IB Mathematics Analysis and Approaches HL

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Contents

1 Topic 1 Number and Algebra

1.1 Sequences and Series

1. Terms: $u_1,\ u_2,\ u_3...$
   Position: n
   Sum: S

2. **Arithmetic Sequence**/Arithmetic Progession (AP):

   - Recursive formula: $u_{n+1} = u_n + d$, d is the common difference.
   - Explicit formula: $u_n = u_1 + d(n-1)$
   - Summation: $S_n = \frac{1}{2}[2u_1 + d(n-1)]$

   Proof 1.1.1 Let $u_1,\ u_2,\ u_3,\ ...,\ u_n$ be an arithmetic sequence with d as common difference.
   Then, $S_n = u_1 + u_2 + u_3 + ... + u_n = u_1 + (u_1 + d) + (u_1 + 2d) + ... + (u_1 + (n-1)d)$
   Also, $S_n = [u_1 + (n-1)d] + ... + (u_1 + d) + u_1$.
   Add two expressions together:

   $$2S_n = [2u_1 + (n-1)d]n$$

   $$\therefore S_n = \frac{n}{2}[2u_1 + (n-1)d].$$

3. **Geometric Sequence**

   - Recursive formula: $u_{n+1} = r \cdot u_n$, r is the common ratio.
   - Explicit formula: $u_n = u_1 \cdot r^{n-1}$
   - $$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = ...$$
   - Summation: $S_n = \frac{u_1(r^n - 1)}{r-1}$

   Proof 1.1.2 Let $u_1,\ u_2,\ u_3,\ ...,\ u_n$ be a geometric sequence with r as common ratio.
   $S_n = u_1 + u_2 + u_3 + ... + u_n = u_1 + (u_1 \cdot r) + (u_1 \cdot r^2) + ... + (u_1 \cdot r^{n-1})$
   Then, $rS_n = (u_1 \cdot r) + (u_1 \cdot r^2) + ... + (u_1 \cdot r^n)$.
   Substract the first expression from the second:

   $$rS_n - S_n = u_1 \cdot r^n - u_1 \Rightarrow (r-1)S_n = u_1(r^n - 1)$$

   $$\therefore S_n = \frac{u_1(r^n - 1)}{r - 1}$$

   - If $r > 1$, the sequence is an exponential growth.
     If $0 < r < 1$, the sequence has an exponential decay.

- When $r > 1$, series approaches ∞.
  When $-1 < r < 1$, or $|r| < 1$, the series converges:

$$S_\infty = \frac{u_1}{1-r}, |r| < 1$$

1.2 Exponents and Logarithms

1. $a^m \cdot a^n = a^{m+n}$
   $a^m \div a^n = a^{m-n}$
   $(a^m)^n = a^{mn}$

2. $x^0 = 1$ $(x^0 = x^{1-1} = \frac{x^1}{x^1} = 1)$
   $x^{-m} = \frac{1}{x^m}$
   $x^{\frac{1}{n}} = \sqrt[n]{x}$ $(x^{\frac{m}{n}} = (\sqrt[n]{x})^m)$

3. If $a = b$, then $a^n = b^n$
   If $m = n$, then $a^m = a^n$
   For $a^b = 1$: $a = 1, b \in \mathbb{R}$; $a \neq 1, b = 0$; OR $a = -1, b = 2n$

4. When solving exponential equations, convert them to the same base.

5. Division Theorem.

   Theorem 1.2.1 If $a^x = b^y$ given $a > 0$ and $b > 0$, then $a = b^{\frac{y}{x}}$.

   Proof 1.2.1

$$a^x = b^y$$

$$(a^x)^{\frac{1}{x}} = (b^y)^{\frac{1}{x}} \Rightarrow a = b^{\frac{y}{x}}$$

6. $a = b^x \Leftrightarrow x = \log_b a$, where $a, b \in \mathbb{R}^+$ and $b \neq 1$.

7. Logarithmic rules:

   - $\log_a x + \log_a y = \log_a(xy)$

     Proof 1.2.2 Let $\log_a x = p$, $\log_a y = q$. $\Rightarrow a^p = x, a^q = y$.
     Then, $x \cdot y = a^p \cdot a^q = a^{p+q}$.

$$\therefore \log_a(xy) = p + q = \log_a x + \log_a y.$$

   - $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

     Proof 1.2.3 Let $\log_a x = p$, $\log_a y = q$. $\Rightarrow a^p = x, a^q = y$.
     Then, $\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$.

$$\therefore \log_a \left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y.$$

- $\log_a x^n = n \log_a x$
- $\log_a 1 = 0$
- $\log_a a = 1$
- $-\log_a x = \log_a \frac{1}{x}$
- $\log_a x = \frac{\log_b x}{\log_b a}$
- $\log_a b = \frac{1}{\log_b a}$

1.3 Proof

1. Direct proof:

   Example 1.3.1 Show that the sum of two even numbers is always even.
   Let m and n be two even positive integers.
   $m = 2p, n = 2q$, where p and $q \in \mathbb{Z}^+$.
   Then, $m + n = 2p + 2q = 2(p + q)$, which is an even number.

   Example 1.3.2 Show that $\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 \equiv x^2 + ax$.

   $$\text{LHS} = x^2 + \frac{a^4}{4} + ax - \frac{a^4}{4} = x^2 + ax = \text{RHS}.$$

   Equations "=": only true from some values.
   Identities "≡": true for all values.

   Example 1.3.3 Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.

   Example 1.3.4 Let n be a 4-digit number: $n = 1000a + 100b + 10c + d$, where $0 \leq a, b, c, d \leq 9$, and $a \neq 0$.
   It is given that $a + b + c + d = 3k, k \in \mathbb{Z}$:

   $$\begin{aligned} n &= 1000a + 100b + 10c + d + 3k - a - b - c - d \\ &= 999a + 99b + 9c + 3k \\ &= 3(333a + 33b + 3c + k) \end{aligned}$$

   Since $(333a + 33b + 3c + k) \in \mathbb{Z}$, it implies that n is divisible by 3.

2. Proof by Contradiction:

   Example 1.3.5 Prove the statement: If the integer n is odd, then n^2 is also odd.
   Let, if possible, n^2 is even and n is odd.
   Then, $n^2 = 2k,\ k \in \mathbb{Z} \Rightarrow n \times n = 2k$, which indicates the product of two odd number is even, and which is not true.
   Hence, there is a contradiction.
   $\therefore$ Our assumption is wrong, and thus given that n is odd, n^2 is also odd.

Example 1.3.6 Show that $\sqrt{2}$ is irrational.
Let us assume, if possible, that $\sqrt{2}$ is rational:
$\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, and p, q have no common factors, $q \neq 0$.
$\therefore 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \quad (1)$.
$\therefore p^2$ is even, and thus p is also even.
As p is an even number, we can write: $p = 2k, k \in \mathbb{Z}. \Rightarrow \therefore p^2 = (2k)^2 = 4k^2 \quad (2)$.
From (1) and (2): $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even, and thus q is also an even number.
But since p and q have no common factors, they cannot have "2" as a common factor. Hence, we have arrived at a contradiction.
$\therefore$ Our assumption is incorrect, and $\sqrt{(2)}$ is irrational.

Definition 1.3.1 A number is **rational** if it can be written as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, and $q \neq 0$.

Example 1.3.7 Prove that there is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$ Assume there is a real number x such that $\frac{1}{x-2} = 1 - x$.
$\therefore (1-x)(x-2) = 1 \Rightarrow x^2 - 3x + 3 = 0$
Solving the equation, we get $x = \frac{3 \pm \sqrt{9-12}}{2}$, which $\notin \mathbb{R}$
$\therefore$ We arrived at a contradiction, and our assumption is incorrect. There is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$

3. Proof by Mathematical Induction

Definition 1.3.2 Principle of Mathematical Induction (PMI):
Suppose P_n is a proposition which is defined for every integer $n \geq a,\ a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true $\forall n \geq a$.

Example 1.3.8 Prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z},\ n \geq 0$, by using PMI.
For $n = 0$, LHS $= 4^0 + 2 = 1 + 2 = 3$, which is divisible by 3.
$\therefore P_0$ (OR denoted as $P(0)$) is true.
Assume that P_k is true: i.e., $4^k + 2$ is divisible by 3. $\Rightarrow 4^k + 2 = 3A,\ A \in \mathbb{Z}^+ \Rightarrow 4^k = 3A - 2$.
Consider P_{k+1}:

$$\begin{aligned} 4^{k+1} + 2 &= 4^k \cdot 4^1 + 2 \\ &= (3A - 2) \cdot 4 + 2 \\ &= 12A - 6 \\ &= 3(4A - 2). \end{aligned}$$

$\therefore 4A - 2$ is an integer as $A \in \mathbb{Z}^+$, $4^{k+1} + 2$ is divisible by 3 whenever $4^k + 2$ is divisible by 3.
Since P_0 is true, and P_{k+1} is true whenever Pk is true, P_n is ture $\forall n \in \mathbb{Z},\ n \geq 0$.

Example 1.3.9 A sequence is defined by $u_{n+1} = 2u_n + 1\ \forall n \in \mathbb{Z}^+$. Prove that $u_n = 2^n - 1$.
For $n = 1,\ u_1 = 2^1 - 1 = 1 \Rightarrow \therefore P_1$ is ture.
Let P_k be true: $u_k = 2^k - 1$ for some $k \in \mathbb{Z}^+$.
Consider P_{k+1}:

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Since P_1 is ture, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

Example 1.3.10 Prove that $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6},\ \forall n \in \mathbb{Z}^+$**.**
For $n=1,$ LHS $=1^2=1,$ RHS $=\frac{1(1+1)(2+1)}{6}=1$
$\therefore$ LHS $=$ RHS $\Rightarrow P_1$ is true.
Assume that P_k is true, $k \in \mathbb{Z}^+:\ 1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$.
Consider P_{k+1}:

$$\begin{aligned}
\text{LHS} &= 1^2+2^2+3^2+\cdots+k^2+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = \text{RHS}.
\end{aligned}$$

Thus, P_{k+1} is true whenever P_k is true.
Since P_1 is true, and P_{k+1} is true whenver P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

Example 1.3.11 Prove that if $x \neq 1$**, the** $\prod_{i=1}^{n}(1+x^{2^{i-1}}) = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x}$**.**
For $n=1$, LHS $=1+x$, RHS $=\frac{1-x^{2^1}}{1-x}=\frac{1-x^2}{1-x}=1+x.\ \Rightarrow \therefore$ LHS $=$ RHS, P_1 is true.
Assume that P_k is true: $(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{k-1}}) = \frac{1-x^{2^k}}{1-x}$.
Conosider P_{k+1}:

$$\begin{aligned}
\text{LHS} &= (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{k-1}})(1+x^{2^k}) \\
&= \frac{1-x^{2^k}}{1-x}(1+x^{2^k}) \\
&= \frac{1+x^{2^k}-x^{2^k}+(x^{2^k})^2}{1-x} \\
&= \frac{1-x^{2^k\cdot 2}}{1-x} \\
&= \frac{1-x^{2^{k+1}}}{1-x} = \text{RHS}.
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

1.4 Counting and Binomial Theorem

1. Choose r from n: $\binom{n}{r} =_n C_r$

- $\binom{n}{m} = \binom{n}{n-m}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Fractorial notation: $n! = n(n-1)(n-2)\cdots 2\cdot 1$
  e.g. $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\times4\times3!}{3!\times2} = 5\times2 = 10.$

Example 1.4.1 Write $\frac{(n!)^2}{(n-1)!(n-2)!}$ without using fractorial notation.

$$(n!)^2 = n!\times n! = n(n-1)!\times n(n-1)(n-2)!$$

$$\therefore \frac{(n!)^2}{(n-1)!(n-2)!} = \frac{n(n-1)!\times n(n-1)(n-2)!}{(n-1)!(n-2)!} = n\cdot n(n-1) = n^3-n^2.$$

2. The number of ways of arranging n distinct objects in a row is $n!$.

3. The number of permutations of r objects out of n distinct objects is given by

$$_nP_r = \frac{n!}{(n-r)!}.$$

4. In permutations, the order matters.
   In combinations, the order does not matter.

5. The Binomial Theorem:

Theorem 1.4.1

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n,\ n\in\mathbb{N}\\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r\end{aligned}$$

Example 1.4.2 Find $(2x+3)^4$.

$$\begin{aligned}(2x+3)^4 &= (2x)^4 + \binom{4}{1}(2x)^3(3)^1 + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)(3)^3 + 3^4\\ &= 16x^4+96x^3+216x^2+216x+81\end{aligned}$$

Example 1.4.3 Find the term independent of x in the expasion of $\left(x-\frac{2}{x^2}\right)^{12}$.

General term: $\binom{12}{r}x^{12-r}\left(-\frac{2}{x^2}\right)^r$
Thus, the general expression for $x: x^{12-r-2r} = x^{12-3r}$
When $12-3r=0$, the term is independent of x: $12-3r=0 \Rightarrow r=4$.

$$\therefore \binom{12}{4}x^{12-4}\left(-\frac{2}{x^2}\right)^4 = 7920.$$

1. The independent term should not involve x in it since the independent term does not vary as x varies. (constant term)
2. The coefficient should not include x as well.

Example 1.4.4 Find the coefficient of x^3y^2 in the expansion of $(2x+y)\left(x+\frac{y}{x}\right)^5$.
Assume $2x\cdot A$ and $y\cdot B$ will yield the term x^3y^2. $\Rightarrow A = x^2y^2,\ B = x^3y$.
General term: $\binom{5}{r}x^{5-r}(\frac{y}{x})^r = \binom{5}{r}x^{5-2r}y^r$.
When $r=2,\ 5-2r=1\neq 2 \Rightarrow x^2y^2$ is not possible.
When $r=1,\ 5-2r=3 \Rightarrow x^3y$ is possible.

$$\therefore \text{Coefficient} = \binom{5}{1} = 5.$$

Example 1.4.5 Find the coefficient of x^2 in the expansion of $(1-2x)(1-4x)^7$.
Assume $1\cdot A = x^2,\ -2x\cdot B = x^2$. $\Rightarrow A = x^2,\ B = x$.
General term: $\binom{7}{r}(-4x)^{7-r}(1)^r$
When $7-r=2,\ r=5:\ \binom{7}{5}(-4x)^2(1)^5 = 336x^2. \Rightarrow 1\cdot 336x^2 = 336x^2$
When $7-r=1,\ r=6:\ \binom{7}{6}(-4x)^1(1)^6 = -28x. \Rightarrow (-2x)\cdot(-28x) = 56x^2$

$$\therefore \text{Coefficient} = 336+56 = 392.$$

6. AHL - Extention of Binomial Theorem:

Theorem 1.4.2

$$\begin{aligned}(a+b)^n &= a^n\left(1+\frac{b}{a}\right)^n \\ &= a^n(1+n\cdot\frac{b}{a}+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\frac{n(n-1)(n-2)}{3!})\left(\frac{b}{a}\right)^3+\cdots),\ n\in\mathbb{Q},\ \left|\frac{b}{a}\right|<1\end{aligned}$$

Example 1.4.6 Expand $\sqrt{1+2x}$ $\left(|x|<\frac{1}{2}\right)$ and $\frac{2}{1-3x}$ $\left(|x|<\frac{1}{3}\right)$ upto x^3 term.

$$\begin{aligned}(1+2x)^{\frac{1}{2}} &= 1+\frac{1}{2}(2x)+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{(2x)^2}{2!}+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{(2x)^3}{3!}+\cdots \\ &= 1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots.\end{aligned}$$

$$\begin{aligned}2(1-3x)^{-1} &= 2(1-(-3x)-(-1-1)\frac{(-3x)^2}{2!}-(-1-1)(-1-2)\frac{(-3x)^3}{3!}+\cdots \\ &= 2(1+3x+x^2+27x^3+\cdots \\ &= 2+6x+18x^2+54x^3+\cdots.\end{aligned}$$

Example 1.4.7 Write the first three terms in the expasion of $(2+x)^{-3}$.

$$
\begin{aligned}
(2+x)^{-3} &= 2^{-3}\left(1+\frac{x}{2}\right)^{-3} \\
&= \frac{1}{8}\left(1+(-3)\frac{x}{2}+(-3)(-3-1)\frac{2^2}{2\cdot 2!}+\cdots\right) \\
&= \frac{1}{8}\left(1-\frac{3}{2}x+\frac{12}{4}x^2+\cdots\right) \\
&= \frac{1}{8}-\frac{3}{16}x+\frac{3}{8}x^2+\cdots.
\end{aligned}
$$

Example 1.4.8 Find square root of 24 correct to 5 decimal places, using the binomial theorem.

$$
\begin{aligned}
24^{\frac{1}{2}} &= (25-1)^{\frac{1}{2}} = 25^{\frac{1}{2}}\left(1-\frac{1}{25}\right)^{\frac{1}{2}} \\
&= 5\left(1+\left(\frac{1}{2}\right)\left(-\frac{1}{25}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{25}\right)^2+\frac{\frac{1}{2\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}}{3!}\left(-\frac{1}{25}\right)^3+\cdots\right) \\
&= 5\left(1-\frac{1}{50}-\frac{1}{5000}-\frac{1}{250000}+\cdots\right) \\
&= 5(1-0.02-0.0002-0.000004) \\
&= 4.89898 \quad (5\ d.p.).
\end{aligned}
$$

1.5 Partial Fraction

1. Proper fractions: The degree of the numerator is less than the degree of the denominator.

2. Partial fraction: A method to separate one complex fraction into two or more simpler fractions.

Example 1.5.1 Find the partial fraction of $\frac{3x}{(x-1)(x+2)}$.
Let $\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2}$.

$$\therefore 3x \equiv A(x+2)+B(x-1).$$

When $x=1$, $3=3A \Rightarrow A=1$.
When $x=-2$, $-6=-3B \Rightarrow B=2$.

$$\therefore \frac{3x}{(x-1)(x+2)} \equiv \frac{1}{x-1}+\frac{2}{x+2}.$$

Example 1.5.2 Find the partial fraction of $\frac{2x+5}{(x-2)(x+1)}$.
Let $\frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2}+\frac{B}{x+1}$.

$$\therefore 2x+5 \equiv A(x+1)+B(x-2).$$

When $x=2$, $9=3A \Rightarrow A=3$.

When $x=-1$, $3=-3B \Rightarrow B=-1$.

$$\therefore \frac{2x+5}{(x-2)(x+1)} \equiv \frac{3}{x-2} - \frac{1}{x+1}.$$

Example 1.5.3 Find the partial fraction of $\frac{34-12x}{3x^2-10x-8}$.
As $\frac{34-12x}{3x^2-10x-8} = \frac{34-12x}{(3x+2)(x-4)}$, let $\frac{34-12x}{(3x+2)(x-4)} = \frac{A}{3x+2} + \frac{B}{x-4}$.

$$\therefore 34-12x \equiv A(x-4)+B(3x+2).$$

When $x=4$, $-14=14A \Rightarrow B=-1$.
When $x=-\frac{2}{3}$, $42=-\frac{14}{3}A \Rightarrow A=-9$.

$$\therefore \frac{34-12x}{(3x+2)(x-4)} \equiv -\frac{9}{3x+2} - \frac{1}{x-4}.$$

1.6 Complex Number

1.6.1 Introduction

1. Complex Number:

   Definition 1.6.1 **Complex Numbers** are numbers in the form of $a+b\mathrm{i}$, where $\mathrm{i}^2=-1$.

   - a is called the **real part**, denoted as $\mathrm{Re}(a+b\mathrm{i})=a$.
   - b is called the **imaginary part**, denoted as $\mathrm{Im}(a+b\mathrm{i})=b$.

   $a+b\mathrm{i}$ is called the Cartesian form of complex number.

2. Basic Calculations of Complex Number:

   - Define $z_1=a+b\mathrm{i}$ and $z_2=c+d\mathrm{i}$:

     $$z_1 \pm z_2 = (a \pm c)+(b \pm d)\mathrm{i}.$$

   - Define $z_1=a+b\mathrm{i}$ and $z_2=c+d\mathrm{i}$:

     $$z_1 z_2 = (ac-bd)+(ad+bc)\mathrm{i}.$$

     Proof 1.6.1

     $$\begin{aligned} z_1 z_2 &= (a+b\mathrm{i})(c+d\mathrm{i}) \\ &= ac+(ad+bc)\mathrm{i}+bd\mathrm{i}^2 \qquad [\mathrm{i}^2=-1] \\ &= (ac-bd)+(ad+bc)\mathrm{i}. \end{aligned}$$

   - Conjugate complex number:

     Definition 1.6.2 We call $a-b\mathrm{i}$ as the **conjugate** of $z=a+b\mathrm{i}$, denoted as $z^*=a-b\mathrm{i}$.

Theorem 1.6.1 Define $z_1 = a + b\mathrm{i}$, and z^* is the conjugate of z_1. Then,

$$z_1 z^* = a^2 + b^2.$$

Proof 1.6.2 By definition, $z^* = a - b\mathrm{i}$. Thus,

$$\begin{aligned} z_1 z^* &= (a + b\mathrm{i})(a - b\mathrm{i}) \\ &= a^2 - (b\mathrm{i})^2 \\ &= a^2 + b^2. \end{aligned}$$

- Define $z_1 = a + b\mathrm{i}$ and $z_2 = c + d\mathrm{i}$:

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}\mathrm{i}.$$

Proof 1.6.3

$$\begin{aligned} \frac{z_1}{z_2} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}} &= \frac{(a + b\mathrm{i})(c - d\mathrm{i})}{(c + d\mathrm{i})(c - d\mathrm{i})} \\ &= \frac{(ac + bd) - (bc - ad)\mathrm{i}}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}\mathrm{i}. \end{aligned}$$

Example 1.6.1 Find $z \in \mathbb{C}$ that satisfies the equation $\frac{z+2}{1-\mathrm{i}} = \frac{z-3\mathrm{i}}{2+\mathrm{i}}$.

