

IB Mathematics Analysis and Approaches HL

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1 Number and Algebra

1.1 Sequences and Series

1. Terms: $u_1, u_2, u_3 \dots$

Position: n

Sum: S

2. **Arithmetic Sequence**/Arithmetic Progression (AP):

- Recursive formula: $u_{n+1} = u_n + d$, d is the common difference.
- Explicit formula: $u_n = u_1 + d(n-1)$
- Summation: $S_n = \frac{1}{2}[2u_1 + d(n-1)]$

Proof 1.1.1 Let $u_1, u_2, u_3, \dots, u_n$ be an arithmetic sequence with d as common difference.

Then, $S_n = u_1 + u_2 + u_3 + \dots + u_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + (u_1 + (n-1)d)$

Also, $S_n = [u_1 + (n-1)d] + \dots + (u_1 + d) + u_1$.

Add two expressions together:

$$2S_n = [2u_1 + (n-1)d]n$$

$$\therefore S_n = \frac{n}{2}[2u_1 + (n-1)d].$$

3. **Geometric Sequence**

- Recursive formula: $u_{n+1} = r \cdot u_n$, r is the common ratio.
- Explicit formula: $u_n = u_1 \cdot r^{n-1}$
- $r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \dots$
- Summation: $S_n = \frac{u_1(r^n - 1)}{r - 1}$

Proof 1.1.2 Let $u_1, u_2, u_3, \dots, u_n$ be a geometric sequence with r as common ratio.

$S_n = u_1 + u_2 + u_3 + \dots + u_n = u_1 + (u_1 \cdot r) + (u_1 \cdot r^2) + \dots + (u_1 \cdot r^{n-1})$

Then, $rS_n = (u_1 \cdot r) + (u_1 \cdot r^2) + \dots + (u_1 \cdot r^n)$.

Subtract the first expression from the second:

$$rS_n - S_n = u_1 \cdot r^n - u_1 \Rightarrow (r-1)S_n = u_1(r^n - 1)$$

$$\therefore S_n = \frac{u_1(r^n - 1)}{r - 1}$$

- If $r > 1$, the sequence is an exponential growth.
If $0 < r < 1$, the sequence has an exponential decay.

- When $r > 1$, series approaches ∞ .

When $-1 < r < 1$, or $|r| < 1$, the series converges:

$$S_\infty = \frac{u_1}{1-r}, |r| < 1$$

1.2 Exponents and Logarithms

$$1. \ a^m \cdot a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$2. \ x^0 = 1 \ (x^0 = x^{1-1} = \frac{x^1}{x^1} = 1)$$

$$x^{-m} = \frac{1}{x^m}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x} \ (x^{\frac{m}{n}} = (\sqrt[n]{x})^m)$$

$$3. \text{ If } a = b, \text{ then } a^n = b^n$$

$$\text{If } m = n, \text{ then } a^m = a^n$$

For $a^b = 1$: $a = 1, b \in \mathbb{R}; a \neq 1, b = 0$; OR $a = -1, b = 2n$

$$4. \text{ When solving exponential equations, convert them to the same base.}$$

$$5. \text{ Division Theorem.}$$

Theorem 1.2.1 If $a^x = b^y$ given $a > 0$ and $b > 0$, then $a = b^{\frac{y}{x}}$.

Proof 1.2.1

$$a^x = b^y$$

$$(a^x)^{\frac{1}{x}} = (b^y)^{\frac{1}{x}} \Rightarrow a = b^{\frac{y}{x}}$$

$$6. \ a = b^x \Leftrightarrow x = \log_b a, \text{ where } a, b \in \mathbb{R}^+ \text{ and } b \neq 1.$$

$$7. \text{ Logarithmic rules:}$$

- $\log_a x + \log_a y = \log_a(xy)$

Proof 1.2.2 Let $\log_a x = p, \log_a y = q \Rightarrow a^p = x, a^q = y$.

Then, $x \cdot y = a^p \cdot a^q = a^{p+q}$.

$$\therefore \log_a(xy) = p + q = \log_a x + \log_a y.$$

- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

Proof 1.2.3 Let $\log_a x = p, \log_a y = q \Rightarrow a^p = x, a^q = y$.

Then, $\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$.

$$\therefore \log_a \left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y.$$

- $\log_a x^n = n \log_a x$
- $\log_a 1 = 0$
- $\log_a a = 1$
- $-\log_a x = \log_a \frac{1}{x}$
- $\log_a x = \frac{\log_b x}{\log_b a}$
- $\log_a b = \frac{1}{\log_b a}$

1.3 Proof

1. Direct proof:

Example 1.3.1 Show that the sum of two even numbers is always even.

Let m and n be two even positive integers.

$m = 2p, n = 2q$, where p and $q \in \mathbb{Z}^+$.

Then, $m + n = 2p + 2q = 2(p + q)$, which is an even number.

Example 1.3.2 Show that $(x + \frac{a}{2})^2 - (\frac{a}{2})^2 \equiv x^2 + ax$.

$$\text{LHS} = x^2 + \frac{a^4}{4} + ax - \frac{a^4}{4} = x^2 + ax = \text{RHS}.$$

Equations " $=$ ": only true from some values.

Identities " \equiv ": true for all values.

Example 1.3.3 Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.

Example 1.3.4 Let n be a 4-digit number: $n = 1000a + 100b + 10c + d$, where $0 \leq a, b, c, d \leq 9$, and $a \neq 0$.

It is given that $a + b + c + d = 3k, k \in \mathbb{Z}$:

$$\begin{aligned} n &= 1000a + 100b + 10c + d + 3k - a - b - c - d \\ &= 999a + 99b + 9c + 3k \\ &= 3(333a + 33b + 3c + k) \end{aligned}$$

Since $(333a + 33b + 3c + k) \in \mathbb{Z}$, it implies that n is divisible by 3.

2. Proof by Contradiction:

Example 1.3.5 Prove the statement: If the integer n is odd, then n^2 is also odd.

Let, if possible, n^2 is even and n is odd.

Then, $n^2 = 2k, k \in \mathbb{Z} \Rightarrow n \times n = 2k$, which indicates the product of two odd number is even, and which is not true.

Hence, there is a contradiction.

\therefore Our assumption is wrong, and thus given that n is odd, n^2 is also odd.

Example 1.3.6 Show that $\sqrt{2}$ is irrational.

Let us assume, if possible, that $\sqrt{2}$ is rational:

$\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, and p, q have no common factors, $q \neq 0$.

$$\therefore 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \quad (1).$$

$\therefore p^2$ is even, and thus p is also even.

As p is an even number, we can write: $p = 2k, k \in \mathbb{Z} \Rightarrow \therefore p^2 = (2k)^2 = 4k^2 \quad (2)$.

From (1) and (2): $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even, and thus q is also an even number.

But since p and q have no common factors, they cannot have "2" as a common factor. Hence, we have arrived at a contradiction.

\therefore Our assumption is incorrect, and $\sqrt{2}$ is irrational.

Definition 1.3.1 A number is **rational** if it can be written as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, and $q \neq 0$.

Example 1.3.7 Prove that there is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$ Assume there is a real number x such that $\frac{1}{x-2} = 1 - x$.

$$\therefore (1-x)(x-2) = 1 \Rightarrow x^2 - 3x + 3 = 0$$

Solving the equation, we get $x = \frac{3 \pm \sqrt{9-12}}{2}$, which $\notin \mathbb{R}$

\therefore We arrived at a contradiction, and our assumption is incorrect. There is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$

3. Proof by Mathematical Induction

Definition 1.3.2 Principle of Mathematical Induction (PMI):

Suppose P_n is a proposition which is defined for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true $\forall n \geq a$.

Example 1.3.8 Prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z}$, $n \geq 0$, by using PMI.

For $n = 0$, LHS = $4^0 + 2 = 1 + 2 = 3$, which is divisible by 3.

$\therefore P_0$ (OR denoted as $P(0)$) is true.

Assume that P_k is true: i.e., $4^k + 2$ is divisible by 3. $\Rightarrow 4^k + 2 = 3A$, $A \in \mathbb{Z}^+$ $\Rightarrow 4^k = 3A - 2$.

Consider P_{k+1} :

$$\begin{aligned} 4^{k+1} + 2 &= 4^k \cdot 4^1 + 2 \\ &= (3A - 2) \cdot 4 + 2 \\ &= 12A - 6 \\ &= 3(4A - 2). \end{aligned}$$

$\therefore 4A - 2$ is an integer as $A \in \mathbb{Z}^+$, $4^{k+1} + 2$ is divisible by 3 whenever $4^k + 2$ is divisible by 3.

Since P_0 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}$, $n \geq 0$.

Example 1.3.9 A sequence is defined by $u_{n+1} = 2u_n + 1 \forall n \in \mathbb{Z}^+$. Prove that $u_n = 2^n - 1$.

For $n = 1$, $u_1 = 2^1 - 1 = 1 \Rightarrow P_1$ is true.

Let P_k be true: $u_k = 2^k - 1$ for some $k \in \mathbb{Z}^+$.

Consider P_{k+1} :

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

Example 1.3.10 Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, $\forall n \in \mathbb{Z}^+$.

For $n = 1$, LHS = $1^2 = 1$, RHS = $\frac{1(1+1)(2+1)}{6} = 1$

\therefore LHS = RHS $\Rightarrow P_1$ is true.

Assume that P_k is true, $k \in \mathbb{Z}^+$: $1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Consider P_{k+1} :

$$\begin{aligned} \text{LHS} &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = \text{RHS}. \end{aligned}$$

Thus, P_{k+1} is true whenever P_k is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

Example 1.3.11 Prove that if $x \neq 1$, the $\prod_{i=1}^n (1+x^{2^{i-1}}) = (1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x}$.

For $n = 1$, LHS = $1+x$, RHS = $\frac{1-x^2}{1-x} = \frac{1-x^2}{1-x} = 1+x$. $\Rightarrow \therefore$ LHS = RHS, P_1 is true.

Assume that P_k is true: $(1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^{k-1}}) = \frac{1-x^{2^k}}{1-x}$.

Consider P_{k+1} :

$$\begin{aligned} \text{LHS} &= (1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^{k-1}})(1+x^{2^k}) \\ &= \frac{1-x^{2^k}}{1-x} (1+x^{2^k}) \\ &= \frac{1+x^{2^k} - x^{2^k} + (x^{2^k})^2}{1-x} \\ &= \frac{1-x^{2^k \cdot 2}}{1-x} \\ &= \frac{1-x^{2^{k+1}}}{1-x} = \text{RHS}. \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

1.4 Counting and Binomial Theorem

- Choose r from n : $\binom{n}{r} =_n C_r$

- $\bullet \quad \binom{n}{m} = \binom{n}{n-m}$

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Factorial notation: $n! = n(n-1)(n-2)\cdots 2 \cdot 1$
e.g. $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 5 \times 2 = 10$.

Example 1.4.1 Write $\frac{(n!)^2}{(n-1)!(n-2)!}$ without using factorial notation.

$$(n!)^2 = n! \times n! = n(n-1)! \times n(n-1)(n-2)!$$

$$\therefore \frac{(n!)^2}{(n-1)!(n-2)!} = \frac{n(n-1)! \times n(n-1)(n-2)!}{(n-1)!(n-2)!} = n \cdot n(n-1) = n^3 - n^2.$$

2. The number of ways of arranging n distinct objects in a row is $n!$.
3. The number of permutations of r objects out of n distinct objects is given by

$${}_nP_r = \frac{n!}{(n-r)!}.$$

4. In permutations, the order matters.
In combinations, the order does not matter.
5. The Binomial Theorem:

Theorem 1.4.1

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n, \quad n \in \mathbb{N} \\ &= \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r\end{aligned}$$

Example 1.4.2 Find $(2x+3)^4$.

$$\begin{aligned}(2x+3)^4 &= (2x)^4 + \binom{4}{1} (2x)^3 (3)^1 + \binom{4}{2} (2x)^2 (3)^2 + \binom{4}{3} (2x) (3)^3 + 3^4 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81\end{aligned}$$

Example 1.4.3 Find the term independent of x in the expansion of $\left(x - \frac{2}{x^2}\right)^{12}$.

General term: $\binom{12}{r} x^{12-r} \left(-\frac{2}{x^2}\right)^r$

Thus, the general expression for x : $x^{12-r-2r} = x^{12-3r}$

When $12 - 3r = 0$, the term is independent of x : $12 - 3r = 0 \Rightarrow r = 4$.

$$\therefore \binom{12}{4} x^{12-4} \left(-\frac{2}{x^2}\right)^4 = 7920.$$

1. The independent term should not involve x in it since the independent term does not vary as x varies. (constant term)
2. The coefficient should not include x as well.

Example 1.4.4 Find the coefficient of x^3y^2 in the expansion of $(2x+y)(x+\frac{y}{x})^5$.

Assume $2x \cdot A$ and $y \cdot B$ will yield the term $x^3y^2 \Rightarrow A = x^2y^2, B = x^3y$.

General term: $\binom{5}{r}x^{5-r}(\frac{y}{x})^r = \binom{5}{r}x^{5-2r}y^r$.

When $r = 2, 5 - 2r = 1 \neq 2 \Rightarrow x^2y^2$ is not possible.

When $r = 1, 5 - 2r = 3 \Rightarrow x^3y$ is possible.

$$\therefore \text{Coefficient} = \binom{5}{1} = 5.$$

Example 1.4.5 Find the coefficient of x^2 in the expansion of $(1-2x)(1-4x)^7$.

Assume $1 \cdot A = x^2, -2x \cdot B = x^2 \Rightarrow A = x^2, B = x$.

General term: $\binom{7}{r}(-4x)^{7-r}(1)^r$

When $7 - r = 2, r = 5 : \binom{7}{5}(-4x)^2(1)^5 = 336x^2 \Rightarrow 1 \cdot 336x^2 = 336x^2$

When $7 - r = 1, r = 6 : \binom{7}{6}(-4x)^1(1)^6 = -28x \Rightarrow (-2x) \cdot (-28x) = 56x^2$

$$\therefore \text{Coefficient} = 336 + 56 = 392.$$

6. AHL - Extension of Binomial Theorem:

Theorem 1.4.2

$$(a+b)^n = a^n \left(1 + \frac{b}{a}\right)^n \\ = a^n \left(1 + n \cdot \frac{b}{a} + \frac{n(n-1)}{2!} \left(\frac{b}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{b}{a}\right)^3 + \dots\right), n \in \mathbb{Q}, \left|\frac{b}{a}\right| < 1$$

Example 1.4.6 Expand $\sqrt{1+2x}$ ($|x| < \frac{1}{2}$) and $\frac{2}{1-3x}$ ($|x| < \frac{1}{3}$) up to x^3 term.

$$(1+2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(2x) + \frac{1}{2} \left(\frac{1}{2}-1\right) \frac{(2x)^2}{2!} + \frac{1}{2} \left(\frac{1}{2}-1\right) \left(\frac{1}{2}-2\right) \frac{(2x)^3}{3!} + \dots \\ = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots$$

$$2(1-3x)^{-1} = 2(1 - (-3x) - (-1-1)) \frac{(-3x)^2}{2!} - (-1-1)(-1-2) \frac{(-3x)^3}{3!} + \dots \\ = 2(1 + 3x + x^2 + 27x^3 + \dots) \\ = 2 + 6x + 18x^2 + 54x^3 + \dots$$

Example 1.4.7 Write the first three terms in the expansion of $(2+x)^{-3}$.

$$(2+x)^{-3} = 2^{-3} \left(1 + \frac{x}{2}\right)^{-3} \\ = \frac{1}{8} \left(1 + (-3)\frac{x}{2} + (-3)(-3-1) \frac{2^2}{2 \cdot 2!} + \dots\right) \\ = \frac{1}{8} \left(1 - \frac{3}{2}x + \frac{12}{4}x^2 + \dots\right) \\ = \frac{1}{8} - \frac{3}{16}x + \frac{3}{8}x^2 + \dots$$

Example 1.4.8 Find square root of 24 correct to 5 decimal places, using the binomial theorem.

$$\begin{aligned}
 24^{\frac{1}{2}} &= (25 - 1)^{\frac{1}{2}} = 25^{\frac{1}{2}} \left(1 - \frac{1}{25}\right)^{\frac{1}{2}} \\
 &= 5 \left(1 + \left(\frac{1}{2}\right) \left(-\frac{1}{25}\right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \left(-\frac{1}{25}\right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \left(-\frac{1}{25}\right)^3 + \dots\right) \\
 &= 5 \left(1 - \frac{1}{50} - \frac{1}{5000} - \frac{1}{250000} + \dots\right) \\
 &= 5(1 - 0.02 - 0.0002 - 0.000004) \\
 &= 4.89898 \quad (5 \text{ d.p.}).
 \end{aligned}$$

1.5 Partial Fraction

1. Proper fractions: The degree of the numerator is less than the degree of the denominator.
2. Partial fraction: A method to separate one complex fraction into two or simpler fractions.

Example 1.5.1 Find the partial fraction of $\frac{3x}{(x-1)(x+2)}$.

$$\begin{aligned}
 \text{Let } \frac{3x}{(x-1)(x+2)} &= \frac{A}{x-1} + \frac{B}{x+2}. \\
 \therefore 3x &\equiv A(x+2) + B(x-1).
 \end{aligned}$$

When $x = 1$, $3 = 3A \Rightarrow A = 1$.

When $x = -2$, $-6 = -3B \Rightarrow B = 2$.

$$\therefore \frac{3x}{(x-1)(x+2)} \equiv \frac{1}{x-1} + \frac{2}{x+2}.$$

Example 1.5.2 Find the partial fraction of $\frac{2x+5}{(x-2)(x+1)}$.

$$\text{Let } \frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

$$\therefore 2x+5 \equiv A(x+1) + B(x-2).$$

When $x = 2$, $9 = 3A \Rightarrow A = 3$.

When $x = -1$, $3 = -3B \Rightarrow B = -1$.

$$\therefore \frac{2x+5}{(x-2)(x+1)} \equiv \frac{3}{x-2} - \frac{1}{x+1}.$$

Example 1.5.3 Find the partial fraction of $\frac{34-12x}{3x^2-10x-8}$.

$$\text{As } \frac{34-12x}{3x^2-10x-8} = \frac{34-12x}{(3x+2)(x-4)}, \text{ let } \frac{34-12x}{(3x+2)(x-4)} = \frac{A}{3x+2} + \frac{B}{x-4}.$$

$$\therefore 34 - 12x \equiv A(x-4) + B(3x+2).$$

When $x = 4$, $-14 = 14A \Rightarrow B = -1$.

When $x = -\frac{2}{3}$, $42 = -\frac{14}{3}A \Rightarrow A = -9$.

$$\therefore \frac{34 - 12x}{(3x+2)(x-4)} \equiv -\frac{9}{3x+2} - \frac{1}{x-4}.$$

1.6 Complex Number

1.6.1 Introduction

1. Complex Number:

Definition 1.6.1 **Complex Numbers** are numbers in the form of $a + bi$, where $i^2 = -1$.

- a is called the **real part**, denoted as $\text{Re}(a + bi) = a$.
- b is called the **imaginary part**, denoted as $\text{Im}(a + bi) = b$.

$a + bi$ is called the **Cartesian form** of complex number.

2. Basic Calculations of Complex Number:

- Define $z_1 = a + bi$ and $z_2 = c + di$:

$$z_1 \pm z_2 = (a \pm c) + (b \pm d)i.$$

- Define $z_1 = a + bi$ and $z_2 = c + di$:

$$z_1 z_2 = (ac - bd) + (ad + bc)i.$$

Proof 1.6.1

$$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) \\ &= ac + (ad + bc)i + bdi^2 \quad [i^2 = -1] \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

- Conjugate complex number:

Definition 1.6.2 We call $a - bi$ as the **conjugate** of $z = a + bi$, denoted as $z^* = a - bi$.

Theorem 1.6.1 Define $z_1 = a + bi$, and z^* is the conjugate of z_1 . Then,

$$z_1 z^* = a^2 + b^2.$$

Proof 1.6.2 By definition, $z^* = a - bi$. Thus,

$$\begin{aligned} z_1 z^* &= (a + bi)(a - bi) \\ &= a^2 - (bi)^2 \\ &= a^2 + b^2. \end{aligned}$$

- Define $z_1 = a + bi$ and $z_2 = c + di$:

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i.$$

Proof 1.6.3

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{(ac + bd) - (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i.\end{aligned}$$

Example 1.6.1 Find $z \in \mathbb{C}$ that satisfies the equation $\frac{z+2}{1-i} = \frac{z-3i}{2+i}$.

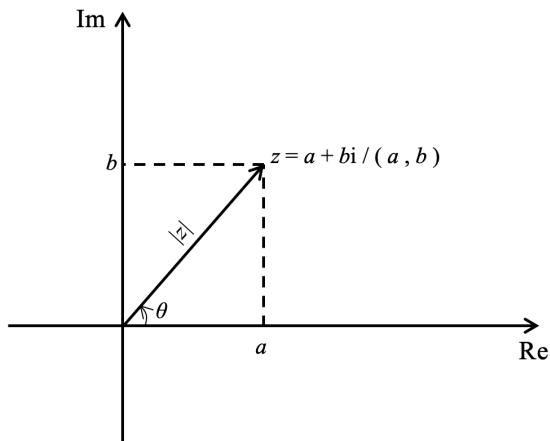
$$\begin{aligned}(z+2)(2+i) &= (z-3i)(1-i) \\ z(2+i) + 4 + 2i &= z(1-i) - 3i + (3i)^2 \\ z(2+i - 1 + i) &= -3i - 3 - 4 - 2i \\ z(1+2i) &= -7 - 5i \\ z &= \frac{-7 - 5i}{1+2i} = -\frac{17}{5} + \frac{9}{5}i.\end{aligned}$$

3. If $s = a + bi$ and $t = c + di$, then:

$$\operatorname{Re}(s) + \operatorname{Re}(t) = \operatorname{Re}(s+t); \text{ and } \operatorname{Im}(i \cdot s) = \operatorname{Re}(s).$$

1.6.2 Argand Diagram

1. The Complex Plane:



$z = a + bi$ can be represented on a complex plane with real coordinate a and imaginary coordinate b . It can also be denoted as $z(a, b)$.

- Modulus of a complex number:

$$|z| = \sqrt{a^2 + b^2}.$$

- Argument of a complex number:

$$\operatorname{Arg}(z) = \arctan\left(\frac{b}{a}\right) (+k\pi) \rightarrow \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

*When determine a complex number, first draw it on the plane to show which quadrant it is in.

The range of argument is $[0, 2\pi]$ or $[-\pi, \pi]$.

