IB Mathematics Analysis and Approaches HL
Topic 5 Calculus

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February 28, 2022

Contents

1 Limits

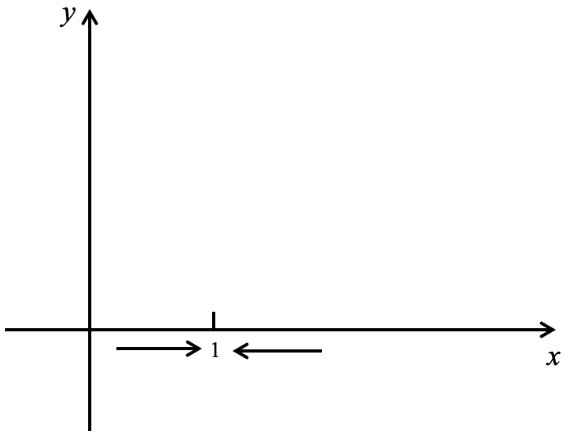
1. Limit

Example: 1.1

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

when x is approaching to 1 (it never equals to 1), the value $\frac{x^2-1}{x-1}$ is appraoching to 2.

- Left-hand and Right-hand Limit



Example: 1.2

The left-hand limit of $\frac{x^2-1}{x-1}$ when $x \to 1$ is

$$\lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = 2.$$

The right-hand limit of $\frac{x^2-1}{x-1}$ when $x \to 1$ is

$$\lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = 2.$$

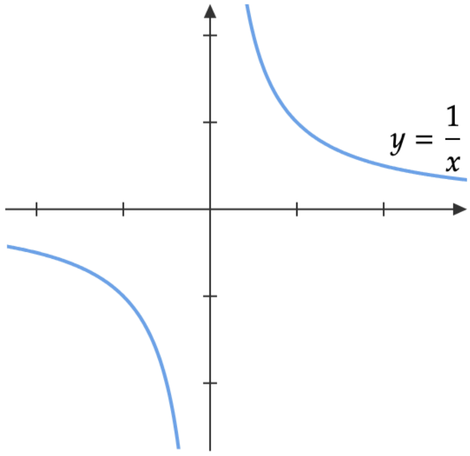
- Only when the left-hand limit and the right-hand limit exist and are the same at the point $x = a$, we say the limit of $f(x)$ exists on $x = a$.

$$\text{i.e., } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = c \Rightarrow \lim_{x \to a} f(x) = c, \ c \text{ is a constant} \in \mathbb{R}$$

Otherwise, the limit does not exist on $x = a$ (OR DNE.).

Example: 1.3

Does $\lim_{x\to 0}\frac{1}{x}$ exist? How about $\lim_{x\to\infty}\frac{1}{x}$?



- $\lim_{x\to 0}\frac{1}{x}$ doest not exist.

$$\because \lim_{x\to 0^+}\frac{1}{x} = +\infty,\ \lim_{x\to 0^-}\frac{1}{x} = -\infty$$

$$\therefore \lim_{x\to 0^+}\frac{1}{x} \neq \lim_{x\to 0^-}\frac{1}{x} \Rightarrow \text{DNE.}$$

- $\lim_{x\to\infty}\frac{1}{x}$ exists.

$$\because \lim_{x\to+\infty}\frac{1}{x} = 0,\ \lim_{x\to-\infty}\frac{1}{x} = 0$$

$$\therefore \lim_{x\to+\infty}\frac{1}{x} = \lim_{x\to-\infty}\frac{1}{x} \Rightarrow \text{Limit exists.}$$

Definition 1: **Horizontal Asymptote (H.A.)**:

$$y = \lim_{x\to\infty} f(x) = c$$

- Limit at ∞:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = c \ \Rightarrow \ \lim_{x\to\infty} f(x) = c.$$

  Note: $+\infty$ and $-\infty$ are not exact values; they should be regarded as a concept.

- Limites does not have to equal to the function value.
  Limit and the function value do not have relationships.

- Generally speaking, if $a \in D_f$, $\lim_{x\to a} f(x) = f(a)$.

2. For a rational function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$, and $Q(x) = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$:

- $\lim_{x \to a} f(x) = f(a)$ as long as $Q(a) \neq 0$.
- $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m}{b_0 x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m} \Rightarrow$ H.A.
  (a) If $m = n$, $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a_0}{b_0} = \frac{a_0}{b_0}$.
  (b) If $m > n$, $\lim_{x \to \infty} f(x)$ DNE.
  (c) If $m < n$, $\lim_{x \to \infty} f(x) = 0$.

3. Continuity and Discontinuity

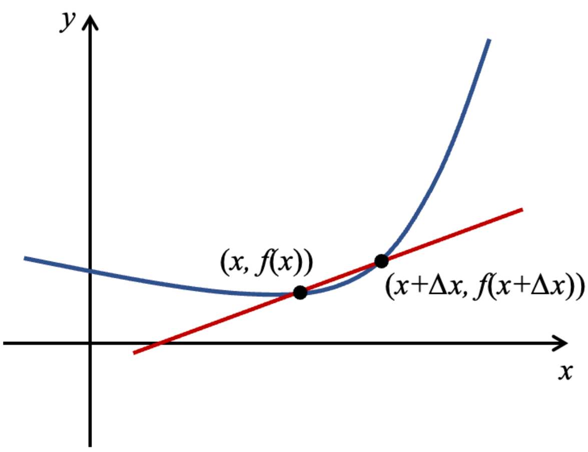
> Definition 2:
> **Continuity**: If the graph of the function does not have any breaks or holes within a certain interval, then the function is continuous within that interval.

