

Emory University  
**MATH 361 Mathematical Statistics I**  
Learning Notes

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# 1 Prerequisites

**Definition 1.0.1 (Geometric Series).** A geometric series has the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

If  $|r| < 1$ , then the series converges to  $\frac{a}{1-r}$ . Otherwise, it diverges.

**Example 1.0.2** Does the series  $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$  converge or diverge?

**Solution 1.**

Note that

$$2^{2n} 3^{1-n} = (2^2)^n 3^{1-n} = 4^n \left(\frac{1}{3}\right)^{n-1} = 4 \cdot 4^{n-1} \left(\frac{1}{3}\right)^{n-1} = 4 \left(\frac{4}{3}\right)^{n-1}.$$

So,

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} 4 \left(\frac{4}{3}\right)^{n-1}$$

is a geometric series, with  $a = 4$  and  $r = \frac{4}{3}$ .

Since  $|r| = \left|\frac{4}{3}\right| = \frac{4}{3} > 1$ , the series diverges. □

**Definition 1.0.3 (Taylor Series).** The Taylor series expanded about  $a$  of a differentiable function  $f$  is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \cdots$$

**Definition 1.0.4 (Maclaurin Series).** The Taylor series expanded about  $a = 0$ .

**Remark.** The Maclaurin Series of  $e^x$  is given by  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

## Theorem 1.0.5 Binomial Expansion

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k},$$

where  $\binom{n}{k}$  is read as “ $n$  choose  $k$ ” and can also be written as  $nCk$ .

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!}.$$

**Theorem 1.0.6 Integration by Parts**

$$\int u \, dv = uv - \int v \, du.$$

**Example 1.0.7** Evaluate  $\int x e^{-x} \, dx$ .

**Solution 2.**

Let  $u = x$ ,  $dv = e^{-x} \, dx$ . So,  $du = dx$  and  $v = \int e^{-x} \, dx = -e^{-x}$ . Then,

$$\int x e^{-x} \, dx = -x e^{-x} - \int -e^{-x} \, dx = -x e^{-x} - e^{-x} + C.$$

□

**Definition 1.0.8 (Type I Improper Integral).** If  $\int_a^t f(x) \, dx$  exists for all  $t > 0$ , then

$$\int_a^\infty f(x) \, dx = \lim_{t \rightarrow \infty} \int_a^t f(x) \, dx.$$

**Example 1.0.9** Evaluate  $\int_0^\infty x e^{-x} \, dx$ .

**Solution 3.**

$$\begin{aligned} \int_0^\infty x e^{-x} \, dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-x} \, dx = \lim_{t \rightarrow \infty} \left[ -x e^{-x} - e^{-x} \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left( -t e^{-t} - e^{-t} + 1 \right) \\ &= -\lim_{t \rightarrow \infty} \left( \frac{t}{e^t} \right) - \lim_{t \rightarrow \infty} e^{-t} + 1 \\ &= -\lim_{t \rightarrow \infty} \left( \frac{1}{e^t} \right) - 0 + 1 = -0 - 0 + 1 = 1. \end{aligned}$$

□

**Example 1.0.10** Double Integrals over Irregular Domains.

Consider

$$\iint_D 4xy - y^4 \, dA,$$

where  $D$  is the region bounded between  $y = \sqrt{x}$  and  $y = x^3$ .

Evaluate this double integral over  $D$ .

**Solution 4.**

Firstly, we draw the diagram representing  $D$  as follows:



$$\begin{aligned}\iint_D 4xy - y^3 \, dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3 \, dy \, dx = \int_0^1 \left[ 2xy^2 - \frac{1}{4}y^4 \right]_{x^3}^{\sqrt{x}} dx \\ &= \int_0^1 2x(x - x^6) - \frac{1}{4}(x^2 - x^{12}) \, dx \\ &= \int_0^1 2x^2 - 2x^7 - \frac{1}{4}x^2 + \frac{1}{4}x^{12} \, dx \\ &= \left[ \frac{2}{3}x^3 - \frac{1}{4}x^8 - \frac{1}{12}x^3 + \frac{1}{52}x^{13} \right]_0^1 \\ &= \frac{2}{3} - \frac{1}{4} - \frac{1}{12} + \frac{1}{52} = \frac{55}{156}.\end{aligned}$$

□

## 2 Probability

### 2.1 Sample Space and Probability

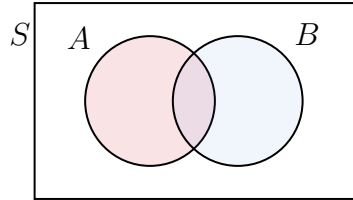
**Definition 2.1.1 (Experiment).** An *experiment* is a procedure with well-defined outcome.

**Definition 2.1.2 (Sample Space/ $S$ ).** The *sample space*, denoted as  $S$  is the set of all possible outcomes of an experiment.

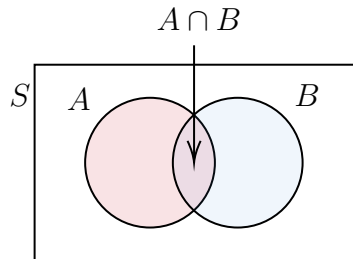
**Definition 2.1.3 (Event).** An *event* is a collection of outcomes.

**Example 2.1.4** Consider flipping two coins. Use  $H$  to represent heads and  $T$  to represent tails. Then,  $S = \{HH, HT, TH, TT\}$ . Event “one heads” =  $\{HT, TH\}$ , and the event “at least one heads” =  $\{HT, TH, HH\}$ .

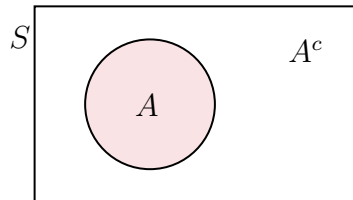
**Definition 2.1.5 (Union/ $\cup$ ).**  $A \cup B$  is the *union* of  $A$  and  $B$ , meaning everything in  $A$  and everything in  $B$ .



**Definition 2.1.6 (Intersection/ $\cap$ ).**  $A \cap B$  is the *intersection* of  $A$  and  $B$ , everything in both  $A$  and  $B$ .



**Definition 2.1.7 (Complement/ $A^c$ ).**  $A^c$  denotes the *complement* of  $A$ , meaning everything in  $S$  that is not in  $A$ .



**Corollary 2.1.8**  $A \cap A^c = \{\} = \emptyset$ .

**Definition 2.1.9 (Mutually Exclusive).** Two sets  $A$  and  $B$  over the same sample space are *mutually exclusive* if they have no outcomes in common. i.e.,  $A \cap B = \emptyset$ .

**Remark.**  $A$  and  $A^c$  are mutually exclusive, but not all sets mutually exclusive are complements of each other.

**Definition 2.1.10 (Probability Function).** Let  $A$  be an event over a sample space  $S$ . Then,  $P(A)$  denotes the *probability* of  $A$  and  $P$  is the *probability function*. The probability function  $P$  assigns a number  $P(A)$  for each event  $A \subseteq S$ .