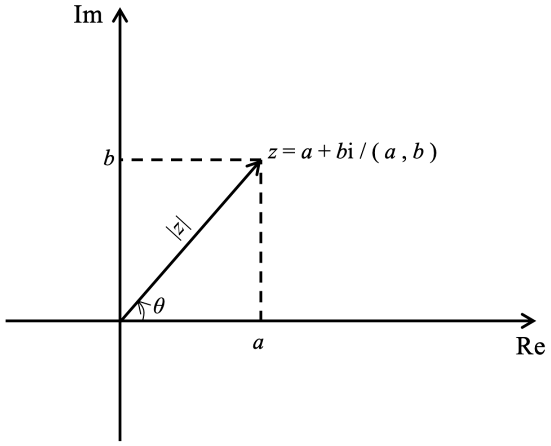
$$\begin{aligned} (z + 2)(2 + \mathrm{i}) &= (z - 3\mathrm{i})(1 - \mathrm{i}) \\ z(2 + \mathrm{i}) + 4 + 2\mathrm{i} &= z(1 - \mathrm{i}) - 3\mathrm{i} + (3\mathrm{i})^2 \\ z(2 + \mathrm{i} - 1 + \mathrm{i}) &= -3\mathrm{i} - 3 - 4 - 2\mathrm{i} \\ z(1 + 2\mathrm{i}) &= -7 - 5\mathrm{i} \\ z &= \frac{-7 - 5\mathrm{i}}{1 + 2\mathrm{i}} = -\frac{17}{5} + \frac{9}{5}\mathrm{i}. \end{aligned}$$

3. If $s = a + b\mathrm{i}$ and $t = c + d\mathrm{i}$, then:

$$\mathrm{Re}(s) + \mathrm{Re}(t) = \mathrm{Re}(s + t); \text{ and } \mathrm{Im}(\mathrm{i} \cdot s) = \mathrm{Re}(s).$$

1.6.2 Argand Diagram

1. The Complex Plane:



$z = a + bi$ can be represented on a complex plane with real coordinate a and imaginary coordinate b. It can also be denoted as $z(a, b)$.

- Modulus of a complex number:

$$|z| = \sqrt{a^2 + b^2}.$$

- Argument of a complex number:

$$\mathrm{Arg}(z) = \arctan\left(\frac{b}{a}\right)(+k\pi) \to \ \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

  *When determine a complex number, first draw it on the plane to show which quadrant it is in.
  The range of arugment is $[0, 2\pi]$ or $[-\pi, \pi]$.

- Use modulus and argument to express a complex number:

$$a = |z| \cdot \cos\theta;$$

$$b = |z| \cdot \sin\theta.$$

2. If $z = a + bi$ and $|z| = 1$, then $z^* = z^{-1}$.

Proof 1.6.4

$$\because |z| = 1$$
$$\therefore \sqrt{a^2 + b^2} = 1$$
$$\therefore a^2 + b^2 = 1$$

$\boxed{\text{Method 1}}$

$$\begin{aligned} \mathrm{RHS} = z^{-1} &= \frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} \\ &= \frac{a - bi}{a^2 + b^2} = a - bi \\ &= z^* = \mathrm{LHS}. \end{aligned}$$

$\boxed{\text{Method 2}}$

$$\begin{aligned} z \cdot z^* &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 = 1 \\ \therefore\ z^* &= z^{-1} \end{aligned}$$

3. When $|z| \neq 1$, $z^* = \frac{|z|^2}{z}$, and $z^{-1} = \frac{z^*}{|z|^2}$.

4. Properties of modulus and arguments:
   For complex number s and $t \in \mathbb{C}$:

   - $$|st| = |s||t|$$

   - $$\left|\frac{s}{t}\right| = \frac{|s|}{|t|}$$

   - $$\operatorname{Arg}(st) = \operatorname{Arg}(s) + \operatorname{Arg}(t) + 2k\pi$$

   - $$\operatorname{Arg}\left(\frac{s}{t}\right) = \operatorname{Arg}(s) - \operatorname{Arg}(t) + 2k\pi$$

1.6.3 Complex Number in Other Forms

1. The Polar Form (Modulus-Argument Form):

   - $$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$$

     Proof 1.6.5 According to the Argand Diagram:

     $$z = x + y\mathrm{i} = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta).$$

   - $$z_1 z_2 = r_1 r_2 \mathrm{cis}(\theta_1 + \theta_2)$$

   - $$\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{cis}(\theta_1 - \theta_2)$$

2. de Movrie's Theorem:

   - By Maclaurin Series:
     $$e^{\mathrm{i}\theta} = \mathrm{cis}\theta = \cos\theta + \mathrm{i}\sin\theta.$$

   - Exponential form of complex number:

     $$z = re^{\mathrm{i}\theta} = r\mathrm{cis}\theta.$$

3. Cartesian Form: Addition and Substraction
   Modulus-Argument Form: Multiply and Division
   Exponential Form: Exponents and Roots

4. Since $\operatorname{cis}\theta = \operatorname{cis}(\theta + 2k\pi)$,

$$re^{\mathrm{i}\theta} = re^{\mathrm{i}(\theta+2k\pi)}.$$

Example 1.6.2 Find $e^{\mathrm{i}\frac{17\pi}{12}}$ in the form of Cartesian.

$$\begin{aligned}
e^{\mathrm{i}\frac{17\pi}{12}} = e^{\mathrm{i}\left(\frac{7\pi}{6}+\frac{\pi}{4}\right)} &= e^{\mathrm{i}\frac{7\pi}{6}} \cdot e^{\mathrm{i}\frac{\pi}{4}} \\
&= \operatorname{cis}\left(\frac{7\pi}{6}\right) \cdot \operatorname{cis}\left(\frac{\pi}{4}\right) \\
&= \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2}-\sqrt{6}}{4} - \frac{\sqrt{2}+\sqrt{6}}{4}\mathrm{i}.
\end{aligned}$$

1.6.4 Power of Complex Number

1. For a complex number $z = re^{\mathrm{i}\theta}$,

$$z^n = r^n e^{\mathrm{i}n\theta}.$$

Example 1.6.3 Find $\left(3\cos\frac{2\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3$

$$\begin{aligned}
\left(3\cos\frac{2\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3 &= \left(-3\cos\frac{\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3 \\
&= \left(-3\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)\right)^3 \\
&= (-3)^3(e^{\mathrm{i}\frac{\pi}{3}})^3 \\
&= -27e^{\mathrm{i}\pi} \\
&= -27(-1) = 27.
\end{aligned}$$

Key learnings:
1. $z = 3$ is only the fundemental root of equation $z^3 = 27$. In $\mathbb{C}$, there are other two complex roots that satisfy the euqation.
2. In $\mathbb{C}$, $\sqrt{4} = \pm 2 = 2 + 0 \cdot \mathrm{i}$ or $-2 + 0 \cdot \mathrm{i}$.

Example 1.6.4 Given a complex number $\omega \neq 1$ is one of the solutions of $z^3 = 1$.
a. Prove $\omega^2 + \omega + 1 = 0$;
b. Calculate $\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022}$.

(a) $\boxed{\text{Approach A}}$

$$\begin{aligned}
&\because \omega^3 = 1 \\
&\therefore \omega^3 - 1 = 0 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0 \\
&\because \omega \neq 1 \\
&\therefore \omega^2 + \omega + 1 = 0.
\end{aligned}$$

$\boxed{\text{Approach B}}$ $\omega^2+\omega+1=0$ is a geometric sequence, $u_1=1,\ r=\omega$:

$$S_3=\frac{u_1(1-r^3)}{1-r}=\frac{1-\omega^3}{1-\omega}=\frac{0}{1-\omega}=0.$$

(b)

$$\begin{aligned}\omega^{2019}+\omega^{2020}+\omega^{2021}+\omega^{2022}&=\omega^{2019}\times(1+\omega+\omega^2+\omega^3)\\&=\omega^{2019}(0+1)=\omega^{2019}\\&=\left(\omega^3\right)^{673}=1.\end{aligned}$$

Example 1.6.5 Find:
a. 1^{i}**;**
b. $\ln(-1)$**;**
c. $\ln(-c)$**, where** c **is a constant.**

(a)

$$1=e^{\mathrm{i}2\pi}\ \Rightarrow\ 1^{\mathrm{i}}=\left(e^{\mathrm{i}2\pi}\right)^{\mathrm{i}}=e^{-2\pi}.\quad (1^{\mathrm{i}}=e^{-2k\pi},k\in\mathbb{Z})$$

(b)

$$-1=e^{\mathrm{i}\pi}\ \Rightarrow\ \ln(-1)=\ln\left(e^{\mathrm{i}\pi}\right)=\mathrm{i}\pi.$$

(c)

$$\ln(-c)=\ln[(-1)\cdot c]=\ln(-1)+\ln(c)=\ln(c)+\mathrm{i}\pi.$$

1.6.5 Polynomial Function with Complex Roots

1. Conjugate Pair Theorem:

   Theorem 1.6.2 If z is a complex root of $P(x)$, then the conjugate of $z(z^*)$ is also a complex root of $P(x)$. ($P(x)$ should be a polynomial with rational coefficients.)

2. Properties of Conjugate.

   - $$(s\pm t)^*=s^*\pm t^*$$
   - $$(st)^*=s^*t^*$$
   - $$\left(\frac{s}{t}\right)^*=\frac{s^*}{t^*}$$

1.6.6 Root of Complex Numbers

1. The Root of Unity:

   Theorem 1.6.3 For any complex equation $\omega^n=1$, there are n distinct roots:

$$1=e^{\mathrm{i}(0+2k\pi)}=\omega^n,\ k\in\mathbb{Z}\quad\Rightarrow\quad \omega=e^{\mathrm{i}\frac{2k\pi}{n}},\ k\in\mathbb{Z}.$$

Example 1.6.6 Solve $z^3 = 8$.

$$z^3 = 8 \cdot 1 = 8e^{i(0+2k\pi)} \;\Rightarrow\; z = 2e^{i\frac{2k\pi}{3}},\ k \in \mathbb{Z}$$

$$\begin{aligned}
k = 0 &:\ z = 2 \\
k = 1 &:\ z = 2e^{i\frac{2\pi}{3}} = 2\mathrm{cis}\left(\frac{2\pi}{3}\right) = -1+\sqrt{3}\mathrm{i} \\
k = 2 &:\ z = 2e^{i\frac{4\pi}{3}} = 2\mathrm{cis}\left(\frac{4\pi}{3}\right) = -1-\sqrt{3}\mathrm{i}
\end{aligned}$$

2. Property of $\mathrm{cis}\theta$:

$$\mathrm{cis}(-\theta) = \cos\theta - \mathrm{i}\sin\theta$$

Proof 1.6.6

$$\begin{aligned}
\cos\theta - \mathrm{i}\sin\theta &= \cos(-\theta) + \mathrm{i}\sin(-\theta) \\
&= \mathrm{cis}(-\theta).
\end{aligned}$$

2 Topic 2 Functions

2.1 Foundations of Functions

1. Relations and functions:

   Definition 2.1.1 A **relation** R is a set of ordered pairs (x,y) such that $x \in A,\ y \in B$, and sets $A,\ B$ are not empty.

   Definition 2.1.2 A **function** f is a relation in which every x-value has a unique y-value.

2. Domain and Range:

   Definition 2.1.3 **Domain** is the set of x-values.

   Definition 2.1.4 **Range** is the set of y-values.

   - Domain and Range should be in inverval notation.
     (a) Using invervals to express the inequalities

     Example 2.1.1

     $$[\,3,4\,[\ \text{ means } 3 \leq x < 4$$

     (b) If the interval will be joint, we use $\cup$ to join the inverval.

     Example 2.1.2

     $$3 < x < 4 \text{ or } x \geq 5 \ \Rightarrow\ ]\,3,4\,] \cup [5,+\infty\,[$$

     Example 2.1.3 Find the interval notation for the domain of $f(x) = \dfrac{1}{x}$.

     $$x \in\ ]-\infty,0\,[\,\cup\,]\,0,+\infty\,[\ \text{ OR } x \in \mathbb{R} \setminus 0$$

     Note: $\setminus$ means "exclude."

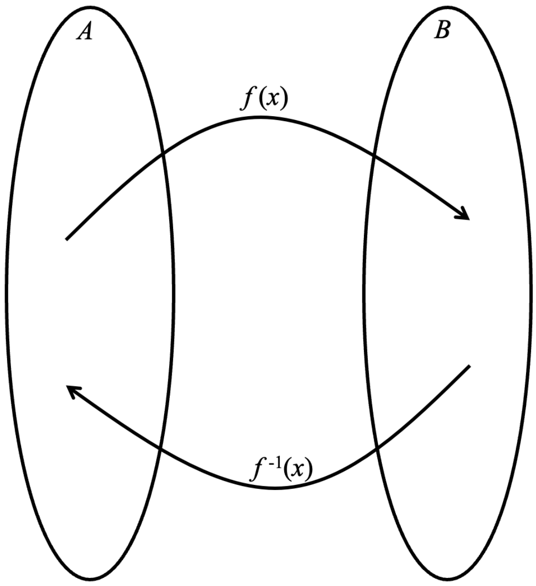
   - Since the y-values (outputs) depend on the x-values (inputs), y is the **dependent variable**, and x is the **independent variable**.
   - The independent vairbale x is also called the **argument** of the function.

3. Vertical Line test:

   - To test whether a relation is a function.
   - Since every x has one and only one value of y, there should be only one intersects.

4. Inverse of a function:

   Definition 2.1.5 $f^{-1}(x)$ is the **inverse function** of $f(x)$.



Example 2.1.4 $f(1) = 3 \Rightarrow f^{-1}(3) = 1$; $f(x) = x+5 \Rightarrow f^{-1}(x) = x-5$

- In inverse function, the input becomes the output, the output becomes the input.
- In inverse function, the domain becomes the range, the range becomes the domain.

Example 2.1.5 (a) **Find the inverse function of** $y = \dfrac{x+2}{3}$.

$$3y = x+2 \Rightarrow x = 3y-2$$
$$f^{-1}(x) = 3x-2$$

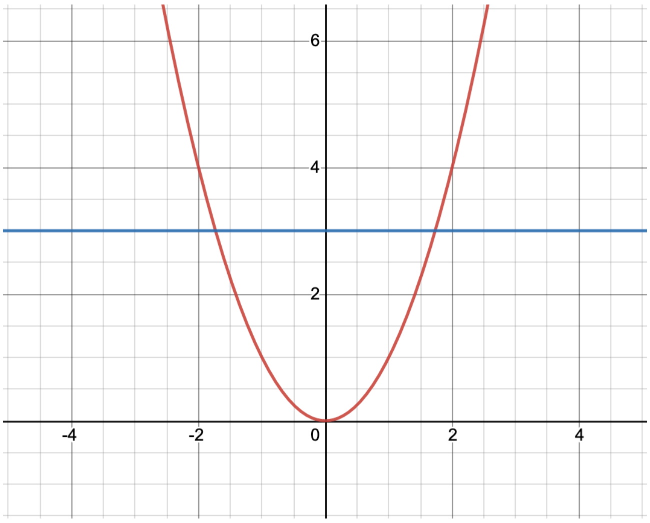
(b) **Find the inverse function of** $f(x) = \dfrac{x}{x+1}$.

$$y = \frac{x}{x+1} \Rightarrow xy+y = x \Rightarrow xy+x = y$$
$$\therefore y(x-1) = -x \Rightarrow y = -\frac{x}{x-1}$$

(c) **Find the inverse of** $\{.(4,2),(0,2),(-2,2)\}$

Inverse: $\{(2,4),(2,0),(2,-2)\}$

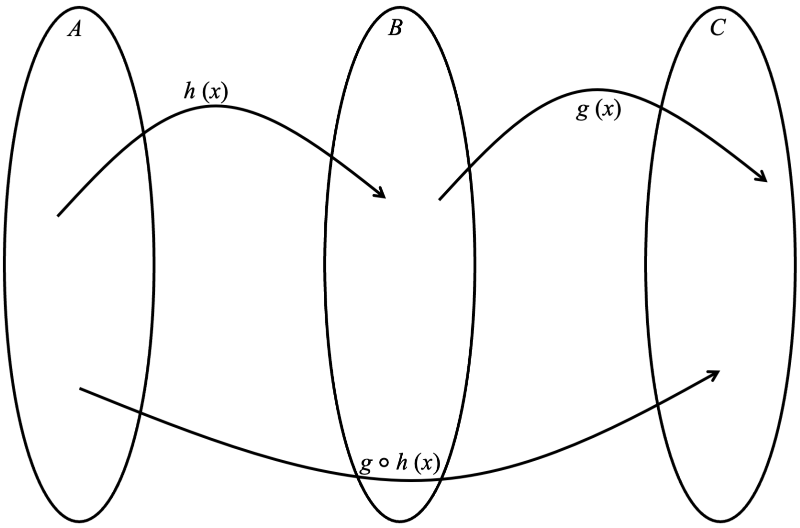
- By restricting the domain, we can find $f^{-1}(x)$ of $f(x)$, if the direct inverse of $f(x)$ is not a function.



Example 2.1.6 Horizontal line test: The largest domain we can find $f^{-1}(x)$ is $x \leq 0$ or $x > 0$.

5. Composite Functions:

Definition 2.1.6 We use $(g \circ h)(x)$ or $g(h(x))$ to represent composite functions.



Example 2.1.7 **Given** $f : x \mapsto 3x-6,\ g : x \mapsto \frac{1}{3}x+2$**. Find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$**.**

$$(f \circ g)(x) = f(g(x)) = 3(\frac{1}{3}x+2)-6 = x.$$

$$(g \circ f)(x) = g(f(x)) = \frac{1}{3}(3x-6)+2 = x.$$

When f and g are inverse functions:

$$(f \circ g)(x)(x) = x = (g \circ f)(x).$$

6. $f(x)$ and $f^{-1}(x)$ are symmetrical to $y = x$ since $D_f = R_{f^{-1}}$, $R_f = D_{f^{-1}}$. That is, if $f(x)$ passes through (a,b), $f^{-1}(x)$ passes through (b,a).

2.2 Quadratic Functions

1. The Standard Form:

$$y = ax^2 + bx + c,$$

where a is the coefficient of x^2, b is the coefficient of x, and c is the constant or y-intercept. $a, b, c \neq 0$.

- Zeros of the function (x-intercepts):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $\Delta = b^2 - 4ac$ is the discriminant of the function.

- Euqation of the line of symmetry & x-coordinate of the vertex

$$x = -\frac{b}{2a}.$$

- Vieta's Formula:

Theorem 2.2.1 Assume x_1, x_2 are two roots for equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then

$$x_1 + x_2 = -\frac{b}{a};$$

$$x_1 \cdot x_2 = \frac{c}{a}.$$

- When $a > 0$, the parabola opens upwards.
When $a < 0$, the parabola opens downwards.

2. Completion of square:

$$x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2.$$

3. The Vertex Form:

$$y = a(x-h)^2 + k, \text{ where } (h,k) \text{ is the vertex}.$$

Example 2.2.1 Given that $f(x) = ax^2 + bx + c$, find the axis of symmetry and vertex.

$$\begin{aligned} f(x) &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}. \end{aligned}$$

$$\begin{aligned} \therefore \text{ axis of symmetry: } x &= -\frac{b}{2a} \\ \text{vertex: } &\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right). \end{aligned}$$

2.3 Higher Order Polynomial Functions

1. Factor Theorem:

   Theorem 2.3.1 If $(x - a)$ is a factor of a polynomial $P(x)$, then $x = a$ must be a root for $P(x) \Rightarrow P(a) = 0$.

   Proof 2.3.1 Assume the quotient when $P(x)$ is divided by $(x - a)$ is $Q(x)$, then $P(x) = Q(x) \cdot (x - a)$. Then, $P(a) = Q(a) \cdot (a - a) = 0$.

2. Long division: solving polynomial equation.

   Example 2.3.1 For a cubic function, $P(x) = 2x^3 + bx^2 + cx + d,\ P(1) = P(2) = P(3) = 2$. **What is $P(0)$?**
   Since $P(1) = P(2) = P(3) = 2, Q(1) = Q(2) = Q(3) = 0,$ where $Q(x) = P(x) - 2$.
   Thus, $Q(x) = 2(x - 1)(x - 2)(x - 3)$.

   $$\therefore P(x) = Q(2) + 2 = 2(x - 1)(x - 2)(x - 3) + 2.$$

   $$\therefore P(0) = 2(-1)(-2)(-3) + 2 = -10.$$

3. Remainder Theorem:

   Theorem 2.3.2 When a polynomial $P(x)$ is divided by $(ax - b)$, the remainder R of this division must be

   $$P\left(\frac{b}{a}\right).$$

   Proof 2.3.2 Assume the quotient is $Q(x)$, and the reminder is R:

   $$P(x) = (ax - b)Q(x) + R.$$

   $$P\left(\frac{b}{a}\right) = 0 \cdot Q(X) + R = R.$$

4. Roots of Cubic Functions:

Theorem 2.3.3 For a cubic function $f(x) = ax^3 + bx^2 + cx + d$, given the roots of it are α, β, and γ. Then,

$$\begin{cases} \alpha+\beta+\gamma = -\frac{b}{a} \Rightarrow \sum\alpha = -\frac{b}{a} \\ \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} \Rightarrow \sum\alpha\beta = \frac{c}{a} \\ \alpha\beta\gamma = -\frac{d}{a} \Rightarrow \sum\alpha\beta\gamma = -\frac{d}{a} \end{cases}$$

Proof 2.3.3 Since α, β, γ are roots of $f(x)$,

$$f(x) = a(x-\alpha)(x-\beta)(x-\gamma).$$

$$\text{So } a(x-\alpha)(x-\beta)(x-\gamma) = ax^3 + bx^2 + cx + d,$$

$$\text{i.e., } ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\alpha\gamma+\beta\gamma)x - a\alpha\beta\gamma = ax^3 + bx^2 + cx + d.$$

$$\Rightarrow \alpha+\beta+\gamma = -\frac{b}{a}, \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}.$$

Theorem 2.3.4

$$\sum\alpha = -\frac{b}{a},\ \sum\alpha\beta = \frac{c}{a},\ \sum\alpha\beta\gamma = -\frac{d}{a},\ \sum\alpha\beta\gamma\delta = \frac{e}{a}.$$

2.4 Rational Functions

1. Reciprocal Functions: $f(x) = \frac{1}{x}$.
   - Domain: $x \in \mathbb{R},\ x \neq 0$
   - As x increases, $\frac{1}{x}$ decreases $\Rightarrow x \to \infty, \frac{1}{x} \to 0$.
   - Range: $y \in \mathbb{R},\ y \neq 0$
   - **Asymptotes**: $x = 0$, $y = 0$.
   - Axis of symmetry: $y = x$, $y = -x$.
   - **Self-inversing function**: have axis of symmetry $y = x$.

