} T' = 0.$$

So, $\dim(\operatorname{range} T)^0 = \dim \operatorname{null} T' = 0$. Then, $(\operatorname{range} T)^0 = \{0\}$. So, $\dim \operatorname{range} T = W$, and thus T is surjective. $\blacksquare$

Theorem 3.5.20 The Range of T'
Suppose V and W are *f-d* and $T \in \mathcal{L}(V, W)$. Then,

1. $\dim \operatorname{range} T' = \dim \operatorname{range} T$
2. $\operatorname{range} T' = (\operatorname{null} T)^0$

Proof 14.

1. By Fundamental Theorem of Linear Map, we have

$$\begin{aligned} \dim \operatorname{range} T' &= \dim W' - \dim \operatorname{null} T' \\ &= \dim W' - \dim(\operatorname{range} T)^0 \\ &= \dim W' - \dim W' + \dim \operatorname{range} T \\ &= \dim \operatorname{range} T. \end{aligned}$$

$\blacksquare$

2. Suppose $\varphi \in \operatorname{range} T' \subseteq V'$. Then, $\exists \psi \in W'$ *s.t.* $T'(\psi) = \psi \circ T = \varphi$. Let $v \in \operatorname{null} T$. Then,

$$\varphi(v) = (\psi \circ T)(v) = \psi(Tv) = \psi(0) = 0.$$

Then, $\varphi \in (\operatorname{null} T)^0$. So, $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. $\square$

Note that

$$\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V - \dim \operatorname{null} T = \dim(\operatorname{null} T)^0.$$

Then, $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$ and $\dim \operatorname{range} T' = \dim(\operatorname{null} T)^0$, so it must be that $\operatorname{range} T' = (\operatorname{null} T)^0$.

$\blacksquare$

Theorem 3.5.21
Suppose V and W are *f-d* and $T \in \mathcal{L}(V, W)$. Then, T is injective if and only if T' is surjective.

Proof 15.

($\Rightarrow$) If T is injective, $\operatorname{null} T = \{0\}$. So,

$$\dim \operatorname{null} T = \dim V - \dim(\operatorname{null} T)^0 = \dim V - \dim \operatorname{range} T' = 0.$$

So, $\dim \operatorname{range} T' = \dim V = \dim V'$. Then, T' is surjective. $\square$

($\Leftarrow$) If T' is surjective, $\dim \operatorname{range} T' = \dim V' = \dim V$. So,

$$\dim \operatorname{null} T = \dim V - \dim(\operatorname{null} T)^0 = \dim V - \dim \operatorname{range} T' = 0.$$

Then, $\operatorname{null} T = \{0\}$, and so T is injective. $\blacksquare$

Definition 3.5.22 (Row Rank & Column Rank). Suppose A is an $m \times n$ matrix with entries in $\mathbb{F}$.

1. The *row rank* of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$.
2. The *column rank* of A is the dimension of the span of the columns of A in $\mathbb{F}^{m,1}$.

Theorem 3.5.23
Suppose V and W are *f-d* and $T \in \mathcal{L}(V, W)$. Then, $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$.

Proof 16. Suppose $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W. Then,

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

and thus

$$\mathcal{M}(Tv_k) = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix} \in \mathbb{F}^{m,1}$$

Therefore, $\mathcal{M}(T) = \begin{pmatrix} \mathcal{M}(Tv_1) & \cdots & \mathcal{M}(Tv_n) \end{pmatrix}$. Note that $\operatorname{range} T = \operatorname{span}(Tv_1, \cdots, Tv_n)$.

Define $\mathcal{M} : \operatorname{span}(Tv_1, \cdots, Tv_n) \to \operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n))$ as $w \mapsto \mathcal{M}(w)$.

1. $\mathcal{M}$ is surjective: Note that

   $$c_1\mathcal{M}(Tv_1) + \cdots + c_n\mathcal{M}(Tv_n) = \mathcal{M}(c_1Tv_1 + \cdots + c_nTv_n).$$

   Since $c_1Tv_1 + \cdots + c_nTv_n \in \operatorname{range} T$, we know $\mathcal{M}$ is surjective. $\square$

2. $\mathcal{M}$ is injective: Let

   $$\mathcal{M}(c_1Tv_1 + \cdots + c_nTv_n) = 0. \tag{10}$$

   We can reduce $c_1Tv_1 + \cdots + c_nTv_n$ to a basis $Tv_{j_1}, \cdots, Tv_{j_m}$. Then, Equation (10) becomes $\mathcal{M}(a_1Tv_{j_1} + \cdots + a_mTv_{j_m}) = 0$. By definition of matrix, we know $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = 0$. So, $a_1 = \cdots = a_m = 0$ and $a_1Tv_{j_1} + \cdots + a_mTv_{j_m} = 0$. So, $\mathcal{M}$ is injective. $\square$

Since $\mathcal{M}$ is both surjective and injective, $\mathcal{M}$ is a bijection. Thus, $\mathcal{M}$ is an isomorphism between $\operatorname{span}(Tv_1, \cdots, Tv_n)$ and $\operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n))$. In other words,

$$\operatorname{span}(Tv_1, \cdots, Tv_n) \cong \operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n)).$$

Then, $\dim \operatorname{span}(Tv_1, \cdots, Tv_n) = \dim \operatorname{span}(\mathcal{M}(Tv_1), \cdots, \mathcal{M}(Tv_n))$. That is,

$$\dim \operatorname{range} T = \text{column rank of } T.$$

■

Theorem 3.5.24 Row Rank Equals Column Rank
Suppose $A \in \mathbb{F}^{m,n}$. Then, the row rank of A equals the column rank of A.

Proof 17. Define $T : \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$. Then, $\mathcal{M}(T) = A$, where $\mathcal{M}(T)$ is computed with respect to the standard basis of $\mathbb{F}^{n,1}$ and $\mathbb{F}^{m,1}$. Note that

$$\begin{aligned}
\text{column rank of } A &= \text{column rank of } \mathcal{M}(T) \\
&= \dim \operatorname{range} T && \text{Theorem 3.5.23} \\
&= \dim \operatorname{range} T' && \text{Theorem 3.5.20(1)} \\
&= \text{column rank of } \mathcal{M}(T') \\
&= \text{column rank of } A^t && \text{Theorem 3.5.13} \\
&= \text{row rank of } A
\end{aligned}$$

■

Definition 3.5.25 (Rank). The *rank* of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of A, denoted as $\operatorname{rank} A$.

3.6 Quotients of Vector Spaces

Definition 3.6.1 ($v+U$/Affine Subset). Suppose $v \in V$ and U is a subspace of V. Then

$$v + U := \{v + u \mid u \in U\}.$$

An *affine subset* of V is a subset of V of the form $v + U$ for some $v \in V$ and some subspace U of V. The affine subset is said to be *parallel* to U.

Definition 3.6.2 (Quotient Space, V/U). Suppose U is a subspace of V. Then the quotient space V/U is the set of all affine subsets of V parallel to U. In other words,

$$V/U := \{v + U \mid v \in V\}.$$

Example 3.6.3 If $U = \left\{(x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ with slope of 2.

Theorem 3.6.4
Suppose U is a subspace of V and $v, w \in V$. Then, the following are equivalent:

1. $v - w \in U$
2. $v + U = w + U$
3. $(v + U) \cap (w + U) \neq \varnothing$

Proof 1.

1. We want to show (1) $\implies$ (2). Suppose $v - w \in U$. Note that $v + u = w + ((v - w) + u)$. Since $v - u$ and $u \in U$, we have $(v - w) + u \in U$. So, $v + u \in w + U$. Similarly, we can show that $w + u \in v + U$. Then, we have $v + U = w + U$. $\square$
2. Now, we want to show (2) $\implies$ (3): Suppose $v + U = w + U$. Then, we have $(v + U) \cap (w + U) \neq \varnothing$, which is evident from the assumption. $\square$
3. Finally, we will show (3) $\implies$ (1). Suppose $(v + U) \cap (w + U) \neq \varnothing$. Then, $\exists u_1, u_2 \in U$ *s.t.* $v + u_1 = w + u_2$. So we have $v - w = u_2 - u_1 \in U$.

■

Definition 3.6.5 (Addition & Scalar Multiplication on V/U). Suppose U is a subspace of V. Then, *addition* and *scalar multiplication* is defined on V/U by

$$(v + U) + (w + U) = (v + w) + U$$

and

$$\lambda(v + U) = (\lambda v) + U$$

for $v, w \in U$ and $\lambda \in \mathbb{F}$.

Theorem 3.6.6
Suppose U is a subspace of V. Then, V/U, with the operations of addition and scalar multiplication defined above, is a vector space.

Proof 2.

1. Addition on V/U makes sense.

   Note the addition can be written in the language of mapping as $+ : V/U \times V/U \to V/U$. So, we have $(v + U, w + U) \mapsto (v + w) + U$. Suppose $\exists\, \hat{v}.\hat{w} \in V$ *s.t.* $v + U = \hat{v} + U$ and $w + U = \hat{w} + U$. Note that $v - \hat{v} \in U$ and $w - \hat{w} \in U$ by Theorem 3.6.4. Then, $(v - \hat{v}) + (w - \hat{w}) \in U$. So, we have $(v + w) - (\hat{v} + \hat{w}) inU$. Further, by Theorem 3.6.4, we have

   $$(v + w) + U = (\hat{v} + \hat{w}) + U. \qquad \square$$

2. Scalar multiplication on V/U makes sense.

   We can write the scalar multiplication on V/U as a mapping: $\cdot : \mathbb{F} \times V/U \to V/U$ defined as $(\lambda, v + U) \mapsto \lambda v + U$. Suppose $\exists\, \hat{v} \in V$ *s.t.* $v + U = \hat{v} + U$. So we know $v - \hat{v} \in U$, and thus $\lambda(v - \hat{v}) = \lambda v - \lambda \hat{v} \in U$. By Theorem 3.6.4, we then have $(\lambda v) + U = (\lambda \hat{v}) + U$. Thus, the scalar multiplication makes sense. $\square$

3. additive identity: $0 + U = U$. $\square$

4. additive inverse: $(-v) + U$. $\square$

5. commutativity:

   $$\begin{aligned}(v + U) + (w + U) = (v + w) + U &= (w + v) + U \\ &= (w + U) + (v + U). \qquad \square\end{aligned}$$

6. associativity:

   $$\begin{aligned}[(v + U) + (w + U)] + (x + U) &= [(v + w) + U] + (x + U) \\ &= [(v + w) + x] + U \\ &= [v + (w + x)] + U \\ &= (v + U) + [(w + x) + U] \\ &= (v + U) + [(x + U) + (x + U)]. \qquad \square\end{aligned}$$

7. multiplicative identity: $1 \cdot (v + U) = (1 \cdot v) + U = v + U$. $\square$

8. distributivity:

   $$\begin{aligned}a[(v + U) + (w + U)] &= a[(v + w) + U] \\ &= a(v + w) + U \\ &= (av + aw) + U \\ &= (av + U) + (aw + U) \\ &= a(v + U) + a(w + U).\end{aligned}$$

$$\begin{aligned}(a+b)(v+U) &= (a+b)v+U \\ &= (av+bv)+U \\ &= (av+U)+(bv+U) \\ &= a(v+U)+b(v+U)\end{aligned}$$

■

Definition 3.6.7 (Quotient Map). Suppose U is a subspace of V. The *quotient map* π is the linear map $\pi : V \to V/U$ defined by $\pi(v) := v+U \quad \forall v \in V$.