- Use modulus and argument to express a complex number:

$$a = |z| \cdot \cos \theta;$$

$$b = |z| \cdot \sin \theta.$$

2. If $z = a + bi$ and $|z| = 1$, then $z^* = z^{-1}$.

Proof 1.6.4

$$\begin{aligned} &\because |z| = 1 \\ &\therefore \sqrt{a^2 + b^2} = 1 \\ &\therefore a^2 + b^2 = 1 \end{aligned}$$

Method 1

$$\begin{aligned} \text{RHS} = z^{-1} &= \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} \\ &= \frac{a-bi}{a^2+b^2} = a-bi \\ &= z^* = \text{LHS}. \end{aligned}$$

$$\begin{aligned} z \cdot z^* &= (a+bi)(a-bi) \\ &= a^2 + b^2 \\ &= |z|^2 = 1 \\ &\therefore z^* = z^{-1} \end{aligned}$$

Method 2

3. When $|z| \neq 1$, $z^* = \frac{|z|^2}{z}$, and $z^{-1} = \frac{z^*}{|z|^2}$.

4. Properties of modulus and arguments:

For complex number s and $t \in \mathbb{C}$:

•

$$|st| = |s||t|$$

•

$$\left|\frac{s}{t}\right| = \frac{|s|}{|t|}$$

•

$$\operatorname{Arg}(st) = \operatorname{Arg}(s) + \operatorname{Arg}(t) + 2k\pi$$

•

$$\operatorname{Arg}\left(\frac{s}{t}\right) = \operatorname{Arg}(s) - \operatorname{Arg}(t) + 2k\pi$$

1.6.3 Complex Number in Other Forms

1. The Polar Form (Modulus-Argument Form):

-

$$z = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$$

Proof 1.6.5 According to the Argand Diagram:

$$z = x + yi = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta).$$

-

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

-

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

2. de Moivre's Theorem:

- By Maclaurin Series:

$$e^{i\theta} = \operatorname{cis} \theta = \cos \theta + i \sin \theta.$$

- Exponential form of complex number:

$$z = r e^{i\theta} = r \operatorname{cis} \theta.$$

3. Cartesian Form: Addition and Subtraction

Modulus-Argument Form: Multiply and Division

Exponential Form: Exponents and Roots

4. Since $\operatorname{cis} \theta = \operatorname{cis}(\theta + 2k\pi)$,

$$r e^{i\theta} = r e^{i(\theta+2k\pi)}.$$

Example 1.6.2 Find $e^{i\frac{17\pi}{12}}$ in the form of Cartesian.

$$\begin{aligned} e^{i\frac{17\pi}{12}} &= e^{i(\frac{7\pi}{6} + \frac{\pi}{4})} = e^{i\frac{7\pi}{6}} \cdot e^{i\frac{\pi}{4}} \\ &= \operatorname{cis}\left(\frac{7\pi}{6}\right) \cdot \operatorname{cis}\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2} - \sqrt{6}}{4} - \frac{\sqrt{2} + \sqrt{6}}{4}i. \end{aligned}$$

1.6.4 Power of Complex Number

1. For a complex number $z = r e^{i\theta}$,

$$z^n = r^n e^{in\theta}.$$

Example 1.6.3 Find $(3 \cos \frac{2\pi}{3} - 3i \sin \frac{\pi}{3})^3$

$$\begin{aligned}
 \left(3 \cos \frac{2\pi}{3} - 3i \sin \frac{\pi}{3}\right)^3 &= \left(-3 \cos \frac{\pi}{3} - 3i \sin \frac{\pi}{3}\right)^3 \\
 &= \left(-3 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)\right)^3 \\
 &= (-3)^3 (e^{i\frac{\pi}{3}})^3 \\
 &= -27 e^{i\pi} \\
 &= -27(-1) = 27.
 \end{aligned}$$

Key learnings:

1. $z = 3$ is only the fundamental root of equation $z^3 = 27$. In \mathbb{C} , there are other two complex roots that satisfy the equation.
2. In \mathbb{C} , $\sqrt{4} = \pm 2 = 2 + 0 \cdot i$ or $-2 + 0 \cdot i$.

Example 1.6.4 Given a complex number $\omega \neq 1$ is one of the solutions of $z^3 = 1$.

- a. Prove $\omega^2 + \omega + 1 = 0$;
- b. Calculate $\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022}$.

(a) Approach A

$$\begin{aligned}
 \because \omega^3 &= 1 \\
 \therefore \omega^3 - 1 &= 0 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0 \\
 \because \omega &\neq 1 \\
 \therefore \omega^2 + \omega + 1 &= 0.
 \end{aligned}$$

Approach B $\omega^2 + \omega + 1 = 0$ is a geometric sequence, $u_1 = 1$, $r = \omega$:

$$S_3 = \frac{u_1(1 - r^3)}{1 - r} = \frac{1 - \omega^3}{1 - \omega} = \frac{0}{1 - \omega} = 0.$$

(b)

$$\begin{aligned}
 \omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022} &= \omega^{2019} \times (1 + \omega + \omega^2 + \omega^3) \\
 &= \omega^{2019}(0 + 1) = \omega^{2019} \\
 &= (\omega^3)^{673} = 1.
 \end{aligned}$$

Example 1.6.5 Find:

- a. 1^i ;
- b. $\ln(-1)$;
- c. $\ln(-c)$, where c is a constant.

(a)

$$1 = e^{i2\pi} \Rightarrow 1^i = \left(e^{i2\pi}\right)^i = e^{-2\pi}. \quad (1^i = e^{-2k\pi}, k \in \mathbb{Z})$$

(b)

$$-1 = e^{i\pi} \Rightarrow \ln(-1) = \ln\left(e^{i\pi}\right) = i\pi.$$

(c)

$$\ln(-c) = \ln[(-1) \cdot c] = \ln(-1) + \ln(c) = \ln(c) + i\pi.$$

1.6.5 Polynomial Function with Complex Roots

1. Conjugate Pair Theorem:

Theorem 1.6.2 If z is a complex root of $P(x)$, then the conjugate of $z(z^*)$ is also a complex root of $P(x)$. ($P(x)$ should be a polynomial with **rational** coefficients.)

2. Properties of Conjugate.

•

$$(s \pm t)^* = s^* \pm t^*$$

•

$$(st)^* = s^*t^*$$

•

$$\left(\frac{s}{t}\right)^* = \frac{s^*}{t^*}$$

1.6.6 Root of Complex Numbers

1. The Root of Unity:

Theorem 1.6.3 For any complex equation $\omega^n = 1$, there are n distinct roots:

$$1 = e^{i(0+2k\pi)} = \omega^n, k \in \mathbb{Z} \Rightarrow \omega = e^{i\frac{2k\pi}{n}}, k \in \mathbb{Z}.$$

Example 1.6.6 Solve $z^3 = 8$.

$$z^3 = 8 \cdot 1 = 8e^{i(0+2k\pi)} \Rightarrow z = 2e^{i\frac{2k\pi}{3}}, k \in \mathbb{Z}$$

$$k = 0 : z = 2$$

$$k = 1 : z = 2e^{i\frac{2\pi}{3}} = 2\text{cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$$

$$k = 2 : z = 2e^{i\frac{4\pi}{3}} = 2\text{cis}\left(\frac{4\pi}{3}\right) = -1 - \sqrt{3}i$$

2. Property of $\text{cis}\theta$:

$$\text{cis}(-\theta) = \cos \theta - i \sin \theta$$

Proof 1.6.6

$$\begin{aligned} \cos \theta - i \sin \theta &= \cos(-\theta) + i \sin(-\theta) \\ &= \text{cis}(-\theta). \end{aligned}$$

2 Functions

2.1 Foundations of Functions

1. Relations and functions:

Definition 2.1.1 A **relation** R is a set of ordered pairs (x, y) such that $x \in A$, $y \in B$, and sets A , B are not empty.

Definition 2.1.2 A **function** f is a relation in which every x -value has a unique y -value.

2. Domain and Range:

Definition 2.1.3 **Domain** is the set of x -values.

Definition 2.1.4 **Range** is the set of y -values.

- Domain and Range should be in interval notation.

- (a) Using intervals to express the inequalities

Example 2.1.1

$$[3, 4[\text{ means } 3 \leq x < 4$$

- (b) If the interval will be joint, we use \cup to join the interval.

Example 2.1.2

$$3 < x < 4 \text{ or } x \geq 5 \Rightarrow]3, 4[\cup [5, +\infty[$$

Example 2.1.3 Find the interval notation for the domain of $f(x) = \frac{1}{x}$.

$$x \in]-\infty, 0[\cup]0, +\infty[\text{ OR } x \in \mathbb{R} \setminus 0$$

Note: \ means "exclude."

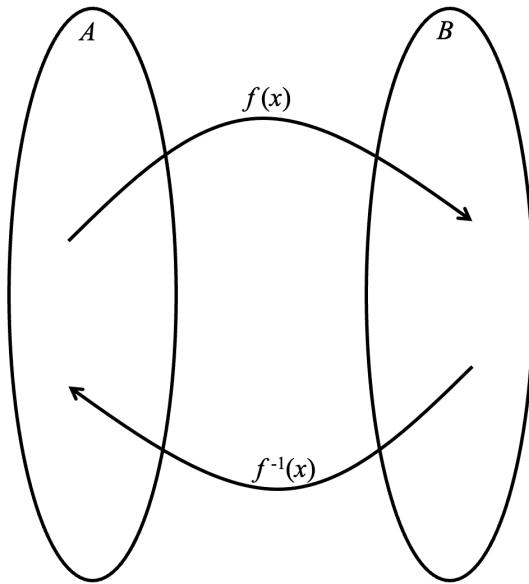
- Since the y -values (outputs) depend on the x -values (inputs), y is the **dependent variable**, and x is the **independent variable**.
- The independent variable x is also called the **argument** of the function.

3. Vertical Line test:

- To test whether a relation is a function.
- Since every x has one and only one value of y , there should be only one intersects.

4. Inverse of a function:

Definition 2.1.5 $f^{-1}(x)$ is the **inverse function** of $f(x)$.



Example 2.1.4 $f(1) = 3 \Rightarrow f^{-1}(3) = 1$; $f(x) = x + 5 \Rightarrow f^{-1}(x) = x - 5$

- In inverse function, the input becomes the output, the output becomes the input.
- In inverse function, the domain becomes the range, the range becomes the domain.

Example 2.1.5 (a) **Find the inverse function of** $y = \frac{x+2}{3}$.

$$3y = x + 2 \Rightarrow x = 3y - 2 \\ f^{-1}(x) = 3x - 2$$

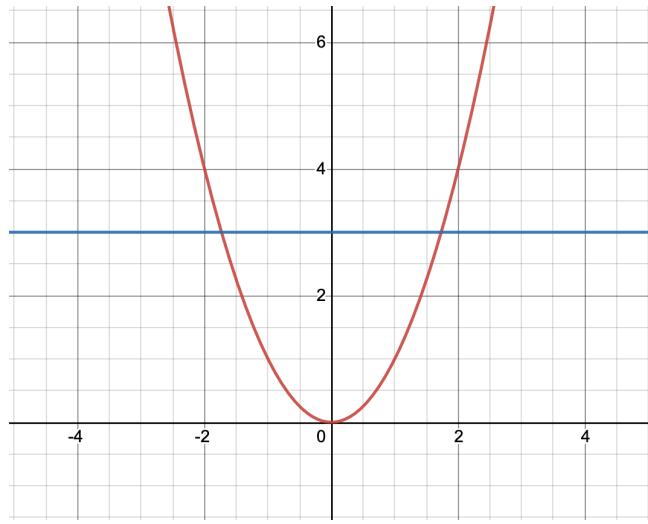
(b) **Find the inverse function of** $f(x) = \frac{x}{x+1}$.

$$y = \frac{x}{x+1} \Rightarrow xy + y = x \Rightarrow xy + x = y \\ \therefore y(x-1) = -x \Rightarrow y = -\frac{x}{x-1}$$

(c) **Find the inverse of** $\{(4, 2), (0, 2), (-2, 2)\}$

$$\text{Inverse: } \{(2, 4), (2, 0), (2, -2)\}$$

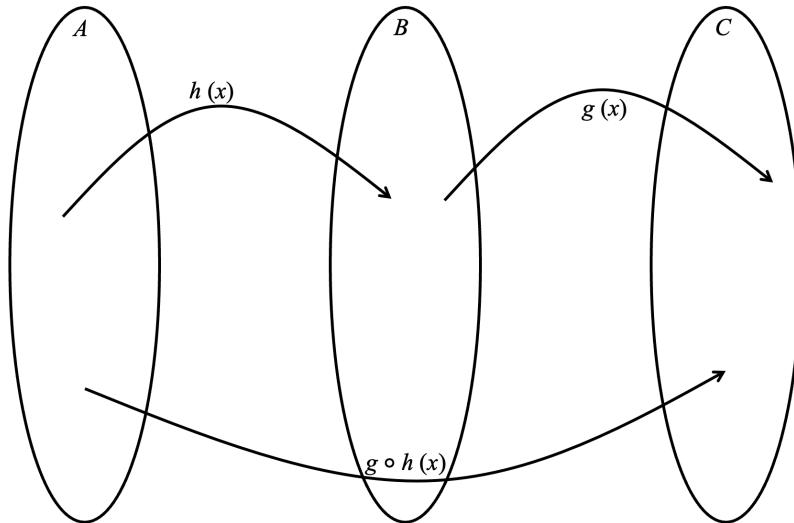
- By restricting the domain, we can find $f^{-1}(x)$ of $f(x)$, if the direct inverse of $f(x)$ is not a function.



Example 2.1.6 **Horizontal line test:** The largest domain we can find $f^{-1}(x)$ is $x \leq 0$ or $x > 0$.

5. Composite Functions:

Definition 2.1.6 We use $(g \circ h)(x)$ or $g(h(x))$ to represent composite functions.



Example 2.1.7 Given $f : x \mapsto 3x - 6$, $g : x \mapsto \frac{1}{3}x + 2$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = f(g(x)) = 3\left(\frac{1}{3}x + 2\right) - 6 = x.$$

$$(g \circ f)(x) = g(f(x)) = \frac{1}{3}(3x - 6) + 2 = x.$$

When f and g are inverse functions:

$$(f \circ g)(x) = x = (g \circ f)(x).$$

6. $f(x)$ and $f^{-1}(x)$ are symmetrical to $y = x$ since $D_f = R_{f^{-1}}$, $R_f = D_{f^{-1}}$. That is, if $f(x)$ passes through (a, b) , $f^{-1}(x)$ passes through (b, a) .

2.2 Quadratic Functions

1. The Standard Form:

$$y = ax^2 + bx + c,$$

where a is the coefficient of x^2 , b is the coefficient of x , and c is the constant or y -intercept. $a, b, c \neq 0$.

- Zeros of the function (x -intercepts):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $\Delta = b^2 - 4ac$ is the discriminant of the function.

- Equation of the line of symmetry & x -coordinate of the vertex

$$x = -\frac{b}{2a}.$$

- Vieta's Formula:

Theorem 2.2.1 Assume x_1, x_2 are two roots for equation $ax^2 + bx + c = 0$ ($a \neq 0$), then

$$x_1 + x_2 = -\frac{b}{a};$$

$$x_1 \cdot x_2 = \frac{c}{a}.$$

- When $a > 0$, the parabola opens upwards.
When $a < 0$, the parabola opens downwards.

2. Completion of square:

$$x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2.$$

3. The Vertex Form:

$$y = a(x - h)^2 + k, \text{ where } (h, k) \text{ is the vertex.}$$

Example 2.2.1 Given that $f(x) = ax^2 + bx + c$, find the axis of symmetry and vertex.

$$\begin{aligned} f(x) &= a \left(x^2 + \frac{b}{a}x \right) + c \\ &= a \left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a} \right)^2 \right] + c - \frac{b^2}{4a} \\ &= a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}. \end{aligned}$$

$$\begin{aligned} \therefore \text{axis of symmetry: } x &= -\frac{b}{2a} \\ \text{vertex: } &\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a} \right). \end{aligned}$$

2.3 Higher Order Polynomial Functions

1. Factor Theorem:

Theorem 2.3.1 If $(x - a)$ is a factor of a polynomial $P(x)$, then $x = a$ must be a root for $P(x) \Rightarrow P(a) = 0$.

Proof 2.3.1 Assume the quotient when $P(x)$ is divided by $(x - a)$ is $Q(x)$, then $P(x) = Q(x) \cdot (x - a)$. Then, $P(a) = Q(a) \cdot (a - a) = 0$.

2. Long division: solving polynomial equation.

Example 2.3.1 For a cubic function, $P(x) = 2x^3 + bx^2 + cx + d$, $P(1) = P(2) = P(3) = 2$.

What is $P(0)$?

Since $P(1) = P(2) = P(3) = 2$, $Q(1) = Q(2) = Q(3) = 0$, where $Q(x) = P(x) - 2$.

Thus, $Q(x) = 2(x - 1)(x - 2)(x - 3)$.

$$\therefore P(x) = Q(x) + 2 = 2(x - 1)(x - 2)(x - 3) + 2.$$

$$\therefore P(0) = 2(-1)(-2)(-3) + 2 = -10.$$

3. Remainder Theorem:

Theorem 2.3.2 When a polynomial $P(x)$ is divided by $(ax - b)$, the remainder R of this division must be

$$P\left(\frac{b}{a}\right).$$

Proof 2.3.2 Assume the quotient is $Q(x)$, and the remainder is R :

$$P(x) = (ax - b)Q(x) + R.$$

$$P\left(\frac{b}{a}\right) = 0 \cdot Q\left(\frac{b}{a}\right) + R = R.$$

4. Roots of Cubic Functions:

Theorem 2.3.3 For a cubic function $f(x) = ax^3 + bx^2 + cx + d$, given the roots of it are α, β , and γ . Then,

$$\begin{cases} \alpha + \beta + \gamma = -\frac{b}{a} \Rightarrow \sum \alpha = -\frac{b}{a} \\ \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} \Rightarrow \sum \alpha\beta = \frac{c}{a} \\ \alpha\beta\gamma = -\frac{d}{a} \Rightarrow \sum \alpha\beta\gamma = -\frac{d}{a} \end{cases}$$

Proof 2.3.3 Since α, β, γ are roots of $f(x)$,

$$f(x) = a(x - \alpha)(x - \beta)(x - \gamma).$$

$$\text{So } a(x - \alpha)(x - \beta)(x - \gamma) = ax^3 + bx^2 + cx + d,$$

$$\text{i.e., } ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \alpha\gamma + \beta\gamma)x - a\alpha\beta\gamma = ax^3 + bx^2 + cx + d.$$

$$\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}, \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}.$$

Theorem 2.3.4

$$\sum \alpha = -\frac{b}{a}, \sum \alpha\beta = \frac{c}{a}, \sum \alpha\beta\gamma = -\frac{d}{a}, \sum \alpha\beta\gamma\delta = \frac{e}{a}.$$

2.4 Rational Functions

1. Reciprocal Functions: $f(x) = \frac{1}{x}$.

- Domain: $x \in \mathbb{R}, x \neq 0$
- As x increases, $\frac{1}{x}$ decreases $\Rightarrow x \rightarrow \infty, \frac{1}{x} \rightarrow 0$.
- Range: $y \in \mathbb{R}, y \neq 0$
- **Asymptotes**: $x = 0, y = 0$.
- Axis of symmetry: $y = x, y = -x$.
- **Self-inversing function**: have axis of symmetry $y = x$.

$$f(x) = f^{-1}(x).$$

2. $y = \frac{a}{bx + c}$

- Vertical asymptotes (V.A.): $bx + c = 0$
- Horizontal asymptotes (H.A.): $y = 0$

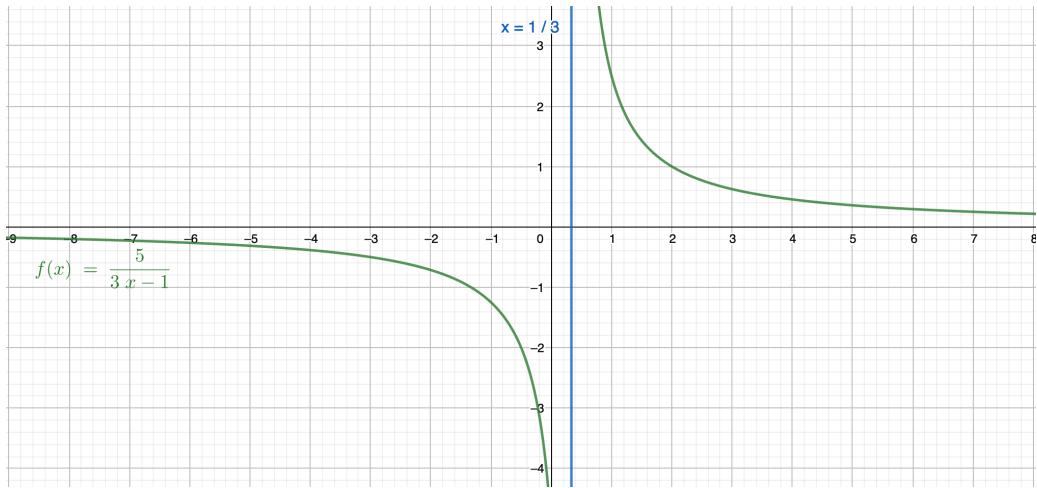
Example 2.4.1 Draw the diagram of $y = \frac{5}{3x - 1}$.

x -intercept: $0 = \frac{5}{3x - 1} \Rightarrow$ no solution, no intercept.

H.A.: $y = 0$

y-intercept: $y = -5$

$$\text{V.A.: } 3x - 1 = 0, x = \frac{1}{3}$$



$$3. \ y = \frac{ax+b}{cx+d}$$

- V.A.: $cx + d = 0$
- H.A.: $y = \frac{a}{c}$

$$4. \ y = \frac{ax+b}{cx^2+dx+e}$$

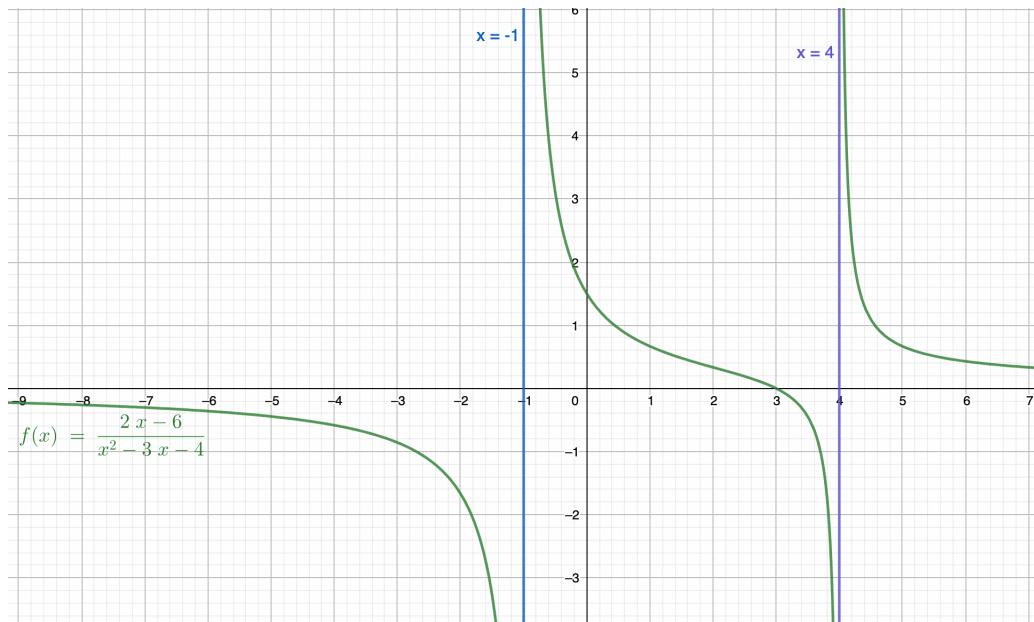
- V.A.: $cx^2 + dx + e = 0$
- H.A.: As $x \rightarrow \pm\infty$, $\frac{ax}{cx^2} \rightarrow 0$, $y = 0$
- Intercepts: $\left(0, \frac{e}{c}\right)$, $\left(-\frac{e}{d}, 0\right)$

Example 2.4.2 Draw the diagram of $y = \frac{2x-6}{x^2-3x-4}$.

Intercept: $\left(0, \frac{3}{2}\right)$, $(3, 0)$

H.A.: $y = 0$

V.A.: $x = -1, x = 4$

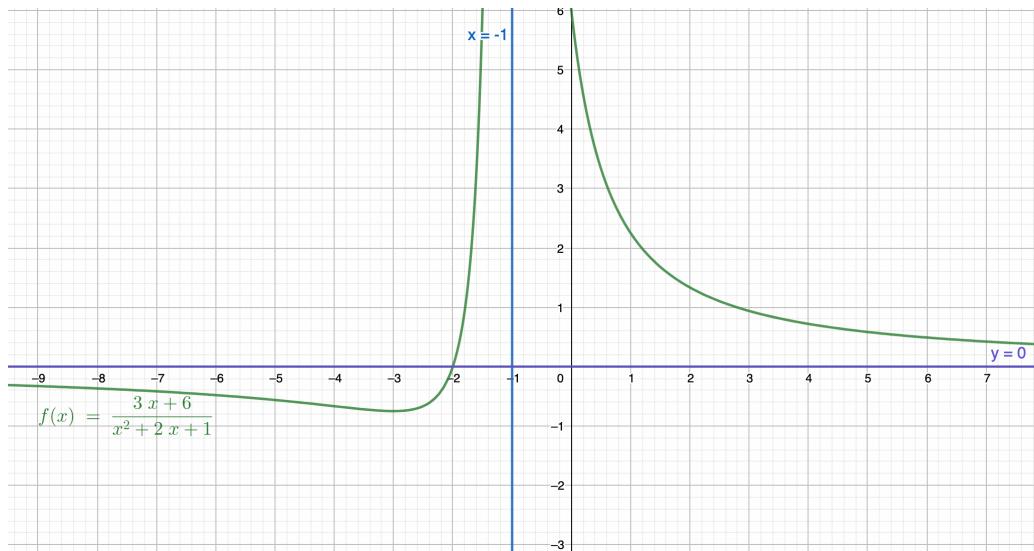


Example 2.4.3 Draw the diagram of $y = \frac{3x+6}{x^2 + 2x + 1}$.

