

# IB Mathematics Analysis and Approaches HL

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# 1 Topic 1 Number and Algebra

## 1.1 Sequences and Series

1. Terms:  $u_1, u_2, u_3\dots$

Position:  $n$

Sum:  $S$

2. **Arithmetic Sequence**/Arithmetic Progression (AP):

- Recursive formula:  $u_{n+1} = u_n + d$ ,  $d$  is the common difference.
- Explicit formula:  $u_n = u_1 + d(n-1)$
- Summation:  $S_n = \frac{1}{2}[2u_1 + d(n-1)]$

**Proof 1.1.1** Let  $u_1, u_2, u_3, \dots, u_n$  be an arithmetic sequence with  $d$  as common difference.

Then,  $S_n = u_1 + u_2 + u_3 + \dots + u_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + (u_1 + (n-1)d)$

Also,  $S_n = [u_1 + (n-1)d] + \dots + (u_1 + d) + u_1$ .

Add two expressions together:

$$2S_n = [2u_1 + (n-1)d]n$$

$$\therefore S_n = \frac{n}{2}[2u_1 + (n-1)d].$$

3. **Geometric Sequence**

- Recursive formula:  $u_{n+1} = r \cdot u_n$ ,  $r$  is the common ratio.
- Explicit formula:  $u_n = u_1 \cdot r^{n-1}$
- $r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \dots$
- Summation:  $S_n = \frac{u_1(r^n - 1)}{r - 1}$

**Proof 1.1.2** Let  $u_1, u_2, u_3, \dots, u_n$  be a geometric sequence with  $r$  as common ratio.

$S_n = u_1 + u_2 + u_3 + \dots + u_n = u_1 + (u_1 \cdot r) + (u_1 \cdot r^2) + \dots + (u_1 \cdot r^{n-1})$

Then,  $rS_n = (u_1 \cdot r) + (u_1 \cdot r^2) + \dots + (u_1 \cdot r^n)$ .

Subtract the first expression from the second:

$$rS_n - S_n = u_1 \cdot r^n - u_1 \Rightarrow (r-1)S_n = u_1(r^n - 1)$$

$$\therefore S_n = \frac{u_1(r^n - 1)}{r - 1}$$

- If  $r > 1$ , the sequence is an exponential growth.  
If  $0 < r < 1$ , the sequence has an exponential decay.

- When  $r > 1$ , series approaches  $\infty$ .

When  $-1 < r < 1$ , or  $|r| < 1$ , the series converges:

$$S_\infty = \frac{u_1}{1-r}, |r| < 1$$

## 1.2 Exponents and Logarithms

$$1. \ a^m \cdot a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$2. \ x^0 = 1 \ (x^0 = x^{1-1} = \frac{x^1}{x^1} = 1)$$

$$x^{-m} = \frac{1}{x^m}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x} \ (x^{\frac{m}{n}} = (\sqrt[n]{x})^m)$$

$$3. \text{ If } a = b, \text{ then } a^n = b^n$$

$$\text{If } m = n, \text{ then } a^m = a^n$$

For  $a^b = 1$ :  $a = 1, b \in \mathbb{R}; a \neq 1, b = 0$ ; OR  $a = -1, b = 2n$

$$4. \text{ When solving exponential equations, convert them to the same base.}$$

$$5. \text{ Division Theorem.}$$

**Theorem 1.2.1** If  $a^x = b^y$  given  $a > 0$  and  $b > 0$ , then  $a = b^{\frac{y}{x}}$ .

### Proof 1.2.1

$$a^x = b^y$$

$$(a^x)^{\frac{1}{x}} = (b^y)^{\frac{1}{x}} \Rightarrow a = b^{\frac{y}{x}}$$

$$6. \ a = b^x \Leftrightarrow x = \log_b a, \text{ where } a, b \in \mathbb{R}^+ \text{ and } b \neq 1.$$

$$7. \text{ Logarithmic rules:}$$

- $\log_a x + \log_a y = \log_a(xy)$

**Proof 1.2.2** Let  $\log_a x = p, \log_a y = q \Rightarrow a^p = x, a^q = y$ .

Then,  $x \cdot y = a^p \cdot a^q = a^{p+q}$ .

$$\therefore \log_a(xy) = p + q = \log_a x + \log_a y.$$

- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

**Proof 1.2.3** Let  $\log_a x = p, \log_a y = q \Rightarrow a^p = x, a^q = y$ .

Then,  $\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$ .

$$\therefore \log_a \left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y.$$

- $\log_a x^n = n \log_a x$
- $\log_a 1 = 0$
- $\log_a a = 1$
- $-\log_a x = \log_a \frac{1}{x}$
- $\log_a x = \frac{\log_b x}{\log_b a}$
- $\log_a b = \frac{1}{\log_b a}$

### 1.3 Proof

1. Direct proof:

**Example 1.3.1 Show that the sum of two even numbers is always even.**

Let  $m$  and  $n$  be two even positive integers.

$m = 2p, n = 2q$ , where  $p$  and  $q \in \mathbb{Z}^+$ .

Then,  $m + n = 2p + 2q = 2(p + q)$ , which is an even number.

**Example 1.3.2 Show that  $(x + \frac{a}{2})^2 - (\frac{a}{2})^2 \equiv x^2 + ax$ .**

$$\text{LHS} = x^2 + \frac{a^4}{4} + ax - \frac{a^4}{4} = x^2 + ax = \text{RHS}.$$

Equations " $=$ ": only true from some values.

Identities " $\equiv$ ": true for all values.

**Example 1.3.3 Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.**

**Example 1.3.4** Let  $n$  be a 4-digit number:  $n = 1000a + 100b + 10c + d$ , where  $0 \leq a, b, c, d \leq 9$ , and  $a \neq 0$ .

It is given that  $a + b + c + d = 3k, k \in \mathbb{Z}$ :

$$\begin{aligned} n &= 1000a + 100b + 10c + d + 3k - a - b - c - d \\ &= 999a + 99b + 9c + 3k \\ &= 3(333a + 33b + 3c + k) \end{aligned}$$

Since  $(333a + 33b + 3c + k) \in \mathbb{Z}$ , it implies that  $n$  is divisible by 3.

2. Proof by Contradiction:

**Example 1.3.5 Prove the statement: If the integer  $n$  is odd, then  $n^2$  is also odd.**

Let, if possible,  $n^2$  is even and  $n$  is odd.

Then,  $n^2 = 2k, k \in \mathbb{Z} \Rightarrow n \times n = 2k$ , which indicates the product of two odd number is even, and which is not true.

Hence, there is a contradiction.

$\therefore$  Our assumption is wrong, and thus given that  $n$  is odd,  $n^2$  is also odd.

**Example 1.3.6 Show that  $\sqrt{2}$  is irrational.**

Let us assume, if possible, that  $\sqrt{2}$  is rational:

$$\sqrt{2} = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z}, \text{ and } p, q \text{ have no common factors, } q \neq 0.$$

$$\therefore 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \quad (1).$$

$\therefore p^2$  is even, and thus  $p$  is also even.

As  $p$  is an even number, we can write:  $p = 2k, k \in \mathbb{Z} \Rightarrow \therefore p^2 = (2k)^2 = 4k^2 \quad (2)$ .

From (1) and (2):  $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$  is even, and thus  $q$  is also an even number.

But since  $p$  and  $q$  have no common factors, they cannot have "2" as a common factor. Hence, we have arrived at a contradiction.

$\therefore$  Our assumption is incorrect, and  $\sqrt{2}$  is irrational.

**Definition 1.3.1** A number is **rational** if it can be written as  $\frac{p}{q}$ , where  $p, q \in \mathbb{Z}$ , and  $q \neq 0$ .

**Example 1.3.7 Prove that there is no  $x \in \mathbb{R}$  such that  $\frac{1}{x-2} = 1 - x$**  Assume there is a real number  $x$  such that  $\frac{1}{x-2} = 1 - x$ .

$$\therefore (1-x)(x-2) = 1 \Rightarrow x^2 - 3x + 3 = 0$$

Solving the equation, we get  $x = \frac{3 \pm \sqrt{9-12}}{2}$ , which  $\notin \mathbb{R}$

$\therefore$  We arrived at a contradiction, and our assumption is incorrect. There is no  $x \in \mathbb{R}$  such that  $\frac{1}{x-2} = 1 - x$

### 3. Proof by Mathematical Induction

**Definition 1.3.2 Principle of Mathematical Induction (PMI):**

Suppose  $P_n$  is a proposition which is defined for every integer  $n \geq a$ ,  $a \in \mathbb{Z}$ . If  $P_a$  is true, and if  $P_{k+1}$  is true whenever  $P_k$  is true, then  $P_n$  is true  $\forall n \geq a$ .

**Example 1.3.8 Prove that  $4^n + 2$  is divisible by 3 for  $n \in \mathbb{Z}$ ,  $n \geq 0$ , by using PMI.**

For  $n = 0$ , LHS =  $4^0 + 2 = 1 + 2 = 3$ , which is divisible by 3.

$\therefore P_0$  (OR denoted as  $P(0)$ ) is true.

Assume that  $P_k$  is true: i.e.,  $4^k + 2$  is divisible by 3.  $\Rightarrow 4^k + 2 = 3A$ ,  $A \in \mathbb{Z}^+$   $\Rightarrow 4^k = 3A - 2$ .

Consider  $P_{k+1}$ :

$$\begin{aligned} 4^{k+1} + 2 &= 4^k \cdot 4^1 + 2 \\ &= (3A - 2) \cdot 4 + 2 \\ &= 12A - 6 \\ &= 3(4A - 2). \end{aligned}$$

$\therefore 4A - 2$  is an integer as  $A \in \mathbb{Z}^+$ ,  $4^{k+1} + 2$  is divisible by 3 whenever  $4^k + 2$  is divisible by 3.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}$ ,  $n \geq 0$ .

**Example 1.3.9 A sequence is defined by  $u_{n+1} = 2u_n + 1 \forall n \in \mathbb{Z}^+$ . Prove that  $u_n = 2^n - 1$ .**

For  $n = 1$ ,  $u_1 = 2^1 - 1 = 1 \Rightarrow P_1$  is true.

Let  $P_k$  be true:  $u_k = 2^k - 1$  for some  $k \in \mathbb{Z}^+$ .

Consider  $P_{k+1}$ :

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}^+$ .

**Example 1.3.10 Prove that**  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ ,  $\forall n \in \mathbb{Z}^+$ .

For  $n = 1$ , LHS =  $1^2 = 1$ , RHS =  $\frac{1(1+1)(2+1)}{6} = 1$

$\therefore$  LHS = RHS  $\Rightarrow P_1$  is true.

Assume that  $P_k$  is true,  $k \in \mathbb{Z}^+$ :  $1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ .

Consider  $P_{k+1}$ :

$$\begin{aligned} \text{LHS} &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = \text{RHS}. \end{aligned}$$

Thus,  $P_{k+1}$  is true whenever  $P_k$  is true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}^+$ .

**Example 1.3.11 Prove that if  $x \neq 1$ , the**  $\prod_{i=1}^n (1+x^{2^{i-1}}) = (1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x}$ .

For  $n = 1$ , LHS =  $1+x$ , RHS =  $\frac{1-x^2}{1-x} = \frac{1-x^2}{1-x} = 1+x$ .  $\Rightarrow \therefore$  LHS = RHS,  $P_1$  is true.

Assume that  $P_k$  is true:  $(1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^{k-1}}) = \frac{1-x^{2^k}}{1-x}$ .

Consider  $P_{k+1}$ :

$$\begin{aligned} \text{LHS} &= (1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^{k-1}})(1+x^{2^k}) \\ &= \frac{1-x^{2^k}}{1-x} (1+x^{2^k}) \\ &= \frac{1+x^{2^k} - x^{2^k} + (x^{2^k})^2}{1-x} \\ &= \frac{1-x^{2^{k+1}}}{1-x} \\ &= \frac{1-x^{2^{k+1}}}{1-x} = \text{RHS}. \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}^+$ .

## 1.4 Counting and Binomial Theorem

1. Choose  $r$  from  $n$ :  $\binom{n}{r} =_n C_r$

- $\binom{n}{m} = \binom{n}{n-m}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Factorial notation:  $n! = n(n-1)(n-2)\cdots 2 \cdot 1$   
e.g.  $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 5 \times 2 = 10.$

**Example 1.4.1** Write  $\frac{(n!)^2}{(n-1)!(n-2)!}$  without using factorial notation.

$$(n!)^2 = n! \times n! = n(n-1)! \times n(n-1)(n-2)!$$

$$\therefore \frac{(n!)^2}{(n-1)!(n-2)!} = \frac{n(n-1)! \times n(n-1)(n-2)!}{(n-1)!(n-2)!} = n \cdot n(n-1) = n^3 - n^2.$$

2. The number of ways of arranging  $n$  distinct objects in a row is  $n!$ .
3. The number of permutations of  $r$  objects out of  $n$  distinct objects is given by

$$nP_r = \frac{n!}{(n-r)!}.$$

4. In permutations, the order matters.  
In combinations, the order does not matter.
5. The Binomial Theorem:

### Theorem 1.4.1

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n, n \in \mathbb{N}$$

$$= \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$$

**Example 1.4.2** Find  $(2x+3)^4$ .

$$(2x+3)^4 = (2x)^4 + \binom{4}{1} (2x)^3 (3)^1 + \binom{4}{2} (2x)^2 (3)^2 + \binom{4}{3} (2x) (3)^3 + 3^4$$

$$= 16x^4 + 96x^3 + 216x^2 + 216x + 81$$

**Example 1.4.3** Find the term independent of  $x$  in the expansion of  $\left(x - \frac{2}{x^2}\right)^{12}$ .

General term:  $\binom{12}{r} x^{12-r} \left(-\frac{2}{x^2}\right)^r$

Thus, the general expression for  $x$ :  $x^{12-r-2r} = x^{12-3r}$

When  $12 - 3r = 0$ , the term is independent of  $x$ :  $12 - 3r = 0 \Rightarrow r = 4$ .

$$\therefore \binom{12}{4} x^{12-4} \left(-\frac{2}{x^2}\right)^4 = 7920.$$

1. The independent term should not involve  $x$  in it since the independent term does not vary as  $x$  varies. (constant term)
2. The coefficient should not include  $x$  as well.

**Example 1.4.4 Find the coefficient of  $x^3y^2$  in the expansion of  $(2x+y)(x+\frac{y}{x})^5$ .**

Assume  $2x \cdot A$  and  $y \cdot B$  will yield the term  $x^3y^2$ .  $\Rightarrow A = x^2y^2$ ,  $B = x^3y$ .

General term:  $\binom{5}{r}x^{5-r}(\frac{y}{x})^r = \binom{5}{r}x^{5-2r}y^r$ .

When  $r = 2$ ,  $5 - 2r = 1 \neq 2 \Rightarrow x^2y^2$  is not possible.

When  $r = 1$ ,  $5 - 2r = 3 \Rightarrow x^3y$  is possible.

$$\therefore \text{Coefficient} = \binom{5}{1} = 5.$$

**Example 1.4.5 Find the coefficient of  $x^2$  in the expansion of  $(1-2x)(1-4x)^7$ .**

Assume  $1 \cdot A = x^2$ ,  $-2x \cdot B = x^2$ .  $\Rightarrow A = x^2$ ,  $B = x$ .

General term:  $\binom{7}{r}(-4x)^{7-r}(1)^r$

When  $7 - r = 2$ ,  $r = 5$ :  $\binom{7}{5}(-4x)^2(1)^5 = 336x^2$ .  $\Rightarrow 1 \cdot 336x^2 = 336x^2$

When  $7 - r = 1$ ,  $r = 6$ :  $\binom{7}{6}(-4x)^1(1)^6 = -28x$ .  $\Rightarrow (-2x) \cdot (-28x) = 56x^2$

$$\therefore \text{Coefficient} = 336 + 56 = 392.$$

## 6. AHL - Extention of Binomial Theorem:

### Theorem 1.4.2

$$\begin{aligned} (a+b)^n &= a^n \left(1 + \frac{b}{a}\right)^n \\ &= a^n \left(1 + n \cdot \frac{b}{a} + \frac{n(n-1)}{2!} \left(\frac{b}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{b}{a}\right)^3 + \dots\right), \quad n \in \mathbb{Q}, \left|\frac{b}{a}\right| < 1 \end{aligned}$$

**Example 1.4.6 Expand  $\sqrt{1+2x}$  ( $|x| < \frac{1}{2}$ ) and  $\frac{2}{1-3x}$  ( $|x| < \frac{1}{3}$ ) upto  $x^3$  term.**

$$\begin{aligned} (1+2x)^{\frac{1}{2}} &= 1 + \frac{1}{2}(2x) + \frac{1}{2} \left(\frac{1}{2}-1\right) \frac{(2x)^2}{2!} + \frac{1}{2} \left(\frac{1}{2}-1\right) \left(\frac{1}{2}-2\right) \frac{(2x)^3}{3!} + \dots \\ &= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots \end{aligned}$$

$$\begin{aligned} 2(1-3x)^{-1} &= 2(1 - (-3x) - (-1-1)) \frac{(-3x)^2}{2!} - (-1-1)(-1-2) \frac{(-3x)^3}{3!} + \dots \\ &= 2(1 + 3x + x^2 + 27x^3 + \dots) \\ &= 2 + 6x + 18x^2 + 54x^3 + \dots \end{aligned}$$

**Example 1.4.7** Write the first three terms in the expansion of  $(2+x)^{-3}$ .

$$\begin{aligned}
 (2+x)^{-3} &= 2^{-3} \left(1 + \frac{x}{2}\right)^{-3} \\
 &= \frac{1}{8} \left(1 + (-3)\frac{x}{2} + (-3)(-3-1)\frac{2^2}{2 \cdot 2!} + \dots\right) \\
 &= \frac{1}{8} \left(1 - \frac{3}{2}x + \frac{12}{4}x^2 + \dots\right) \\
 &= \frac{1}{8} - \frac{3}{16}x + \frac{3}{8}x^2 + \dots.
 \end{aligned}$$

**Example 1.4.8** Find square root of 24 correct to 5 decimal places, using the binomial theorem.

$$\begin{aligned}
 24^{\frac{1}{2}} &= (25-1)^{\frac{1}{2}} = 25^{\frac{1}{2}} \left(1 - \frac{1}{25}\right)^{\frac{1}{2}} \\
 &= 5 \left(1 + \left(\frac{1}{2}\right) \left(-\frac{1}{25}\right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \left(-\frac{1}{25}\right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \left(-\frac{1}{25}\right)^3 + \dots\right) \\
 &= 5 \left(1 - \frac{1}{50} - \frac{1}{5000} - \frac{1}{250000} + \dots\right) \\
 &= 5(1 - 0.02 - 0.0002 - 0.000004) \\
 &= 4.89898 \quad (5 \text{ d.p.}).
 \end{aligned}$$

## 1.5 Partial Fraction

1. Proper fractions: The degree of the numerator is less than the degree of the denominator.
2. Partial fraction: A method to separate one complex fraction into two or more simpler fractions.

**Example 1.5.1** Find the partial fraction of  $\frac{3x}{(x-1)(x+2)}$ .

$$\begin{aligned}
 \text{Let } \frac{3x}{(x-1)(x+2)} &= \frac{A}{x-1} + \frac{B}{x+2}. \\
 \therefore 3x &\equiv A(x+2) + B(x-1).
 \end{aligned}$$

When  $x = 1$ ,  $3 = 3A \Rightarrow A = 1$ .

When  $x = -2$ ,  $-6 = -3B \Rightarrow B = 2$ .

$$\therefore \frac{3x}{(x-1)(x+2)} \equiv \frac{1}{x-1} + \frac{2}{x+2}.$$

**Example 1.5.2** Find the partial fraction of  $\frac{2x+5}{(x-2)(x+1)}$ .

$$\text{Let } \frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

$$\therefore 2x+5 \equiv A(x+1) + B(x-2).$$

When  $x = 2$ ,  $9 = 3A \Rightarrow A = 3$ .

When  $x = -1$ ,  $3 = -3B \Rightarrow B = -1$ .

$$\therefore \frac{2x+5}{(x-2)(x+1)} \equiv \frac{3}{x-2} - \frac{1}{x+1}.$$

**Example 1.5.3 Find the partial fraction of  $\frac{34-12x}{3x^2-10x-8}$ .**

As  $\frac{34-12x}{3x^2-10x-8} = \frac{34-12x}{(3x+2)(x-4)}$ , let  $\frac{34-12x}{(3x+2)(x-4)} = \frac{A}{3x+2} + \frac{B}{x-4}$ .

$$\therefore 34 - 12x \equiv A(x-4) + B(3x+2).$$

When  $x = 4$ ,  $-14 = 14A \Rightarrow A = -1$ .

When  $x = -\frac{2}{3}$ ,  $42 = -\frac{14}{3}A \Rightarrow A = -9$ .

$$\therefore \frac{34-12x}{(3x+2)(x-4)} \equiv -\frac{9}{3x+2} - \frac{1}{x-4}.$$

## 1.6 Complex Number

### 1.6.1 Introduction

1. Complex Number:

**Definition 1.6.1 Complex Numbers** are numbers in the form of  $a + bi$ , where  $i^2 = -1$ .

- $a$  is called the **real part**, denoted as  $\text{Re}(a+bi) = a$ .
- $b$  is called the **imaginary part**, denoted as  $\text{Im}(a+bi) = b$ .

$a + bi$  is called the **Cartesian form of complex number**.