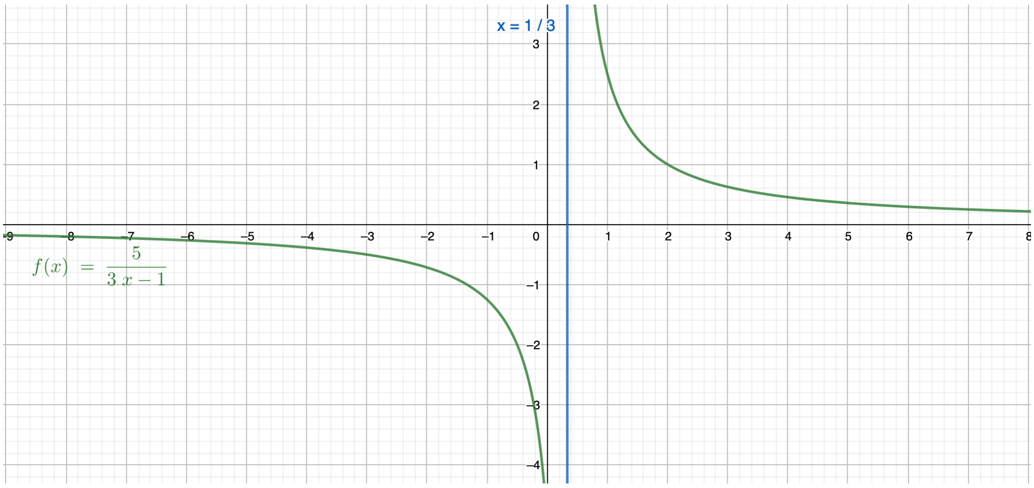
$$f(x) = f^{-1}(x).$$

2. $y = \dfrac{a}{bx+c}$
   - Vertical asymptotes (V.A.): $bx + c = 0$
   - Horizontal asymptotes (H.A.): $y = 0$

   Example 2.4.1 Draw the diagram of $y = \frac{5}{3x-1}$.
   x-intercept: $0 = \frac{5}{3x-1} \Rightarrow$ no solution, no intercept.
   H.A.: $y = 0$

y-intercept: $y = -5$

V.A.: $3x - 1 = 0,\ x = \dfrac{1}{3}$



3. $y = \dfrac{ax+b}{cx+d}$

   - V.A.: $cx + d = 0$
   - H.A.: $y = \dfrac{a}{c}$

4. $y = \dfrac{ax+b}{cx^2+dx+e}$

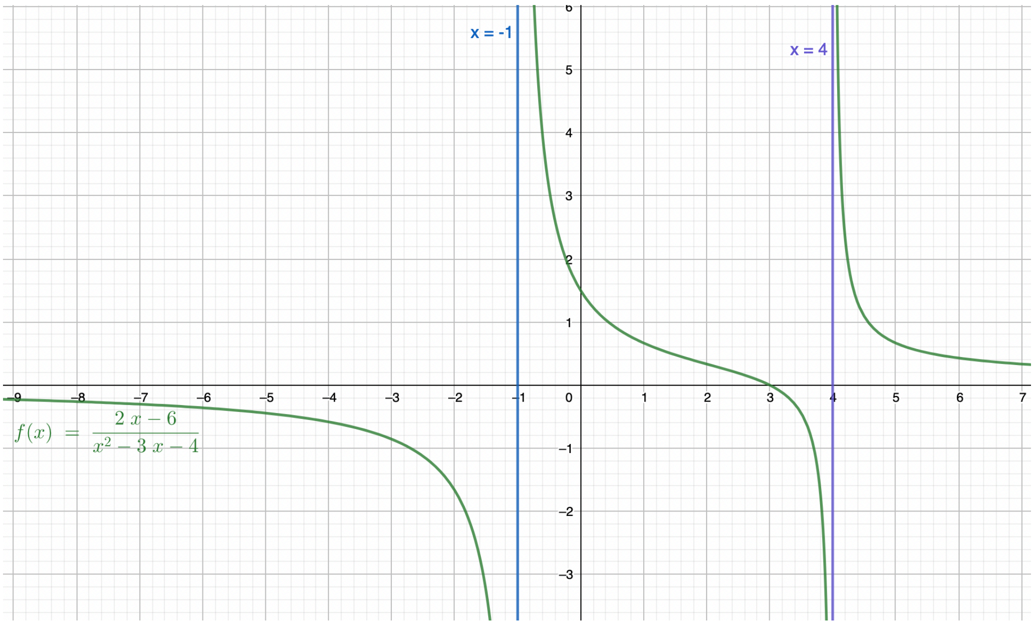
   - V.A.: $cx^2 + dx + e = 0$
   - H.A.: As $x \to \pm\infty$, $\dfrac{ax}{cx^2} \to 0$, $y = 0$
   - Intercepts: $\left(0, \dfrac{e}{c}\right)$, $\left(-\dfrac{e}{d}, 0\right)$

Example 2.4.2 Draw the diagram of $y = \dfrac{2x-6}{x^2-3x-4}$**.**

Intercept: $\left(0, \dfrac{3}{2}\right)$, $(3, 0)$

H.A.: $y = 0$

V.A.: $x = -1,\ x = 4$

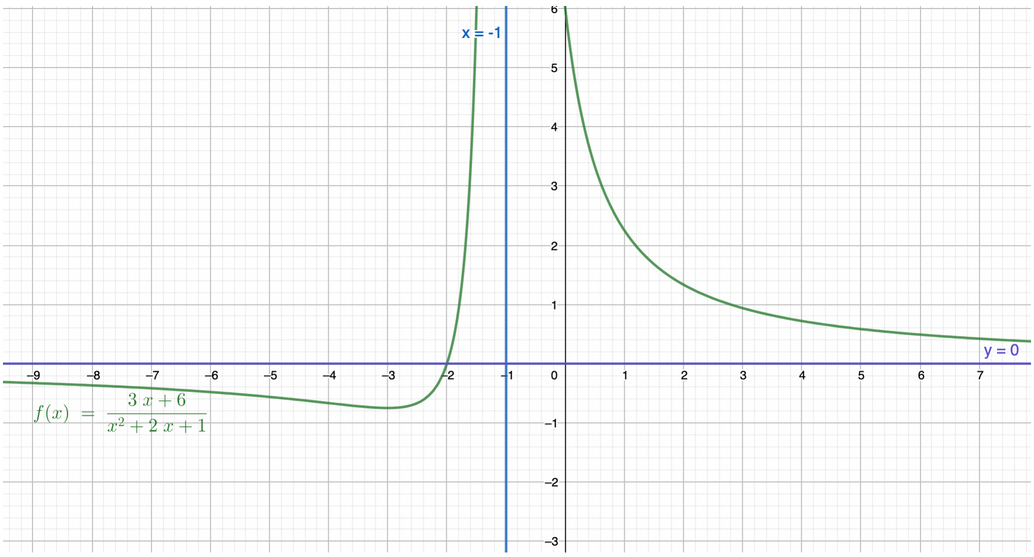


Example 2.4.3 Draw the diagram of $y = \dfrac{3x+6}{x^2+2x+1}$**.**

Intercept: $(0,6)$, $(-2,0)$

H.A.: $y = 0$

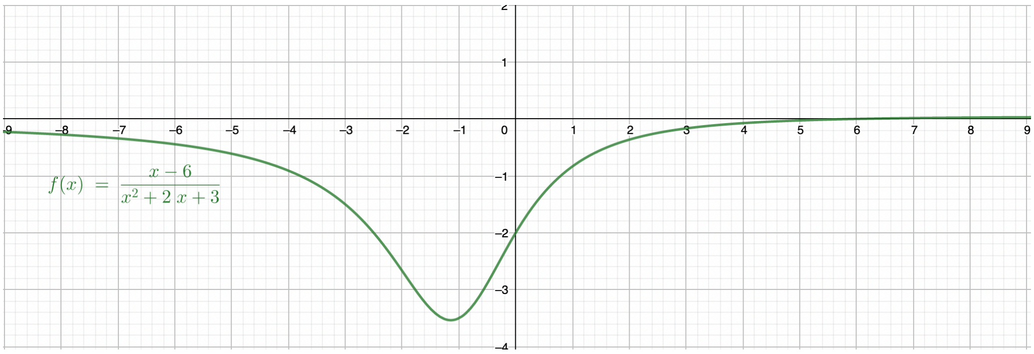
V.A.: $x = -1$



Example 2.4.4 Draw the diagram of $y = \dfrac{x-6}{x^2+2x+3}$**.**

Intercept: $(6,0)$, $(0,-2)$

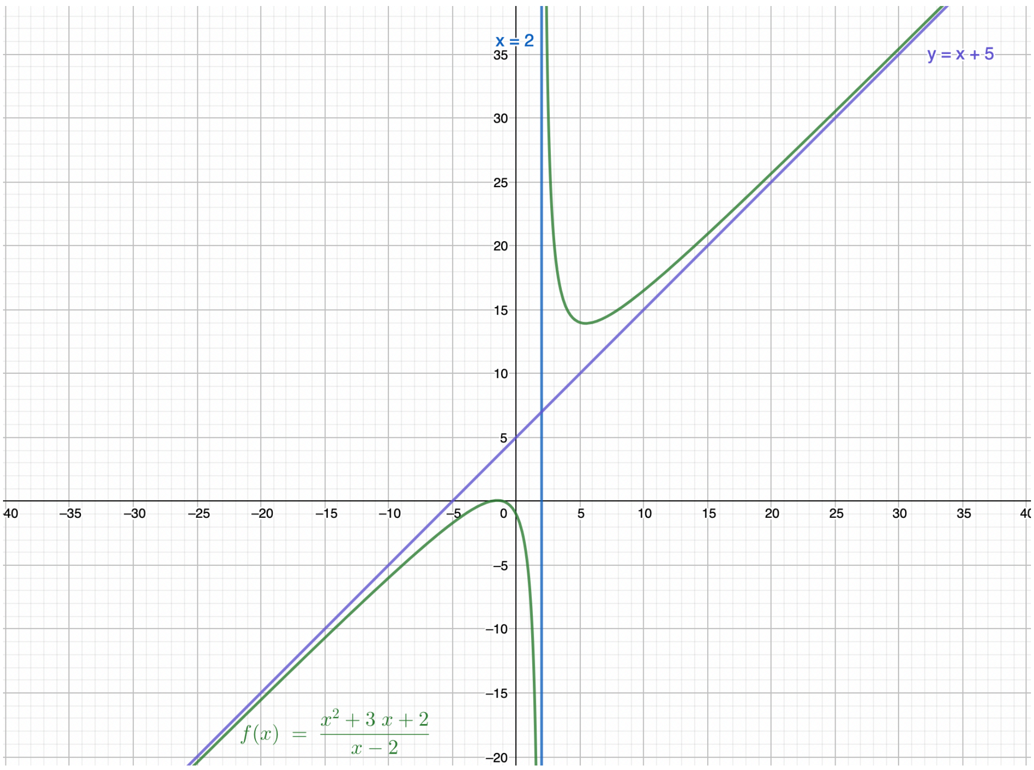
When $x \to \infty$, $f(x)$ is positive. When $x \to -\infty$, $f(x)$ is negative.



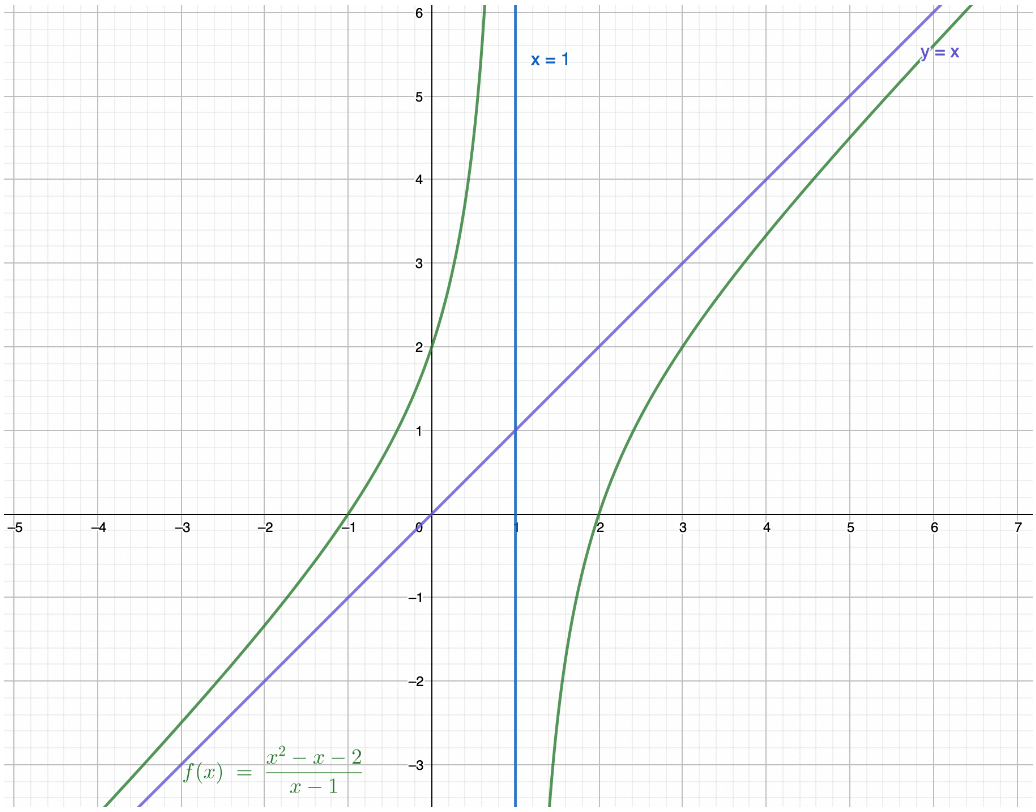
5. $y = \dfrac{ax^2+bx+c}{dx+e}$

   - V.A.: $dx+e=0$
   - **Oblique Asymptote**: Quotient of (ax^2+bx+c) dividied by $(dx+e)$.
   - Intercepts: $\left(0, \dfrac{c}{e}\right)$, $ax^2+bx+c=0$

   Example 2.4.5 Draw the diagram of $y = \dfrac{x^2+3x+2}{x-2}$**.**
   Intercept: $(0,-1)$, $(-1,0)$, $(-2,0)$
   V.A.: $x=2$
   O.A.: $y=x+5$ (Use long division)



   Example 2.4.6 Draw the diagram of $y = \dfrac{x^2-x-2}{x-1}$**.**
   Intercept: $(0,2)$, $(2,0)$, $(-1,0)$
   V.A.: $x=1$
   O.A.: $y=x$ (Use long division)



6. When the function has asymptotes:
   - Denominator$= 0$;
   - $\log_a 0$ (argument of a logarithm is 0)

2.5 Transformation of Functions

1. **Translation**:
   - $f(x+n)$ means translate $f(x)$ n units to the left.
   - $f(x-n)$ means translate $f(x)$ n units to the right.
   - $f(x)+n$ means translate $f(x)$ n units upwards.
   - $f(x)-n$ means translate $f(x)$ n units downwards.
2. Use translation vector to represent translation:
   A vector $\begin{pmatrix} a \\ b \end{pmatrix}$ means a units in the horizontal axis and b units in the vertical axis.

   Example 2.5.1 A translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ means $f(x+2)+3$, 2 units to the left and 3 units upwards.
3. **Reflections**:
   - $f(-x)$ reflects in the y-axis.
   - $-f(x)$ reflects in the x-axis.

- $f^{-1}(x)$ reflects in the $y = x$.
- $-f(-x)$ reflects in the origin.

4. **Stretches**:

- $f(qx)$ is a horizontal stretch of a scale factor of $\frac{1}{q}$.
- $pf(x)$ is a vertical stretch of a scale factor of p.

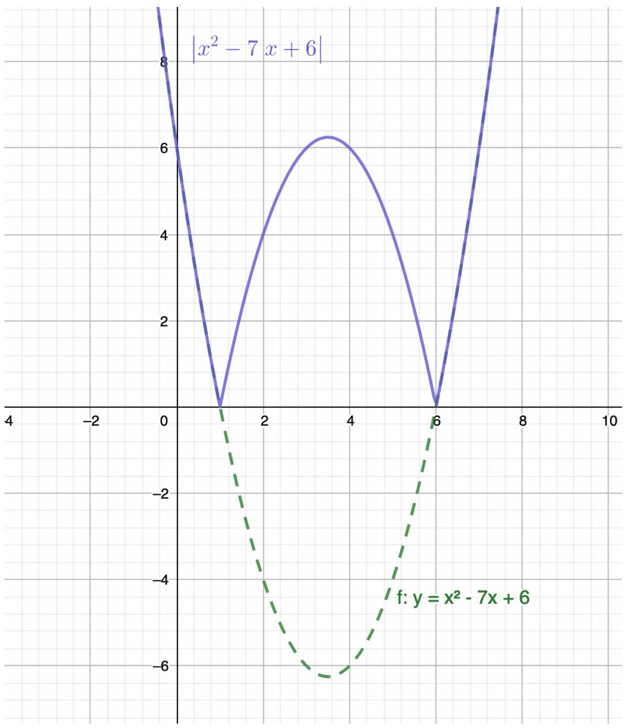
5. When a graph is transforming, the points shift but the connection remains.

6. Sequence of transformation:

- Do the horizontal translation before the horizontal stretch.
- The vertical translation is always after the vertical stretch.
- Vertical stretch $\to$ Reflection $\to$ Horizontal translation $\to$ Horizontal stretch $\to$ Vertical translation

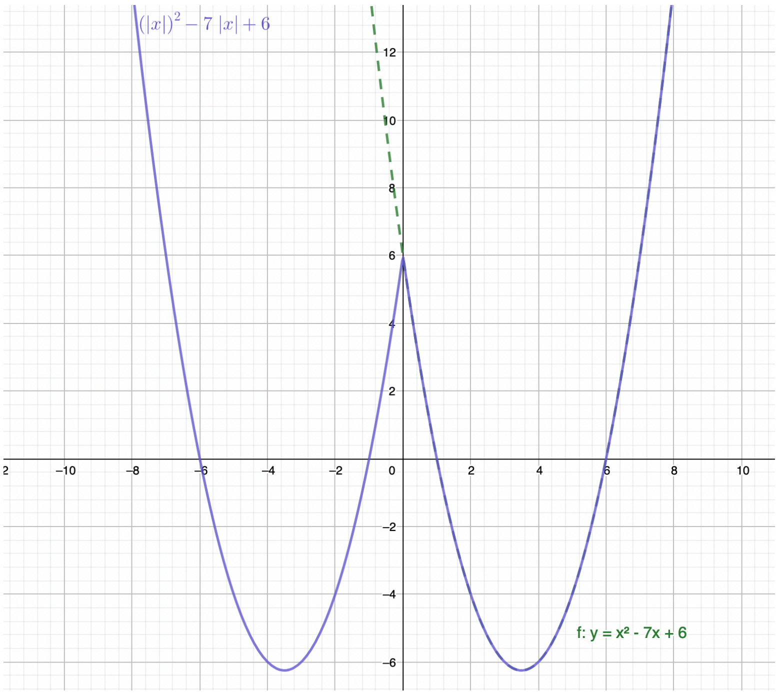
7. **Modulus Function**

- $|f(x)|$: Fold everything below x-axis above x-axis.



Example 2.5.2

- $f(|x|)$: Reflect everthing on the right of y-axis to the left. Since $|x|$ must be positive, $|x| = |-x| \Rightarrow f(-x) = f(x)$, which is an even function.



Example 2.5.3

8. **Reciprocal of $f(x)$**

   - Table of Summary:

$f(x)$	$g(x) = \frac{1}{x}$
$f(a) = 0$	Line $x = a$ is vertical asymptote
Line $x = a$ is vertical asymptote	$g(a) = 0$
$f(x) \to \infty$	$g(x) \to 0$
$f(x) \to 0$	$g(x) \to \infty$
Line $y = b$ is horizontal asymptote	Line $y = \frac{1}{b}$ is horizontal asymptote
$f(x) = a$	$g(x) = \frac{1}{a}$

   - When $f(x)$ increases, $g(x)$ decreases.

2.6 Exponential and Logarithmic Functions

1. Exponential functions:

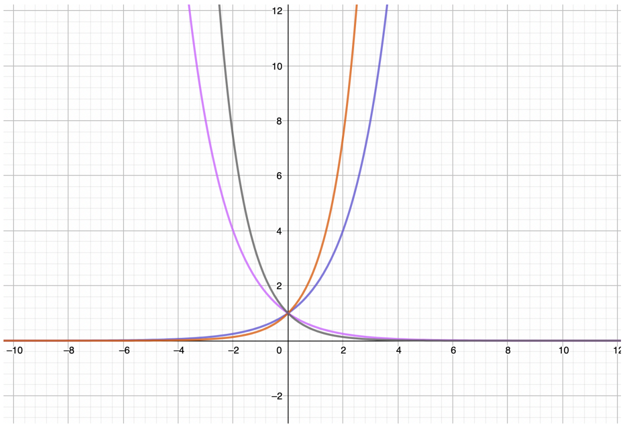
   - $f(x) = a^x$, $a > 1$ (increasing) and $0 < a < 1$ (decreasing).
   - $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$ are symmetric to the y-axis.