Intercept: $(0, 6)$, $(-2, 0)$

H.A.: $y = 0$

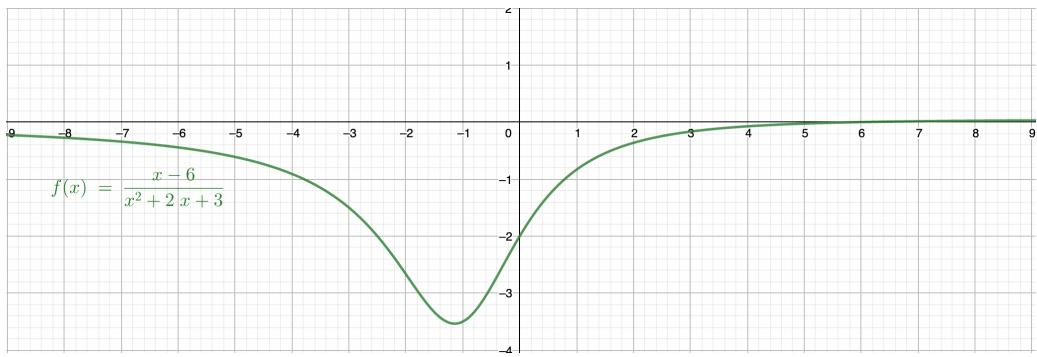
V.A.: $x = -1$



Example 2.4.4 Draw the diagram of $y = \frac{x-6}{x^2 + 2x + 3}$.

Intercept: $(6, 0)$, $(0, -2)$

When $x \rightarrow \infty$, $f(x)$ is positive. When $x \rightarrow -\infty$, $f(x)$ is negative.



$$5. \ y = \frac{ax^2 + bx + c}{dx + e}$$

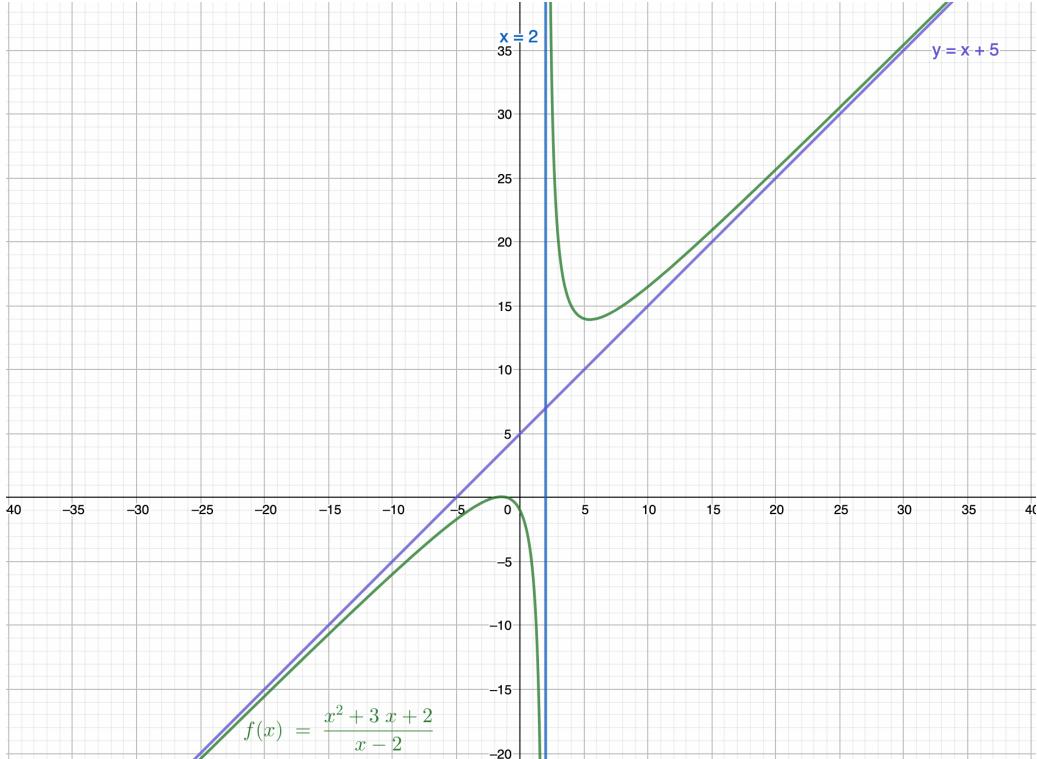
- V.A.: $dx + e = 0$
- **Oblique Asymptote:** Quotient of $(ax^2 + bx + c)$ divided by $(dx + e)$.
- Intercepts: $\left(0, \frac{c}{e}\right)$, $ax^2 + bx + c = 0$

Example 2.4.5 Draw the diagram of $y = \frac{x^2 + 3x + 2}{x - 2}$.

Intercept: $(0, -1)$, $(-1, 0)$, $(-2, 0)$

V.A.: $x = 2$

O.A.: $y = x + 5$ (Use long division)

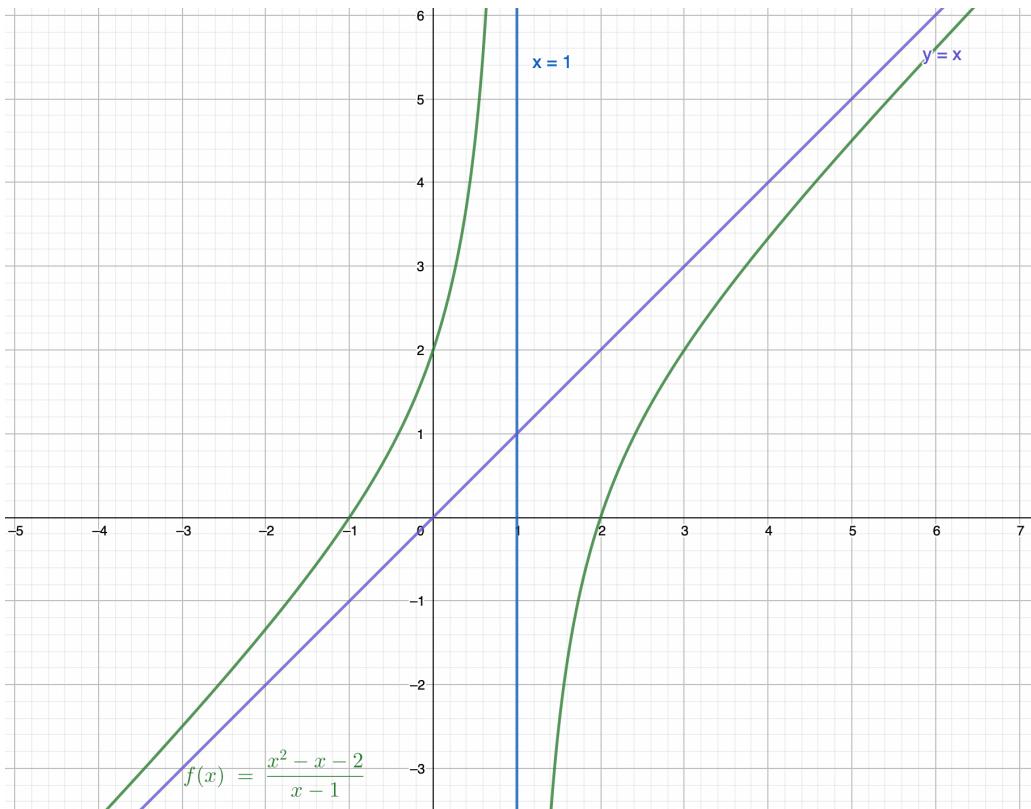


Example 2.4.6 Draw the diagram of $y = \frac{x^2 - x - 2}{x - 1}$.

Intercept: $(0, 2)$, $(2, 0)$, $(-1, 0)$

V.A.: $x = 1$

O.A.: $y = x$ (Use long division)



6. When the function has asymptotes:

- Denominator= 0;
- $\log_a 0$ (argument of a logarithm is 0)

2.5 Transformation of Functions

1. Translation:

- $f(x+n)$ means translate $f(x)$ n units to the left.
- $f(x-n)$ means translate $f(x)$ n units to the right.
- $f(x)+n$ means translate $f(x)$ n units upwards.
- $f(x)-n$ means translate $f(x)$ n units downwards.

2. Use translation vector to represent translation:

A vector $\begin{pmatrix} a \\ b \end{pmatrix}$ means a units in the horizontal axis and b units in the vertical axis.

Example 2.5.1 A translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ means $f(x+2)+3$, 2 units to the left and 3 units upwards.

3. Reflections:

- $f(-x)$ reflects in the y -axis.
- $-f(x)$ reflects in the x -axis.

- $f^{-1}(x)$ reflects in the $y = x$.
- $-f(-x)$ reflects in the origin.

4. Stretches:

- $f(qx)$ is a horizontal stretch of a scale factor of $\frac{1}{q}$.
- $pf(x)$ is a vertical stretch of a scale factor of p .

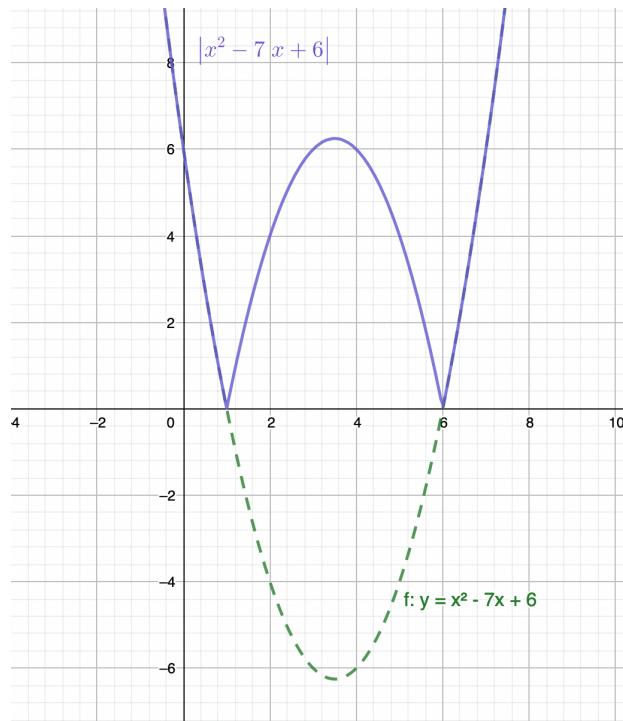
5. When a graph is transforming, the points shift but the connection remains.

6. Sequence of transformation:

- Do the horizontal translation before the horizontal stretch.
- The vertical translation is always after the vertical stretch.
- Vertical stretch → Reflection → Horizontal translation → Horizontal stretch → Vertical translation

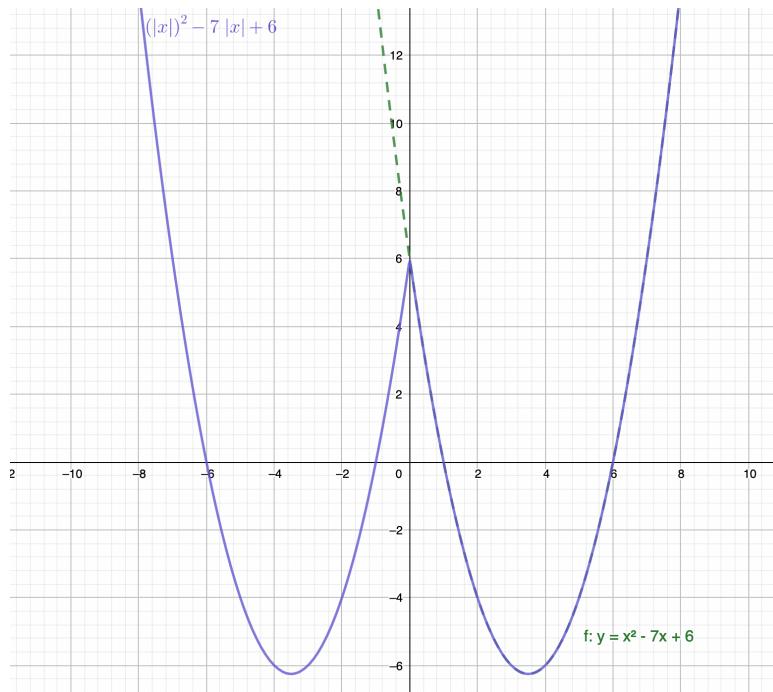
7. Modulus Function

- $|f(x)|$: Fold everything below x -axis above x -axis.



Example 2.5.2

- $f(|x|)$: Reflect everything on the right of y -axis to the left. Since $|x|$ must be positive, $|x| = |-x| \Rightarrow f(-x) = f(x)$, which is an even function.



Example 2.5.3

8. Reciprocal of $f(x)$

- Table of Summary:

$f(x)$	$g(x) = \frac{1}{x}$
$f(a) = 0$	Line $x = a$ is vertical asymptote
Line $x = a$ is vertical asymptote	$g(a) = 0$
$f(x) \rightarrow \infty$	$g(x) \rightarrow 0$
$f(x) \rightarrow 0$	$g(x) \rightarrow \infty$
Line $y = b$ is horizontal asymptote	Line $y = \frac{1}{b}$ is horizontal asymptote
$f(x) = a$	$g(x) = \frac{1}{a}$

- When $f(x)$ increases, $g(x)$ decreases.

2.6 Exponential and Logarithmic Functions

1. Exponential functions:

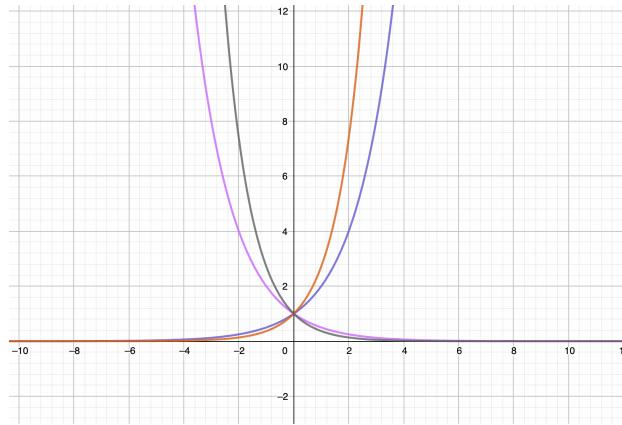
- $f(x) = a^x$, $a > 1$ (increasing) and $0 < a < 1$ (decreasing).
- $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$ are symmetric to the y -axis.

Proof 2.6.1

$$g(x) = \left(\frac{1}{a}\right)^x = (a^{-1})^x = a^{-x} = f(-x).$$

- Domain: $x \in \mathbb{R}$, Range: $y > 0$
- Common point: $(0, 1)$; common H.A.: $y = 0$

- Graph:



2. Logarithmic functions:

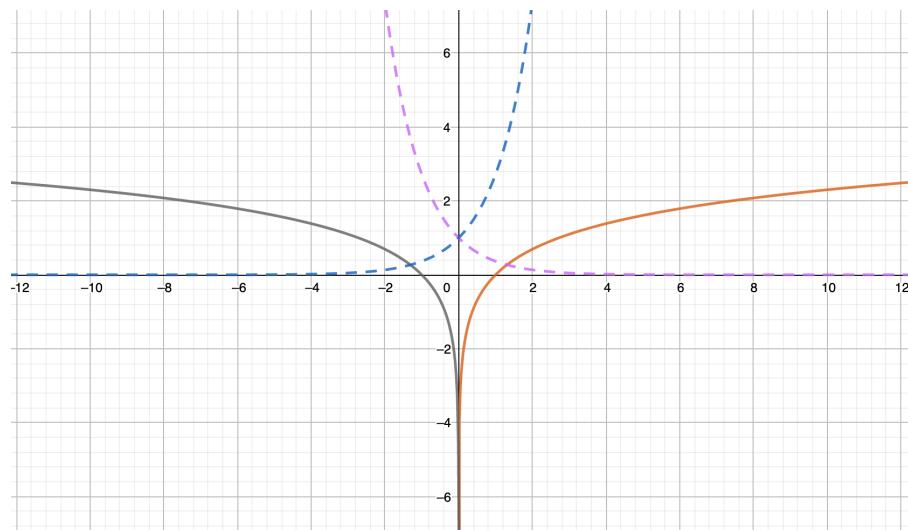
- $f(x) = \log_a x = g^{-1}(x)$, $g(x) = a^x$.
- Common point: $(1, 0)$; common V.A.: $x = 0$.
- $f(x) = \log_a x$ and $g(x) = \log_{\frac{1}{a}} x$ are symmetric to the x -axis.

Proof 2.6.2

$$\log_{\frac{1}{a}} x = \frac{\log_a x}{\log_{\frac{1}{a}} a} = \frac{\log_a x}{-1} = -\log_a x,$$

$$\therefore g(x) = \log_{\frac{1}{a}} x = -\log_a x = -f(x).$$

- When $a > 1$, increasing function; when $0 < a < 1$, decreasing function.
- Domain: $x > 0$, Range: $y \in \mathbb{R}$
- Graph:



3. Solving logarithmic equations.

4. Solving exponential equations: take logarithm on both sides.

3 Trigonometry and Geometry

3.1 Trigonometry

3.1.1 Radian

1. Radian as the unit of angle:

-

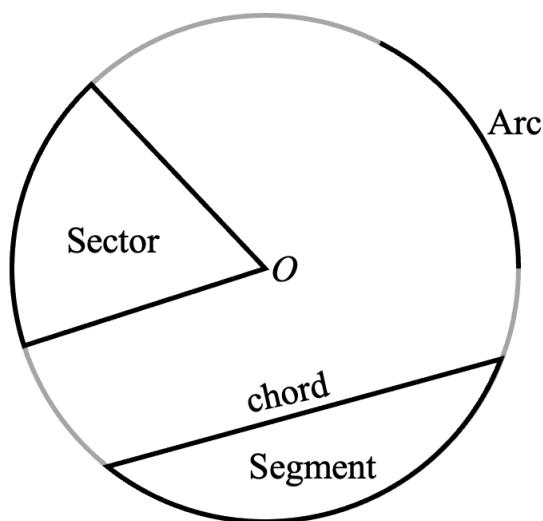
$$\pi \text{ rad} = 180^\circ$$

- rad can be omitted. i.e., $\widehat{A} = 1$ means angle A is 1 radian.
- Unit conversion:

$$\text{degree} \times \frac{\pi}{180^\circ} = \text{radian}; \text{radian} \times \frac{180^\circ}{\pi} = \text{degree}.$$

2. Arc:

- The **circumference** (perimeter) is $2\pi r$.



- If the angle of the arc is θ (in radian), the length of arc(l) = $r \cdot \theta$.
- The area of a sector:

$$A = \frac{1}{2}r^2\theta.$$

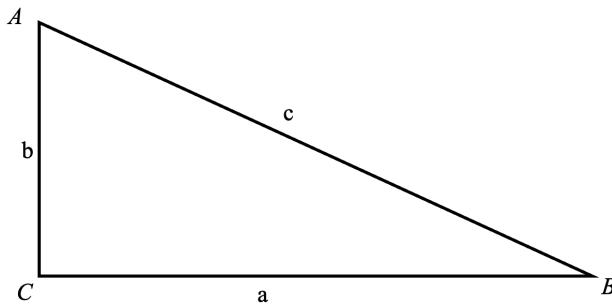
- The area of a segment:

$$A = \frac{1}{2}r^2(\theta - \sin \theta).$$

(Proof: the area of the triangle according to the sine rule is $\frac{1}{2}ab \sin C$)

3.1.2 Solution of Triangle

1. Define sine, cosine, and tangent:



Definition 3.1.1

$$\sin A = \frac{a}{c}, \sin B = \frac{b}{c};$$

$$\cos A = \frac{b}{c}, \cos B = \frac{a}{c};$$

$$\tan A = \frac{a}{b}, \tan B = \frac{b}{a}.$$

2. The Sine Rule:

Theorem 3.1.1

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

- The bigger the angle, the longer the side.
- Area of a triangle:

$$A = \frac{1}{2}ab \sin C.$$

3. The Cosine Rule:

Theorem 3.1.2

$$b^2 + c^2 - a^2 = 2bc \cdot \cos A;$$

$$a^2 + c^2 - b^2 = 2ac \cdot \cos B;$$

$$a^2 + b^2 - c^2 = 2ab \cdot \cos C.$$

4. Inverse Trigonometric Functions:

Definition 3.1.2

$$\sin^{-1} \theta = \arcsin \theta;$$

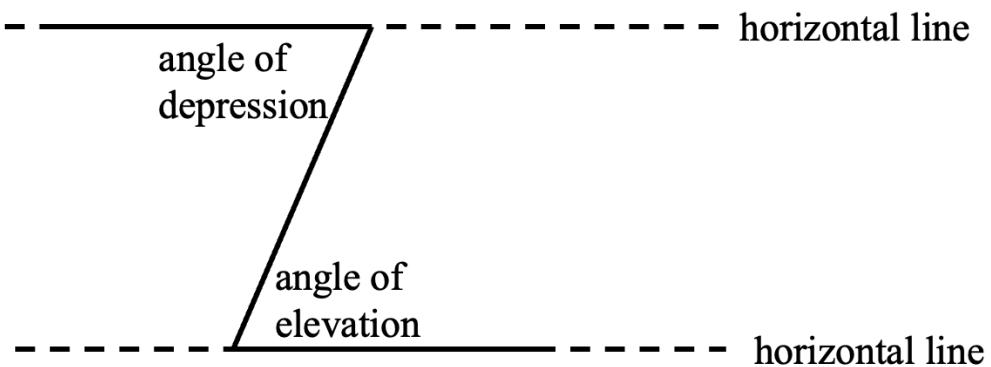
$$\cos^{-1} \theta = \arccos \theta;$$

$$\tan^{-1} \theta = \arctan \theta.$$

5. Ambiguity of Sine Rule:

$$\sin \theta = \sin(180^\circ - \theta) \text{ OR } \sin \theta = \sin(\pi - \theta).$$

6. Angle of Elevation and Depression:

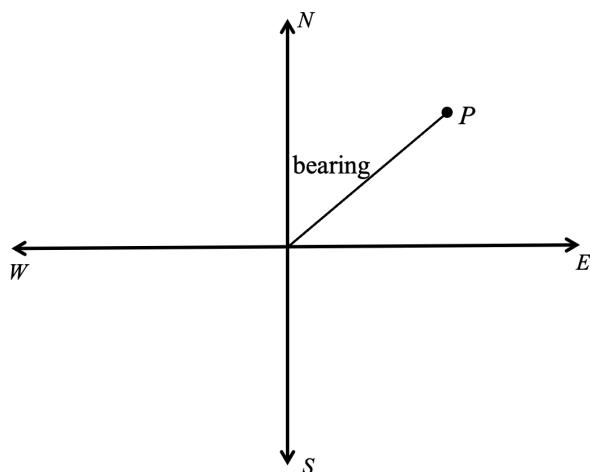


Definition 3.1.3 • **Angle of Elevation** is the angle "up" from horizontal.

- **Angle of Depression** is the angle "down" from horizontal.

7. Bearing:

- Bearing is a way of describing direction.
- All bearings are measured **clockwise** from the **North** direction.



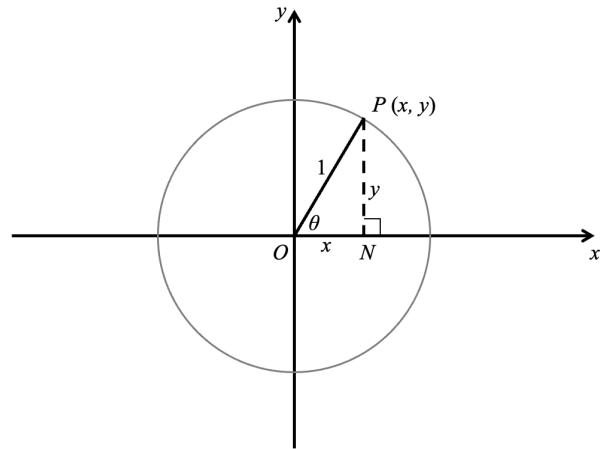
- Bearing of A from B: construct at B.

N.B.: Bearing of A from B is different from bearing of B from A.

3.1.3 Definition of Trigonometric Function

1. Unit Circle:

- Center at $(0,0)$ with a radius of 1.