2. Basic Calculations of Complex Number:

- Define  $z_1 = a + bi$  and  $z_2 = c + di$ :

$$z_1 \pm z_2 = (a \pm c) + (b \pm d)i.$$

- Define  $z_1 = a + bi$  and  $z_2 = c + di$ :

$$z_1 z_2 = (ac - bd) + (ad + bc)i.$$

### Proof 1.6.1

$$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) \\ &= ac + (ad + bc)i + bdi^2 \quad [i^2 = -1] \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

- Conjugate complex number:

**Definition 1.6.2** We call  $a - bi$  as the **conjugate** of  $z = a + bi$ , denoted as  $z^* = a - bi$ .

**Theorem 1.6.1** Define  $z_1 = a + bi$ , and  $z^*$  is the conjugate of  $z_1$ . Then,

$$z_1 z^* = a^2 + b^2.$$

**Proof 1.6.2** By definition,  $z^* = a - bi$ . Thus,

$$\begin{aligned} z_1 z^* &= (a + bi)(a - bi) \\ &= a^2 - (bi)^2 \\ &= a^2 + b^2. \end{aligned}$$

- Define  $z_1 = a + bi$  and  $z_2 = c + di$ :

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i.$$

**Proof 1.6.3**

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{(ac + bd) - (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i. \end{aligned}$$

**Example 1.6.1** Find  $z \in \mathbb{C}$  that satisfies the equation  $\frac{z+2}{1-i} = \frac{z-3i}{2+i}$ .

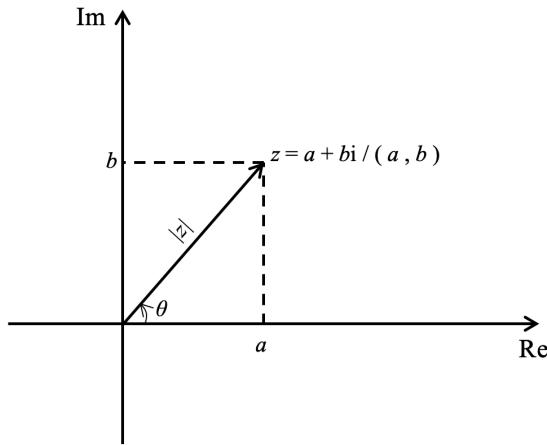
$$\begin{aligned} (z+2)(2+i) &= (z-3i)(1-i) \\ z(2+i) + 4 + 2i &= z(1-i) - 3i + (3i)^2 \\ z(2+i - 1 + i) &= -3i - 3 - 4 - 2i \\ z(1 + 2i) &= -7 - 5i \\ z &= \frac{-7 - 5i}{1 + 2i} = -\frac{17}{5} + \frac{9}{5}i. \end{aligned}$$

3. If  $s = a + bi$  and  $t = c + di$ , then:

$$\operatorname{Re}(s) + \operatorname{Re}(t) = \operatorname{Re}(s + t); \text{ and } \operatorname{Im}(i \cdot s) = \operatorname{Re}(s).$$

## 1.6.2 Argand Diagram

1. The Complex Plane:



$z = a + bi$  can be represented on a complex plane with real coordinate  $a$  and imaginary coordinate  $b$ . It can also be denoted as  $z(a, b)$ .

- Modulus of a complex number:

$$|z| = \sqrt{a^2 + b^2}.$$

- Argument of a complex number:

$$\text{Arg}(z) = \arctan\left(\frac{b}{a}\right) (+k\pi) \rightarrow \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

\*When determine a complex number, first draw it on the plane to show which quadrant it is in.

The range of argument is  $[0, 2\pi]$  or  $[-\pi, \pi]$ .

- Use modulus and argument to express a complex number:

$$a = |z| \cdot \cos \theta;$$

$$b = |z| \cdot \sin \theta.$$

2. If  $z = a + bi$  and  $|z| = 1$ , then  $z^* = z^{-1}$ .

#### Proof 1.6.4

$$\begin{aligned} \therefore |z| &= 1 \\ \therefore \sqrt{a^2 + b^2} &= 1 \\ \therefore a^2 + b^2 &= 1 \end{aligned}$$

Method 1

$$\begin{aligned} \text{RHS} = z^{-1} &= \frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} \\ &= \frac{a - bi}{a^2 + b^2} = a - bi \\ &= z^* = \text{LHS}. \end{aligned}$$

Method 2

$$\begin{aligned} z \cdot z^* &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 = 1 \\ \therefore z^* &= z^{-1} \end{aligned}$$

3. When  $|z| \neq 1$ ,  $\bar{z}^* = \frac{|z|^2}{z}$ , and  $\bar{z}^{-1} = \frac{\bar{z}^*}{|z|^2}$ .

4. Properties of modulus and arguments:

For complex number  $s$  and  $t \in \mathbb{C}$ :

- 

$$|st| = |s||t|$$

- 

$$\left| \frac{s}{t} \right| = \frac{|s|}{|t|}$$

- 

$$\operatorname{Arg}(st) = \operatorname{Arg}(s) + \operatorname{Arg}(t) + 2k\pi$$

- 

$$\operatorname{Arg}\left(\frac{s}{t}\right) = \operatorname{Arg}(s) - \operatorname{Arg}(t) + 2k\pi$$

### 1.6.3 Complex Number in Other Forms

1. The Polar Form (Modulus-Argument Form):

- 

$$z = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$$

**Proof 1.6.5** According to the Argand Diagram:

$$z = x + yi = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta).$$

- 

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

- 

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

2. de Moivre's Theorem:

- By Maclaurin Series:

$$e^{i\theta} = \operatorname{cis} \theta = \cos \theta + i \sin \theta.$$

- Exponential form of complex number:

$$z = r e^{i\theta} = r \operatorname{cis} \theta.$$

3. Cartesian Form: Addition and Subtraction

Modulus-Argument Form: Multiply and Division

Exponential Form: Exponents and Roots

4. Since  $\text{cis}\theta = \text{cis}(\theta + 2k\pi)$ ,

$$re^{i\theta} = re^{i(\theta+2k\pi)}.$$

**Example 1.6.2 Find  $e^{i\frac{17\pi}{12}}$  in the form of Cartesian.**

$$\begin{aligned} e^{i\frac{17\pi}{12}} &= e^{i(\frac{7\pi}{6} + \frac{\pi}{4})} = e^{i\frac{7\pi}{6}} \cdot e^{\frac{\pi}{4}} \\ &= \text{cis}\left(\frac{7\pi}{6}\right) \cdot \text{cis}\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2} - \sqrt{6}}{4} - \frac{\sqrt{2} + \sqrt{6}}{4}i. \end{aligned}$$

#### 1.6.4 Power of Complex Number

1. For a complex number  $z = re^{i\theta}$ ,

$$z^n = r^n e^{in\theta}.$$

**Example 1.6.3 Find  $(3 \cos \frac{2\pi}{3} - 3i \sin \frac{\pi}{3})^3$**

$$\begin{aligned} \left(3 \cos \frac{2\pi}{3} - 3i \sin \frac{\pi}{3}\right)^3 &= \left(-3 \cos \frac{\pi}{3} - 3i \sin \frac{\pi}{3}\right)^3 \\ &= \left(-3 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)\right)^3 \\ &= (-3)^3 (e^{i\frac{\pi}{3}})^3 \\ &= -27e^{i\pi} \\ &= -27(-1) = 27. \end{aligned}$$

Key learnings:

1.  $z = 3$  is only the fundamental root of equation  $z^3 = 27$ . In  $\mathbb{C}$ , there are other two complex roots that satisfy the equation.
2. In  $\mathbb{C}$ ,  $\sqrt{4} = \pm 2 = 2 + 0 \cdot i$  or  $-2 + 0 \cdot i$ .

**Example 1.6.4 Given a complex number  $\omega \neq 1$  is one of the solutions of  $z^3 = 1$ .**

a. Prove  $\omega^2 + \omega + 1 = 0$ ;

b. Calculate  $\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022}$ .

(a) Approach A

$$\begin{aligned} \because \omega^3 &= 1 \\ \therefore \omega^3 - 1 &= 0 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0 \\ \because \omega &\neq 1 \\ \therefore \omega^2 + \omega + 1 &= 0. \end{aligned}$$

**Approach B**  $\omega^2 + \omega + 1 = 0$  is a geometric sequence,  $u_1 = 1$ ,  $r = \omega$ :

$$S_3 = \frac{u_1(1-r^3)}{1-r} = \frac{1-\omega^3}{1-\omega} = \frac{0}{1-\omega} = 0.$$

(b)

$$\begin{aligned}\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022} &= \omega^{2019} \times (1 + \omega + \omega^2 + \omega^3) \\ &= \omega^{2019}(0 + 1) = \omega^{2019} \\ &= (\omega^3)^{673} = 1.\end{aligned}$$

**Example 1.6.5 Find:**

- a.  $1^i$ ;
- b.  $\ln(-1)$ ;
- c.  $\ln(-c)$ , where  $c$  is a constant.

(a)

$$1 = e^{i2\pi} \Rightarrow 1^i = \left(e^{i2\pi}\right)^i = e^{-2\pi}. \quad (1^i = e^{-2k\pi}, k \in \mathbb{Z})$$

(b)

$$-1 = e^{i\pi} \Rightarrow \ln(-1) = \ln\left(e^{i\pi}\right) = i\pi.$$

(c)

$$\ln(-c) = \ln[(-1) \cdot c] = \ln(-1) + \ln(c) = \ln(c) + i\pi.$$

### 1.6.5 Polynomial Function with Complex Roots

1. Conjugate Pair Theorem:

**Theorem 1.6.2** If  $z$  is a complex root of  $P(x)$ , then the conjugate of  $z(z^*)$  is also a complex root of  $P(x)$ . ( $P(x)$  should be a polynomial with rational coefficients.)

2. Properties of Conjugate.

•

$$(s \pm t)^* = s^* \pm t^*$$

•

$$(st)^* = s^*t^*$$

•

$$\left(\frac{s}{t}\right)^* = \frac{s^*}{t^*}$$

### 1.6.6 Root of Complex Numbers

1. The Root of Unity:

**Theorem 1.6.3** For any complex equation  $\omega^n = 1$ , there are  $n$  distinct roots:

$$1 = e^{i(0+2k\pi)} = \omega^n, \quad k \in \mathbb{Z} \quad \Rightarrow \quad \omega = e^{i\frac{2k\pi}{n}}, \quad k \in \mathbb{Z}.$$

**Example 1.6.6 Solve  $z^3 = 8$ .**

$$z^3 = 8 \cdot 1 = 8e^{i(0+2k\pi)} \Rightarrow z = 2e^{i\frac{2k\pi}{3}}, k \in \mathbb{Z}$$

$$k = 0 : z = 2$$

$$k = 1 : z = 2e^{i\frac{2\pi}{3}} = 2\text{cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$$

$$k = 2 : z = 2e^{i\frac{4\pi}{3}} = 2\text{cis}\left(\frac{4\pi}{3}\right) = -1 - \sqrt{3}i$$

2. Property of  $\text{cis}\theta$ :

$$\text{cis}(-\theta) = \cos \theta - i \sin \theta$$

**Proof 1.6.6**

$$\begin{aligned} \cos \theta - i \sin \theta &= \cos(-\theta) + i \sin(-\theta) \\ &= \text{cis}(-\theta). \end{aligned}$$

## 2 Topic 2 Functions

### 2.1 Foundations of Functions

1. Relations and functions:

**Definition 2.1.1** A **relation**  $R$  is a set of ordered pairs  $(x, y)$  such that  $x \in A$ ,  $y \in B$ , and sets  $A$ ,  $B$  are not empty.

**Definition 2.1.2** A **function**  $f$  is a relation in which every  $x$ -value has a unique  $y$ -value.

2. Domain and Range:

**Definition 2.1.3** **Domain** is the set of  $x$ -values.

**Definition 2.1.4** **Range** is the set of  $y$ -values.

- Domain and Range should be in interval notation.

- (a) Using intervals to express the inequalities

**Example 2.1.1**

$$[3, 4[ \text{ means } 3 \leq x < 4$$

- (b) If the interval will be joint, we use  $\cup$  to join the interval.

**Example 2.1.2**

$$3 < x < 4 \text{ or } x \geq 5 \Rightarrow ]3, 4] \cup [5, +\infty[$$

**Example 2.1.3** Find the interval notation for the domain of  $f(x) = \frac{1}{x}$ .

$$x \in ]-\infty, 0[ \cup ]0, +\infty[ \text{ OR } x \in \mathbb{R} \setminus 0$$

Note: \ means "exclude."

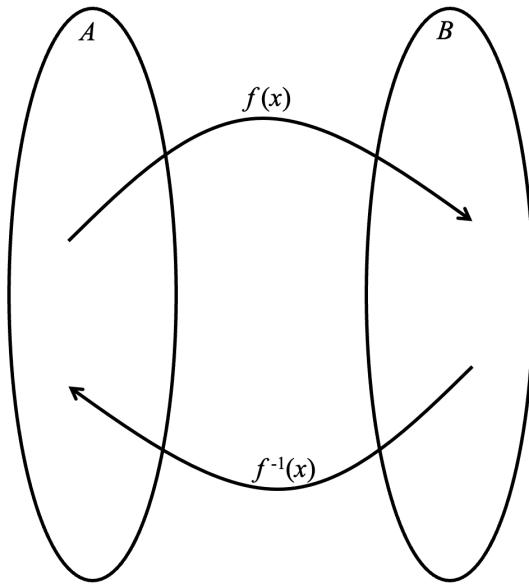
- Since the  $y$ -values (outputs) depend on the  $x$ -values (inputs),  $y$  is the **dependent variable**, and  $x$  is the **independent variable**.
- The independent variable  $x$  is also called the **argument** of the function.

3. Vertical Line test:

- To test whether a relation is a function.
- Since every  $x$  has one and only one value of  $y$ , there should be only one intersects.

4. Inverse of a function:

**Definition 2.1.5**  $f^{-1}(x)$  is the **inverse function** of  $f(x)$ .



**Example 2.1.4**  $f(1) = 3 \Rightarrow f^{-1}(3) = 1$ ;  $f(x) = x + 5 \Rightarrow f^{-1}(x) = x - 5$

- In inverse function, the input becomes the output, the output becomes the input.
- In inverse function, the domain becomes the range, the range becomes the domain.

**Example 2.1.5** (a) **Find the inverse function of**  $y = \frac{x+2}{3}$ .

$$3y = x + 2 \Rightarrow x = 3y - 2 \\ f^{-1}(x) = 3x - 2$$

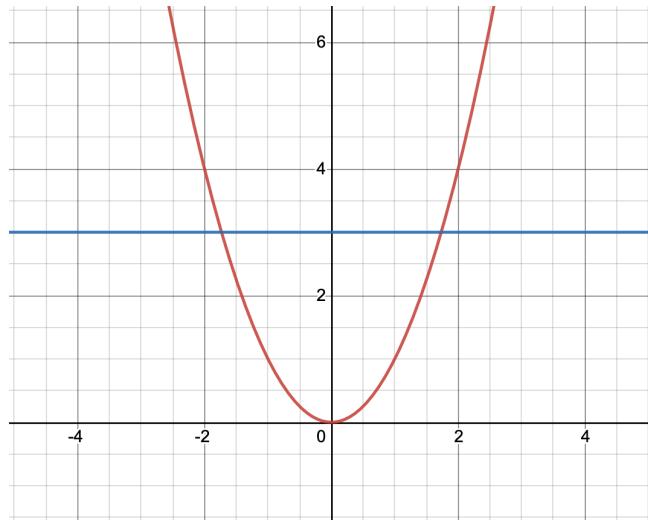
(b) **Find the inverse function of**  $f(x) = \frac{x}{x+1}$ .

$$y = \frac{x}{x+1} \Rightarrow xy + y = x \Rightarrow xy + x = y \\ \therefore y(x-1) = -x \Rightarrow y = -\frac{x}{x-1}$$

(c) **Find the inverse of**  $\{(4, 2), (0, 2), (-2, 2)\}$

$$\text{Inverse: } \{(2, 4), (2, 0), (2, -2)\}$$

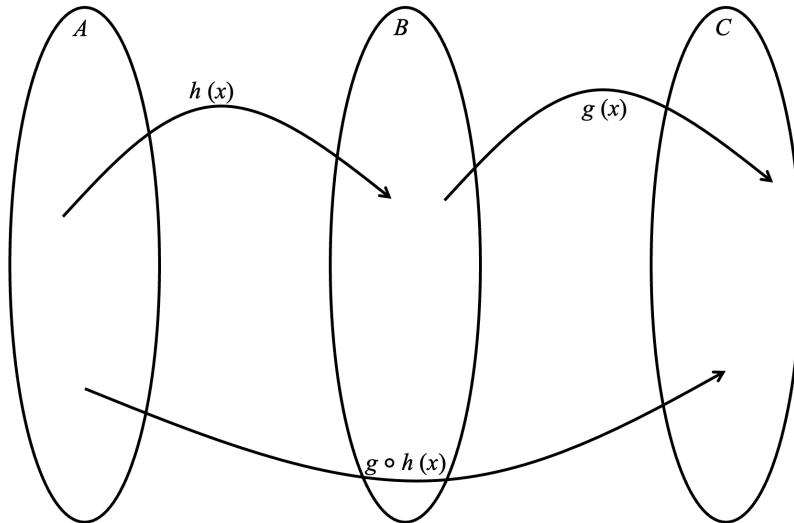
- By restricting the domain, we can find  $f^{-1}(x)$  of  $f(x)$ , if the direct inverse of  $f(x)$  is not a function.



**Example 2.1.6** **Horizontal line test:** The largest domain we can find  $f^{-1}(x)$  is  $x \leq 0$  or  $x > 0$ .

## 5. Composite Functions:

**Definition 2.1.6** We use  $(g \circ h)(x)$  or  $g(h(x))$  to represent composite functions.



**Example 2.1.7** Given  $f : x \mapsto 3x - 6$ ,  $g : x \mapsto \frac{1}{3}x + 2$ . Find  $(f \circ g)(x)$  and  $(g \circ f)(x)$ .

$$(f \circ g)(x) = f(g(x)) = 3\left(\frac{1}{3}x + 2\right) - 6 = x.$$

$$(g \circ f)(x) = g(f(x)) = \frac{1}{3}(3x - 6) + 2 = x.$$

When  $f$  and  $g$  are inverse functions:

$$(f \circ g)(x) = x = (g \circ f)(x).$$

6.  $f(x)$  and  $f^{-1}(x)$  are symmetrical to  $y = x$  since  $D_f = R_{f^{-1}}$ ,  $R_f = D_{f^{-1}}$ . That is, if  $f(x)$  passes through  $(a, b)$ ,  $f^{-1}(x)$  passes through  $(b, a)$ .

## 2.2 Quadratic Functions

1. The Standard Form:

$$y = ax^2 + bx + c,$$

where  $a$  is the coefficient of  $x^2$ ,  $b$  is the coefficient of  $x$ , and  $c$  is the constant or  $y$ -intercept.  $a, b, c \neq 0$ .

- Zeros of the function ( $x$ -intercepts):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where  $\Delta = b^2 - 4ac$  is the discriminant of the function.

- Equation of the line of symmetry &  $x$ -coordinate of the vertex

$$x = -\frac{b}{2a}.$$

- Vieta's Formula:

**Theorem 2.2.1** Assume  $x_1, x_2$  are two roots for equation  $ax^2 + bx + c = 0$  ( $a \neq 0$ ), then

$$x_1 + x_2 = -\frac{b}{a};$$

$$x_1 \cdot x_2 = \frac{c}{a}.$$

- When  $a > 0$ , the parabola opens upwards.  
When  $a < 0$ , the parabola opens downwards.

2. Completion of square:

$$x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2.$$

3. The Vertex Form:

$$y = a(x - h)^2 + k, \text{ where } (h, k) \text{ is the vertex.}$$

**Example 2.2.1** Given that  $f(x) = ax^2 + bx + c$ , find the axis of symmetry and vertex.

$$\begin{aligned} f(x) &= a \left( x^2 + \frac{b}{a}x \right) + c \\ &= a \left[ x^2 + \frac{b}{a}x + \left( \frac{b}{2a} \right)^2 \right] + c - \frac{b^2}{4a} \\ &= a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}. \end{aligned}$$

$$\begin{aligned} \therefore \text{axis of symmetry: } x &= -\frac{b}{2a} \\ \text{vertex: } &\left( -\frac{b}{2a}, \frac{4ac - b^2}{4a} \right). \end{aligned}$$

## 2.3 Higher Order Polynomial Functions

1. Factor Theorem:

**Theorem 2.3.1** If  $(x - a)$  is a factor of a polynomial  $P(x)$ , then  $x = a$  must be a root for  $P(x) \Rightarrow P(a) = 0$ .

**Proof 2.3.1** Assume the quotient when  $P(x)$  is divided by  $(x - a)$  is  $Q(x)$ , then  $P(x) = Q(x) \cdot (x - a)$ . Then,  $P(a) = Q(a) \cdot (a - a) = 0$ .

2. Long division: solving polynomial equation.

**Example 2.3.1 For a cubic function,**  $P(x) = 2x^3 + bx^2 + cx + d$ ,  $P(1) = P(2) = P(3) = 2$ .

**What is  $P(0)$ ?**

Since  $P(1) = P(2) = P(3) = 2$ ,  $Q(1) = Q(2) = Q(3) = 0$ , where  $Q(x) = P(x) - 2$ .