   Proof 2.6.1

$$g(x) = \left(\frac{1}{a}\right)^x = (a^{-}1)^x = a^{-x} = f(-x).$$

   - Domain: $x \in \mathbb{R}$, Range: $y > 0$
   - Common point: $(0,1)$; common H.A.: $y = 0$

- Graph:



2. Logarithmic functions:

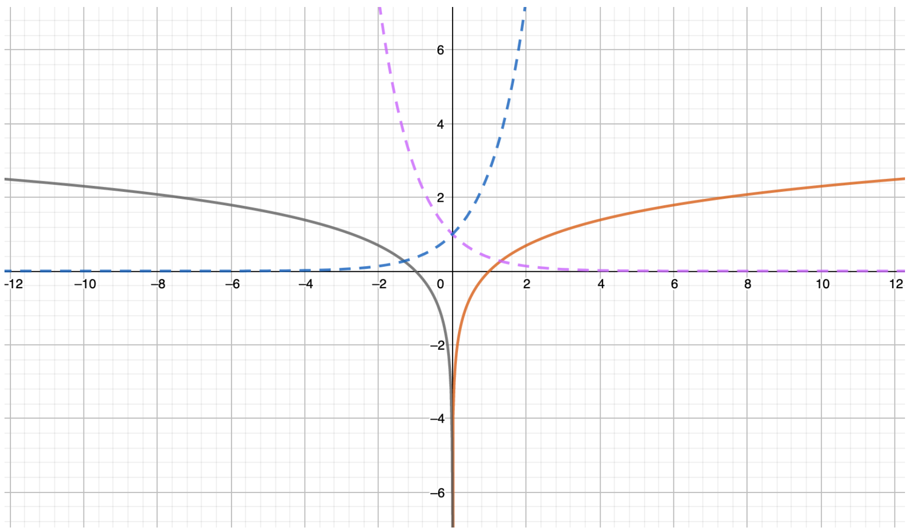
- $f(x) = \log_a x = g^{-1}(x), g(x) = a^x$.
- Common point: $(1,0)$; common V.A.: $x = 0$.
- $f(x) = \log_a x$ and $g(x) = \log_{\frac{1}{a}} x$ are symmetric to the x-axis.

Proof 2.6.2

$$\log_{\frac{1}{a}} x = \frac{\log_a x}{\log_{\frac{1}{a}} a} = \frac{\log_a x}{-1} = -\log_a x,$$

$$\therefore g(x) = \log_{\frac{1}{a}} x = -\log_a x = -f(x).$$

- When $a > 1$, increasing function; when $0 < a < 1$, decreasing function.
- Domain: $x > 0$, Range: $y \in \mathbb{R}$
- Graph:



3. Solving logarithmic equations.

4. Solving exponential equations: take logarithm on both sides.

3 Topic 3 Trigonometry and Geometry

3.1 Trigonometry

3.1.1 Radian

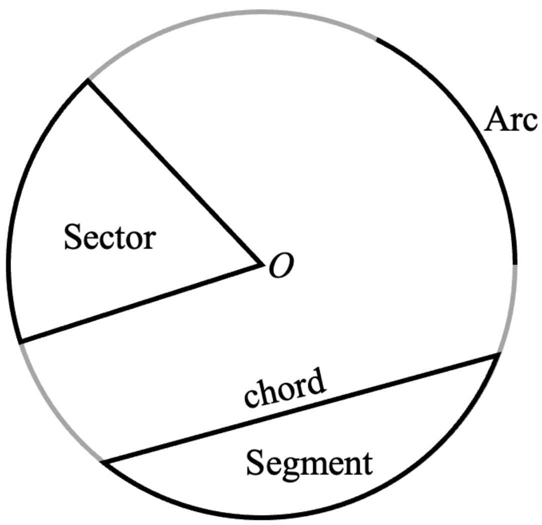
1. Radian as the unit of angle:

   -
   $$\pi \text{ rad} = 180^\circ$$

   - rad can be omitted. i.e., $\widehat{A} = 1$ means angle A is 1 radian.
   - Unit coversion:
   $$\text{degree} \times \frac{\pi}{180^\circ} = \text{radian; radian} \times \frac{180^\circ}{\pi} = \text{degree.}$$

2. Arc:

   - The **circumference** (perimeter) is $2\pi r$.

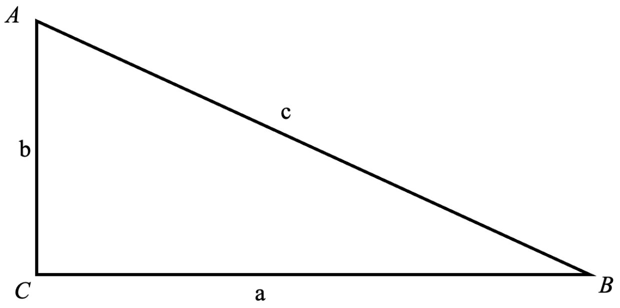


   - If the angle of the arc is θ (in radian), the length of arc$(l) = r \cdot \theta$.
   - The area of a sector:
   $$A = \frac{1}{2} r^2 \theta.$$

   - The area of a segment:
   $$A = \frac{1}{2} r^2 (\theta - \sin\theta).$$
   (Proof: the area of the triangle according to the sine rule is $\frac{1}{2}ab\sin C$)

3.1.2 Solution of Triangle

1. Define sine, cosine, and tangent:



Definition 3.1.1

$$\sin A = \frac{a}{c},\ \sin B = \frac{b}{c};$$
$$\cos A = \frac{b}{c},\ \cos B = \frac{a}{c};$$
$$\tan A = \frac{a}{b},\ \tan B = \frac{b}{a}.$$

2. The Sine Rule:

   Theorem 3.1.1

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

   - The bigger the angle, the longer the side.
   - Area of a triangle:

$$A = \frac{1}{2}ab\sin C.$$

3. The Consine Rule:

   Theorem 3.1.2

$$b^2 + c^2 - a^2 = 2bc \cdot \cos A;$$
$$a^2 + c^2 - b^2 = 2ac \cdot \cos B;$$
$$a^2 + b^2 - c^2 = 2ab \cdot \cos C.$$

4. Inverse Trigonometric Functions:

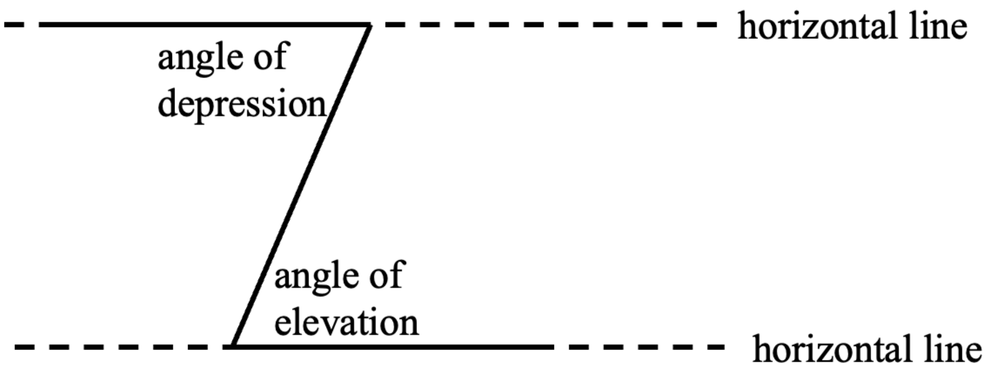
   Definition 3.1.2

$$\sin^{-1}\theta = \arcsin\theta;$$
$$\cos^{-1}\theta = \arccos\theta;$$
$$\tan^{-1}\theta = \arctan\theta.$$

5. Ambiguity of Sine Rule:

$$\sin\theta = \sin(180^\circ - \theta)\ \text{OR}\ \sin\theta = \sin(\pi - \theta).$$

6. Angle of Elevation and Depression:

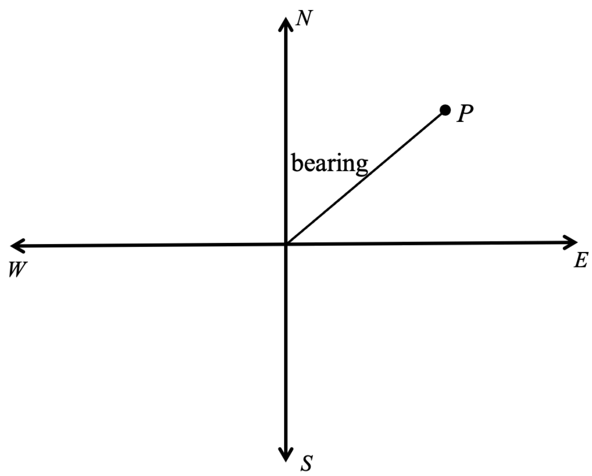


Definition 3.1.3 • **Angle of Elevation** is the angle "up" from horizontal.

- **Angle of Depression** is the angle "down" from horizontal.

7. Bearing:

   - Bearing is a way of describing direction.
   - All bearings are measured clockwise from the North direction.

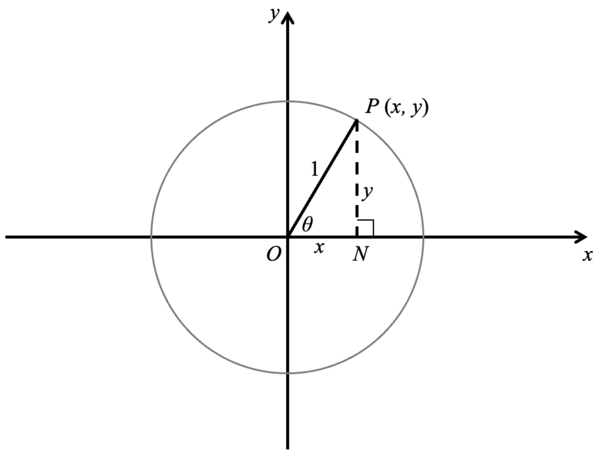


   - Bearing of A from B: construct at B.
     N.B.: Bearing of A from B is different from bearing of B from A.

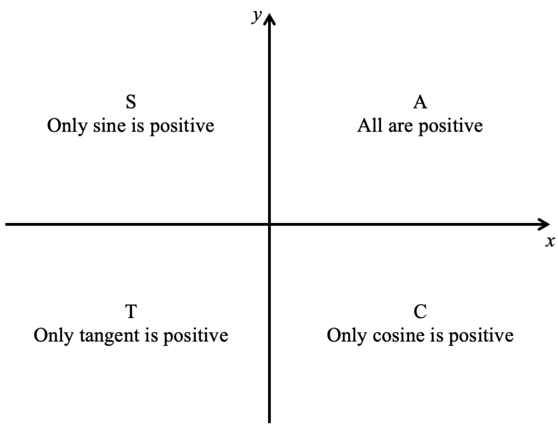
3.1.3 Definition of Trigonometric Function

1. Unit Circle:

   - Center at $(0, 0)$ with a radius of 1.



- If an angel θ opens in a counterclockwise direction, then θ is positive.
  If an angle θ opens in a clockwise direction, then θ is negative.
- In the diagram, $\theta = \theta + 2k\pi,\ k \in \mathbb{Z}$.
- $$\sin\theta = \frac{PN}{OP} = \frac{y}{1} = y;$$
  $$\cos\theta = \frac{ON}{OP} = \frac{x}{1} = x;$$
  $$\tan\theta = \frac{PN}{ON} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta};$$
- In Q_1 and Q_2, $\sin\theta$ will be positive.
  In Q_1 and Q_4, $\cos\theta$ will be positive.
  In Q_1 and Q_3, $\tan\theta$ will be positive.
  $\Rightarrow$ CAST:



2. Special Angles:

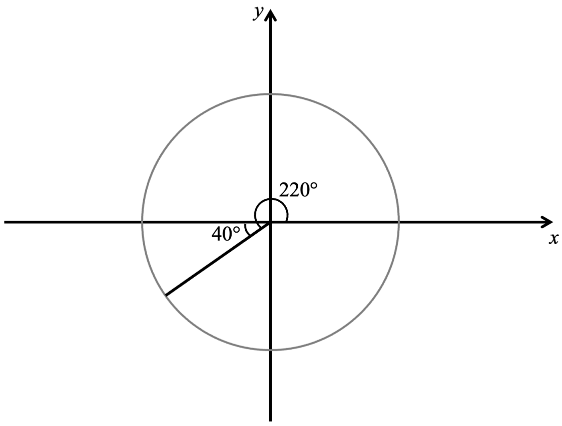
$$
\begin{aligned}
\sin 0^\circ &= 0 = \cos 90^\circ \\
\sin 30^\circ &= \frac{1}{2} = \cos 60^\circ \\
\sin 45^\circ &= \frac{\sqrt{2}}{2} = \cos 45^\circ \\
\sin 60^\circ &= \frac{\sqrt{3}}{2} = \cos 30^\circ \\
\sin 90^\circ &= 1 = \cos 0^\circ
\end{aligned}
\qquad
\begin{aligned}
\tan 0^\circ &= \frac{\sin 0^\circ}{\cos 0^\circ} = 0 \\
\tan 30^\circ &= \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\sqrt{3}}{3} \\
\tan 45^\circ &= \frac{\sin 45^\circ}{\cos 45^\circ} = 1 \\
\tan 60^\circ &= \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3} \\
\tan 90^\circ &= \frac{\sin 90^\circ}{\cos 90^\circ} = \infty
\end{aligned}
$$

3. Relative Acute Angles (RAA):

   - Acute angle is the angle with x-axis.
   - The absolute value of angles have the same acute angle is the same.

   Example 3.1.1 (a) $30^\circ,\ 150^\circ,\ 210^\circ,\ 330^\circ$ have the same acute angle.
   $\therefore |\sin 30^\circ| = |\sin 150^\circ| = |\sin 210^\circ| = |\sin 330^\circ|$.

   (b)
   $$\tan 220^\circ = \tan 40^\circ;\ \cos 215^\circ = -\cos 35^\circ$$



3.1.4 Trigonometric Identity

1. Pythagorean's Identity:
   $$\sin^2\theta + \cos^2\theta \equiv 1.$$

   Proof 3.1.1
   $$
   \begin{aligned}
   a^2 + b^2 = c^2 &\Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \\
   &\Rightarrow \sin^2\theta + \cos^2\theta = 1.
   \end{aligned}
   $$

2. Definition of Tangent:

- $$\tan\theta = \frac{\sin\theta}{\cos\theta};$$
- $$\cot\theta = \frac{1}{\tan\theta};$$
- $$\sec\theta = \frac{1}{\cos\theta};$$
- $$\csc\theta = \frac{1}{\sin\theta}.$$

3. Extended Pythagorean's Identity:

$$\tan^2\theta + 1 = \sec^2\theta;$$

$$\cot^2\theta + 1 = \csc^2\theta.$$

Proof 3.1.2

$$\begin{aligned}\sin^2\theta + \cos^2\theta = 1 \Rightarrow & \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \tan^2 + 1 = \sec^2\theta; \\ & \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow \cot^2\theta + 1 = csc^2\theta.\end{aligned}$$

N.B.: a reflex angle is an angle bigger than 180°, smaller than 360°.

4. Compound Angle Formula:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B;$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B;$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B;$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B;$$

Example 3.1.2 Find the exact value of $\cos\frac{\pi}{12}$**.**

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\frac{\pi}{4} - \frac{\pi}{6} \\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Proof 3.1.3

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$
$$= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$$
$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

5. In the linear function $y = mx + b$, $m = \tan\theta$, where θ is the angle between the line and the positive x-axis.

6. Double Angle Formula:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta;$$

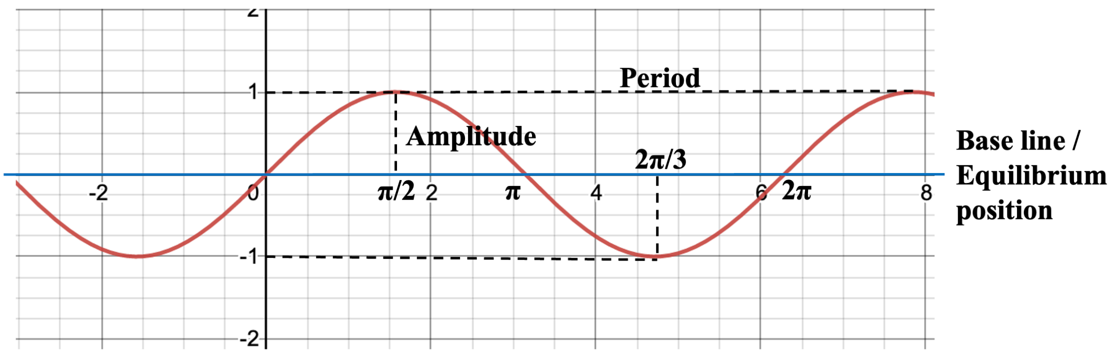
$$\sin(2\theta) = 2\sin\theta\cos\theta;$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

7. Proving Identities.

3.1.5 Trigonometric Functions and Transformation

1. Sine: Odd function: $\sin(-x) = -\sin x$.



$$\text{T(Period)} = 2\pi;$$

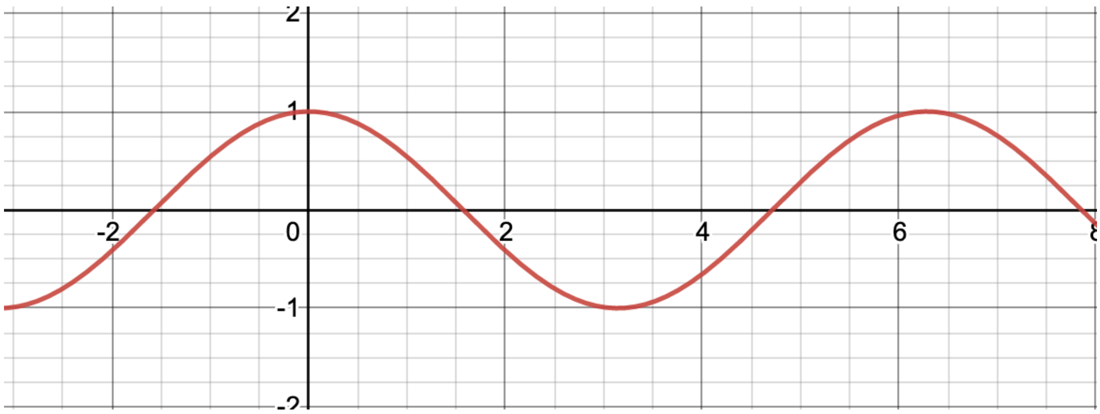
$$\text{Base line} = 0;$$

$$\text{Amplitude} = \left|\frac{y_{\max} - y_{\min}}{2}\right| = 1;$$

$$\text{Range: } \sin x \in [-1,\ 1];$$

$$\text{Domain: } x \in \mathbb{R}.$$

2. Cosine: Even function: $\cos(-x) = \cos x$.



$$\text{T(Period)} = 2\pi;$$

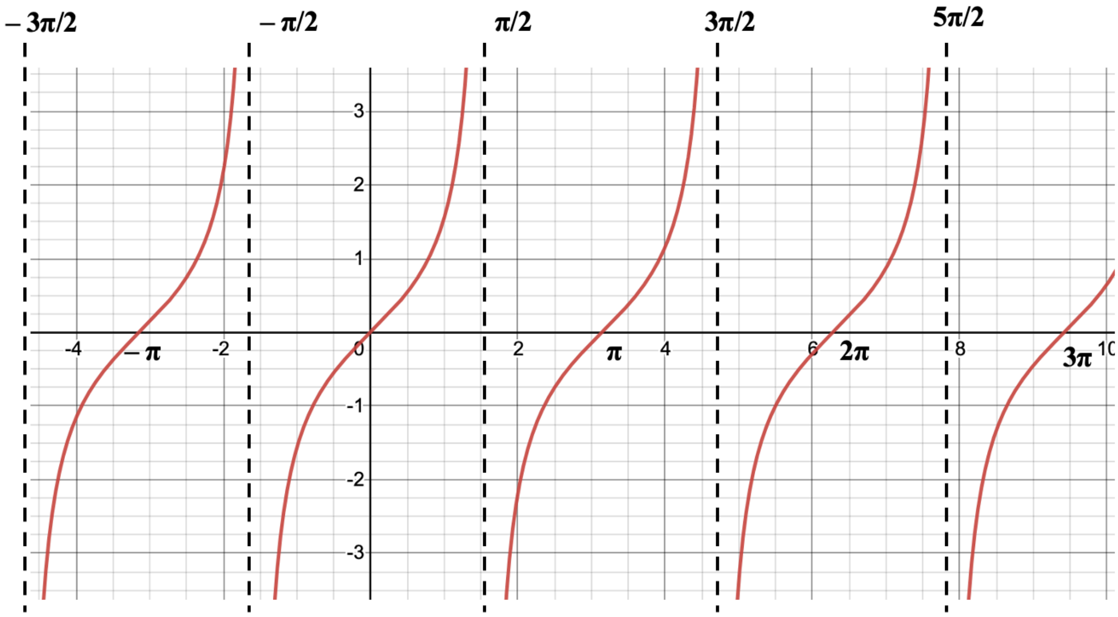
$$\text{Base line} = 0;$$

$$\text{Amplitude} = \left|\frac{y_{\max} - y_{\min}}{2}\right| = 1;$$

$$\text{Range: } \cos x \in [-1,\ 1];$$

$$\text{Domain: } x \in \mathbb{R}.$$

3. Tangent:



$$\text{T(Period)} = \pi;$$

$$\text{No amplitude(A)};$$

$$\text{V.A.: } x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z};$$

$$\text{Range: } \tan x \in \mathbb{R};$$

$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}.$$

4. Transformation of Sine and Cosine:

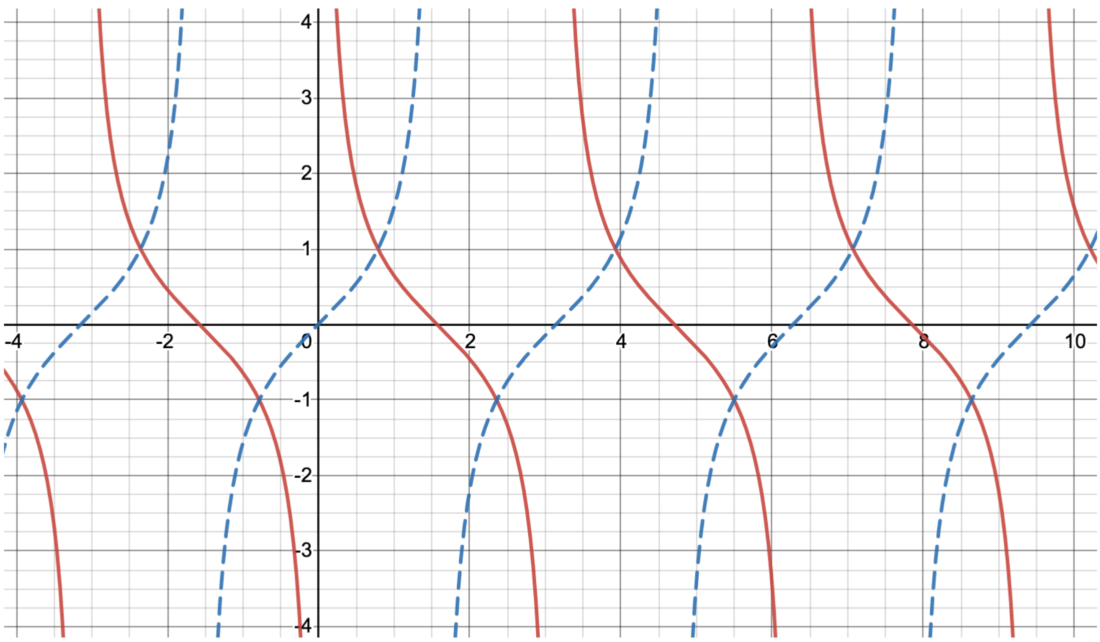
$$y = A\sin\left(\omega(x-\varphi)\right) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. $\Rightarrow$ changes $T = \frac{\pi}{\omega}$.
- Horizontal translate to the right φ units. $\Rightarrow$ changes the initial point to $(\varphi, 0)$.
- Vertical stretch with a scale factor of A. $\Rightarrow$ changes the amplitude$= |A|$.
- Vertical translation of h units upwards. $\Rightarrow$ changes the equilibrium position $y = h$.
- Range of $y = A\sin\left(\omega(x-\varphi)\right) + h$: $y \in [h-A, h+A]$.

$$y = A\cos\left(\omega(x-\varphi)\right) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. $\Rightarrow$ changes $T = \frac{\pi}{\omega}$.
- Horizontal translate to the right φ units. $\Rightarrow$ changes the initial point to $(\varphi, 1)$.
- Vertical stretch with a scale factor of A. $\Rightarrow$ changes the amplitude$= |A|$, initial point (φ, A).
- Vertical translation of h units upwards. $\Rightarrow$ changes the equilibrium position $y = h$, initial point $(\varphi, A+h)$.