- If an angle θ opens in a counterclockwise direction, then θ is positive.
If an angle θ opens in a clockwise direction, then θ is negative.
- In the diagram, $\theta = \theta + 2k\pi$, $k \in \mathbb{Z}$.

•

$$\sin \theta = \frac{PN}{OP} = \frac{y}{1} = y;$$

$$\cos \theta = \frac{ON}{OP} = \frac{x}{1} = x;$$

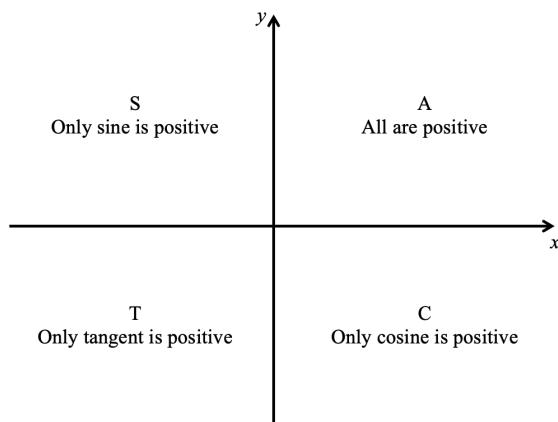
$$\tan \theta = \frac{PN}{ON} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta};$$

- In Q_1 and Q_2 , $\sin \theta$ will be positive.

In Q_1 and Q_4 , $\cos \theta$ will be positive.

In Q_1 and Q_3 , $\tan \theta$ will be positive.

\Rightarrow CAST:



2. Special Angles:

$$\sin 0^\circ = 0 = \cos 90^\circ$$

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = 0$$

$$\sin 30^\circ = \frac{1}{2} = \cos 60^\circ$$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\sqrt{3}}{3}$$

$$\sin 45^\circ = \frac{\sqrt{2}}{2} = \cos 45^\circ$$

$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \cos 30^\circ$$

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}$$

$$\sin 90^\circ = 1 = \cos 0^\circ$$

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \infty$$

3. Relative Acute Angles (RAA):

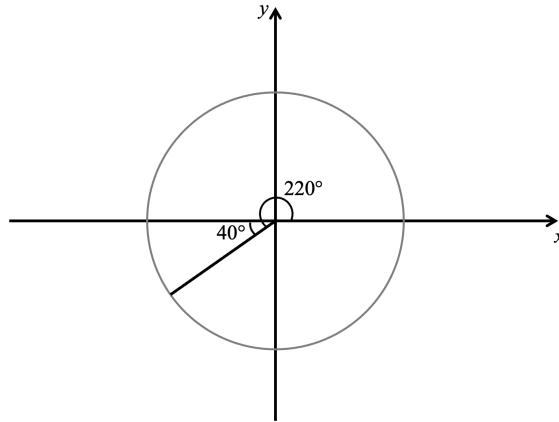
- Acute angle is the angle with x -axis.
- The absolute value of angles have the same acute angle is the same.

Example 3.1.1 (a) $30^\circ, 150^\circ, 210^\circ, 330^\circ$ have the same acute angle.

$$\therefore |\sin 30^\circ| = |\sin 150^\circ| = |\sin 210^\circ| = |\sin 330^\circ|.$$

(b)

$$\tan 220^\circ = \tan 40^\circ; \cos 215^\circ = -\cos 35^\circ$$



3.1.4 Trigonometric Identity

1. Pythagorean's Identity:

$$\sin^2 \theta + \cos^2 \theta \equiv 1.$$

Proof 3.1.1

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \\ &\Rightarrow \sin^2 \theta + \cos^2 \theta = 1. \end{aligned}$$

2. Definition of Tangent:

- $\tan \theta = \frac{\sin \theta}{\cos \theta};$
- $\cot \theta = \frac{1}{\tan \theta};$
- $\sec \theta = \frac{1}{\cos \theta};$
- $\csc \theta = \frac{1}{\sin \theta}.$

3. Extended Pythagorean's Identity:

$$\tan^2 \theta + 1 = \sec^2 \theta;$$

$$\cot^2 \theta + 1 = \csc^2 \theta.$$

Proof 3.1.2

$$\begin{aligned}\sin^2 \theta + \cos^2 \theta &= 1 \Rightarrow \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \Rightarrow \tan^2 \theta + 1 = \sec^2 \theta; \\ \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} &= \frac{1}{\sin^2 \theta} \Rightarrow \cot^2 \theta + 1 = \csc^2 \theta.\end{aligned}$$

N.B.: a reflex angle is an angle bigger than 180° , smaller than 360° .

4. Compound Angle Formula:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B;$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B;$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B;$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B;$$

Example 3.1.2 Find the exact value of $\cos \frac{\pi}{12}$.

$$\begin{aligned}\cos \frac{\pi}{12} &= \cos \frac{\pi}{4} - \frac{\pi}{6} \\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.\end{aligned}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Proof 3.1.3

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B}.\end{aligned}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

5. In the linear function $y = mx + b$, $m = \tan \theta$, where θ is the angle between the line and the positive x -axis.
6. Double Angle Formula:

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta;$$

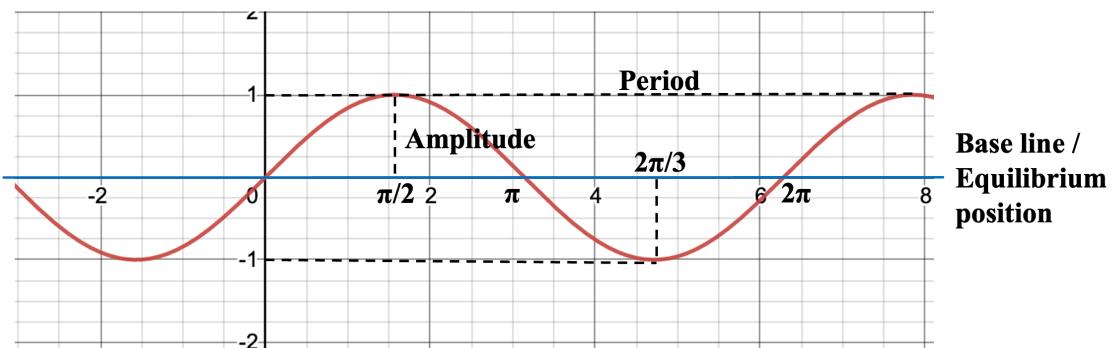
$$\sin(2\theta) = 2\sin \theta \cos \theta;$$

$$\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}.$$

7. Proving Identities.

3.1.5 Trigonometric Functions and Transformation

1. Sine: Odd function: $\sin(-x) = -\sin x$.



$$T(\text{Period}) = 2\pi;$$

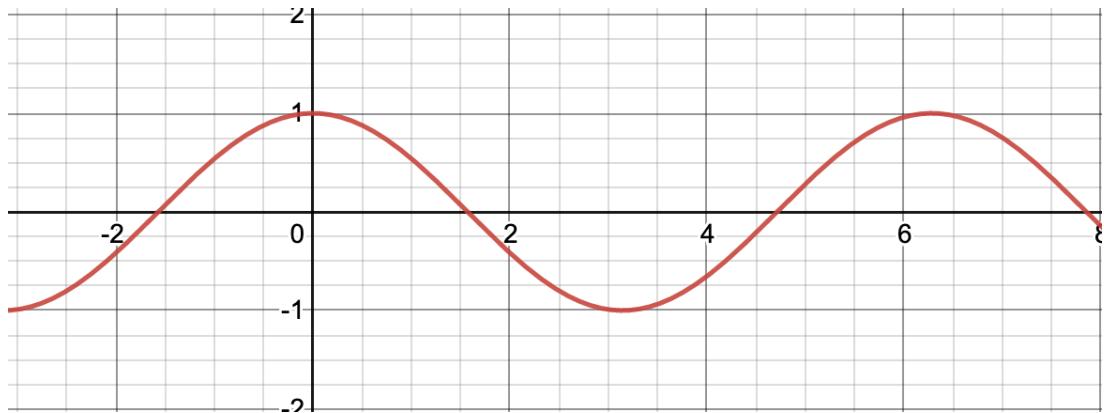
$$\text{Base line} = 0;$$

$$\text{Amplitude} = \left| \frac{y_{\max} - y_{\min}}{2} \right| = 1;$$

$$\text{Range: } \sin x \in [-1, 1];$$

$$\text{Domain: } x \in \mathbb{R}.$$

2. Cosine: Even function: $\cos(-x) = \cos x$.



$$T(\text{Period}) = 2\pi;$$

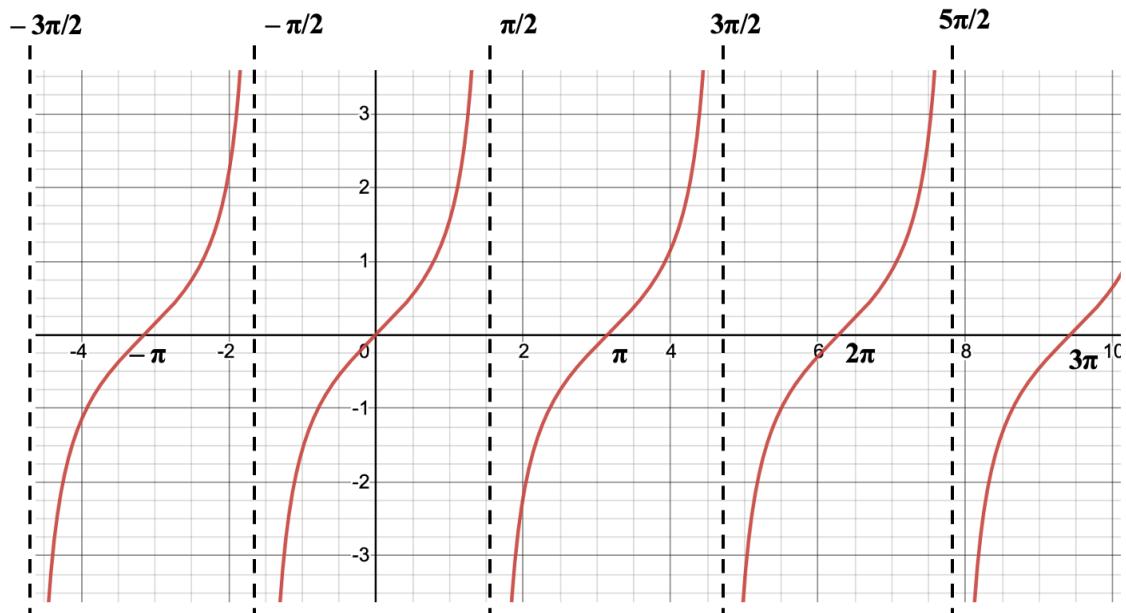
Base line = 0;

$$\text{Amplitude} = \left| \frac{y_{\max} - y_{\min}}{2} \right| = 1;$$

$$\text{Range: } \cos x \in [-1, 1];$$

Domain: $x \in \mathbb{R}$.

3. Tangent:



$$T(\text{Period}) = \pi;$$

No amplitude(A);

$$\text{V.A.: } x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z};$$

Range: $\tan x \in \mathbb{R}$;

Domain: $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

4. Transformation of Sine and Cosine:

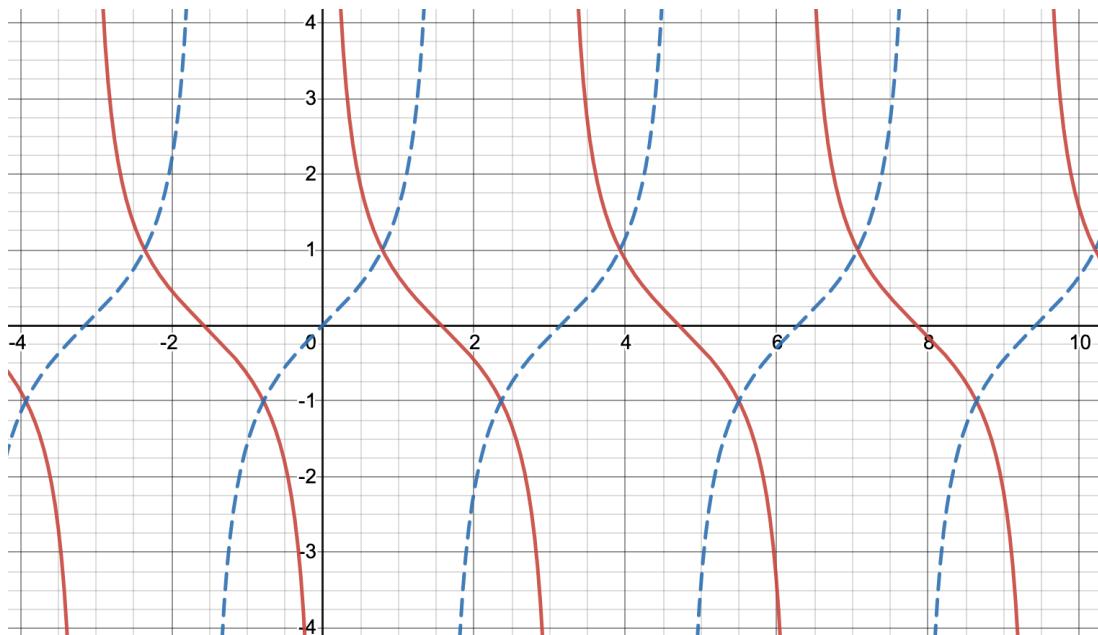
$$y = A \sin(\omega(x - \varphi)) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. \Rightarrow changes $T = \frac{\pi}{\omega}$.
- Horizontal translate to the right φ units. \Rightarrow changes the initial point to $(\varphi, 0)$.
- Vertical stretch with a scale factor of A . \Rightarrow changes the amplitude = $|A|$.
- Vertical translation of h units upwards. \Rightarrow changes the equilibrium position $y = h$.
- Range of $y = A \sin(\omega(x - \varphi)) + h$: $y \in [h - A, h + A]$.

$$y = A \cos(\omega(x - \varphi)) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. \Rightarrow changes $T = \frac{\pi}{\omega}$.
- Horizontal translate to the right φ units. \Rightarrow changes the initial point to $(\varphi, 1)$.
- Vertical stretch with a scale factor of A . \Rightarrow changes the amplitude = $|A|$, initial point (φ, A) .
- Vertical translation of h units upwards. \Rightarrow changes the equilibrium position $y = h$, initial point $(\varphi, A + h)$.

5. Cotangent:

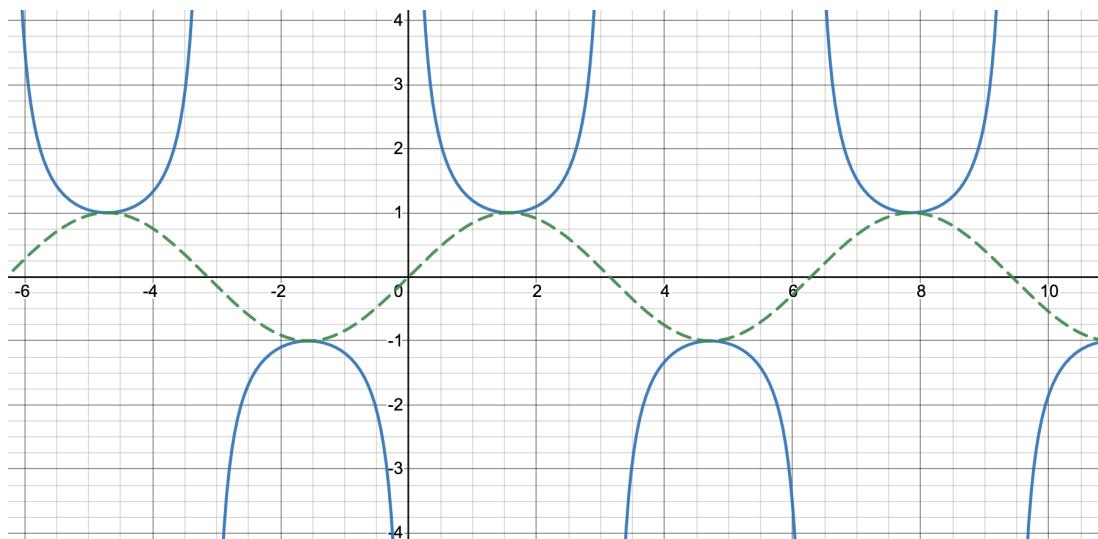


V.A.: $x = k\pi$

Period: π

Pass through $\left(\frac{\pi}{2} + k\pi, 0\right)$

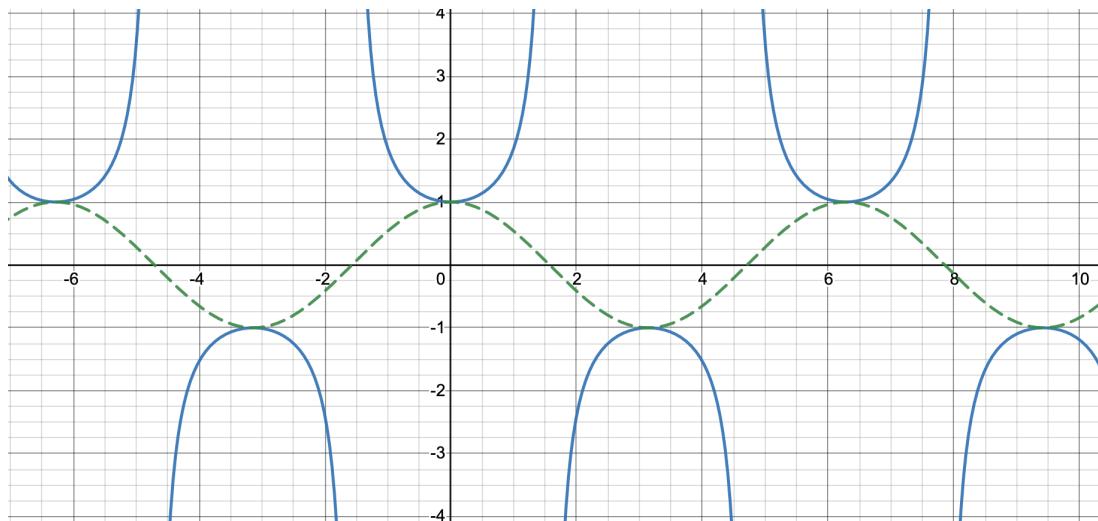
6. Cosecant:



$$\text{Domain: } x \neq k\pi$$

$$\text{Range: } y \in]-\infty, -1[\cup]1, +\infty[$$

7. Secant:



$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi$$

$$\text{Range: } y \in]-\infty, -1[\cup]1, +\infty[$$

8. When drawing the graph of $\sec x$ and $\csc x$, draw $\cos x$ and $\sin x$ first.

9. Conversion between sine and cosine:

•

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\cos\left(\frac{\pi}{2} + x\right) = \cos\left[\pi - \left(\frac{\pi}{2} - x\right)\right] = -\cos\left(\frac{\pi}{2} - x\right) = -\sin x$
- $\sin\left(\frac{\pi}{2} + x\right) = \sin\left[\pi - \left(\frac{\pi}{2} - x\right)\right] = \sin\left(\frac{\pi}{2} - x\right) = \cos x$

3.1.6 Solving Trigonometric Functions

1. Solving Trigonometric Functions in Paper 1:

- Values of special angles
- From relative acute angles and CAST rule
- Modification of period
- Check the solution with domain

Example 3.1.3 Solve for $\cos x = \frac{\sqrt{3}}{2}$ for $0 < x < 3\pi$.

Consider $x \in [0, 2\pi]$

$$x = \frac{\pi}{6}, \frac{11\pi}{6}.$$

In the domain of $x \in [0, 3\pi]$,

Another solution is $\frac{13\pi}{6}$.

2. Transformed Trigonometric Equations:

Example 3.1.4 Solve $6\sin\left(2\left(x - \frac{\pi}{6}\right)\right) - 2 = 1$, $\frac{\pi}{6} < x < 2\pi$.

$$\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = \frac{1}{2}.$$

Let $t = 2\left(x - \frac{\pi}{6}\right)$:

$$\therefore \frac{\pi}{6} < x < 2\pi,$$

$$\therefore 0 < 2\left(x - \frac{\pi}{6}\right) < \frac{11\pi}{3}, \quad 0 < t < \frac{11\pi}{3}.$$

$$\sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6};$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}.$$

3. Solving Trigonometric Functions in Paper 2:

- Change mode to RADIANS.

- Plot the functions.
- Adjust the window.
- Calculate the intersects.
- Repeat step 4 if necessary.

3.1.7 Inverse Trigonometric Functions

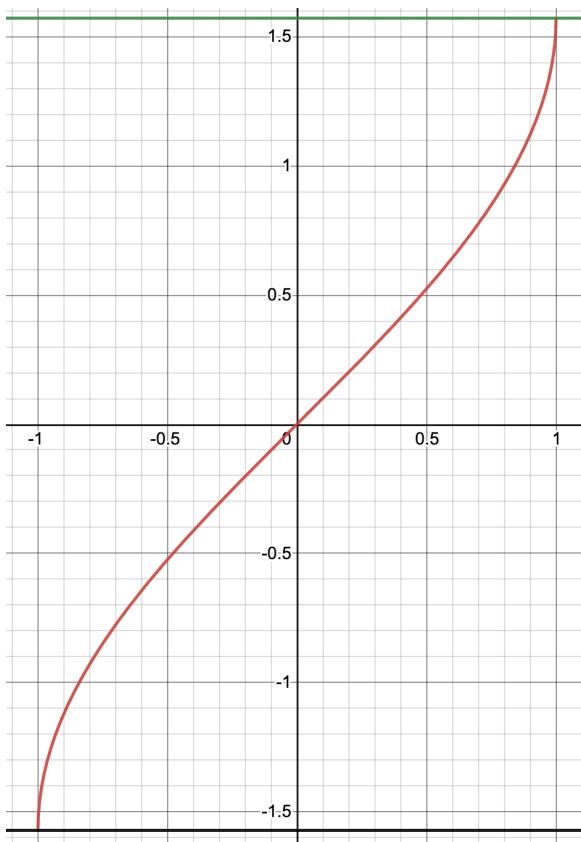
1. Inverse Trigonometric Function:

- $y = \arcsin x$
- $y = \arccos x$
- $y = \arctan x$
- $\text{arcsec}x = \arccos\left(\frac{1}{x}\right)$
- $\text{arccsc}x = \arcsin\left(\frac{1}{x}\right)$
- $\text{arccot}x = \arctan\left(\frac{1}{x}\right)$

2. One-to-one Function:

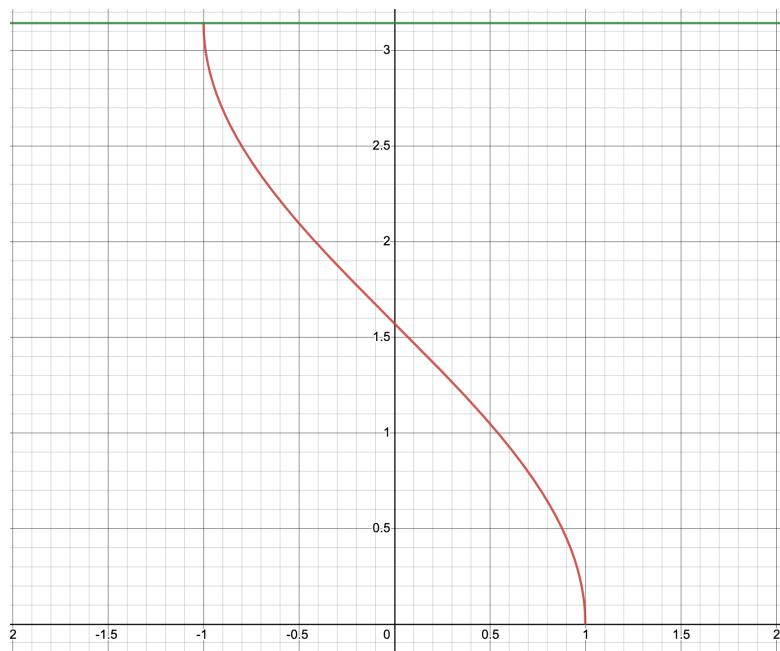
- In order for functions to have the inverse function, it must be so-called **one-to-one** function (bijection).
- One x value to one (and only one) y value.
One y value to one (and only one) x value.