Remark. *Here are some properties of the quotient map:*

1. $\pi(v)$ *is defined* $\forall v \in V$*. Thus,* π *is surjective.*
2. $\operatorname{null}\pi = \{v \in V \mid \pi(v) = 0\}$*. If* $\pi(v) = 0$*, then* $v+U = U = 0+U$*. So,* $v-0 \in U$ *by Theorem 3.6.4. Then,* $v \in U$*. So,* $\operatorname{null}\pi \subseteq U$*. Further,* $\forall v \in U$*, if* $\pi(v) = 0$*, then* $v \in \operatorname{null}\pi$*, then* $U \subseteq \operatorname{null}\pi$*. So,* $U = \operatorname{null}\pi$*.*
3. $\pi(v+w) = (v+w)+U = (v+U)+(w+U) = \pi(v)+\pi(w)$.
4. $\pi(\lambda v) = (\lambda v)+U = \lambda(v+U) = \lambda\pi(v)$.

Theorem 3.6.8
Suppose V is *f-d* and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U.$$

Proof 3. By Fundamental Theorem of Linear Map, we have

$$\dim V = \dim \operatorname{null}\pi + \dim \operatorname{range}\pi. \tag{11}$$

Since $\operatorname{null}\pi = U$ from the Remark, we have $\dim\operatorname{null}\pi = \dim U$. Further, since π is surjective as mentioned in the Remark, $\operatorname{range}\pi = V/U$. Hence, $\dim\operatorname{range}\pi = \dim V/U$. Therefore, Equation (11) becomes

$$\dim V = \dim U + \dim V/U,$$

or we have

$$\dim V/U = \dim V - \dim U$$

■

Definition 3.6.9 ($\tilde{T}$). Suppose $T \in \mathcal{L}(V,W)$. Define $\tilde{T} : V/(\operatorname{null} T) \to W$ by $\tilde{T}(v+\operatorname{null} T) - Tv$.

Proof 4.

1. This definition makes sense

   Suppose $u, v \in V$ *s.t.* $u+\operatorname{null}T = v+\operatorname{null}T$. By Theorem 3.6.4, we know $u-v \in \operatorname{null}T$. Then, $T(u-v) = 0$, or $Tu = Tv$. □

2. $\tilde{T}$ is a linear map.

$$\begin{aligned}\tilde{T}[(u+\operatorname{null}T)+(v+\operatorname{null}T)] &= \tilde{T}[(u+v)+\operatorname{null}T] \\ &= T(u+v) \\ &= Tu+Tv = \tilde{T}(u+\operatorname{null}T)+\tilde{T}(v+\operatorname{null}T). \qquad \square\end{aligned}$$

$$\begin{aligned}\tilde{T}[\lambda(u+\operatorname{null}T)] &= \tilde{T}(\lambda u+\operatorname{null}T)\\ &= T(\lambda u)\\ &= \lambda Tu\\ &= \lambda T(u+\operatorname{null}T).\end{aligned}$$

■

Theorem 3.6.10
Suppose $T \in \mathcal{L}(V,W)$. Then,

1. $\tilde{T}$ is injective.
2. $\operatorname{range}\tilde{T} = \operatorname{range}T$.
3. $V/(\operatorname{null}T) \cong \operatorname{range}T$.

Proof 5.

1. Suppose $v \in V$ and $\tilde{T}(v+\operatorname{null}T) = 0$. Then, $Tv = 0$. So, $v \in \operatorname{null}T$, or $v - 0 \in \operatorname{null}T$. By Theorem 3.6.4, we then have $v + \operatorname{null}T = 0 + \operatorname{null}T$. Then, it implies $\operatorname{null}\tilde{T} = 0$. So, $\tilde{T}$ is injective. □
2. By definition of $\tilde{T}$, it must be $\operatorname{range}\tilde{T} = \operatorname{range}T$. □
3. Note that $\dim V/(\operatorname{null}T) = \dim\operatorname{null}\tilde{T} + \dim\operatorname{range}\tilde{T} = 0 + \dim\operatorname{range}T$. Then, by Theorem 3.4.9, we know two vector spaces are isomorphic if and only if their dimensions are equal. Then,

$$V(\operatorname{null}T) \cong \operatorname{range}T.$$

■

4 Eigenvectors and Invariant Subspaces

4.1 Invariant Subspaces

Theorem 4.1.1
Suppose V is *f-d* with $\dim V = n \geq 1$. Then, $\exists\, 1-$dimensional subspaces $U_1, \cdots, U_n$ of V *s.t.*

$$V = U_1 \oplus \cdots \oplus U_n.$$

Proof 1. Choose a basis $v_1, \cdots, v_n$ of V. Then, we know $V = \text{span}(v_1) + \cdots + \text{span}(v_n)$. Also, $\forall v \in V$, we have $v = a_1 v_1 + \cdots + a_n v_n$ with $a_j v_j \in \text{span}(v_j)$. Set $a_1 v_1 + \cdots + a_n v_n = 0$. Since $v_1, \cdots, v_n$ is a basis, it must be $a_1 = \cdots = a_n = 0$. Then,

$$V = \text{span}(v_1) \oplus \cdots \oplus \text{span}(v_n).$$

■

Theorem 4.1.2
Suppose $U_1, \cdots, U_m$ are *f-d* subspaces of V *s.t.* $U_1 + \cdots + U_m$ is a direct sum. Then, $U_1 \oplus \cdots \oplus U_m$ is *f-d* and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

Proof 2. Suppose $u_{k,1}, \cdots, u_{k.j_k}$ is a basis of the subspace U_k. Then, any vector in $\bigoplus_{i=1}^{m} U_i$ is in the form of $u_1 + \cdots + u_m, \quad u_j \in U_j$. Also,

$$u_i = \sum_{k=1}^{j_i} a_{i,k} u_{i,k}.$$

So,

$$u_1 + \cdots + u_m = \sum_{k=1}^{j_1} a_{1,k} u_{1,k} + \cdots + \sum_{k=1}^{j_m} a_{m,k} u_{m,k}.$$

Then, $u_1 + \cdots + u_m$ is a linear combination of $u_{1,1}, \cdots, u_{j,m}$. So, the direct sum is *f-d*. □

Further, suppose

$$\sum_{k=1}^{j_1} a_{1,k} u_{1,k} + \cdots + \sum_{k=1}^{j_m} a_{m,k} u_{m,k} = 0.$$

Since $U_1 + \cdots + U_m$ is a direct sum, it must be

$$\sum_{k=1}^{j_1} a_{1,k} u_{1,k} = \cdots = \sum_{k=1}^{j_m} a_{a,k} u_{m,k} = 0.$$

Since we selected bases, $a_{1,k} = \cdots = a_{m,k} = 0$. So, $u_{1,1}, \cdots, u_{m,j_m}$ is a basis of $U_1 \oplus \cdots \oplus U_m$. Then,

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

■

Definition 4.1.3 (Invariant Subspace). Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if $u \in U$ implies $Tu \in U$.

Example 4.1.4 Suppose $T \in \mathcal{L}(V)$. Show that each of the following subspaces of V is invariant under T:

1. $\{0\}$

   Proof 3. $T0 = 0 \in \{0\}$ ■

2. V

   Proof 4. $u \in V \implies Tu \in V$ ■

3. $\text{null}\, T$

   Proof 5. $u \in \text{null}\, T \implies Tu = 0 \in \text{range}\, T$ ■

4. $\text{range}\, T$

   Proof 6. $u \in \text{range}\, T \implies Tu \in \text{range}\, T$ ■

Example 4.1.5 Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is defined by $Tp = p'$. Then, $\mathcal{P}_4(\mathbb{R})$ is invariant under T.

Proof 7. Note that $Tp_4) \in \mathcal{P}_4(\mathbb{R})$. Then, $\mathcal{P}_4(\mathbb{R})$ is invariant under T. ■

Definition 4.1.6 (Eigenvalue). Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an *eigenvalue* of T if $\exists\, v \in V$ *s.t.* $v \neq 0$ and $Tv = \lambda v$.

Corollary 4.1.7 T has a $1-$dimensional invariant subspace if and only if T has an eigenvalue.

Proof 8.

($\Rightarrow$) Suppose $\text{span}(v)$ is invariant under T. Let U be defined as $U = \{\lambda v \mid \lambda \in \mathbb{F}\} = \text{span}(v)$. Then. U is the invariant subspace under T and $\dim U = 1$. Then, $\forall v \in V$, we have $Tv \in U$. Hence, $\exists \lambda \in \mathbb{F}$ *s.t.* $Tv = \lambda v$. Then, λ is an eigenvalue. □

($\Leftarrow$) Suppose $\lambda \in \mathbb{F}$ is an eigenvalue. Then, $Tv = \lambda v$. Hence, $\text{span}(v)$ is a 1 =dimensional invariant subspace under T. ■

Theorem 4.1.8 Equivalent Conditions to be an Eigenvalue
Suppose V is *f-d*, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then, the following are equivalent:

1. λ is an eigenvalue of T.
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

Proof 9.

1. (1) $\implies$ (2): Suppose λ is an eigenvalue of T. Then, $\exists v \in V$ *s.t.* $v \neq 0$ and $Tv - \lambda v$. So, $Tv - \lambda v = (T - \lambda I)v = 0$. Since $v \neq 0$, $\text{null}\,(T - \lambda I) \neq \{0\}$, and thus T is not injective. □

2. Note that $T - \lambda I$ is an operator by itself. By Theorem 3.4.17, we know (2), (3), and (4) are equivalent.

3. (4) $\implies$ (1): Suppose $T - \lambda I$ is not invertible. Then, it is not injective. So, $\exists v \neq 0$ *s.t.* $(T - \lambda I)v = 0$. That is, $Tv - \lambda Iv = Tv - \lambda v = 0$. So, $Tv = \lambda v$. Then, λ is an eigenvalue of T.

■

Definition 4.1.9 (Eigenvector). Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Corollary 4.1.10 A vector $v \in V$ with $v \neq 0$ is an eigenvector of T with respect to λ if and only if $v \in \operatorname{null}(T - \lambda I)$.

Proof 10. Note that $Tv = \lambda v$ if and only if $(T - \lambda I)v = 0$. ■

Example 4.1.11 Suppose $T \in \mathcal{L}(\mathbb{F}^2)$ is defined by $T(w, z) = (-z, w)$.