**Axiom 2.1.11 Kolmogorov Axioms**

1. Let  $A$  be an event in  $S$ , then  $P(A) \geq 0$ .
2.  $P(S) = 1$ .
3. If  $A$  and  $B$  are mutually exclusive, then  $P(A \cup B) = P(A) + P(B)$ .
4. If  $A_1, \dots, A_n, \dots$  are mutually exclusive sets, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

**Proposition 2.1.12**  $P(A^c) = 1 - P(A)$ .

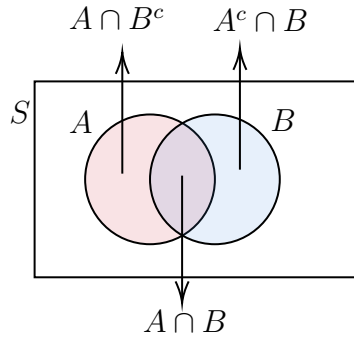
**Proof 1.** Note that  $P(S) = 1$ . Since  $A^c \cup A = S$ , we have  $P(A \cap A^c) = 0$ . Since  $A$  and  $A^c$  are mutually exclusive,  $P(A \cup A^c) = P(A) + P(A^c) = 1$ . So,  $P(A^c) = 1 - P(A)$ . ■

**Proposition 2.1.13**  $P(\emptyset) = 0$ .

**Proof 2.** Note that  $P(S) = 1$ . Then,  $P(S^c) = 1 - P(S)$ . By definition, we know  $S^c = \emptyset$ . So,  $P(\emptyset) = 1 - P(S) = 1 - 1 = 0$ . ■

**Proposition 2.1.14**  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

**Proof 3.** Consider the following Venn diagram:



Note that  $P(A) = P(A \cap B) + P(A \cap B^c)$  and  $P(B) = P(A \cap B) + P(A^c \cap B)$ . So, we have

$$P(A) + P(B) = \boxed{P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)} + P(A \cap B). \quad (1)$$

From the Venn diagram, we notice that  $P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$  is exactly  $P(A \cup B)$ . So, Eq. (1) becomes  $P(A) + P(B) = P(A \cup B) + P(A \cap B)$ . That is exactly what is required:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ . ■

**Definition 2.1.15 (Classical Probability).** In a discrete and finite case,  $S$  is finite and all outcomes are equally likely, and the probability function is defined as

$$P(A) = \frac{|A|}{|S|},$$

where  $|A|$  is the cardinality of  $A$  and  $|S|$  is the cardinality of  $S$ .

**Example 2.1.16** Despite the definition of classical probability (probability function defined for a discrete and finite case), there are other definitions of probability functions:

1. Discrete and Countably Infinite:

Let  $S = \mathbb{N}$  be the set of natural numbers. Then,

$$P(k) = \frac{1}{2^k}.$$

It can also be verified that

$$P(S) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

2. Continuous and Uncountably Infinite:

Let  $S = [0, 1]$ . Suppose  $E$  is a subset of  $[0, 1]$  such that  $\int_E dx$  is defined. Then,

$$P(E) = \int_E dx,$$

and it can also be verified that  $P(S) = 1$ .

## 2.2 Conditional Probability and Independence

**Definition 2.2.1 (Conditional Probability).** We read  $P(A|B)$  as the probability of  $A$  given  $B$ . Knowing  $B$  occurs, we create a new sample space, in which the probability of  $A$  occurs changes:

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|}{|B|} \cdot \frac{1/|S|}{1/|S|} = \frac{|A \cap B|/|S|}{|B|/|S|} = \frac{P(A \cap B)}{P(B)}.$$

**Corollary 2.2.2**  $P(A \cap B) = P(A|B)P(B)$

**Example 2.2.3** Find the probability of dealing  $A$  first, 2 second, and 3 third.

**Solution 1.**

$$\begin{aligned} P(\text{dealing } A, 2, 3) &= P(A \text{ first})P(2 \text{ second}|A \text{ first})P(3 \text{ third}|A \text{ first} \cap 2 \text{ second}) \\ &= \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} \end{aligned}$$

□

**Corollary 2.2.4**  $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_2 \cap A_1).$

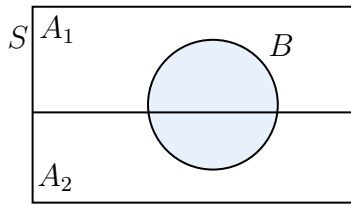
### Theorem 2.2.5 The Law of Total Probability

Suppose the sample space  $S = A_1 \cup A_2 \cup \cdots \cup A_n$ , with  $A_i \cap A_j = \emptyset \quad \forall i \neq j$ . Then,

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n).$$

**Remark.** This theorem gives us a nice way to partition the sample space.

### Example 2.2.6



As represented in the diagram above,  $P(B) = P(B \cap A_1) + P(B \cap A_2).$

### Theorem 2.2.7 Bayes Theorem

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$



**Example 2.2.8** Coronary Artery Disease (CAD)

The probability of someone having CAD is 60%. In a study of 101 patients, 37 of them are known to NOT have CAD and 64 are known to have CAD. Of the 37 patients without CAD, 34 had negative tests while 3 had positive tests. Of the 64 with CAD, 54 had positive tests and 10 had negative tests. Find the probability of a patient has CAD given positive test.

**Solution 2.**

Let  $T+$  be positive test,  $T-$  be negative test,  $D+$  be presence of CAD, and  $D-$  be absence of CAD. Then, from the problem, we have

$$P(D+) = 0.6; \quad P(D-) = 1 - P(D+) = 0.4$$

and

$$P(T+|D+) = \frac{54}{64} \approx 0.84; \quad P(T-|D-) = \frac{34}{37} \approx 0.92; \quad P(T+|D-) = \frac{3}{37} \approx 0.08.$$

Then, by Bayes Theorem,

$$\begin{aligned} P(D+|T+) &= \frac{P(T+|D+)P(D+)}{P(T+|D+)P(D+) + P(T+|D-)P(D-)} \\ &= \frac{0.84 \times 0.6}{0.84 \times 0.6 + 0.08 \times 0.4} \approx \boxed{0.94}. \end{aligned}$$

□

**Definition 2.2.9 (Independence).** Events  $A$  and  $B$  are *independent* if  $P(A|B) = P(A)$ , meaning the occurrence of  $B$  does not affect the occurrence of  $A$ .

**Corollary 2.2.10** If  $A$  and  $B$  are independent, then  $P(A \cap B) = P(A|B)P(B) = P(A)P(B)$ .

**Example 2.2.11** Draw a card from 52 card deck

Let  $A$ : The card is an Ace and  $H$ : The card is a hearts. Then,

$$P(A \cap H) = P(\text{The card is an Ace of hearts}) = \frac{1}{52} = \frac{1}{4} \cdot \frac{1}{13} = P(H)P(A).$$

So, ranks and suits are independent.