3. Domain and range for $\arcsin x$:



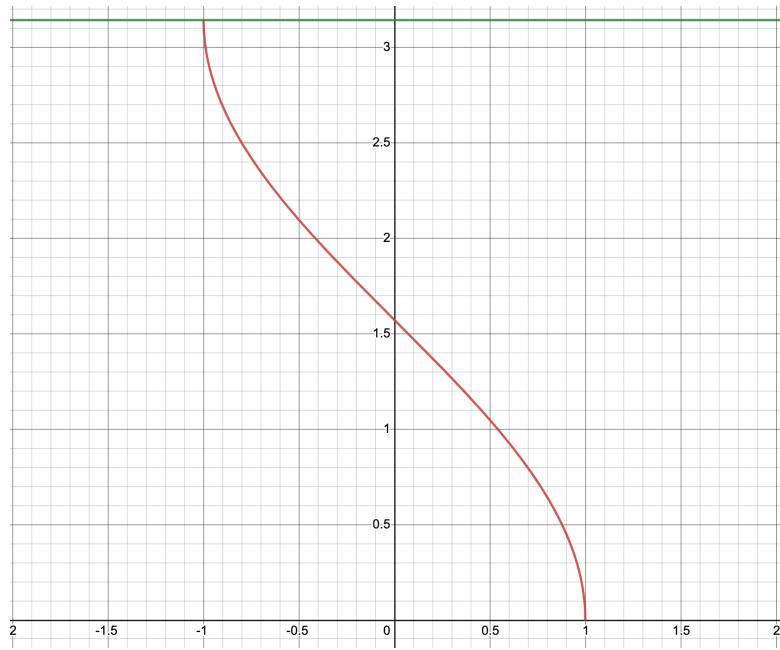
- Domain: $x \in [-1, 1]$ (Range $\sin x \in [-1, 1]$).
- Range: $\arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Domain $\sin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

4. Domain and range for $\arccos x$:



- Domain: $x \in [-1, 1]$.
- Range: $\arccos x \in [0, \pi]$.

5. Domain and range for $\arctan x$:



- Domain: $x \in \mathbb{R}$
- Range: $y \in]-\frac{\pi}{2}, \frac{\pi}{2}[$

3.2 Vectors

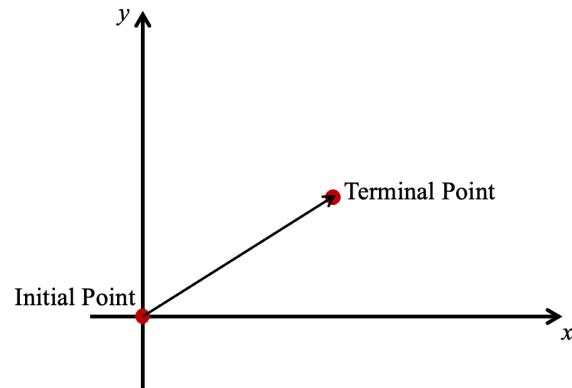
3.2.1 Introduction to Vectors

1. Vector:

Definition 3.2.1 A **vector** is a quantity with a direction and magnitude. It is noted as \vec{a} .

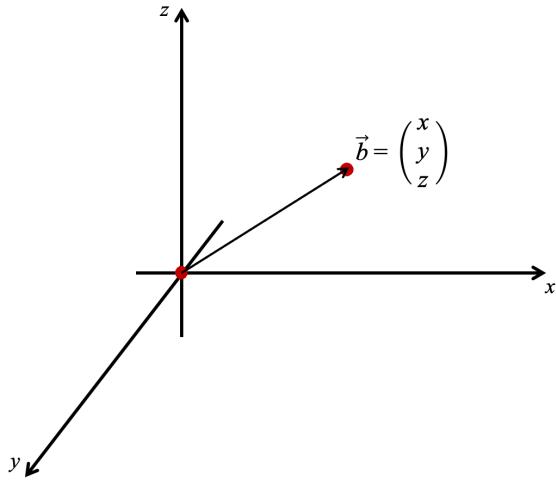
2. Components of a vector:

- 2-D:



Example 3.2.1 The vector $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ means 3 units in the horizontal direction and 2 units in the vertical direction.

- 3D:



3. **Magnitude/Modulus** of vector:

- 2D:

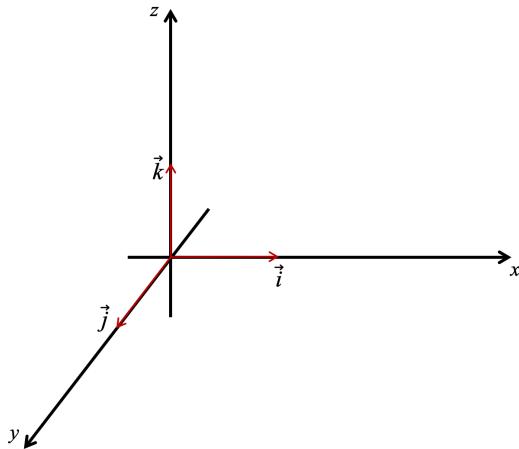
$$\text{For } \vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}, |\vec{a}| = \sqrt{x^2 + y^2}.$$

- 3D:

$$\text{For } \vec{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, |\vec{b}| = \sqrt{x^2 + y^2 + z^2}.$$

4. **Unit Vector**: A vector of length 1:

- \vec{i} : unit vector on the x -axis.
- \vec{j} : unit vector on the y -axis.
- \vec{k} : unit vector on the z -axis.



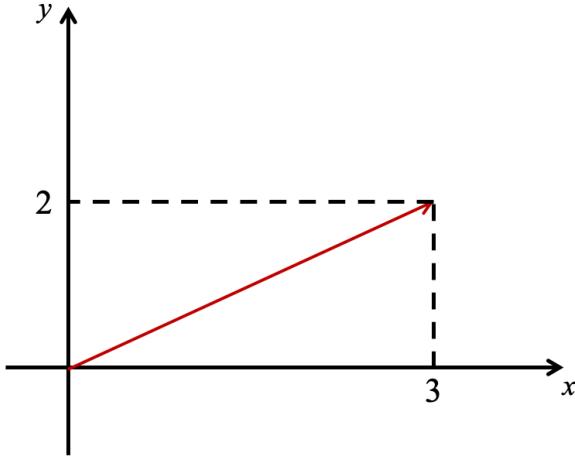
5. Sum of vectors:

- **Position vector**: A vector that has an initial point at the origin.

Example 3.2.2

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\vec{a} = 3\vec{i} + 2\vec{j}.$$



- Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$
- $$\vec{a} + \vec{b} = \begin{pmatrix} x+m \\ y+n \end{pmatrix}.$$

6. Multiplication of vectors by a scalar:

Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and n be a scalar:

$$n\vec{a} = n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} nx \\ ny \end{pmatrix}.$$

$n\vec{a}$ and \vec{a} are in the same direction \Rightarrow parallel.

7. Subtracting a vector:

Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$.

$$\vec{a} - \vec{b} = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

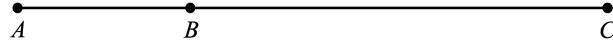
Proof 3.2.1

$$-\vec{b} = (-1)\vec{b} = \begin{pmatrix} -m \\ -n \end{pmatrix}$$

$$\vec{a} - \vec{b} = \vec{a} + \begin{pmatrix} -\vec{b} \end{pmatrix} = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

8. **Zero vector:** $\vec{0}$.

9. **Collinear points:** three points, A , B , and C , are said to be collinear if $\vec{AB} = t\vec{AC}$.

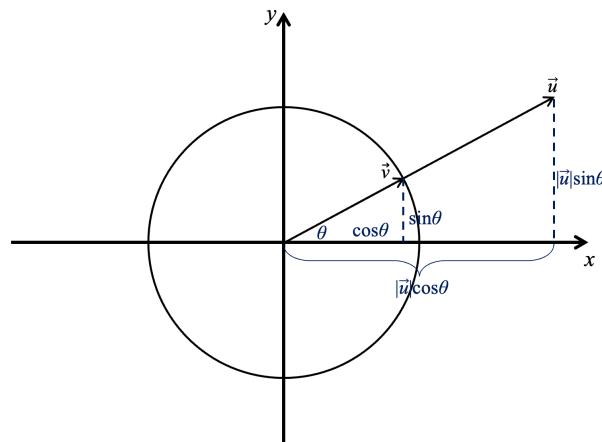


10. Find a unit vector parallel to $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$.

- Find the value $|\vec{u}|$.
- Then, the unit vector parallel to \vec{u} is

$$\vec{v} = \frac{\vec{u}}{|\vec{u}|}.$$

11. Vectors and unit circle:



θ is the angle with the horizontal axis. The unit vector \vec{v} , in the same direction as \vec{u} is:

$$\vec{v} = \cos \theta \cdot \vec{i} + \sin \theta \cdot \vec{j}$$

$$\begin{aligned} \vec{v} &= \frac{1}{|\vec{u}|} \cdot \vec{u} \Rightarrow \vec{u} = |\vec{u}| \cdot \vec{v} = |\vec{u}| \cos \theta \cdot \vec{i} + |\vec{u}| \sin \theta \cdot \vec{j} \\ &= |\vec{u}| \left(\cos \theta \cdot \vec{i} + \sin \theta \cdot \vec{j} \right). \end{aligned}$$

3.2.2 Scalar Product and Its Properties

1. The **scalar product** of two vectors is a real number (scalar).

- The algebraic definition:

For $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

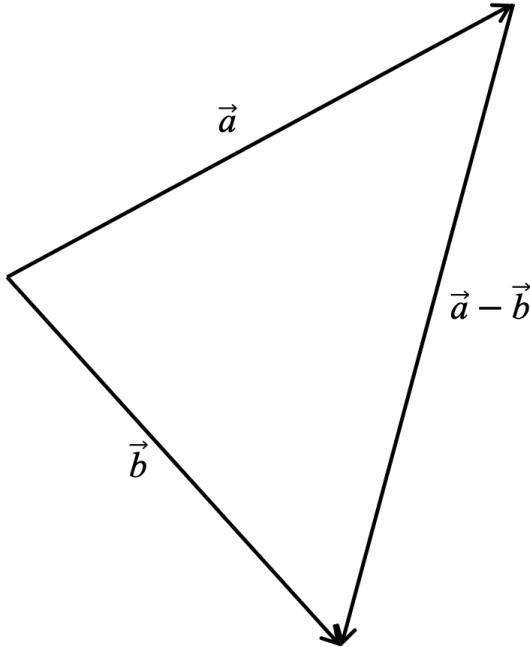
$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2.$$

The scalar product is also called the dot product.

- The geometric definition:

For \vec{a} and \vec{b} ,

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta, \quad \theta \text{ is the angle between the two vectors.}$$



Proof 3.2.2 By cosine rule:

$$\begin{aligned} |\vec{b} - \vec{a}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos \theta \\ |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{a}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos \theta \\ \therefore \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos \theta. \end{aligned}$$

- Combining the two definitions:

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2}{\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}}.$$

2. 3-D vectors: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

•

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}.$$

3. Properties of scalar product:

- If $\vec{a} \cdot \vec{b} = 0 \Rightarrow \begin{cases} \vec{a} = 0 \\ \vec{b} = 0 \\ \vec{a} \text{ and } \vec{b} \text{ are perpendicular (orthogonal)} \Rightarrow \theta = \frac{\pi}{2} \end{cases}$
 - If \vec{a} and \vec{b} are collinear,
- $$\vec{a} \cdot \vec{b} = \pm |\vec{a}| |\vec{b}|.$$

Proof 3.2.3 Angel between \vec{a} and \vec{b} is 0° .

$\cos 0^\circ = 1 \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$ for \vec{a}, \vec{b} at the same direction.

OR $\vec{a} \cdot \vec{b} = -|\vec{a}| |\vec{b}|$ for \vec{a} and \vec{b} at opposite directions.

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

Proof 3.2.4

$$\vec{a} \cdot \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = a_1^2 + a_2^2 = |\vec{a}|^2.$$

- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.
- $\lambda (\vec{a} \cdot \vec{b}) = (\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b})$.

3.2.3 Vector Equation of a Line

1. There is only one line that passes through two distinct points.

Theorem 3.2.1 In the coordinate plane, the equation can be found as:

For $A(x_1, y_1)$ and $B(x_2, y_2)$, the line passes through A, B is given by

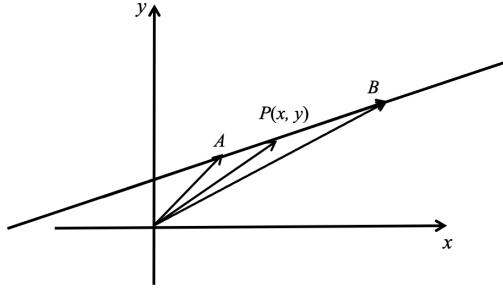
$$y = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) + y_1.$$

2. **Slope, y-intercept form:** $y = mx + k$, where m is the slope, and k is the y -intercept.

It can be rearranged to $ax + by = c$; $a, b, c \in \mathbb{R}$, where a and b cannot be equal to 0 at the same time.

3. Vector form of a line:

- For every point $P(x, y)$ that lies on the line AB , the vector \vec{AP} must be collinear or parallel to \vec{AB} : $\vec{AP} = k\vec{AB}$, $k \in \mathbb{R}$.



(a) The vector \vec{AB} is called a **direction vector** of the line.

All the vectors that are parallel to \vec{AB} can also define the same line.

(b) Assume $\vec{OA} = \vec{a}$, $\vec{OP} = \vec{p}$, \vec{AB} is the direction vector \vec{d} . Then, $\vec{AP} = \vec{p} - \vec{a} = k\vec{AB} = k\vec{d}$

$$\vec{p} = \vec{a} + k\vec{d}, k \in \mathbb{R}.$$

- Vector equation of a line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, k \in \mathbb{R}.$$

- Parametric form:

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases}, k \in \mathbb{R}.$$

- Cartesian form:

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}.$$

Proof 3.2.5

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases} \Rightarrow \begin{cases} k = \frac{x - x_1}{d_1} \\ k = \frac{y - y_1}{d_2} \end{cases}.$$

(a) Cartesian form can be further rearranged to slope-intercept form

$$\begin{aligned} \frac{x - x_1}{d_1} &= \frac{y - y_1}{d_2} \\ \frac{d_2}{d_1} (x - x_1) &= y - y_1 \\ y &= \frac{d_2}{d_1} (x - x_1) + y_1, \end{aligned}$$

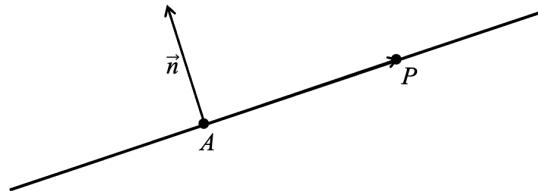
where $\frac{d_2}{d_1}$ is the slope.

(b) Another way of interpretation:

$$\begin{aligned}
 \overrightarrow{AP} = k\overrightarrow{AB} &\Rightarrow \vec{p} - \vec{a} = k(\vec{b} - \vec{a}) \\
 &\Rightarrow \vec{p} = (1-k)\vec{a} + k\vec{b}, \quad k \in \mathbb{R}. \\
 &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = (1-k) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad k \in \mathbb{R}. \\
 &\Rightarrow \begin{cases} x = (1-k)x_1 + kx_2 = x_1 + k(x_2 - x_1) \\ y = (1-k)y_1 + ky_2 = y_1 + k(y_2 - y_1) \end{cases}, \quad k \in \mathbb{R}. \\
 &\Rightarrow \begin{cases} k = \frac{x-x_1}{x_2-x_1} \\ y = y_1 + k(y_2 - y_1) \end{cases} \\
 &\Rightarrow y = y_1 + \frac{x-x_1}{x_2-x_1}(y_2 - y_1) \\
 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.
 \end{aligned}$$

4. Orthogonal / Perpendicular vector of a line.

- There is one and only one line in the plane that is perpendicular to a given line at a particular point on that line.
- Normal Vector:



Definition 3.2.2 A **normal vector** is perpendicular or **orthogonal** to any vector on the lines.

$$\text{i.e., } \vec{n} \cdot \overrightarrow{AP} = 0.$$

Theorem 3.2.2

$$\vec{n} \cdot (\vec{p} - \vec{a}) = 0 \Rightarrow \vec{n} \cdot \vec{p} = \vec{n} \cdot \vec{a}.$$

- If the direction vector $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, then one possible normal vector would be $\vec{n} = \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$ or any other vectors parallel to it.

- The vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$$

$$\Rightarrow xd_2 - yd_1 = x_1d_2 - y_1d_1$$

$$(x - x_1)d_2 = yd_1 - y_1d_1$$

$$\therefore y = \frac{d_2}{d_1}(x - x_1) + y_1.$$

5. Direction vectors:

- **Parallel lines** have **collinear** direction vectors.
- **Perpendicular lines** have **orthogonal** direction vectors, such that the scalar product is equal to 0.

6. Vector equation of lines in 3-D spaces:

-

$$\vec{r} = \vec{a} + \lambda \vec{d}, \lambda \in \mathbb{R}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

- The parametric form:

$$\begin{cases} x = a_1 + \lambda d_1 \\ y = a_2 + \lambda d_2 \\ z = a_3 + \lambda d_3 \end{cases}, \lambda \in \mathbb{R}.$$

- The Cartesian form:

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$

7. Two lines:

- 2-D spaces: two distinctive lines can either be parallel or they can intersect.
- 3-D spaces:
 - (a) Lines are parallel.
 - (b) Lines intersect at one common points.
 - (c) Lines are **skewed** (do not intersect and are not parallel).

3.2.4 Vector Product and Properties

1. The vector product is an operation that takes two vectors and results in another **vector**.
 - Definition

Definition 3.2.3 Given the two vectors and their components, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then the **vector product** is given by:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

- The vector product of two vectors is another vector that is perpendicular to both vectors.
- Magnitude of the vector product:

Theorem 3.2.3 The magnitude of the vector product is given by the formula

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin \theta,$$

where θ is the angle between those two vectors. If $\vec{a} \times \vec{b} = 0$, then \vec{a} and \vec{b} are parallel/-collinear.

- The geometrical definition of cross product (vector product):

Theorem 3.2.4 Given two vectors \vec{a} and \vec{b} , then the vector product is given by

$$\vec{a} \times \vec{b} = (|\vec{a}| |\vec{b}| \sin \theta) \hat{n},$$

where \hat{n} is the unit vector whose direction is given by the right-hand screw rule to both \vec{a} and \vec{b} and the vectors \vec{a} , \vec{b} , and \hat{n} follows the right-hand rule.

- Geometrical meaning of the magnitude of the vector product:
It is equal to the area of the parallelogram enclosed by those two vectors.

2. Properties of the vector product:

- $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
- $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$
- $\lambda (\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}), \lambda \in \mathbb{R}$
- $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}).$

3. Mixed product:

- An operation with three vectors \vec{a} , \vec{b} , and \vec{c} combining both the vector and scalar product is called a **mixed product**:

$$(\vec{a} \times \vec{b}) \cdot \vec{c}.$$

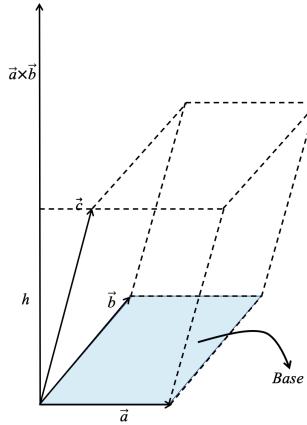
- Given $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, the mixed product is given by:

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot \vec{c} &= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \end{aligned}$$

- Geometric meaning of mixed products:

The volume of a parallelepiped formed by three non-coplanar vectors, \vec{a} , \vec{b} , and \vec{c} is given by:

$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|.$$



Proof 3.2.6

$$V = \text{Base} \times h$$

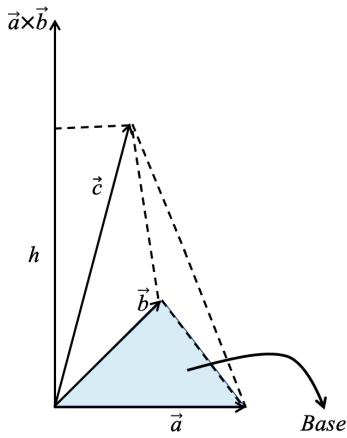
Base=magnitude of cross product of \vec{a} and \vec{b} .

= perpendicular projection of \vec{c} to $\vec{a} \times \vec{b}$.

$$\therefore V = \text{Base} \times h = |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cdot |\cos \theta| = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

- Three or more vectors are said to be coplanar if they lie in the same plane.
- Using mixed product to find the volume of a triangular pyramid:

$$V = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|.$$



Proof 3.2.7 Since the base is not a parallelogram but a triangle, that is half an area of the parallelogram, we multiply $\frac{1}{2}$ in front of the expression of the cross product.

$$\text{Base} = \frac{1}{2} |\vec{a} \times \vec{b}|.$$

The volume of a pyramid is $\frac{1}{3}$ of the product of the base and the height.

$$\therefore V = \frac{1}{3} \text{Base} \cdot h = \frac{1}{3} \cdot \frac{1}{2} |\vec{a} \times \vec{b}| |\vec{c}| |\cos \theta| = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

4. Proving vector product using matrix.

Proof 3.2.8 Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Convert into a 3×3 matrix: $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$.

Find the determinant: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$

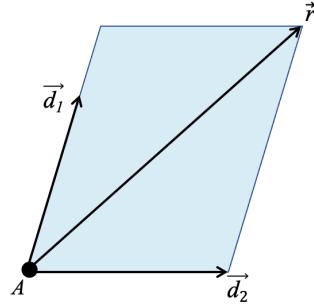
$$\Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

3.2.5 Vector Equation of a Plane

1. A plane is uniquely determined by **three points** (or a line and a point outside the line).
→ A plane can also be determined by two intersecting lines and a point outside the lines.
2. Vector equation of a plane:

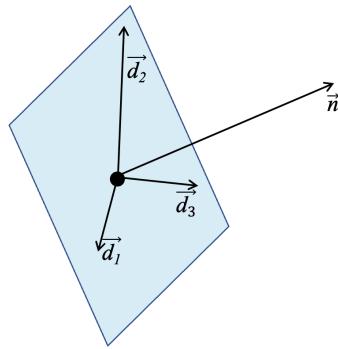
$$\vec{r} = \vec{a} + \lambda \vec{d}_1 + \mu \vec{d}_2, \lambda, \mu \in \mathbb{R}.$$

where \vec{d}_1 and \vec{d}_2 are direction vectors, and \vec{a} is the position vector.



3. The scalar product form:

- **Normal vector** is a vector that is perpendicular to every line in the plane.



- All planes with the same normal vector are parallel to each other.
- If R is any other point on the plane, then AR lies in the plane, and it is perpendicular to the normal vector n.

Theorem 3.2.5

$$\overrightarrow{AR} \cdot \vec{n} = 0 \Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\therefore \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

where \vec{a} is the position vector, and \vec{n} is the normal vector.

4. The Cartesian equation of a plane:

$$n_1x + n_2y + n_3z = d, \text{ where } n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, d = \vec{a} \cdot \vec{n}.$$

Proof 3.2.9

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, d = \vec{a} \cdot \vec{n}, \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The scalar product form converts to:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \vec{a} \cdot \vec{n}$$

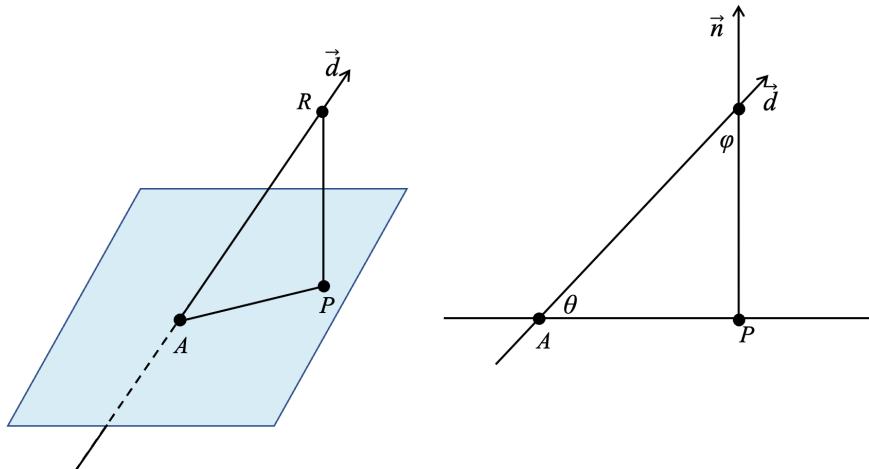
$$\Rightarrow n_1x + n_2y + n_3z = d.$$

5. A plane with the vector equation $\vec{r} = \vec{a} + \lambda\vec{d}_1 + \mu\vec{d}_2$ has a normal vector $\vec{n} = \vec{d}_1 \times \vec{d}_2$.