Theroem: 1.1 Continuity Test

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$, then the function f is **continuous** at $x = a$.

2 Differentiation and Derivatives

1. Gradient of Secant:



- $$\text{Slope } m = \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

> Definition 3: **Derivative of a function**:
>
> $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ is the derivative of a function, denoted as $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$.

- The graphic meaning of derivative is the gradient of tangent of the function.

Example: 2.1

By definition, find the derivative of $f(x) = x^2 + 1$ and hance find the gradient of the tangent line when $x = 3$.

$$
\begin{aligned}
f'(x) = \lim_{x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{x\to 0} \frac{\left[(x+\Delta x)^2 + 1\right] - (x^2+1)}{\Delta x} \\
&= \lim_{x\to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x} \\
&= \lim_{x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{x\to 0} (2x + \Delta x) \\
&= 2x.
\end{aligned}
$$

At $x = 3$, $f'(3) = 2 \times 3 = 6$. The gradient is 6.

2. Derivative of x^n

Theroem: 2.1 Power Rule

If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}, \text{ for any } n \in \mathbb{R}.$$

Note: The derivative of any constant is 0.

Example: 2.2

$$
\begin{aligned}
f(x) = \frac{1}{x} = x^{-1} &\Rightarrow f'(x) = (-1)x^{-1-1} = -x^{-2}; \\
f(x) = \sqrt{x} = x^{\frac{1}{2}} &\Rightarrow f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}; \\
f(x) = c = cx^0 &\Rightarrow f'(x) = 0 \times cx^{0-1} = 0.
\end{aligned}
$$

3. Rules of Differentiation:
   Name $f(x)$ and $g(x)$ as two functions with derivatives of $f'(x)$ and $g'(x)$, respectively. Then

$$(cf(x))' = cf'(x)$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

4. More Derivatives:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

5. Differentiablity:

> Definition 4:
> A function has to be **continuous** and **no sharp turning point** to be **differentiable**.
> Note: Smooth turning point on the graph is allowed.

6. More Rules of Differentiation:

Theroem: 2.2 Product Rule

Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$(f(x) \times g(x))' = f'(x)g(x) + f(x)g'(x).$$

Theroem: 2.3 Quotient Rule

Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Theroem: 2.4 Chain Rule

For a composite function $f(g(x))$ or $(f \circ g)(x)$, the derivative will be

$$f'(g(x)) \times g'(x).$$

OR
If $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

7. Higher Order Differentiation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2},\ f''(x),\ f'''(x),\ f^{(4)}(x),\ f^{(5)}(x),\ \cdots$$

3 Applications of Derivatives

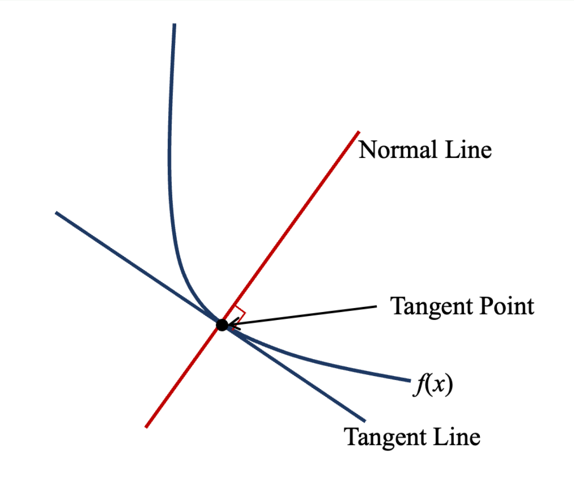
1. Equation of Tangent Line:
Via the original functions, we could get the tangent point (x_0, y_0). Then, the expression of the

tangent line is

$$y - y_0 = m(x - x_0),$$

where m is the derivative.

2. Normal and Tangent Lines:

> Definition 5:
> **Normal** is perdendicular to the tangent and passes through the same tangent point.
>
> 
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3. Increasing and Decreasing Function:

> Definition 6:
> **Increasing Function**: As x is getting larger, y is getting larger.
> i.e.,
> $$\frac{\mathrm{d}y}{\mathrm{d}x} > 0.$$
> **Decreasing Function**: As x is getting larger, y is getting smaller.
> i.e.,
> $$\frac{\mathrm{d}y}{\mathrm{d}x} < 0.$$

4. Local Extrema: $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ Stationary point
   Global extrema is the maximum and the minimum points of the entire function.
   $f''(x)$ is used to determine if the local extrema is maxima or minima.

   - Minima: $f''(x) > 0$ Concave up.
   - Maxima: $f''(x) < 0$ Concave down.
   - Point of Inflection (the point that is changing from concaving up to concaving down, or vice visa): $f''(x) = 0$

5. With local extrema, x-intercepts, y-intercepts, concavity, and asympotes, draw approxiate diagrams of a function.

4 Implicit Differentiation

1. When differentiating something with y, multiply $\frac{dy}{dx}$ at the end.

2. $(y^2)' = 2y\frac{dy}{dx}$.