Thus,  $Q(x) = 2(x - 1)(x - 2)(x - 3)$ .

$$\therefore P(x) = Q(x) + 2 = 2(x - 1)(x - 2)(x - 3) + 2.$$

$$\therefore P(0) = 2(-1)(-2)(-3) + 2 = -10.$$

3. Remainder Theorem:

**Theorem 2.3.2** When a polynomial  $P(x)$  is divided by  $(ax - b)$ , the remainder  $R$  of this division must be

$$P\left(\frac{b}{a}\right).$$

**Proof 2.3.2** Assume the quotient is  $Q(x)$ , and the remainder is  $R$ :

$$P(x) = (ax - b)Q(x) + R.$$

$$P\left(\frac{b}{a}\right) = 0 \cdot Q\left(\frac{b}{a}\right) + R = R.$$

#### 4. Roots of Cubic Functions:

**Theorem 2.3.3** For a cubic function  $f(x) = ax^3 + bx^2 + cx + d$ , given the roots of it are  $\alpha, \beta$ , and  $\gamma$ . Then,

$$\begin{cases} \alpha + \beta + \gamma = -\frac{b}{a} \Rightarrow \sum \alpha = -\frac{b}{a} \\ \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} \Rightarrow \sum \alpha\beta = \frac{c}{a} \\ \alpha\beta\gamma = -\frac{d}{a} \Rightarrow \sum \alpha\beta\gamma = -\frac{d}{a} \end{cases}$$

**Proof 2.3.3** Since  $\alpha, \beta, \gamma$  are roots of  $f(x)$ ,

$$f(x) = a(x - \alpha)(x - \beta)(x - \gamma).$$

$$\text{So } a(x - \alpha)(x - \beta)(x - \gamma) = ax^3 + bx^2 + cx + d,$$

$$\text{i.e., } ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \alpha\gamma + \beta\gamma)x - a\alpha\beta\gamma = ax^3 + bx^2 + cx + d.$$

$$\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}, \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}.$$

#### Theorem 2.3.4

$$\sum \alpha = -\frac{b}{a}, \sum \alpha\beta = \frac{c}{a}, \sum \alpha\beta\gamma = -\frac{d}{a}, \sum \alpha\beta\gamma\delta = \frac{e}{a}.$$

## 2.4 Rational Functions

1. Reciprocal Functions:  $f(x) = \frac{1}{x}$ .

- Domain:  $x \in \mathbb{R}, x \neq 0$
- As  $x$  increases,  $\frac{1}{x}$  decreases  $\Rightarrow x \rightarrow \infty, \frac{1}{x} \rightarrow 0$ .
- Range:  $y \in \mathbb{R}, y \neq 0$
- **Asymptotes**:  $x = 0, y = 0$ .
- Axis of symmetry:  $y = x, y = -x$ .
- **Self-inversing function**: have axis of symmetry  $y = x$ .

$$f(x) = f^{-1}(x).$$

2.  $y = \frac{a}{bx + c}$

- Vertical asymptotes (V.A.):  $bx + c = 0$
- Horizontal asymptotes (H.A.):  $y = 0$

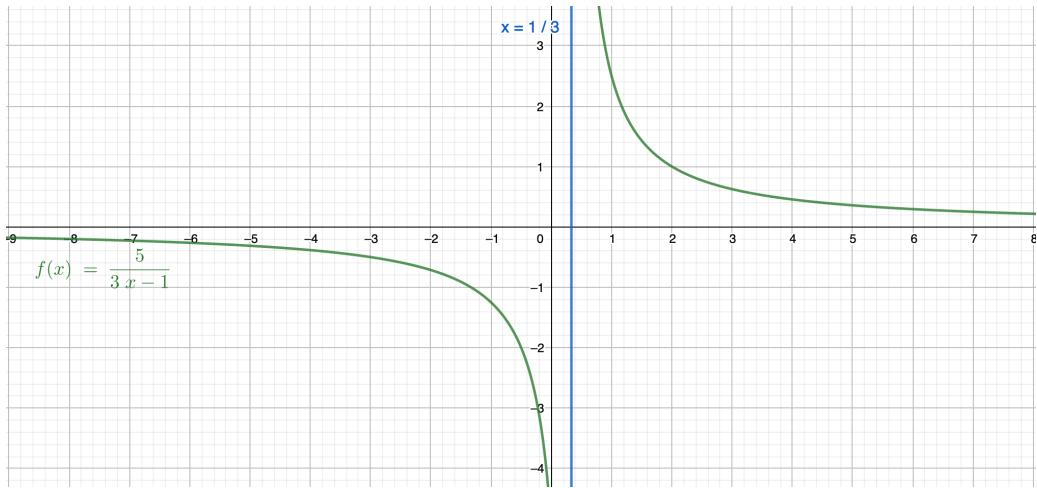
**Example 2.4.1** Draw the diagram of  $y = \frac{5}{3x - 1}$ .

$x$ -intercept:  $0 = \frac{5}{3x - 1} \Rightarrow$  no solution, no intercept.

H.A.:  $y = 0$

y-intercept:  $y = -5$

$$\text{V.A.: } 3x - 1 = 0, x = \frac{1}{3}$$



$$3. \ y = \frac{ax+b}{cx+d}$$

- V.A.:  $cx + d = 0$
- H.A.:  $y = \frac{a}{c}$

$$4. \ y = \frac{ax+b}{cx^2+dx+e}$$

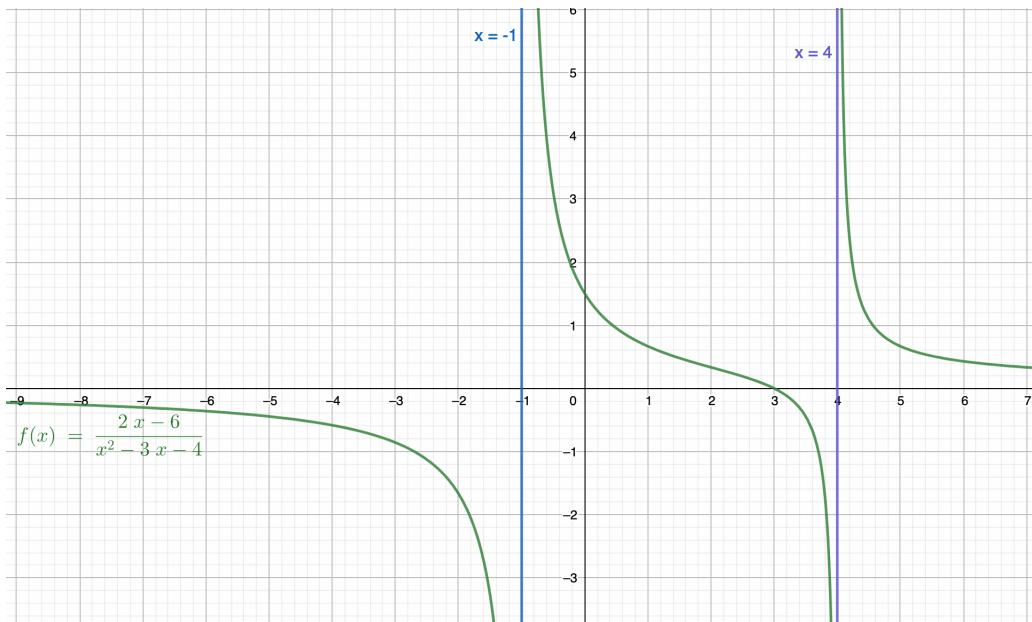
- V.A.:  $cx^2 + dx + e = 0$
- H.A.: As  $x \rightarrow \pm\infty$ ,  $\frac{ax}{cx^2} \rightarrow 0$ ,  $y = 0$
- Intercepts:  $\left(0, \frac{e}{c}\right)$ ,  $\left(-\frac{e}{d}, 0\right)$

**Example 2.4.2** Draw the diagram of  $y = \frac{2x-6}{x^2-3x-4}$ .

Intercept:  $\left(0, \frac{3}{2}\right)$ ,  $(3, 0)$

H.A.:  $y = 0$

V.A.:  $x = -1, x = 4$

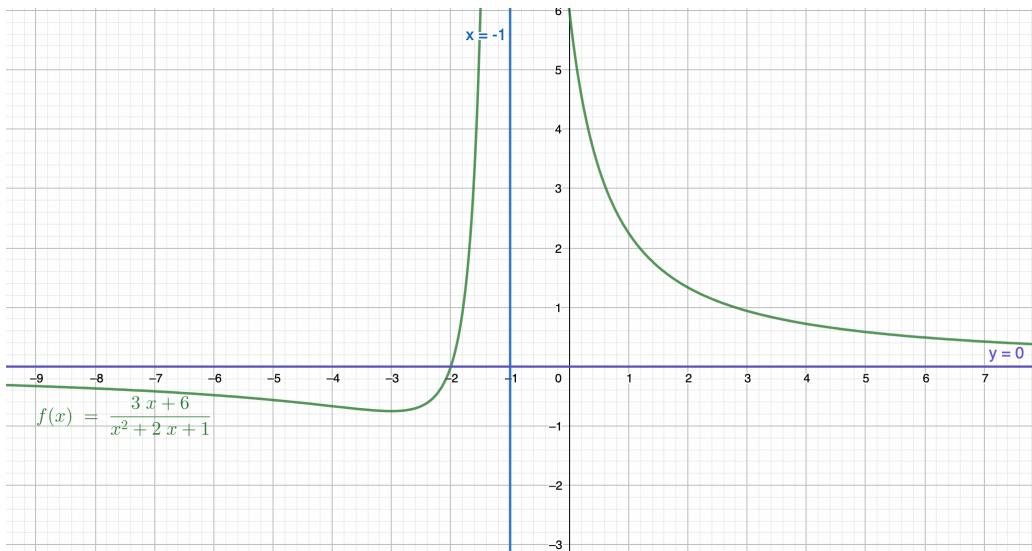


**Example 2.4.3** Draw the diagram of  $y = \frac{3x+6}{x^2 + 2x + 1}$ .

Intercept:  $(0, 6)$ ,  $(-2, 0)$

H.A.:  $y = 0$

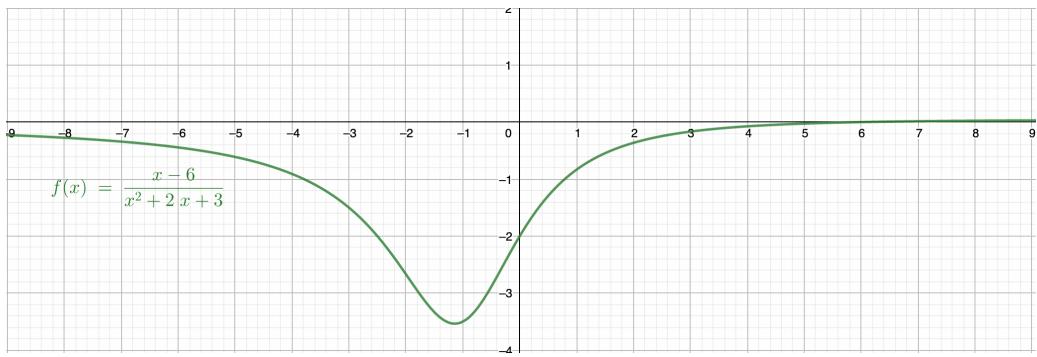
V.A.:  $x = -1$



**Example 2.4.4** Draw the diagram of  $y = \frac{x-6}{x^2 + 2x + 3}$ .

Intercept:  $(6, 0)$ ,  $(0, -2)$

When  $x \rightarrow \infty$ ,  $f(x)$  is positive. When  $x \rightarrow -\infty$ ,  $f(x)$  is negative.



$$5. \ y = \frac{ax^2 + bx + c}{dx + e}$$

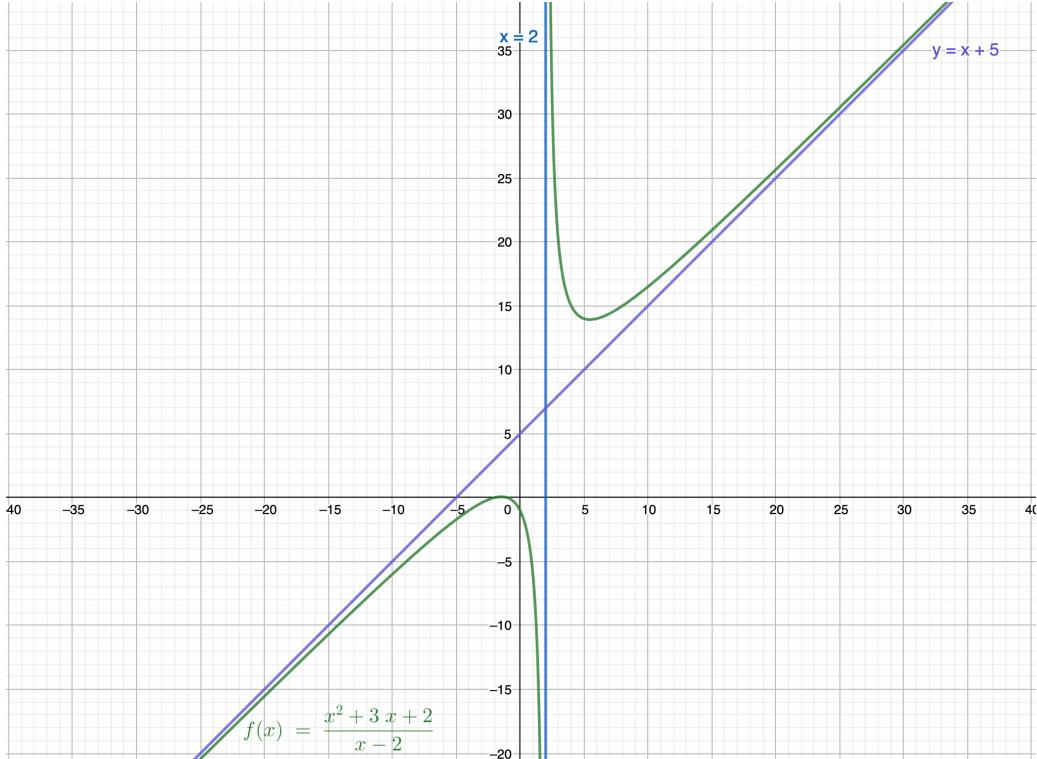
- V.A.:  $dx + e = 0$
- **Oblique Asymptote:** Quotient of  $(ax^2 + bx + c)$  divided by  $(dx + e)$ .
- Intercepts:  $\left(0, \frac{c}{e}\right)$ ,  $ax^2 + bx + c = 0$

**Example 2.4.5 Draw the diagram of  $y = \frac{x^2 + 3x + 2}{x - 2}$ .**

Intercept:  $(0, -1)$ ,  $(-1, 0)$ ,  $(-2, 0)$

V.A.:  $x = 2$

O.A.:  $y = x + 5$  (Use long division)

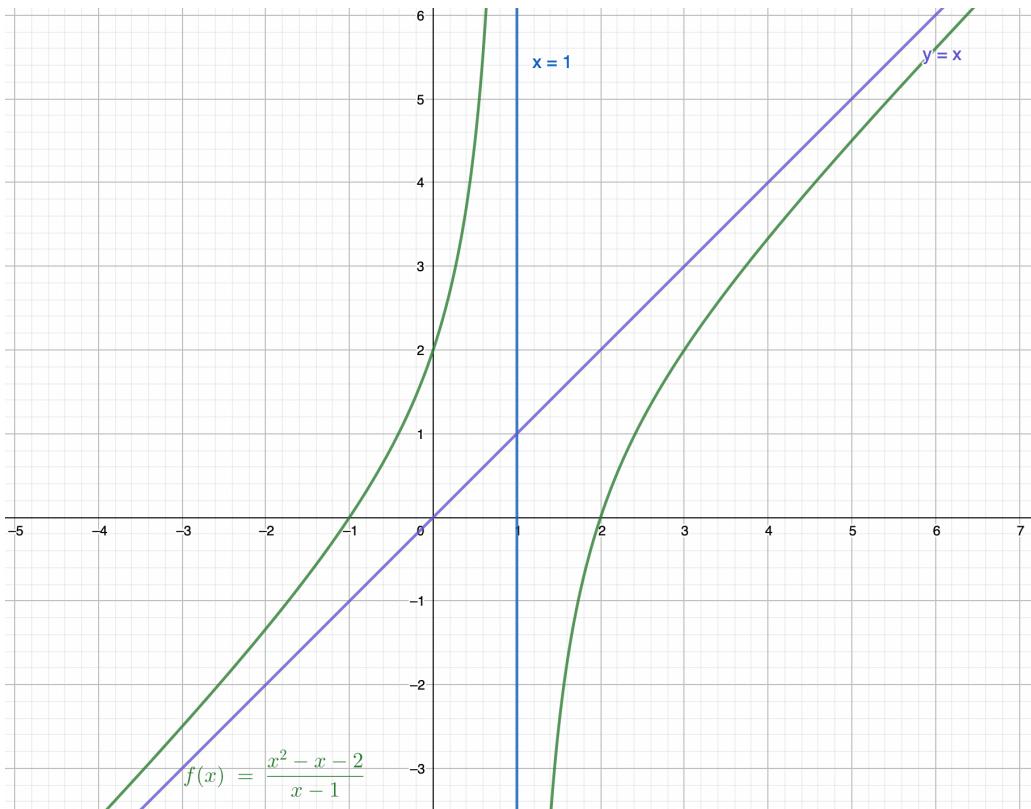


**Example 2.4.6 Draw the diagram of  $y = \frac{x^2 - x - 2}{x - 1}$ .**

Intercept:  $(0, 2)$ ,  $(2, 0)$ ,  $(-1, 0)$

V.A.:  $x = 1$

O.A.:  $y = x$  (Use long division)



6. When the function has asymptotes:

- Denominator= 0;
- $\log_a 0$  (argument of a logarithm is 0)

## 2.5 Transformation of Functions

### 1. Translation:

- $f(x + n)$  means translate  $f(x)$   $n$  units to the left.
- $f(x - n)$  means translate  $f(x)$   $n$  units to the right.
- $f(x) + n$  means translate  $f(x)$   $n$  units upwards.
- $f(x) - n$  means translate  $f(x)$   $n$  units downwards.

2. Use translation vector to represent translation:

A vector  $\begin{pmatrix} a \\ b \end{pmatrix}$  means  $a$  units in the horizontal axis and  $b$  units in the vertical axis.

**Example 2.5.1** A translation vector  $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$  means  $f(x + 2) + 3$ , 2 units to the left and 3 units upwards.

### 3. Reflections:

- $f(-x)$  reflects in the  $y$ -axis.
- $-f(x)$  reflects in the  $x$ -axis.

- $f^{-1}(x)$  reflects in the  $y = x$ .
- $-f(-x)$  reflects in the origin.

#### 4. Stretches:

- $f(qx)$  is a horizontal stretch of a scale factor of  $\frac{1}{q}$ .
- $pf(x)$  is a vertical stretch of a scale factor of  $p$ .

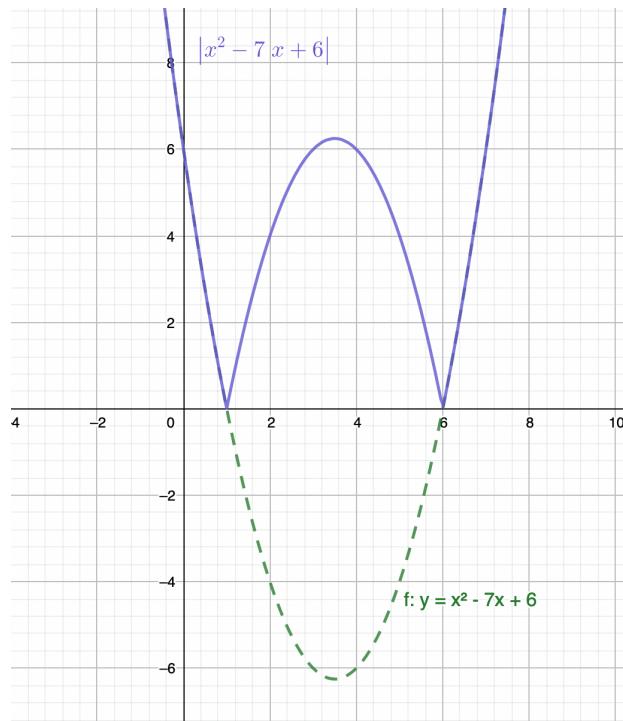
5. When a graph is transforming, the points shift but the connection remains.

#### 6. Sequence of transformation:

- Do the horizontal translation before the horizontal stretch.
- The vertical translation is always after the vertical stretch.
- Vertical stretch → Reflection → Horizontal translation → Horizontal stretch → Vertical translation

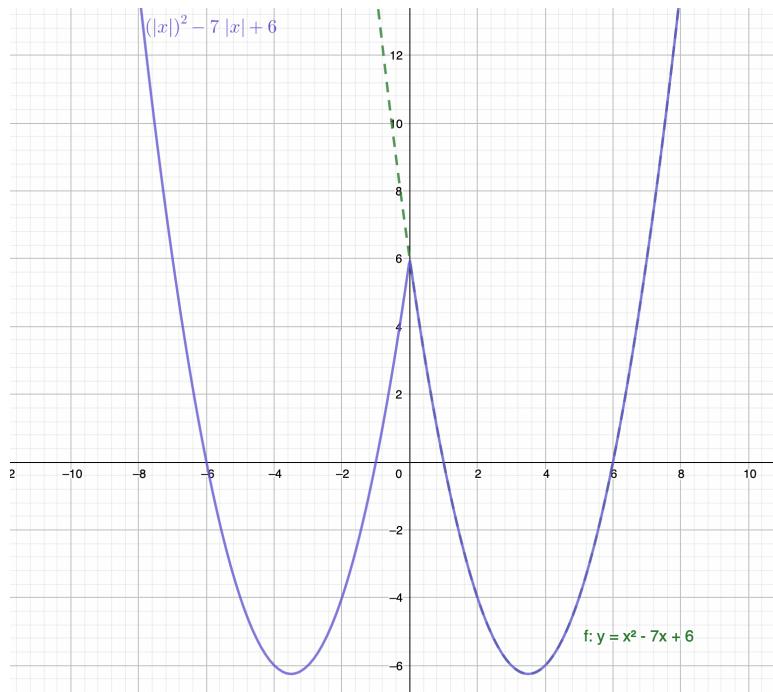
#### 7. Modulus Function

- $|f(x)|$ : Fold everything below  $x$ -axis above  $x$ -axis.



#### Example 2.5.2

- $f(|x|)$ : Reflect everything on the right of  $y$ -axis to the left. Since  $|x|$  must be positive,  $|x| = |-x| \Rightarrow f(-x) = f(x)$ , which is an even function.



### Example 2.5.3

#### 8. Reciprocal of $f(x)$

- Table of Summary:

$f(x)$	$g(x) = \frac{1}{x}$
$f(a) = 0$	Line $x = a$ is vertical asymptote
Line $x = a$ is vertical asymptote	$g(a) = 0$
$f(x) \rightarrow \infty$	$g(x) \rightarrow 0$
$f(x) \rightarrow 0$	$g(x) \rightarrow \infty$
Line $y = b$ is horizontal asymptote	Line $y = \frac{1}{b}$ is horizontal asymptote
$f(x) = a$	$g(x) = \frac{1}{a}$

- When  $f(x)$  increases,  $g(x)$  decreases.