1. Find the eigenvalues and eigenvectors of T if $\mathbb{F} = \mathbb{R}$.

   Answer 11.

   Let $T(2, z) = \lambda(w, z)$. So, $(-z, w) = (\lambda w, \lambda z)$. Then, solve $\begin{cases} -z = \lambda w \\ w = \lambda z \end{cases}$.

   Then, we have $\lambda^2 z + z = 0$. If $z \neq 0$, $\lambda^2 + 1 = 0$. This equation has no solutions on $\mathbb{R}$. So T has no eigenvalues. If $w = 0, z = 0$, then $T(w, z) = T(0.0) = T0$. By definition, T has no eigenvalues. □

2. Find the eigenvalues and eigenvectors of T if $\mathbb{F} = \mathbb{C}$.

   Answer 12.

   Applying similar rational, $z \neq 0$ and solve $\lambda^2 + 1 = 0$. Then, we have $\lambda = \pm \mathrm{i}$. If $\lambda = \mathrm{i}$, then $-z = \mathrm{i}w$. So, $v = (w, z) = (w, -\mathrm{i}w)$. If $\lambda - \mathrm{i}$, then $-z = -\mathrm{i}w$, or $z = \mathrm{i}w$. So, $v = (w, \mathrm{i}w)$. □

Theorem 4.1.12
Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T and $v_1, \cdots, v_m$ are corresponding eigenvectors. Then, $v_1, \cdots, v_m$ is L.I..

Proof 13. Suppose for the sake of contradiction that $v_1, \cdots, v_m$ is linearly dependent. Let k be the smallest positive integer *s.t.* $v_k \in \operatorname{span}(v_1, \cdots, v_{k-1})$. Then, $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$. Applying T, we have

$$\lambda_k v_k = a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}. \tag{12}$$

Since $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$, we also have

$$\lambda_k v_k = a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}. \tag{13}$$

So, by Equation (13)-(12), we have

$$0 = a_1(\lambda_k - \lambda_1)v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1}.$$

By assumption, $v_1, \cdots, v_{k-1}$ is L.I.. Then, it must be that $a_1 = \cdots = a_{k-1} = 0$ since $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues. Therefore, $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1} = 0$. ⋇ This contradicts with the fact that v_k is an eigenvector, which cannot be 0. So,it must be that $v_1, \cdots, v_m$ are L.I. ■

Theorem 4.1.13
Suppose V is *f-d*. Then, each operator on V has at most $\dim V$ distinct eigenvalues.

Proof 14. Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T. Let $v_1, \cdots, v_m$ be corresponding eigenvectors. By Theorem 4.1.12, we know $v_1, \cdots, v_m$ is $\mathrm{L.I.}$. Further by Theorem 2.3.5, we know $\dim \operatorname{span}(v_1, \cdots, v_m) \leq \dim V$. That is, $m \leq \dim V$ as desired. ■

4.2 Eigenvectors and Upper-Triangular Matrices

Definition 4.2.1 (T^m). Suppose $T \in \mathcal{L}(V)$ and m is a positive integer. Then, T^m is defined by

$$T^m := \underbrace{T \cdots T}_{m \text{ times}}.$$

Specially, T^0 is defined to be the identity operator I on V. Further, if T is invertible with inverse T^{-1}, then T^{-m} is defined by $T^{-m} := (T^{-1})^m$.

Theorem 4.2.2

$$T^m T^n = T^{m+n}; \qquad (T^m)^n = T^{mn}.$$

Definition 4.2.3 ($p(T)$). Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m, \quad z \in \mathbb{F}.$$

Then, $p(T)$ is the operator defined by

$$p(T) := a_o I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

Example 4.2.4 Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation operator defined by $Dq = q'$ and p is the polynomuial defined by $p(x) = 7 - 3x + 5x^2$. Find $p(D)$ and $(p(D))q$.

Answer 1.

$$\begin{aligned} p(D) &= 7I - 3D + 5D^2 \\ (p(D))q &= (7I - 3D + 5D^2)q \\ &= 7Iq - 3Dq + 5D^2 q \\ &= 7q - 3q' + 5q''. \end{aligned}$$

□

Theorem 4.2.5
If we fix an operator $T \in \mathcal{L}(V)$, then the function from $\mathcal{P}(\mathbb{F})$ to $\mathcal{L}(V)$ given by $p \mapsto p(T)$ is linear.

Proof 2. Suppose $f : \mathcal{P}(\mathbb{F}) \to \mathcal{L}(V)$ is defined by $p \mapsto p(T)$. Suppose

$$p = a_0 + a_1 z + \cdots + a_m z^m \mapsto a_0 I + a_1 T + \cdots + a_m T^m$$

and

$$q = b_0 + b_1 z + \cdots + b_m z^m \mapsto b_0 I + b_1 T + \cdots + b_m T^m.$$

Then,

$$\begin{aligned} f(p+q) &= (a_0 + b_0)I + (a_1 + b_1)T + \cdots + (a_m + b_m)T^m \\ &= (a_0 I + a_1 T + \cdots + a_m T^m) + (b_0 I + b_1 T + \cdots + b_m T^m) \\ &= f(p) + f(q). \end{aligned}$$

Further, suppose $\lambda \in \mathbb{F}$, then

$$\begin{aligned} f(\lambda p) &= \lambda a_0 I + \lambda a_1 T + \cdots + \lambda a_m T^m \\ &= \lambda(a_0 I + a_1 T + \cdots + a_m T^m) \\ &= \lambda f(p). \end{aligned}$$

■

Definition 4.2.6 (Product of Polynomials). If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by $(pq)(z) := p(z)q(z)$ for $z \in \mathbb{F}$.

Remark. $(pq)(z) = p(z)q(z) = q(z)p(z) = (qp)(z)$ *for* $z \in \mathbb{F}$.

Theorem 4.2.7 Multiplicative Properties
Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$. Then

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

Proof 3.

1. Suppose $p(z) = \sum_{j=0}^{m} a_j z^j$ and $q(z) = \sum_{k=0}^{n} b_k z^k$. Then

$$(pq)(z) = p(z)q(z) = \sum_{j=0}^{m} a_j z^j \sum_{k=0}^{n} b_k z^k = \sum_{j=0}^{m} \sum_{k=0}^{n} a_j b_k z^{j+k}$$

So, by definition, we have

$$p(T)q(T) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_j b_k T^{j+k} = \left(\sum_{j=0}^{m} a_j T^j \right) \cdot \left(\sum_{k=0}^{n} b_k T^k \right) = p(T)q(T). \qquad \square$$

2. Similar to the Remark,

$$p(T)q(T) = (pq)(T) = (qp)(T) = q(T)p(T).$$

■

Theorem 4.2.8 Fundamental Theorem of Algebra
Every non-constant polynomial with complex coefficients has a zero.

Theorem 4.2.9 Existence of Eigenvalues
Every operator on a *f-d*, non-zero, complex vector space has an eigenvalue.

Proof 4. Let V be a complex vector space with dimension $n > 0$. Suppose $T \in \mathcal{L}(V)$. Choose $v \in V$ *s.t.* $v \neq 0$. Then, $v, Tv, T^2 v, \cdots, T^n v$ is linearly dependent because $\dim V = n$ but the length of the list is $n + 1 > n$. Hence, $\exists\, a_0, a_1, \cdots, a_n$ not all $0 \in \mathbb{C}$ *s.t.*

$$0 = a_0 v + a_1 T v + \cdots + a_n T^n v \tag{14}$$

By Fundamental Theorem of Algebra (Theorem 4.2.8), we have

$$a_0 + a_1 z + \cdots + a_n z^n = c(z - \lambda_1) \cdots (z - \lambda_m)$$

with $c \in \mathbb{C},\ c \neq 0$, and $\lambda_j \in \mathbb{C}$. Then, Equation (14) becomes

$$\begin{aligned} 0 &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= (a_0 I + a_1 T + \cdots + a_n T^n) v \\ &= c(T - \lambda_1 I) \cdots (T - \lambda_m I) v \end{aligned}$$

Since $v \neq 0$ and $c \neq 0$, it must be some $T - \lambda_i I = 0$. Thus, $T = \lambda_i I$, and λ_i is an eigenvalue of T. ■

Definition 4.2.10 (Diagonal of a Matrix). The *diagonal of a square matrix* consists of the entires along the line from the upper left corner to the bottom right corner.

Definition 4.2.11 (Upper-Triangular Matrix). A matrix is called *upper-triangular* if all the entires below the diagonal equal 0. Typically, we present an upper triangular matrix in the form

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Theorem 4.2.12 Conditions for Upper-Triangular Matrix
Suppose $T \in \mathcal{L}(V)$ and $v_1, \cdots, v_n$ is a basis of V. Then, the following are equivalent:

1. the matrix of T with respect to $v_1, \cdots, v_n$ is upper triangular.
2. $Tv_j \in \operatorname{span}(v_1, \cdots, v_j)$ for each $j = 1, \cdots, n$
3. $\operatorname{span}(1, \cdots, v_j)$ is invariant under T for each $j = 1, \cdots, n$.

Proof 5.

1. First, we will show (1) $\iff$ (2).

   Suppose $\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ & \ddots & \vdots \\ 0 & & A_{n,n} \end{pmatrix}$. Then,

   $$\begin{aligned} Tv_1 &= A_{1,1} v_1 \\ Tv_2 &= A_{1,2} v_1 + a_{2,2} v_2 \\ &\vdots \\ Tv_j &= A_{1,j} v_1 + \cdots + A_{j,j} v_j. \end{aligned}$$

   So, $Tv_j \in \operatorname{span}(v_1, \cdots, v_j)$. The reverse implication is trivial to prove. □

2. (3) $\implies$ (2) is obvious and trivial to prove.

3. Lastly, we want to show (2) $\implies$ (3).

Note that for each fixed $j = 1, \cdots, n$, we have

$$\begin{aligned} Tv_1 &\in \operatorname{span}(v_1) \subseteq \operatorname{span}(v_1, \cdots, v_j) \\ Tv_2 &\in \operatorname{span}(v_1, v_2) \subseteq \operatorname{span}(v_1, \cdots, v_j) \\ &\vdots \\ Tv_j &\in \operatorname{span}(v_1, \cdots, v_j) \end{aligned}$$

Let $v \in \operatorname{span}(v_1, \cdots, v_j)$. Then, v is a linear combination of $v_1, \cdots, v_j$, then

$$Tv \in \operatorname{span}(v_1, \cdots, v_j).$$

That is, $\operatorname{span}(v_1, \cdots, v_j)$ is invariant under T.

■

Definition 4.2.13 (Quotient Operator). Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by $(T/U)(v+U) \coloneqq Tv + U$.