**Example 2.2.12** Mutually Exclusive v.s. Independence

A coin is flipped twice:  $S = \{HH, TH, HT, TT\}$ . Let  $A =$  The first flip is  $H = \{HH, HT\}$  and  $B =$  The second flip is  $T = \{HT, TT\}$ .

- $A$  and  $B$  are independent:  $A \cap B = \{HT\}$ . So,  $P(A \cap B) = \frac{1}{4}$ . Since  $P(A \cap B) \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$ , we know  $A$  and  $B$  are independent.

- $A$  and  $B$  are not mutually exclusive because  $P(A \cap B) = \frac{1}{4} \neq 0$ .

**Definition 2.2.13 (Repeated Trials).** A sequence of events  $A_1, \dots, A_n$  is called independent if for any combination

$$P(A_{i1} \cap A_{i2} \cap \dots \cap A_{ik}) = P(A_{i1})P(A_{i2}) \dots P(A_{ik}).$$

In this case, each individual event is called a *trial*.

**Example 2.2.14** Roll a fair die repeatedly. What is the probability that the first 6 appears on the roll  $k$ ? If I win when 6 is rolled, what is the probability that I win?

**Solution 3.**

Let  $A_j$  = the first 6 is rolled on roll  $j$ .

$$j = 1 \quad P(A_1) = \frac{1}{6}$$

$$j = 2 \quad P(A_2) = \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$$

$$j = 3 \quad P(A_3) = \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)$$

$$j = 4 \quad P(A_4) = \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)$$

$\vdots$

$$j = k \quad P(A_k) = \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)$$

So,

$$\begin{aligned} P(\text{I win}) &= P(A_1) + P(A_2) + \dots + P(A_k) + \dots \\ &= \frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \dots + \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right) + \dots \\ &= \left(\frac{1}{6}\right) \left(1 + \left(\frac{5}{6}\right) + \dots + \left(\frac{5}{6}\right)^{k-1} + \dots\right) \\ &= \left(\frac{1}{6}\right) \sum_{i=0}^{\infty} \left(\frac{5}{6}\right)^i = \frac{1}{6} \cdot \frac{1}{1 - \frac{5}{6}} = \frac{1}{6} \cdot 6 = \boxed{1}. \end{aligned}$$

□

**Example 2.2.15** Three people  $A$ ,  $B$  and  $C$  take turn to flip a coin. Whoever gets a heads wins. Find the probability of each individual winning.

**Solution 4.**

First consider the case when Player  $A$  wins. Let  $A_j$  = Player  $A$  wins on the  $j$ -th turn.

$$j = 1 \quad P(A_1) = \frac{1}{2}$$

$$j = 2 \quad P(A_2) = \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) \left(\frac{1}{2}\right)$$

$$j = 3 \quad P(A_3) = \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) \left(\frac{1}{2}\right) = \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right)^2 \left(\frac{1}{2}\right)$$

$$j = 4 \quad P(A_4) = \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right)^3 \left(\frac{1}{2}\right)$$

$\vdots$

$$j = k \quad P(A_k) = \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right)^{k-1} \left(\frac{1}{2}\right) = \left(\frac{1}{8}\right)^{k-1} \left(\frac{1}{2}\right)$$

So,

$$\begin{aligned} P(A \text{ wins}) &= \sum_{j=1}^{\infty} P(A_j) = \frac{1}{2} + \left(\frac{1}{8}\right) \left(\frac{1}{2}\right) + \cdots + \left(\frac{1}{8}\right)^{k-1} \left(\frac{1}{2}\right) + \cdots \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{1}{8}\right)^i \\ &= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{1}{2} \cdot \frac{8}{7} = \boxed{\frac{4}{7}}. \end{aligned}$$

Similarly, we can get the probability of player  $B$  wins to be  $P(B \text{ wins}) = \frac{2}{7}$ . Finally, we can compute the probability of player  $C$  wins by

$$P(C \text{ wins}) = 1 - P(A \text{ wins}) - P(B \text{ wins}) = 1 - \frac{4}{7} - \frac{2}{7} = \frac{1}{7}.$$

□

## 2.3 Combinatorics

### Theorem 2.3.1 Multiplication Rule

If operation  $A$  can be performed in  $n$  ways and operation  $B$  in  $m$  ways, then the sequence (operation  $A$ , operation  $B$ ) can be performed in  $n \times m$  ways.

**Corollary 2.3.2 Ordered Sequence** Consider a set  $A$  and  $|A| = n$ . Then, an *ordered sequence* of  $A$ ,  $(x_1, x_2, \dots, x_k)$  s.t.  $x_i \in A$ , is picked with replacement of elements. Then,

$$|(x_1, x_2, \dots, x_k)| = n^k.$$

**Remark.** *In this situation, repetition is allowed.*

**Definition 2.3.3 (Permutation).** *Permutation* is an ordered sequence without replacement of elements. That is,  $(x_1, x_2, \dots, x_k)$  s.t.  $x_i \in A$  and  $x_i \neq x_j \forall i \neq j$ . Then,

$$|(x_1, x_2, \dots, x_k)| = n(n-1) \cdots (n-k+1).$$

It is also written as  ${}_nP_k = \frac{n!}{(n-k)!}$ .

**Definition 2.3.4 (Combination).** *Combination* is an unordered permutation (no order, no replacement of elements). So, we have

permutation = combination  $\times$  orderings

$${}_nP_k = {}_nC_k \times k!$$

$${}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

**Remark.** *People are always distinct. Letter or coins are not usually distinct.*

**Example 2.3.5** How many ways can we scramble the letters in STATISTICS?

**Solution 1.**

If the letter are distinct, then  $10!$  ways to scramble the word. However, they are not distinct:

Non-distinct Letters	Ways to Scramble
$S - 3$	$3!$
$T - 3$	$3!$
$I - 2$	$2!$

So, ways to scramble the word  $N$  satisfies

$$10! = N \cdot 3! \cdot 3! \cdot 2!$$

$$N = \frac{10!}{3! \cdot 3! \cdot 2!} \quad \text{Multinomial Coefficient}$$

□

**Definition 2.3.6 (Multinomial Coefficient).** The *multinomial coefficient* is the number of ways that  $n$  objects with  $n_j$  of type  $j$ , where  $j = 1, \dots, r$ , can be distinctly ordered. So,

$$\sum_{j=1}^r n_j = n$$

and

$$\text{Multinomial Coefficient} = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_r!}$$

**Remark.** *Tips for Counting:*

1. Draw a picture of the structure
2. Construct a smaller problem when there are large numbers or variables.
3. If the structure of the problem falls into different categories, then add instead of multiple.

## 2.4 Combinatorial Probabilities

**Remark.** *Probability Tips*

1. Avoid multiplying probabilities. Always set up quotient.
2. Keep track of order. If we have order in the sample space, we will need order in the event.
3. Know some basic sample spaces:
  - Rolling  $n$  fair die:  $|S| = 6^n$  (ordered).
  - Flipping  $n$  coins:  $|S| = 2^n$
  - Dealing a hand of  $n$  cards:  $\binom{52}{n}$

**Example 2.4.1** Roll 5 Fair Die. What is the size of the sample space? What is the probability that the first three have one face and the last two another? What is the probability that two faces show up exactly twice?