Proof: 4.1

If $u = y^2$, then

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}. \qquad \text{[Chain Rule]}$$

Example: 4.1

Find $\frac{dy}{dx}$ for the circle $x^2 + y^2 = 16$.

$$(x^2)' + (y^2)' = (16)' \Rightarrow 2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

Example: 4.2

Find $\frac{dy}{dx}$ for $e^x + x\sin y = \cos 2y$.

$$(e^x)' + (x\sin y)' = (\cos 2y)'$$
$$e^x + \left(\sin y + x\cos y\frac{dy}{dx}\right) = -2\sin 2y\frac{dy}{dx}$$
$$(-x\cos y - 2\sin 2y)\frac{dy}{dx} = e^x + \sin y$$
$$\frac{dy}{dx} = \frac{e^x + \sin y}{-x\cos y - 2\sin 2y}.$$

3. Second Order Differentiation of Implicit functions*: Differentiate the first order differentiation.

Example: 4.3

Find $\frac{d^2y}{dx^2}$ for the circle $x^2 + y^2 = 16$. (From Ex. 4.1: $2x + 2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$.)

$$(2x)' + \left(2y\frac{dy}{dx}\right)' = (0)' \Rightarrow 2 + \left((2y)'\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right)'\right) = 0 \Rightarrow 2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{-2 - 2\left(\frac{dy}{dx}\right)^2}{2y} = \frac{-2 - 2\left(-\frac{x}{y}\right)^2}{2y}.$$

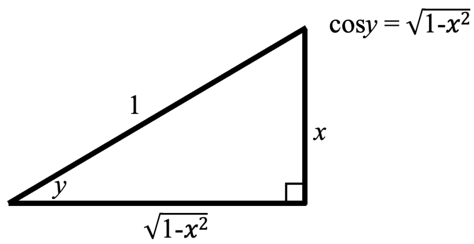
4. Derivative of Inverse Trignometry Functions

Theroem: 4.1

$$y = \arcsin x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \ \arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] (\cos y > 0).$$

Proof: 4.1

From $y = \arcsin x$, we get $\sin y = x$. This situation can be illustrated by the figure below:



$$\therefore (\sin y)' = (x)' \Rightarrow \cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

Theroem: 4.2

$$y = \arccos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}, \ \arccos x \in [0, \pi] \, (\sin y > 0).$$

$$y = \arctan x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}.$$

Proof: 4.2

(Hint: Try to visualize a similar diagram as in proof 4.1.)

From $y = \arccos x$, we get $\cos y = x$.

$$\therefore (\cos y)' = (x)' \Rightarrow -\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

From $y = \arctan x$, we get $\tan y = x$.

$$\therefore (\tan y)' = (x)' \Rightarrow \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{1+x^2}.$$

5 Related Rate of Change

1. When finding a rate of change of x, we are finding the $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example: 5.1

Area of circle is increasing at a rate of 10π per second. When the raduis is 2, what is the rate of change of radius?

Known: $\frac{\mathrm{d}A}{\mathrm{d}t} = 10\pi,\ r = 2$. Find: $\frac{\mathrm{d}r}{\mathrm{d}t}$.

$$A = \pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t} = 10\pi$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{10\pi}{2\pi r} = \frac{5}{r}$$

$$\text{When } r = 2,\ \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{2}.$$

Example: 5.2

A spherical balloon is expanding at a rate of 60π per second. How fast is the surface area of the balloon expanding when the radius is 4?

Known: $\frac{\mathrm{d}V}{\mathrm{d}t} = 60\pi,\ r = 4$. Find $\frac{\mathrm{d}A}{\mathrm{d}t}$.

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 3 \cdot \frac{4}{3}\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$
$$\therefore 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = 60\pi \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{60\pi}{4\pi r^2} = \frac{15}{r^2}$$
$$A = 4\pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t} = 8\pi r \cdot \frac{15}{r^2} = \frac{120\pi}{r}.$$

$$\text{When } r = 4,\ \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{120\pi}{4} = 30\pi.$$

2. Kinematics:

   - Velocity, displacement, and acceleration are vector variables that have a value and a direction.
   - Speed only has a value and no direction. It is a scalar variable. No sign should be reported in the answer.
   - If s is the displacement, v is the velocity, a is the acceleration:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v;\ \frac{\mathrm{d}v}{\mathrm{d}t} = a.$$

6 More Limits - L'Hopital's Rule

Theroem: 6.1 L'Hopital's Rule

When the limit is in the **indeterminant form** ($\frac{0}{0}$ or $\frac{\infty}{\infty}$),

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] \lim_{x \to a} \left[\frac{f'(x)}{g'(x)} \right],$$

where $f'(x)$ and $g'(x)$ are the first derivatives of $f(x)$ and $g(x)$, respectively.

Example: 6.1

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = 1$$

7 Indefinite Integration

1. Regard Integration as Anti-differentiation:

$$f'(x) = x \quad \Rightarrow \quad f(x) = \frac{1}{2}x^2 + C, \text{ where } C \text{ is a constant.}$$

$$f'(x) = x^2 \quad \Rightarrow \quad f(x) = \frac{1}{3}x^3 + C, \text{ where } C \text{ is a constant.}$$

$$f'(x) = x^n \quad \Rightarrow \quad f(x) = \frac{1}{n+1}x^{n+1} + C, \text{ where } C \text{ is a constant.}$$

> Definition 7:
> Anti-differentiation is also called **indefinited integration**. It is denoted by $\int \mathrm{d}x$.
>
> e.g. $\int x^n \, \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$.