Proof 6. The definition makes sense, and here is the proof. If $v + U = w + U$, then $v - w \in U$. So, $T(v-w) \in U$ since U is invariant. That is, $Tv - Tw \in U$. Then, $Tv + U = Tw + U$. ■

Theorem 4.2.14

Suppose U is a subspace of V. Let $v_1 + U, \cdots, v_m + U$ be a basis of V/U and $u_1, \cdots, u_n$ be a basis of U. Then, $v_1, \cdots, v_m, u_1, \cdots, u_n$ is a basis of V.

Proof 7. Let $v \in V$. Then $v + U \in V/U$. So, $v + U = a_1v_1 + \cdots + a_mv_m + U$, uniquely. Then, by Theorem 3.6.4, we have $v - (a_1v_1 + \cdots + a_mv_m) \in U$. Therefore, $v - (a_1v_1 + \cdots + a_mv_m) = b_1u_1 + \cdots + b_nu_n$, uniquely. So, $v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n$. uniquely. By definition, $v_1, \cdots, v_m, u_1, \cdots, u_n$ is a basis of V. ■

Theorem 4.2.15

Suppose V is a *f-d* complex vector space and $T \in \mathcal{L}(V)$. Then, T has an upper-triangular matrix with respect to some basis of V.

Proof 8.

$\boxed{\text{Base Case}}$ When $\dim V = 1$, the implication holds.

$\boxed{\text{Inductive Steps}}$ Suppose the implication is true for some complex vector space with dimension of $n-1$. Let $\dim V = n$ and v_1 be any eigenvector of T. Suppose $U = \operatorname{span}(v_1)$. Then, U is invariant under T. Note that $\dim V/U = \dim V - \dim U = n - 1$, so we can use the inductive hypothesis on the quotient operator $T/U \in \mathcal{L}(V/U)$. Then, $\exists$ a basis $v_2 + U, \cdots, v_n + U \in V/U$ *s.t.* T/U has an upper-triangular matrix. By Theorem 4.2.12, we have

$$(T/U)(v_j + U) \in \operatorname{span}(v_2 + U, \cdots, c = v_j + U) \quad \text{for } j \in \{2, \cdots, n\}.$$

So, $Tv_j + U = (c_2v_2 + \cdots + c_jv_j) + U$. Then,

$$Tv_j - (c_2v_2 + \cdots + c_jv_j) \in U = \operatorname{span}(v_1).$$

So, $Tv_j - (c_2v_2 + \cdots + c_jv_j) = c_1v_1$ for some $c_1 \in \mathbb{F}$. Then, $Tv_j = c_1v_1 + c_2v_2 + \cdots + c_jv_j$. So, $Tv_j \in \operatorname{span}(v_1, \cdots, v_j)$ for $j \in \{1, \cdots, n\}$. Since by Theorem 4.2.14, $v_1, \cdots, v_n$ is a basis of V, further by Theorem 4.2.12, T has an upper-triangular matrix with respect to $v_1, \cdots, v_n$. So, the implication is true for $\dim V = n$.

Since the implication is true for $\dim V = 1$ and is true for $\dim V = n$ whenever it is hold for $\dim V = n - 1$, by the Principle of Mathematical Induction, the implication is true for all positive integers n. Hence, the proof is complete. ■

4.3 Eigenspaces and Diagonal Matrices

Definition 4.3.1 (Diagonal Matrix). A *diagonal matrix* is a square matrix that is 0 everywhere except possibly along the diagonal.

Definition 4.3.2 (Eigenspace, $E(\lambda, T)$). Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The *eigenspace* of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \text{null}\,(T - \lambda I).$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

Theorem 4.3.3 Sum of Eigenspaces is a Direct Sum
Suppose V is *f-d* and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Further

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

Proof 1. Suppose $u_1 + \cdots + u_m = 0$, where $u_j \in E(\lambda_j, T)$. If some $u_i \neq 0$, then $u_1 + \cdots + u_m$ can never be 0 because $u_1, \cdots, u_m$, as eigenvectors corresponding to distinct eigenvalues, is L.I.. Hence, the only way for $u_1 + \cdots + u_m$ to be 0 is by taking $u_1 = \cdots = u_m = 0$. Hence, we know $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum. □

By Theorem 4.1.2, we know

$$\begin{aligned} \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) &= \dim E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T) \\ &\leq \dim V. \end{aligned}$$

■

Definition 4.3.4 (Diagonalizable). An operator $T \in \mathcal{L}(V)$ is called *diagonalizable* if the operator has a diagonal matrix with respect to some basis of V.

Theorem 4.3.5 Conditions Equivalent to Diagonalizability
Suppose V is *f-d* and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ denote the distinct eigenvalues of T. Then, the following are equivalent:

1. T is diagonalizable.
2. V has a basis consisting of eigenvectors of T.
3. $\exists\, 1-$dimensional subspaces $U_1, \cdots, U_n$ of V, each invariant under T, *s.t.* $V = U_1 \oplus \cdots \oplus U_n$.
4. $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.
5. $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Remark. *To prove this theorem, we will prove (1) $\iff$ (2), (2) $\iff$ (3), (2) $\implies$ (4), (4) $\implies$ (5), and (5) $\implies$ (2).*

Proof 2.

1. (1) $\iff$ (2): By definition, we know T is diagonalizable if and only if $\exists$ a basis $v_1, \cdots, v_n$ of T *s.t.*

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

   which holds if and only if we have $Tv_1 = \lambda_1 v_1, \cdots, Tv_n = \lambda_n v_n$. i.e., $v_1, \cdots, v_n$ are eigenvectors of T. $\square$

2. (2) $\implies$ (3): Suppose $v_1, \cdots, v_n$ is a basis of V. Then, for some $v \in V$, we have $v = a_1 v_1 + \cdots + a_n v_n$. So, we know $V = \operatorname{span}(v_1) + \cdots + \operatorname{span}(v_n)$. Further, let $a_1 v_1 + \cdots + a_m v_m = 0$. Since $v_1, \cdots, v_n$ is a basis, it must be $a_1 = \cdots = a_m = 0$. So, there is only one way to represent 0. So,

$$V = \operatorname{span}(v_1) \oplus \cdots \oplus \operatorname{span}(v_n).$$

   Now, we want to show each $\operatorname{span}(v_j)$ is invariant. Consider $T(c_j v_j) = c_j T v_j = c_j \lambda_j v_j \in \operatorname{span}(v_j)$. So, $\operatorname{span}(v_j)$ is invariant. $\square$

3. (3) $\implies$ (2): Suppose $\exists$ $1-$dimensional subspaces $U_1, \cdots, U_n$ of V, each invariant under T, *s.t.* $V = U_1 \oplus \cdots \oplus U_n$. Then, $\forall v \in V$, we have $v = a_1 u_1 + \cdots + a_n u_n$ uniquely. Then, $u_1, \cdots, u_n$ is a basis of V. Since $U_1, \cdots, U_n$ are $1-$dimensional invariant subspaces, $u_1, \cdots, u_n$ are the eigenvalues. $\square$

4. (2) $\implies$ (4): Suppose V has a basis consisting of eigenvectors of T. Then, $v = a_1 v_1 + \cdots + a_n v_n$ is a linear combination of eigenvectors of T. By definition, $E(\lambda_j, T)$ contains the eigenvectors corresponding to λ_j. Further since $\lambda_1, \cdots, \lambda_m$ is distinct, corresponding eigenvectors are $\mathrm{L.I.}$. Then, $E(\lambda_j, T) \cap E(\lambda_i, T) = \{0\}$ if $i \neq j$. Then, we have

$$v = a_1 v_1 + \cdots + a_n v_n \in E(\lambda_1, T) + \cdots + E(\lambda_m, T).$$

   Hence, $V = \mathbb{E}(\lambda_1, T) + \cdots + E(\lambda_m, T)$. Further by Theorem 4.3.3, we have

$$V = \mathbb{E}(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T). \qquad \square$$

5. (4) $\implies$ (5): This conclusion can be deduced from Theorem 4.3.3 and its proof.

6. (5) $\implies$ (2): Suppose $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$. Select B_j, the basis of $E(\lambda_j, T)$ for $j = 1, \cdots, m$. Denote $\dim V = n$. Then, if we put these bases together as $B_1, \cdots, B_m$, we can write the collection as $v_1, \cdots, v_n$. Suppose $a_1 v_1 + \cdots + a_n v_n = 0$. Let u_j denote the sum of all the terms $a_k v_k$ *s.t.* $v_k \in E(\lambda_j, T)$. Then, the equation becomes $u_1 + \cdots + u_m = 0$ and each $u_j \in E(\lambda_j, T)$. Since eigenvectors corresponding to distinct eigenvalues are $\mathrm{L.I.}$, it must be that $u_1 = \cdots = u_m = 0$. Further, by definition of u_j, and since $u_k's$ are bases of $E(\lambda_j, T)$, it must be $a_1 = \cdots = a_n = 0$. So, we know $v_1, \cdots, v_n$ is $\mathrm{L.I.}$. Further, since $\operatorname{len}(v_1, \cdots, v_n) = n = \dim V$, we know that $v_1, \cdots, v_n$ is a basis of V.

$\blacksquare$

Theorem 4.3.6
If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Proof 3. Suppose $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues: $\lambda_1, \cdots, \lambda_{\dim V}$. Then, it has $v_1, \cdots, v_{\dim V}$ as corresponding eigenvectors and is L.I.. Note that $\operatorname{len}(v_1, \cdots, v_{\dim V}) = \dim V$. So, $v_1, \cdots, v_{\dim V}$ is a basis of V. By Theorem 4.3.5, with respect to this basis consisting of eigenvectors, T has a diagonal matrix. ■

Example 4.3.7 The *Fibonacci Sequence* $F_1, F_2, \cdots$ is defined by

$$F_1 = F_2 = 3 \qquad \text{and} \qquad F_n = F_{n-2} + F_{n-1} \quad \text{for } n \geq 3.$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

1. Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each $n \in \mathbb{Z}^+$.

   Proof 4.

   - Base Case: Note that $T(0, 1) = (1, 1) = (F_1, F_2)$.
   - Inductive Process: Suppose $T^{n-1}(0, 1) = (F_{n-1}, F_n)$. Then,

   $$\begin{aligned} T^n = [T(T^{n-1})](0, 1) &= T[T^{n-1}(0, 1)] \\ &= T(F_{n-1}, F_n) \\ &= (F_n, F_{n-1} + F_n) \\ &= (F_n, F_{n+1}). \end{aligned}$$

   So, $T^n(0, 1) = (F_n, F_{n+1}) \quad \forall\, n \in \mathbb{Z}^+$ by Principle of Mathematical Induction.

   ■

2. Find the eigenvalues of T.

   Answer 5.