**Solution 1.**

Size of the sample space:  $|S| = 6^5$ .

Let  $A$  = the probability that the first three have one face and the last two another.

$$|A| = (6 \times 1 \times 1) \times (5 \times 1) = 30. \text{ So, } P(A) = \frac{30}{6^5} = \frac{5}{6^4}.$$

Let  $B$  = the probability that two faces show up exactly twice. Note that we use  $\binom{6}{2}$  to give the faces of the pairs.  $\binom{4}{1}$  to the last one.  $\frac{5!}{2! \cdot 2! \cdot 1!}$  ways to order the faces. So,

$$|B| = \binom{6}{2} \binom{4}{1} \frac{5!}{2! \cdot 2! \cdot 1!}. \text{ Then, } P(B) = \frac{\binom{6}{2} \binom{4}{1} \frac{5!}{2! \cdot 2! \cdot 1!}}{6^5}. \quad \square$$

### 3 Random Variables

#### 3.1 Discrete RV: Binomial & Hypergeometric

**Definition 3.1.1 (Random Variable).** A *random variable* is a number determined by the outcome of an experiment,  $X : S \rightarrow \mathbb{R}$ . Usually, we are not particularly interested in  $S$  but in the distribution of the outcomes for  $X$ . We want to describe the probability associated with different values of  $X$ .

**Example 3.1.2** Flip three coins. Count the number of heads ( $H$ ):

$$S = \{HHH, HHT, HTH, THT, THT, HTT, TTT\}$$

$$\tilde{S} = \{3H, 2H, 1H, 0H\}.$$

Note in  $\tilde{S}$ , not every outcome is equally likely. We would need to define a function for or a list of the values for each outcome in  $\tilde{S}$ . This is an example of a *discrete random variable*.

**Example 3.1.3** Pick a student. Let the random variable  $Y$  = height of the students in cm. Then,  $Y$  is an example of a *continuous random variable*. Continuous random variables can take on an interval of values.

**Notation 3.1.4.**  $X$  is a random variable and has a distribution (it is still abstract and unrealized).  $x$  is a number and a realized random variable  $X$ .

**Definition 3.1.5 (Discrete Random Variable).** A *discrete random variable* is a random variable whose range is finite or countable.

**Definition 3.1.6 (Probability Density (Mass) Function / pdf).**

$$P_X(x) = P(X = x) = P(\{s \in S \mid X(s) = x\}).$$

**Definition 3.1.7 (Cumulative Density Function / cdf).**

$$F_X(x) = P(X \leq x).$$

**Remark.** CDFs and PDFs can be represented by functions, graphs, or tables.

**Example 3.1.8** Roll three fair die. Let  $X$  be the largest value of the three die. Find the pdf.

**Solution 1.**

Note the pdf

$$P_X(x) = P(X = x) = P(X \leq x) - P(X \leq x - 1).$$

Find the cdf of  $x$ . The die that take on at most the value  $x$ , so each die have  $x$  possible outcomes, and considering order, we know

$$F_X(x) = \frac{x^3}{6^3}.$$

Therefore,

$$P_X(x) = P(X \leq x) - P(X \leq x-1) = F_X(x) - F_X(x-1) = \frac{x^3}{6^3} - \frac{(x-1)^3}{6^3}.$$

□

**Definition 3.1.9 (Bernoulli Distribution).** The *Bernoulli distribution* is the classic “flip one coin,” where  $X$  is the number of heads. Let  $X \sim \text{Bernoulli}(p)$ , where  $p$  stands for the probability of success.  $x = 1$  for success and  $x = 0$  for failure. The pdf of Bernoulli distribution is

$$P_X(x) = p^x(1-p)^{1-x}$$

So,

$$P_X(1) = p; \quad P_X(0) = (1-p).$$

**Definition 3.1.10 (Binomial Distribution).** The *binomial distribution* is adding Bernoulli trials together. Let  $Y = X_1 + \cdots + X_n$  be the number of success with  $X \sim \text{Bernoulli}(p)$  and  $Y \sim \text{Binomial}(n, p)$ .  $n$  is the number of trials and  $p$  is the probability of success. The pdf of binomial distribution is

$$P_Y(y) = \binom{n}{y} p^y (1-p)^{n-y}.$$

**Definition 3.1.11 (Hypergeometric Distribution).** Suppose we have a bag of red ( $r$ ) and white ( $w$ ) chips and  $r + w = N$ . Let  $X$  = the number of red chips when choosing  $n$  chips *without replacement*. Then,  $X \sim \text{Hypergeometric}(r, w, n)$ , and the pdf is given by

$$P_X(x) = P(X = x) = \frac{\binom{r}{x} \binom{w}{n-x}}{\binom{r+w}{n}},$$

where  $\binom{r}{x}$  is the number of ways to get  $x$  red,  $\binom{w}{n-x}$  is the number of ways to get  $n-x$  white, and  $\binom{r+w}{n}$  is the total number of picking a size of  $x$ .

**Remark.** If we choose sequence (ordered choose without replacement), we should get the exact same answer:  ${}_r P_x$  is the number of ways the red chips can be selected in order,  ${}_w P_{n-x}$  is the ways to choose the white chips,  $\binom{n}{x}$  is the locations of the red chips, and  ${}_N P_n$  is the

orders of  $n$  chips. Then,

$$\begin{aligned}
 P_X(x) = P(X = x) &= \frac{({}_rP_x) \cdot ({}_wP_{n-x}) \cdot \binom{n}{x}}{{}_NP_n} = \frac{\frac{r!}{(r-x)!} \cdot \frac{w!}{(w-(n-x))!} \cdot \frac{n!}{(n-x)!x!}}{\frac{N!}{(N-n)!}} \\
 &= \frac{\frac{r!}{(r-x)!x!} \cdot \frac{w!}{(w-(n-x))!(n-x)!}}{\frac{N!}{(N-n)!n!}} \\
 &= \frac{\binom{r}{x} \binom{w}{n-x}}{\binom{N}{n}}.
 \end{aligned}$$

### 3.2 Continuous Random Variables

**Definition 3.2.1 (Random Variable).** A *random variable* is an outcome of an experiment mapped to a number:  $X : S \rightarrow \mathbb{R}$ . We are most interested describing the probability associated with different values of  $X$ .

**Example 3.2.2** In a chemistry experiment that depends on temperature. Let  $Y$  = the temperature measured for experiment, then the unit of temperature will/may lead to different “looking” results.

