

Emory University

MATH 362 Mathematical Statistics II

Learning Notes

Jiuru Lyu

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1 Estimation

1.1 Introduction

Definition 1.1.1 (Model). A *model* is a distribution with certain parameters.

Example 1.1.2 The normal distribution: $N(\mu, \sigma^2)$.

Definition 1.1.3 (Population). The *population* is all the objects in the experiment.

Definition 1.1.4 (Data, Sample, and Random Sample). *Data* refers to observed value from sample. The *sample* is a subset of the population. A *random sample* is a sequence of independent, identical (*i.i.d.*) random variables.

Definition 1.1.5 (Statistics). *Statistics* refers to a function of the random sample.

Example 1.1.6 The sample mean is a function of the sample:

$$\bar{Y} = \frac{1}{n}(Y_1 + \cdots + Y_n).$$

Example 1.1.7 Central Limit Theorem

We randomly toss $n = 200$ fair coins on the table. Calculate, using the central limit theorem, the probability that at least 110 coins have turned on the same side.

$$\bar{X} = \frac{X_1 + \cdots + X_{200}}{200} \stackrel{\text{CLT}}{\sim} N(\mu, \sigma^2),$$

where

$$\mu = \mathbf{E}(\bar{X}) = \frac{\sum_{i=1}^{200} \mathbf{E}(X_i)}{200},$$

$$\sigma^2 = \mathbf{Var}(\bar{X}) = \mathbf{Var}\left(\frac{X_1 + \cdots + X_{200}}{200}\right) = \frac{\sum_{i=1}^{200} \mathbf{Var}(X_i)}{200^2}.$$

Definition 1.1.8 (Statistical Inference). The process of *statistical inference* is defined to be the process of using data from a sample to gain information about the population.

Example 1.1.9 Goals in statistical inference

1. **Definition 1.1.10 (Estimation).** To obtain values of the parameters from the data.
2. **Definition 1.1.11 (Hypothesis Testing).** To test a conjecture about the parameters.
3. **Definition 1.1.12 (Goodness of Fit).** How well does the data fit a given distribution.
4. Linear Regression

1.2 The Method of Maximum Likelihood and the Method of Moments

Example 1.2.1 Given an unfair coin, or p -coin, such that

$$X = \begin{cases} 1 & \text{head with probability } p, \\ 0 & \text{tail with probability } 1 - p. \end{cases}$$

How can we determine the value p ?

Solution 1.

1. Try to flip the coin several times, say, three times. Suppose we get HHT.
2. Draw a conclusion from the experiment.

Key idea: The choice of the parameter p should be the value that maximizes the probability of the sample.

$$\mathbf{P}(X_1 = 1, X_2 = 1, X_3 = 0) = \mathbf{P}(X_1 = 1)\mathbf{P}(X_2 = 1)\mathbf{P}(X_3 = 0) = p^2(1 - p) := f(p).$$

Solving the optimization problem $\max_{p>0} f(p)$, we find it is most likely that $p = \frac{2}{3}$. This method is called the *likelihood maximization method*. □

Definition 1.2.2 (Likelihood Function). For a random sample of size n from the discrete (or continuous) pdf $p_X(k; \theta)$ (or $f_Y(y; \theta)$), the *likelihood function*, $L(\theta)$, is the product of the pdf evaluated at $X_i = k_i$ (or $Y_i = y_i$). That is,

$$L(\theta) := \prod_{i=1}^n p_X(k_i; \theta) \quad \text{or} \quad L(\theta) := \prod_{i=1}^n f_Y(y_i; \theta).$$

Definition 1.2.3 (Maximum Likelihood Estimate). Let $L(\theta)$ be as defined in Definition 1.2.2. If θ_e is a value of the parameter such that $L(\theta_e) \geq L(\theta)$ for all possible values of θ , then we call θ_e the *maximum likelihood estimate* for θ .

Theorem 1.2.4 The Method of Maximum Likelihood

Given random samples X_1, \dots, X_N and a density function $p_X(x)$ (or $f_X(x)$), then we have the likelihood function defined as

$$\begin{aligned} L(\theta) &= p_X(X; \theta) = \mathbf{P}(X_1, X_2, \dots, X_N) \\ &= \mathbf{P}(X_1)\mathbf{P}(X_2) \cdots \mathbf{P}(X_N) && [\text{independent}] \\ &= \prod_{i=1}^N p_X(X_i; \theta) && [\text{identical}] \end{aligned}$$

Then, the maximum likelihood estimate for θ is given by

$$\theta^* = \arg \max_{\theta} L(\theta),$$

where

$$L\left(\arg \max_{\theta} L(\theta)\right) = L^*(\theta) = \max_{\theta} L(\theta).$$

Example 1.2.5 Consider the Poisson distribution $X = 0, 1, \dots$, with $\lambda > 0$. Then, the pdf is given by

$$p_X(k, \lambda) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots$$

Given data k_1, \dots, k_n , we have the likelihood function

$$L(\lambda) = \prod_{i=1}^n p_X(X = k_i; \lambda) = \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{k_i}}{k_i!} = e^{-n\lambda} \frac{\lambda^{\sum k_i}}{k_1! \cdots k_n!}$$

Then, to find the maximum likelihood estimate of λ , we need to $\max_{\lambda} L(\lambda)$. That is to solve

$$\frac{\partial L(\lambda)}{\partial \lambda} = 0 \text{ and } \frac{\partial^2 L(\lambda)}{\partial \lambda^2} < 0.$$

Example 1.2.6 Waiting Time.