   Suppose $T(x, y) = \lambda(x, y)$. So, $(y, x+y) = (\lambda x, \lambda y)$. Solve $\begin{cases} y = \lambda x \\ x + y = \lambda y \end{cases}$. That is, $x + \lambda x = \lambda^2 x$, or $\lambda^2 x - \lambda x - x = 0$. It follows $x \neq 0$, so solving $\lambda^2 - \lambda - 1 = 0$, we get

   $$\lambda_1 = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1 - \sqrt{5}}{2}.$$

   □

3. Since T has two eigenvalues, T should have a basis of $\mathbb{R}^2$ consisting of eigenvectors. Find the basis.

   Answer 6.

When $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$, we have $y = \lambda x = \dfrac{1+\sqrt{5}}{2}x$. So, $v_1 = \left(x, \dfrac{1+\sqrt{5}}{2}x\right) = x\left(1, \dfrac{1+\sqrt{5}}{2}\right)$. That is,

$$v_1 = \left(1, \frac{1+\sqrt{5}}{2}\right).$$

Similarly, we have

$$v_2 = \left(1, \frac{1-\sqrt{5}}{2}\right).$$

Further, it follows that

$$\mathcal{M}(T, v_1, v_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

□

4. Find F_n using an expression of n only.

Answer 7.

Note that $(0,1) = \dfrac{1}{\sqrt{5}}(v_1 - v_2)$. So, we have

$$\begin{aligned}
T^n(0,1) &= T^n\left(\frac{1}{\sqrt{5}}(v_1 - v_2)\right) \\
&= \frac{1}{\sqrt{5}}T^n(v_1 - v_2) \\
&= \frac{1}{\sqrt{5}}(T^n v_1 - T^n v_2) \\
&= \frac{1}{\sqrt{5}}(\lambda_1^n v_1 - \lambda_2^n v_2) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n \left(1, \frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n \left(1, \frac{1-\sqrt{5}}{2}\right)\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n, \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \\
&= (F_n, F_{n+1}).
\end{aligned}$$

So, we have

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

□

5 Inner Product Spaces

5.1 Inner Products and Norms

Definition 5.1.1 (Dot Product). For $x, y \in \mathbb{R}^n$, the *dot product* of x and y, denoted $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.

Theorem 5.1.2 Properties of dot Product

1. $x \cdot x = x_1^2 + \cdots + x_n^2 \geq 0 \quad \forall x \in \mathbb{R}^n$.
2. $x \cdot x = 0$ if and only if $x = 0$.
3. For $y \in \mathbb{R}^n$, define $f : \mathbb{R}^n \to \mathbb{R}$ as $x \mapsto x \cdot y$. Then, f is linear.
4. $\forall x, y \in \mathbb{R}^n, \ x \cdot y = y \cdot x$.

Proof 1. Properties #1, #2, and #4 are trivial to prove, so the proof is omitted. Here we complete a proof for property #3, linearity of dot product. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $x \mapsto x \cdot y$ for a fixed $y \in \mathbb{R}^n$. Note that

$$\begin{aligned} f(a+b) = (a+b) \cdot y &= (a_1 + b_1) y_1 + \cdots + (a_n + b_n) y_n) \\ &= (a_1 y_1 + \cdots + a_n y_n) + (b_1 y_1 + \cdots + b_n y_n) \\ &= f(a) + f(b). \end{aligned}$$

Further, notice that

$$\begin{aligned} f(\lambda x) = (\lambda x) \cdot y &= (\lambda x_1) y_1 + \cdots + (\lambda x_n) y_n \\ &= \lambda (x_1 y_x + \cdots + x_n y_n = \lambda f(x). \end{aligned}$$

■

Remark. *For $w, z \in \mathbb{C}^n$, we define the dot product of w and z, denoted as $\langle w, z \rangle$, as*

$$\langle w, z \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}.$$

Definition 5.1.3 (Inner Product). An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbb{F}$ and has the following properties:

1. positivity: $\langle v, v \rangle \geq 0 \quad \forall v \in V$.
2. definiteness: $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. additivity in first slot: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \quad \forall u, v, w \in V$.
4. homogeneity in first slot: $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \quad \forall \lambda \in \mathbb{F}$ and $\forall u, v \in V$.
5. conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle} \quad \forall u, v \in V$.

Example 5.1.4 Here, we offer some examples of inner product. Note that there might be multiple inner products over a vector space, as long as the following the definition and properties given in Definition 5.1.3.

1. Consider $\mathbb{C}[-1,1]$, the set of continuous real-valued functions on the interval $[-1,1]$. An inner product can be defined as $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$.

   Proof 2.

   (a) $\langle f, f\rangle = \int_{-1}^{1} f^2(x)\,\mathrm{d}x \geq 0$.

   (b) $\langle f, f\rangle = 0$ if and only if $f(x) = 0$.

   (c) Note that
$$\begin{aligned}\langle f+g, h\rangle &= \int_{-1}^{1}[f(x)+g(x)]h(x)\,\mathrm{d}x\\ &= \int_{-1}^{1} f(x)h(x)+g(x)h(x)\,\mathrm{d}x\\ &= \int_{-1}^{1} f(x)h(x)\,\mathrm{d}x + \int_{-1}^{1} g(x)h(x)\,\mathrm{d}x\\ &= \langle f, h\rangle + \langle g, h\rangle.\end{aligned}$$

   (d) $\langle \lambda f, g\rangle = \int_{-1}^{1} \lambda f(x)g(x)\,\mathrm{d}x = \lambda\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x = \lambda\langle f, g\rangle$.

   (e) $\langle g, f\rangle = \int_{-1}^{1} g(x)f(x)\,\mathrm{d}x = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x = \langle f, g\rangle = \overline{\langle f, g\rangle}$.

   ■

2. An inner product on $\mathcal{P}(\mathbb{R})$ can be defined as $\langle p, q\rangle = \int_{0}^{\infty} p(x)q(x)e^{-x}\,\mathrm{d}x$

   Proof 3. The definition makes sense. Consider the inner product as $\langle\ \rangle : \mathcal{P}(\mathbb{R})\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$. Note that $\infty\notin\mathbb{R}$. So we need to show the improper integral converges to a finite number under any circumstances. Consider
$$\frac{x^2p(x)q(x)}{e^x} = \frac{p(x)q(x)e^{-x}}{\frac{1}{x^2}}.$$
   Note that
$$\lim_{x\to\infty}\frac{p(x)q(x)e^{-x}}{\frac{1}{x^2}} = 0$$
   Further since $\int_{0}^{\infty}\frac{1}{x^2}\,\mathrm{d}x$ converges as it is a p-series with $p = 2 > 1$, we know it must be $\int_{0}^{\infty} p(x)q(x)e^{-x}\,\mathrm{d}x$ converges as well, by the comparison test. The remaining job is to show this definition of $\langle\ \rangle$ indeed retain the five properties as required in Definition 5.1.3, which is trivial and so is omitted. ■

Definition 5.1.5 (Inner Product Space). An *inner product space* is a vector space V along with an inner product on V.

Example 5.1.6 Euclidean Inner Product on $\mathbb{F}^n$ is defined as

$$\langle (w_1, \cdots, w_n), (z_1, \cdots, z_n) \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n},$$

where $(w_1, \cdots, w_n), (z_1, \cdots, z_n) \in \mathbb{F}^n$.

Notation 5.1.7. For the rest of this Chapter, without otherwise specification, V denotes an inner product space over $\mathbb{F}$.

Remark. *If not explicitly defined, the inner product is the Euclidean inner product as defined in Example 5.1.6.*

Theorem 5.1.8 Basic Properties of an Inner Product

1. For each fixed $u \in V$, the function that takes v to $\langle v, u \rangle$ is a linear map from V to $\mathbb{F}$.
2. $\langle 0, u \rangle = 0$ for every $u \in V$.
3. $\langle u, 0 \rangle = 0$ for ever $u \in V$.
4. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.
5. $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle \quad \forall \lambda \in \mathbb{F}$ and $u, v \in V$.

Proof 4.

1. Define $f : V \to \mathbb{F}$ as $v \mapsto \langle v, u \rangle$ for some fixed $u \in V$. Then,
$$f(v + w) = \langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = f(v) + f(w).$$
$$f(\lambda v) = \langle \lambda v, u \rangle = \lambda \langle v, u \rangle = \lambda f(v). \qquad \square$$
2. Since f is a linear map, then $f(0) = \langle 0, u \rangle = 0. \qquad \square$
3. Note that $\langle u, 0 \rangle = \overline{\langle 0, u \rangle} = \overline{0} = 0. \qquad \square$
4. Notice
$$\begin{aligned} \langle u, v + w \rangle = \overline{\langle v + w, u \rangle} &= \overline{\langle v, u \rangle + \langle w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle. \qquad \square \end{aligned}$$
5. Observe that
$$\begin{aligned} \langle u, \lambda v \rangle = \overline{\langle \lambda v, u \rangle} &= \overline{\lambda \langle v, u \rangle} \\ &= \overline{\lambda} \cdot \overline{\langle v, u \rangle} = \overline{\lambda} \langle u, v \rangle. \end{aligned}$$

■

Definition 5.1.9 (Norm). Suppose V is a vector space. Then, the *norm* of v is a real-valued function $\| \, \| : V \to \mathbb{R}$ satisfying the following properties:

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
2. $\|\alpha v\| = |\alpha|\|v\| \quad \forall \alpha \in \mathbb{F}$ and $v \in V$.
3. triangle inequality: $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in \mathbb{F}$.

Definition 5.1.10 (Norm Induced by An Inner Product). For $v \in V$, $\|v\| = \sqrt{\langle v, v\rangle}$ is a *norm* on V.

Remark. *We will prove Definition 5.1.10 is indeed a definition of norm that satisfies the conditions indicated by Definition 5.1.9 throughout the rest of this section.*

Theorem 5.1.11 Basic Properties of Norms
Let $v \in V$. Then,

1. $\|v\| = 0$ if and only if $v = 0$.
2. $\|\lambda v\| = |\lambda|\|v\| \quad \forall \lambda \in \mathbb{F}$.

Proof 5.

1. $\|v\| = 0$ if and only if $\sqrt{\langle v, v\rangle} = 0$, which is equivalent to $\langle v, v\rangle = 0$. By properties of an inner product, $\langle v, v\rangle = 0$ if and only if $v = 0$. So, the proof is complete. $\square$
2. Consider
$$\|\lambda v\|^2 = \langle \lambda v, \lambda v\rangle = \lambda \cdot \overline{\lambda}\langle v, v\rangle = |\lambda|^2\langle v, v\rangle.$$
So, $\|\lambda v\| = \sqrt{|\lambda|^2\langle v, v\rangle} = |\lambda|\|v\|$.

$\blacksquare$

Definition 5.1.12 (Orthogonal). Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v\rangle = 0$.

Theorem 5.1.13 Orthogonality and 0

1. 0 is orthogonal to every vector in V.
2. 0 is the only vector in V that is orthogonal to itself.

Proof 6.