**Definition 3.2.3 (Continuous Random Variables).** A *continuous random variable* is a function from a sample space  $S$  to the real numbers *s.t.* for any values  $a, b$  with  $a < b$ , there exists a function  $f_Y(y)$  *s.t.*

$$P(a < Y < b) = \int_a^b f_Y(y) dy,$$

where  $f_Y(y)$  is called the *pdf (probability density function)*, and (1)  $f_Y(y) \geq 0 \quad \forall y \in \mathbb{R}$ ; (2)  $\int_{-\infty}^{\infty} f_Y(y) dy = 1$ . Meanwhile, the *cdf (cumulative distribution function)* is defined as

$$P(Y \leq y) = F_Y(y) = \int_{-\infty}^y f_Y(t) dt.$$

The  $q$ -th quantile  $c$  can be defined as  $F(c) = q \quad \forall q \in (0, 1)$ .

**Remark.** The probability for  $a < Y < b$  can be regarded as the area under  $f_Y(y)$  over the interval  $(a, b)$ . Therefore,  $P(Y = a) = 0$ .

**Definition 3.2.4 (Uniform Distribution).** Suppose  $Y$  is a continuous random variable, and  $Y \sim \text{Uniform}(a, b)$ , then

$$f_Y(y) = \begin{cases} \frac{1}{b-a} & y \in [a, b]; \\ 0 & \text{otherwise.} \end{cases}$$



**Remark.**  $Y$  is continuous does not imply  $f_Y(y)$  is also continuous.

**Definition 3.2.5 (Exponential Distribution).** Suppose  $Y$  is a continuous random variable, and  $Y \sim \text{Exponential}(\lambda)$  with  $\lambda > 0$  is defined as

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0.$$

**Theorem 3.2.6 Fundamental Theorem of Calculus**

- $\frac{d}{dx} \int_a^x f(t) dt = f(x)$ ; and
- $\int_a^b f(x) dx = F(b) - F(a)$  s.t.  $\frac{d}{dx} F(x) = f(x)$ .

**Example 3.2.7 The Temperature Example - Cont'd.**

Let  $X$  = temperature in °F and  $Y$  = temperature in °C. Given  $f_X(x)$ ,  $F_X(x)$ , and  $Y = \frac{5}{9}(X - 32)$ . Use the cdf→pdf method to find  $f_Y(y)$ .

**Solution 1.**

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P\left(Y = \frac{5}{9}(X - 32) \leq y\right) \\ &= P\left(X \leq \frac{9}{5}y + 32\right) \\ &= F_X\left(\frac{9}{5}y + 32\right). \end{aligned}$$

The derivative of cdf gives pdf:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{9}{5}y + 32\right) \\ &= f_X\left(\frac{9}{5}y + 32\right) \frac{d}{dy} \left(\frac{9}{5}y + 32\right) \quad [Chain Rule] \\ &= \frac{9}{5} f_X\left(\frac{9}{5}y + 32\right). \end{aligned}$$

□

### 3.3 Expected Values and Variances

**Definition 3.3.1 (Expected Values).** For discrete random variables,

$$E(X) = \sum_{\text{all values of } x} x P_X(x) = \sum_{\text{all values of } x} x P(X = x).$$

For continuous random variables,

$$E(Y) = \int_{-\infty}^{\infty} yf(y) \, dy$$

**Remark.**  $E(X) = \mu_X = \mu$  is the balancing point of the distribution, also known as the **first moment**. Another center of the distribution is the **median**,  $m$ , such that

$$P(X \geq m) = P(X < m) = \frac{1}{2}.$$

### Theorem 3.3.2 Properties of Expected Values

- Let  $Y = g(X)$ . Then,

$$E(Y) = E(g(X)) = \begin{cases} \sum_{\text{all } x} g(x)P_X(x), & \text{discrete} \\ \int_{-\infty}^{\infty} g(x)f(x) \, dx, & \text{continuous} \end{cases}$$

- Special Case:  $Y = aX + b$ .

$$E(Y) = E(aX + b) = aE(X) + b.$$

This special case also indicates that  $E(X)$  is linear.

**Definition 3.3.3 (Variance).** The width of a distribution can be described by the variance. The *variance* is the *second centered moment*:

$$\text{Var}(X) = E((X - \mu)^2)$$

Another way to write variance is

$$\text{Var}(X) = E(X^2) - E(X)^2$$

### Theorem 3.3.4 Properties of Variance

- Variance is not linear.
- Variance is translation invariant:

$$\text{Var}(X + b) = \text{Var}(X).$$

**Remark.** *Variance of a line:*

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

**Proof 1.**

$$\begin{aligned}
 \text{Var}(aX + b) &= E((aX + b)^2) - E(aX + b)^2 \\
 &= E(a^2X^2 + b^2 + 2abX) - (aE(X) + b)^2 \\
 &= a^2E(X^2) + b^2 + 2abE(X) - a^2E(X)^2 - b^2 - 2abE(X) \\
 &= a^2(E(X^2) - E(X)^2) \\
 &= a^2\text{Var}(X)
 \end{aligned}$$

■

### 3.4 Joint Densities

**Definition 3.4.1 (Joint pdf of  $X, Y$ ).**

- Discrete:  $p_{X,Y}(k_1, k_2) = P(X = k_1, Y = k_2)$ .
- Continuous:  $f_{X,Y} = P(X, Y \in R) = \iint_R f_{X,Y}(x, y) \, dA$ .

**Definition 3.4.2 (Marginal pdf).** We can recover the single variable pdf from the joint pdf. This is called the *marginal pdf*.

- Discrete:  $p_X(x) = P(X = x, Y = \text{anything})$ .
- Continuous:  $P(a \leq X \leq b) = \int_a^b \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx \implies f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy$ .

#### Theorem 3.4.3 Independence

Two random variable  $X, Y$  are *independent* if

- Discrete:  $p_{X,Y}(x, y) = P(X = x, Y = y) = P(X = x)P(Y = y) = p_X(x)p_Y(y)$ .
- Continuous:  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ .

**Proof 1.** We will prove the continuous case using the cdf  $\rightarrow$  pdf method. If  $X$  and  $Y$  are independent, the events  $X \in [a, b]$  and  $Y \in [c, d]$  are independent  $\forall a, b, c, d$ . Then,

$$\begin{aligned}
 F_{X,Y}(x, y) &= P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(s, t) \, dt \, ds \\
 &= \int_{-\infty}^x f_X(s) \int_{-\infty}^y f_Y(t) \, dt \, ds \quad [Independence]
 \end{aligned}$$

So,

$$f_{x,y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = f_X(x)f_Y(y) \quad [Fund. Thm. of Calculus]$$

■

**Corollary 3.4.4** Independent random variables must have a rectangle domain.

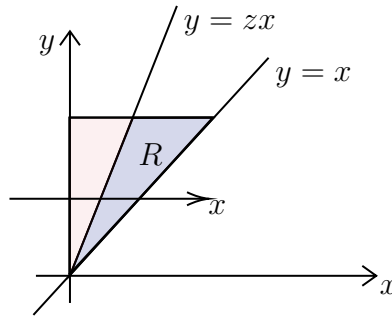
**Example 3.4.5** Given  $X < 0, Y < 0$ , there's a chance that they can be independent. However, given  $0 < X \leq Y \leq 1$ , there is no chance that they will be independent.