3.2.6 Lines, Planes, and Angles

1. Angles and intersections between lines and planes:

- When a line intersects a plane, the angle between them is defined as the **smallest possible angle** that the line makes with any of the lines in the plane.



- (a) \overrightarrow{AR} : the direction vector of the line, \vec{d} .
- (b) Point P is the projection of point R onto the plane. \overrightarrow{AP} is the shadow of \overrightarrow{AR} on the plane.
- (c) \overrightarrow{PR} is in the direction of \vec{n} since it is perpendicular to the plane.
- (d) φ is the angle between \vec{n} and \vec{d} .
- (e)

$$\theta = 90^\circ - \varphi, \cos \varphi = \frac{|\vec{n} \cdot \vec{d}|}{|\vec{n}| |\vec{d}|}.$$

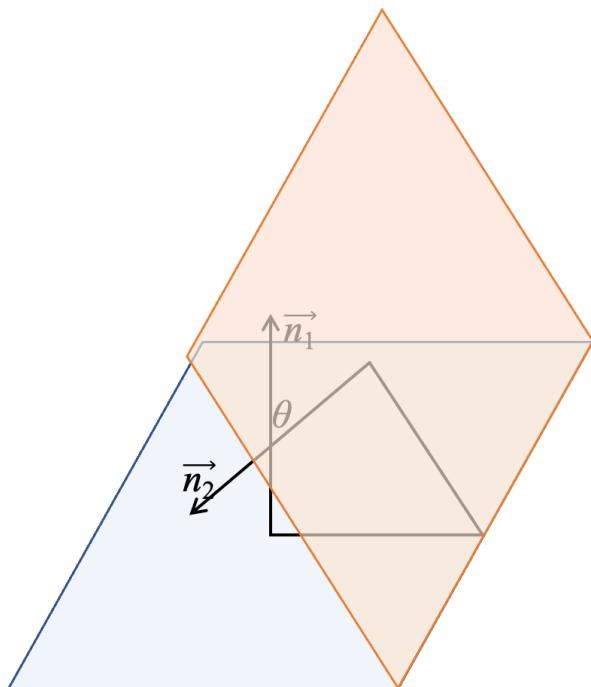
- A line that is not parallel to a plane intersects a plane at one point. The coordinates of this point of intersection satisfies both the equation of the line and the equation of the plane.

2. Relationship of two planes:

- Two planes can either intersect at a line or they can be parallel.
- When two planes are parallel, their normal vectors are **collinear**; otherwise they intersect at a line.

3. Angles between two planes:

- The angle between two planes is **the angle between their normal vectors**.
 -
- $$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}.$$

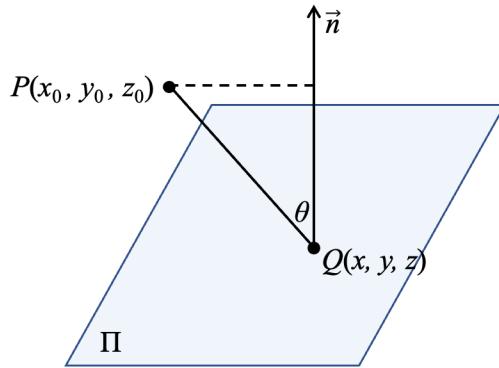


4. Two non-parallel planes intersect along a line. The equation of this line is formed by treating the Cartesian equation of two planes as simultaneous equations and finding the general solution.
5. Distance between a point and a plane.

- The distance, d , between a point $P(x_0, y_0, z_0)$, and a plane with equation $Ax + By + Cz = D$ where $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$, is given by:

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

- Proof:



Proof 3.2.10 Let $Q(x, y, z)$ be any point on the plane Π .

The distance, d , is the projection of the distance of point P to the plane on the normal vector, \vec{n} .

$$\begin{aligned}
 d &= |\overrightarrow{QP}| \cdot |\cos \theta| = |\overrightarrow{QP}| \cdot \frac{\overrightarrow{QP} \cdot \vec{n}}{|\overrightarrow{QP}| \cdot |\vec{n}|} \\
 &= \frac{\overrightarrow{QP} \cdot \vec{n}}{|\vec{n}|} = \frac{|\langle A, B, C \rangle \cdot \langle (x_0 - x), (y_0 - y), (z_0 - z) \rangle|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{|Ax_0 + By_0 + Cz_0 - (Ax + By + Cz)|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}
 \end{aligned}$$

6. Intersection of three points:

Unique solution	Infinitely many solutions	No solutions (inconsistent system)		
		No normals parallel	Two normals parallel	Three normals parallel
Three planes intersect at a point	Three planes intersect along a line	Three planes form a prism	One plane cutting two parallel planes	Three parallel planes

- The plane intersect:
 - (a) At a point: the system of equations will have a unique solution.
 - (b) Along a line: the system of equations will have infinitely many solutions
- The systems of equations have no solutions:
 - (a) No normals are parallel (the planes from a prism)
 - (b) 2 normals are parallel or three normal are parallel (the planes are parallel)

4 Statistics and Probability

4.1 An Introduction to Statistics

1. Statistical inferences: when the sample data is well chosen and described, an analysis will allow us to draw conclusions about the population based on this sample.

- **Discrete data** is data that can be counted; it gives the number of times something occurs or the number of items that exists.
- **Continuous data** is data that is measured; however, the values of the actual data cannot be determined exactly, and the data may be limited to a range.

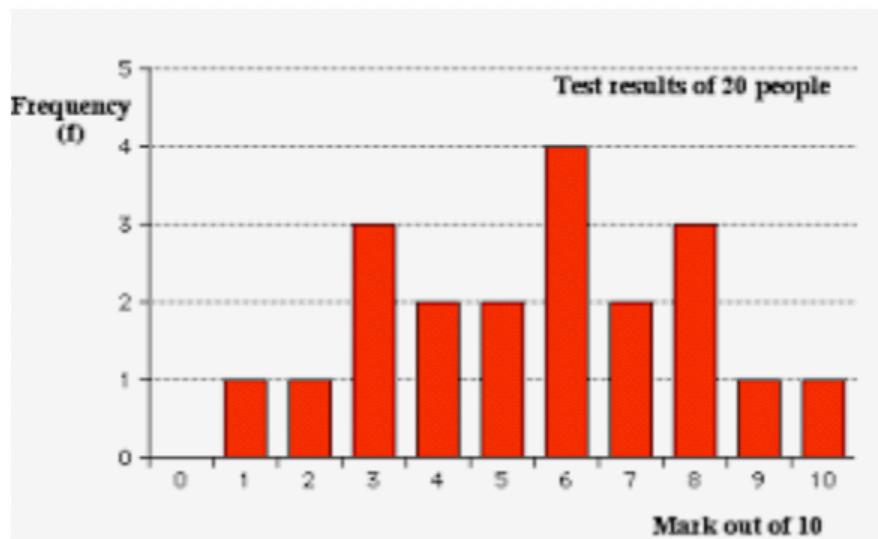
2. Reliability and Bias.

3. Data Sampling (Sampling Methods):

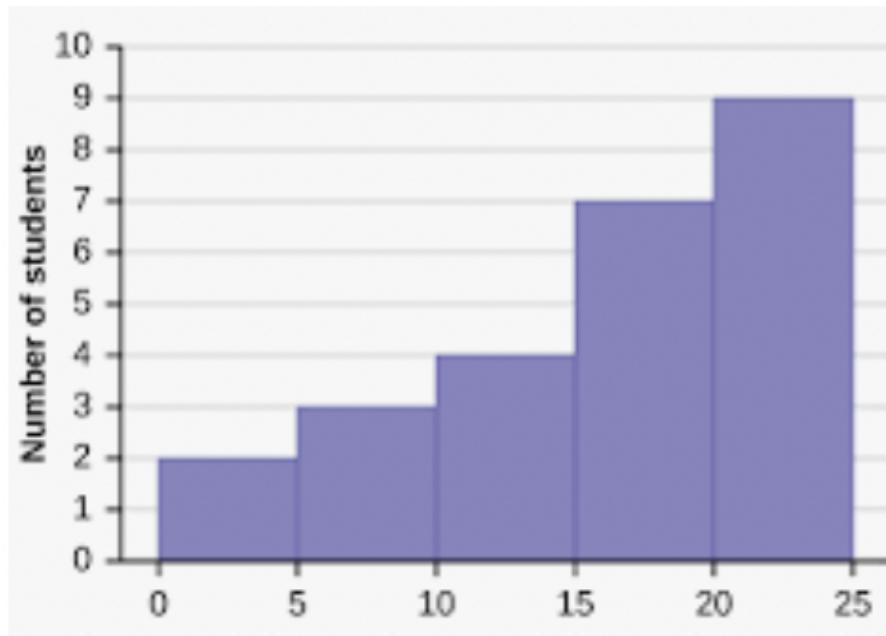
- **Simple:** Achieving randomness by a simple, completely random process.
- **Convenience:** Choosing a sample based on how easy it is to find the data.
- **Systematic:** If data is listed, selecting a random starting point and then choosing the rest of the sample at a consistent interval in the list.
- **Quota:** Choosing a sample that is only comprised of members of the population that fit certain characteristics.
- **Satisfied:** Choosing a random sample in a way that the population of certain characteristics matches the proportion of those characteristics in the population.

4. Grouped Data:

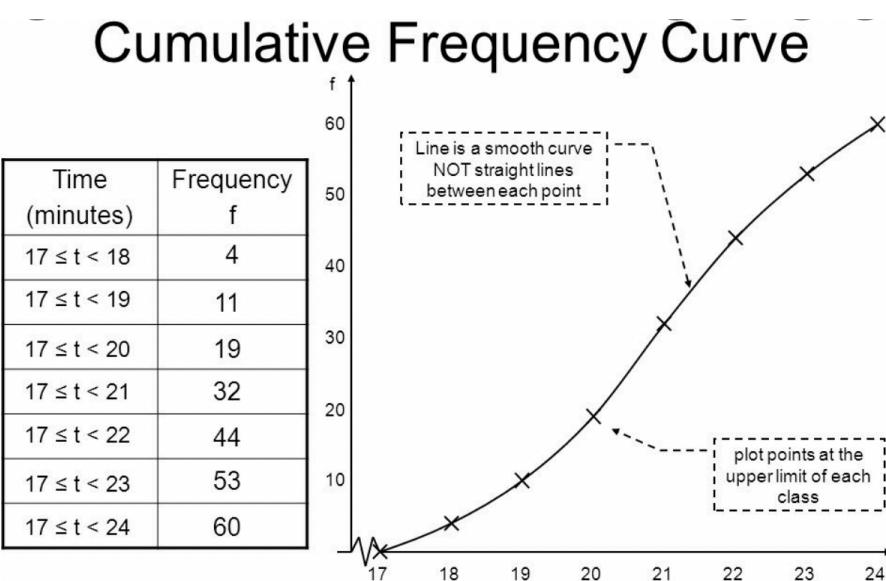
- Key terms: intervals/classes, frequency distribution table, frequency diagram.
- Frequency diagram of discrete data:



- Frequency diagram of continuous data - **Histogram**



- Cumulative frequency table and cumulative frequency graph:



- For data grouped into intervals or classes, we can identify:
 - Mid-interval values
 - Interval width
 - Lower interval boundaries
 - Higher interval boundaries
 - Modal class: the class with the highest frequency

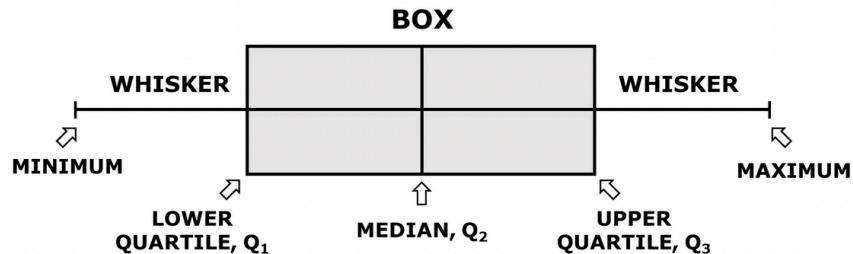
5. Quartiles:

- Minimum: the lowest value
- Q_1 : the 25th percentile
- Median (Q_2): the 50th percentile

- Q_3 : the 75th percentile
- Maximum: the highest value
-

$$\text{Interquartile Range (IQR)} = Q_3 - Q_1.$$

- Box-and-whisker plots: → spread of data:



6. Normal distribution, negatively skewed, and positively skewed:

7. Normal distribution: mean = mode = median

8. Positively skewed: median < mean

9. Negatively skewed: median > mean

10. Outliers:

$$\text{Outlier} < Q_1 - 1.5\text{IQR}$$

$$\text{OR} \quad \text{Outlier} > Q_3 + 1.5\text{IQR}.$$

11. Measuring central tendency:

- **Definition 4.1.1 Mode**: the value with the greatest frequency.
- **Definition 4.1.2 Median(m)**: the middle value when the data is arranged in order.
- **Definition 4.1.3 Mean(μ)**: the arithmetic mean, the sum of the numerical data divided by the number of data points.

$$\mu = \frac{\sum_{i=1}^n x_i}{n}.$$

12. Measures of dispersion:

- **Definition 4.1.4 Range**:

$$\text{Range} = x_{\max} - x_{\min}.$$

- **Definition 4.1.5 Interquartile Range (IQR)**:

$$\text{IQR} = Q_3 - Q_1.$$

- **Definition 4.1.6 Variance σ^2 and standard deviation σ** :

$$\sigma^2 = \frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - \mu^2$$

$$\sigma = \sqrt{\sigma^2}$$

Proof 4.1.1

$$\begin{aligned}
 \sigma^2 &= \frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - 2\mu \frac{\sum_{i=1}^k f_i x_i}{n} + \mu^2 \frac{\sum_{i=1}^k f_i}{n} \\
 &= \frac{\sum_{i=1}^k f_i x_i^2}{n} - 2\mu^2 + \mu^2 \\
 &= \frac{\sum_{i=1}^k f_i x_i^2}{n} - \mu^2
 \end{aligned}$$

4.2 Linear Correlation of Bivariate Data

1. Scatter Plot

2. Linear correlation:

- **Positive linear correlation:** the line as a general upward trend.
- Negative correlation.
- Quadratic trend
- Exponential trend

3. Estimate a line of best fit:

Theorem 4.2.1 The line must pass through the mean point of the data set (\bar{x}, \bar{y}) , where \bar{x} is the mean of x and \bar{y} is the mean of y .

4. Pearson's correlation coefficient (r):

- Range: $-1 \leq r \leq 1$
 - (a) $r = -1$: perfect negative linear correlation
 - (b) $r = 1$: perfect positive linear correlation
 - (c) $r = 0$: no linear correlation
 - (d) $0.7 < |r| \leq 1$: strong linear correlation
 - (e) $0.3 < |r| \leq 0.7$: weak to moderate linear correlation

• Formula:

$$\begin{aligned}
 r &= \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}} \\
 r &= \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{\sum x^2 - n\bar{x}^2} \sqrt{\sum y^2 - n\bar{y}^2}}
 \end{aligned}$$

5. Prediction using the least square linear regression:

- The dangers of extrapolation.
- We cannot always reliably make a prediction of x from a value of y , using a y on x line.

4.3 Probability and Expected Outcomes

1. **Definition 4.3.1 Random Events:** although we don't know the exact result in advance, we do know the set of all possible results.
 - The set of all possible results is called the **sample space** (U).
 - The number of all possible outcomes that make up the sample space is denoted as $n(U)$.
 - Any subset of a sample space is called an **event**. The event consists of one or more outcomes.
 - Each time an experiment is repeated, it is considered a **trial** of the experiment.
2. **Experimental probability** (relative frequency) is found by repeating an experiment a number of times and counting the number of times that particular outcome occurs.
3. **Definition 4.3.2 Mutually exclusive:** two events that do not share any outcomes → They cannot occur together.
4. **Definition 4.3.3 Theoretical probability:** When running an experiment for N trials that results in an event A occurring $n(A)$ times, the probability of A happening, $P(A)$ is

$$P(A) = \lim_{N \rightarrow \infty} \frac{n(A)}{N}.$$

If an experiment has equally likely outcomes, the probability of event A occurring is defined as

$$P(A) = \frac{n(A)}{n(U)} = \frac{\text{Number of outcomes in which } A \text{ occurs}}{\text{Total number of outcomes in the sample space}}.$$

5. Axioms:

Axiom 4.3.1 For any event A ,

$$0 \leq P(A) \leq 1.$$

Axiom 4.3.2 The probability of nothing (\emptyset) occurring is zero:

$$P(\emptyset) = 0.$$

The probability of one of all the outcomes in the sample space occurring is one:

$$P(U) = 1.$$

Axiom 4.3.3 If A and B are $\in U$ and are mutually exclusive, then the probability of either A or B happening is

$$P(A \cup B) = P(A) + P(B).$$

Axiom 4.3.4 If $P(A')$ is the probability of event A not happening,

$$P(A') = 1 - P(A),$$

where $P(A)$ is the probability of event A occurring.

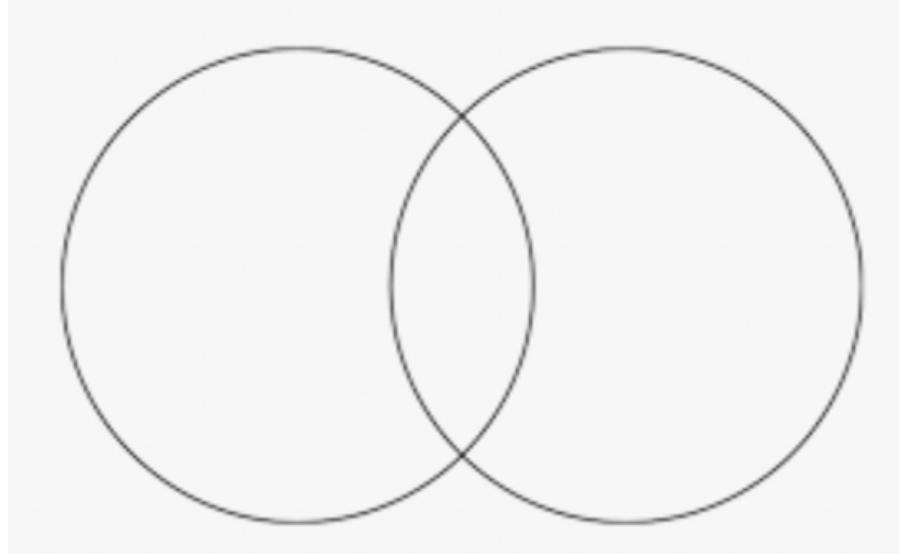
6. **Definition 4.3.4** Expectation: The formula for expected number of members of group A in a sample of size n is

$$nP(A).$$

If choosing n from the population, $nP(A)$ gives an approximation of how many of them would be members of group A .

4.4 Probability Calculations

1. **Definition 4.4.1** A **Venn diagram** is a model illustrating two or more sets of data using overlapping circles to show elements of each set.



- **Definition 4.4.2** The probability of event A and B occurring at the same time:

$$P(A \cap B)$$

- **Definition 4.4.3** The probability of event A or B occurring (A union B):

$$P(A \cup B)$$

- **Theorem 4.4.1** If $P(A \cup B) = \emptyset$, then A and B are **mutually exclusive**.

2. The tree diagram
3. **Theorem 4.4.2** The probability of a union:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof 4.4.1

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B), \\ P(A \cup B) &= \frac{n(A \cup B)}{n(U)} = \frac{n(A) + n(B) - n(A \cap B)}{n(U)} \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

4. Conditional probability:

- Conditional probability shrinks the sample space and, therefore, increases the probability of an event occurring, unless the given information renders the event impossible.
- **Theorem 4.4.3**

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

where $P(A|B)$ is read as probability of event A occurring, given event B occurring.

Proof 4.4.2 If we know that B has occurred, the sample space now contains all the elements of B but no more. Now, we select events from the sample space that falls in A :

$$\begin{aligned} P(A|B) &= \frac{n(A \cap B)}{n(B)} \\ &= \frac{\frac{n(A \cap B)}{n(U)}}{\frac{n(B)}{n(U)}} = \frac{P(A \cap B)}{P(B)}. \end{aligned}$$

5. The probability of independent events:

- **Definition 4.4.4** A and B are **independent events** if the occurrence of A has no effect on the probability of B occurring.

$$P(B|A) = P(B) \quad \text{OR} \quad P(A|B) = P(A).$$

- **Theorem 4.4.4** A and B are independent if and only if

$$P(A \cap B) = P(A) \times P(B).$$

6. **Theorem 4.4.5** Probability of union when A and B are mutually exclusive:

$$P(A \cup B) = P(A) + P(B).$$

4.5 Discrete Random Variables

1. **Definition 4.5.1** If a random variable can take exactly N values, each of which corresponds to a unique outcome in the sample space of an experiment, then this variable is a **discrete random variable**.

- The probability that x takes on any of the N values is written as $P(X = x)$
- A **probability distribution** is a combination of the sample space of a random experiment with the probabilities of each of the events in the sample space.
- PDF: probability distribution function
- **Axiom 4.5.1** For each value $\{x_i\}$ of the random variable X , we have that

$$0 \leq P(X = x_i) \leq 1.$$

- **Axiom 4.5.2** As X may take on any of the N values of the sample space, the sum of the probabilities of each of them as an outcome of the experiment must equal one:

$$\sum_{i=1}^N P(X = x_i) = P(X = x_1) + P(X = x_2) + P(X = x_3) + \cdots + P(X = x_N) = 1.$$

- **Definition 4.5.2 Well-defined probability distribution:** a probability distribution that satisfies both

$$\begin{cases} 0 \leq P(X = x_i) \leq 1 \\ \sum_{i=1}^N P(X = x_i) = 1 \end{cases}$$

2. Calculating expected value:

- **Definition 4.5.3** The **expected value** (expected mean) is the value you would expect to obtain on average if you performed an experiment many times.
- The expected value of a random variable X with a probability distribution function $P(X = x)$ is written as $E(X)$ or μ .
- Formula:

$$\begin{aligned} E(X) &= \mu = \sum_{i=1}^N x_i P(X = x_i) \\ &= x_1 P(X = x_1) + x_2 P(X = x_2) + x_3 P(X = x_3) + \cdots + x_N P(X = x_N) \end{aligned}$$

- Application: fairness of a game

A fair game is perfectly balanced so that the expected mean pay-out is 0 for both players.