1. As $\langle 0, u\rangle = 0 \quad \forall u \in V$, the proof is complete. $\square$
2. Note that $\langle v, v\rangle = 0$ if and only if $v = 0$, so we complete the proof. $\square$

$\blacksquare$

Theorem 5.1.14 Pythagorean Theorem
Suppose u and v are orthogonal vectors in V, then
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

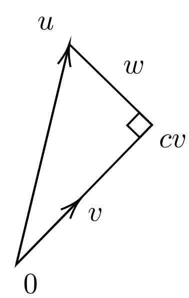
Proof 7. Note that

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle.\end{aligned}$$

Since u and v are orthogonal, $\langle u,v\rangle = \langle v,u\rangle = 0$. So, $\|u+v\|^2 = \langle u,u\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2$. ■

Theorem 5.1.15 An Orthogonal Decomposition
Suppose $u, v \in V$, with $v \neq 0$. Set $c = \dfrac{\langle u,v\rangle}{\|v\|^2}$ and $w = u - \dfrac{\langle u,v\rangle}{\|v\|^2}v$. Then, $\rangle w, v\rangle = 0$ and $u = cv + w$.

Proof 8.



The idea is the find c, w *s.t.* $\langle v, w\rangle = 0$ and $w = u - cv$. That is, $u = w + cv$. Since $\langle v,w\rangle = 0$, then we have

$$\langle v, u - cv\rangle = 0 = \langle u - cv, v\rangle = \langle u, v\rangle - c\|v\|^2.$$

So,

$$c = \frac{\langle u,v\rangle}{\|v\|^2}$$

and

$$w = u - cv = u - \frac{\langle u,v\rangle}{\|v\|^2}v.$$

■

Theorem 5.1.16 Cauchy-Schwarz Inequality
Suppose $u, v \in V$. Then,

$$|\langle u,v\rangle| \leq \|u\|\|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiples of the other.

Proof 9. If $v = 0$, then $|\langle u,v\rangle| = 0 = \|u\|\|v\|$. So, we can assume $v \neq 0$. Consider the orthogonal decomposition,

$$u = \frac{\langle u,v\rangle}{\|v\|^2}\cdot v + w.$$

Then, by the Pythagorean Theorem, we have

$$\begin{aligned}\|u\|^2 = \left\|\frac{\langle u,v\rangle}{\|v\|^2}\cdot v\right\|^2 + \|w\|^2 &= \frac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2 + \|w\|^2 \\ &= \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \|w\|^2 \geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}\end{aligned}$$

As $\|v\|^2 > 0$, we have $\|u\|^2\|v\|^2 \geq |\langle u, v\rangle|^2$. Further since $\|u\| \geq 0,\ \|v\| \geq 0$, and $|\langle u, v\rangle| \geq 0$, then

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

The equality holds if and only if $\|w\|^2 = 0$. That is, $w = 0$ from the orthogonal decomposition. In other words, u and v are linearly dependent. ■

Theorem 5.1.17 Triangle Inequality
Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a non-negative multiple of the other.

Proof 10. Note that

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\
&= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \overline{\langle u, v\rangle} \\
&= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}(\langle u, v\rangle) \\
&\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v\rangle| \\
&\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| && \text{Cauchy-Schwarz Inequality} \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

Since $\|u+v\| \geq 0,\ \|u\| \geq 0$, and $\|v\| \geq 0$, we have

$$\|u + v\| \leq \|u\| + \|v\|.$$

The equality holds if and only if $\langle u, v\rangle = \|u\|\|v\|$. That is, when u and v are linearly dependent to each other. ■

Remark. *After proving this triangle inequality, we finally, and officially, complete our proof to show the norm induced by an inner product as stated in Definition 5.1.10 is indeed a norm satisfying the formal definition of norms as stated in Definition 5.1.9.*

Theorem 5.1.18 Parallelogram Equality
Suppose $u, v \in V$. Then

$$\|u+v\|^2 + \|u-v\|^2 = 2\big(\|u\|^2 + \|v\|^2\big).$$

Proof 11. Note that

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle u, u\rangle + \langle v, v\rangle - \langle u, v\rangle - \langle v, u\rangle \\
&= \|u\|^2 + \|u\|^2 + \|v\|^2 + \|v\|^2 \\
&= 2\big(\|u\|^2 + \|v\|^2\big).
\end{aligned}$$

■

Theorem 5.1.19
Suppose V is a real inner product space. Then,

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

Proof 12. Note that

$$\begin{aligned}
\|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\
&= \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle - \big(\|u\|^2 + \|v\|^2 - 2\langle u, v\rangle\big) \\
&= 4\langle u, v\rangle.
\end{aligned}$$

So, we have

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

■

Theorem 5.1.20
Suppose V is a complex inner product space. Then,

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+\mathrm{i}v\|^2\mathrm{i} - \|u-\mathrm{i}v\|^2\mathrm{i}}{4}.$$

Proof 13. Note that

$$\begin{aligned}
&\langle u+v, u+v\rangle - \langle u-v, u-v\rangle + \langle u+\mathrm{i}v, u+\mathrm{i}v\rangle\mathrm{i} - \langle u-\mathrm{i}v, u-\mathrm{i}v\rangle\mathrm{i} \\
&= 2\langle u, v\rangle + 2\langle v, u\rangle + (2\langle u, \mathrm{i}v\rangle + 2\langle \mathrm{i}v, u\rangle)\mathrm{i} \\
&= 2\langle u, v\rangle + 2\langle v, u\rangle + (-2\mathrm{i}\langle u, v\rangle + 2\mathrm{i}\langle v, u\rangle)\mathrm{i} \\
&= 2\langle u, v\rangle + 2\langle v, u\rangle + 2\langle u, v\rangle - 2\langle v, u\rangle \\
&= 4\langle u, v\rangle.
\end{aligned}$$

so, we have

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+\mathrm{i}v\|^2\mathrm{i} - \|u-\mathrm{i}v\|^2\mathrm{i}}{4}.$$

■

Theorem 5.1.21
Let U be a vector space. If $\| \, \|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle \, \rangle$ on U *s.t.* $\|u\| = \sqrt{\langle u, u\rangle} \quad \forall u \in U$.

5.2 Orthonormal Bases

Definition 5.2.1 (Orthonormal). A list of vectors is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list. In other words, a list $e_1, \cdots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

Theorem 5.2.2
If $e_1, \cdots, e_m$ is an orthonormal list of vectors in V, then

$$\|a_1e_1 + \cdots + a_me_m\|^2 = |a_1|^2 + \cdots + |a_m|^2 \quad \forall a_1, \cdots, a_m \in \mathbb{F}.$$

Proof 1. Note that

$$\langle a_1e_1, a_2e_2 + \cdots + a_me_m \rangle = \langle a_1e_1, a_2e_2 \rangle + \cdots + \langle a_1e_1, a_me_m \rangle = 0.$$

So, by the Pythagorean Theorem, we have

$$\begin{aligned} \|a_1e_1 + \cdots + a_me_m\|^2 &= \|a_1e_1\|^2 + \|a_2e_2 + \cdots + a_me_m\|^2 \\ &= \|a_1e_1\|^2 + \|a_2e_2\|^2 + \cdots + \|a_me_m\|^2 \\ &= |a_1|^2 + |a_2|^2 + \cdots + |a_m|^2. \end{aligned}$$

■

Theorem 5.2.3
Every orthonormal list of vectors is L.I..

Proof 2. Suppose $e_1, \cdots, e_m$ is an orthonormal list of vectors in V. Then, $\|a_1e_1 + \cdots + a_me_m\|^2 = 0$. By Theorem 5.2.2, it is equivalent to $|a_1|^2 + \cdots + |a_m|^2 = 0$. Since each $|a_j| \geq 0$, it must be $a_j = 0$ for all $j = 1, \cdots, m$. Therefore, the orthonormal list is L.I.. ■

Definition 5.2.4 (Orthonormal Basis). An *orthonormal basis* of Vc is an orthonormal list of vectors in V that is also a basis of V.

Theorem 5.2.5
Every orthonormal list of vectors in V with length $\dim V$c is an orthonormal basis of V.

Proof 3. By Theorem 5.2.3, any orthonormal list of vectors must be L.I.. Further since it has length $\dim V$, it is a basis of V. So, by definition, it is an orthonormal basis of V. ■

Theorem 5.2.6
Suppose $e_1, \cdots, e_n$ is an orthonormal basis of V and $v \in V$. Then, $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$, and $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$.

Proof 4. Suppose $v \in V$ and $v = a_1e_1 + \cdots + a_ne_n$. Then,

$$\langle v, e_j \rangle = \langle a_1e_1 + \cdots + a_ne_n, e_j \rangle = \langle a_je_j, e_j \rangle = a_j.$$

So, we have

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n.$$

Further, by Theorem 5.2.2, we have

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2.$$

■

Theorem 5.2.7 Gram-Schmidt Procedure
Suppose $v_1, \cdots, v_m$ is L.I. list of vectors in V. Let $e_1 = \frac{v_1}{\|v_1\|}$. For $j = 2, \cdots, m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}. \quad (15)$$

Then, $e_1, \cdots, e_m$ is an orthonormal list of vectors in V *s.t.* $\text{span}(v_1, \cdots, v_j) = \text{span}(e_1, \cdots, e_j)$ for $j = 1, \cdots, m$.

Proof 5. To prove that Gram-Schmidt Procedure indeed produces an orthonormal list of vectors in V, we will use prove by mathematical induction.

$\boxed{\text{Base Case}}$ Suppose $j = 1$, then $\text{span}(v_1) = \text{span}(e_1)$ since v_1 is a positive multiple of e_1. So, the conclusion holds when $j = 1$.

$\boxed{\text{Inductive Steps}}$ Suppose for some $1 < j < m$, we have $\text{span}(v_1, \cdots, v_{j-1}) = \text{span}(e_1, \cdots, e_{j-1})$. Since$v_1, \cdots, v_m$ is L.I., we know $v_j \notin \text{span}(v_1, \cdots, v_{j-1})$. That is, $v_j \notin \text{span}(e_1, \cdots, e_{j-1})$. (*If* $v_j \in \text{span}(e_1, \cdots, e_{j-1})$, *then* $v_j = \langle v_j, e_1\rangle e_1 + \cdots + \langle v_j, e_{j-1}\rangle e_{j-1}$.*) Then, we are dividing by* 0 *in Equation (15).* So, we are not dividing by 0 in Equation (15). Dividing a vector by its norm produces a new vector with norm 1, so $\|e_j\| = 1$. Now, we want to verify e_j is orthogonal to $e_1, \cdots, e_{j-1}$. Pick some k *s.t.* $1 \leq k < j$. Then

$$\begin{aligned}
\langle e_j, e_k\rangle &= \left\langle \frac{v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}, e_k \right\rangle \\
&= \frac{\langle v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}, e_k\rangle}{\|\langle v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|} \\
&= \frac{\langle v_j, e_k\rangle - \langle v_j, e_k\rangle}{\|\langle v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|} \\
&= 0
\end{aligned}$$

Then, $e_1, \cdots, e_j$ is an orthonormal basis, and $v_j \in \text{span}(e_1, \cdots, e_j)$ since v_j is a linear combination of $e_1, \cdots, e_j$ by Equation (15). Further, we have

$$\dim \text{span}(v_1, \cdots, v_j) = \dim \text{span}(e_1, \cdots, e_j)$$

and

$$\text{span}(v_1, \cdots, v_j) \subseteq \text{span}(e_1, \cdots, e_j).$$

That is, exactly, $\text{span}(v_1, \cdots, v_j) = \text{span}(e_1, \cdots, e_j)$. ■

Theorem 5.2.8
Every *f-d* inner product space has an orthonormal basis.