**Example 3.4.6** Suppose  $X$  and  $Y$  have the joint pdf

$$f_{X,Y}(x, y) = 2 \quad 0 < x \leq y \leq 1$$

Find the pdf of  $Z = \frac{Y}{X}$ .

**Solution 2.**



$$\begin{aligned}
 P_Z(z) &= P(Z \leq z) = P\left(\frac{Y}{X} \leq z\right) = P(Y \leq zX) = \iint_R f_{X,Y}(x, y) \, dx \, dy \\
 &= \int_0^1 \int_{y/z}^y 2 \, dx \, dy \\
 &= \int_0^1 2\left(y - \frac{y}{z}\right) \, dy \\
 &= \int_0^1 2\left(1 - \frac{1}{z}\right)y \, dy \\
 &= 2 \cdot \left(1 - \frac{1}{z}\right) \cdot \frac{1}{2} \left[y^2\right]_0^1 \\
 &= 1 - \frac{1}{z}
 \end{aligned}$$

Therefore,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \left(1 - \frac{1}{z}\right) = z^{-2}, \quad z > 1.$$

□

### 3.5 Combining Independent Random Variables

**Remark.** First check for dependencies of the domain: We need rectangular relationship between  $X$  and  $Y$ .

**Theorem 3.5.1 Review**

- Joint cdf:  $F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(s, t) \, dt \, ds$

- If  $Y = aX + b$ , then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

**Theorem 3.5.2 Strategies for Finding New pdfs**

- Discrete pdf: compute the probability directly
- Continuous pdf: use the cdf  $\rightarrow$  pdf method.

**Example 3.5.3** Let  $X \sim \text{Binomial}(n, p)$  and  $Y \sim \text{Binomial}(m, p)$ . Let  $X$  and  $Y$  be independent. Find the pdf for  $W = X + Y$ .

**Solution 1.**

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}; \quad P_Y(y) = \binom{m}{y} p^y (1-p)^{m-y}$$

$$\begin{aligned} P_W(w) &= P(W = w) = \sum_x \underbrace{P_X(x) P_Y(w-x)}_{\text{Joint pdf } P_{X,Y}(x,y), w=x+y} \\ &= \sum_x \binom{n}{x} p^x (1-p)^{n-x} \binom{m}{w-x} p^{w-x} (1-p)^{m-(w-x)} \\ &= \sum_x \binom{n}{x} \binom{m}{w-x} p^{x+w-x} (1-p)^{n-x+m-w+x} \\ &= \sum_x \binom{n}{x} \binom{m}{w-x} p^w (1-p)^{n+m-w} \\ &= p^w (1-p)^{n+m-w} \left( \sum_x \binom{n}{x} \binom{m}{w-x} \right) \\ &= \binom{m+n}{w} p^w (1-p)^{n+m-w} \end{aligned}$$

So,  $W = X + Y \sim \text{Binomial}(n + m, p)$

□

**Theorem 3.5.4 Sum of Continuous Independent Random Variable,  $W = X + Y$** 

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x) \, dx$$

**Proof 2.** By independence, we have

$$\begin{aligned}
 F_W(w) &= P(W \leq w) \\
 &= P(X + Y \leq w) \\
 &= P(Y \leq w - X) \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_X(x) f_Y(y) dy dx
 \end{aligned}$$

Then, by Fundamental Theorem of Calculus, we have

$$\begin{aligned}
 f_W(w) &= \frac{d}{dw} \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_X(x) f_Y(y) dy dx \\
 &= \int_{-\infty}^{\infty} f_X(x) \frac{d}{dw} \int_{-\infty}^{w-x} f_Y(y) dy dx \\
 \text{convolution} &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x) (1) dx
 \end{aligned}$$

■

**Theorem 3.5.5 Quotient of Continuous Independent Random Variables,  $W = \frac{Y}{X}$**

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx) dx; \quad \frac{d}{dw}(wx) = |x|$$

**Theorem 3.5.6 Product of Continuous Independent Random Variables,  $W = XY$**

$$f_W(w) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y\left(\frac{w}{x}\right) dx, \quad x \neq 0; \quad \frac{d}{dw}\left(\frac{w}{x}\right) = \frac{1}{|x|}.$$

### 3.6 Further Properties of Mean and Variance

**Lemma 3.6.1** Assume  $X, Y$  are continuous random variable with joint pdf  $f_{X,Y}(x, y)$  and let  $g(X, Y)$  be a function of random variable  $X, Y$ , then

$$E(g(X, Y)) = \iint_{\mathbb{R}^2} g(x, y) f_{X,Y}(x, y) dx dy.$$

**Theorem 3.6.2 Linearity of Expected Values**

$$E(X + Y) = E(X) + E(Y)$$

**Proof 1.** Suppose  $X$  and  $Y$  are two continuous random variables, we want to examine

the sum of their expected values.

$$\begin{aligned}
 E(X + Y) &= \iint_{\mathbb{R}^2} (x + y) f_{X,Y}(x, y) \, dx dy \\
 &= \iint_{\mathbb{R}^2} [x f_{X,Y}(x, y) + y f_{X,Y}(x, y)] \, dx dy \\
 &= \iint_{\mathbb{R}^2} x f_{X,Y}(x, y) \, dy dx + \iint_{\mathbb{R}^2} y f_{X,Y}(x, y) \, dx dy \\
 &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx + \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy \\
 &= \int_{-\infty}^{\infty} x f_X(x) \, dx + \int_{-\infty}^{\infty} y f_Y(y) \, dy = E(X) + E(Y).
 \end{aligned}$$

**Conjecture 3.6.3** We hope the expected value of a product to be  $E(XY) = E(X)E(Y)$ . ■

**Disproof 2.** Take an urn with two chips numbered 1 and 2. Draw 2 chips without replacement. Let  $X_1$  = the value on draw 1 and  $X_2$  = the value on draw 2.

$$E(X_1) = \sum_{i=1}^2 iP(X = i) = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$P(X_2 = 1) = P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1) + P(X_2 = 1 \mid X_1 = 2)P(X_1 = 2) = 0 + 1\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$P(X_2 = 2) = P(X_2 = 2 \mid X_1 = 1)P(X_1 = 1) + P(X_2 = 2 \mid X_1 = 2)P(X_1 = 2) = 1\left(\frac{1}{2}\right) + 0 = \frac{1}{2}$$

$$E(X_2) = \sum_{i=1}^2 iP(X = i) = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$E(X_1)E(X_2) = \left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}; \quad E(X_1X_2) = E(2) = 2.$$

Therefore,  $E(X_1)E(X_2) \neq E(X_1X_2)$ . ■

#### Theorem 3.6.4 Expected Value of a Product

If  $X, Y$  are independent, then

$$E(XY) = E(X)E(Y).$$

**Proof 3.** Suppose  $X$  and  $Y$  are independent. Then,

$$\begin{aligned}
 E(X, Y) &= \iint_{\mathbb{R}^2} xy f_{X,Y}(x, y) \, dx dy \\
 &= \iint_{\mathbb{R}^2} xy f_X(x) f_Y(y) \, dx dy && \text{Independence} \\
 &= \left( \int_{-\infty}^{\infty} x f_X(x) \, dx \right) \left( \int_{-\infty}^{\infty} y f_Y(y) \, dy \right) = E(X)E(Y).
 \end{aligned}$$