Consider the exponential distribution $f_Y(y) = \lambda e^{-\lambda y}$ for $y \geq 0$. Find the MLE λ_e of λ .

Solution 2.

The likelihood function of the exponential distribution is given by

$$\mathbf{L}(\lambda) = \prod_{i=1}^n \lambda e^{-\lambda y_i} = \lambda^n \exp \left(-\lambda \sum_{i=1}^n y_i \right).$$

Now, define

$$\ell(\lambda) = \ln \mathbf{L}(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^n y_i.$$

To optimize $\ell(\lambda)$, we compute

$$\frac{d}{d\lambda} \ell(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^n y_i \stackrel{\text{set}}{=} 0$$

So,

$$\frac{n}{\lambda} = \sum_{i=1}^n y_i \implies \lambda_e = \frac{n}{\sum_{i=1}^n y_i} =: \frac{1}{\bar{y}},$$

where \bar{y} is the sample mean. □

Example 1.2.7 Given the exponential distribution $f_Y(y) = \lambda e^{-\lambda y}$ for $y \geq 0$. Find the MLE of λ^2 .

Solution 3.

Define $\tau = \lambda^2$. Then, $\lambda = \sqrt{\tau}$, and so

$$f_Y(y) = \sqrt{\tau} e^{-\sqrt{\tau} y}, \quad y \geq 0.$$

Then, the likelihood function becomes

$$\mathbf{L}(\tau) = \prod_{i=1}^n f_Y(y) = \tau^{\frac{n}{2}} \exp \left(-\sqrt{\tau} \sum_{i=1}^n y_i \right).$$

Similarly, after maximization, we find

$$\tau_e = \frac{1}{(\bar{y})^2}.$$

□

Theorem 1.2.8 Invariant Property for MLE

Suppose λ_e is the MLE of λ . Define $\tau := h(\lambda)$. Then, $\tau_e = h(\lambda_e)$.

Proof 4. In this proof, we will prove the case when h is a one-to-one function. The case of h being a many-to-one function is beyond the scope of this course.

Suppose $h(\cdot)$ is a one-to-one function. Then, $\lambda = h^{-1}(\tau)$ is well-defined. Then,

$$\max_{\lambda} \mathbf{L}(\lambda; y_1, \dots, y_n) = \max_{\tau} \mathbf{L}(h^{-1}(\tau); y_1, \dots, y_n) = \max_{\tau} \mathbf{L}(\tau; y_1, \dots, y_n).$$

■

Example 1.2.9 Waiting Time with an unknown Threshold.

Let $\lambda = 1$ in exponential but there is an unknown threshold θ , that is, $f_Y(y) = e^{-(y-\theta)}$ for $y \geq \theta$, $\theta > 0$.

Solution 5.

Note that the likelihood function is given by

$$\begin{aligned} \mathbf{L}(\theta; y_1, \dots, y_n) &= \prod_{i=1}^n f_Y(y_i) = \exp \left(- \sum_{i=1}^n (y_i - \theta) \right), \quad y_i \geq \theta, \theta > 0 \\ &= \exp \left(- \sum_{i=1}^n (y_i - \theta) \right) \cdot \mathbb{1}_{[y_i \geq \theta, \theta > 0]}, \end{aligned}$$

where

$$\mathbb{1}_{x \in A} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Using order statistics,

$$\begin{aligned} \mathbf{L}(\theta) &= \exp \left(- \sum_{i=1}^n (y_i - \theta) \right) \cdot \mathbb{1}_{[y_{(n)} \geq y_{(n-1)} \geq \dots \geq y_{(1)} \geq \theta, \theta > 0]} \\ &= \exp \left(- \sum_{i=1}^n y_i + n\theta \right) \mathbb{1}_{[y_{(n)} \geq \dots \geq y_{(1)} \geq \theta, \theta > 0]}. \end{aligned}$$

So, we know $\theta \leq y_{(1)} = y_{\min}$.

To maximize the likelihood function, we want to maximize $-\sum y_i + n\theta$. That is, to maximize θ , as $\theta \leq y_{\min}$, it must be that $\theta_{\max} = y_{\min}$. Therefore, the MLE is $\theta^* = y_{\min}$. \square

Example 1.2.10 Suppose $Y_1, \dots, Y_n \sim \text{Uniform}[0, a]$. That is, $f_Y(y; a) = \frac{1}{a}$ for $y \in [0, a]$. Find MLE a_e of a .

Solution 6.

Note that

$$\begin{aligned} f_Y(y; a) &= \frac{1}{a} \cdot \mathbb{1}_{\{y \in [0, a]\}} \\ &= \frac{1}{a} \cdot \mathbb{1}_{\{0 \leq y_{(1)} \leq \dots \leq y_{(n)} \leq a\}} \end{aligned} \quad \text{where } y_{(1)} = \min y_i \text{ and } y_{(n)} = \max y_i$$

Then,

$$\mathbf{L}(a) = \frac{1}{a^n} \mathbb{1}_{\{0 \leq y_{(1)} \leq \dots \leq y_{(n)} \leq a\}}$$

To maximize $\mathbf{L}(a)$, we want to minimize a^n . Since $a \geq y_{(n)}$, it must be that $a_e = y_{(n)}$. Here, we call $a_e = y_{(n)}$ an *estimate*, and $\widehat{a_{\text{MLE}}} = Y_{(n)}$ an *estimator*. \square

Example 1.2.11 MLE that Does Not Exist

Suppose $f_Y(y; a) = \frac{1}{a}$, $y \in [0, a)$. Find the MLE.