2. General Rules of Integration.

- $$\int x^n \, \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$$
- $$\int k \, \mathrm{d}x = kx + C$$
- $$\int kf(x) \, \mathrm{d}x = k \int f(x) \, \mathrm{d}x$$
- $$\int f(x) \pm g(x) \, \mathrm{d}x = \int f(x) \, \mathrm{d}x \pm \int g(x) \, \mathrm{d}x$$

3. $\int f'(x)\ \mathrm{d}x = f(x) + C$. Therefore, if we know the $f'(x)$ and a point on the $f(x)$, which is to determine the constant C, then we can deduce the original function $f(x)$.

4. More Rules of Integration:

Differentiation	Integration
$(e^x)' = e^x$	$\int e^x\ \mathrm{d}x = e^x + C$
$(\ln x)' = frac1x$	$\int \frac{1}{x}\ \mathrm{d}x = \ln x + C$
$(\sin x)' = \cos x$	$\int \cos x\ \mathrm{d}x = \sin x + C$
$(\cos x)' = -\sin x$	$\int \sin x\ \mathrm{d}x = -\cos x + C$
$(\tan x)' = \sec^2 x$	$\int \sec^2 x\ \mathrm{d}x = \tan x + C$
$(\cot x)' = -\csc^2 x$	$\int \csc^2 x\ \mathrm{d}x = -\cot x + C$
$(\sec x)' = \sec x \tan x$	$\int \sec x \tan x\ \mathrm{d}x = \sec x + C$
$(\csc x)' = -\csc x \cot x$	$\int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

5. Anti-chain Rule in Integration: We must divide the chain rule factor.

Example: 7.1

$$\int (ax+b)^n\ \mathrm{d}x = \frac{1}{a}\left(\frac{1}{n+1}(ax+b)^{n+1}\right) + C$$

$$\int e^{(ax+b)}\ \mathrm{d}x = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{1}{(ax+b)}\ \mathrm{d}x = \frac{1}{a}\ln(ax+b) + C$$

$$\int \sin(ax+b)\ \mathrm{d}x = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\ \mathrm{d}x = \frac{1}{a}\sin(ax+b) + C$$

6. Integration Techniques: by Substitution and by Parts:

Theroem: 7.1 Integration By Substitution

Whenever we have an integration like: $\int g'(x) \times f(g(x))\ \mathrm{d}x$, we can always assume $u = g(x)$. Therefore, $\mathrm{d}u = g'(x) \cdot \mathrm{d}x$. $()u = g(x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = g'(x))$:

$$\int g'(x) \times f(g(x))\ \mathrm{d}x = \int f(u\ \mathrm{d}u).$$

Theroem: 7.2 Integration By Parts

If $f(x)$ and $g(x)$ are two functions, and $f'(x)$ and $g'(x)$ are their derivatives, respectively, integration by parts can be written as following:

$$\int f(x)g'(x)\ \mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\ \mathrm{d}x.$$

Example: 7.2

Find $\int 2x(x^2+3)^5\,\mathrm{d}x$**.**

Since $2x = (x^2+3)'$, we consider to use integration by substitution.
Assume $u = x^2+3$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = (x^2+3)' = 2x \Rightarrow \mathrm{d}u = 2x \cdot \mathrm{d}x$.

$$\begin{aligned}
\therefore \int 2x(x^2+3)^5\,\mathrm{d}x &= \int (x^2+3)^5 \cdot (2x\cdot \mathrm{d}x) \\
&= \int u^5\,\mathrm{d}u \\
&= \frac{1}{6}u^6 + C \\
&= \frac{1}{6}(x^2+3)^6 + C.
\end{aligned}$$

7. Integration of Inverse Trignometric Functions:

- $$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C$$
- $$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C$$
- $$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{arcsec} x + C$$

Example: 7.3

Find $\int \frac{\mathrm{d}x}{x^2+4x+5}$

$$\int \frac{\mathrm{d}x}{x^2+4x+5} = \frac{\mathrm{d}x}{(x+2)^2+1}$$

$$\text{Assume } u = x+2,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \Rightarrow \mathrm{d}u = \mathrm{d}x$$

$$\begin{aligned}
\therefore \int \frac{\mathrm{d}x}{x^2+4x+5} &= \frac{\mathrm{d}u}{u^2+1} \\
&= \arctan u + C \\
&= \arctan(x+2) + C.
\end{aligned}$$

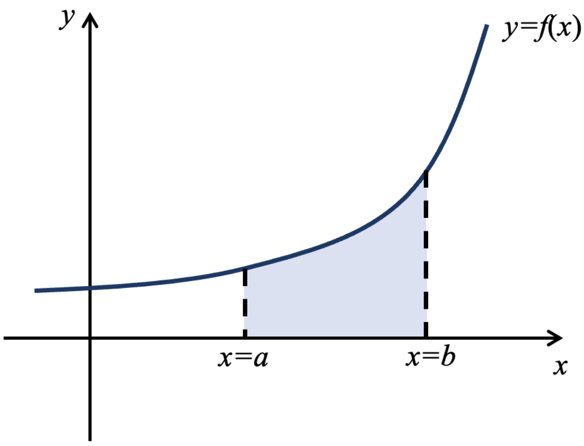
8 Approximiating the Area Under a Curve

1. The definite integral is equal to the limit at infinity of the Riemann sum, and hence gives the exact area under the curve between $x = a$ and $x = b$. i.e.,

$$\lim_{n\to\infty} = \sum_{i}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x,$$

where a is the lower limit and b is the upper limit.