5. Cotangent:

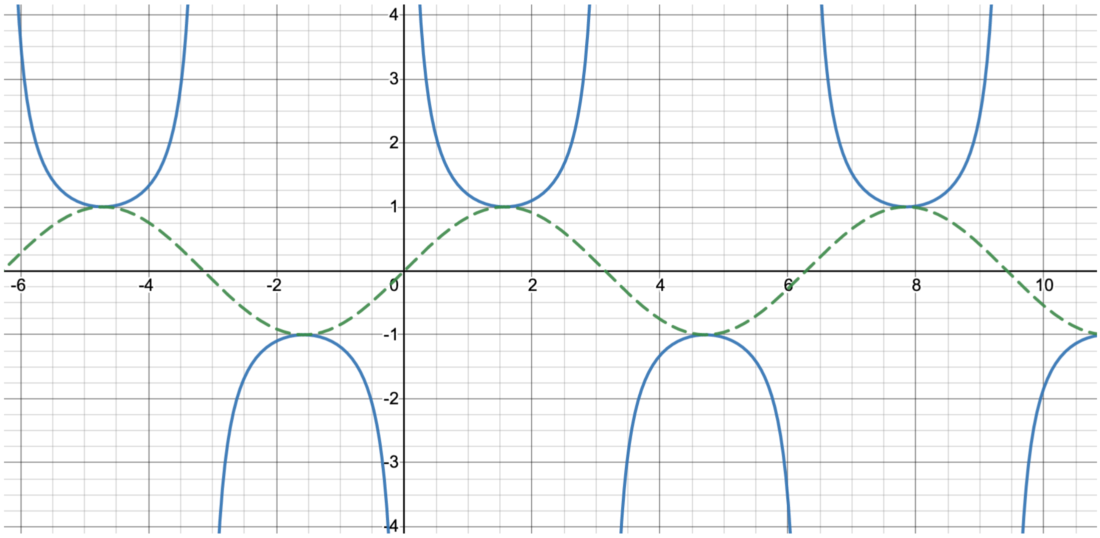


$$\text{V.A.: } x = k\pi$$

$$\text{Period: } \pi$$

$$\text{Pass through } \left(\frac{\pi}{2} + k\pi, 0\right)$$

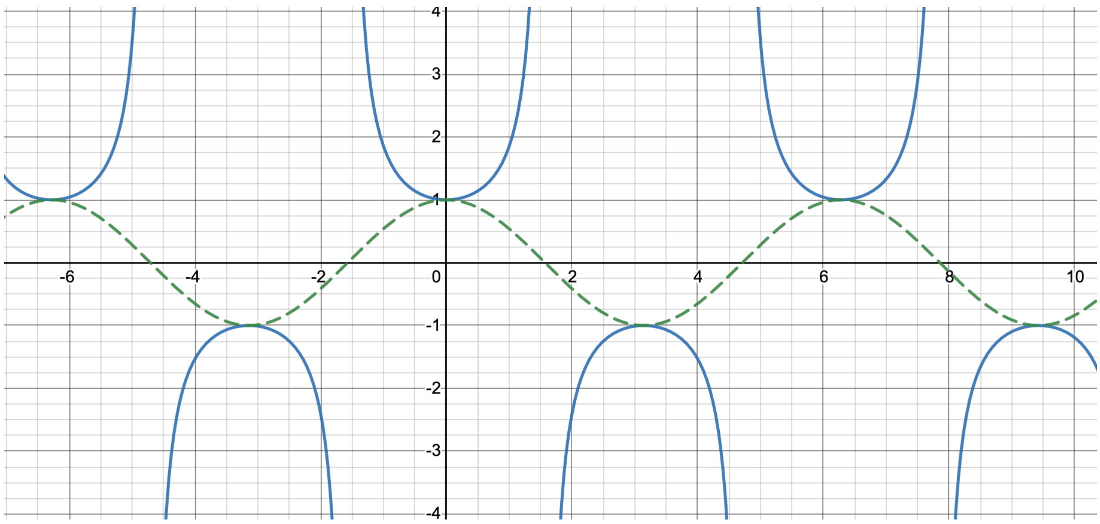
6. Cosecant:



$$\text{Domain: } x \neq k\pi$$

$$\text{Range: } y \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[$$

7. Secant:



$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi$$

$$\text{Range: } y \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[$$

8. When drawing the graph of *secx* and $\csc x$, draw *cosx* and $\sin x$ first.

9. Conversion between sine and cosine:

- $$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

- $$\cos\left(\frac{\pi}{2}-x\right)=\sin x$$
- $$\cos\left(\frac{\pi}{2}+x\right)=\cos\left[\pi-\left(\frac{\pi}{2}-x\right)\right]=-\cos\left(\frac{\pi}{2}-x\right)=-\sin x$$
- $$\sin\left(\frac{\pi}{2}+x\right)=\sin\left[\pi-\left(\frac{\pi}{2}-x\right)\right]=\sin\left(\frac{\pi}{2}-x\right)=\cos x$$

3.1.6 Solving Trigonometric Functions

1. Solving Trigonometric Functions in Paper 1:

   - Values of sepcial angles
   - From relative acute angles and CAST rule
   - Modification of period
   - Check the solution with domain

   Example 3.1.3 Solve for $\cos x=\frac{\sqrt{3}}{2}$ for $0<x<3\pi$.

   $$\text{Consider } x\in[0,2\pi]$$
   $$x=\frac{\pi}{6},\frac{11\pi}{6}.$$
   $$\text{In the domain of } x\in[0,3\pi],$$
   $$\text{Another solution is } \frac{13\pi}{6}.$$

2. Transformed Trigonometric Equations:

   Example 3.1.4 Solve $6\sin\left(2\left(x-\frac{\pi}{6}\right)\right)-2=1,\ \frac{\pi}{6}<x<2\pi$.

   $$\sin\left(2\left(x-\frac{\pi}{6}\right)\right)=\frac{1}{2}.$$
   $$\text{Let } t=2\left(x-\frac{\pi}{6}\right):$$
   $$\because \frac{\pi}{6}<x<2\pi,$$
   $$\therefore 0<2\left(x-\frac{\pi}{6}\right)<\frac{11\pi}{3},\ 0<t<\frac{11\pi}{3}.$$
   $$\sin t=\frac{1}{2}\ \Rightarrow\ t=\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{13\pi}{6},\ \frac{17\pi}{6};$$
   $$\Rightarrow x=\frac{\pi}{4},\ \frac{7\pi}{12},\ \frac{5\pi}{4},\ \frac{19\pi}{12}.$$

3. Solving Trigonometric Functions in Paper 2:

   - Change mode to RADIAN.

- Plot the functions.
- Adjust the window.
- Calculate the intersects.
- Repeat step 4 if necessary.

3.1.7 Inverse Trigonometric Functions

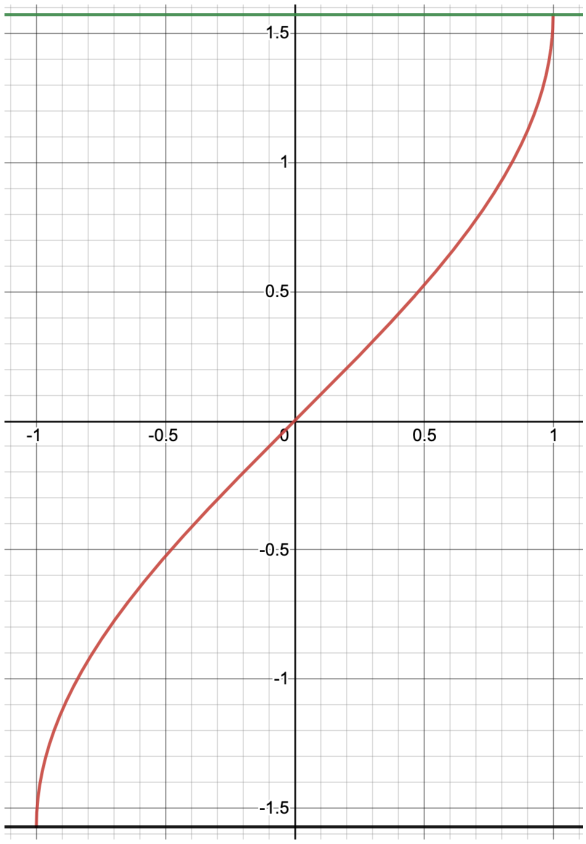
1. Inverse Trigonometric Function:

   - $$y = \arcsin x$$
   - $$y = \arccos x$$
   - $$y = \arctan x$$
   - $$\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$$
   - $$\operatorname{arccsc} x = \arcsin\left(\frac{1}{x}\right)$$
   - $$\operatorname{arccot} x = \arctan\left(\frac{1}{x}\right)$$

2. One-to-one Function:

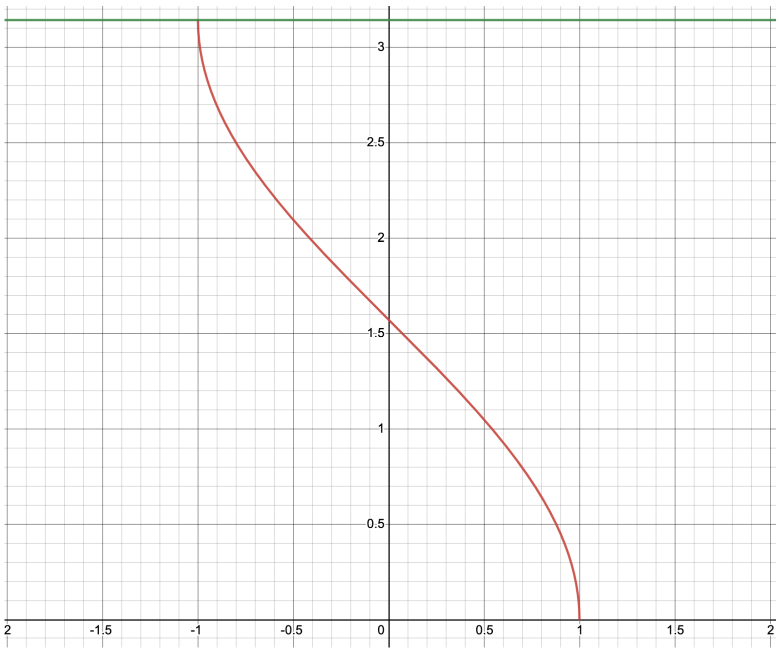
   - In order for functions to have the inverse function, it must be so called **one-to-one** function (bijection).
   - One x value to one (and only one) y value.
     One y value to one (and only one) x value.

3. Domain and range for $\arcsin x$:



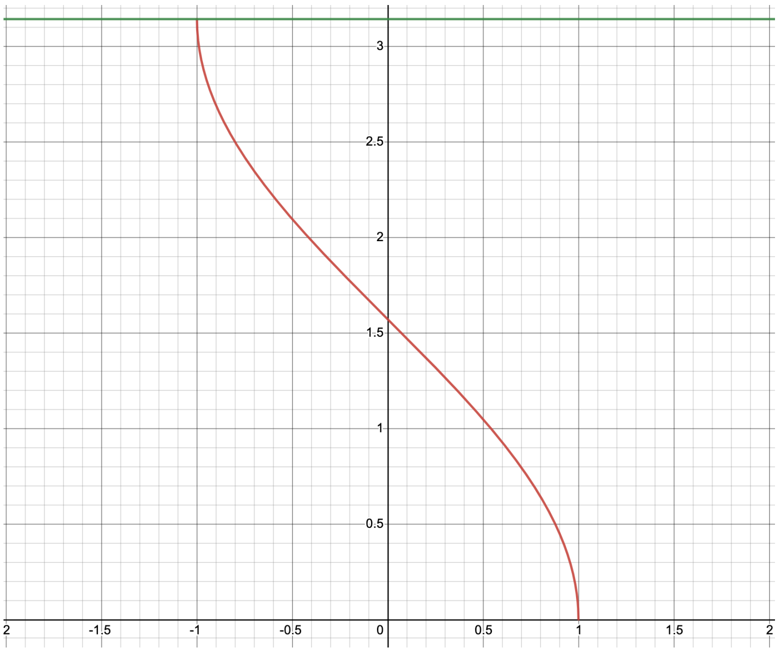
- Domain: $x \in [-1, 1]$ (Range $\sin x \in [-1, 1]$).
- Range: $\arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Domain $\sin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

4. Domain and range for $\arccos x$:



- Domain: $x \in [-1, 1]$.
- Range: $\arccos x \in [0, \pi]$.

5. Domain and range for $\arctan x$:



- Domain: $x \in \mathbb{R}$
- Range: $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

3.2 Vectors

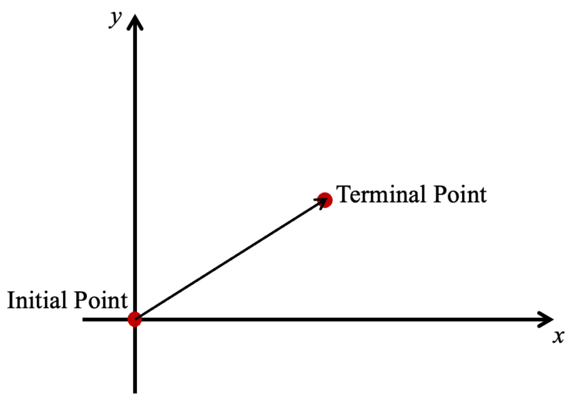
3.2.1 Introduction to Vectors

1. Vector:

   Definition 3.2.1 A **vector** is a quantity with a direction and magnitude. It is noted as $\vec{a}$.

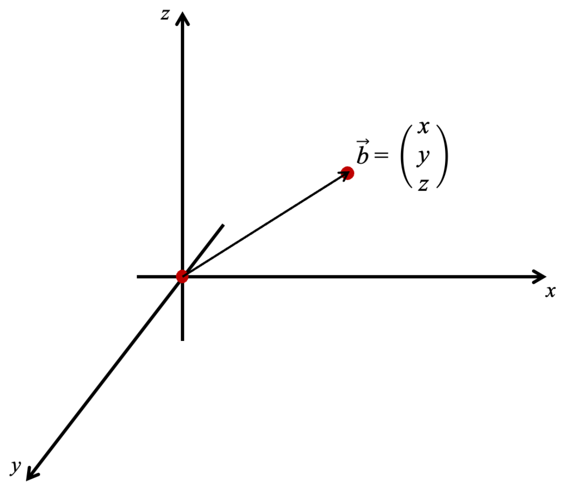
2. Components of a vector:

   - 2-D:



Example 3.2.1 The vector $\vec{a} = \binom{3}{2}$ menas 3 units in the horizontal direction and 2 units in the vertical direction.

   - 3D:



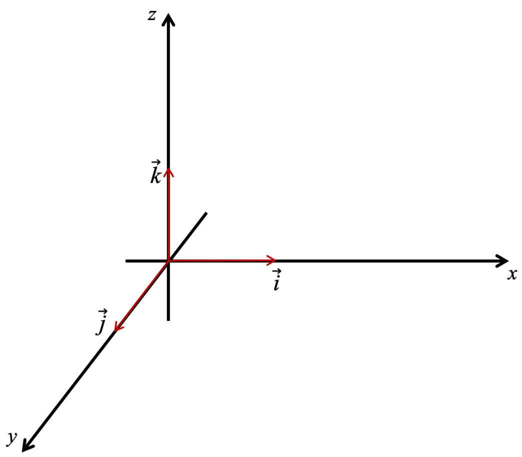
3. **Magnitude/Modulus** of vector:

   - 2D:
   $$\text{For } \vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \ |\vec{a}| = \sqrt{x^2 + y^2}.$$

   - 3D:
   $$\text{For } \vec{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ \left|\vec{b}\right| = \sqrt{x^2 + y^2 + z^2}.$$

4. **Unit Vector**: A vector of length 1:

   - $\vec{i}$: unit vector on the x-axis.
   - $\vec{j}$: unit vector on the y-axis.
   - $\vec{k}$: unit vector on the z-axis.



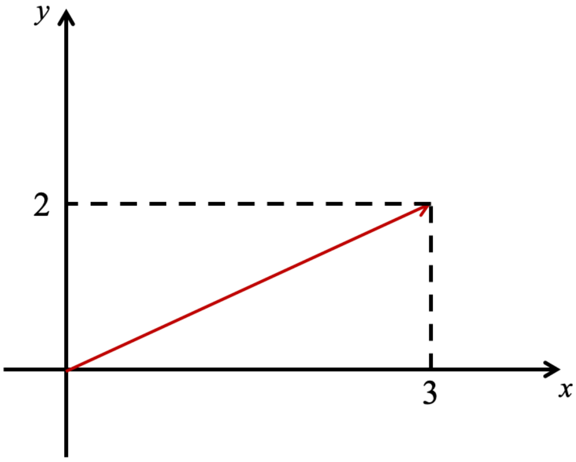
5. Sum of vectors:

   - **Position vector**: A vector that has an initial point at the origin.

Example 3.2.2

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\vec{a} = 3\vec{i} + 2\vec{j}.$$



- Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$

$$\vec{a} + \vec{b} = \begin{pmatrix} x+m \\ y+n \end{pmatrix}.$$

6. Multiplication of vectors by a scalar:

   Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and n be a scalar:

$$n\vec{a} = n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} nx \\ ny \end{pmatrix}.$$

   $n\vec{a}$ and $\vec{a}$ are in the same direction $\Rightarrow$ parallel.

7. Subtracting a vector:

   Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}.$

$$\vec{a} - \vec{b} = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

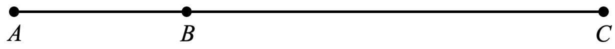
Proof 3.2.1

$$-\vec{b} = (-1)\vec{b} = \begin{pmatrix} -m \\ -n \end{pmatrix}$$

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right) = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

8. **Zero vector**: $\vec{0}$.

9. **Collinear points**: three points, A, B, and C, are said to be collinear if $\vec{AB} = t\vec{AC}$.

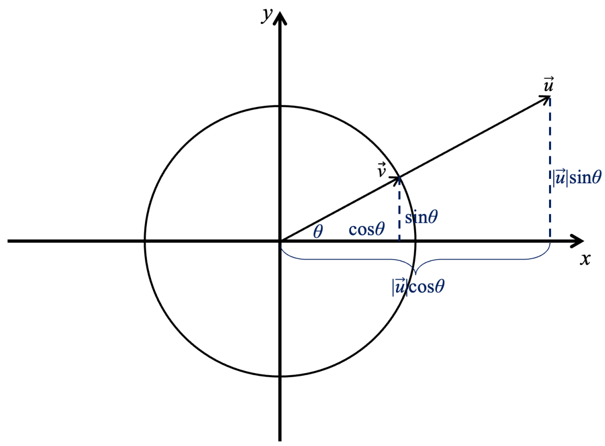


10. Find a unit vector parralel to $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$.

    - Find the value $|\vec{u}|$.
    - Then, the unit vector parralel to $\vec{u}$ is

$$\vec{v} = \frac{\vec{u}}{|\vec{u}|}.$$

11. Vectors and unit circle:



θ is the angle with the horizontal axis. The unit vector $\vec{v}$, in the same direction as $\vec{u}$ is:

$$\vec{v} = \cos\theta \cdot \vec{i} + \sin\theta \cdot \vec{j}$$

$$\vec{v} = \frac{1}{|\vec{u}|} \cdot \vec{u} \Rightarrow \vec{u} = |\vec{u}| \cdot \vec{v} = |\vec{u}|\cos\theta \cdot \vec{i} + |\vec{u}|\sin\theta \cdot \vec{j}$$
$$= |\vec{u}|\left(\cos\theta \cdot \vec{i} + \sin\theta \cdot \vec{j}\right).$$

3.2.2 Scalar Product and Its Properties

1. The **scalar product** of two vectors is a real number (scalar).

    - The algebraic definition:

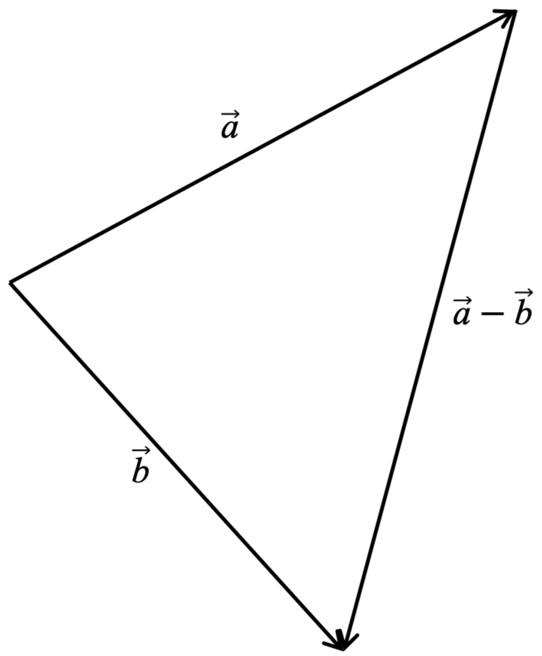
      For $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2.$$

The scalar product is also called the dot product.