Solution 7.

The likelihood function is the same:

$$\mathbf{L}(a) = \frac{1}{a^n} \mathbb{1}_{\{0 \leq y_{(1)} \leq \dots \leq y_{(n)} < a\}}.$$

However, since $[0, a)$ is not a closed set, the optimization problem $\max_{a \in [0, a)} \mathbf{L}(a)$ does not have a solution. Hence, the estimate does not exist. \square

Remark 1.1 MLE may not be unique all the time.

Example 1.2.12 Multiple MLE Values

Suppose $X_1, \dots, X_n \sim \text{Uniform}\left[a - \frac{1}{2}, a + \frac{1}{2}\right]$, where $f_X(x; a) = 1$, $x \in \left[a - \frac{1}{2}, a + \frac{1}{2}\right]$. Find the MLE.

Solution 8.

In the indicator function notation, we can rewrite the pdf to be

$$f_X(x; a) = \mathbb{1}_{\{a - \frac{1}{2} \leq x \leq a + \frac{1}{2}\}} = \mathbb{1}_{\{a - \frac{1}{2} \leq x_{(1)} \leq \dots \leq x_{(n)} \leq a + \frac{1}{2}\}}.$$

So, the likelihood function will be

$$L(a) = \prod_{i=1}^n f_x(x_i; a) = \begin{cases} 1, & a \in \left[x_{(n)} - \frac{1}{2}, x_{(1)} + \frac{1}{2} \right] \\ 0, & \text{otherwise.} \end{cases}$$

So, the $L(a)$ will be maximized whenever $a \in \left[x_{(n)} - \frac{1}{2}, x_{(1)} + \frac{1}{2} \right]$. Therefore, MLE can be any value in the range $\left[x_{(n)} - \frac{1}{2}, x_{(1)} + \frac{1}{2} \right]$. Say,

$$a_e = x_{(n)} - \frac{1}{2} \quad \text{or} \quad a_e = x_{(1)} + \frac{1}{2} \quad \text{or} \quad a_e = \frac{x_{(n)} - \frac{1}{2} + x_{(1)} + \frac{1}{2}}{2} = \frac{x_{(n)} + x_{(1)}}{2}.$$

□

Theorem 1.2.13 MLE for Multiple Parameters

In general, we have the likelihood function $L(\theta)$, where $\theta = (\theta_1, \dots, \theta_p)$. To find the MLE, we need

$$\frac{\partial L(\theta)}{\partial \theta_i} = 0 \quad i = 1, \dots, p,$$

and the Hessian matrix

$$\left(\frac{\partial^2 L(\theta)}{\partial \theta_i \partial \theta_j} \right)_{i,j=1,\dots,p} := \begin{pmatrix} \frac{\partial^2 L(\theta)}{\partial \theta_1^2} & \cdots & \frac{\partial^2 L(\theta)}{\partial \theta_1 \partial \theta_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L(\theta)}{\partial \theta_p \partial \theta_1} & \cdots & \frac{\partial^2 L(\theta)}{\partial \theta_p^2} \end{pmatrix}$$

should be negative definite.

Example 1.2.14 MLE for Multiple Parameters: Normal Distribution

Suppose $Y_1, \dots, Y_n \sim N(\mu, \sigma)$. Then,

$$f_{Y_i}(u; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y_i - \mu)^2 / (2\sigma^2)}.$$

Find the MLE for μ and σ .

Solution 9.

The likelihood function will be

$$\mathbf{L}(\mu, \sigma) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-(y_i - \mu)^2 / (2\sigma^2)}.$$

Then, we define

$$\ell(\mu, \sigma) = \ln \mathbf{L}(\mu, \sigma) = -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 - \frac{1}{2} (\sigma^2)^{-1} \sum_{i=1}^n (y_i - \mu)^2.$$

Set

$$\begin{cases} \frac{\partial \ell(\mu, \sigma)}{\partial \mu} = 0 & \textcircled{1} \\ \frac{\partial \ell(\mu, \sigma)}{\partial \sigma} = 0 & \textcircled{2} \end{cases}$$

From ①, we have

$$\begin{aligned} \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \mu) &= 0 \\ \sum_{i=1}^n y_i &= n\mu \implies \boxed{\mu_e = \frac{\sum y_i}{n} = \bar{y}} \end{aligned}$$

From ②, by the invariant property of MLE, we instead set

$$\begin{aligned} \frac{\partial \ell(\mu, \sigma)}{\partial \sigma^2} &= 0 \\ -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2} \left(\frac{1}{\sigma^2} \right)^2 \sum_{i=1}^n (y_i - \mu)^2 &= 0 \\ \frac{1}{2\sigma^2} \left(-n + \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 \right) &= 0 \\ -n\sigma^2 + \sum_{i=1}^n (y_i - \mu)^2 &= 0 \quad (\mu_e = \bar{y}) \\ \sum_{i=1}^n (y_i - \bar{y})^2 &= n\sigma^2 \\ \sigma_e^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2 &\implies \boxed{\sigma_e = \sqrt{\frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2}} \end{aligned}$$

□

1.3 The Method of Moment

Definition 1.3.1 (Moment Generating Function). The *Moment Generating Function (MGF)* is defined as

$$\mathbf{M}_X(t) = \mathbf{E}[e^{tX}],$$

and it uniquely determines a probability distribution.