2. If $f(x) \geq 0\ \forall x \in [a,b]$, then $\int_a^b f(x)\,\mathrm{d}x$ is defined as the shaded area:



This is known as the Riemann integral.

3. The Fundamental Theorem of Calculus:

Theroem: 8.1 Fundamental Theorem of Calculus

For a continuous function $f(x)$ with antiderivative $F(x)$:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

This theorem explains the link between differential calculus and the definite integral.

4. Properties of Definite Integrals:

- $$\int_a^a f(x)\,\mathrm{d}x = 0$$

- $$\int_a^b d\,\mathrm{d}x = k(b-a),\ (k \text{ is a constant}).$$

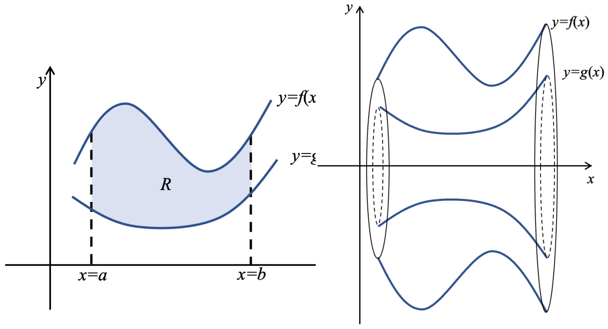
- $$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

- $$\int_a^b kf(x)\ \mathrm{d}x = k\int_a^b f(x)\ \mathrm{d}x$$
- $$\int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x$$
- $$\int_a^b [f(x) \pm g(x)]\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x \pm \int_a^b g(x)\ \mathrm{d}x$$

5. When the function $f(x)$ is negative for $x \in [a,b]$, then the area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$ is given by
$$\left| \int_a^b f(x)\ \mathrm{d}x \right|.$$
6. Finding Areas Between Two Functions:
   - Sketch: find the intersections and determine which function is above.
   - Integration.

9 Volumes of Revolution

1. The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $[a,b]$, is rotated 2π radians about the x-axis is given by
$$V = \pi \int_a^b y^2\ \mathrm{d}x.$$
2. The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $y = c$ to $y = d$, is rotated 2π radians about the y-axis is given by
$$V = \pi \int_c^d x^2\ \mathrm{d}y.$$
3. Consider a region R between two curves, $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, when $f(x) > g(x)$.
   - Rotating R about the x-axis generates a solid of revolution S. The criss-section of this shape looks like a washer whose area is given by:
   $$A = \pi(R^2 - r^2) = \pi\left((f(x))^2 - (g(x))^2\right).$$
   So the volume of S is given by:
   $$\begin{aligned} V &= \int_a^b A(x)\ \mathrm{d}x \\ &= \int_a^b \left((f(x))^2 - (g(x))^2\right)\ \mathrm{d}x. \end{aligned}$$



(a) (b)

- Rotating R about the y-axis in the interval $c \leq y \leq d$:

$$V = \pi \int_c^d \left((x_1)^2 - (x_2)^2\right) \, \mathrm{d}y,$$

where x_1 and x_2 are expression of x with respect to y of $f(x)$ and $g(x)$.

10 Differential Equation

1. Differential Equation:

> Definition 8:
> A **differential equation** is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables. Equation that involves the derivatives of one or more functions.

E.g.

$$y' = 6x + 1 \qquad \text{Lagrage notation}$$

$$3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{dx} - y = 3 \qquad \text{Leibriz notation}$$

$$f'(x) = 6x + 1 \qquad \text{Function notation}$$

The independent variable is x, and the dependent variable is y. The solution to a differential equation is a function or a set of functions.

2. Two Types of Differential Equations:

> Definition 9:
> **Ordinary Differential Equations (ODEs)**: deals with functions of a single variable and ordinary derivatives.
> **Partial Differential Equations (PDEs)**: deals with multivariable equations and their partial derivatives (with more than one independent variables).

3. Order of Differential Equations:

Definition 10:
The **order of the differential equation** is the highest order derivative in the equation.

4. Linearity of ODEs:

Theroem: 10.1

A differential equation is said to be **linear** if:

- All the terms with dependent variabels are in first-order.
- The coefficients of all the terms in the dependent variable and its derivatives depend only on the independent variable x.

5. Linear First-Order ODEs:

Definition 11:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = b(x), \text{ where } a(x) \text{ and } b(x) \text{ are functions of } x.$$

6. Solutions of ODEs:

- The solution to an ODE is a function or a set of functions.
- General solution to the differential equation:
  For a differential equation of order n, a solution is a function that satisfies the equation on some interval I. The function should havee at least its first n derivatives on this invertal I.
- To find particular solutions, we need to initial conditions for the problem.
  (a) Initial Value Problem (IVP): where initial values are given to solve the differential equations depending on the order of the ODE.
  E.g. $y(0)$, $t(0)$, $(0, y)$.
  (b) Boundary Value Problem: where a certain boundary is given.
  E.g. (x, y).