## 2.6 Exponential and Logarithmic Functions

### 1. Exponential functions:

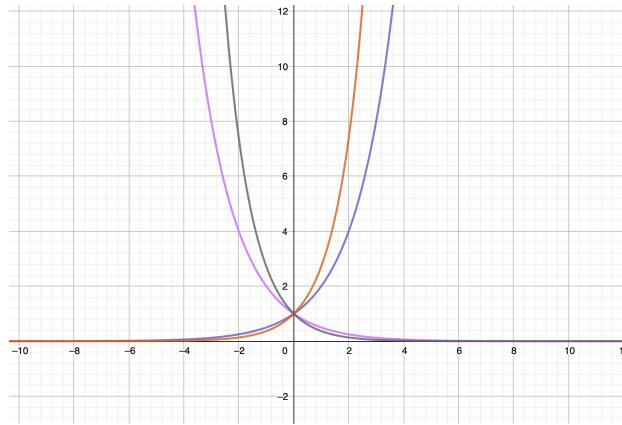
- $f(x) = a^x$ ,  $a > 1$  (increasing) and  $0 < a < 1$  (decreasing).
- $f(x) = a^x$  and  $g(x) = \left(\frac{1}{a}\right)^x$  are symmetric to the  $y$ -axis.

#### Proof 2.6.1

$$g(x) = \left(\frac{1}{a}\right)^x = (a^{-1})^x = a^{-x} = f(-x).$$

- Domain:  $x \in \mathbb{R}$ , Range:  $y > 0$
- Common point:  $(0, 1)$ ; common H.A.:  $y = 0$

- Graph:



## 2. Logarithmic functions:

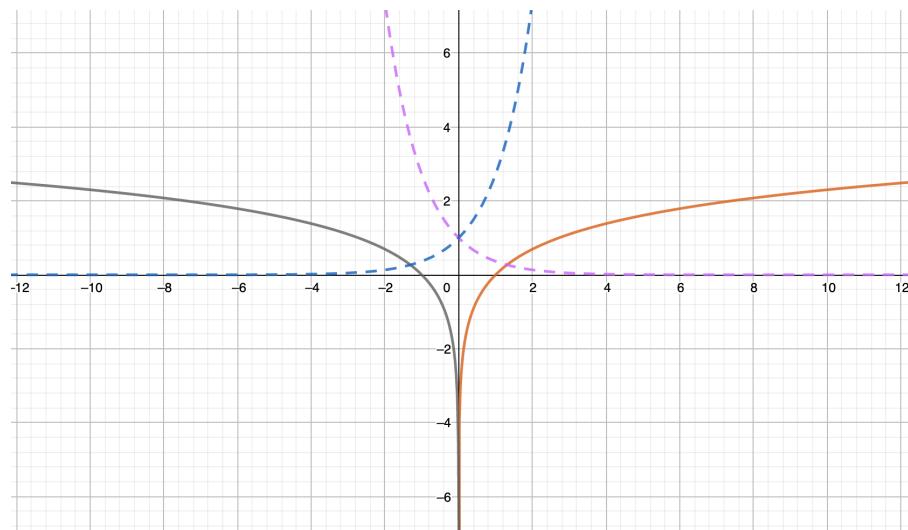
- $f(x) = \log_a x = g^{-1}(x)$ ,  $g(x) = a^x$ .
- Common point:  $(1, 0)$ ; common V.A.:  $x = 0$ .
- $f(x) = \log_a x$  and  $g(x) = \log_{\frac{1}{a}} x$  are symmetric to the  $x$ -axis.

### Proof 2.6.2

$$\log_{\frac{1}{a}} x = \frac{\log_a x}{\log_{\frac{1}{a}} a} = \frac{\log_a x}{-1} = -\log_a x,$$

$$\therefore g(x) = \log_{\frac{1}{a}} x = -\log_a x = -f(x).$$

- When  $a > 1$ , increasing function; when  $0 < a < 1$ , decreasing function.
- Domain:  $x > 0$ , Range:  $y \in \mathbb{R}$
- Graph:



## 3. Solving logarithmic equations.

## 4. Solving exponential equations: take logarithm on both sides.

### 3 Topic 3 Trigonometry and Geometry

#### 3.1 Trigonometry

##### 3.1.1 Radian

1. Radian as the unit of angle:

- 

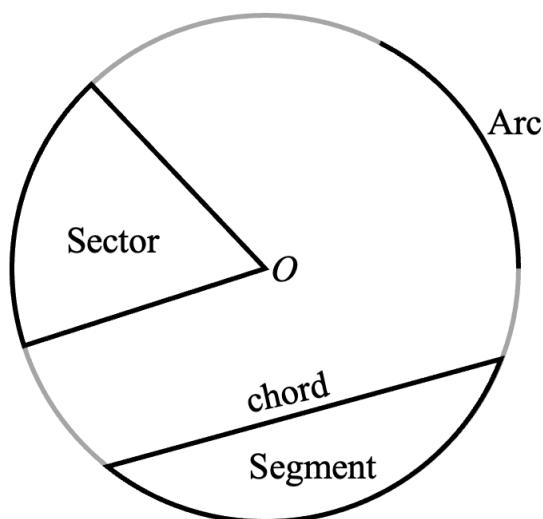
$$\pi \text{ rad} = 180^\circ$$

- rad can be omitted. i.e.,  $\widehat{A} = 1$  means angle  $A$  is 1 radian.
- Unit conversion:

$$\text{degree} \times \frac{\pi}{180^\circ} = \text{radian}; \text{radian} \times \frac{180^\circ}{\pi} = \text{degree}.$$

2. Arc:

- The **circumference** (perimeter) is  $2\pi r$ .



- If the angle of the arc is  $\theta$  (in radian), the length of arc( $l$ ) =  $r \cdot \theta$ .
- The area of a sector:

$$A = \frac{1}{2}r^2\theta.$$

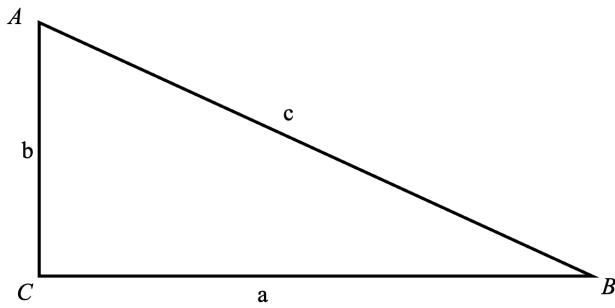
- The area of a segment:

$$A = \frac{1}{2}r^2(\theta - \sin \theta).$$

(Proof: the area of the triangle according to the sine rule is  $\frac{1}{2}ab \sin C$ )

##### 3.1.2 Solution of Triangle

1. Define sine, cosine, and tangent:



**Definition 3.1.1**

$$\sin A = \frac{a}{c}, \sin B = \frac{b}{c};$$

$$\cos A = \frac{b}{c}, \cos B = \frac{a}{c};$$

$$\tan A = \frac{a}{b}, \tan B = \frac{b}{a}.$$

2. The Sine Rule:

**Theorem 3.1.1**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

- The bigger the angle, the longer the side.
- Area of a triangle:

$$A = \frac{1}{2}ab \sin C.$$

3. The Consine Rule:

**Theorem 3.1.2**

$$b^2 + c^2 - a^2 = 2bc \cdot \cos A;$$

$$a^2 + c^2 - b^2 = 2ac \cdot \cos B;$$

$$a^2 + b^2 - c^2 = 2ab \cdot \cos C.$$

4. Inverse Trigonometric Functions:

**Definition 3.1.2**

$$\sin^{-1} \theta = \arcsin \theta;$$

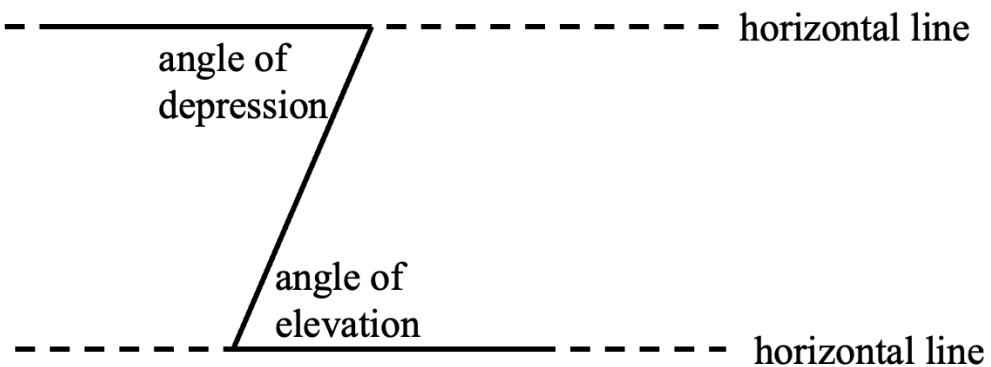
$$\cos^{-1} \theta = \arccos \theta;$$

$$\tan^{-1} \theta = \arctan \theta.$$

5. Ambiguity of Sine Rule:

$$\sin \theta = \sin(180^\circ - \theta) \text{ OR } \sin \theta = \sin(\pi - \theta).$$

6. Angle of Elevation and Depression:

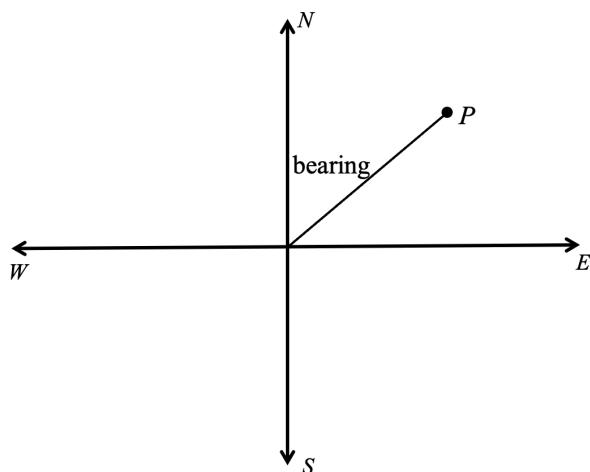


**Definition 3.1.3** • **Angle of Elevation** is the angle "up" from horizontal.

- **Angle of Depression** is the angle "down" from horizontal.

7. Bearing:

- Bearing is a way of describing direction.
- All bearings are measured **clockwise** from the **North** direction.



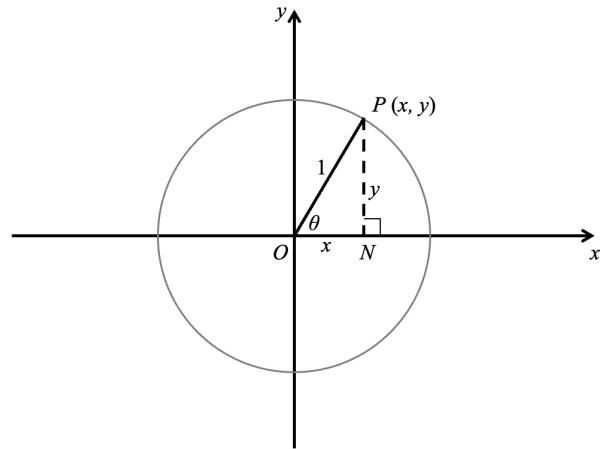
- Bearing of A from B: construct at B.

**N.B.:** Bearing of A from B is different from bearing of B from A.

### 3.1.3 Definition of Trigonometric Function

1. Unit Circle:

- Center at  $(0,0)$  with a radius of 1.



- If an angle  $\theta$  opens in a counterclockwise direction, then  $\theta$  is positive.  
If an angle  $\theta$  opens in a clockwise direction, then  $\theta$  is negative.
- In the diagram,  $\theta = \theta + 2k\pi$ ,  $k \in \mathbb{Z}$ .

•

$$\sin \theta = \frac{PN}{OP} = \frac{y}{1} = y;$$

$$\cos \theta = \frac{ON}{OP} = \frac{x}{1} = x;$$

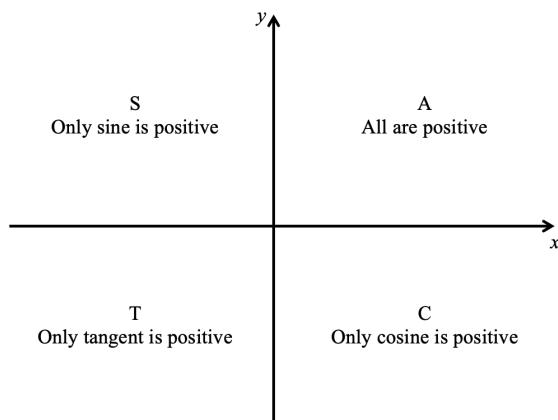
$$\tan \theta = \frac{PN}{ON} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta};$$

- In  $Q_1$  and  $Q_2$ ,  $\sin \theta$  will be positive.

In  $Q_1$  and  $Q_4$ ,  $\cos \theta$  will be positive.

In  $Q_1$  and  $Q_3$ ,  $\tan \theta$  will be positive.

$\Rightarrow$  CAST:



## 2. Special Angles:

$$\sin 0^\circ = 0 = \cos 90^\circ$$

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = 0$$

$$\sin 30^\circ = \frac{1}{2} = \cos 60^\circ$$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\sqrt{3}}{3}$$

$$\sin 45^\circ = \frac{\sqrt{2}}{2} = \cos 45^\circ$$

$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \cos 30^\circ$$

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}$$

$$\sin 90^\circ = 1 = \cos 0^\circ$$

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \infty$$

### 3. Relative Acute Angles (RAA):

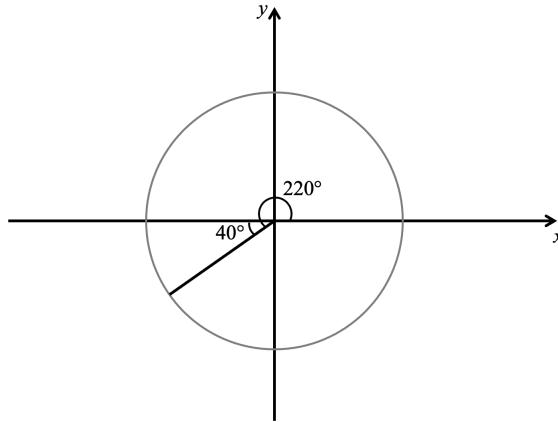
- Acute angle is the angle with  $x$ -axis.
- The absolute value of angles have the same acute angle is the same.

**Example 3.1.1** (a)  $30^\circ, 150^\circ, 210^\circ, 330^\circ$  have the same acute angle.

$$\therefore |\sin 30^\circ| = |\sin 150^\circ| = |\sin 210^\circ| = |\sin 330^\circ|.$$

(b)

$$\tan 220^\circ = \tan 40^\circ; \cos 215^\circ = -\cos 35^\circ$$



### 3.1.4 Trigonometric Identity

#### 1. Pythagorean's Identity:

$$\sin^2 \theta + \cos^2 \theta \equiv 1.$$

#### Proof 3.1.1

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \\ &\Rightarrow \sin^2 \theta + \cos^2 \theta = 1. \end{aligned}$$

#### 2. Definition of Tangent:

- $\tan \theta = \frac{\sin \theta}{\cos \theta};$
- $\cot \theta = \frac{1}{\tan \theta};$
- $\sec \theta = \frac{1}{\cos \theta};$
- $\csc \theta = \frac{1}{\sin \theta}.$

3. Extended Pythagorean's Identity:

$$\tan^2 \theta + 1 = \sec^2 \theta;$$

$$\cot^2 \theta + 1 = \csc^2 \theta.$$

### Proof 3.1.2

$$\begin{aligned}\sin^2 \theta + \cos^2 \theta &= 1 \Rightarrow \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \Rightarrow \tan^2 \theta + 1 = \sec^2 \theta; \\ \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} &= \frac{1}{\sin^2 \theta} \Rightarrow \cot^2 \theta + 1 = \csc^2 \theta.\end{aligned}$$

N.B.: a reflex angle is an angle bigger than  $180^\circ$ , smaller than  $360^\circ$ .

4. Compound Angle Formula:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B;$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B;$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B;$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B;$$

**Example 3.1.2 Find the exact value of  $\cos \frac{\pi}{12}$ .**

$$\begin{aligned}\cos \frac{\pi}{12} &= \cos \frac{\pi}{4} - \frac{\pi}{6} \\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.\end{aligned}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

### Proof 3.1.3

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B}.\end{aligned}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

5. In the linear function  $y = mx + b$ ,  $m = \tan \theta$ , where  $\theta$  is the angle between the line and the positive  $x$ -axis.
6. Double Angle Formula:

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta;$$

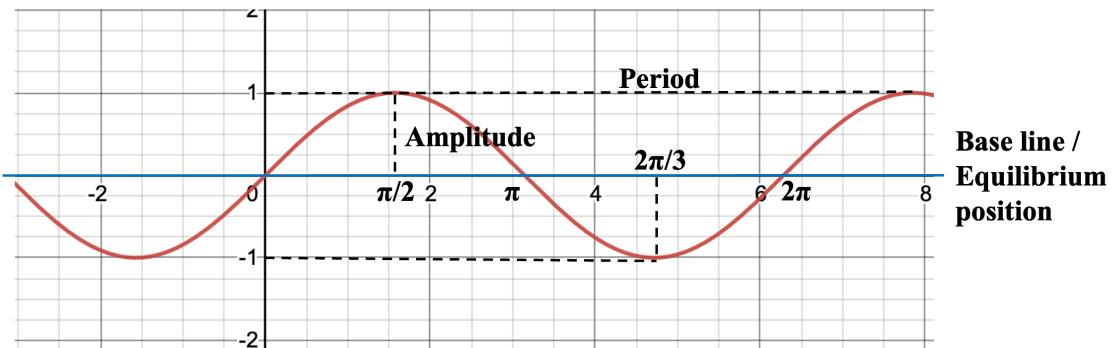
$$\sin(2\theta) = 2\sin \theta \cos \theta;$$

$$\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}.$$

7. Proving Identities.

### 3.1.5 Trigonometric Functions and Transformation

1. Sine: Odd function:  $\sin(-x) = -\sin x$ .



$$T(\text{Period}) = 2\pi;$$

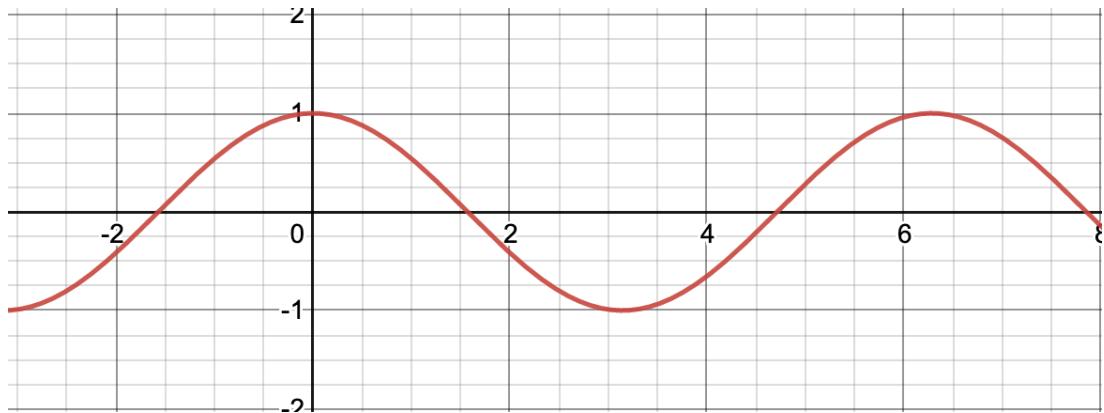
$$\text{Base line} = 0;$$

$$\text{Amplitude} = \left| \frac{y_{\max} - y_{\min}}{2} \right| = 1;$$

$$\text{Range: } \sin x \in [-1, 1];$$

$$\text{Domain: } x \in \mathbb{R}.$$

2. Cosine: Even function:  $\cos(-x) = \cos x$ .



$$T(\text{Period}) = 2\pi;$$

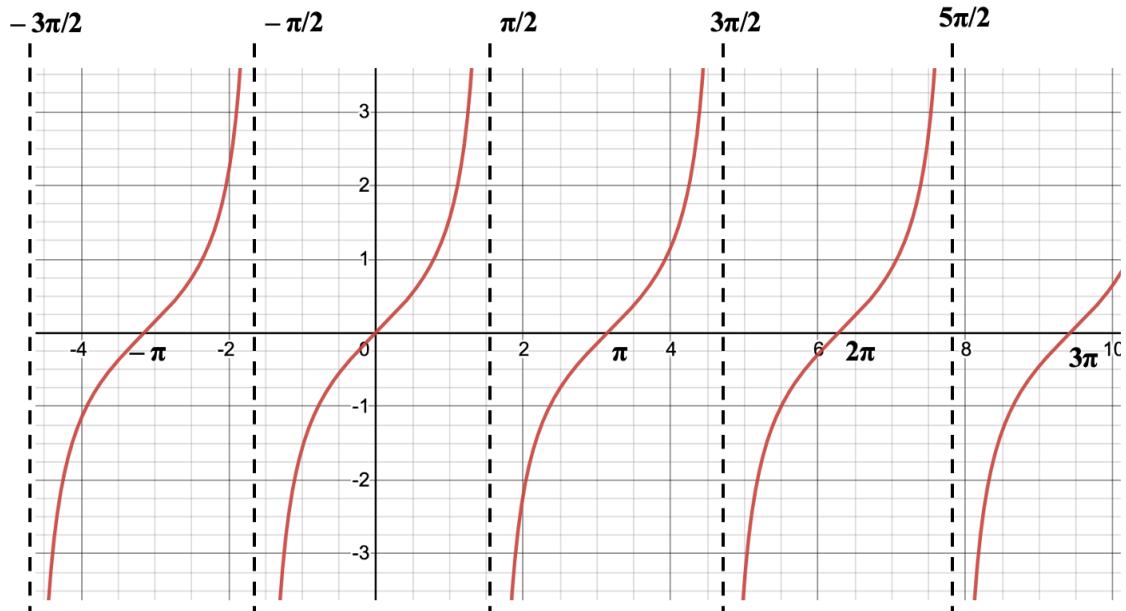
Base line = 0;

$$\text{Amplitude} = \left| \frac{y_{\max} - y_{\min}}{2} \right| = 1;$$

$$\text{Range: } \cos x \in [-1, 1];$$

Domain:  $x \in \mathbb{R}$ .