Definition 1.3.2 (Moment). The *k-th order moment* of X is $\mathbf{E}[X^k]$.

Example 1.3.3 Meaning of Different Moments

- $\mathbf{E}[X]$: location of a distribution
- $\mathbf{E}[X^2] = \text{Var}(X) + \mathbf{E}[X]^2$: width of a distribution
- $\mathbf{E}[X^3]$: skewness – positively skewed / negatively skewed
- $\mathbf{E}[X^4]$: kurtosis / tailedness – speed decaying to 0.

Example 1.3.4 Moment Estimate: Moments of Population and Sample

Population	Sample, X_1, \dots, X_n
$\mathbf{E}[X] = \mu$	$\hat{\mu} = \overline{X} = \frac{X_1 + \dots + X_n}{n}$
$\mathbf{E}[X^2] = \mu^2 + \sigma^2$	$\hat{\mu}^2 + \hat{\sigma}^2 = \frac{X_1^2 + \dots + X_n^2}{n}$
\vdots	\vdots
$\mathbf{E}[X^k]$	$\frac{X_1^k + \dots + X_n^k}{n}$

Rationale: The population moments should be close to the sample moments.

Example 1.3.5

- Consider $N(\mu, \sigma^2)$, where σ is given. Estimate μ .

By the method of moment estimate, we have $\mu_e = \overline{X}$.

- Consider $N(\mu, \sigma^2)$. Estimate μ and σ .

We have $\mu_e = \overline{X}$ and $\mu_e^2 + \sigma_e^2 = \frac{X_1^2 + \dots + X_n^2}{n}$.

- Consider $N(\theta, \sigma^2)$. Given $E(X^4) = 3\sigma^4$, estimate μ and σ .

We have $\mu_e = \bar{X}$, $\mu_e^2 + \sigma_e^2 = \frac{X_1^2 + \cdots + X_n^2}{n}$, and $3\sigma^4 = \frac{X_1^4 + \cdots + X_n^4}{n}$. We have three equations but only two unknowns, then a solution is not guaranteed. So, we need some restrictions on this method (see Remark 1.2).

Theorem 1.3.6 Method of Moments Estimates

For a random sample of size n from the discrete (or continuous) population/pdf $p_X(k; \theta_1, \dots, \theta_s)$ (or $f_Y(y; \theta_1, \dots, \theta_s)$), solutions to the system

$$\begin{cases} E(Y) = \frac{1}{n} \sum_{i=1}^n y_i \\ \vdots \\ E(Y^s) = \frac{1}{n} \sum_{i=1}^n y_i^s \end{cases}$$

which are denoted by $\theta_{1e}, \dots, \theta_{se}$, are called the **method of moments estimates** of $\theta_1, \dots, \theta_s$.

Remark 1.2 To estimate k parameters with the method of moments estimates, we will only match the first k orders of moments.

Example 1.3.7 Consider the Gamma distribution:

$$f_Y(y; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y} \quad \text{for } y \geq 0.$$

Given $E(Y) = \frac{r}{\lambda}$ and $E(Y^2) = \frac{r}{\lambda^2} + \frac{r^2}{\lambda^2}$. Estimate r and λ .

Solution 1.

$$E(Y) = \frac{r}{\lambda} \implies \frac{r_e}{\lambda_e} = \frac{y_1 + \cdots + y_n}{n} = \bar{y} \quad \text{①}$$

$$E(Y^2) = \frac{r}{\lambda^2} + \frac{r^2}{\lambda^2} \implies \frac{r_e}{\lambda_e^2} + \frac{r_e^2}{\lambda_e^2} = \frac{y_1^2 + \cdots + y_n^2}{n} \quad \text{②}$$

Substitute ① into ②, we have

$$\frac{\bar{y}}{\lambda_e} + (\bar{y})^2 = \frac{1}{n} \sum_{i=1}^n y_i^2 \implies \boxed{\lambda_e = \frac{\bar{y}}{\frac{1}{n} \sum y_i^2 - \bar{y}^2}} \quad \text{③}$$

Substitute ③ into ①, we have

$$r_e = \bar{y}\lambda_e = \frac{\bar{y}^2}{\frac{1}{n} \sum y_i^2 - \bar{y}^2}.$$

□

Remark 1.3 *The sample variance is defined as*

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2 &= \frac{1}{n} \sum_{i=1}^n (y_i^2 - 2y_i\bar{y} + \bar{y}^2) \\ &= \frac{1}{n} \sum_{i=1}^n y_i^2 - 2\bar{y} \cdot \frac{\sum y_i}{n} + \frac{1}{n} \cdot n\bar{y}^2 \\ &= \frac{1}{n} \sum_{i=1}^n y_i^2 - 2\bar{y}^2 + \bar{y}^2 \\ &= \frac{1}{n} \sum_{i=1}^n y_i^2 - \bar{y}^2. \end{aligned} \quad \bar{y} = \frac{\sum y_i}{n}$$

So, in Example 1.3.7, if we define $\hat{\sigma}^2$ to be the sample variance, we can further simplify our estimate as follows:

$$\lambda_e = \frac{\bar{y}}{\hat{\sigma}^2}, \quad r_e = \frac{\bar{y}^2}{\hat{\sigma}^2}.$$

1.4 Interval Estimation

Example 1.4.1 Estimate μ , where $X \sim N(\mu, 1)$.