7. Separable Differential Equations:

Definition 12:
A differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ is **separable** if it can be expressed as a product of a function in x and a function in y:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = g(x)h(y).$$

- Particularly, if $h(x) \neq 0$, the variable can be separated to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y) \Rightarrow \frac{\mathrm{d}y}{h(y)} = g(x)\,\mathrm{d}x$$
$$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x$$

- Solving differential equations using separation of variables:
  (a) Separate the variables such that everything involving y is on one side and everything involving x is on the other side.
  (b) Integrate both sides and combine the constant of integration on one side of the equation (normally the right side).

Example: 10.1

Solve for y if $\frac{\mathrm{d}y}{\mathrm{d}x} = x(1+y)e^x$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1+y)e^x$$
$$\frac{1}{1+y}\mathrm{d}y = xe^x\mathrm{d}x$$
$$\int \frac{1}{1+y}\,\mathrm{d}y = \int xe^x\,\mathrm{d}x$$
$$\left(= xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x \text{ [Integration by Parts]}\right)$$
$$\ln|1+y| = xe^x - e^x + C$$
$$1+y = e^{xe^x - e^x + C} = e^{xe^x - e^x} \cdot e^C$$
$$y = Ae^{xe^x - e^x} - 1 \quad (A = e^C).$$

8. The Standard Logistic Equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = kn(a-n); \quad a, k \in \mathbb{R}.$$

where t is the time during which a population grows,
n is the population after time t,
k is the relative growth, and
a is a constant.

9. Homogeneous Differential Equations:

> Definition 13:
> Differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$, where $y = y(x)$, are known as **homogeneous differential equations**.

Theroem: 10.2

Homogeneous differential equations can be solved by using the substituion $y = vx$, where v is a function of x. The substitution will always reduce the differential equation to a separable differential equation.

Proof: Theorem 10.2

If $y = vx$, where v is a function of x, then:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}v}{\mathrm{d}x}x + v \qquad \text{[Product Rule]}\\
\because \frac{\mathrm{d}y}{\mathrm{d}x} &= f\left(\frac{y}{x}\right),\\
\therefore \frac{\mathrm{d}v}{\mathrm{d}x}x + v &= f\left(\frac{y}{x}\right) = f(v)\\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{f(v) - v}{x}\\
\Rightarrow \frac{1}{f(v) - v}\,\mathrm{d}v &= \frac{1}{x}\,\mathrm{d}x
\end{aligned}$$

Example: 10.2

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y}{x}$, given $y(3) = \frac{3}{2}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 2\frac{y}{x} \quad \rightarrow \text{ homogenous differential equation}$$

$$\text{Let } y = vx,\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}x + v \ \Rightarrow\ \frac{\mathrm{d}v}{\mathrm{d}x}x + v = 1 + 2\frac{y}{x} = 1 + 2v.$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+v}{x}\\
\frac{1}{1+v}\,\mathrm{d}v &= \frac{1}{x}\,\mathrm{d}x\\
\int \frac{1}{1+v}\,\mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x\\
\ln|1+v| &= \ln|x| + C = ln|Ax|\\
1 + v = Ax \ &\Rightarrow\ \frac{y}{x} + 1 = Ax\\
y &= Ax^2 - x.
\end{aligned}$$

Substituting $y = \frac{3}{2}$, $x = 3$: $\frac{3}{2} = A(3)^2 - 3 \ \Rightarrow\ A = \frac{1}{2}$

$$\therefore y = \frac{1}{2}x^2 - x.$$

Example: 10.3

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x} \ \rightarrow \text{ homogenous differential equation}$$

$$\text{Assume } v = \frac{y}{x} : \ y = vx \ \Rightarrow \ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}x + v.$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + v &= \frac{\mathrm{d}v}{\mathrm{d}x}x + v \\
\mathrm{d}v &= \frac{1}{x}\ \mathrm{d}x \\
\int \ \mathrm{d}v &= \int \frac{1}{x}\ \mathrm{d}x \\
v &= \ln|x| + C = \ln|Ax| \\
\frac{y}{x} &= \ln|Ax| \\
y &= x\ln|Ax|.
\end{aligned}$$

10. Using the Integrating Factor $I(x)$:

Definition 14:

$$I(x) = e^{\int P(x)\ \mathrm{d}x}$$

is the **integrating factor** for $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$, where P and Q are continuous functions of x on a given interval.

Theroem: 10.3

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y &= Q(x) \\
I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)P(x)y &= I(x)Q(x) \text{ [Multiply both sides by } I(x)] \\
&\left(\frac{\mathrm{d}}{\mathrm{d}x}(I(x)y) = I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)P(x)y \text{ [Product Rule]}\right) \\
\Big[(I(x))' = (e^{\int P(x)\ \mathrm{d}x})' &= e^{\int P(x)\ \mathrm{d}x} \cdot (\int P(x)\ \mathrm{d}x)' = e^{\int P(x)\ \mathrm{d}x} \cdot P(x) = I(x)P(x)\Big] \\
\therefore \frac{\mathrm{d}}{\mathrm{d}x}(I(x)y) &= I(x)Q(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}(I(x)y)\ \mathrm{d}x &= \int I(x)Q(x)\ \mathrm{d}x \\
I(x)y &= \int I(x)Q(x)\ \mathrm{d}x.
\end{aligned}$$