4.6 The Binomial Distribution

1. The binomial distribution discrete the probability distribution of different outcomes of repeated binary events.

- Since there are only two outcomes, we use p to represent the probability of success, meaning the probability that the event we are looking for occurs.
- We use $X \sim$ to denote a random variable X that is distributed in a certain way.
- $X \sim B(n, p)$ represents the binomial distribution, where $mathrm{B}$ stands for binomial, n represents the number of trials in the binomial experiment, and p is the probability of success.
- The probability of failure is q :

$$q = 1 - p.$$

- **Theorem 4.6.1** Expected value of a binomial distribution:

$$E(X) = np.$$

- **Theorem 4.6.2** Variance of a binomial distribution:

$$\text{Var}(X) = np(1 - p) = npq.$$

2. Probabilities within the binomial distribution:

Theorem 4.6.3 A random variable that is binomially distributed and takes on the value of the number of success in n trials is written as $X \sim B(n, p)$, where p is the probability of success in one trial. Then, the probability that X takes on an actual value of x successes is given by:

$$X \sim B(n, p) \Rightarrow P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, 2, 3, \dots, n.$$

3. Finding binomial probabilities using GDC:

- **Binomial PDF** returns a specific value of $P(X = x)$
- **Binomial CDP** (cumulative distribution function) gives the value of

$$P(0 \leq X \leq \text{upper bound}).$$

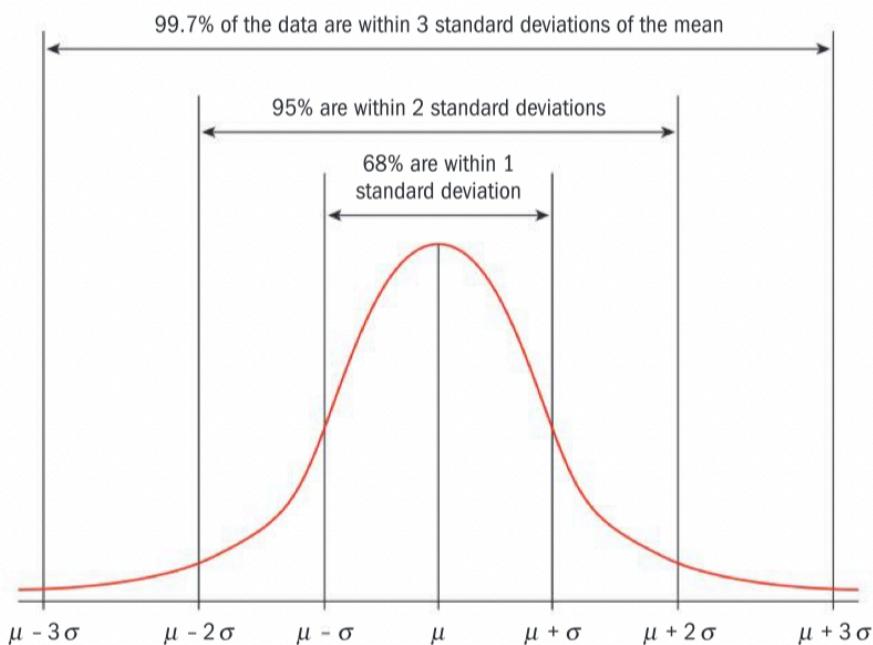
- **Theorem 4.6.4**

$$P(a \leq X \leq b) = P(0 \leq X \leq b) - P(0 \leq X \leq a - 1).$$

4.7 The Normal Distribution and Curve

1. The normal (bell) curve:

- The normal distribution is modeled graphically with a **normal curve**:



- The normal curve always has its highest point in the center, and that point is the mean (μ), median, and mode.
- The axis is often labeled using the standard deviation σ .
- The normal distribution deals with continuous random variables.

2. Expressing the normal distribution algebraically:

- Normal distribution is denoted as

$$X \sim N(\mu, \sigma^2),$$

where N means normal distribution, μ is the mean, and σ is the standard deviation.

- Normal probability density function:

(a) A probability density function is the equation of a curve that has the probabilities as the area underneath.

(b) **Definition 4.7.1** For $X \sim N(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \text{ where } -\infty < x < \infty.$$

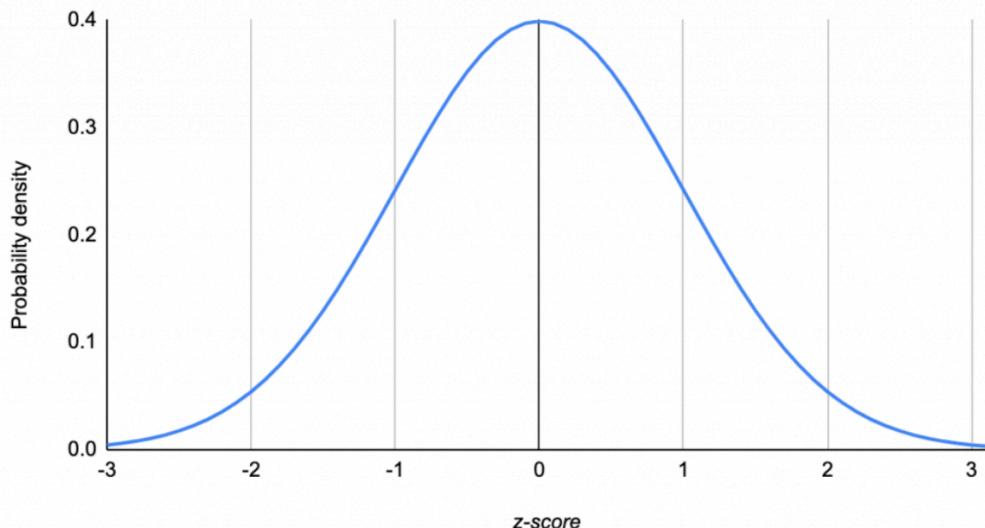
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx = 1.$$

3. Calculating normal distributed probabilities with GDC:

- Normal PDF and Normal CDF.
- Inverse Normal Function.

4. Standard Normal Distribution:

- Transform $X \sim N(\mu, \sigma^2)$ into $Z \sim N(0, 1)$:



- Z-score:

$$z = \frac{x - \mu}{\sigma}.$$

- Finding probabilities with z-scores:

(a)

$$P(-1 < z < 1) \approx 0.68$$

$$P(-2 < z < 2) \approx 0.95$$

$$P(-3 < z < 3) \approx 0.997$$

(b) Normal CDP on GDC

- Using z-score to find μ and σ .

4.8 Probability Density Function (PDF)

1. For a **continuous random variable** X with **probability density function** $f(x)$, it holds

$$\begin{cases} f(x) \geq 0 & \text{i.e., the function is non-negative} \\ \int_{-\infty}^{\infty} f(x)dx = 1 & \text{i.e., the total area under the curve is 1} \end{cases}$$

2. The probability that X takes values between a and b is

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

N.B.: $P(a \leq X \leq b) = P(a < X < b)$ because $P(X = a) = 0$.

3. The **mean μ** , or the **expected value $E(X)$** , is defined by

$$\mu = E(x) = \int_{-\infty}^{\infty} xf(x)dx.$$

4. The **Variance $Var(X)$** is defined by

$$Var(X) = E(X - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx.$$

An equivalent and more practical definition is

$$Var(X) = E(X^2) - \mu^2, \text{ where } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx.$$

5. Mode: the value of x where $f(x)$ has its maximum.

6. Median: The value of m where $P(X \leq m) = 0.5$

Example 4.8.1

$$\text{Let } f(x) = \begin{cases} f_1(x), a \leq x \leq b \\ f_2(x), b < x \leq c \end{cases}, \text{ we check } \int_a^b f_1(x)dx = A :$$

(a) If $A > 0.5$, the median $m \in [a, b]$, solve

$$\int_a^m f_1(x)dx = 0.5.$$

(b) If $A < 0.5$, the median $m \in [b, c]$, solve

$$\int_m^c f_2(x)dx = 0.5.$$

7. Quartiles:

- The **lower quartile Q_1** is defined by $P(X \leq Q_1) = 0.25$, i.e., solve

$$\int_{-\infty}^{Q_1} f(x)dx = 0.25.$$

- The **upper quartile Q_3** is defined by $P(X \leq Q_3) = 0.75$, i.e., solve

$$\int_{-\infty}^{Q_3} f(x)dx = 0.75.$$

8. The effects of linear transformations on the random variable X :

- $E(aX + b) = aE(X) + b$.
- $\text{Var}(aX + b) = a^2\text{Var}(X)$.

4.9 The Bayes' Theorem

1. Bayes' Theorem:

Theorem 4.9.1 Bayes' Theorem

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}.$$

Proof 4.9.1

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B|A)P(A)$$

$$\because P(A \cap B) = P(B \cap A),$$

$$\therefore P(A|B)P(B) = P(B|A)P(A).$$

$$\therefore P(A|B) = \frac{P(A)P(B|A)}{P(B)}.$$

2. Other versions of the Bayes' theorem:

•

$$P(B) = P(A \cap B) + P(A' \cap B) = P(A)P(B|A) + P(A')P(B|A').$$

- Substitute $P(B)$:

$$P(A|B) = \frac{P(B|A)P(A)}{P(A)P(B|A) + P(A')P(B|A')};$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(B)P(A|B) + P(B')P(A|B')}.$$

3. Bayes' theorem for three events:

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3). \end{aligned}$$

$$\begin{aligned} \therefore P(A_i|B) &= \frac{P(A_i)P(B|A_i)}{P(B)} \\ &= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}. \end{aligned}$$

5 Calculus

5.1 Limits

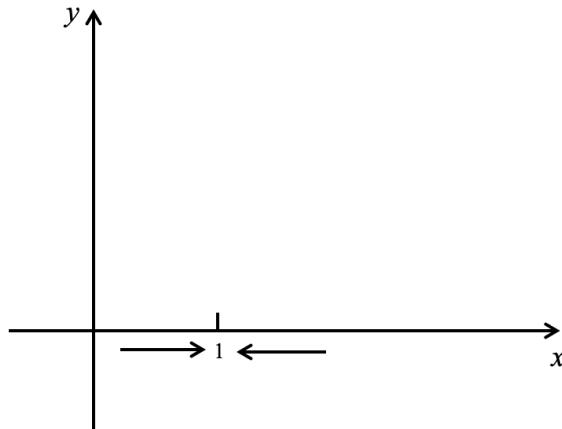
1. Limit

Example 5.1.1

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

when x is approaching to 1 (it never equals to 1), the value $\frac{x^2 - 1}{x - 1}$ is approaching to 2.

- Left-hand and Right-hand Limit



Example 5.1.2 The left-hand limit of $\frac{x^2 - 1}{x - 1}$ when $x \rightarrow 1$ is

$$\lim_{x \rightarrow 1^-} \frac{x^2 - 1}{x - 1} = 2.$$

The right-hand limit of $\frac{x^2 - 1}{x - 1}$ when $x \rightarrow 1$ is

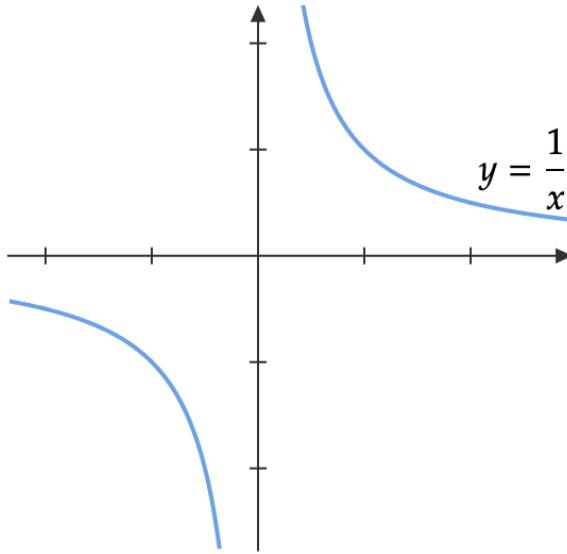
$$\lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = 2.$$

- Only when the left-hand limit and the right-hand limit exist and are the same at the point $x = a$, we say the limit of $f(x)$ exists on $x = a$.

i.e., $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = c \Rightarrow \lim_{x \rightarrow a} f(x) = c$, c is a constant $\in \mathbb{R}$

Otherwise, the limit does not exist on $x = a$ (OR DNE.).

Example 5.1.3 Does $\lim_{x \rightarrow 0} \frac{1}{x}$ exist? How about $\lim_{x \rightarrow \infty} \frac{1}{x}$?



- $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

$$\because \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty, \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$\therefore \lim_{x \rightarrow 0^+} \frac{1}{x} \neq \lim_{x \rightarrow 0^-} \frac{1}{x} \Rightarrow \text{DNE.}$$

- $\lim_{x \rightarrow \infty} \frac{1}{x}$ exists.

$$\because \lim_{x \rightarrow +\infty} \frac{1}{x} = 0, \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x \rightarrow +\infty} \frac{1}{x} = \lim_{x \rightarrow -\infty} \frac{1}{x} \Rightarrow \text{Limit exists.}$$

Definition 5.1.1 Horizontal Asymptote (H.A.):

$$y = \lim_{x \rightarrow \infty} f(x) = c$$

- Limit at ∞ :

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = c \Rightarrow \lim_{x \rightarrow \infty} f(x) = c.$$

Note: $+\infty$ and $-\infty$ are not exact values; they should be regarded as a concept.

- Limits do not have to equal to the function value.
Limit and the function value do not have relationships.
- Generally speaking, if $a \in D_f$, $\lim_{x \rightarrow a} f(x) = f(a)$.

2. For a rational function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x) = a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m$, and $Q(x) = b_0x^m + b_1x^{m-1} + \dots + b_{m-1}x + b_m$:

- $\lim_{x \rightarrow a} f(x) = f(a)$ as long as $Q(a) \neq 0$.
- $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m}{b_0x^m + b_1x^{m-1} + \dots + b_{m-1}x + b_m} \Rightarrow \text{H.A.}$
- (a) If $m = n$, $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_0}{b_0} = \frac{a_0}{b_0}$.

- (b) If $m > n$, $\lim_{x \rightarrow \infty} f(x)$ DNE.
(c) If $m < n$, $\lim_{x \rightarrow \infty} f(x) = 0$.

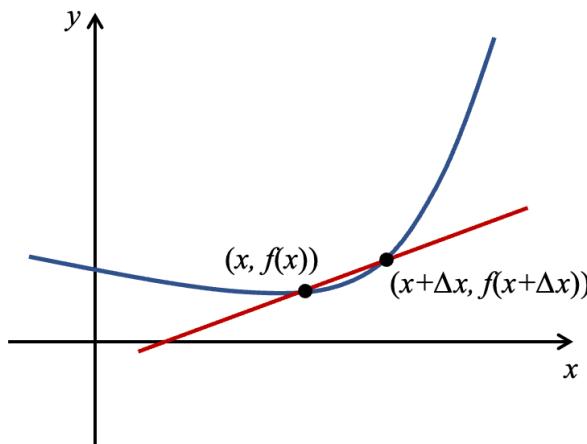
3. Continuity and Discontinuity

Definition 5.1.2 Continuity: If the graph of the function does not have any **breaks or holes** within a certain interval, then the function is continuous within that interval.

Theorem 5.1.1 If $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$, then the function f is **continuous** at $x = a$.

5.2 Differentiation and Derivatives

1. Gradient of Secant:



- Slope $m = \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

Definition 5.2.1 Derivative of a function:

$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ is the derivative of a function, denoted as $\frac{dy}{dx}$ or $f'(x)$.

- The graphic meaning of derivative is the gradient of tangent of the function.

Example 5.2.1 By definition, find the derivative of $f(x) = x^2 + 1$ and hence find the gradient of the tangent line when $x = 3$.

$$\begin{aligned}
f'(x) &= \lim_{x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{x \rightarrow 0} \frac{[(x + \Delta x)^2 + 1] - (x^2 + 1)}{\Delta x} \\
&= \lim_{x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x} \\
&= \lim_{x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{x \rightarrow 0} (2x + \Delta x) \\
&= 2x.
\end{aligned}$$

At $x = 3$, $f'(3) = 2 \times 3 = 6$. The gradient is 6.

2. Derivative of x^n

Theorem 5.2.1 If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}, \text{ for any } n \in \mathbb{R}.$$

Note: The derivative of any **constant** is **0**.

Example 5.2.2

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = (-1)x^{-1-1} = -x^{-2};$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}};$$

$$f(x) = c = cx^0 \Rightarrow f'(x) = 0 \times cx^{0-1} = 0.$$

3. Rules of Differentiation:

Name $f(x)$ and $g(x)$ as two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

Then

$$(cf(x))' = cf'(x)$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

4. More Derivatives:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

5. Differentiability:

Definition 5.2.2 A function has to be **continuous** and **no sharp turning point** to be **differentiable**.

Note: Smooth turning point on the graph is allowed.

6. More Rules of Differentiation:

Theorem 5.2.2 Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$(f(x) \times g(x))' = f'(x)g(x) + f(x)g'(x).$$

Theorem 5.2.3 Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$\left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Theorem 5.2.4 For a composite function $f(g(x))$ or $(f \circ g)(x)$, the derivative will be

$$f'(g(x)) \times g'(x).$$

OR

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

7. Higher Order Differentiation:

$$\frac{d^2y}{dx^2}, f''(x), f'''(x), f^{(4)}(x), f^{(5)}(x), \dots$$

5.3 Applications of Derivatives

1. Equation of Tangent Line:

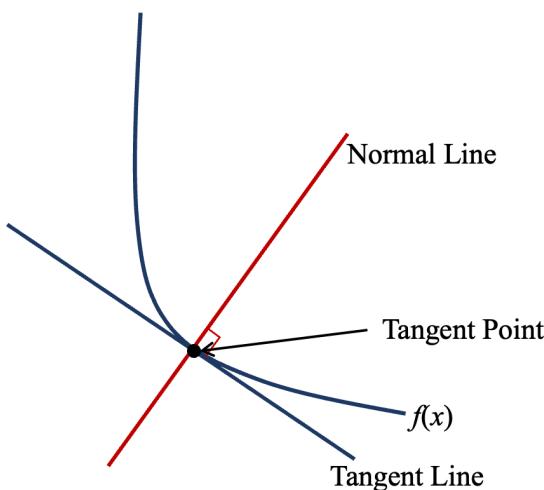
Via the original functions, we could get the tangent point (x_0, y_0) . Then, the expression of the tangent line is

$$y - y_0 = m(x - x_0),$$

where m is the derivative.

2. Normal and Tangent Lines:

Definition 5.3.1 **Normal** is perpendicular to the tangent and passes through the same tangent point.



3. Increasing and Decreasing Function:

Definition 5.3.2 Increasing Function: As x is getting larger, y is getting larger.

i.e.,

$$\frac{dy}{dx} > 0.$$

Decreasing Function: As x is getting larger, y is getting smaller.

i.e.,

$$\frac{dy}{dx} < 0.$$

4. Local Extrema: $\frac{dy}{dx} = 0$ Stationary point

Global extrema is the maximum and the minimum points of the entire function.

$f''(x)$ is used to determine if the local extrema are maxima or minima.

- Minima: $f''(x) > 0$ Concave up.
- Maxima: $f''(x) < 0$ Concave down.
- Point of Inflection (the point that is changing from concaving up to concaving down, or vice versa): $f''(x) = 0$

5. With local extrema, x -intercepts, y -intercepts, concavity, and asymptotes, draw approximate diagrams of a function.

5.4 Implicit Differentiation

1. When differentiating something with y , multiply $\frac{dy}{dx}$ at the end.
2. $(y^2)' = 2y \frac{dy}{dx}$.

Proof 5.4.1 If $u = y^2$, then

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}. \quad [\text{Chain Rule}]$$

Example 5.4.1 Find $\frac{dy}{dx}$ for the circle $x^2 + y^2 = 16$.

$$\begin{aligned} (x^2)' + (y^2)' &= (16)' \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \\ \frac{dy}{dx} &= -\frac{2x}{2y} = -\frac{x}{y}. \end{aligned}$$

Example 5.4.2 Find $\frac{dy}{dx}$ for $e^x + x \sin y = \cos 2y$.

$$\begin{aligned} (e^x)' + (x \sin y)' &= (\cos 2y)' \\ e^x + \left(\sin y + x \cos y \frac{dy}{dx} \right) &= -2 \sin 2y \frac{dy}{dx} \\ (-x \cos y - 2 \sin 2y) \frac{dy}{dx} &= e^x + \sin y \\ \frac{dy}{dx} &= \frac{e^x + \sin y}{-x \cos y - 2 \sin 2y}. \end{aligned}$$

3. Second Order Differentiation of Implicit functions*: Differentiate the first order differentiation.

Example 5.4.3 Find $\frac{d^2y}{dx^2}$ for the circle $x^2 + y^2 = 16$. (From Ex. 4.1: $2x + 2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$.)

$$(2x)' + \left(2y\frac{dy}{dx}\right)' = (0)' \Rightarrow 2 + \left((2y)'\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right)'\right) = 0 \Rightarrow 2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$$

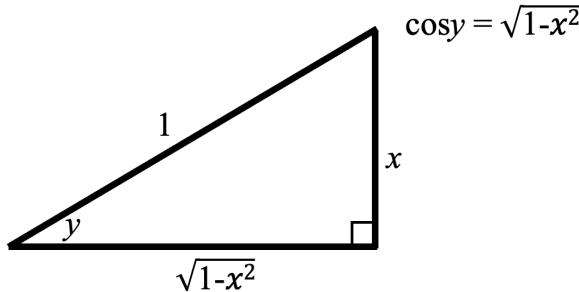
$$\frac{d^2y}{dx^2} = \frac{-2 - 2\left(\frac{dy}{dx}\right)^2}{2y} = \frac{-2 - 2\left(-\frac{x}{y}\right)^2}{2y}.$$

4. Derivative of Inverse Trigonometry Functions

Theorem 5.4.1

$$y = \arcsin x \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \quad \arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] (\cos y > 0).$$

Proof 5.4.2 From $y = \arcsin x$, we get $\sin y = x$. This situation can be illustrated by the figure below:



$$\begin{aligned} \therefore (\sin y)' &= (x)' \Rightarrow \cos y \frac{dy}{dx} = 1 \\ \therefore \frac{dy}{dx} &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

Theorem 5.4.2

$$y = \arccos x \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}, \quad \arccos x \in [0, \pi] (\sin y > 0).$$

$$y = \arctan x \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}.$$

Proof 5.4.3 (Hint: Try to visualize a similar diagram as in proof 4.1.)

From $y = \arccos x$, we get $\cos y = x$.

$$\therefore (\cos y)' = (x)' \Rightarrow -\sin y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

From $y = \arctan x$, we get $\tan y = x$.

$$\therefore (\tan y)' = (x)' \Rightarrow \sec^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \left(\frac{1}{\sqrt{1+x^2}} \right) = \frac{1}{1+x^2}.$$

5.5 Related Rate of Change

- When finding a rate of change of x , we are finding the $\frac{dy}{dx}$.

Example 5.5.1 Area of circle is increasing at a rate of 10π per second. When the radius is 2, what is the rate of change of radius?

Known: $\frac{dA}{dt} = 10\pi$, $r = 2$. Find: $\frac{dr}{dt}$.

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 10\pi$$

$$\frac{dr}{dt} = \frac{10\pi}{2\pi r} = \frac{5}{r}$$

$$\text{When } r = 2, \frac{dr}{dt} = \frac{5}{2}.$$

Example 5.5.2 A spherical balloon is expanding at a rate of 60π per second. How fast is the surface area of the balloon expanding when the radius is 4?

Known: $\frac{dV}{dt} = 60\pi$, $r = 4$. Find $\frac{dA}{dt}$.

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 3 \cdot \frac{4}{3}\pi r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\therefore 4\pi r^2 \frac{dr}{dt} = 60\pi \Rightarrow \frac{dr}{dt} = \frac{60\pi}{4\pi r^2} = \frac{15}{r^2}$$

$$A = 4\pi r^2 \Rightarrow \frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \cdot \frac{15}{r^2} = \frac{120\pi}{r}.$$

$$\text{When } r = 4, \frac{dA}{dt} = \frac{120\pi}{4} = 30\pi.$$

- Kinematics:

- Velocity, displacement, and acceleration are vector variables that have a value and a direction.
- Speed only has a value and no direction. It is a scalar variable. No sign should be reported in the answer.
- If s is the displacement, v is the velocity, a is the acceleration:

$$\frac{ds}{dt} = v; \frac{dv}{dt} = a.$$

5.6 More Limits - L'Hopital's Rule

Theorem 5.6.1 When the limit is in the **indeterminant form** ($\frac{0}{0}$ or $\frac{\infty}{\infty}$),

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] \lim_{x \rightarrow a} \left[\frac{f'(x)}{g'(x)} \right],$$

where $f'(x)$ and $g'(x)$ are the first derivatives of $f(x)$ and $g(x)$, respectively.

Example 5.6.1

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sec^2 x}{1} = 1$$

5.7 Indefinite Integration

1. Regard Integration as Anti-differentiation:

$$f'(x) = x \Rightarrow f(x) = \frac{1}{2}x^2 + C, \text{ where } C \text{ is a constant.}$$

$$f'(x) = x^2 \Rightarrow f(x) = \frac{1}{3}x^3 + C, \text{ where } C \text{ is a constant.}$$

$$f'(x) = x^n \Rightarrow f(x) = \frac{1}{n+1}x^{n+1} + C, \text{ where } C \text{ is a constant.}$$

Definition 5.7.1 Anti-differentiation is also called **indefinite integration**. It is denoted by $\int dx$.

$$\text{e.g. } \int x^n dx = \frac{1}{n+1}x^{n+1} + C.$$

2. General Rules of Integration.

- $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$
- $\int k dx = kx + C$
- $\int kf(x) dx = k \int f(x) dx$
- $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

3. $\int f'(x) dx = f(x) + C$. Therefore, if we know the $f'(x)$ and a point on the $f(x)$, which is to determine the constant C , then we can deduce the original function $f(x)$.