**Theorem 3.6.5 Variance of a Sum**

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y),$$

where

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y).$$

Specially, if  $X$  and  $Y$  are independent,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y),$$

and

$$\text{Cov}(X, Y) = 0.$$

**Proof 4.** Recall  $\text{Var}(X) = E(X^2) - E(X)^2$ . Then,

$$\begin{aligned} \text{Var}(X + Y) &= E((X + Y)^2) - E(X + Y)^2 \\ &= E(X^2 + Y^2 + 2XY) - [E(X) + E(Y)]^2 \\ &= E(X^2) + E(Y^2) + 2E(XY) - E(X)^2 - E(Y)^2 - 2E(X)E(Y) \\ &= [E(X^2) - E(X)^2] + [E(Y^2) - E(Y)^2] + 2[E(XY) - E(X)E(Y)] \\ &= \text{Var}(X) + \text{Var}(Y) + 2[E(XY) - E(X)E(Y)] \\ &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \end{aligned}$$

When  $X$  and  $Y$  are independent,  $E(XY) = E(X)E(Y)$ , so  $E(XY) - E(X)E(Y) = 0$ . Then,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

■

**Remark.**  $\text{Cov}(X, Y)$  does not imply  $X$  and  $Y$  are independent. When  $E(X)$  or  $E(Y)$  is 0, we would have  $\text{Cov}(X, Y) = 0$  as well.

**Remark.** *Summary of Properties:*

• *Always true:*

- $E(aX + b) = aE(X) + b$
- $E(X + Y) = E(X) + E(Y)$
- $\text{Var}(aX + b) = a^2\text{Var}(X)$
- $\text{Var}(X) = E(X^2) - E(X)^2$

• *When  $X$  and  $Y$  are independent:*

- $E(XY) = E(X)E(Y)$
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$



### 3.7 Order Statistics

**Definition 3.7.1 (Median).** The point where half data lies above and half below.

**Example 3.7.2** When all the GPAs are ordered, a number is assigned by placement in the list

**Definition 3.7.3 (Max/Min).** The largest and smallest values.

**Definition 3.7.4 (Percentiles).** A score that tells you the percent of people that you scored better than.

**Definition 3.7.5 ( $i$ -th Order Statistics).** Let  $Y$  be a continuous random variable for which we have drawn a *random sample* (independent and identically distributed/*i.i.d.*), say we have  $y_1, y_2, \dots, y_n$ . We re-order them from smallest to largest:

$$y_{\min} = y'_1 \leq y'_2 \leq \dots \leq y'_n = y_{\max}$$

Define a new random variable  $Y'_i$ , and  $Y'_i$  is called the  $i$ -th order statistics. Given a random sample  $Y_1, \dots, Y_n$ , we define

$$Y_{\min} = \min(Y_1, \dots, Y_n); \quad Y_{\max} = \max(Y_1, \dots, Y_n).$$

**Definition 3.7.6 (Percentiles).** For any value  $p$  between 0 and 1, the  $100p$ -th percentile is the observation such that  $np$  observations are less than that value, and  $n(1 - p)$  are greater.

**Example 3.7.7** Suppose a random sample  $y_1 = 3.2, y_2 = 4, y_3 = 1.1, y_4 = 0$ . Then,

$$0 \leq 1.1 \leq 3.2 \leq 4.$$

So,  $Y_{\min} = 0 = y_{\min}, y'_2 = 1.1, y'_3 = 3.2$ , and  $y_{\max} = 4 = Y_{\max}$ .

**Example 3.7.8** Let  $Y_1, \dots, Y_n$  be a random sample from  $Y \sim \text{Uniform}(0, a)$  where we do not know  $a$ . We can estimate  $a$  by different methods.

**Method 1** Use the sample mean:

$$E(Y) = \mu = \frac{a}{2}$$

So we solve for  $\hat{a} \approx 2\mu \approx 2\bar{Y}$ . Then, we can estimate  $a$  with  $\bar{Y}$ . However, we might have  $y_{\max} \geq 2\bar{Y}$  which leads to a better method.

**Method 2** Use observed  $y_{\max}$ . We know the following inequality must hold:

$$a \geq Y_{\max}.$$

To find  $E(Y_{\max})$ , we first need to find the pdf of  $Y_{\max}$ . We consider the cdf→pdf method:

$$\begin{aligned} F_{Y_{\max}}(y) &= P(Y_{\max} \leq y) = P(Y_1 \leq y, Y_2 \leq y, Y_3 \leq y, \dots, Y_n \leq y) \\ &= P(Y_1 \leq y)P(Y_2 \leq y) \cdots P(Y_n \leq y) \\ &= \left(\frac{y}{a}\right)^n \end{aligned}$$

So, we know

$$f_{Y_{\max}}(y) = \frac{d}{dy} F_{Y_{\max}}(y) = \frac{d}{dy} \left[ \left(\frac{y}{a}\right)^n \right] = \frac{n}{a} \left(\frac{y}{a}\right)^{n-1}.$$

So, we can find the expected value:

$$\begin{aligned} E(Y_{\max}) &= \int_0^a y f_{Y_{\max}}(y) dy = \int_0^a y \frac{n}{a} \left(\frac{y}{a}\right)^{n-1} dy \\ &= \frac{n}{a^n} \int_0^a y^n dy \\ &= \frac{n}{a^n} \left[ \frac{1}{n+1} y^{n+1} \right]_0^a \\ &= \frac{n}{a^n} \cdot \frac{1}{n+1} a^{n+1} \\ &= \frac{n}{n+1} a \end{aligned}$$

Therefore, we can estimate  $y_{\max} = \frac{n}{n+1}a$ . So we get  $a \approx \frac{n+1}{n}y_{\max}$ .