We take some samples and compute their sample means:

$$\bar{X}^1 = \frac{x_1 + \cdots + x_n}{n}, \bar{X}^2 = \frac{\tilde{x}_1 + \cdots + \tilde{x}_n}{n}, \dots$$

Finding the distribution of \bar{X} , we can find an interval $[\hat{\theta}_L, \hat{\theta}_U]$ such that

$$\mathbf{P}(\hat{\theta}_L \leq \bar{X} \leq \hat{\theta}_U) = 1 - \alpha.$$

Remark 1.4 *By using the variance of the estimator, one can construct an interval such that with a high probability that the interval contains the unknown parameter.*

Definition 1.4.2 (Confidence Interval). The interval, $[\hat{\theta}_L, \hat{\theta}_U]$ is called the *confidence interval*, and the high probability is $1 - \alpha$, where α is given.

Remark 1.5 Take $\alpha = 5\%$, then $[\hat{\theta}_L, \hat{\theta}_U]$ is the 95% confidence interval of μ . It does not mean that μ has 95% chance to be in $[\hat{\theta}_L, \hat{\theta}_U]$. However, if we construct 1000 such intervals, 950 of them will contain μ .

Example 1.4.3 A random sample of size 4, ($Y_1 = 6.5, Y_2 = 9.2, Y_3 = 9.9, Y_4 = 12.4$), from a normal population:

$$f_Y(y; \mu) = \frac{1}{\sqrt{2\pi}0.8} e^{-\frac{1}{2}\left(\frac{y-\mu}{0.8}\right)^2} \sim N(\mu, \sigma^2 = 0.64).$$

Both MLE and MME give $\mu_e = \bar{y} = 9.5$. The estimator $\hat{\mu} = \bar{Y}$ follows normal distribution. Construct 95%-confidence interval for μ .

Solution 1.

$E(\bar{Y}) = \mu$ and $\text{Var}(\bar{Y}) = \frac{\sigma^2}{n} = \frac{0.64}{4}$. By the Central Limit Theorem, \bar{Y} approximately follow $N\left(\mu, \frac{\sigma^2}{n}\right)$. So, $\frac{\bar{Y} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0, 1)$. Then,

$$\mathbf{P}\left(z_1 \leq \frac{\bar{Y} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \leq z_2\right) = 0.95 \implies \mathbf{P}\left(\bar{Y} - z_2 \sqrt{\frac{\sigma^2}{n}} \leq \mu \leq \bar{Y} - z_1 \sqrt{\frac{\sigma^2}{n}}\right) = 0.95$$

There are infinite many ways to construct a confidence interval by selecting different z_1 and z_2 . However, since we don't have any prior knowledge on μ , it is good for us to choose z_1 and z_2 symmetrically. Moreover, symmetric z_1 and z_2 will yield a smaller interval. We know the symmetric z_1, z_2 pair will be $z_1 = -1.96$ and $z_2 = 1.96$. Therefore,

$$\mathbf{P}\left(\bar{Y} - 1.96 \sqrt{\frac{0.64}{4}} \leq \mu \leq \bar{Y} + 1.96 \sqrt{\frac{0.64}{4}}\right) = 0.95.$$

Then, 95% confidence interval is $[9.5 - 1.96 \times 0.4, 9.5 + 1.96 \times 0.4]$. □

Theorem 1.4.4 Confidence Interval

In general, for a normal population with σ known, the $100(1 - \alpha)\%$ *two-sided confidence interval* for μ is

$$\left(\bar{y} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$$

Theorem 1.4.5 Variation of Confidence Interval

- One-sided interval:

$$\left(\bar{y} - z_{\alpha} \frac{\sigma}{\sqrt{n}}, \bar{y} \right) \text{ or } \left(\bar{y}, \bar{y} + z_{\alpha} \frac{\sigma}{\sqrt{n}} \right)$$

- σ is unknown and sample size is small: z -score $\rightarrow t$ -score.
- σ is unknown and sample size is large: z -score by CLT.
- Non Gaussian population but sample size is large: z -score by CLT.

Theorem 1.4.6

Let k be the number of successes in n independent trials, where n is large and $p = P(\text{success})$ is unknown. An approximate $100(1 - \alpha)\%$ confidence interval for p is the set of numbers

$$\left(\frac{k}{n} - z_{\alpha/2} \sqrt{\frac{(k/n)(1 - k/n)}{n}}, \frac{k}{n} + z_{\alpha/2} \sqrt{\frac{(k/n)(1 - k/n)}{n}} \right).$$

Definition 1.4.7 (Margin of Error). The *margin of error*, denoted by d , is the quantity

$$d = z_{\alpha/2} \sqrt{\frac{(k/n)(1 - k/n)}{n}}.$$

Remark 1.6 *Stating the sample mean and the margin of error is equivalent to stating the confidence interval. Note that C.I. = $\hat{p} \pm d$.*

Theorem 1.4.8 Estimate Margin of Error

When p is close to $\frac{1}{2}$, then $d \approx d_m = \frac{z_{\alpha/2}}{2\sqrt{n}}$, which is equivalent to $\sigma_n \approx \frac{1}{2\sqrt{n}}$. However, if p is away from $\frac{1}{2}$, d and d_m are very different.

Remark 1.7 *Theorem 1.4.8 gives a conservative estimation of the margin of error, which is d_m .*

Proposition 1.9 : Given d , we can estimate the sample size.