3. Tangent:



$$T(\text{Period}) = \pi;$$

No amplitude(A);

$$\text{V.A.: } x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z};$$

Range:  $\tan x \in \mathbb{R}$ ;

Domain:  $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ .

#### 4. Transformation of Sine and Cosine:

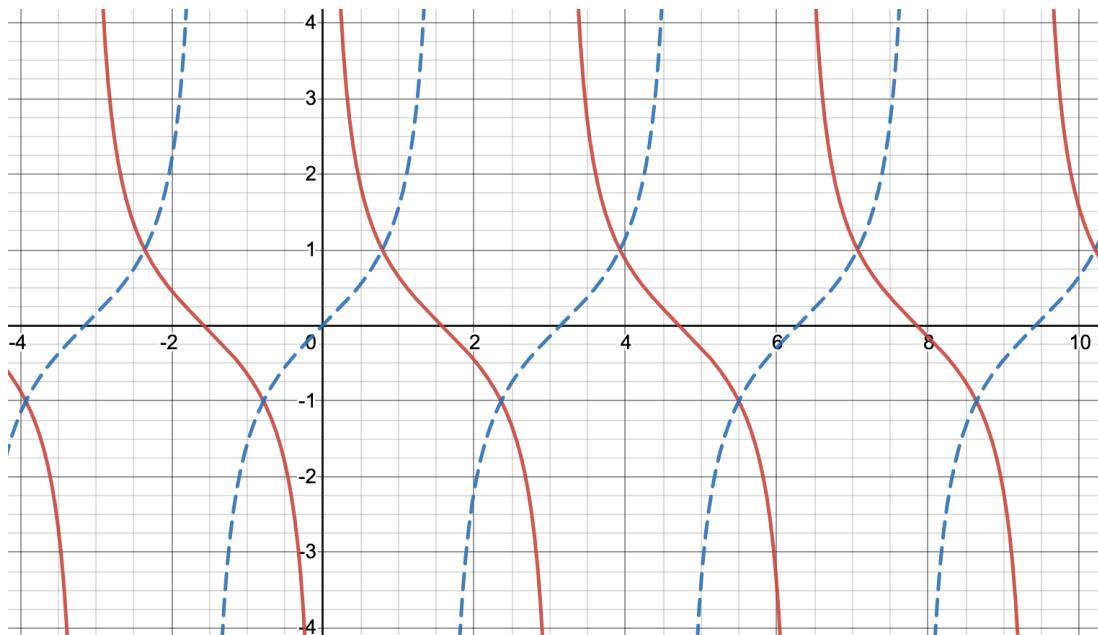
$$y = A \sin(\omega(x - \varphi)) + h.$$

- Horizontal stretch with the scale factor of  $\frac{1}{\omega}$ .  $\Rightarrow$  changes  $T = \frac{\pi}{\omega}$ .
- Horizontal translate to the right  $\varphi$  units.  $\Rightarrow$  changes the initial point to  $(\varphi, 0)$ .
- Vertical stretch with a scale factor of  $A$ .  $\Rightarrow$  changes the amplitude =  $|A|$ .
- Vertical translation of  $h$  units upwards.  $\Rightarrow$  changes the equilibrium position  $y = h$ .
- Range of  $y = A \sin(\omega(x - \varphi)) + h$ :  $y \in [h - A, h + A]$ .

$$y = A \cos(\omega(x - \varphi)) + h.$$

- Horizontal stretch with the scale factor of  $\frac{1}{\omega}$ .  $\Rightarrow$  changes  $T = \frac{\pi}{\omega}$ .
- Horizontal translate to the right  $\varphi$  units.  $\Rightarrow$  changes the initial point to  $(\varphi, 1)$ .
- Vertical stretch with a scale factor of  $A$ .  $\Rightarrow$  changes the amplitude =  $|A|$ , initial point  $(\varphi, A)$ .
- Vertical translation of  $h$  units upwards.  $\Rightarrow$  changes the equilibrium position  $y = h$ , initial point  $(\varphi, A + h)$ .

#### 5. Cotangent:

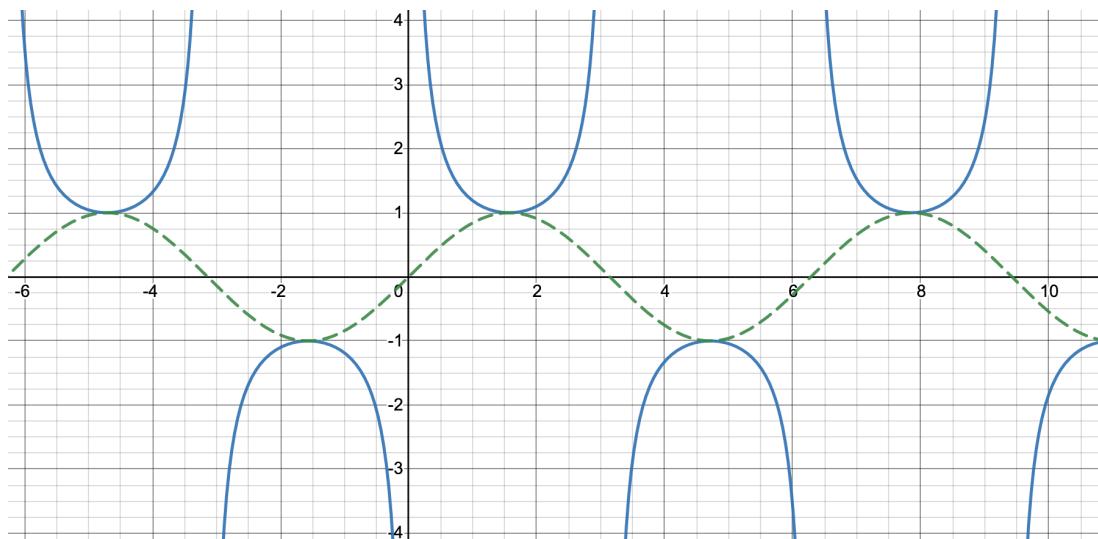


V.A.:  $x = k\pi$

Period:  $\pi$

Pass through  $\left(\frac{\pi}{2} + k\pi, 0\right)$

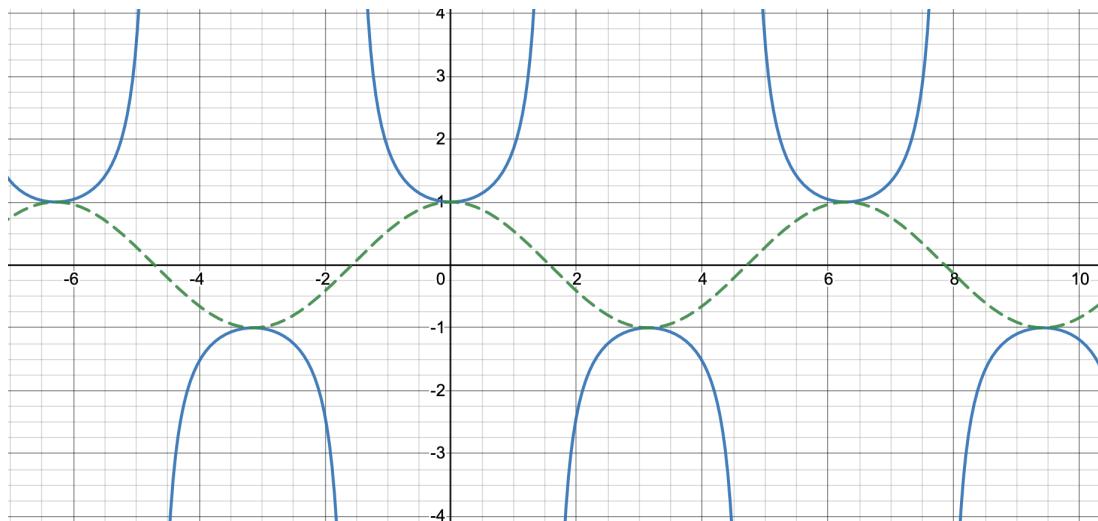
6. Cosecant:



$$\text{Domain: } x \neq k\pi$$

$$\text{Range: } y \in ]-\infty, -1[ \cup ]1, +\infty[$$

7. Secant:



$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi$$

$$\text{Range: } y \in ]-\infty, -1[ \cup ]1, +\infty[$$

8. When drawing the graph of  $\sec x$  and  $\csc x$ , draw  $\cos x$  and  $\sin x$  first.

9. Conversion between sine and cosine:

•

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\cos\left(\frac{\pi}{2} + x\right) = \cos\left[\pi - \left(\frac{\pi}{2} - x\right)\right] = -\cos\left(\frac{\pi}{2} - x\right) = -\sin x$
- $\sin\left(\frac{\pi}{2} + x\right) = \sin\left[\pi - \left(\frac{\pi}{2} - x\right)\right] = \sin\left(\frac{\pi}{2} - x\right) = \cos x$

### 3.1.6 Solving Trigonometric Functions

1. Solving Trigonometric Functions in Paper 1:

- Values of special angles
- From relative acute angles and CAST rule
- Modification of period
- Check the solution with domain

**Example 3.1.3** Solve for  $\cos x = \frac{\sqrt{3}}{2}$  for  $0 < x < 3\pi$ .

Consider  $x \in [0, 2\pi]$

$$x = \frac{\pi}{6}, \frac{11\pi}{6}.$$

In the domain of  $x \in [0, 3\pi]$ ,

Another solution is  $\frac{13\pi}{6}$ .

2. Transformed Trigonometric Equations:

**Example 3.1.4** Solve  $6\sin\left(2\left(x - \frac{\pi}{6}\right)\right) - 2 = 1$ ,  $\frac{\pi}{6} < x < 2\pi$ .

$$\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = \frac{1}{2}.$$

Let  $t = 2\left(x - \frac{\pi}{6}\right)$ :

$$\therefore \frac{\pi}{6} < x < 2\pi,$$

$$\therefore 0 < 2\left(x - \frac{\pi}{6}\right) < \frac{11\pi}{3}, \quad 0 < t < \frac{11\pi}{3}.$$

$$\sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6};$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}.$$

3. Solving Trigonometric Functions in Paper 2:

- Change mode to RADIANS.

- Plot the functions.
- Adjust the window.
- Calculate the intersects.
- Repeat step 4 if necessary.

### 3.1.7 Inverse Trigonometric Functions

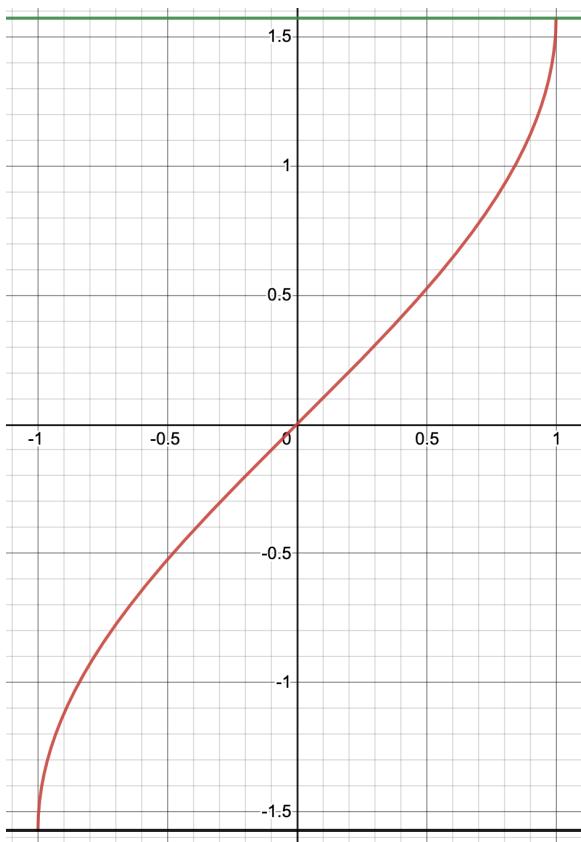
1. Inverse Trigonometric Function:

- $y = \arcsin x$
- $y = \arccos x$
- $y = \arctan x$
- $\text{arcsec}x = \arccos\left(\frac{1}{x}\right)$
- $\text{arccsc}x = \arcsin\left(\frac{1}{x}\right)$
- $\text{arccot}x = \arctan\left(\frac{1}{x}\right)$

2. One-to-one Function:

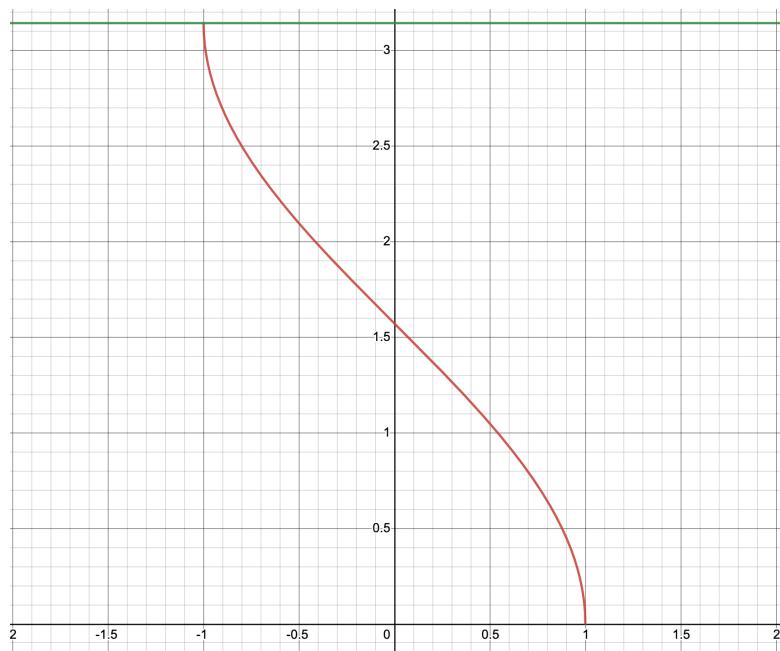
- In order for functions to have the inverse function, it must be so called **one-to-one** function (bijection).
- One  $x$  value to one (and only one)  $y$  value.  
One  $y$  value to one (and only one)  $x$  value.

3. Domain and range for  $\arcsin x$ :



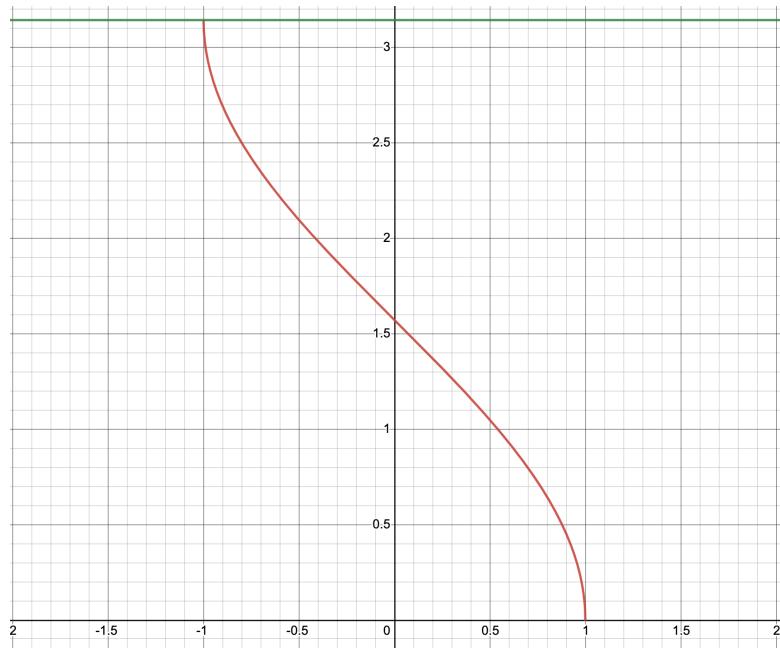
- Domain:  $x \in [-1, 1]$  (Range  $\sin x \in [-1, 1]$ ).
- Range:  $\arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  (Domain  $\sin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ).

4. Domain and range for  $\arccos x$ :



- Domain:  $x \in [-1, 1]$ .
- Range:  $\arccos x \in [0, \pi]$ .

5. Domain and range for  $\arctan x$ :



- Domain:  $x \in \mathbb{R}$
- Range:  $y \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$

## 3.2 Vectors

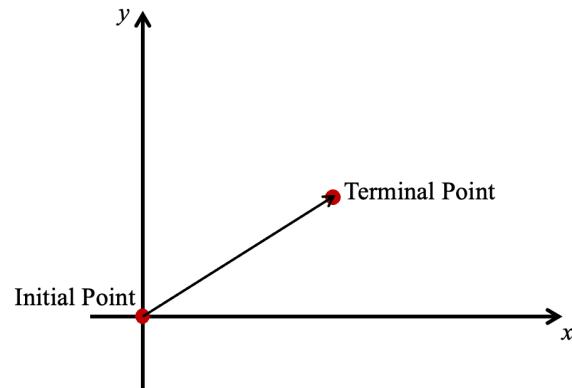
### 3.2.1 Introduction to Vectors

1. Vector:

**Definition 3.2.1** A **vector** is a quantity with a direction and magnitude. It is noted as  $\vec{a}$ .

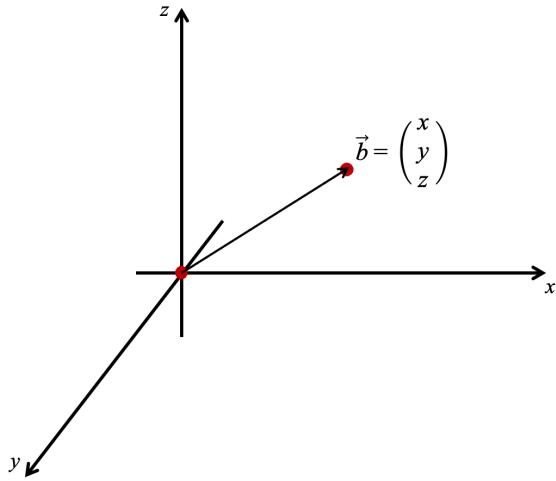
2. Components of a vector:

- 2-D:



**Example 3.2.1** The vector  $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$  means 3 units in the horizontal direction and 2 units in the vertical direction.

- 3D:



3. **Magnitude/Modulus** of vector:

- 2D:

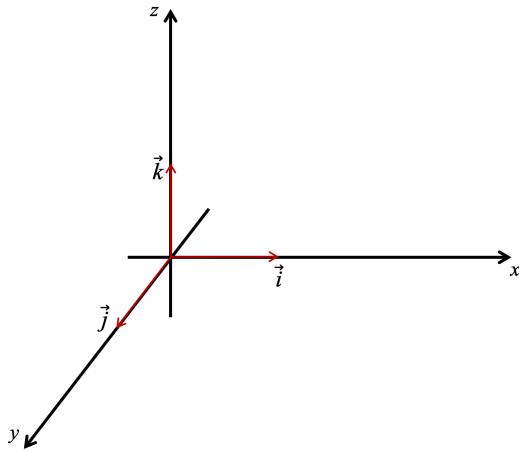
$$\text{For } \vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}, |\vec{a}| = \sqrt{x^2 + y^2}.$$

- 3D:

$$\text{For } \vec{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, |\vec{b}| = \sqrt{x^2 + y^2 + z^2}.$$

4. **Unit Vector**: A vector of length 1:

- $\vec{i}$ : unit vector on the  $x$ -axis.
- $\vec{j}$ : unit vector on the  $y$ -axis.
- $\vec{k}$ : unit vector on the  $z$ -axis.



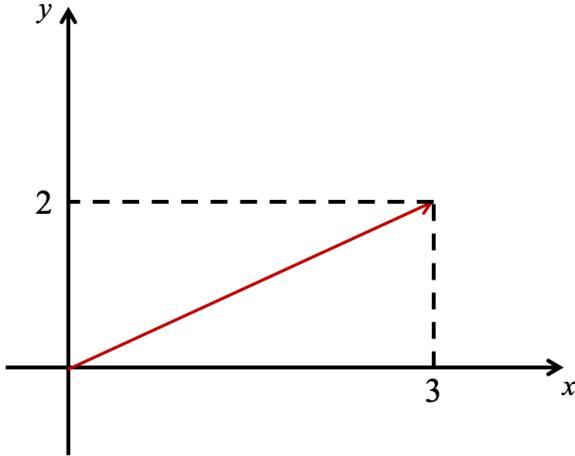
5. Sum of vectors:

- **Position vector**: A vector that has an initial point at the origin.

**Example 3.2.2**

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\vec{a} = 3\vec{i} + 2\vec{j}.$$



- Let  $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$
- $$\vec{a} + \vec{b} = \begin{pmatrix} x+m \\ y+n \end{pmatrix}.$$

6. Multiplication of vectors by a scalar:

Let  $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$  and  $n$  be a scalar:

$$n\vec{a} = n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} nx \\ ny \end{pmatrix}.$$

$n\vec{a}$  and  $\vec{a}$  are in the same direction  $\Rightarrow$  parallel.

7. Subtracting a vector:

Let  $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$ .

$$\vec{a} - \vec{b} = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

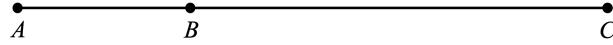
**Proof 3.2.1**

$$-\vec{b} = (-1)\vec{b} = \begin{pmatrix} -m \\ -n \end{pmatrix}$$

$$\vec{a} - \vec{b} = \vec{a} + \begin{pmatrix} -\vec{b} \end{pmatrix} = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

8. **Zero vector:**  $\vec{0}$ .

9. **Collinear points:** three points,  $A$ ,  $B$ , and  $C$ , are said to be collinear if  $\vec{AB} = t\vec{AC}$ .