- The geometric definition:
  For $\vec{a}$ and $\vec{b}$,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta,\ \theta \text{is the angle between the two vectors.}$$



Proof 3.2.2 By cosine rule:

$$\left|\vec{b}-\vec{a}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2|\vec{a}|\left|\vec{b}\right|\cos\theta$$
$$\left|\vec{b}\right|^2 - 2\vec{a}\vec{b} + |\vec{a}|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2|\vec{a}|\left|\vec{b}\right|\cos\theta$$
$$\therefore \vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta.$$

- Combining the two definitions:

$$\cos\theta = \frac{a_1b_1 + a_2b_2}{\sqrt{\left(a_1^2+a_2^2\right)\left(b_1^2+b_2^2\right)}}.$$

2. 3-D vectors: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

- $$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

- $$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)}}.$$

3. Properties of scalar product:

   - If $\vec{a}\cdot\vec{b} = 0 \;\Rightarrow\; \begin{cases} \vec{a} = 0 \\ \vec{b} = 0 \\ \vec{a} \text{ and } \vec{b} \text{ are perpendicular (orthogonal)} \;\Rightarrow\; \theta = \frac{\pi}{2} \end{cases}.$

   - If $\vec{a}$ and $\vec{b}$ are colinear,
   $$\vec{a}\cdot\vec{b} = \pm|\vec{a}|\left|\vec{b}\right|.$$

   Proof 3.2.3 Angel between $\vec{a}$ and $\vec{b}$ is $0°$.
   $\cos 0° = 1 \;\Rightarrow\; \vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|$ for $\vec{a}$, $\vec{b}$ at the same direction.
   OR $\vec{a}\cdot\vec{b} = -|\vec{a}|\left|\vec{b}\right|$ for $\vec{a}$ and $\vec{b}$ at opposite directions.

   - $$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$$

   - $$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$

   Proof 3.2.4
   $$\vec{a}\cdot\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = a_1^2 + a_2^2 = |\vec{a}|^2.$$

   - $$\vec{a}\cdot\left(\vec{b}+\vec{c}\right) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}.$$

   - $$\lambda\left(\vec{a}\cdot\vec{b}\right) = (\lambda\vec{a})\cdot\vec{b} = \vec{a}\cdot\left(\lambda\vec{b}\right).$$

3.2.3 Vector Equation of a Line

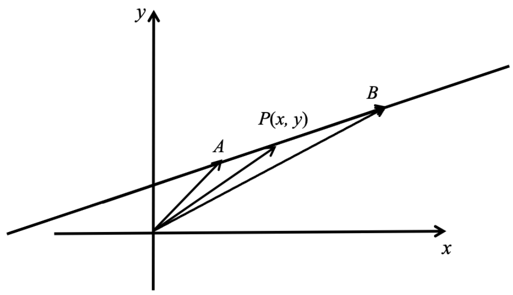
1. There is only one line that passes through two distinct points.

   Theorem 3.2.1 In the coordinate plane, the equation can be found as:
   For $A(x_1, y_1)$ and $B(x_2, y_2)$, the line passes through A, B is given by
   $$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.$$

2. **Slope, y-intercept form**: $y = mx + k$, where m is the slope, and k is the y-intercept.
   It can be rearranged to $ax + by = c$; $a, b, c \in \mathbb{R}$, where a and b cannot be equal to 0 at the same time.

3. **Vector form** of a line:

- For every point $P(x,y)$ that lies on the line AB, the vector $\overrightarrow{AP}$ must be collinear or parallel to $\overrightarrow{AB}$: $\overrightarrow{AP} = k\overrightarrow{AB}$, $k \in \mathbb{R}$.



(a) The vector $\overrightarrow{AB}$ is called a **direction vector** of the line.
All the vectors that are parralel to $\overrightarrow{AB}$ can also define the same line.

(b) Assume $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{AB}$ is the direction vector $\vec{d}$. Then, $\overrightarrow{AP} = \vec{p} - \vec{a} = k\overrightarrow{AB} = k\vec{d}$

$$\vec{p} = \vec{a} + k\vec{d}, \ k \in \mathbb{R}.$$

- Vector equation of a line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} + k \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \ k \in \mathbb{R}.$$

- Parametric form:

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases}, \ k \in \mathbb{R}.$$

- Cartesian form:

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}.$$

Proof 3.2.5

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases} \Rightarrow \begin{cases} k = \frac{x-x_1}{d_1} \\ k = \frac{y-y_1}{d_2} \end{cases}.$$

(a) Cartesian form can be further rearranged to slope-intercept form

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}$$
$$\frac{d_2}{d_1}(x - x_1) = y - y_1$$
$$y = \frac{d_2}{d_1}(x - x_1) + y_1,$$

where $\frac{d_2}{d_1}$ is the slope.

(b) Another way of interpretation:

$$\overrightarrow{AP} = k\overrightarrow{AB} \Rightarrow \vec{p} - \vec{a} = k\left(\vec{b} - \vec{a}\right)$$

$$\vec{p} = (1-k)\vec{a} + k\vec{b},\ k \in \mathbb{R}.$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = (1-k)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k\begin{pmatrix} x_2 \\ y_2 \end{pmatrix},\ k \in \mathbb{R}.$$

$$\Rightarrow \begin{cases} x = (1-k)x_1 + kx_2 = x_1 + k(x_2 - x_1) \\ y = (1-k)y_1 + ky_2 = y_1 + k(y_2 - y_1) \end{cases},\ k \in \mathbb{R}.$$

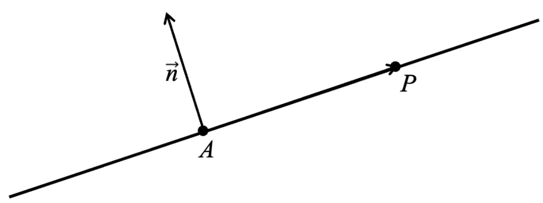
$$\Rightarrow \begin{cases} k = \frac{x - x_1}{x_2 - x_1} \\ y = y_1 + k(y_2 - y_1) \end{cases}$$

$$\Rightarrow\ y = y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)$$

$$= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.$$

4. Orthogonal / Perpendicular vector of a line.

- There is one and only one line in the plane that is perpendicular to a given line at a particular point on that line.
- Normal Vector:



Definition 3.2.2 A **normal vector** is perpendicular or **orthogonal** to any vector on the lines.

$$\text{i.e., } \vec{n} \cdot \overrightarrow{AP} = 0.$$

Theorem 3.2.2

$$\vec{n} \cdot (\vec{p} - \vec{a}) = 0 \ \Rightarrow\ \vec{n} \cdot \vec{p} = \vec{n} \cdot \vec{a}.$$

- If the direction vector $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, then one possible normal vector would be $\vec{n} = \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$ or any other vectors parallel to it.

- The vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$$
$$\Rightarrow xd_2 - yd_1 = x_1 d_2 - y_1 d_1$$
$$(x - x_1)d_2 = yd_1 - y_1 d_1$$
$$\therefore y = \frac{d_2}{d_1}(x - x_1) + y_1.$$

5. Direction vectors:

   - **Parallel lines** have **collinear** direction vectors.
   - **Perpendicular lines** have **orthogonal** direction vectors, such that the scalar product is equal to 0.

6. Vector equation of lines in 3-D spaces:

   - $$\vec{r} = \vec{a} + \lambda \vec{d}, \ \lambda \in \mathbb{R}.$$
   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$
   - The parametric form:
   $$\begin{cases} x = a_1 + \lambda d_1 \\ y = a_2 + \lambda d_2 \\ z = a_3 + \lambda d_3 \end{cases}, \ \lambda \in \mathbb{R}.$$
   - The cartesian form:
   $$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$

7. Two lines:

   - 2-D spaces: two distinctive lines can either be parallel or they can intersect.
   - 3-D spaces:
     (a) Lines are parallel.
     (b) Lines intersect at one common points.
     (c) Lines are **skewed** (do not intersect and they are not parallel).

3.2.4 Vector Product and Properties

1. The vector product is an operation that takes two vectors and results in another vector.

   - Definition

Definition 3.2.3 Given the two vectors and their components, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then the **vector product** is given by:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

- The vector product of two vectors is another vector that is perpendicular to both vectors.
- Magnitude of the vector product:

  Theorem 3.2.3 The magnitude of the vector product is given by the formula

  $$\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \sin\theta,$$

  where θ is the angle between those two vectors. If $\vec{a} \times \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are parallel/co-linear.

- The geometrical definition of cross product (vector product):

  Theorem 3.2.4 Given two vectors $\vec{a}$ and $\vec{b}$, then the vector product is given by

  $$\vec{a} \times \vec{b} = \left(|\vec{a}|\left|\vec{b}\right| \sin\theta\right)\widehat{n},$$

  where $\widehat{n}$ is the unit vector whose direction is given by the right-hand screw rule to both $\vec{a}$ and $\vec{b}$ and the vectors $\vec{a}$, $\vec{b}$, and $\widehat{n}$ follows the right-hand rule.

- Geometrical meaning of the magnitude of the vector product:
  It is equal to the area of the parallelogram enclosed by those two vectors.

2. Properties of the vector product:

   - $$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$
   - $$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \vec{a} \times \left(\vec{b} \times \vec{c}\right)$$
   - $$\lambda\left(\vec{a} \times \vec{b}\right) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times \left(\lambda\vec{b}\right),\ \lambda \in \mathbb{R}$$
   - $$\left(\vec{a} + \vec{b}\right) \times \vec{c} = (\vec{a} \times \vec{c}) + \left(\vec{b} \times \vec{c}\right).$$

3. Mixed product:

   - An operation with three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ combining both the vector and scalar product is called a **mixed product**:

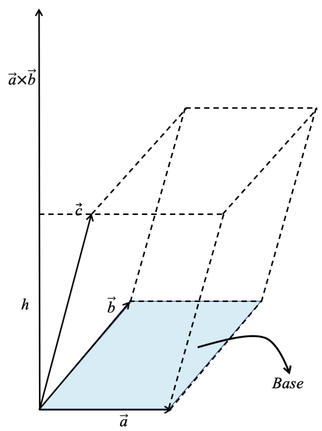
     $$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}.$$

- Given $\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$, $\vec{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, and $\vec{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$, the mixed product is given by:

$$\begin{aligned}\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} &= \begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}\cdot\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\\ &= a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1\end{aligned}$$

- Geometric meaning of mixed products:
  The volume of a parallelepiped fromed by three non-coplanar vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by:

$$V=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|.$$



Proof 3.2.6

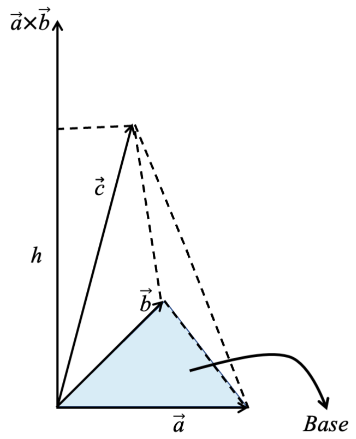
$$V=\text{Base}\times h$$

Base=magnitude of cross product of $\vec{a}$ and $\vec{b}$.
$=$ perpendicular projection of $\vec{c}$ to $\vec{a}\times\vec{b}$.

$$\therefore V=\text{Base}\times h=\left|\vec{a}\times\vec{b}\right|\cdot|\vec{c}|\cdot|\cos\theta|=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$$

- Three or more vectors are said to be coplanar if they lie in the same plane.
- Using mixed product to find the volume of a triangular pyramid:

$$V=\frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|.$$



Proof 3.2.7 Since the base is not a parallelogram but a triangle, that is half an area of the parallelogram, we multiply $\frac{1}{2}$ in front of the expression of the cross product.

$$\text{Base} = \frac{1}{2}\left|\vec{a}\times\vec{b}\right|.$$

The volume of a pyramid is $\frac{1}{3}$ of the product of the base and the height.

$$\therefore V = \frac{1}{3}\text{Base}\cdot h = \frac{1}{3}\cdot\frac{1}{2}\left|\vec{a}\times\vec{b}\right||\vec{c}||\cos\theta| = \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$$

4. Proving vector product using matrix.

Proof 3.2.8 Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Convert into a 3×3 matrix: $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$.

Find the determinant: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$

$$\Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

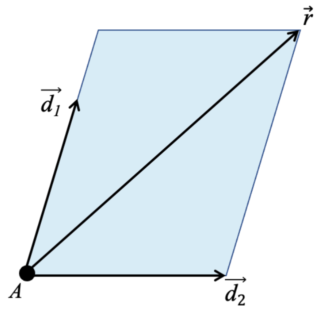
3.2.5 Vector Equation of a Plane

1. A plane is uniquely determined by three points (or a line and a point outside the line).
   $\rightarrow$ A plane can also be determined by two intersecting lines and a point outside the lines.

2. Vector equation of a plane:

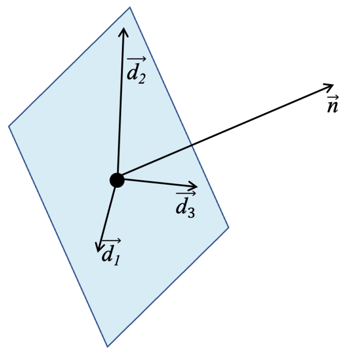
$$\vec{r} = \vec{a} + \lambda\vec{d}_1 + \mu\vec{d}_2,\ \lambda,\mu\in\mathbb{R}.$$

   where $\vec{d}_1$ and $\vec{d}_2$ are direction vectors, and $\vec{a}$ is the position vector.



3. The scalar product form:

   - **Normal vector** is a vector that is perpendicular to every line in the plane.



   - All planes with the same normal vector are parallel to each other.
   - If R is any other point on the plane, then $\overrightarrow{AR}$ lies in the plane, and it is perpendicular to the normal vector $\vec{n}$.

   Theorem 3.2.5

   $$\overrightarrow{AR}\cdot\vec{n}=0 \;\Rightarrow\; (\vec{r}-\vec{a})\cdot\vec{n}=0$$

   $$\therefore \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

   where $\vec{a}$ is the position vector, and $\vec{n}$ is the normal vector.

4. The Cartesian equation of a plane:

   $$n_1x+n_2y+n_3z=d, \text{ where } n=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix},\ d=\vec{a}\cdot\vec{n}.$$

Proof 3.2.9

$$\vec{n}=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix},\ d=\vec{a}\cdot\vec{n},\ \vec{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

The scalar product form converts to:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \vec{a} \cdot \vec{n}$$

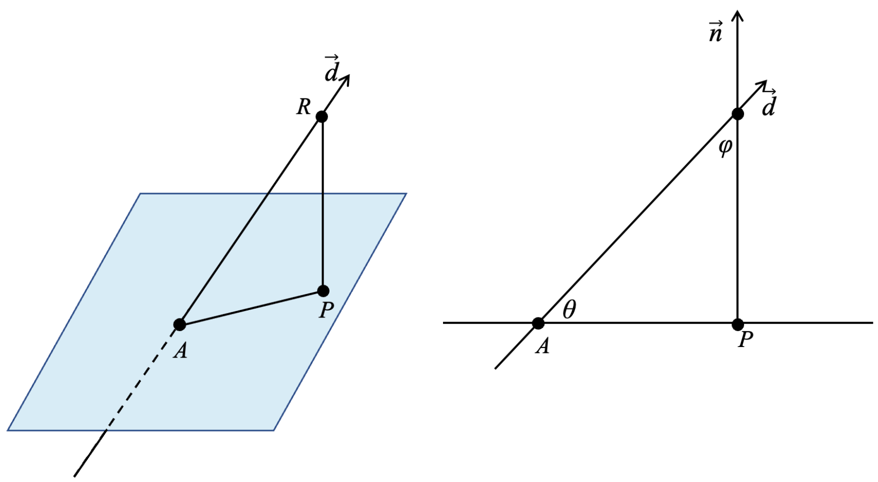
$$\Rightarrow n_1 x + n_2 y + n_3 z = d.$$

5. A plane with the vector equation $\vec{r} = \vec{a} + \lambda \vec{d_1} + \mu \vec{d_2}$ has a normal vector $\vec{n} = \vec{d_1} \times \vec{d_2}$.

3.2.6 Lines, Planes, and Angles

1. Angles and intersections between lines and planes:

   - When a line intersects a plane, the angle between them is defined as the smallest possible angle that the line makes with any of the lines in the plane.



   (a) $\overrightarrow{AR}$: the direction vector of the line, $\vec{d}$.
   (b) Point P is the projection of point R onto the plane. $\overrightarrow{AP}$ is the shadow of $\overrightarrow{AR}$ on the plane.
   (c) $\overrightarrow{PR}$ is in the direction of $\vec{n}$ since it is perpendicular to the plane.
   (d) φ is the angle between $\vec{n}$ and $\vec{d}$.
   (e)
   $$\theta = 90^\circ - \varphi, \ \cos\varphi = \frac{\left|\vec{n} \cdot \vec{d}\right|}{|\vec{n}| \left|\vec{d}\right|}.$$

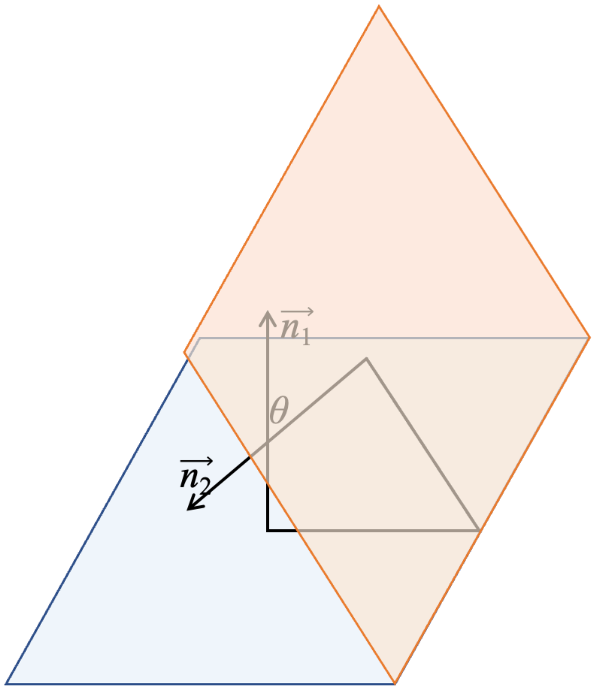
   - A line that is not parallel to a plane intersects a plane at one point. The coordinates of this point of intersection satisfies both the equation of the line and the equation of the plane.

2. Relationship of two planes:

- Two planes can either intersect at a line or they can be parallel.
- When two planes are parallel, their normal vectors are colinear; otherwise they intersect at a line.

3. Angles between two planes:

   - The angle between two planes is **the angle between their normal vectors**.
   - $$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}.$$



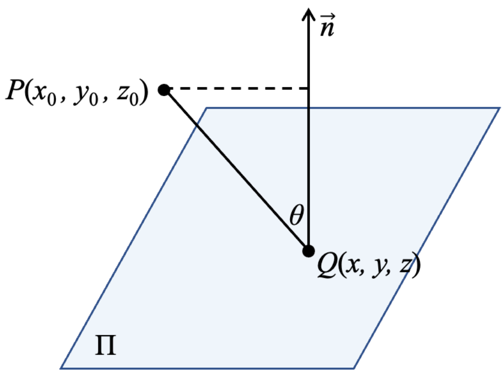
4. Two non-parallel planes intersect along a line. The equation of this line is formed by treating the Cartesian equation of two planes as simultaneous equations and finding the general solution.