Proof 2.

$$d = z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \implies n \approx \hat{p}(1 - \hat{p}) / \left(\frac{d}{z_{\alpha/2}} \right)^2.$$

However, since n is unknown, \hat{p} is also unknown. We, therefore, need information on the actual p to conclude an estimation of the sample size.

- If p is known,

$$n = \frac{p(1-p)}{\left(\frac{d}{z_{\alpha/2}}\right)^2}.$$

- If p is unknown. Let $f(p) = p(1-p)$. f will be maximized when $p = 0.5$. So, $f(p) = p(1-p) \leq 0.25$. Then,

$$n \leq \frac{0.25}{\left(\frac{d}{z_{\alpha/2}}\right)^2}.$$

Since we are conservative, take $n = \frac{\frac{1}{4}z_{\alpha/2}^2}{d^2} = \frac{z_{\alpha/2}^2}{4d^2}$. This estimation is a conservative estimation of the sample size.

■

1.5 Properties of Estimation

The main question is that estimators are not unique in general. How do we choose a good estimator?

Definition 1.5.1 (Unbiasedness). Given a random sample of size n when whose population distribution depends on an unknown parameter θ . Let $\hat{\theta}$ be an estimator of θ . Then,

- $\hat{\theta}$ is called *unbiased* if $\mathbf{E}(\hat{\theta}) = \theta$.
- $\hat{\theta}$ is called *asymptotically unbiased* if $\lim_{n \rightarrow \infty} \mathbf{E}(\hat{\theta}) = \theta$.
- If θ is biased, then the *bias* is given by the quantity $\mathbf{B}(\hat{\theta}) = \mathbf{E}(\hat{\theta}) - \theta$.

Example 1.5.2 Consider the exponential distribution: $f_Y(y; \lambda) = \lambda e^{-\lambda y}$ for $y \geq 0$. Determine if the estimator $\hat{\lambda} = \frac{1}{\bar{Y}}$ is biased or not.

Hint: $n\bar{Y} = \sum_{i=1}^n Y_i \sim \text{Gamma}(n, \lambda)$.

Solution 1.

Recall that $\mathbf{E}[g(x)] = \int_x g(x) f_X(x) dx$. Define $X = \sum_{i=1}^n Y_i \sim \text{Gamma}(n, \lambda)$. Also, recall the following facts:

$$\Gamma(n) = (n-1)! = (n-1)\Gamma(n-1)$$

and the integration over any probability density function will yield a result of 1 by definition.

Then,

$$\begin{aligned}
 \mathbf{E}(\hat{\lambda}) &= \mathbf{E}\left(\frac{1}{\bar{Y}}\right) = \mathbf{E}\left(\frac{n}{\sum Y_i}\right) = n\mathbf{E}\left(\frac{1}{\sum Y_i}\right) \\
 &= n\mathbf{E}\left(\frac{1}{X}\right) \\
 &= n \int_x \frac{1}{x} \cdot \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} dx \\
 &= n \int_x \frac{\lambda^n}{(n-1)!} x^{n-2} e^{-\lambda x} dx \\
 &= \frac{n\lambda}{(n-1)} \underbrace{\int_x \frac{\lambda^{n-1}}{\Gamma(n-1)} x^{n-2} e^{-\lambda x} dx}_{=1} \\
 &= \frac{n}{n-1} \lambda.
 \end{aligned}$$

Therefore, $\mathbf{E}(\hat{\lambda}) \neq \lambda$, and so $\hat{\lambda}$ is biased. However, note that

$$\lim_{n \rightarrow \infty} \mathbf{E}(\hat{\lambda}) = \lim_{n \rightarrow \infty} \frac{n}{n-1} \lambda = \lambda.$$

By definition, then $\hat{\lambda}$ is asymptotically unbiased. □

Example 1.5.3 Consider the exponential distribution $f(y; \theta) = \frac{1}{\theta} e^{-y/\theta}$ for $y \geq 0$. Then, $\hat{\theta} = \bar{Y}$ is unbiased.

Remark 1.8 Suppose $\{X_1, \dots, X_n\}$ are i.i.d. random variables, and $\mathbf{E}(X_i) = \mu$ for $i = 1, \dots, n$. Then, \bar{X} , the sample mean, is always an unbiased estimator:

$$\mathbf{E}(\bar{X}) = \mathbf{E}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n \mathbf{E}(X_i) = \frac{1}{n} \cdot n \cdot \mu = \mu.$$

Theorem 1.5.4 Sample Variance is Biased

Suppose $\{X_1, \dots, X_n\}$ are i.i.d. random variables, and $\mathbf{E}(X_i) = \mu$, $\text{Var}(X_i) = \sigma^2$ for $i = 1, \dots, n$. Then, the sample variance $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$ is biased.