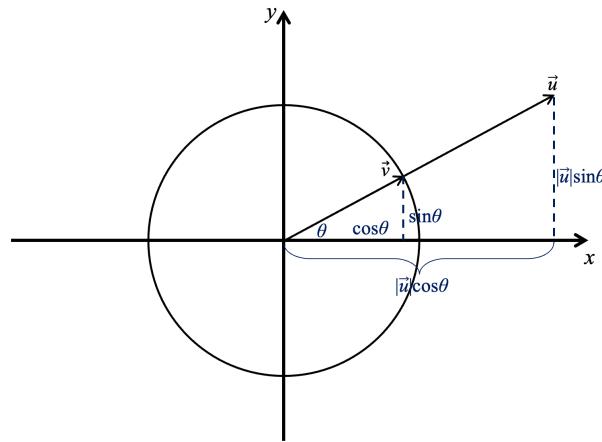


10. Find a unit vector parallel to  $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ .

- Find the value  $|\vec{u}|$ .
- Then, the unit vector parallel to  $\vec{u}$  is

$$\vec{v} = \frac{\vec{u}}{|\vec{u}|}.$$

11. Vectors and unit circle:



$\theta$  is the angle with the horizontal axis. The unit vector  $\vec{v}$ , in the same direction as  $\vec{u}$  is:

$$\vec{v} = \cos \theta \cdot \vec{i} + \sin \theta \cdot \vec{j}$$

$$\begin{aligned} \vec{v} &= \frac{1}{|\vec{u}|} \cdot \vec{u} \Rightarrow \vec{u} = |\vec{u}| \cdot \vec{v} = |\vec{u}| \cos \theta \cdot \vec{i} + |\vec{u}| \sin \theta \cdot \vec{j} \\ &= |\vec{u}| \left( \cos \theta \cdot \vec{i} + \sin \theta \cdot \vec{j} \right). \end{aligned}$$

### 3.2.2 Scalar Product and Its Properties

1. The **scalar product** of two vectors is a real number (scalar).

- The algebraic definition:

For  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ ,

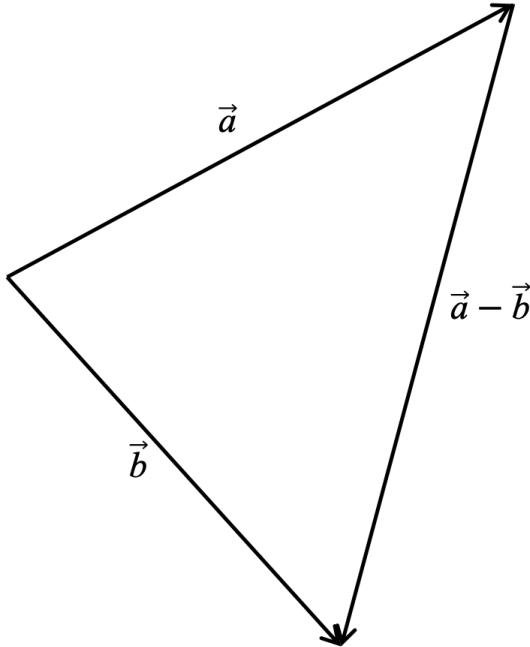
$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2.$$

The scalar product is also called the dot product.

- The geometric definition:

For  $\vec{a}$  and  $\vec{b}$ ,

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta, \quad \theta \text{ is the angle between the two vectors.}$$



**Proof 3.2.2** By cosine rule:

$$\begin{aligned} |\vec{b} - \vec{a}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos \theta \\ |\vec{b}|^2 - 2\vec{a}\vec{b} + |\vec{a}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos \theta \\ \therefore \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos \theta. \end{aligned}$$

- Combining the two definitions:

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2}{\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}}.$$

2. 3-D vectors:  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ :

•

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}.$$

3. Properties of scalar product:

- If  $\vec{a} \cdot \vec{b} = 0 \Rightarrow \begin{cases} \vec{a} = 0 \\ \vec{b} = 0 \\ \vec{a} \text{ and } \vec{b} \text{ are perpendicular (orthogonal)} \Rightarrow \theta = \frac{\pi}{2} \end{cases}$
  - If  $\vec{a}$  and  $\vec{b}$  are colinear,
- $$\vec{a} \cdot \vec{b} = \pm |\vec{a}| |\vec{b}|.$$

**Proof 3.2.3** Angel between  $\vec{a}$  and  $\vec{b}$  is  $0^\circ$ .

$\cos 0^\circ = 1 \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$  for  $\vec{a}, \vec{b}$  at the same direction.

OR  $\vec{a} \cdot \vec{b} = -|\vec{a}| |\vec{b}|$  for  $\vec{a}$  and  $\vec{b}$  at opposite directions.

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ .
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

**Proof 3.2.4**

$$\vec{a} \cdot \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = a_1^2 + a_2^2 = |\vec{a}|^2.$$

- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ .
- $\lambda (\vec{a} \cdot \vec{b}) = (\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b})$ .

### 3.2.3 Vector Equation of a Line

1. There is only one line that passes through two distinct points.

**Theorem 3.2.1** In the coordinate plane, the equation can be found as:

For  $A(x_1, y_1)$  and  $B(x_2, y_2)$ , the line passes through  $A, B$  is given by

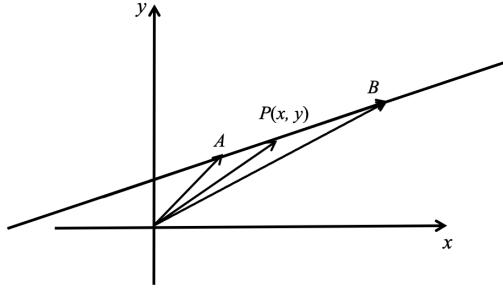
$$y = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) + y_1.$$

2. **Slope, y-intercept form:**  $y = mx + k$ , where  $m$  is the slope, and  $k$  is the  $y$ -intercept.

It can be rearranged to  $ax + by = c$ ;  $a, b, c \in \mathbb{R}$ , where  $a$  and  $b$  cannot be equal to 0 at the same time.

### 3. Vector form of a line:

- For every point  $P(x, y)$  that lies on the line  $AB$ , the vector  $\vec{AP}$  must be collinear or parallel to  $\vec{AB}$ :  $\vec{AP} = k\vec{AB}$ ,  $k \in \mathbb{R}$ .



(a) The vector  $\vec{AB}$  is called a **direction vector** of the line.

All the vectors that are parallel to  $\vec{AB}$  can also define the same line.

(b) Assume  $\vec{OA} = \vec{a}$ ,  $\vec{OP} = \vec{p}$ ,  $\vec{AB}$  is the direction vector  $\vec{d}$ . Then,  $\vec{AP} = \vec{p} - \vec{a} = k\vec{AB} = k\vec{d}$

$$\vec{p} = \vec{a} + k\vec{d}, k \in \mathbb{R}.$$

- Vector equation of a line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, k \in \mathbb{R}.$$

- Parametric form:

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases}, k \in \mathbb{R}.$$

- Cartesian form:

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}.$$

#### Proof 3.2.5

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases} \Rightarrow \begin{cases} k = \frac{x - x_1}{d_1} \\ k = \frac{y - y_1}{d_2} \end{cases}.$$

(a) Cartesian form can be further rearranged to slope-intercept form

$$\begin{aligned} \frac{x - x_1}{d_1} &= \frac{y - y_1}{d_2} \\ \frac{d_2}{d_1} (x - x_1) &= y - y_1 \\ y &= \frac{d_2}{d_1} (x - x_1) + y_1, \end{aligned}$$

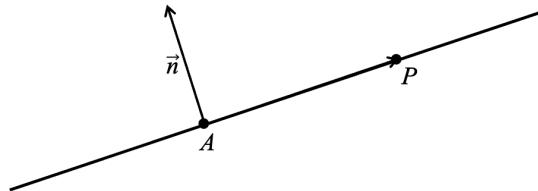
where  $\frac{d_2}{d_1}$  is the slope.

(b) Another way of interpretation:

$$\begin{aligned}
 \overrightarrow{AP} = k\overrightarrow{AB} &\Rightarrow \vec{p} - \vec{a} = k(\vec{b} - \vec{a}) \\
 &\Rightarrow \vec{p} = (1-k)\vec{a} + k\vec{b}, \quad k \in \mathbb{R}. \\
 &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = (1-k) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad k \in \mathbb{R}. \\
 &\Rightarrow \begin{cases} x = (1-k)x_1 + kx_2 = x_1 + k(x_2 - x_1) \\ y = (1-k)y_1 + ky_2 = y_1 + k(y_2 - y_1) \end{cases}, \quad k \in \mathbb{R}. \\
 &\Rightarrow \begin{cases} k = \frac{x-x_1}{x_2-x_1} \\ y = y_1 + k(y_2 - y_1) \end{cases} \\
 &\Rightarrow y = y_1 + \frac{x-x_1}{x_2-x_1}(y_2 - y_1) \\
 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.
 \end{aligned}$$

#### 4. Orthogonal / Perpendicular vector of a line.

- There is one and only one line in the plane that is perpendicular to a given line at a particular point on that line.
- Normal Vector:



**Definition 3.2.2** A **normal vector** is perpendicular or **orthogonal** to any vector on the lines.

$$\text{i.e., } \vec{n} \cdot \overrightarrow{AP} = 0.$$

#### Theorem 3.2.2

$$\vec{n} \cdot (\vec{p} - \vec{a}) = 0 \Rightarrow \vec{n} \cdot \vec{p} = \vec{n} \cdot \vec{a}.$$

- If the direction vector  $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ , then one possible normal vector would be  $\vec{n} = \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$  or any other vectors parallel to it.

- The vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$$

$$\Rightarrow xd_2 - yd_1 = x_1d_2 - y_1d_1$$

$$(x - x_1)d_2 = yd_1 - y_1d_1$$

$$\therefore y = \frac{d_2}{d_1}(x - x_1) + y_1.$$

## 5. Direction vectors:

- **Parallel lines** have **collinear** direction vectors.
- **Perpendicular lines** have **orthogonal** direction vectors, such that the scalar product is equal to 0.

## 6. Vector equation of lines in 3-D spaces:

- 

$$\vec{r} = \vec{a} + \lambda \vec{d}, \lambda \in \mathbb{R}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

- The parametric form:

$$\begin{cases} x = a_1 + \lambda d_1 \\ y = a_2 + \lambda d_2 \\ z = a_3 + \lambda d_3 \end{cases}, \lambda \in \mathbb{R}.$$

- The cartesian form:

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$

## 7. Two lines:

- 2-D spaces: two distinctive lines can either be parallel or they can intersect.
- 3-D spaces:
  - (a) Lines are parallel.
  - (b) Lines intersect at one common points.
  - (c) Lines are **skewed** (do not intersect and they are not parallel).

### 3.2.4 Vector Product and Properties

1. The vector product is an operation that takes two vectors and results in another **vector**.
  - Definition

**Definition 3.2.3** Given the two vectors and their components,  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ , then the **vector product** is given by:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

- The vector product of two vectors is another vector that is perpendicular to both vectors.
- Magnitude of the vector product:

**Theorem 3.2.3** The magnitude of the vector product is given by the formula

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin \theta,$$

where  $\theta$  is the angle between those two vectors. If  $\vec{a} \times \vec{b} = 0$ , then  $\vec{a}$  and  $\vec{b}$  are parallel/co-linear.

- The geometrical definition of cross product (vector product):

**Theorem 3.2.4** Given two vectors  $\vec{a}$  and  $\vec{b}$ , then the vector product is given by

$$\vec{a} \times \vec{b} = (|\vec{a}| |\vec{b}| \sin \theta) \hat{n},$$

where  $\hat{n}$  is the unit vector whose direction is given by the right-hand screw rule to both  $\vec{a}$  and  $\vec{b}$  and the vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\hat{n}$  follows the right-hand rule.

- Geometrical meaning of the magnitude of the vector product:  
It is equal to the area of the parallelogram enclosed by those two vectors.

## 2. Properties of the vector product:

- $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
- $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$
- $\lambda (\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}), \lambda \in \mathbb{R}$
- $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}).$

## 3. Mixed product:

- An operation with three vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  combining both the vector and scalar product is called a **mixed product**:

$$(\vec{a} \times \vec{b}) \cdot \vec{c}.$$

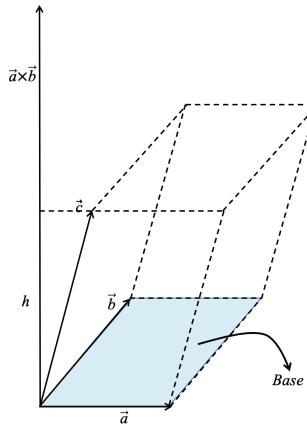
- Given  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ , and  $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ , the mixed product is given by:

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot \vec{c} &= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \end{aligned}$$

- Geometric meaning of mixed products:

The volume of a parallelepiped formed by three non-coplanar vectors,  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  is given by:

$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|.$$



### Proof 3.2.6

$$V = \text{Base} \times h$$

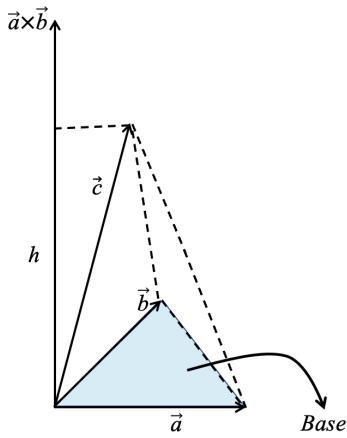
Base=magnitude of cross product of  $\vec{a}$  and  $\vec{b}$ .

= perpendicular projection of  $\vec{c}$  to  $\vec{a} \times \vec{b}$ .

$$\therefore V = \text{Base} \times h = |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cdot |\cos \theta| = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

- Three or more vectors are said to be coplanar if they lie in the same plane.
- Using mixed product to find the volume of a triangular pyramid:

$$V = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|.$$



**Proof 3.2.7** Since the base is not a parallelogram but a triangle, that is half an area of the parallelogram, we multiply  $\frac{1}{2}$  in front of the expression of the cross product.

$$\text{Base} = \frac{1}{2} |\vec{a} \times \vec{b}|.$$

The volume of a pyramid is  $\frac{1}{3}$  of the product of the base and the height.

$$\therefore V = \frac{1}{3} \text{Base} \cdot h = \frac{1}{3} \cdot \frac{1}{2} |\vec{a} \times \vec{b}| |\vec{c}| |\cos \theta| = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

4. Proving vector product using matrix.

**Proof 3.2.8** Let  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ . Convert into a  $3 \times 3$  matrix:  $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ .

Find the determinant:  $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$

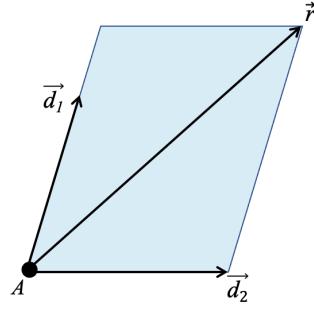
$$\Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

### 3.2.5 Vector Equation of a Plane

1. A plane is uniquely determined by **three points** (or a line and a point outside the line).  
→ A plane can also be determined by two intersecting lines and a point outside the lines.
2. Vector equation of a plane:

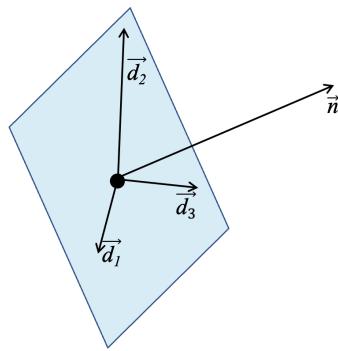
$$\vec{r} = \vec{a} + \lambda \vec{d}_1 + \mu \vec{d}_2, \lambda, \mu \in \mathbb{R}.$$

where  $\vec{d}_1$  and  $\vec{d}_2$  are direction vectors, and  $\vec{a}$  is the position vector.



3. The scalar product form:

- **Normal vector** is a vector that is perpendicular to every line in the plane.



- All planes with the same normal vector are parallel to each other.
- If R is any other point on the plane, then AR lies in the plane, and it is perpendicular to the normal vector n.

### Theorem 3.2.5

$$\overrightarrow{AR} \cdot \vec{n} = 0 \Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\therefore \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

where  $\vec{a}$  is the position vector, and  $\vec{n}$  is the normal vector.

4. The Cartesian equation of a plane:

$$n_1x + n_2y + n_3z = d, \text{ where } n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, d = \vec{a} \cdot \vec{n}.$$

### Proof 3.2.9

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, d = \vec{a} \cdot \vec{n}, \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The scalar product form converts to:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \vec{a} \cdot \vec{n}$$

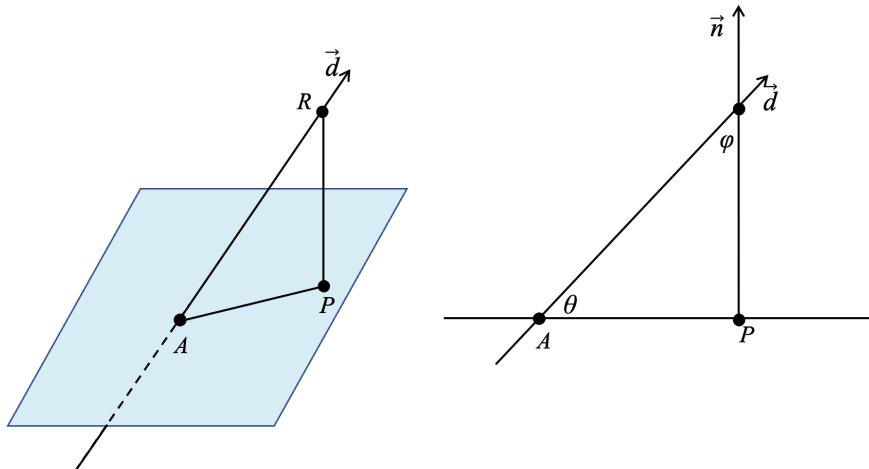
$$\Rightarrow n_1x + n_2y + n_3z = d.$$

5. A plane with the vector equation  $\vec{r} = \vec{a} + \lambda\vec{d}_1 + \mu\vec{d}_2$  has a normal vector  $\vec{n} = \vec{d}_1 \times \vec{d}_2$ .

### 3.2.6 Lines, Planes, and Angles

1. Angles and intersections between lines and planes:

- When a line intersects a plane, the angle between them is defined as the **smallest possible angle** that the line makes with any of the lines in the plane.



- (a)  $\overrightarrow{AR}$ : the direction vector of the line,  $\vec{d}$ .
- (b) Point  $P$  is the projection of point  $R$  onto the plane.  $\overrightarrow{AP}$  is the shadow of  $\overrightarrow{AR}$  on the plane.
- (c)  $\overrightarrow{PR}$  is in the direction of  $\vec{n}$  since it is perpendicular to the plane.
- (d)  $\varphi$  is the angle between  $\vec{n}$  and  $\vec{d}$ .
- (e)

$$\theta = 90^\circ - \varphi, \cos \varphi = \frac{|\vec{n} \cdot \vec{d}|}{|\vec{n}| |\vec{d}|}.$$

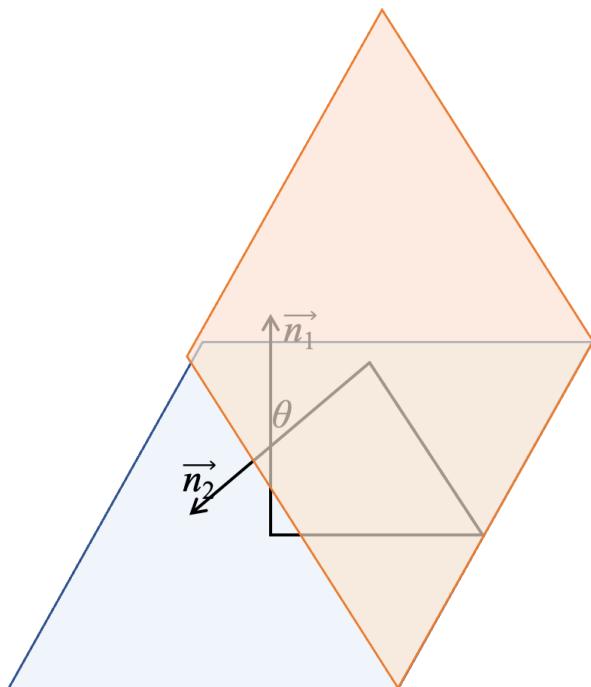
- A line that is not parallel to a plane intersects a plane at one point. The coordinates of this point of intersection satisfies both the equation of the line and the equation of the plane.

2. Relationship of two planes:

- Two planes can either intersect at a line or they can be parallel.
- When two planes are parallel, their normal vectors are **colinear**; otherwise they intersect at a line.

3. Angles between two planes:

- The angle between two planes is **the angle between their normal vectors**.
  -
- $$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}.$$

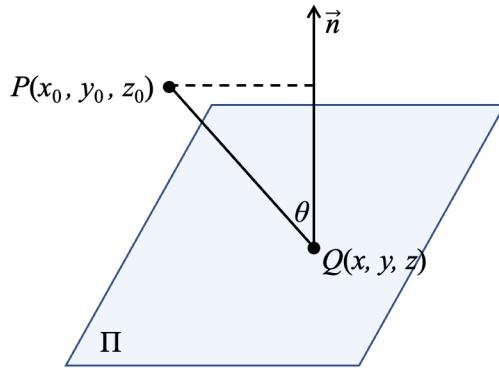


4. Two non-parallel planes intersect along a line. The equation of this line is formed by treating the Cartesian equation of two planes as simultaneous equations and finding the general solution.
5. Distance between a point and a plane.