### Theorem 3.7.9 Order Statistics

$$Y_{\max} : F_{Y_{\max}}(y) = F_Y(y)^n \implies f_{Y_{\max}}(y) = nF_Y(y)^{n-1}f_Y(y).$$

$$Y_{\min} : F_{Y_{\min}}(y) = 1 - [1 - F_Y(y)]^n \implies f_{Y_{\min}}(y) = n[1 - F_Y(y)]^{n-1}f_Y(y).$$

**Proof 1.** The formula of  $Y_{\min}$  will proven here.

$$\begin{aligned} F_{Y_{\min}}(y) &= P(Y_{\min} \leq y) = 1 - P(Y_{\min} > y) \\ &= 1 - P(Y_1 > y, Y_2 > y, \dots, Y_n > y) \\ &= 1 - P(Y_1 > y)P(Y_2 > y) \cdots P(Y_n > y) \\ &= 1 - [1 - F_Y(y)][1 - F_Y(y)] \cdots [1 - F_Y(y)] \\ &= 1 - [1 - F_Y(y)]^n \end{aligned}$$

So, we would also have  $f_{Y_{\min}}(y) = n[1 - F_Y(y)]^{n-1}f_Y(y)$ . ■

### Theorem 3.7.10 $i$ -th Order Statistics

$$Y'_i : f_{Y'_i}(y) = \frac{n!}{(i-1)!(n-i)!} [F_Y(y)]^{i-1} [1 - F_Y(y)]^{n-i} f_Y(y), \quad 1 \leq i \leq n.$$

### 3.8 Conditional Densities

**Definition 3.8.1 (Conditional Probability).** Let  $X, Y$  be discrete random variables with joint pdf  $p_{X,Y}(x, y)$ , then the conditional probability

$$p_{Y|X}(y) = P(Y = y | X = x) = \frac{p_{X,Y}(x, y)}{p_X(x)}.$$

**Example 3.8.2** A bag with 5 fair coins and one 2 headed coin. Choose a coin and flip it  $n$  times. Let  $X = \begin{cases} 1 & \text{if the coin is fair} \\ 2 & \text{if the two headed coin is selected} \end{cases}$  and  $Y =$  the number of heads in  $n$  tosses. Find  $P_Y(y)$ .

**Solution 1.**

Our plan: find  $p_X(x) \implies p_{Y|X=1}(y), p_{Y|X=2}(y) \implies p_{X,Y}(x, y) = p_{Y|X}(y)p_X(x) \implies$  sum over all values of  $x$  to get the  $p_Y(y)$ .

$$p_X(x) = \begin{cases} \frac{5}{6} & X = 1 \\ \frac{1}{6} & X = 2 \end{cases}$$

$$p_{Y|1}(y) = \binom{n}{y} \left(\frac{1}{2}\right)^y \left(1 - \frac{1}{2}\right)^{n-y} = \binom{n}{y} \left(\frac{1}{2}\right)^n$$

$$p_{Y|2}(y) = \begin{cases} 0 & y < n \\ 1 & y = n \end{cases}.$$

Therefore, the joint pdf

	$Y = 0$	$Y = 1$	$Y = k$	$Y = n$
$X = 1$	$\left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right)$	$\binom{n}{1} \left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right)$	$\binom{n}{k} \left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right)$	$1 \left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right)$
$X = 2$	0	0	0	$1 \left(\frac{1}{6}\right)$
$p_Y(y)$	$\left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right)$	$\binom{n}{1} \left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right)$	$\binom{n}{k} \left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right)$	$\left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)$

In other words,

$$p_Y(y) = \begin{cases} \left(\frac{5}{6}\right) \binom{n}{y} \left(\frac{1}{2}\right)^n & 0 \leq y < n \\ \left(\frac{1}{2}\right)^n \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right) & y = n \end{cases}$$

□

**Remark.** If we define the conditional density in the same way as in the discrete case, we shall have

$$\frac{P(X = x, Y = y)}{P(X = x)}.$$

However,  $P(X = x) = 0$  when  $X$  is continuous. Thus, we need an alternative definition when  $X$  is continuous.

**Definition 3.8.3 (Conditional Probability - Continuous).** Let  $X$  and  $Y$  be two continuous random variables. Then, the conditional probability

$$f_{Y|X}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

**Proof 2.** We will use the cdf→pdf method. Define the cdf of the conditional probability to be

$$\begin{aligned} F_{Y|X}(y) &= \lim_{h \rightarrow 0} P(Y \leq y \mid X \in [x, x+h]) \\ &= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} \int_{-\infty}^y P_{X,Y}(s, t) \, dt \, ds}{\int_x^{x+h} f_X(s) \, ds} && \frac{0}{0} \implies L'Hopitals' \\ &= \lim_{h \rightarrow 0} \frac{\frac{d}{dh} \int_x^{x+h} \int_{-\infty}^y P_{X,Y}(s, t) \, dt \, ds}{\frac{d}{dh} \int_x^{x+h} f_X(s) \, ds} \\ &= \lim_{h \rightarrow 0} \frac{\int_{-\infty}^y P_{X,Y}(x+h, t) \, dt}{f_X(x+h)} \\ &= \frac{\int_{-\infty}^y P_{X,Y}(x, t) \, dt}{f_X(x)} \end{aligned}$$

So, we know

$$\begin{aligned} f_{Y|X}(y) &= \frac{d}{dy} \frac{\int_{-\infty}^y P_{X,Y}(x, t) \, dt}{f_X(x)} \\ &= \frac{f_{X,Y}(x, y)}{f_X(x)}. \end{aligned}$$

■

**Example 3.8.4** A stick of unit length, and break it at a random point. Let  $X$  = length of the larger piece. Then, break the larger piece, and let  $Y$  = the length of the larger piece after the second break. Find  $E(Y)$ .

**Solution 3.**

Find the pdf of  $X$ : let  $W =$  the breaking point. Assume  $W \sim \text{Uniform}(0, 1)$ . Then,  $X = \max(W, 1 - W)$ . So we have

$$\begin{aligned}
 F_X(x) &= P(X \leq x) = P(W \leq x \text{ and } 1 - x \leq W) \\
 &= P(W \in [1 - x, x]) & X \in \left[\frac{1}{2}, 1\right] \\
 &= \int_{1-x}^x 1 \, dw = 2x - 1 \\
 f_X(x) &= \frac{d}{dx} F_X(x) = \frac{d}{dx} [2x - 1] = 2.
 \end{aligned}$$

Find the joint pdf: repeat the same process for  $Y \mid X \sim \text{Uniform}\left(\frac{X}{2}, X\right)$ . We have

$$\begin{aligned}
 f_{Y|X}(y) &= \frac{1}{x - \frac{x}{2}} = \frac{2}{x} & y \in \left[\frac{x}{2}, x\right] \\
 f_{X,Y}(x, y) &= f_{Y|X}(y) f_X(x) \\
 &= \frac{2}{x} \cdot 2 = \frac{4}{x}. & x \in \left[\frac{1}{2}, 1\right], y \in \left[\frac{x}{2}, x\right]
 \end{aligned}$$

Now, we can find the expected value:

$$\begin{aligned}
 E(Y) &= \int_{1/2}^1 \int_{x/2}^x y f_{X,Y}(x, y) \, dy \, dx \\
 &= \int_{1/2}^1 \int_{x/2}^x y \frac{4}{x} \, dy \, dx \\
 &= \int_{1/2}^1 \frac{4}{x} \left[ \frac{1}{2} y^2 \right]_{x/2}^x \, dx \\
 &= \int_{1/2}^1 \frac{2}{x} \left( x^2 - \frac{x^2}{4} \right) \, dx \\
 &= \int_{1/2}^1 \frac{3}{2} x \, dx \\
 &= \frac{3}{2} \left[ \frac{1}{2} x^2 \right]_{1/2}^1 \\
 &= \frac{3}{4} \left( 1 - \frac{1}{4} \right) = \frac{9}{16}.
 \end{aligned}$$

□