Proof2. Note that

$$\begin{aligned}
 \mathbf{E}(\hat{\sigma}^2) &= \mathbf{E}\left(\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2\right) \\
 &= \mathbf{E}\left(\frac{1}{n} \sum_{i=1}^n (X_i - \mu + \mu - \bar{X})^2\right) \\
 &= \frac{1}{n} \sum_{i=1}^n \mathbf{E}\left[(X_i - \mu)^2 + (\mu - \bar{X})^2 + 2(X_i - \mu)(\mu - \bar{X})\right] \\
 &= \frac{1}{n} \sum_{i=1}^n \left\{ \underbrace{\mathbf{E}(X_i - \mu)^2}_{\text{Var}(X_i)} + \mathbf{E}(\mu - \bar{X})^2 + 2\mathbf{E}[(\mu - \bar{X})(X_i - \mu)] \right\} \\
 &\quad \left| \begin{array}{l} \text{Hint: } \frac{1}{n} \sum_{i=1}^n (X_i - \mu) = \frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=1}^n \mu = \bar{X} - \mu \end{array} \right. \\
 &= \frac{1}{n} \sum_{i=1}^n \text{Var}(X_i) + \frac{1}{n} \cdot n \mathbf{E}(\mu - \bar{X})^2 + 2\mathbf{E}\left[(\mu - \bar{X}) \frac{1}{n} \sum_{i=1}^n (X_i - \mu)\right] \\
 &= \frac{1}{n} \sum_{i=1}^n \sigma^2 + \mathbf{E}(\mu - \bar{X})^2 + 2\mathbf{E}[(\mu - \bar{X})(\bar{X} - \mu)] \\
 &= \frac{1}{n} \cdot n \cdot \sigma^2 + \mathbf{E}(\mu - \bar{X})^2 - 2\mathbf{E}[(\mu - \bar{X})^2] \\
 &= \sigma^2 - \mathbf{E}(\mu - \bar{X})^2 \\
 &= \sigma^2 - \underbrace{\mathbf{E}(\bar{X} - \mu)^2}_{=\text{Var}(\bar{X})} \\
 &= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n} \sigma^2 \neq \sigma^2
 \end{aligned}$$

Therefore, $\hat{\sigma}^2$ is not an unbiased estimator. ■

Theorem 1.5.5 Adjusted Sample Variance is Unbiased

With the same set up in Theorem 1.5.4, define the adjusted sample variance to be

$$S^2 = \frac{n}{n-1} \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2.$$

Then, S^2 is an unbiased estimator of σ^2 .

Definition 1.5.6 (Decision Theory). Minimize the error of an estimator (sample statistics) relative to the true parameter (population parameter) using a loss function.

Definition 1.5.7 (Mean Squared Error). The *mean squared error* (MSE) is defined by

$$\text{MSE}(\hat{\theta}) = \mathbf{E}[(\hat{\theta} - \theta)^2]$$

Theorem 1.5.8 Decomposition of MSE

Generally,

$$\text{MSE}(\theta) = \text{Var}(\hat{\theta}) + \mathbf{B}(\hat{\theta})^2$$

If $\hat{\theta}$ is unbiased, $\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta})$. $\text{Var}(\theta)$ measures the precision of the estimator.

Proof 3. Note that we will the following:

$$\begin{aligned} \text{MSE}(\hat{\theta}) &= \mathbf{E}[(\hat{\theta} - \theta)^2] \\ &= \mathbf{E}(\hat{\theta}^2 + \theta^2 - 2\hat{\theta}\theta) \\ &= \mathbf{E}(\hat{\theta}^2) - 2\theta\mathbf{E}(\hat{\theta}) + \theta^2 \\ &= \underbrace{\mathbf{E}(\hat{\theta}^2) - \mathbf{E}(\hat{\theta})^2}_{\text{Var}(\hat{\theta})} + \underbrace{\mathbf{E}(\hat{\theta})^2 - 2\theta\mathbf{E}(\hat{\theta}) + \theta^2}_{[\mathbf{E}(\hat{\theta}) - \theta]^2} \\ &= \text{Var}(\hat{\theta}) + [\mathbf{E}(\hat{\theta}) - \theta]^2 \\ &= \text{Var}(\theta) + \mathbf{B}(\hat{\theta})^2 \end{aligned}$$

If $\hat{\theta}$ is unbiased, $\mathbf{B}(\hat{\theta}) = 0$, and so $\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta})$. ■

Definition 1.5.9 (Efficiency). Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators for a parameter θ . If we have $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$, then we say that $\hat{\theta}_1$ is *more efficient* than $\hat{\theta}_2$. The *relative efficiency* of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is the ratio $\frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)}$.

1.6 Best Unbiased Estimator

Definition 1.6.1 (Best/Minimum-Variance Estimator). Let Θ be the set of all estimators $\hat{\theta}$ that are unbiased for the parameter θ . We say that $\hat{\theta}^*$ is a *best* or *minimum-variance estimator* (MVE) if $\hat{\theta}^* \in \Theta$ and $\text{Var}(\hat{\theta}^*) \leq \text{Var}(\hat{\theta}) \quad \forall \hat{\theta} \in \Theta$.

Definition 1.6.2 (Fisher's Information). The *Fisher's information* of a continuous random variable Y with pdf $f_Y(y; \theta)$ is defined as

$$\mathbf{I}(\theta) = \mathbf{E} \left[\left(\frac{\partial \ln f_Y(y; \theta)}{\partial \theta} \right)^2 \right] = -\mathbf{E} \left[\frac{\partial^2}{\partial \theta^2} \ln f_Y(y; \theta) \right].$$

Remark 1.9 The Fisher's information measures the amount of information that a sample Y contains about the unknown parameter θ . If $\mathbf{I}(\theta)$ is big, then the curvature of $f_Y(y; \theta)$ is big, and

thus it is more likely that we can find a region where $\hat{\theta}$ is concentrated.