Example: 10.4

Solve $\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2y=6x^2$.

$$\because P(x)=3x^2,\ Q(x)=6x^2,$$

$$\therefore I(x)=e^{\int P(x)\ \mathrm{d}x}=e^{\int 3x^2\ \mathrm{d}x}=e^{x^3}.$$

$$\text{Multiply both sides by } I(x):$$

$$e^{x^3}\frac{\mathrm{d}y}{\mathrm{d}x}+e^{x^3}\cdot 3x^2y=e^{x^3}\cdot 6x^2$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right)=e^{x^3}\cdot 6x^2$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right)\ \mathrm{d}x=\int e^{x^3}\cdot 6x^2\ \mathrm{d}x$$

$$\left[\text{Let } x^3=u,\ \frac{\mathrm{d}u}{\mathrm{d}x}=2x^2,\ \mathrm{d}u=2x^2\ \mathrm{d}x \Rightarrow\ 2\int e^u\ \mathrm{d}u=2e^u+C=2e^{x^3}+C\right]$$

$$e^{x^3}y=2e^{x^3}+C$$

$$y=2+Ce^{-x^3}.$$

11. Euler's Method:

- For $y=f(x)$, $y_{n+1}=y_n+hf'(x_0)$, h is a constant.

$$y-y_n=f'(x_n)(x-x_n).$$

Example: 10.5

$y=x^2,\ \frac{\mathrm{d}y}{\mathrm{d}x}=2x,\ h=0.1$

---

n	x_n	y_n	Actual
0	1	1	1
1	1.1	1.2	1.21
2	1.2	1.42	1.44
3	1.3	1.66	1.69
4	1.4	1.92	1.96
5	1.5	2.2	2.25

- The smaller the h, the more accurate the approximation.
- Consider a differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$, given an initial condition. The derivative at any point on the curve $(x_0,y(x_0))$ can be approximated using the gradient of the tangent to the curve at x_0:

$$y'(x_0)=\frac{y(x_0+h)-y(x_0)}{h}.$$

Rearranging the formula, we get:

$$y(x_0+h) = y(x_0) + hy'(x+0).$$

This is the **linearization** or **Euler's method** and becomes more accurate over small increments and as long as the function does not change too rapidly.

- If $\frac{dy}{dx} = f(x_n, y-n)$ and $x_{n+1} = x_n + h$, we have

$$y_{n+1} = y_n + hf(x_n, y_n).$$

11 Maclaurin Series

1. The Maclaurin Polynomial:

Definition 15:
If $f(x)$ has n derivatives at $x=0$, then $P(x)$, the **Maclaurin polynomial** of degreen n for $f(x)$ centered at $x=0$, is the unique polynomial of degree n that satisfies:

- $$f(0) = P(0);$$
- $$f^{(n)}(0) = P^{(n)}(0);$$
- $$a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!},\ a_3 = \frac{f'''(0)}{3!},\ \cdots\ a_n = \frac{f^{(n)}(0)}{n!};$$
- $$P_n(x) = f(0) + + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

2. Maclaurin polynomials approximate the behavior of functions around a certain interval. The more terms we take, the better the approximation.

3. The Maclaurin Series:

Definition 16:
If $f(x)$ has derivatives of all orders throughout an open interval I such that $0 \in I$, then the **Maclaurin series** generated by f at $x=0$ is:

$$P_n(x) = f(0) + + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

A series converges when the sum of them is a constant (a limit can be found).

Example: 11.1

Find the Maclaurin series for $f(x) = \frac{1}{2+x}$.

$$
\begin{array}{r|r}
f(x) = (2+x)^{-1} & f(0) = 2^{-1} = \frac{1}{2} \\
f'(x) = -(2+x)^{-2} & f'(0) = -2(2)^{-2} = -\frac{1}{4} \\
f''(x) = 2(2+x)^{-3} & f''(0) = 2(2)^{-3} = 2 \times \frac{1}{8} \\
f'''(x) = -6(2+x)^{-4} & f'''(0) = -6(2)^{-4} = -6 \times \frac{1}{16} \\
f^{(4)}(x) = 24(2+x)^{-5} & f^{(4)}(0) = 24(2)^{-5} = 24 \times \frac{1}{32}
\end{array}
$$

$$
\begin{aligned}
P(x) &= \frac{1}{2} + \frac{-\frac{1}{4}}{1!}x + \frac{2 \times \frac{1}{8}}{2!}x^2 + \frac{-6 \times \frac{1}{16}}{3!}x^3 + \frac{24(2)^{-5} = 24 \times \frac{1}{32}}{4!}x^4 + \cdots \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (-x)^n.
\end{aligned}
$$

4. The Binomial series is the Maclaurin expansion for $f(x) = (1+x)^p$:

$$
(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n,\ 1 \le n \le p,\ \binom{p}{n} = \frac{p!}{n!(p-n)!} = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.
$$

Example: 11.2

Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{(x+2)^2}$.

$$
\begin{aligned}
f(x) &= (1+x)^{-2} \\
\because \binom{-2}{n} &= \frac{-2(-2-1)(-2-2)\cdots(-2-(n-1))}{n!} \\
&= (-1)^n \frac{2(3)(4)\cdots(n+1)}{n!} = (-1)^n(n+1) \\
\therefore P(x) &= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \\
&= 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^n(n+1)x^n + \cdots
\end{aligned}
$$

Example: 11.3

Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{\sqrt{2-x}}$.

$$
f(x) = (2-x)^{-\frac{1}{2}} = (2)^{-\frac{1}{2}} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}}
$$

$$
\therefore P(x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2} \binom{-\frac{1}{2}}{n} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}}.
$$

5. Applications of Maclaurin Series:

- Approximation of sin, cos, tan, ...