Proof 6. Suppose V is *f-d*, and select a basis of V. Apply Gram-Schmidt Procedure (Theorem 5.2.7) to this basis, we then have an orthonormal basis of V. ■

Theorem 5.2.9
Suppose V is *f-d*. Then, every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Proof 7. Suppose $e_1, \cdots, e_m$ is an orthonormal list of vectors in V. Then, $e_1, \cdots, e_m$ is L.I. and can be extended to a basis $e_1, \cdots, e_m, v_1, \cdots, v_n$ of V. Apply Gram-Schmidt Procedure to this basis, we get an orthonormal list $e_1, \cdots, e_m, f_1, \cdots, f_n$. Here, $e_1, \cdots, e_m$ is unchanged since they are already orthonormal. Then, $e_1, \cdots, e_m, f_1, \cdots, f_n$ is an orthonormal basis of V. ■

Theorem 5.2.10
Suppose $T \in \mathcal{L}(V)$. If T has an upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some orthonormal basis of V.

Proof 8. Suppose $\mathcal{M}(T)$ is upper-triangular with respect to a basis $v_1, \cdots, v_n$ of V. Then, we know $\mathrm{span}(v_1, \cdots, v_j)$ is invariant under T for $j = 1, \cdots, n$. Apply Gram-Schmidt Procedure to $v_1, \cdots, v_n$, we will get an orthonormal basis $e_1, \cdots, e_n$ of V. Further, since $\mathrm{span}(e_1, \cdots, e_j) = \mathrm{span}(v_1, \cdots, v_j)$ for $j = 1, \cdots, n$, we know $\mathrm{span}(e_1, \cdots, e_j)$ is invariant under T. Therefore, T has an upper-triangular matrix with respect to the orthonormal basis $e_1, \cdots, e_n$. ■

Theorem 5.2.11 Schur's Theorem
Suppose V is a *f-d* complex vector space and $T \in \mathcal{L}(V)$. Then, T has an upper-triangular matrix with respect to some orthonormal basis of V.

Proof 9. Since V is a *f-d* complex vector space, T must have an upper-triangular matrix with respect to some basis of V. Further, by Theorem 5.2.10, T must have an upper-triangular matrix with respect to an orthonormal basis of V. ■

Example 5.2.12 The function $\varphi : \mathbb{F}^3 \to \mathbb{F}$ defined by

$$\varphi(z_1, z_2, z_3) = 2z_1 - 5z_2 + z_3$$

is a linear functional on $\mathbb{F}^3$. We could write this linear functional in the form $\varphi(z) = \langle z, u\rangle$ for every $z \in \mathbb{F}^3$, where $u = \langle 2, -5, 1\rangle$.

Theorem 5.2.13 Riesz Representation Theorem
Suppose V is *f-d* and φ is a linear functional on V. Then, there is a unique vector $u \in V$ *s.t.* $\varphi(v) = \langle v, u\rangle$ for every $v \in V$.

Proof 10. Let $e_1, \cdots, e_n$ be an orthonormal basis of V. Then, for all $v \in V$, we have

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n.$$

So,

$$\begin{aligned}\varphi(v) &= \varphi(\langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n) \\ &= \langle v, e_1 \rangle \varphi(e_1) + \cdots + \langle v, e_n \rangle \varphi(e_n) \\ &= \langle v, \overline{\varphi(e_1)} e_1 \rangle + \cdots + \langle v, \overline{\varphi(e_n)} e_n \rangle \\ &= \langle v, \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n \rangle.\end{aligned}$$

Suppose $\exists u_1, u_2 \in V$ *s.t.* $\varphi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle$. Then, $\langle v, u_1 \rangle - \langle v, u_2 \rangle = \langle v, u_1 - u_2 \rangle = 0$. Let $v = u_1 - u_2$, then we have $\langle u_1 - u_2, u_1 - u_2 \rangle = 0$. So, it must be $u_1 = u_2$. Therefore, $\exists$ a unique $u \in V$ and

$$u = \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n \text{ s.t. } \varphi(v) = \langle v, u \rangle \quad \forall v \in V.$$

■

Example 5.2.14 Find $u \in \mathcal{P}_2(\mathbb{R})$ *s.t.* $\displaystyle\int_{-1}^{1} p(t)(\cos(\pi t)) \, \mathrm{d}t = \int_{-1}^{1} p(t)u(t) \, \mathrm{d}t$ for every $p \in \mathcal{P}_2(\mathbb{R})$.

Remark. *Define an inner product on* $\mathcal{P}_2(\mathbb{R})$ *as* $\langle p, q \rangle = \displaystyle\int_{-1}^{1} p(x)q(x) \, \mathrm{d}x$ *to solve this problem.*

Answer 11.

Let $\varphi \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}), \mathbb{R})$ be defined as $\varphi(t) = \displaystyle\int_{-1}^{1} p(t)(\cos(\pi t)) \, \mathrm{d}t$. Note that $1, x, x^2$ is a basis of $\mathcal{P}_2(\mathbb{R})$. To find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, apply Gram-Schmidt Procedure, we have

$$e_1 = \frac{1}{\|1\|} = \frac{1}{\displaystyle\int_{-1}^{1} 1 \cdot 1 \, \mathrm{d}t} = \sqrt{\frac{1}{2}}.$$

□

5.3 Orthogonal Complements and Minimization Problems

6 Operators on Inner Product Spaces

6.1 Self-Adjoint and Normal Operators

6.2 The Spectral Theorem

6.3 Positive Operators and Isometries

6.4 Polar Decomposition and SVD

7 Operators on Complex Vector Spaces

7.1 Generalized Eigenvectors, Nilpotent Operators

7.2 Decomposition of an Operator

7.3 Characteristic and Minimal Polynomials

7.4 Jordan Form

8 Operators on Real Vectors Spaces

8.1 Complexification

8.2 Operators on Real Inner Product Spaces

9 Trace and Determinant

9.1 Trace

9.2 Determinant

10 Exercises

10.1 Span and Linear Independence

1. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ also spans V.

2. Prove that if $\mathbb{C}$ is a vector space on $\mathbb{R}$, then the list $1 + \mathrm{i}, 1 - \mathrm{i}$ is L.I..

3. Prove that if $\mathbb{C}$ is a vector space on $\mathbb{C}$, then the list $1 + \mathrm{i}, 1 - \mathrm{i}$ is linearly dependent.

4. Prove or give a counterexample: Suppose $v_1, v_2, \cdots, v_m$ is L.I. in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Then $\lambda v_1, \lambda v_2, \cdots, \lambda v_m$ is L.I..

5. Suppose $v_1, \cdots, v_m$ is L.I. in V and $w \in V$. Prove that if $v_1 + w, \cdots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \cdots, v_m)$.

10.2 Bases

1. Find all the vectors spaces that consist of only one basis.

   Hint. $\{0\}$.

2. Suppose U is a subspace of $\mathbb{R}^5$ *s.t.* $U = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2, x_3 = 7x_4\right\}$. Find a basis of U. Extend this basis into a basis of $\mathbb{R}^5$. Then, find a subspace W of $\mathbb{R}^5$ *s.t.* $\mathbb{R}^5 = U \oplus W$.

3. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of V.

4. **Prove** or disprove: $\mathcal{P}_3(\mathbb{F})$ has a basis p_0, p_1, p_2, p_3 *s.t.* no one from p_0, p_1, p_2, p_3 has a degree of 2.

   Hint. *Use the conclusion from #3.*

5. Prove or **give a counterexample**: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V *s.t.* $v_1, v_2 \in U, v_3 \notin U, v_4 \notin U$, then v_1, v_2 is basis of U.

10.3 Dimension

1. Suppose V is *f-d* and U is a subspace of V *s.t.* $\dim U = \dim V$. Prove that $U = V$.

2. Prove that the subspaces of $\mathbb{R}^2$ are exactly the following: $\{0\}, \mathbb{R}^2$, and all the lines passing through the origin in $\mathbb{R}^2$.

3. Suppose $v_1, \cdots, v_m$ is L.I. in V and $w \in V$. Prove $\dim \operatorname{span}(v_1 + w, \cdots, v_m + w) \geq m - 1$.

4. Suppose $p_0, p_1, \cdots, p_m \in \mathcal{P}(\mathbb{F})$ *s.t.* $\deg p_j = j$. Prove $p_0, p_1, \cdots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

5. Suppose U and W are subspaces of $\mathbb{R}^8$ *s.t.* $\dim U = 3, \dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

6. Suppose U and W are 5-dimensional subspaces of $\mathbb{R}^9$. Prove $U \cap W \neq \{0\}$.

7. Suppose U and W are 4-dimensional subspaces of $\mathbb{C}^6$. Prove that $\exists$ two vectors in $U \cap W$ *s.t.* any one of which is not a scalar multiple of another one.

8. Suppose $U_1, \cdots, U_m$ are *f-d* vector spaces of V. Prove that $U_1 + \cdots + U_m$ is *f-d* and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

9. Suppose V is *f-d* and $\dim V = n \geq 1$. Prove that $\exists$ 1-dimensional subspaces of V, $U_1, \cdots, U_n$ *s.t.*

$$V = U_1 \oplus \cdots \oplus U_n.$$

10. Suppose $U_1, \cdots, U_m$ are *f-d* vector subspaces of V *s.t.* $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is *f-d* and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

Hint. *Use mathematical induction.*

Remark. *This problem deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this problem to the following obvious statement: if a set is written as a disjoint union of finite subsets, then the number of elements in the set equals the sum of the numbers of elements in the disjoint subsets.*

11. Prove or give a counter example:

$$\begin{aligned}\dim(U_1 + U_2 + U_3) = &\dim U_1 + \dim U_2 + \dim U_3 \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Hint. *Consider* $U_1 = \{(x, 0) \mid x \in \mathbb{R}\},\ U_2 = \{(0, y) \mid y \in \mathbb{R}\},\ U_3 = \{(x, x) \mid x \in \mathbb{R}\}$.