- The distance,  $d$ , between a point  $P(x_0, y_0, z_0)$ , and a plane with equation  $Ax + By + Cz = D$  where  $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ , is given by:

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

- Proof:



**Proof 3.2.10** Let  $Q(x, y, z)$  be any point on the plane  $\Pi$ .

The distance,  $d$ , is the projection of the distance of point  $P$  to the plane on the normal vector,  $\vec{n}$ .

$$\begin{aligned}
 d &= |\overrightarrow{QP}| \cdot |\cos \theta| = |\overrightarrow{QP}| \cdot \frac{\overrightarrow{QP} \cdot \vec{n}}{|\overrightarrow{QP}| \cdot |\vec{n}|} \\
 &= \frac{\overrightarrow{QP} \cdot \vec{n}}{|\vec{n}|} = \frac{|\langle A, B, C \rangle \cdot \langle (x_0 - x), (y_0 - y), (z_0 - z) \rangle|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{|Ax_0 + By_0 + Cz_0 - (Ax + By + Cz)|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}
 \end{aligned}$$

6. Intersection of three points:

Unique solution	Infinitely many solutions	No solutions (inconsistent system)		
		No normals parallel	Two normals parallel	Three normals parallel
Three planes intersect at a point	Three planes intersect along a line	Three planes form a prism	One plane cutting two parallel planes	Three parallel planes

- The plane intersect:
  - (a) At a point: the system of equations will have a unique solution.
  - (b) Along a line: the system of equations will have infinitely many solutions
- The systems of equations have no solutions:
  - (a) No normals are parallel (the planes from a prism)
  - (b) 2 normals are parallel or three normal are parallel (the planes are parallel)

## 4 Topic 4 Statistics

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## 5 Topic 5 Calculus

### 5.1 Limits

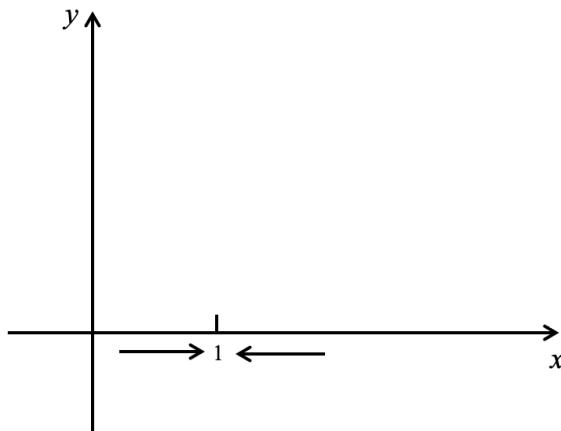
#### 1. Limit

##### Example 5.1.1

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

when  $x$  is approaching to 1 (it never equals to 1), the value  $\frac{x^2 - 1}{x - 1}$  is approaching to 2.

- Left-hand and Right-hand Limit



**Example 5.1.2** The left-hand limit of  $\frac{x^2 - 1}{x - 1}$  when  $x \rightarrow 1$  is

$$\lim_{x \rightarrow 1^-} \frac{x^2 - 1}{x - 1} = 2.$$

The right-hand limit of  $\frac{x^2 - 1}{x - 1}$  when  $x \rightarrow 1$  is

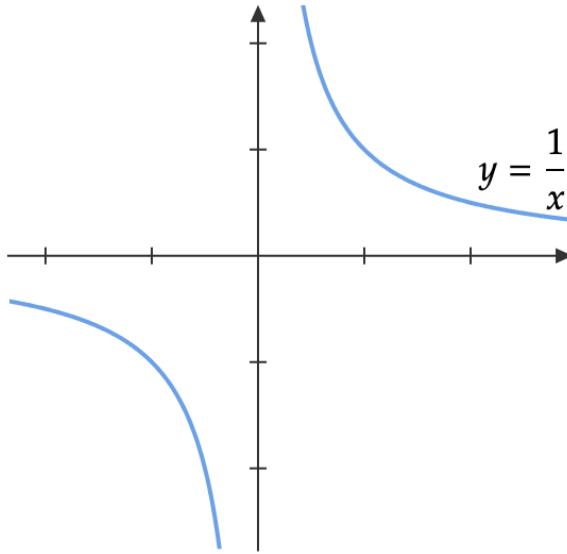
$$\lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = 2.$$

- Only when the left-hand limit and the right-hand limit exist and are the same at the point  $x = a$ , we say the limit of  $f(x)$  exists on  $x = a$ .

i.e.,  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = c \Rightarrow \lim_{x \rightarrow a} f(x) = c$ ,  $c$  is a constant  $\in \mathbb{R}$

Otherwise, the limit does not exist on  $x = a$  (OR DNE.).

**Example 5.1.3 Does  $\lim_{x \rightarrow 0} \frac{1}{x}$  exist? How about  $\lim_{x \rightarrow \infty} \frac{1}{x}$ ?**



-  $\lim_{x \rightarrow 0} \frac{1}{x}$  does not exist.

$$\because \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty, \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$\therefore \lim_{x \rightarrow 0^+} \frac{1}{x} \neq \lim_{x \rightarrow 0^-} \frac{1}{x} \Rightarrow \text{DNE.}$$

-  $\lim_{x \rightarrow \infty} \frac{1}{x}$  exists.

$$\because \lim_{x \rightarrow +\infty} \frac{1}{x} = 0, \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x \rightarrow +\infty} \frac{1}{x} = \lim_{x \rightarrow -\infty} \frac{1}{x} \Rightarrow \text{Limit exists.}$$

### Definition 5.1.1 Horizontal Asymptote (H.A.):

$$y = \lim_{x \rightarrow \infty} f(x) = c$$

- Limit at  $\infty$ :

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = c \Rightarrow \lim_{x \rightarrow \infty} f(x) = c.$$

Note:  $+\infty$  and  $-\infty$  are not exact values; they should be regarded as a concept.

- Limits does not have to equal to the function value.  
Limit and the function value do not have relationships.
- Generally speaking, if  $a \in D_f$ ,  $\lim_{x \rightarrow a} f(x) = f(a)$ .

2. For a rational function  $f(x) = \frac{P(x)}{Q(x)}$  where  $P(x) = a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m$ , and  $Q(x) = b_0x^m + b_1x^{m-1} + \dots + b_{m-1}x + b_m$ :

- $\lim_{x \rightarrow a} f(x) = f(a)$  as long as  $Q(a) \neq 0$ .
- $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m}{b_0x^m + b_1x^{m-1} + \dots + b_{m-1}x + b_m} \Rightarrow \text{H.A.}$
- (a) If  $m = n$ ,  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_0}{b_0} = \frac{a_0}{b_0}$ .

- (b) If  $m > n$ ,  $\lim_{x \rightarrow \infty} f(x)$  DNE.  
(c) If  $m < n$ ,  $\lim_{x \rightarrow \infty} f(x) = 0$ .

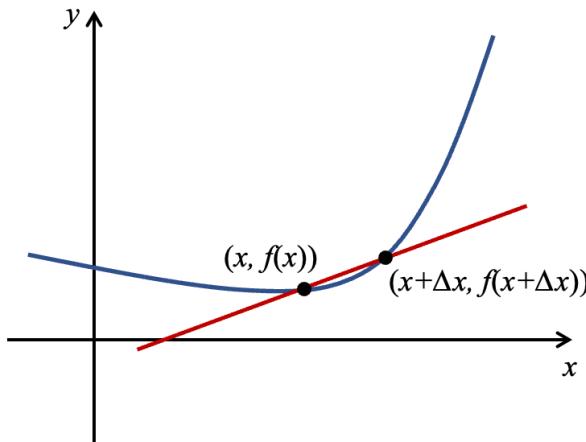
### 3. Continuity and Discontinuity

**Definition 5.1.2 Continuity:** If the graph of the function does not have any **breaks or holes** within a certain interval, then the function is continuous within that interval.

**Theorem 5.1.1** If  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$ , then the function  $f$  is **continuous** at  $x = a$ .

## 5.2 Differentiation and Derivatives

### 1. Gradient of Secant:



- Slope  $m = \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$ .

**Definition 5.2.1 Derivative of a function:**

$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$  is the derivative of a function, denoted as  $\frac{dy}{dx}$  or  $f'(x)$ .

- The graphic meaning of derivative is the gradient of tangent of the function.

**Example 5.2.1 By definition, find the derivative of  $f(x) = x^2 + 1$  and hence find the gradient of the tangent line when  $x = 3$ .**

$$\begin{aligned}
f'(x) &= \lim_{x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{x \rightarrow 0} \frac{[(x + \Delta x)^2 + 1] - (x^2 + 1)}{\Delta x} \\
&= \lim_{x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x} \\
&= \lim_{x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{x \rightarrow 0} (2x + \Delta x) \\
&= 2x.
\end{aligned}$$

At  $x = 3$ ,  $f'(3) = 2 \times 3 = 6$ . The gradient is 6.

## 2. Derivative of $x^n$

**Theorem 5.2.1** If  $f(x) = x^n$ , then

$$f'(x) = nx^{n-1}, \text{ for any } n \in \mathbb{R}.$$

Note: The derivative of any **constant** is **0**.

**Example 5.2.2**

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = (-1)x^{-1-1} = -x^{-2};$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}};$$

$$f(x) = c = cx^0 \Rightarrow f'(x) = 0 \times cx^{0-1} = 0.$$

## 3. Rules of Differentiation:

Name  $f(x)$  and  $g(x)$  as two functions with derivatives of  $f'(x)$  and  $g'(x)$ , respectively.

Then

$$(cf(x))' = cf'(x)$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

## 4. More Derivatives:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\ln x$	$\frac{1}{x}$
$e^x$	$e^x$

## 5. Differentiability:

**Definition 5.2.2** A function has to be **continuous** and **no sharp turning point** to be **differentiable**.

Note: Smooth turning point on the graph is allowed.

## 6. More Rules of Differentiation:

**Theorem 5.2.2** Let  $f(x)$  and  $g(x)$  be two functions with derivatives of  $f'(x)$  and  $g'(x)$ , respectively.

$$(f(x) \times g(x))' = f'(x)g(x) + f(x)g'(x).$$

**Theorem 5.2.3** Let  $f(x)$  and  $g(x)$  be two functions with derivatives of  $f'(x)$  and  $g'(x)$ , respectively.

$$\left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

**Theorem 5.2.4** For a composite function  $f(g(x))$  or  $(f \circ g)(x)$ , the derivative will be

$$f'(g(x)) \times g'(x).$$

OR

If  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

7. Higher Order Differentiation:

$$\frac{d^2y}{dx^2}, f''(x), f'''(x), f^{(4)}(x), f^{(5)}(x), \dots$$

### 5.3 Applications of Derivatives

1. Equation of Tangent Line:

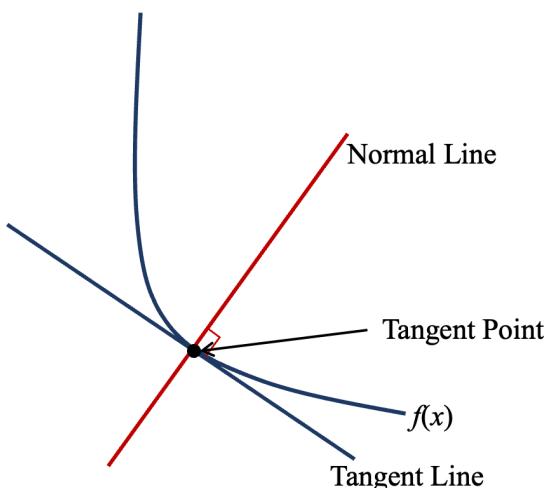
Via the original functions, we could get the tangent point  $(x_0, y_0)$ . Then, the expression of the tangent line is

$$y - y_0 = m(x - x_0),$$

where  $m$  is the derivative.

2. Normal and Tangent Lines:

**Definition 5.3.1** **Normal** is perpendicular to the tangent and passes through the same tangent point.



3. Increasing and Decreasing Function:

**Definition 5.3.2 Increasing Function:** As  $x$  is getting larger,  $y$  is getting larger.

i.e.,

$$\frac{dy}{dx} > 0.$$

**Decreasing Function:** As  $x$  is getting larger,  $y$  is getting smaller.

i.e.,

$$\frac{dy}{dx} < 0.$$

4. Local Extrema:  $\frac{dy}{dx} = 0$  Stationary point

Global extrema is the maximum and the minimum points of the entire function.

$f''(x)$  is used to determine if the local extrema is maxima or minima.

- Minima:  $f''(x) > 0$  Concave up.
  - Maxima:  $f''(x) < 0$  Concave down.
  - Point of Inflection (the point that is changing from concaving up to concaving down, or vice versa):  $f''(x) = 0$
5. With local extrema,  $x$ -intercepts,  $y$ -intercepts, concavity, and asymptotes, draw approximate diagrams of a function.

## 5.4 Implicit Differentiation

1. When differentiating something with  $y$ , multiply  $\frac{dy}{dx}$  at the end.
2.  $(y^2)' = 2y \frac{dy}{dx}$ .

**Proof 5.4.1** If  $u = y^2$ , then

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}. \quad [\text{Chain Rule}]$$

**Example 5.4.1** Find  $\frac{dy}{dx}$  for the circle  $x^2 + y^2 = 16$ .

$$\begin{aligned} (x^2)' + (y^2)' &= (16)' \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \\ \frac{dy}{dx} &= -\frac{2x}{2y} = -\frac{x}{y}. \end{aligned}$$

**Example 5.4.2** Find  $\frac{dy}{dx}$  for  $e^x + x \sin y = \cos 2y$ .

$$\begin{aligned} (e^x)' + (x \sin y)' &= (\cos 2y)' \\ e^x + \left( \sin y + x \cos y \frac{dy}{dx} \right) &= -2 \sin 2y \frac{dy}{dx} \\ (-x \cos y - 2 \sin 2y) \frac{dy}{dx} &= e^x + \sin y \\ \frac{dy}{dx} &= \frac{e^x + \sin y}{-x \cos y - 2 \sin 2y}. \end{aligned}$$

3. Second Order Differentiation of Implicit functions\*: Differentiate the first order differentiation.

**Example 5.4.3** Find  $\frac{d^2y}{dx^2}$  for the circle  $x^2 + y^2 = 16$ . (From Ex. 4.1:  $2x + 2y\frac{dy}{dx} = 0$ ,  $\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{y}$ .)

$$(2x)' + \left(2y\frac{dy}{dx}\right)' = (0)' \Rightarrow 2 + \left((2y)'\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right)'\right) = 0 \Rightarrow 2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$$

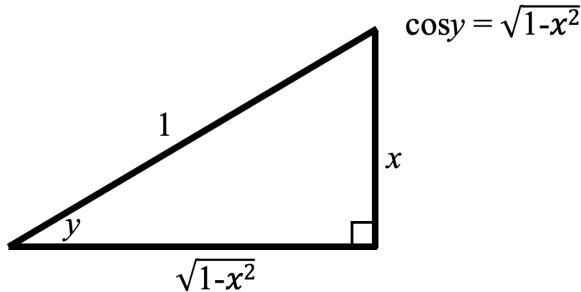
$$\frac{d^2y}{dx^2} = \frac{-2 - 2\left(\frac{dy}{dx}\right)^2}{2y} = \frac{-2 - 2\left(-\frac{x}{y}\right)^2}{2y}.$$

4. Derivative of Inverse Trigonometry Functions

### Theorem 5.4.1

$$y = \arcsin x \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \quad \arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] (\cos y > 0).$$

**Proof 5.4.2** From  $y = \arcsin x$ , we get  $\sin y = x$ . This situation can be illustrated by the figure below:



$$\begin{aligned} \therefore (\sin y)' &= (x)' \Rightarrow \cos y \frac{dy}{dx} = 1 \\ \therefore \frac{dy}{dx} &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

### Theorem 5.4.2

$$y = \arccos x \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}, \quad \arccos x \in [0, \pi] (\sin y > 0).$$

$$y = \arctan x \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}.$$

**Proof 5.4.3** (Hint: Try to visualize a similar diagram as in proof 4.1.)

From  $y = \arccos x$ , we get  $\cos y = x$ .

$$\therefore (\cos y)' = (x)' \Rightarrow -\sin y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

From  $y = \arctan x$ , we get  $\tan y = x$ .

$$\therefore (\tan y)' = (x)' \Rightarrow \sec^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \left( \frac{1}{\sqrt{1+x^2}} \right) = \frac{1}{1+x^2}.$$

## 5.5 Related Rate of Change

- When finding a rate of change of  $x$ , we are finding the  $\frac{dy}{dx}$ .

**Example 5.5.1** Area of circle is increasing at a rate of  $10\pi$  per second. When the radius is 2, what is the rate of change of radius?

Known:  $\frac{dA}{dt} = 10\pi$ ,  $r = 2$ . Find:  $\frac{dr}{dt}$ .

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 10\pi$$

$$\frac{dr}{dt} = \frac{10\pi}{2\pi r} = \frac{5}{r}$$

$$\text{When } r = 2, \frac{dr}{dt} = \frac{5}{2}.$$

**Example 5.5.2** A spherical balloon is expanding at a rate of  $60\pi$  per second. How fast is the surface area of the balloon expanding when the radius is 4?

Known:  $\frac{dV}{dt} = 60\pi$ ,  $r = 4$ . Find  $\frac{dA}{dt}$ .

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 3 \cdot \frac{4}{3}\pi r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\therefore 4\pi r^2 \frac{dr}{dt} = 60\pi \Rightarrow \frac{dr}{dt} = \frac{60\pi}{4\pi r^2} = \frac{15}{r^2}$$

$$A = 4\pi r^2 \Rightarrow \frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \cdot \frac{15}{r^2} = \frac{120\pi}{r}.$$

$$\text{When } r = 4, \frac{dA}{dt} = \frac{120\pi}{4} = 30\pi.$$

- Kinematics:

- Velocity, displacement, and acceleration are vector variables that have a value and a direction.
- Speed only has a value and no direction. It is a scalar variable. No sign should be reported in the answer.
- If  $s$  is the displacement,  $v$  is the velocity,  $a$  is the acceleration:

$$\frac{ds}{dt} = v; \frac{dv}{dt} = a.$$

## 5.6 More Limits - L'Hopital's Rule

**Theorem 5.6.1** When the limit is in the **indeterminant form** ( $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ ),

$$\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] \lim_{x \rightarrow a} \left[ \frac{f'(x)}{g'(x)} \right],$$

where  $f'(x)$  and  $g'(x)$  are the first derivatives of  $f(x)$  and  $g(x)$ , respectively.

### Example 5.6.1

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sec^2 x}{1} = 1$$

## 5.7 Indefinite Integration

1. Regard Integration as Anti-differentiation:

$$f'(x) = x \Rightarrow f(x) = \frac{1}{2}x^2 + C, \text{ where } C \text{ is a constant.}$$

$$f'(x) = x^2 \Rightarrow f(x) = \frac{1}{3}x^3 + C, \text{ where } C \text{ is a constant.}$$

$$f'(x) = x^n \Rightarrow f(x) = \frac{1}{n+1}x^{n+1} + C, \text{ where } C \text{ is a constant.}$$

**Definition 5.7.1** Anti-differentiation is also called **indefinite integration**. It is denoted by  $\int dx$ .

$$\text{e.g. } \int x^n dx = \frac{1}{n+1}x^{n+1} + C.$$

2. General Rules of Integration.

- $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$
- $\int k dx = kx + C$
- $\int kf(x) dx = k \int f(x) dx$
- $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

3.  $\int f'(x) dx = f(x) + C$ . Therefore, if we know the  $f'(x)$  and a point on the  $f(x)$ , which is to determine the constant  $C$ , then we can deduce the original function  $f(x)$ .

4. More Rules of Integration:

Differentiation	Integration
$(e^x)' = e^x$	$\int e^x dx = e^x + C$
$(\ln x)' = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + C$
$(\sin x)' = \cos x$	$\int \cos x dx = \sin x + C$
$(\cos x)' = -\sin x$	$\int \sin x dx = -\cos x + C$
$(\tan x)' = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$(\cot x)' = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$(\sec x)' = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$(\csc x)' = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$

5. Anti-chain Rule in Integration: We must **divide** the chain rule factor.

**Example 5.7.1**

$$\int (ax+b)^n \, dx = \frac{1}{a} \left( \frac{1}{n+1} (ax+b)^{n+1} \right) + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

$$\int \frac{1}{(ax+b)} \, dx = \frac{1}{a} \ln(ax+b) + C$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$$

$$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$$

6. Integration Techniques: by Substitution and by Parts:

**Theorem 5.7.1** Whenever we have an integration like:  $\int g'(x) \times f(g(x)) \, dx$ , we can always assume  $u = g(x)$ . Therefore,  $du = g'(x) \cdot dx$ . ( $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$ ):

$$\int g'(x) \times f(g(x)) \, dx = \int f(u \, du).$$

**Theorem 5.7.2** If  $f(x)$  and  $g(x)$  are two functions, and  $f'(x)$  and  $g'(x)$  are their derivatives, respectively, integration by parts can be written as following:

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.$$

**Example 5.7.2** Find  $\int 2x(x^2 + 3)^5 \, dx$ .

Since  $2x = (x^2 + 3)'$ , we consider to use integration by substitution.

Assume  $u = x^2 + 3$ , then  $\frac{du}{dx} = (x^2 + 3)' = 2x \Rightarrow du = 2x \cdot dx$ .

$$\begin{aligned} \therefore \int 2x(x^2 + 3)^5 \, dx &= \int (x^2 + 3)^5 \cdot (2x \cdot dx) \\ &= \int u^5 \, du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(x^2 + 3)^6 + C. \end{aligned}$$

7. Integration of Inverse Trigonometric Functions:

•

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

•

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

•

$$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arcsec} x + C$$

**Example 5.7.3** Find  $\int \frac{dx}{x^2+4x+5}$

$$\begin{aligned} \int \frac{dx}{x^2+4x+5} &= \frac{dx}{(x+2)^2+1} \\ \text{Assume } u = x+2, \frac{du}{dx} &= 1 \Rightarrow du = dx \\ \therefore \int \frac{dx}{x^2+4x+5} &= \frac{du}{u^2+1} \\ &= \arctan u + C \\ &= \arctan(x+2) + C. \end{aligned}$$

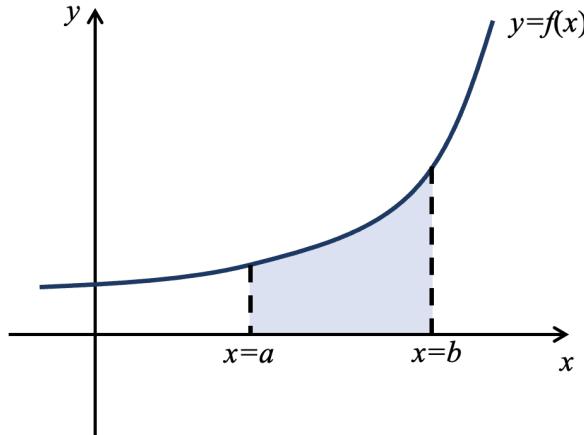
## 5.8 Approximating the Area Under a Curve

- The **definite integral** is equal to the limit at infinity of the Riemann sum, and hence gives the exact area under the curve between  $x = a$  and  $x = b$ . i.e.,

$$\lim_{n \rightarrow \infty} \sum_i^n f(x_i) \Delta x_i = \int_a^b f(x) dx,$$

where  $a$  is the lower limit and  $b$  is the upper limit.