5. Distance between a point and a plane.

   - The distance, d, between a point $P(x_0, y_0, z_0)$, and a plane with equation $Ax+By+Cz=D$ where $\vec{n} = A\vec{i}+B\vec{j}+C\vec{k}$, is given by:

   $$d = \frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}.$$

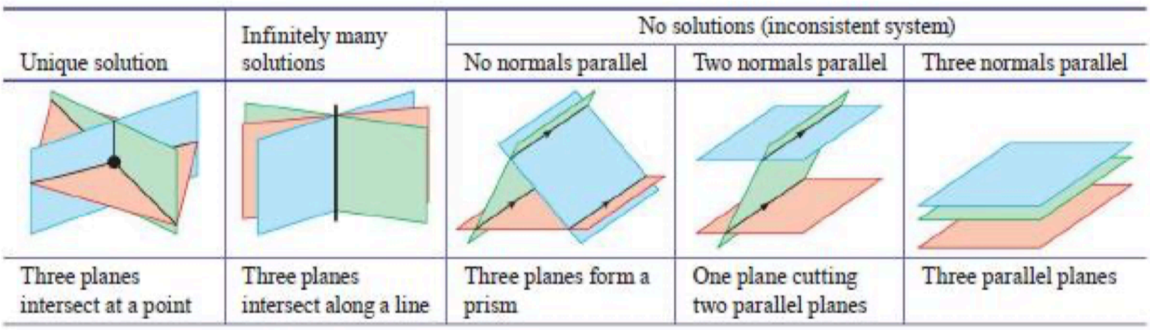
   - Proof:



Proof 3.2.10 Let $Q(x,y,z)$ be any point on the plane Π.
The distance, d, is the projection of the distance of point P to the plane on the normal vector, $\vec{n}$.

$$
\begin{aligned}
d &= \left|\overrightarrow{QP}\right| \cdot |\cos\theta| = \left|\overrightarrow{QP}\right| \cdot \frac{\overrightarrow{QP}\cdot\vec{n}}{\left|\overrightarrow{QP}\right| \cdot |\vec{n}|} \\
&= \frac{\overrightarrow{QP}\cdot\vec{n}}{|\vec{n}|} = \frac{|\langle A,B,C\rangle \cdot \langle (x_0-x),(y_0-y),(z_0-z)\rangle|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|Ax_0+By_0+Cz_0-(Ax+By+Cz)|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{Ax_0+Bx_0+Cz_0-D}{\sqrt{A^2+B^2+C^2}}
\end{aligned}
$$

6. Intersection of three points:



Unique solution	Infinitely many solutions	No solutions (inconsistent system)		
		No normals parallel	Two normals parallel	Three normals parallel
Three planes intersect at a point	Three planes intersect along a line	Three planes form a prism	One plane cutting two parallel planes	Three parallel planes

- The plane intersect:
  (a) At a point: the system of equations will have a unique solution.
  (b) Alone a line: the system of equations will have infinitely many solutions
- The systems of equations have no solutions:
  (a) No normals are parallel (the planes from a prism)
  (b) 2 normals are parallel or three normal are parallel (the planes are parallel)

4 Topic 4 Statistics

Content Pending

5 Topic 5 Calculus

5.1 Limits

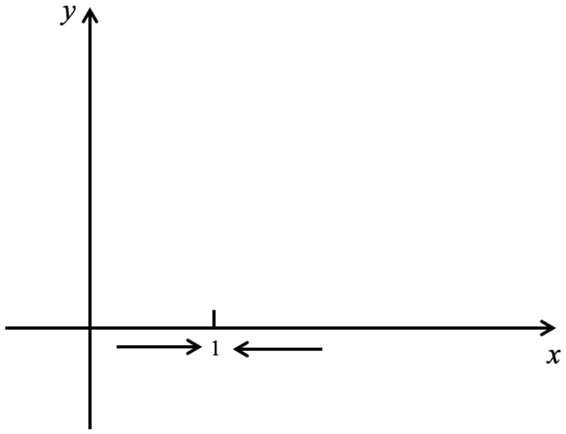
1. Limit

Example 5.1.1

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2$$

when x is approaching to 1 (it never equals to 1), the value $\frac{x^2-1}{x-1}$ is appraoching to 2.

- Left-hand and Right-hand Limit



Example 5.1.2 The left-hand limit of $\frac{x^2-1}{x-1}$ when $x\to 1$ is

$$\lim_{x\to 1^-}\frac{x^2-1}{x-1}=2.$$

The right-hand limit of $\frac{x^2-1}{x-1}$ when $x\to 1$ is

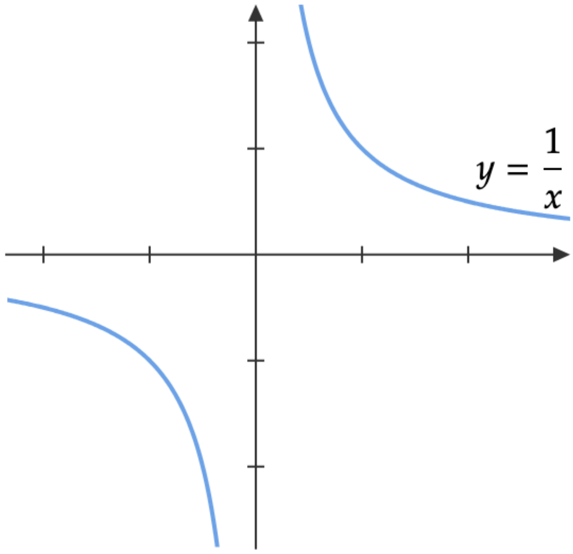
$$\lim_{x\to 1^+}\frac{x^2-1}{x-1}=2.$$

- Only when the left-hand limit and the right-hand limit exist and are the same at the point $x=a$, we say the limit of $f(x)$ exists on $x=a$.

$$\text{i.e., } \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=c \Rightarrow \lim_{x\to a} f(x)=c,\ c \text{ is a constant} \in \mathbb{R}$$

Otherwise, the limit does not exist on $x=a$ (OR DNE.).

Example 5.1.3 Does $\lim_{x\to 0}\frac{1}{x}$ exist? How about $\lim_{x\to\infty}\frac{1}{x}$?



- $\lim_{x \to 0} \frac{1}{x}$ doest not exist.

$$\because \lim_{x \to 0^+} \frac{1}{x} = +\infty, \; \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\therefore \lim_{x \to 0^+} \frac{1}{x} \neq \lim_{x \to 0^-} \frac{1}{x} \Rightarrow \text{DNE.}$$

- $\lim_{x \to \infty} \frac{1}{x}$ exists.

$$\because \lim_{x \to +\infty} \frac{1}{x} = 0, \; \lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x \to +\infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} \Rightarrow \text{Limit exists.}$$

Definition 5.1.1 **Horizontal Asymptote (H.A.)**:

$$y = \lim_{x \to \infty} f(x) = c$$

- Limit at ∞:

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = c \;\Rightarrow\; \lim_{x \to \infty} f(x) = c.$$

Note: $+\infty$ and $-\infty$ are not exact values; they should be regarded as a concept.

- Limites does not have to equal to the function value.
  Limit and the function value do not have relationships.
- Generally speaking, if $a \in D_f$, $\lim_{x \to a} f(x) = f(a)$.

2. For a rational function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$, and $Q(x) = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$:

   - $\lim_{x \to a} f(x) = f(a)$ as long as $Q(a) \neq 0$.
   - $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m} \;\Rightarrow\;$ H.A.

     (a) If $m = n$, $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a_0}{b_0} = \frac{a_0}{b_0}$.

(b) If $m > n$, $\lim_{x\to\infty} f(x)$ DNE.

(c) If $m < n$, $\lim_{x\to\infty} f(x) = 0$.

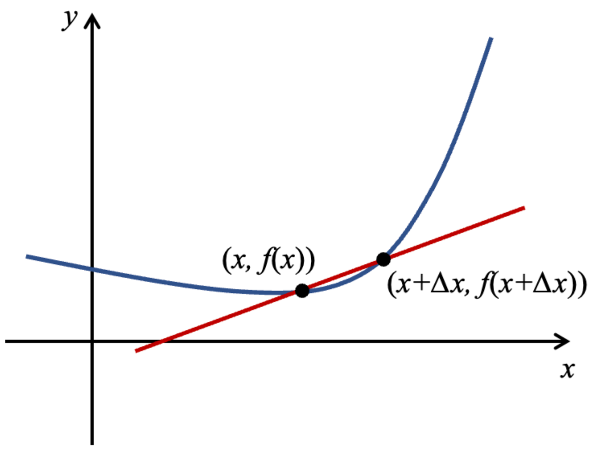
3. Continuity and Discontinuity

Definition 5.1.2 Continuity: If the graph of the function does not have any breaks or holes within a certain interval, then the function is continuous within that interval.

Theorem 5.1.1 If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$, then the function f is **continuous** at $x = a$.

5.2 Differentiation and Derivatives

1. Gradient of Secant:



- $$\text{Slope } m = \frac{f(x+\Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Definition 5.2.1 Derivative of a function:

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \text{ is the derivative of a function, denoted as } \frac{\mathrm{d}y}{\mathrm{d}x} \text{ or } f'(x).$$

- The graphic meaning of derivative is the gradient of tangent of the function.

Example 5.2.1 By definition, find the derivative of $f(x) = x^2 + 1$ and hance find the gradient of the tangent line when $x = 3$.

$$\begin{aligned}
f'(x) = \lim_{x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{x\to 0} \frac{\left[(x+\Delta x)^2 + 1\right] - (x^2+1)}{\Delta x} \\
&= \lim_{x\to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x} \\
&= \lim_{x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{x\to 0} (2x + \Delta x) \\
&= 2x.
\end{aligned}$$

At $x = 3$, $f'(3) = 2 \times 3 = 6$. The gradient is 6.

2. Derivative of x^n

Theorem 5.2.1 If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}, \text{ for any } n \in \mathbb{R}.$$

Note: The derivative of any constant is 0.

Example 5.2.2

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = (-1)x^{-1-1} = -x^{-2};$$
$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}};$$
$$f(x) = c = cx^0 \Rightarrow f'(x) = 0 \times cx^{0-1} = 0.$$

3. Rules of Differentiation:
Name $f(x)$ and $g(x)$ as two functions with derivatives of $f'(x)$ and $g'(x)$, respectively. Then

$$(cf(x))' = cf'(x)$$
$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

4. More Derivatives:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

5. Differentiablity:

Definition 5.2.2 A function has to be **continuous** and **no sharp turning point** to be **differentiable**.
Note: Smooth turning point on the graph is allowed.

6. More Rules of Differentiation:

Theorem 5.2.2 Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$(f(x) \times g(x))' = f'(x)g(x) + f(x)g'(x).$$

Theorem 5.2.3 Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Theorem 5.2.4 For a composite function $f(g(x))$ or $(f \circ g)(x)$, the derivative will be

$$f'(g(x)) \times g'(x).$$

OR

If $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

7. Higher Order Differentiation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2},\ f''(x),\ f'''(x),\ f^{(4)}(x),\ f^{(5)}(x),\ \cdots$$

5.3 Applications of Derivatives

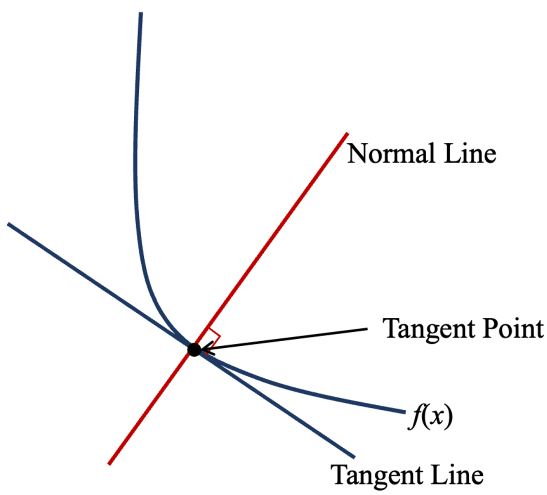
1. Equation of Tangent Line:
   Via the original functions, we could get the tangent point (x_0, y_0). Then, the expression of the tangent line is

$$y - y_0 = m(x - x_0),$$

   where m is the derivative.

2. Normal and Tangent Lines:

   Definition 5.3.1 **Normal** is perdendicular to the tangent and passes through the same tangent point.



3. Increasing and Decreasing Function:

Definition 5.3.2 **Increasing Function**: As x is getting larger, y is getting larger.
i.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}x} > 0.$$

Decreasing Function: As x is getting larger, y is getting smaller.
i.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}x} < 0.$$

4. Local Extrema: $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ Stationary point
   Global extrema is the maximum and the minimum points of the entire function.
   $f''(x)$ is used to determine if the local extrema is maxima or minima.
   - Minima: $f''(x) > 0$ Concave up.
   - Maxima: $f''(x) < 0$ Concave down.
   - Point of Inflection (the point that is changing from concaving up to concaving down, or vice visa): $f''(x) = 0$
5. With local extrema, x-intercepts, y-intercepts, concavity, and asympotes, draw approxiate diagrams of a function.

5.4 Implicit Differentiation

1. When differentiating something with y, multiply $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the end.
2. $(y^2)' = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$.

   Proof 5.4.1 If $u = y^2$, then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}. \qquad \text{[Chain Rule]}$$

Example 5.4.1 **Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the circle $x^2 + y^2 = 16$.**

$$\begin{aligned}(x^2)' + (y^2)' = (16)' \;\Rightarrow\; 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{2y} &= -\frac{x}{y}.\end{aligned}$$

Example 5.4.2 **Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $e^x + x\sin y = \cos 2y$.**

$$\begin{aligned}(e^x)' + (x\sin y)' &= (\cos 2y)' \\ e^x + \left(\sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= -2\sin 2y\frac{\mathrm{d}y}{\mathrm{d}x} \\ (-x\cos y - 2\sin 2y)\frac{\mathrm{d}y}{\mathrm{d}x} &= e^x + \sin y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{e^x + \sin y}{-x\cos y - 2\sin 2y}.\end{aligned}$$

3. Second Order Differentiation of Implicit functions*: Differentiate the first order differentiation.

Example 5.4.3 Find $\frac{d^2y}{dx^2}$ for the circle $x^2+y^2=16$. (From Ex. 4.1: $2x+2y\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}$.)

$$(2x)'+\left(2y\frac{dy}{dx}\right)'=(0)'\Rightarrow 2+\left((2y)'\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)'\right)=0\Rightarrow 2+2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}=0$$

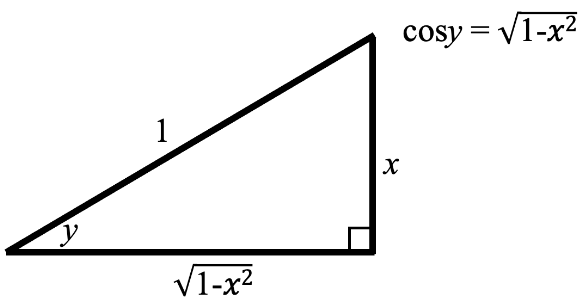
$$\frac{d^2y}{dx^2}=\frac{-2-2\left(\frac{dy}{dx}\right)^2}{2y}=\frac{-2-2\left(-\frac{x}{y}\right)^2}{2y}.$$

4. Derivative of Inverse Trignometry Functions

Theorem 5.4.1

$$y=\arcsin x\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}},\ \arcsin x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right](\cos y>0).$$

Proof 5.4.2 From $y=\arcsin x$, we get $\sin y=x$. This situation can be illustrated by the figure below:



$$\therefore (\sin y)'=(x)'\ \Rightarrow\ \cos y\frac{dy}{dx}=1$$

$$\therefore \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}.$$

Theorem 5.4.2

$$y=\arccos x\ \Rightarrow\ \frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-x^2}},\ \arccos x\in[0,\pi]\,(\sin y>0).$$

$$y=\arctan x\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{1+x^2}.$$

Proof 5.4.3 (Hint: Try to visualize a similar diagram as in proof 4.1.)
From $y=\arccos x$, we get $\cos y=x$.

$$\therefore (\cos y)'=(x)'\ \Rightarrow\ -\sin y\frac{dy}{dx}=1\ \Rightarrow\ \frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-x^2}}.$$

From $y = \arctan x$, we get $\tan y = x$.

$$\therefore (\tan y)' = (x)' \Rightarrow \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{1+x^2}.$$

5.5 Related Rate of Change

1. When finding a rate of change of x, we are finding the $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   Example 5.5.1 Area of circle is increasing at a rate of 10π per second. When the raduis is 2, what is the rate of change of radius?
   Known: $\frac{\mathrm{d}A}{\mathrm{d}t} = 10\pi,\ r = 2$. Find: $\frac{\mathrm{d}r}{\mathrm{d}t}$.

$$A = \pi r^2 \ \Rightarrow \ \frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t} = 10\pi$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{10\pi}{2\pi r} = \frac{5}{r}$$

$$\text{When } r = 2,\ \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{2}.$$

   Example 5.5.2 A spherical balloon is expanding at a rate of 60π per second. How fast is the surface area of the balloon expanding when the radius is 4?
   Known: $\frac{\mathrm{d}V}{\mathrm{d}t} = 60\pi,\ r = 4$. Find $\frac{\mathrm{d}A}{\mathrm{d}t}$.

$$V = \frac{4}{3}\pi r^3 \ \Rightarrow \ \frac{\mathrm{d}V}{\mathrm{d}t} = 3 \cdot \frac{4}{3}\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$
$$\therefore 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = 60\pi \ \Rightarrow \ \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{60\pi}{4\pi r^2} = \frac{15}{r^2}$$
$$A = 4\pi r^2 \ \Rightarrow \ \frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t} = 8\pi r \cdot \frac{15}{r^2} = \frac{120\pi}{r}.$$

$$\text{When } r = 4,\ \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{120\pi}{4} = 30\pi.$$

2. Kinematics:

   - Velocity, displacement, and acceleration are vector variables that have a value and a direction.
   - Speed only has a value and no direction. It is a scalar variable. No sign should be reported in the answer.
   - If s is the displacement, v is the velocity, a is the acceleration:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v;\ \frac{\mathrm{d}v}{\mathrm{d}t} = a.$$

5.6 More Limits - L'Hopital's Rule

Theorem 5.6.1 When the limit is in the **indeterminant form** ($\frac{0}{0}$ or $\frac{\infty}{\infty}$),

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right],$$

where $f'(x)$ and $g'(x)$ are the first derivatives of $f(x)$ and $g(x)$, respectively.

Example 5.6.1

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sec^2 x}{1}=1$$

5.7 Indefinite Integration

1. Regard Integration as Anti-differentiation:

$$f'(x)=x \quad\Rightarrow\quad f(x)=\frac{1}{2}x^2+C,\ \text{where } C \text{ is a constant.}$$

$$f'(x)=x^2 \quad\Rightarrow\quad f(x)=\frac{1}{3}x^3+C,\ \text{where } C \text{ is a constant.}$$

$$f'(x)=x^n \quad\Rightarrow\quad f(x)=\frac{1}{n+1}x^{n+1}+C,\ \text{where } C \text{ is a constant.}$$

Definition 5.7.1 Anti-differentiation is also called **indefinited integration**. It is denoted by $\int\ \mathrm{d}x$.

$$\text{e.g. } \int x^n\ \mathrm{d}x=\tfrac{1}{n+1}x^{n+1}+C.$$

2. General Rules of Integration.

- $$\int x^n\ \mathrm{d}x=\frac{1}{n+1}x^{n+1}+C$$
- $$\int k\ \mathrm{d}x=kx+C$$
- $$\int kf(x)\ \mathrm{d}x=k\int f(x)\ \mathrm{d}x$$
- $$\int f(x)\pm g(x)\ \mathrm{d}x=\int f(x)\ \mathrm{d}x\pm\int g(x)\ \mathrm{d}x$$

3. $\int f'(x)\ \mathrm{d}x=f(x)+C$. Therefore, if we know the $f'(x)$ and a point on the $f(x)$, which is to determine the constant C, then we can deduce the original function $f(x)$.

4. More Rules of Integration:

Differentiation	Integration
$(e^x)'=e^x$	$\int e^x\ \mathrm{d}x=e^x+C$
$(\ln x)'=frac1x$	$\int \frac{1}{x}\ \mathrm{d}x=\ln x+C$
$(\sin x)'=\cos x$	$\int \cos x\ \mathrm{d}x=\sin x+C$
$(\cos x)'=-\sin x$	$\int \sin x\ \mathrm{d}x=-\cos x+C$
$(\tan x)'=\sec^2 x$	$\int \sec^2 x\ \mathrm{d}x=\tan x+C$
$(\cot x)'=-\csc^2 x$	$\int \csc^2 x\ \mathrm{d}x=-\cot x+C$
$(\sec x)'=\sec x\tan x$	$\int \sec x\tan x\ \mathrm{d}x=\sec x+C$
$(\csc x)'=-\csc x\cot x$	$\int \csc x\cot x\ \mathrm{d}x=-\csc x+C$

5. Anti-chain Rule in Integration: We must divide the chain rule factor.

Example 5.7.1

$$\int (ax+b)^n \,\mathrm{d}x = \frac{1}{a}\left(\frac{1}{n+1}(ax+b)^{n+1}\right)+C$$

$$\int e^{(ax+b)} \,\mathrm{d}x = \frac{1}{a}e^{ax+b}+C$$

$$\int \frac{1}{(ax+b)} \,\mathrm{d}x = \frac{1}{a}\ln(ax+b)+C$$

$$\int \sin(ax+b) \,\mathrm{d}x = -\frac{1}{a}\cos(ax+b)+C$$

$$\int \cos(ax+b) \,\mathrm{d}x = \frac{1}{a}\sin(ax+b)+C$$

6. Integration Techniques: by Substitution and by Parts:

Theorem 5.7.1 Whenever we have an integration like: $\int g'(x)\times f(g(x))\,\mathrm{d}x$, we can always assume $u=g(x)$. Therefore, $\mathrm{d}u = g'(x)\cdot\mathrm{d}x$. $()u=g(x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=g'(x))$:

$$\int g'(x)\times f(g(x))\,\mathrm{d}x = \int f(u\,\mathrm{d}u).$$

Theorem 5.7.2 If $f(x)$ and $g(x)$ are two functions, and $f'(x)$ and $g'(x)$ are their derivatives, respectively, integration by parts can be written as following:

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x.$$

Example 5.7.2 Find $\int 2x(x^2+3)^5\,\mathrm{d}x$.
Since $2x=(x^2+3)'$, we consider to use integration by substitution.
Assume $u=x^2+3$, then $\frac{\mathrm{d}u}{\mathrm{d}x}=(x^2+3)'=2x \Rightarrow \mathrm{d}u=2x\cdot\mathrm{d}x$.

$$\begin{aligned}\therefore \int 2x(x^2+3)^5\,\mathrm{d}x &= \int (x^2+3)^5\cdot(2x\cdot\mathrm{d}x)\\ &= \int u^5\,\mathrm{d}u\\ &= \frac{1}{6}u^6+C\\ &= \frac{1}{6}(x^2+3)^6+C.\end{aligned}$$

7. Integration of Inverse Trignometric Functions:

- $$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C$$
- $$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C$$
- $$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{arcsec} x + C$$

Example 5.7.3 **Find** $\int \frac{\mathrm{d}x}{x^2+4x+5}$

$$\int \frac{\mathrm{d}x}{x^2+4x+5} = \frac{\mathrm{d}x}{(x+2)^2+1}$$

$$\text{Assume } u = x+2,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \ \Rightarrow\ \mathrm{d}u = \mathrm{d}x$$

$$\begin{aligned}\therefore \int \frac{\mathrm{d}x}{x^2+4x+5} &= \frac{\mathrm{d}u}{u^2+1}\\ &= \arctan u + C\\ &= \arctan(x+2) + C.\end{aligned}$$

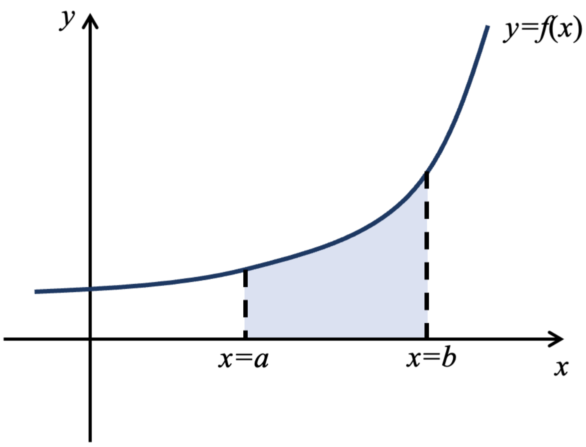
5.8 Approximiating the Area Under a Curve

1. The definite integral is equal to the limit at infinity of the Riemann sum, and hence gives the exact area under the curve between $x = a$ and $x = b$. i.e.,

$$\lim_{n\to\infty} = \sum_{i}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\ \mathrm{d}x,$$

where a is the lower limit and b is the upper limit.