Extension 1.1 (Joint Fisher's Information) Suppose Y_1, \dots, Y_n are continuous i.i.d. random variables, each has a Fisher's information of $\mathbf{I}(\theta)$. Then,

$$\mathbf{E} \left[\left(\frac{\partial}{\partial \theta} \ln f_{Y_1, \dots, Y_n}(y_1, \dots, y_n; \theta) \right)^2 \right] = n\mathbf{I}(\theta).$$

Theorem 1.6.3 Properties of Fisher's Information

Define the *Fisher's Score Function* $\frac{\partial}{\partial \theta} \ln f_Y(y; \theta)$. Then,

$$\mathbf{E}_Y \left[\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right] = 0.$$

Proof 1. Note that by chain rule, we have

$$\begin{aligned} \mathbf{E}_Y \left[\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right] &= \int_Y \left(\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right) f_Y(y; \theta) \, dy \\ &= \int_Y \frac{1}{f_Y(y; \theta)} \left(\frac{\partial}{\partial \theta} f_Y(y; \theta) \right) f_Y(y; \theta) \, dy \\ &= \int_Y \frac{\partial}{\partial \theta} f_Y(y; \theta) \, dy \\ &= \frac{\partial}{\partial \theta} \int_Y f_Y(y; \theta) \, dy = \frac{\partial}{\partial \theta} (1) = 0. \end{aligned}$$

■

Corollary 1.4 :

$$\mathbf{I}(\theta) = \mathbf{Var} \left(\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right).$$

Proof 2. By definition, we have

$$\begin{aligned} \mathbf{Var} \left(\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right) &= \mathbf{E} \left[\left(\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right)^2 \right] - \underbrace{\left(\mathbf{E} \left(\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right) \right)^2}_{=0, \text{ by Theorem 1.6.3.}} \\ &= \mathbf{E} \left[\left(\frac{\partial}{\partial \theta} \ln f_Y(y; \theta) \right)^2 \right] \\ &= \mathbf{I}(\theta). \end{aligned}$$

■

Theorem 1.6.5 Cramér-Rao Inequality

Under regular condition, let Y_1, \dots, Y_n be a random sample of size n form the continuous population pdf $f_Y(y; \theta)$. Let $\hat{\theta} = \hat{\theta}(Y_1, \dots, Y_n)$ be any unbiased estimator for θ . Then,

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n\mathbf{I}(\theta)}.$$

Remark 1.10 A similar statement holds for the discrete case $p_X(k; \theta)$.

Definition 1.6.6 (Efficiency of Unbiased Estimator). An unbiased estimator $\hat{\theta}$ is *efficient* if $\text{Var}(\hat{\theta})$ is equal to the Cramér-Rao lower bound. That is, $\text{Var}(\hat{\theta}) = (n\mathbf{I}(\theta))^{-1}$. Such an estimator is the MVE defined in Definition 1.6.1. The *efficiency* of an unbiased estimator $\hat{\theta}$ is defined to be the quantity

$$\left(n\mathbf{I}(\theta)\text{Var}(\hat{\theta})\right)^{-1}.$$

Example 1.6.7 Suppose $X \sim \text{Bernoulli}(p)$. Is $\hat{p} = \bar{X}$ efficient?

Solution 3.

Note that we have the following

$$\begin{aligned} f_X(x; p) &= p^x(1-p)^{1-x}, \quad x = 0, 1 \\ \ln f_X(x; p) &= x \ln p + (1-x) \ln(1-p) \\ \frac{\partial}{\partial p} \ln f_X(x; p) &= \frac{x}{p} - \frac{1-x}{1-p} \\ \frac{\partial^2}{\partial p^2} \ln f_X(x; p) &= -\frac{x}{p^2} - \frac{1-x}{(1-p)^2} \end{aligned}$$

Therefore, the Fisher's information can be computed by

$$\begin{aligned} \mathbf{I}(p) &= -\mathbf{E} \left[\frac{\partial^2}{\partial p^2} \ln f_X(x; p) \right] = -\mathbf{E} \left[-\frac{x}{p^2} - \frac{1-x}{(1-p)^2} \right] \\ &= \mathbf{E} \left[\frac{x}{p^2} \right] + \mathbf{E} \left[\frac{1-x}{(1-p)^2} \right] \\ &= \frac{\mathbf{E}(x)}{p^2} + \frac{1 - \mathbf{E}(x)}{(1-p)^2} \\ &= \frac{p}{p^2} + \frac{1-p}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}. \end{aligned}$$

Note that

$$\text{Var}(\bar{X}) = \text{Var} \left(\frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n} \text{Var}(X_i) = \frac{1}{n} \cdot p(1-p).$$

So, we have

$$\text{Var}(\bar{X}) = \frac{p(1-p)}{n} = \frac{1}{n \left(\frac{1}{p(1-p)} \right)} = \frac{1}{n\mathbf{I}(p)}.$$

Therefore, \hat{p} is efficient. □

Example 1.6.8 Suppose $X \sim N(\mu, \sigma^2)$, with σ^2 is known. What is $\mathbf{I}(\mu)$?

Solution 4.

Note that

$$\frac{d^2}{d\mu^2} \ln f_X(x; \mu) = -\frac{1}{\sigma^2}.$$

Then,

$$\mathbf{I}(\mu) = -\mathbf{E} \left[\frac{d^2}{d\mu^2} \ln f_X(x; \mu) \right] = -\mathbf{E} \left[-\frac{1}{\sigma^2} \right] = \frac{1}{\sigma^2}.$$

□