- If  $f(x) \geq 0 \forall x \in [a, b]$ , then  $\int_a^b f(x) dx$  is defined as the shaded area:



This is known as the **Riemann integral**.

- The **Fundamental Theorem of Calculus**:

**Theorem 5.8.1** For a continuous function  $f(x)$  with antiderivative  $F(x)$ :

$$\int_a^b f(x) dx = F(b) - F(a).$$

This theorem explains the link between differential calculus and the definite integral.

- Properties of Definite Integrals:

- $\int_a^a f(x) \, dx = 0$
- $\int_a^b d \, dx = k(b-a)$ , ( $k$  is a constant).
- $\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx$
- $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$
- $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$
- $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

5. When the function  $f(x)$  is **negative** for  $x \in [a, b]$ , then the area bounded by the curve, the  $x$ -axis and the lines  $x = a$  and  $x = b$  is given by

$$\left| \int_a^b f(x) \, dx \right|.$$

6. Finding Areas Between Two Functions:

- Sketch: find the intersections and determine which function is above.
- Integration.

## 5.9 Volumes of Revolution

1. The volume of a solid of revolution formed when  $y = f(x)$ , which is continuous in the interval  $[a, b]$ , is rotated  $2\pi$  radians about the  $x$ -axis is given by

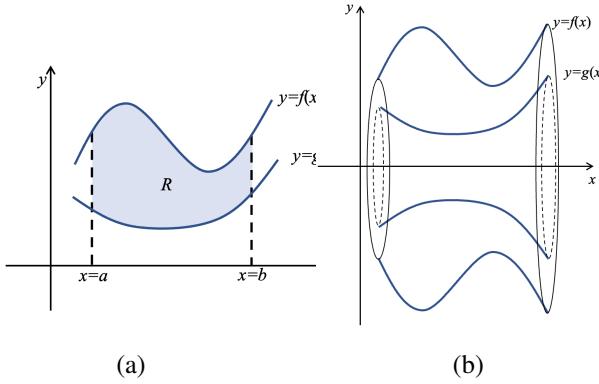
$$V = \pi \int_a^b y^2 \, dx.$$

2. The volume of a solid of revolution formed when  $y = f(x)$ , which is continuous in the interval  $y = c$  to  $y = d$ , is rotated  $2\pi$  radians about the  $y$ -axis is given by

$$V = \pi \int_c^d x^2 \, dy.$$

3. Consider a region  $R$  between two curves,  $y = f(x)$  and  $y = g(x)$ , from  $x = a$  to  $x = b$ , when  $f(x) > g(x)$ .

- Rotating  $R$  about the  $x$ -axis generates a solid of revolution  $S$ . The criss-section of this



shape looks like a washer whose area is given by:

$$A = \pi(R^2 - r^2) = \pi((f(x))^2 - (g(x))^2).$$

So the volume of  $S$  is given by:

$$\begin{aligned} V &= \int_a^b A(x) \, dx \\ &= \int_a^b ((f(x))^2 - (g(x))^2) \, dx. \end{aligned}$$

- Rotating  $R$  about the  $y$ -axis in the interval  $c \leq y \leq d$ :

$$V = \pi \int_c^d ((x_1)^2 - (x_2)^2) \, dy,$$

where  $x_1$  and  $x_2$  are expression of  $x$  with respect to  $y$  of  $f(x)$  and  $g(x)$ .

## 5.10 Differential Equation

1. Differential Equation:

**Definition 5.10.1** A **differential equation** is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables. **Equation that involves the derivatives of one or more functions.**

E.g.

$$y' = 6x + 1 \quad \text{Lagrange notation}$$

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - y = 3 \quad \text{Leibniz notation}$$

$$f'(x) = 6x + 1 \quad \text{Function notation}$$

The independent variable is  $x$ , and the dependent variable is  $y$ . The solution to a differential equation is a function or a set of functions.

2. Two Types of Differential Equations:

**Definition 5.10.2 Ordinary Differential Equations (ODEs):** deals with functions of a single variable and ordinary derivatives. **Partial Differential Equations (PDEs):** deals with multi-variable equations and their partial derivatives (with more than one independent variables).

### 3. Order of Differential Equations:

**Definition 5.10.3** The **order of the differential equation** is the highest order derivative in the equation.

### 4. Linearity of ODEs:

**Theorem 5.10.1** A differential equation is said to be **linear** if:

- All the terms with dependent variables are in first-order.
- The coefficients of all the terms in the dependent variable and its derivatives depend only on the independent variable  $x$ .

### 5. Linear First-Order ODEs:

**Definition 5.10.4**

$$\frac{dy}{dx} + a(x)y = b(x), \text{ where } a(x) \text{ and } b(x) \text{ are functions of } x.$$

### 6. Solutions of ODEs:

- The solution to an ODE is a function or a set of functions.
- **General solution** to the differential equation:  
For a differential equation of order  $n$ , a solution is a function that satisfies the equation on some interval  $I$ . The function should have at least its first  $n$  derivatives on this interval  $I$ .
- To find **particular solutions**, we need to initial conditions for the problem.
  - (a) **Initial Value Problem (IVP):** where initial values are given to solve the differential equations depending on the order of the ODE.  
E.g.  $y(0)$ ,  $t(0)$ ,  $(0,y)$ .
  - (b) **Boundary Value Problem:** where a certain boundary is given.  
E.g.  $(x,y)$ .

### 7. Separable Differential Equations:

**Definition 5.10.5** A differential equation  $\frac{dy}{dx} = f(x,y)$  is **separable** if it can be expressed as a product of a function in  $x$  and a function in  $y$ :

$$\frac{dy}{dx} = f(x,y) = g(x)h(y).$$

- Particularly, if  $h(x) \neq 0$ , the variable can be separated to

$$\begin{aligned} \frac{dy}{dx} = g(x)h(y) &\Rightarrow \frac{dy}{h(y)} = g(x) dx \\ \int \frac{dy}{h(y)} &= \int g(x) dx \end{aligned}$$

- Solving differential equations using separation of variables:
  - Separate the variables such that everything involving  $y$  is on one side and everything involving  $x$  is on the other side.
  - Integrate both sides and combine the constant of integration on one side of the equation (normally the right side).

**Example 5.10.1** Solve for  $y$  if  $\frac{dy}{dx} = x(1+y)e^x$ .

$$\begin{aligned}\frac{dy}{dx} &= x(1+y)e^x \\ \frac{1}{1+y} dy &= xe^x dx \\ \int \frac{1}{1+y} dy &= \int xe^x dx \\ (= xe^x - \int e^x dx) &= xe^x - e^x \quad [\text{Integration by Parts}] \\ \ln|1+y| &= xe^x - e^x + C \\ 1+y &= e^{xe^x - e^x + C} = e^{xe^x - e^x} \cdot e^C \\ y &= Ae^{xe^x - e^x} - 1 \quad (A = e^C).\end{aligned}$$

#### 8. The Standard Logistic Equation:

$$\frac{du}{dt} = kn(a-n); \quad a, k \in \mathbb{R}.$$

where  $t$  is the time during which a population grows,  
 $n$  is the population after time  $t$ ,  
 $k$  is the relative growth, and  
 $a$  is a constant.

#### 9. Homogeneous Differential Equations:

**Definition 5.10.6** Differential equations of the form  $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ , where  $y = y(x)$ , are known as **homogeneous differential equations**.

**Theorem 5.10.2** Homogeneous differential equations can be solved by using the substitution  $y = vx$ , where  $v$  is a function of  $x$ . The substitution will always reduce the differential equation to a separable differential equation.

**Proof 5.10.1** If  $y = vx$ , where  $v$  is a function of  $x$ , then:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dv}{dx}x + v \quad [\text{Product Rule}] \\ \therefore \frac{dy}{dx} &= f\left(\frac{y}{x}\right), \\ \therefore \frac{dv}{dx}x + v &= f\left(\frac{y}{x}\right) = f(v) \\ \frac{dv}{dx} &= \frac{f(v) - v}{x} \\ \Rightarrow \frac{1}{f(v) - v} dv &= \frac{1}{x} dx\end{aligned}$$

**Example 5.10.2** Solve for  $\frac{dy}{dx} = \frac{x+2y}{x}$ , given  $y(3) = \frac{3}{2}$ .

$$\frac{dy}{dx} = 1 + 2\frac{y}{x} \rightarrow \text{homogenous differential equation}$$

$$\begin{aligned}\text{Let } y = vx, \frac{dy}{dx} &= \frac{dv}{dx}x + v \Rightarrow \frac{dv}{dx}x + v = 1 + 2\frac{y}{x} = 1 + 2v. \\ \frac{dv}{dx} &= \frac{1+v}{x} \\ \frac{1}{1+v} dv &= \frac{1}{x} dx \\ \int \frac{1}{1+v} dv &= \int \frac{1}{x} dx \\ \ln|1+v| &= \ln|x| + C = \ln|Ax| \\ 1+v &= Ax \Rightarrow \frac{y}{x} + 1 = Ax \\ y &= Ax^2 - x.\end{aligned}$$

Substituting  $y = \frac{3}{2}$ ,  $x = 3$ :  $\frac{3}{2} = A(3)^2 - 3 \Rightarrow A = \frac{1}{2}$

$$\therefore y = \frac{1}{2}x^2 - x.$$

**Example 5.10.3** Solve for  $\frac{dy}{dx} = \frac{x+y}{x}$ .

$$\frac{dy}{dx} = 1 + \frac{y}{x} \rightarrow \text{homogenous differential equation}$$

$$\text{Assume } v = \frac{y}{x} : y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v.$$

$$\begin{aligned}
\frac{dy}{dx} &= 1 + v = \frac{dv}{dx}x + v \\
\frac{dv}{dx} &= \frac{1}{x} \\
\int dv &= \int \frac{1}{x} dx \\
v &= \ln|x| + C = \ln|Ax| \\
\frac{y}{x} &= \ln|Ax| \\
y &= x \ln|Ax|.
\end{aligned}$$

10. Using the Integrating Factor  $I(x)$ :

**Definition 5.10.7**

$$I(x) = e^{\int P(x) dx}$$

is the **integrating factor** for  $\frac{dy}{dx} + P(x)y = Q(x)$ , where P and Q are continuous functions of  $x$  on a given interval.

**Theorem 5.10.3**

$$\begin{aligned}
&\frac{dy}{dx} + P(x)y = Q(x) \\
&I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x) \quad [\text{Multiply both sides by } I(x)] \\
&\left( \frac{d}{dx}(I(x)y) = I(x)\frac{dy}{dx} + I(x)P(x)y \quad [\text{Product Rule}] \right) \\
&\left[ (I(x))' = (e^{\int P(x) dx})' = e^{\int P(x) dx} \cdot (\int P(x) dx)' = e^{\int P(x) dx} \cdot P(x) = I(x)P(x) \right] \\
&\therefore \frac{d}{dx}(I(x)y) = I(x)Q(x) \\
&\int \frac{d}{dx}(I(x)y) dx = \int I(x)Q(x) dx \\
&I(x)y = \int I(x)Q(x) dx.
\end{aligned}$$

**Example 5.10.4 Solve**  $\frac{dy}{dx} + 3x^2y = 6x^2$ .

$$\therefore P(x) = 3x^2, Q(x) = 6x^2,$$

$$\therefore I(x) = e^{\int P(x) dx} = e^{\int 3x^2 dx} = e^{x^3}.$$

Multiply both sides by  $I(x)$ :

$$\begin{aligned}
e^{x^3} \frac{dy}{dx} + e^{x^3} \cdot 3x^2 y &= e^{x^3} \cdot 6x^2 \\
\therefore \frac{d}{dx} (e^{x^3} y) &= e^{x^3} \cdot 6x^2 \\
\int \frac{d}{dx} (e^{x^3} y) \, dx &= \int e^{x^3} \cdot 6x^2 \, dx \\
\left[ \text{Let } x^3 = u, \frac{du}{dx} = 2x^2, du = 2x^2 \, dx \Rightarrow 2 \int e^u \, du = 2e^u + C = 2e^{x^3} + C \right] \\
e^{x^3} y &= 2e^{x^3} + C \\
y &= 2 + Ce^{-x^3}.
\end{aligned}$$

### 11. Euler's Method:

- For  $y = f(x)$ ,  $y_{n+1} = y_n + hf'(x_0)$ ,  $h$  is a constant.

$$y - y_n = f'(x_n)(x - x_n).$$

**Example 5.10.5**  $y = x^2$ ,  $\frac{dy}{dx} = 2x$ ,  $h = 0.1$

$n$	$x_n$	$y_n$	Actual
0	1	1	1
1	1.1	1.2	1.21
2	1.2	1.42	1.44
3	1.3	1.66	1.69
4	1.4	1.92	1.96
5	1.5	2.2	2.25

- The smaller the  $h$ , the more accurate the approximation.
- Consider a differential equation of the form  $\frac{dy}{dx} = f(x, y)$ , given an initial condition. The derivative at any point on the curve  $(x_0, y(x_0))$  can be approximated using the gradient of the tangent to the curve at  $x_0$ :

$$y'(x_0) = \frac{y(x_0 + h) - y(x_0)}{h}.$$

Rearranging the formula, we get:

$$y(x_0 + h) = y(x_0) + hy'(x_0).$$

This is the **linearization** or **Euler's method** and becomes more accurate over small increments and as long as the function does not change too rapidly.

- If  $\frac{dy}{dx} = f(x_n, y_n)$  and  $x_{n+1} = x_n + h$ , we have

$$y_{n+1} = y_n + hf(x_n, y_n).$$

## 5.11 Maclaurin Series

1. The Maclaurin Polynomial:

**Definition 5.11.1** If  $f(x)$  has  $n$  derivatives at  $x = 0$ , then  $P(x)$ , the **Maclaurin polynomial** of degree  $n$  for  $f(x)$  centered at  $x = 0$ , is the unique polynomial of degree  $n$  that satisfies:

- $f(0) = P(0);$
- $f^{(n)}(0) = P^{(n)}(0);$
- $a_1 = \frac{f'(0)}{1!}, a_2 = \frac{f''(0)}{2!}, a_3 = \frac{f'''(0)}{3!}, \dots a_n = \frac{f^{(n)}(0)}{n!};$
- $P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k.$

2. Maclaurin polynomials approximate the behavior of functions around a certain interval. The more terms we take, the better the approximation.
3. The Maclaurin Series:

**Definition 5.11.2** If  $f(x)$  has derivatives of all orders throughout an open interval  $I$  such that  $0 \in I$ , then the **Maclaurin series** generated by  $f$  at  $x = 0$  is:

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

A series converges when the sum of them is a constant (a limit can be found).

**Example 5.11.1** Find the Maclaurin series for  $f(x) = \frac{1}{2+x}$ .

$$\begin{array}{l|l} \begin{array}{l} f(x) = (2+x)^{-1} \\ f'(x) = -(2+x)^{-2} \\ f''(x) = 2(2+x)^{-3} \\ f'''(x) = -6(2+x)^{-4} \\ f^{(4)}(x) = 24(2+x)^{-5} \end{array} & \begin{array}{l} f(0) = 2^{-1} = \frac{1}{2} \\ f'(0) = -2(2)^{-2} = -\frac{1}{4} \\ f''(0) = 2(2)^{-3} = 2 \times \frac{1}{8} \\ f'''(0) = -6(2)^{-4} = -6 \times \frac{1}{16} \\ f^{(4)}(0) = 24(2)^{-5} = 24 \times \frac{1}{32} \end{array} \end{array}$$

$$\begin{aligned} P(x) &= \frac{1}{2} + \frac{-\frac{1}{4}}{1!}x + \frac{2 \times \frac{1}{8}}{2!}x^2 + \frac{-6 \times \frac{1}{16}}{3!}x^3 + \frac{24(2)^{-5}}{4!}x^4 + \dots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (-x)^n. \end{aligned}$$

4. The Binomial series is the Maclaurin expansion for  $f(x) = (1+x)^p$ :

$$(1+x)^p = \sum_{n=0}^p \binom{p}{n} x^n, \quad 1 \leq n \leq p, \quad \binom{p}{n} = \frac{p!}{n!(p-n)!} = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.$$

**Example 5.11.2 Use the Binomial series to find the Maclaurin series for  $f(x) = \frac{1}{(x+2)^2}$ .**

$$\begin{aligned}
f(x) &= (1+x)^{-2} \\
\therefore \binom{-2}{n} &= \frac{-2(-2-1)(-2-2)\cdots(-2-(n-1))}{n!} \\
&= (-1)^n \frac{2(3)(4)\cdots(n+1)}{n!} = (-1)^n(n+1) \\
\therefore P(x) &= \sum_{n=0}^{\infty} (-1)^n(n+1)x^n \\
&= 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^n(n+1)x^n + \cdots
\end{aligned}$$

**Example 5.11.3 Use the Binomial series to find the Maclaurin series for  $f(x) = \frac{1}{\sqrt{2-x}}$ .**

$$\begin{aligned}
f(x) &= (2-x)^{-\frac{1}{2}} = (2)^{-\frac{1}{2}} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}} \\
\therefore P(x) &= \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2} \binom{-\frac{1}{2}}{n} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}}.
\end{aligned}$$

## 5. Applications of Maclaurin Series:

- Approximation of sin, cos, tan, ...

**Example 5.11.4 Approximate  $\sin 3^\circ$  using the first four terms of Maclaurin series.**

$$\begin{aligned}
3^\circ &= \frac{\pi}{60}, \text{ For } \sin x, P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\therefore P\left(\frac{\pi}{60}\right) &= x - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \frac{\left(\frac{\pi}{60}\right)^7}{7!} + \cdots \approx 0.052336 \text{ (6 d.p.)}.
\end{aligned}$$

- More Complicated Functions

**Example 5.11.5 Find the Maclaurin series of  $f(x) = e^{x^2}$ .**

Let  $u = x^2, f(u) = e^u$ :

$$P(x) = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

**Example 5.11.6 Find the Maclaurin series of  $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ .**

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\begin{aligned}
\therefore P(x) &= -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \left( x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \right) \\
&= 2\left(x + \frac{x^3}{3} + \cdots\right) \\
&= 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.
\end{aligned}$$

**Example 5.11.7 Find the Maclaurin series of  $f(x) = \frac{x}{(1+x)^2}$ .**

$$\begin{aligned}
f(x) &= x(1+x)^{-2} \\
&= x \sum_{n=0}^{\infty} \binom{-2}{n} x^n \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1) x^{n+1}.
\end{aligned}$$

- Evaluate Limits

**Example 5.11.8 Find  $\lim_{x \rightarrow 0} \frac{1-e^{x^2}}{1-\cos x}$ .**

$$\begin{aligned}
e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\therefore \lim_{x \rightarrow 0} \frac{1-e^{x^2}}{1-\cos x} &= \lim_{x \rightarrow 0} \frac{1 - \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right)}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)} \\
&= \lim_{x \rightarrow 0} \frac{-x^2 - \frac{x^4}{2!} - \frac{x^6}{3!} - \cdots}{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots} \\
&= \lim_{x \rightarrow 0} \frac{-x^2}{\frac{x^2}{2!}} \\
&= -2.
\end{aligned}$$

[Consider only the smallest power of  $x$ , as higher powers will go to zero much quicker.]

- Solve Differential Equations

**Example 5.11.9 Use the first six terms of a Maclaurin series to approximate the solution of  $y' = y^2 - x$  on an open interval centered at  $x = 0$  if  $y(0) = 1$ .**

	$y(0) = 1$
$y' = y^2 - x$	$y'(0) = 1$
$y'' = 2yy' - 1$	$y''(0) = 2 - 1 = 1$
$y''' = 2yy'' + 2(y')^2$	$y'''(0) = 2 + 2 = 4$
$y^{(4)} = 2yy''' + 6y'y''$	$y^{(4)}(0) = 14$
$y^{(5)} = 2yy^{(4)} + 8y'y''' + 6(y'')^2$	$y^{(5)}(0) = 66$

$$\therefore P(x) = 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \dots$$

- Binomial Theorem

**Theorem 5.11.1** Function  $f(x) = (1+x)^p$ ,  $p \in \mathbb{R}$  is equal to its Binomial series using the initial condition  $y(0) = 1$ .

### Proof 5.11.1

$$f(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \dots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \dots$$

$$\therefore f'(x) = p + p(p-1)x + \frac{p(p-1)(p-2)}{2!}x^2 + \dots$$

$$xf'(x) = px + p(p-1)x^2 + \frac{p(p-1)(p-2)}{2!}x^3 + \dots$$

$$\begin{aligned}\therefore f'(x) + xf'(x) &= p + [p(p-1) + p]x + \left[ \frac{p(p-1)}{2!}p(p-1) + \right] x^2 + \dots \\ &= p + p^2x + \frac{p^2(p-1)}{2!}x^2 + \dots \\ &= p(1 + px + \frac{p(p-1)}{2!}x^2) + \dots \\ &= pf(x)\end{aligned}$$

$$\therefore f'(x) + xf'(x) = pf(x) \Rightarrow (1+x)f'(x) = pf(x)$$

$$f'(x) - \frac{p}{1+x}f(x) = 0 \Rightarrow P(x) = -\frac{p}{1+x}, Q(x) = 0$$

$$\therefore I(x) = e^{\int -\frac{p}{1+x} dx} = A(1+x)^{-p}$$

$$\therefore \frac{d}{dx} (A(1+x)^{-p}f(x)) = 0 \cdot I(x)$$

$$\int \frac{d}{dx} (A(1+x)^{-p}f(x)) dx = \int 0 dx$$

$$A(1+x)^{-p}f(x) = C$$

$$f(x) = C \cdot A(1+x)^{x-p} = B(1+x)^{x-p} \quad [\text{Let } B = C \cdot A.]$$

$$\text{Let } f(0) = 1 : B = 1$$

$$\therefore f(x) = (1+x)^p.$$