2. If $f(x) \geq 0\ \forall x \in [a,b]$, then $\int_a^b f(x)\ \mathrm{d}x$ is defined as the shaded area:



This is known as the Riemann integral.

3. The Fundamental Theorem of Calculus:

Theorem 5.8.1 For a continuous function $f(x)$ with antiderivative $F(x)$:

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a).$$

This theorem explains the link between differential calculus and the definite integral.

4. Properties of Definite Integrals:

- $$\int_a^a f(x)\ \mathrm{d}x = 0$$

- $$\int_a^b d\ \mathrm{d}x = k(b-a),\ (k \text{ is a constant}).$$

- $$\int_b^a f(x)\ \mathrm{d}x = -\int_a^b f(x)\ \mathrm{d}x$$

- $$\int_a^b kf(x)\ \mathrm{d}x = k\int_a^b f(x)\ \mathrm{d}x$$

- $$\int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x$$

- $$\int_a^b [f(x) \pm g(x)]\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x \pm \int_a^b g(x)\ \mathrm{d}x$$

5. When the function $f(x)$ is negative for $x \in [a,b]$, then the area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\left|\int_a^b f(x)\ \mathrm{d}x\right|.$$

6. Finding Areas Between Two Functions:

   - Sketch: find the intersections and determine which function is above.
   - Integration.

5.9 Volumes of Revolution

1. The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $[a,b]$, is rotated 2π radians about the x-axis is given by

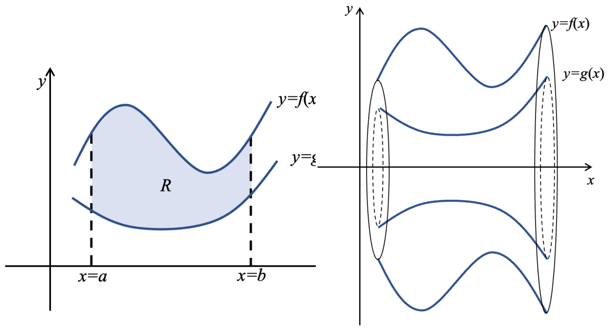
$$V = \pi\int_a^b y^2\ \mathrm{d}x.$$

2. The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $y = c$ to $y = d$, is rotated 2π radians about the y-axis is given by

$$V = \pi\int_c^d x^2\ \mathrm{d}y.$$

3. Consider a region R between two curves, $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, when $f(x) > g(x)$.

   - Rotating R about the x-axis generates a solid of revolution S. The criss-section of this



(a) (b)

shape looks like a washer whose area is given by:

$$A = \pi(R^2 - r^2) = \pi\left((f(x))^2 - (g(x))^2\right).$$

So the volume of S is given by:

$$\begin{aligned} V &= \int_a^b A(x)\,\mathrm{d}x \\ &= \int_a^b \left((f(x))^2 - (g(x))^2\right)\,\mathrm{d}x. \end{aligned}$$

- Rotating R about the y-axis in the interval $c \le y \le d$:

$$V = \pi \int_c^d \left((x_1)^2 - (x_2)^2\right)\,\mathrm{d}y,$$

where x_1 and x_2 are expression of x with respect to y of $f(x)$ and $g(x)$.

5.10 Differential Equation

1. Differential Equation:

Definition 5.10.1 A **differential equation** is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables. Equation that involves the derivatives of one or more functions.

E.g.

$$y' = 6x + 1 \qquad \text{Lagrage notation}$$

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{dx} - y = 3 \qquad \text{Leibriz notation}$$

$$f'(x) = 6x + 1 \qquad \text{Function notation}$$

The independent variable is x, and the dependent variable is y. The solution to a differential equation is a function or a set of functions.

2. Two Types of Differential Equations:

Definition 5.10.2 **Ordinary Differential Equations (ODEs)**: deals with functions of a single variable and ordinary derivatives. **Partial Differential Equations (PDEs)**: deals with multi-variable equations and their partial derivatives (with more than one independent variables).

3. Order of Differential Equations:

   Definition 5.10.3 The **order of the differential equation** is the highest order derivative in the equation.

4. Linearity of ODEs:

   Theorem 5.10.1 A differential equation is said to be **linear** if:

   - All the terms with dependent variabels are in first-order.
   - The coefficients of all the terms in the dependent variable and its derivatives depend only on the independent variable x.

5. Linear First-Order ODEs:

   Definition 5.10.4

   $$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = b(x), \text{ where } a(x) \text{ and } b(x) \text{ are functions of } x.$$

6. Solutions of ODEs:

   - The solution to an ODE is a function or a set of functions.
   - General solution to the differential equation:
     For a differential equation of order n, a solution is a function that satisfies the equation on some interval I. The function should havee at least its first n derivatives on this invertal I.
   - To find particular solutions, we need to initial conditions for the problem.
     (a) Initial Value Problem (IVP): where initial values are given to solve the differential equations depending on the order of the ODE.
     E.g. $y(0),\ t(0),\ (0,y)$.
     (b) Boundary Value Problem: where a certain boundary is given.
     E.g. (x,y).

7. Separable Differential Equations:

   Definition 5.10.5 A differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$ is **separable** if it can be expressed as a product of a function in x and a function in y:

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = g(x)h(y).$$

   - Particularly, if $h(x) \neq 0$, the variable can be separated to

   $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y) \ \Rightarrow \frac{\mathrm{d}y}{h(y)} &= g(x)\ \mathrm{d}x \\ \int \frac{\mathrm{d}y}{h(y)} &= \int g(x)\ \mathrm{d}x \end{aligned}$$

- Solving differential equations using separation of variables:
  (a) Separate the variables such that everything involving y is on one side and everything involving x is on the other side.
  (b) Integrate both sides and combine the constant of integration on one side of the equation (normally the right side).

Example 5.10.1 Solve for y if $\frac{\mathrm{d}y}{\mathrm{d}x} = x(1+y)e^x$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x(1+y)e^x \\
\frac{1}{1+y}\mathrm{d}y &= xe^x \mathrm{d}x \\
\int \frac{1}{1+y}\,\mathrm{d}y &= \int xe^x\,\mathrm{d}x \\
&\left(= xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x \text{ [Integration by Parts]}\right) \\
\ln|1+y| &= xe^x - e^x + C \\
1+y &= e^{xe^x - e^x + C} = e^{xe^x - e^x} \cdot e^C \\
y &= Ae^{xe^x - e^x} - 1 \quad (A = e^C).
\end{aligned}$$

8. The Standard Logistic Equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = kn(a-n); \quad a, k \in \mathbb{R}.$$

where t is the time during which a population grows,
n is the population after time t,
k is the relative growth, and
a is a constant.

9. Homogeneous Differential Equations:

Definition 5.10.6 Differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$, where $y = y(x)$, are known as **homogeneous differential equations**.

Theorem 5.10.2 Homogeneous differential equations can be solved by using the substituion $y = vx$, where v is a function of x. The substitution will always reduce the differential equation to a separable differential equation.

Proof 5.10.1 If $y = vx$, where v is a function of x, then:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}v}{\mathrm{d}x}x + v \qquad \text{[Product Rule]} \\
\because \frac{\mathrm{d}y}{\mathrm{d}x} &= f\left(\frac{y}{x}\right), \\
\therefore \frac{\mathrm{d}v}{\mathrm{d}x}x + v &= f\left(\frac{y}{x}\right) = f(v) \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{f(v) - v}{x} \\
\Rightarrow \frac{1}{f(v) - v}\,\mathrm{d}v &= \frac{1}{x}\,\mathrm{d}x
\end{aligned}$$

Example 5.10.2 Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y}{x}$, given $y(3) = \frac{3}{2}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 2\frac{y}{x} \;\rightarrow\; \text{homogenous differential equation}$$

$$\text{Let } y = vx,\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}x + v \;\Rightarrow\; \frac{\mathrm{d}v}{\mathrm{d}x}x + v = 1 + 2\frac{y}{x} = 1 + 2v.$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+v}{x} \\
\frac{1}{1+v}\,\mathrm{d}v &= \frac{1}{x}\,\mathrm{d}x \\
\int \frac{1}{1+v}\,\mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x \\
\ln|1+v| &= \ln|x| + C = ln|Ax| \\
1 + v = Ax \;&\Rightarrow \frac{y}{x} + 1 = Ax \\
y &= Ax^2 - x.
\end{aligned}$$

Substituting $y = \frac{3}{2}$, $x = 3$: $\frac{3}{2} = A(3)^2 - 3 \;\Rightarrow\; A = \frac{1}{2}$

$$\therefore y = \frac{1}{2}x^2 - x.$$

Example 5.10.3 Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x} \;\rightarrow\; \text{homogenous differential equation}$$

$$\text{Assume } v = \frac{y}{x} :\; y = vx \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}x + v.$$

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 1+v &= \frac{\mathrm{d}v}{\mathrm{d}x}x+v \\
\mathrm{d}v &= \frac{1}{x}\ \mathrm{d}x \\
\int\ \mathrm{d}v &= \int \frac{1}{x}\ \mathrm{d}x \\
v &= \ln|x|+C = \ln|Ax| \\
\frac{y}{x} &= \ln|Ax| \\
y &= x\ln|Ax|.
\end{aligned}
$$

10. Using the Integrating Factor $I(x)$:

Definition 5.10.7

$$I(x) = e^{\int P(x)\ \mathrm{d}x}$$

is the **integrating factor** for $\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y = Q(x)$, where P and Q are continuous functions of x on a given interval.

Theorem 5.10.3

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y &= Q(x) \\
I(x)\frac{\mathrm{d}y}{\mathrm{d}x}+I(x)P(x)y &= I(x)Q(x) \text{ [Multiply both sides by } I(x)\text{]} \\
\left(\frac{\mathrm{d}}{\mathrm{d}x}(I(x)y)\right. &= \left. I(x)\frac{\mathrm{d}y}{\mathrm{d}x}+I(x)P(x)y \text{ [Product Rule]}\right) \\
\Big[(I(x))' = (e^{\int P(x)\ \mathrm{d}x})' &= e^{\int P(x)\ \mathrm{d}x}\cdot\Big(\int P(x)\ \mathrm{d}x\Big)' = e^{\int P(x)\ \mathrm{d}x}\cdot P(x) = I(x)P(x)\Big] \\
\therefore \frac{\mathrm{d}}{\mathrm{d}x}(I(x)y) &= I(x)Q(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}(I(x)y)\ \mathrm{d}x &= \int I(x)Q(x)\ \mathrm{d}x \\
I(x)y &= \int I(x)Q(x)\ \mathrm{d}x.
\end{aligned}
$$

Example 5.10.4 Solve $\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2y = 6x^2$.

$$\because P(x) = 3x^2,\ Q(x) = 6x^2,$$

$$\therefore I(x) = e^{\int P(x)\ \mathrm{d}x} = e^{\int 3x^2\ \mathrm{d}x} = e^{x^3}.$$

Multiply both sides by $I(x)$:

$$
\begin{aligned}
e^{x^3}\frac{\mathrm{d}y}{\mathrm{d}x}+e^{x^3}\cdot 3x^2y &= e^{x^3}\cdot 6x^2\\
\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right) &= e^{x^3}\cdot 6x^2\\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right)\,\mathrm{d}x &= \int e^{x^3}\cdot 6x^2\,\mathrm{d}x\\
\left[\text{Let } x^3=u,\ \frac{\mathrm{d}u}{\mathrm{d}x}=2x^2,\ \mathrm{d}u=2x^2\,\mathrm{d}x \Rightarrow\ 2\int e^u\,\mathrm{d}u=2e^u+C=2e^{x^3}+C\right]\\
e^{x^3}y &= 2e^{x^3}+C\\
y &= 2+Ce^{-x^3}.
\end{aligned}
$$

11. Euler's Method:

- For $y=f(x)$, $y_{n+1}=y_n+hf'(x_0)$, h is a constant.

$$y-y_n=f'(x_n)(x-x_n).$$

Example 5.10.5 $y=x^2,\ \frac{\mathrm{d}y}{\mathrm{d}x}=2x,\ h=0.1$

n	x_n	y_n	Actual
0	1	1	1
1	1.1	1.2	1.21
2	1.2	1.42	1.44
3	1.3	1.66	1.69
4	1.4	1.92	1.96
5	1.5	2.2	2.25

- The smaller the h, the more accurate the approximation.
- Consider a differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$, given an initial condition. The derivative at any point on the curve $(x_0,y(x_0))$ can be approximated using the gradient of the tangent to the curve at x_0:

$$y'(x_0)=\frac{y(x_0+h)-y(x_0)}{h}.$$

Rearranging the formula, we get:

$$y(x_0+h)=y(x_0)+hy'(x+0).$$

This is the **linearization** or **Euler's method** and becomes more accurate over small increments and as long as the function does not change too rapidly.

- If $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x_n,y-n)$ and $x_{n+1}=x_n+h$, we have

$$y_{n+1}=y_n+hf(x_n,y_n).$$

5.11 Maclaurin Series

1. The Maclaurin Polynomial:

   Definition 5.11.1 If $f(x)$ has n derivatives at $x = 0$, then $P(x)$, the **Maclaurin polynomial** of degreen n for $f(x)$ centered at $x = 0$, is the unique polynomial of degree n that satisfies:

   - $$f(0) = P(0);$$
   - $$f^{(n)}(0) = P^{(n)}(0);$$
   - $$a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!},\ a_3 = \frac{f'''(0)}{3!},\ \cdots\ a_n = \frac{f^{(n)}(0)}{n!};$$
   - $$P_n(x) = f(0) + + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

2. Maclaurin polynomials approximate the behavior of functions around a certain interval. The more terms we take, the better the approximation.

3. The Maclaurin Series:

   Definition 5.11.2 If $f(x)$ has derivatives of all orders throughout an open interval I such that $0 \in I$, then the **Maclaurin series** generated by f at $x = 0$ is:

   $$P_n(x) = f(0) + + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

   A series converges when the sum of them is a constant (a limit can be found).

   Example 5.11.1 Find the Maclaurin series for $f(x) = \frac{1}{2+x}$.

   $$\begin{array}{r|r} f(x) = (2+x)^{-1} & f(0) = 2^{-1} = \frac{1}{2} \\ f'(x) = -(2+x)^{-2} & f'(0) = -2(2)^{-2} = -\frac{1}{4} \\ f''(x) = 2(2+x)^{-3} & f''(0) = 2(2)^{-3} = 2 \times \frac{1}{8} \\ f'''(x) = -6(2+x)^{-4} & f'''(0) = -6(2)^{-4} = -6 \times \frac{1}{16} \\ f^{(4)}(x) = 24(2+x)^{-5} & f^{(4)}(0) = 24(2)^{-5} = 24 \times \frac{1}{32} \end{array}$$

   $$\begin{aligned} P(x) &= \frac{1}{2} + \frac{-\frac{1}{4}}{1!}x + \frac{2 \times \frac{1}{8}}{2!}x^2 + \frac{-6 \times \frac{1}{16}}{3!}x^3 + \frac{24(2)^{-5} = 24 \times \frac{1}{32}}{4!}x^4 + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (-x)^n. \end{aligned}$$

4. The Binomial series is the Maclaurin expansion for $f(x) = (1+x)^p$:

   $$(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n,\ 1 \le n \le p,\ \binom{p}{n} = \frac{p!}{n!(p-n)!} = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.$$

Example 5.11.2 Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{(x+2)^2}$.

$$
\begin{aligned}
f(x) &= (1+x)^{-2} \\
\because \binom{-2}{n} &= \frac{-2(-2-1)(-2-2)\cdots(-2-(n-1))}{n!} \\
&= (-1)^n \frac{2(3)(4)\cdots(n+1)}{n!} = (-1)^n(n+1) \\
\therefore P(x) &= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \\
&= 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^n(n+1)x^n + \cdots
\end{aligned}
$$

Example 5.11.3 Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{\sqrt{2-x}}$.

$$
f(x) = (2-x)^{-\frac{1}{2}} = (2)^{-\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}
$$

$$
\therefore P(x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2}\binom{-\frac{1}{2}}{n}\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}.
$$

5. Applications of Maclaurin Series:

- Approximation of sin, cos, tan,...

  Example 5.11.4 Approximate $\sin 3^\circ$ using the first four terms of Maclaurin series.

$$
3^\circ = \frac{\pi}{60}, \text{ For } \sin x,\ P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
$$

$$
\therefore P\left(\frac{\pi}{60}\right) = x - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \frac{\left(\frac{\pi}{60}\right)^7}{7!} + \cdots \approx 0.052336 \text{ (6 d.p.)}.
$$

- More Complicated Functions

  Example 5.11.5 Find the Maclaurin series of $f(x) = e^{x^2}$.
  Let $u = x^2, f(x) = e^u$:

$$
P(x) = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{u^n}{n!} = \sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}.
$$

  Example 5.11.6 Find the Maclaurin series of $f(x) = \ln\left(\frac{1+x}{1-x}\right)$.

$$
\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)
$$

$$
\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
\ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots
\end{aligned}
$$

$$\begin{aligned}\therefore P(x) &= -\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots-\left(x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right)\\ &= 2(x+\frac{x^3}{3}+\cdots)\\ &= 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}.\end{aligned}$$

Example 5.11.7 Find the Maclaurin series of $f(x)=\frac{x}{(1+x)^2}$.

$$\begin{aligned}f(x) &= x(1+x)^{-2}\\ &= x\sum_{n=0}^{\infty}\binom{-2}{n}x^n\\ &= \sum_{n=0}^{\infty}(-1)^n(n+1)x^{n+1}.\end{aligned}$$

- Evaluate Limits

Example 5.11.8 Find $\lim\limits_{x\to 0}\frac{1-e^{x^2}}{1-\cos x}$.

$$e^{x^2} = 1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$\begin{aligned}\therefore \lim_{x\to 0}\frac{1-e^{x^2}}{1-\cos x} &= \lim_{x\to 0}\frac{1-\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots\right)}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}\\ &= \lim_{x\to 0}\frac{-x^2-\frac{x^4}{2!}-\frac{x^6}{3!}-\cdots}{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}\\ &= \lim_{x\to 0}\frac{-x^2}{\frac{x^2}{2!}}\\ &= -2.\end{aligned}$$

[Consider only the smallest power of x, as higher powers will go to zero much quicker.]

- Solve Differential Equations

Example 5.11.9 Use the first six terms of a Maclaurin series to approximate the solution of $y'=y^2-x$ on an open interval centered at $x=0$ if $y(0)=1$.

	$y(0)=1$
$y'=y^2-x$	$y'(0)=1$
$y''=2yy'-1$	$y''(0)=2-1=1$
$y'''=2yy''+2(y')^2$	$y'''(0)=2+2=4$
$y^{(4)}=2yy'''+6y'y''$	$y^{(4)}(0)=14$
$y^{(5)}=2yy^{(4)}+8y'y'''+6(y'')^2$	$y^{(5)}(0)=66$

$$\therefore P(x) = 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \cdots$$

- Binomial Theorem

Theorem 5.11.1 Function $f(x) = (1+x)^p,\ p \in \mathbb{R}$ is equal to its Binomial series using the initial condition $y(0) = 1$.

Proof 5.11.1

$$f(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots$$

$$\therefore f'(x) = p + p(p-1)x + \frac{p(p-1)(p-2)}{2!}x^2 + \cdots$$

$$xf'(x) = px + p(p-1)x^2 + \frac{p(p-1)(p-2)}{2!}x^3 + \cdots$$

$$\begin{aligned}
\therefore f'(x) + xf'(x) &= p + [p(p-1) + p]x + \left[\frac{p(p-1)}{2!}p(p-1) + \right]x^2 + \cdots \\
&= p + p^2x + \frac{p^2(p-1)}{2!}x^2 + \cdots \\
&= p(1 + px + \frac{p(p-1)}{2!}x^2) + \cdots \\
&= pf(x)
\end{aligned}$$

$$\therefore f'(x) + xf'(x) = pf(x) \ \Rightarrow\ (1+x)f'(x) = pf(x)$$

$$f'(x) - \frac{p}{1+x}f(x) = 0 \ \Rightarrow\ P(x) = -\frac{p}{1+x},\ Q(x) = 0$$

$$\therefore I(x) = e^{\int -\frac{p}{1+x}\,\mathrm{d}x} = A(1+x)^{-p}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(A(1+x)^{-p}f(x)\right) = 0 \cdot I(x)$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left(A(1+x)^{-p}f(x)\right)\,\mathrm{d}x = \int 0\,\mathrm{d}x$$

$$A(1+x)^{-p}f(x) = C$$

$$f(x) = C \cdot A(1+x)^{x-p} = B(1+x)^{x-p} \qquad \text{[Let } B = C \cdot A.\text{]}$$

$$\text{Let } f(0) = 1 :\ B = 1$$

$$\therefore f(x) = (1+x)^p.$$