Example: 11.4

Approximate $\sin 3^\circ$ **using the first four terms of Maclaurin series.**

$$3^\circ = \frac{\pi}{60},\ \text{For } \sin x,\ P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\therefore P\left(\frac{\pi}{60}\right) = x - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \frac{\left(\frac{\pi}{60}\right)^7}{7!} + \cdots \approx 0.052336 \text{ (6 d.p.)}.$$

- More Complicated Functions

Example: 11.5

Find the Maclaurin series of $f(x) = e^{x^2}$**.**

Let $u = x^2, f(x) = e^u$:

$$P(x) = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Example: 11.6

Find the Maclaurin series of $f(x) = \ln\left(\frac{1+x}{1-x}\right)$**.**

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\begin{aligned}\therefore P(x) &= -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \left(x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right) \\ &= 2\left(x + \frac{x^3}{3} + \cdots\right) \\ &= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.\end{aligned}$$

Example: 11.7 Question

Find the Maclaurin series of $f(x) = \frac{x}{(1+x)^2}$**.**

Example: 11.7 Answer

$$\begin{aligned}
f(x) &= x(1+x)^{-2} \\
&= x\sum_{n=0}^{\infty}\binom{-2}{n}x^n \\
&= \sum_{n=0}^{\infty}(-1)^n(n+1)x^{n+1}.
\end{aligned}$$

- Evaluate Limits

Example: 11.8

Find $\lim_{x\to 0}\frac{1-e^{x^2}}{1-\cos x}$**.**

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\begin{aligned}
\therefore \lim_{x\to 0}\frac{1-e^{x^2}}{1-\cos x} &= \lim_{x\to 0}\frac{1-\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots\right)}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)} \\
&= \lim_{x\to 0}\frac{-x^2-\frac{x^4}{2!}-\frac{x^6}{3!}-\cdots}{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots} \\
&= \lim_{x\to 0}\frac{-x^2}{\frac{x^2}{2!}} \\
&= -2.
\end{aligned}$$

[Consider only the smallest power of x, as higher powers will go to zero much quicker.]

- Solve Differential Equations

Example: 11.9

Use the first six terms of a Maclaurin series to approximate the solution of $y' = y^2 - x$ **on an open interval centered at** $x = 0$ **if** $y(0) = 1$**.**

	$y(0) = 1$
$y' = y^2 - x$	$y'(0) = 1$
$y'' = 2yy' - 1$	$y''(0) = 2 - 1 = 1$
$y''' = 2yy'' + 2(y')^2$	$y'''(0) = 2 + 2 = 4$
$y^{(4)} = 2yy''' + 6y'y''$	$y^{(4)}(0) = 14$
$y^{(5)} = 2yy^{(4)} + 8y'y''' + 6(y'')^2$	$y^{(5)}(0) = 66$

Example: 11.9 *Continued*

$$\therefore P(x) = 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \cdots$$

- Binomial Theorem

Theroem: 11.1

Function $f(x) = (1+x)^p,\ p \in \mathbb{R}$ is equal to its Binomial series using the initial condition $y(0) = 1$.

Proof: Theorem 11.1

$$f(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots$$

$$\therefore f'(x) = p + p(p-1)x + \frac{p(p-1)(p-2)}{2!}x^2 + \cdots$$

$$xf'(x) = px + p(p-1)x^2 + \frac{p(p-1)(p-2)}{2!}x^3 + \cdots$$

$$\therefore f'(x) + xf'(x) = p + [p(p-1) + p]x + \left[\frac{p(p-1)}{2!}p(p-1) + \right]x^2 + \cdots$$

$$= p + p^2x + \frac{p^2(p-1)}{2!}x^2 + \cdots$$

$$= p(1 + px + \frac{p(p-1)}{2!}x^2) + \cdots$$

$$= pf(x)$$

$$\therefore f'(x) + xf'(x) = pf(x) \ \Rightarrow \ (1+x)f'(x) = pf(x)$$

$$f'(x) - \frac{p}{1+x}f(x) = 0 \ \Rightarrow \ P(x) = -\frac{p}{1+x},\ Q(x) = 0$$

$$\therefore I(x) = e^{\int -\frac{p}{1+x}\,\mathrm{d}x} = A(1+x)^{-p}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(A(1+x)^{-p}f(x)\right) = 0 \cdot I(x)$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left(A(1+x)^{-p}f(x)\right)\,\mathrm{d}x = \int 0\,\mathrm{d}x$$

$$A(1+x)^{-p}f(x) = C$$

$$f(x) = C \cdot A(1+x)^{x-p} = B(1+x)^{x-p} \qquad [\text{Let } B = C \cdot A.]$$

$$\text{Let } f(0) = 1: \ B = 1$$

$$\therefore f(x) = (1+x)^p.$$