4. More Rules of Integration:

Differentiation	Integration
$(e^x)' = e^x$	$\int e^x dx = e^x + C$
$(\ln x)' = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + C$
$(\sin x)' = \cos x$	$\int \cos x dx = \sin x + C$
$(\cos x)' = -\sin x$	$\int \sin x dx = -\cos x + C$
$(\tan x)' = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$(\cot x)' = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$(\sec x)' = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$(\csc x)' = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$

5. Anti-chain Rule in Integration: We must **divide** the chain rule factor.

Example 5.7.1

$$\int (ax+b)^n \, dx = \frac{1}{a} \left(\frac{1}{n+1} (ax+b)^{n+1} \right) + C$$

$$\int e^{(ax+b)} \, dx = \frac{1}{a} e^{ax+b} + C$$

$$\int \frac{1}{(ax+b)} \, dx = \frac{1}{a} \ln(ax+b) + C$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$$

$$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$$

6. Integration Techniques: by Substitution and by Parts:

Theorem 5.7.1 Whenever we have an integration like: $\int g'(x) \times f(g(x)) \, dx$, we can always assume $u = g(x)$. Therefore, $du = g'(x) \cdot dx$. ($u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$):

$$\int g'(x) \times f(g(x)) \, dx = \int f(u \, du).$$

Theorem 5.7.2 If $f(x)$ and $g(x)$ are two functions, and $f'(x)$ and $g'(x)$ are their derivatives, respectively, integration by parts can be written as following:

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.$$

Example 5.7.2 Find $\int 2x(x^2 + 3)^5 \, dx$.

Since $2x = (x^2 + 3)'$, we consider to use integration by substitution.

Assume $u = x^2 + 3$, then $\frac{du}{dx} = (x^2 + 3)' = 2x \Rightarrow du = 2x \cdot dx$.

$$\begin{aligned} \therefore \int 2x(x^2 + 3)^5 \, dx &= \int (x^2 + 3)^5 \cdot (2x \cdot dx) \\ &= \int u^5 \, du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(x^2 + 3)^6 + C. \end{aligned}$$

7. Integration of Inverse Trigonometric Functions:

- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C$
- $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arcsec} x + C$

Example 5.7.3 Find $\int \frac{dx}{x^2+4x+5}$

$$\begin{aligned} \int \frac{dx}{x^2+4x+5} &= \frac{dx}{(x+2)^2+1} \\ \text{Assume } u = x+2, \frac{du}{dx} &= 1 \Rightarrow du = dx \\ \therefore \int \frac{dx}{x^2+4x+5} &= \frac{du}{u^2+1} \\ &= \arctan u + C \\ &= \arctan(x+2) + C. \end{aligned}$$

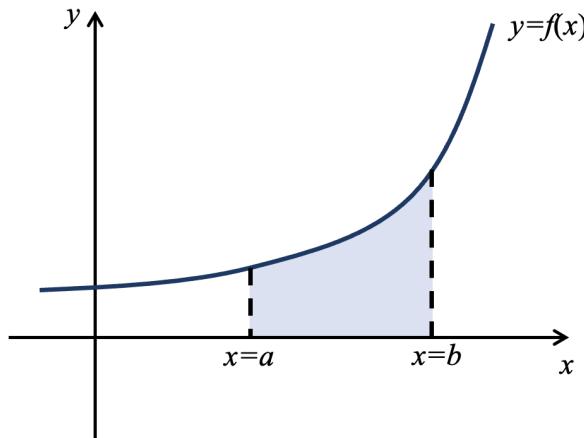
5.8 Approximating the Area Under a Curve

- The **definite integral** is equal to the limit at infinity of the Riemann sum, and hence gives the exact area under the curve between $x = a$ and $x = b$. i.e.,

$$\lim_{n \rightarrow \infty} \sum_i^n f(x_i) \Delta x_i = \int_a^b f(x) \, dx,$$

where a is the lower limit and b is the upper limit.

- If $f(x) \geq 0 \forall x \in [a, b]$, then $\int_a^b f(x) \, dx$ is defined as the shaded area:



This is known as the **Riemann integral**.

- The **Fundamental Theorem of Calculus**:

Theorem 5.8.1 For a continuous function $f(x)$ with antiderivative $F(x)$:

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

This theorem explains the link between differential calculus and the definite integral.

- Properties of Definite Integrals:

- $\int_a^a f(x) \, dx = 0$
- $\int_a^b d \, dx = k(b-a)$, (k is a constant).
- $\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx$
- $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$
- $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$
- $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

5. When the function $f(x)$ is **negative** for $x \in [a, b]$, then the area bounded by the curve, the x -axis and the lines $x = a$ and $x = b$ is given by

$$\left| \int_a^b f(x) \, dx \right|.$$

6. Finding Areas Between Two Functions:

- Sketch: find the intersections and determine which function is above.
- Integration.

5.9 Volumes of Revolution

1. The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $[a, b]$, is rotated 2π radians about the x -axis is given by

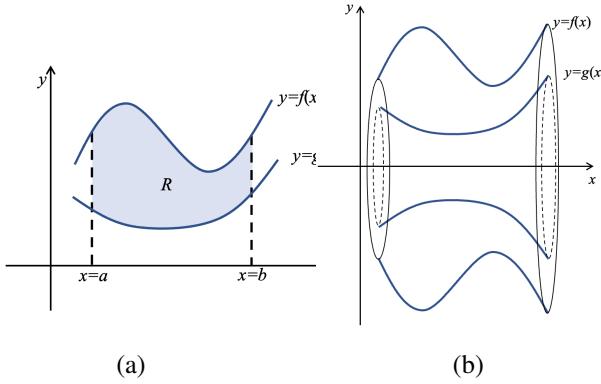
$$V = \pi \int_a^b y^2 \, dx.$$

2. The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $y = c$ to $y = d$, is rotated 2π radians about the y -axis is given by

$$V = \pi \int_c^d x^2 \, dy.$$

3. Consider a region R between two curves, $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, when $f(x) > g(x)$.

- Rotating R about the x -axis generates a solid of revolution S . The criss-section of this



shape looks like a washer whose area is given by:

$$A = \pi(R^2 - r^2) = \pi((f(x))^2 - (g(x))^2).$$

So the volume of S is given by:

$$\begin{aligned} V &= \int_a^b A(x) \, dx \\ &= \int_a^b ((f(x))^2 - (g(x))^2) \, dx. \end{aligned}$$

- Rotating R about the y -axis in the interval $c \leq y \leq d$:

$$V = \pi \int_c^d ((x_1)^2 - (x_2)^2) \, dy,$$

where x_1 and x_2 are expression of x with respect to y of $f(x)$ and $g(x)$.

5.10 Differential Equation

1. Differential Equation:

Definition 5.10.1 A **differential equation** is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables. **Equation that involves the derivatives of one or more functions.**

E.g.

$$y' = 6x + 1 \quad \text{Lagrange notation}$$

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - y = 3 \quad \text{Leibniz notation}$$

$$f'(x) = 6x + 1 \quad \text{Function notation}$$

The independent variable is x , and the dependent variable is y . The solution to a differential equation is a function or a set of functions.

2. Two Types of Differential Equations:

Definition 5.10.2 Ordinary Differential Equations (ODEs): deals with functions of a single variable and ordinary derivatives. **Partial Differential Equations (PDEs):** deals with multi-variable equations and their partial derivatives (with more than one independent variables).

3. Order of Differential Equations:

Definition 5.10.3 The **order of the differential equation** is the highest order derivative in the equation.

4. Linearity of ODEs:

Theorem 5.10.1 A differential equation is said to be **linear** if:

- All the terms with dependent variables are in first-order.
- The coefficients of all the terms in the dependent variable and its derivatives depend only on the independent variable x .

5. Linear First-Order ODEs:

Definition 5.10.4

$$\frac{dy}{dx} + a(x)y = b(x), \text{ where } a(x) \text{ and } b(x) \text{ are functions of } x.$$

6. Solutions of ODEs:

- The solution to an ODE is a function or a set of functions.
- **General solution** to the differential equation:
For a differential equation of order n , a solution is a function that satisfies the equation on some interval I . The function should have at least its first n derivatives on this interval I .
- To find **particular solutions**, we need to initial conditions for the problem.
 - (a) **Initial Value Problem (IVP):** where initial values are given to solve the differential equations depending on the order of the ODE.
E.g. $y(0)$, $t(0)$, $(0,y)$.
 - (b) **Boundary Value Problem:** where a certain boundary is given.
E.g. (x,y) .

7. Separable Differential Equations:

Definition 5.10.5 A differential equation $\frac{dy}{dx} = f(x,y)$ is **separable** if it can be expressed as a product of a function in x and a function in y :

$$\frac{dy}{dx} = f(x,y) = g(x)h(y).$$

- Particularly, if $h(x) \neq 0$, the variable can be separated to

$$\begin{aligned} \frac{dy}{dx} = g(x)h(y) &\Rightarrow \frac{dy}{h(y)} = g(x) dx \\ \int \frac{dy}{h(y)} &= \int g(x) dx \end{aligned}$$

- Solving differential equations using separation of variables:
 - Separate the variables such that everything involving y is on one side and everything involving x is on the other side.
 - Integrate both sides and combine the constant of integration on one side of the equation (normally the right side).

Example 5.10.1 Solve for y if $\frac{dy}{dx} = x(1+y)e^x$.

$$\begin{aligned}\frac{dy}{dx} &= x(1+y)e^x \\ \frac{1}{1+y} dy &= xe^x dx \\ \int \frac{1}{1+y} dy &= \int xe^x dx \\ (= xe^x - \int e^x dx) &= xe^x - e^x \quad [\text{Integration by Parts}] \\ \ln|1+y| &= xe^x - e^x + C \\ 1+y &= e^{xe^x - e^x + C} = e^{xe^x - e^x} \cdot e^C \\ y &= Ae^{xe^x - e^x} - 1 \quad (A = e^C).\end{aligned}$$

8. The Standard Logistic Equation:

$$\frac{du}{dt} = kn(a-n); \quad a, k \in \mathbb{R}.$$

where t is the time during which a population grows,
 n is the population after time t ,
 k is the relative growth, and
 a is a constant.

9. Homogeneous Differential Equations:

Definition 5.10.6 Differential equations of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, where $y = y(x)$, are known as **homogeneous differential equations**.

Theorem 5.10.2 Homogeneous differential equations can be solved by using the substitution $y = vx$, where v is a function of x . The substitution will always reduce the differential equation to a separable differential equation.

Proof 5.10.1 If $y = vx$, where v is a function of x , then:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dv}{dx}x + v \quad [\text{Product Rule}] \\ \therefore \frac{dy}{dx} &= f\left(\frac{y}{x}\right), \\ \therefore \frac{dv}{dx}x + v &= f\left(\frac{y}{x}\right) = f(v) \\ \frac{dv}{dx} &= \frac{f(v) - v}{x} \\ \Rightarrow \frac{1}{f(v) - v} dv &= \frac{1}{x} dx\end{aligned}$$

Example 5.10.2 Solve for $\frac{dy}{dx} = \frac{x+2y}{x}$, given $y(3) = \frac{3}{2}$.

$$\frac{dy}{dx} = 1 + 2\frac{y}{x} \rightarrow \text{homogeneous differential equation}$$

$$\begin{aligned}\text{Let } y = vx, \frac{dy}{dx} &= \frac{dv}{dx}x + v \Rightarrow \frac{dv}{dx}x + v = 1 + 2\frac{y}{x} = 1 + 2v. \\ \frac{dv}{dx} &= \frac{1+v}{x} \\ \frac{1}{1+v} dv &= \frac{1}{x} dx \\ \int \frac{1}{1+v} dv &= \int \frac{1}{x} dx \\ \ln|1+v| &= \ln|x| + C = \ln|Ax| \\ 1+v &= Ax \Rightarrow \frac{y}{x} + 1 = Ax \\ y &= Ax^2 - x.\end{aligned}$$

Substituting $y = \frac{3}{2}$, $x = 3$: $\frac{3}{2} = A(3)^2 - 3 \Rightarrow A = \frac{1}{2}$

$$\therefore y = \frac{1}{2}x^2 - x.$$

Example 5.10.3 Solve for $\frac{dy}{dx} = \frac{x+y}{x}$.

$$\frac{dy}{dx} = 1 + \frac{y}{x} \rightarrow \text{homogeneous differential equation}$$

$$\text{Assume } v = \frac{y}{x} : y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v.$$

$$\begin{aligned}
\frac{dy}{dx} &= 1 + v = \frac{dv}{dx}x + v \\
\frac{dv}{dx} &= \frac{1}{x} \\
\int dv &= \int \frac{1}{x} dx \\
v &= \ln|x| + C = \ln|Ax| \\
\frac{y}{x} &= \ln|Ax| \\
y &= x \ln|Ax|.
\end{aligned}$$

10. Using the Integrating Factor $I(x)$:

Definition 5.10.7

$$I(x) = e^{\int P(x) dx}$$

is the **integrating factor** for $\frac{dy}{dx} + P(x)y = Q(x)$, where P and Q are continuous functions of x on a given interval.

Theorem 5.10.3

$$\begin{aligned}
&\frac{dy}{dx} + P(x)y = Q(x) \\
&I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x) \quad [\text{Multiply both sides by } I(x)] \\
&\left(\frac{d}{dx}(I(x)y) = I(x)\frac{dy}{dx} + I(x)P(x)y \quad [\text{Product Rule}] \right) \\
&\left[(I(x))' = (e^{\int P(x) dx})' = e^{\int P(x) dx} \cdot (\int P(x) dx)' = e^{\int P(x) dx} \cdot P(x) = I(x)P(x) \right] \\
&\therefore \frac{d}{dx}(I(x)y) = I(x)Q(x) \\
&\int \frac{d}{dx}(I(x)y) dx = \int I(x)Q(x) dx \\
&I(x)y = \int I(x)Q(x) dx.
\end{aligned}$$

Example 5.10.4 Solve $\frac{dy}{dx} + 3x^2y = 6x^2$.

$$\therefore P(x) = 3x^2, Q(x) = 6x^2,$$

$$\therefore I(x) = e^{\int P(x) dx} = e^{\int 3x^2 dx} = e^{x^3}.$$

Multiply both sides by $I(x)$:

$$\begin{aligned}
e^{x^3} \frac{dy}{dx} + e^{x^3} \cdot 3x^2 y &= e^{x^3} \cdot 6x^2 \\
\therefore \frac{d}{dx} (e^{x^3} y) &= e^{x^3} \cdot 6x^2 \\
\int \frac{d}{dx} (e^{x^3} y) \, dx &= \int e^{x^3} \cdot 6x^2 \, dx \\
\left[\text{Let } x^3 = u, \frac{du}{dx} = 2x^2, du = 2x^2 \, dx \Rightarrow 2 \int e^u \, du = 2e^u + C = 2e^{x^3} + C \right] \\
e^{x^3} y &= 2e^{x^3} + C \\
y &= 2 + Ce^{-x^3}.
\end{aligned}$$

11. Euler's Method:

- For $y = f(x)$, $y_{n+1} = y_n + hf'(x_0)$, h is a constant.

$$y - y_n = f'(x_n)(x - x_n).$$

Example 5.10.5 $y = x^2$, $\frac{dy}{dx} = 2x$, $h = 0.1$

n	x_n	y_n	Actual
0	1	1	1
1	1.1	1.2	1.21
2	1.2	1.42	1.44
3	1.3	1.66	1.69
4	1.4	1.92	1.96
5	1.5	2.2	2.25

- The smaller the h , the more accurate the approximation.
- Consider a differential equation of the form $\frac{dy}{dx} = f(x, y)$, given an initial condition. The derivative at any point on the curve $(x_0, y(x_0))$ can be approximated using the gradient of the tangent to the curve at x_0 :

$$y'(x_0) = \frac{y(x_0 + h) - y(x_0)}{h}.$$

Rearranging the formula, we get:

$$y(x_0 + h) = y(x_0) + hy'(x_0).$$

This is the **linearization** or **Euler's method** and becomes more accurate over small increments and as long as the function does not change too rapidly.

- If $\frac{dy}{dx} = f(x_n, y_n)$ and $x_{n+1} = x_n + h$, we have

$$y_{n+1} = y_n + hf(x_n, y_n).$$

5.11 Maclaurin Series

1. The Maclaurin Polynomial:

Definition 5.11.1 If $f(x)$ has n derivatives at $x = 0$, then $P(x)$, the **Maclaurin polynomial** of degree n for $f(x)$ centered at $x = 0$, is the unique polynomial of degree n that satisfies:

- $f(0) = P(0);$
- $f^{(n)}(0) = P^{(n)}(0);$
- $a_1 = \frac{f'(0)}{1!}, a_2 = \frac{f''(0)}{2!}, a_3 = \frac{f'''(0)}{3!}, \dots a_n = \frac{f^{(n)}(0)}{n!};$
- $P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k.$

2. Maclaurin polynomials approximate the behavior of functions around a certain interval. The more terms we take, the better the approximation.
3. The Maclaurin Series:

Definition 5.11.2 If $f(x)$ has derivatives of all orders throughout an open interval I such that $0 \in I$, then the **Maclaurin series** generated by f at $x = 0$ is:

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

A series converges when the sum of them is a constant (a limit can be found).

Example 5.11.1 Find the Maclaurin series for $f(x) = \frac{1}{2+x}$.

$$\begin{array}{l|l} \begin{array}{l} f(x) = (2+x)^{-1} \\ f'(x) = -(2+x)^{-2} \\ f''(x) = 2(2+x)^{-3} \\ f'''(x) = -6(2+x)^{-4} \\ f^{(4)}(x) = 24(2+x)^{-5} \end{array} & \begin{array}{l} f(0) = 2^{-1} = \frac{1}{2} \\ f'(0) = -2(2)^{-2} = -\frac{1}{4} \\ f''(0) = 2(2)^{-3} = 2 \times \frac{1}{8} \\ f'''(0) = -6(2)^{-4} = -6 \times \frac{1}{16} \\ f^{(4)}(0) = 24(2)^{-5} = 24 \times \frac{1}{32} \end{array} \end{array}$$

$$\begin{aligned} P(x) &= \frac{1}{2} + \frac{-\frac{1}{4}}{1!}x + \frac{2 \times \frac{1}{8}}{2!}x^2 + \frac{-6 \times \frac{1}{16}}{3!}x^3 + \frac{24(2)^{-5}}{4!}x^4 + \dots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (-x)^n. \end{aligned}$$

4. The Binomial series is the Maclaurin expansion for $f(x) = (1+x)^p$:

$$(1+x)^p = \sum_{n=0}^p \binom{p}{n} x^n, \quad 1 \leq n \leq p, \quad \binom{p}{n} = \frac{p!}{n!(p-n)!} = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.$$

Example 5.11.2 Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{(x+2)^2}$.

$$\begin{aligned}
f(x) &= (1+x)^{-2} \\
\therefore \binom{-2}{n} &= \frac{-2(-2-1)(-2-2)\cdots(-2-(n-1))}{n!} \\
&= (-1)^n \frac{2(3)(4)\cdots(n+1)}{n!} = (-1)^n(n+1) \\
\therefore P(x) &= \sum_{n=0}^{\infty} (-1)^n(n+1)x^n \\
&= 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^n(n+1)x^n + \cdots
\end{aligned}$$

Example 5.11.3 Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{\sqrt{2-x}}$.

$$\begin{aligned}
f(x) &= (2-x)^{-\frac{1}{2}} = (2)^{-\frac{1}{2}} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}} \\
\therefore P(x) &= \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2} \binom{-\frac{1}{2}}{n} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(1-\frac{x}{2}\right)^{-\frac{1}{2}}.
\end{aligned}$$

5. Applications of Maclaurin Series:

- Approximation of sin, cos, tan, ...

Example 5.11.4 Approximate $\sin 3^\circ$ using the first four terms of Maclaurin series.

$$\begin{aligned}
3^\circ &= \frac{\pi}{60}, \text{ For } \sin x, P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\therefore P\left(\frac{\pi}{60}\right) &= x - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \frac{\left(\frac{\pi}{60}\right)^7}{7!} + \cdots \approx 0.052336 \text{ (6 d.p.)}.
\end{aligned}$$

- More Complicated Functions

Example 5.11.5 Find the Maclaurin series of $f(x) = e^{x^2}$.

Let $u = x^2, f(u) = e^u$:

$$P(x) = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Example 5.11.6 Find the Maclaurin series of $f(x) = \ln\left(\frac{1+x}{1-x}\right)$.

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\begin{aligned}
\therefore P(x) &= -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \left(x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \right) \\
&= 2\left(x + \frac{x^3}{3} + \cdots\right) \\
&= 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.
\end{aligned}$$

Example 5.11.7 Find the Maclaurin series of $f(x) = \frac{x}{(1+x)^2}$.

$$\begin{aligned}
f(x) &= x(1+x)^{-2} \\
&= x \sum_{n=0}^{\infty} \binom{-2}{n} x^n \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1) x^{n+1}.
\end{aligned}$$

- Evaluate Limits

Example 5.11.8 Find $\lim_{x \rightarrow 0} \frac{1-e^{x^2}}{1-\cos x}$.

$$\begin{aligned}
e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\therefore \lim_{x \rightarrow 0} \frac{1-e^{x^2}}{1-\cos x} &= \lim_{x \rightarrow 0} \frac{1 - \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right)}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)} \\
&= \lim_{x \rightarrow 0} \frac{-x^2 - \frac{x^4}{2!} - \frac{x^6}{3!} - \cdots}{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots} \\
&= \lim_{x \rightarrow 0} \frac{-x^2}{\frac{x^2}{2!}} \\
&= -2.
\end{aligned}$$

[Consider only the smallest power of x , as higher powers will go to zero much quicker.]

- Solve Differential Equations

Example 5.11.9 Use the first six terms of a Maclaurin series to approximate the solution of $y' = y^2 - x$ on an open interval centered at $x = 0$ if $y(0) = 1$.

	$y(0) = 1$
$y' = y^2 - x$	$y'(0) = 1$
$y'' = 2yy' - 1$	$y''(0) = 2 - 1 = 1$
$y''' = 2yy'' + 2(y')^2$	$y'''(0) = 2 + 2 = 4$
$y^{(4)} = 2yy''' + 6y'y''$	$y^{(4)}(0) = 14$
$y^{(5)} = 2yy^{(4)} + 8y'y''' + 6(y'')^2$	$y^{(5)}(0) = 66$

$$\therefore P(x) = 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \dots$$

- Binomial Theorem

Theorem 5.11.1 Function $f(x) = (1+x)^p$, $p \in \mathbb{R}$ is equal to its Binomial series using the initial condition $y(0) = 1$.

Proof 5.11.1

$$f(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \dots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \dots$$

$$\therefore f'(x) = p + p(p-1)x + \frac{p(p-1)(p-2)}{2!}x^2 + \dots$$

$$xf'(x) = px + p(p-1)x^2 + \frac{p(p-1)(p-2)}{2!}x^3 + \dots$$

$$\begin{aligned}\therefore f'(x) + xf'(x) &= p + [p(p-1) + p]x + \left[\frac{p(p-1)}{2!}p(p-1) + \right] x^2 + \dots \\ &= p + p^2x + \frac{p^2(p-1)}{2!}x^2 + \dots \\ &= p(1 + px + \frac{p(p-1)}{2!}x^2) + \dots \\ &= pf(x)\end{aligned}$$

$$\therefore f'(x) + xf'(x) = pf(x) \Rightarrow (1+x)f'(x) = pf(x)$$

$$f'(x) - \frac{p}{1+x}f(x) = 0 \Rightarrow P(x) = -\frac{p}{1+x}, Q(x) = 0$$

$$\therefore I(x) = e^{\int -\frac{p}{1+x} dx} = A(1+x)^{-p}$$

$$\therefore \frac{d}{dx} (A(1+x)^{-p}f(x)) = 0 \cdot I(x)$$

$$\int \frac{d}{dx} (A(1+x)^{-p}f(x)) dx = \int 0 dx$$

$$A(1+x)^{-p}f(x) = C$$

$$f(x) = C \cdot A(1+x)^{x-p} = B(1+x)^{x-p} \quad [\text{Let } B = C \cdot A.]$$

$$\text{Let } f(0) = 1 : B = 1$$

$$\therefore f(x) = (1+x)^p.$$