10.4 The Vector Space of Linear Maps

1. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_m$ is a vector list in V *s.t.* $Tv_1, \cdots, Tv_m$ is L.I. in W. Prove that $v_1, \cdots, v_m$ is L.I..
2. Prove that $\mathcal{L}(V, W)$ is a vector space.
3. Prove the algebraic properties of products of linear maps.
4. Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then $\exists \lambda \in \mathbb{F}$ *s.t.* $Tv = \lambda v \quad \forall v \in V$.

10.5 Null Spaces and Range

1. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ *s.t.* $\operatorname{range} S \subset \operatorname{null} T$. Prove that $(ST)^2 = 0$.
2. Prove that $\nexists$ a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ *s.t.* $\operatorname{range} T = \operatorname{null} T$.
3. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \cdots, v_n$ is L.I. in V. Prove that $Tv_1, \cdots, Tv_n$ is L.I. in W.

4. Suppose $v_1, \cdots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that $Tv_1, \cdots, Tv_n$ spans $\operatorname{range} T$.

5. Suppose U is a 3-dimensional subspace of $\mathbb{R}^8$ and T is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ *s.t.* $\operatorname{null} T = U$. Prove that T is surjective.

6. Suppose V and W are *f-d*. Prove that $\exists$ an injective linear map from V to $W \iff \dim V \leq \dim W$.

7. Suppose U and V are *f-d* vector spaces, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(U, V)$. Prove

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

8. Suppose U and V are *f-d* vector spaces, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(U, V)$. Prove

$$\dim \operatorname{range} ST \leq \min \{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

9. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ *s.t.* $\deg Dp = (\deg p) - 1 \, \forall$ non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$.

   Remark. *The notation D is used above to remind you of the differentiation map that sends a polynomial p to p'. Without knowing the formula for the derivative of a polynomial (except that it reduces the degree by 1), you can use the exercise above to show that for every polynomial $q \in \mathcal{P}(\mathbb{R})$, $\exists$ a polynomial $p \in \mathcal{P}(\mathbb{R})$* s.t. $p' = q$.

10. Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that $\exists q \in \mathcal{P}(\mathbb{R})$ *s.t.* $5q'' + 3q' = p$.

    Remark. *This problem can be solved without using knowledge in Linear Algebra, but it is more interesting to solve with Linear Algebra.*

11. Suppose $T \in \mathcal{L}(V, W)$ and let $w_1, \cdots, w_m$ be a basis of $\operatorname{range} T$. Prove that $\exists \, \varphi_1, \cdots, \varphi_m \in \mathcal{L}(V, \mathbb{F})$ *s.t.* $Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m \quad \forall v \in V$.

10.6 Matrices

1. Suppose V and W are *f-d* and $T \in \mathcal{L}(V, W)$. Prove that for any basis in V and W, the matrix for T has at least $\dim \operatorname{range} T$ non-zero entries.

2. If $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

3. Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

10.7 Invertibility and Isomorphism

1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

2. Suppose V is *f-d* and $\dim V > 1$. Prove that the set of non-invertible operators on V is not a subspace of $\mathcal{L}(V)$.

3. Suppose V is *f-d* and U is a subspace of V. Let $S \in \mathcal{L}(U, V)$. Prove that $\exists$ invertible operator $T \in \mathcal{L}(V)$ *s.t.* $Tu = Su \quad \forall u \in U \iff S$ is injective.

4. Suppose W is *f-d* and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2 \iff \exists$ invertible operator $S \in \mathcal{L}(W)$ *s.t.* $T_1 = ST_2$.

5. Suppose V is *f-d* and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2 \iff \exists$ invertible operator $S \in \mathcal{L}(V)$ *s.t.* $T_1 = T_2 S$.

6. Suppose V is *f-d* and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible $\iff$ both S and T are invertible.

7. Suppose V is *f-d* and $S, T \in \mathcal{L}(V)$. Prove $ST = I \iff TS = I$.

8. Suppose V is *f-d* and $S, T, U \in \mathcal{L}(V)$ *s.t.* $STU = I$. Prove T is invertible and $T^{-1} = US$.

9. Suppose V is *f-d* and $R, S, T \in \mathcal{L}(V)$ *s.t.* RST is a surjection. Prove that S is an injection.

10. Suppose $v_1, \cdots, v_n$ is a basis of V. Define a linear map $T : V \to \mathbb{F}^{n,1}$ as $Tv = \mathcal{M}(v)$, where $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \cdots, v_n$. Prove that T is an isomorphism from V to $\mathbb{F}^{n,1}$.

11. Prove that $V \cong \mathcal{L}(\mathbb{F}, V)$.

10.8 Duality

1. Let V be *f-d* and U is a subspace of V *s.t.* $U \neq V$. Prove that $\exists\, \varphi \in V'$ *s.t.* $\forall u \in U,\ \varphi(u) = 0$ but $\varphi \neq 0$.

2. Suppose V is *f-d,* $v_1, \cdots, v_m \in V$. Define a linear map $\Gamma : V' \to \mathbb{F}^m$ as follows:

$$\Gamma(\varphi) = (\varphi(v_1), \cdots, \varphi(v_m)).$$

   (a) Prove that $v_1, \cdots, v_m$ spans V if and only if Γ is injective.

   (b) Prove that $v_1, \cdots, v_m$ is L.I. if and only if Γ is surjective.

3. Suppose $v_1, \cdots, v_n$ is a basis of V and $\varphi_1, \cdots, \varphi_n$ is the corresponding dual basis of v'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

4. Suppose W is *f-d* and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

5. Suppose V and W are *f-d.* Prove that the linear map from $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism between $L(V, W)$ and $L(W', V')$.

6. Suppose V is *f-d* and $U \subset V$. Prove that $U = \{0\}$ if and only if $U^0 = V'$.

7. Suppose V is *f-d* and U is a subspace of V. Prove that $U = V$ if and only if $U^0 = \{0\}$.

8. Suppose U and V are subsets of V with $U \subset W$. Prove that $W^0 \subset U^0$.

9. Suppose V is *f-d* and U and W are subspaces of V with $W^0 \subset U^0$. Prove that $U \subseteq W$.

10. Suppose U and W are subspaces of V. Prove that $(U + W)^0 = U^0 \cap W^0$.

11. Suppose V is *f-d* and U and W are subspaces of V. Prove that $(U \cap W)^0 = U^0 + W^0$.

10.9 Quotient Space

1. Suppose $v, x \in V$ and U and W are subspaces of V *s.t.* $v + U = x + W$. Prove that $U = W$.

10.10 Invariant Spaces

1. Suppose $T \in \mathcal{L}(V)$ and U is the subspace of V.
   (a) Prove that if $U \subset \operatorname{null} T$, then U is invariant under T.
   (b) Prove that if $\operatorname{range} T \subset U$ then U is invariant under T.
2. Suppose $S, T \in \mathcal{L}(V)$ *s.t.* $ST = TS$. Prove that $\operatorname{null} S$ is invariant under T.
3. Suppose $S, T \in \mathcal{L}(V)$ *s.t.* $ST = TS$. Prove that $\operatorname{range} S$ is invariant under T.
4. Suppose $T \in \mathcal{L}(V)$ and $U_1, \cdots, U_m$ are invariant subspaces of V under T. Prove that $U_1 + \cdots + U_m$ are invariant under T.
5. Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of any collection of invariant subspaces of V under T is also invariant under T.
6. Prove or give a counterexample: If V is *f-d* and U is a subspace of V *s.t.* U is invariant under every operators of V, then $U = \{0\}$ or $U = V$.
7. Define $T \in \mathcal{L}(\mathbb{R}^2)$ as $T(x, y) = (-3y, x)$. Find the eigenvalues of T.
8. Define $T \in \mathcal{L}(\mathbb{F}^2)$ as $T(w, z) = (z, w)$. Find all the eigenvalues and eigenvectors of T.
9. Define $T \in \mathcal{L}(\mathbb{F}^3)$ as $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$, Find all the eigenvalues and eigenvectors of T.
10. Suppose $T \in \mathcal{L}(V)$. Let $S \in \mathcal{L}(V)$ is invertible.
    (a) Prove that T and $S^{-1}TS$ have identical eigenvalues.
    (b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?
11. Define $T \in \mathcal{L}(\mathbb{C}^\infty)$ as $T(z_1, z_2, \cdots) = (0, z_1, z_2, \cdots)$. Prove that T has no eigenvalues.
12. Suppose $T \in \mathcal{L}(V)$ is invertible.
    (a) Suppose $\lambda \in \mathbb{F}$ and $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\dfrac{1}{\lambda}$ is an eigenvalue of T^{-1}.
    (b) Prove that T and T^{-1} have identical eigenvectors.
13. Suppose $T \in \mathcal{L}(V)$ and $\exists$ nonzero $v, w \in V$ *s.t.* $Tv = 3w$ and $Tw = 3v$. Prove that 3 or -3 is an eigenvalue of T.
14. Suppose V is *f-d* and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have identical eigenvalues.

10.11 Eigenvalues and Upper-Triangular Matrices

1. Suppose $T \in \mathcal{L}(V)$ and $\exists n \in \mathbb{Z}^+$ *s.t.* $T^n = 0$.

   (a) Prove that $(I - T)$ is invertible and $(I - T)^{-1} = I + T + \cdots + T^{n-1}$.

   (b) Give a rationale of coming up with the formula above.

2. Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$, $\lambda = 3$, or $\lambda = 4$.

3. Suppose $T \in \mathcal{L}(V)$, $T^2 = I$, and -1 is not an eigenvalue of T. Prove that $T = I$.

4. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

5. Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

6. Find a $T \in \mathcal{L}(\mathbb{R}^2)$ *s.t.* $T^4 = -I$.

7. Suppose $T \in \mathcal{L}(V)$ and v is the eigenvector of T corresponding to the eigenvalue λ. Let $p \in \mathcal{P}(\mathbb{F})$. Prove that $p(T)v = p(\lambda)v$.

10.12 Eigenspace and Diagonal Matrices

1. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{range} T$.

2. Suppose V is *f-d* and $T \in \mathcal{L}(V)$. Prove that the following conditions are equivalent:

   (a) $V = \operatorname{null} T \oplus \operatorname{range} T$;

   (b) $V = \operatorname{null} T + \operatorname{range} T$;

   (c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

3. Suppose V is *f-d*, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has identical eigenvectors with T (but not necessarily corresponding to the same eigenvalues). Prove that $ST = TS$.

4. Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix corresponding to some basis of V, denoted as A. Let $\lambda \in \mathbb{F}$. Prove that λ will appear on the diagonal of A exactly $\dim E(\lambda, T)$ times.

5. Suppose $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim E(8, T) = 4$. Prove that $(T - 2I)$ or $(T - 6I)$ is invertible.

6. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that for every nonzero $\lambda \in \mathbb{F}$, we have $E(\lambda, T) = \left(\frac{1}{\lambda}, T^{-1}\right)$.

7. Suppose V is *f-d* and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ be distinct nonzero eigenvalues. Prove that

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range} T.$$