

# **IB Mathematics Analysis and Approaches HL**

## **Topic 1 Number and Algebra**

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# 1 Sequences and Series

1. Terms:  $u_1, u_2, u_3 \dots$

Position:  $n$

Sum:  $S$

2. **Arithmetic Sequence**/Arithmetic Progression (AP):

- Recursive formula:  $u_{n+1} = u_n + d$ ,  $d$  is the common difference.
- Explicit formula:  $u_n = u_1 + d(n-1)$
- Summation:  $S_n = \frac{1}{2}[2u_1 + d(n-1)]$

## Proof: 1.1

Let  $u_1, u_2, u_3, \dots, u_n$  be an arithmetic sequence with  $d$  as common difference.

Then,  $S_n = u_1 + u_2 + u_3 + \dots + u_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + (u_1 + (n-1)d)$

Also,  $S_n = [u_1 + (n-1)d] + \dots + (u_1 + d) + u_1$ .

Add two expressions together:

$$2S_n = [2u_1 + (n-1)d]n$$

$$\therefore S_n = \frac{n}{2}[2u_1 + (n-1)d].$$

3. **Geometric Sequence**

- Recursive formula:  $u_{n+1} = r \cdot u_n$ ,  $r$  is the common ratio.
- Explicit formula:  $u_n = u_1 \cdot r^{n-1}$

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$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \dots$$

- Summation:  $S_n = \frac{u_1(r^n - 1)}{r - 1}$

## Proof: 1.2

Let  $u_1, u_2, u_3, \dots, u_n$  be a geometric sequence with  $r$  as common ratio.

$S_n = u_1 + u_2 + u_3 + \dots + u_n = u_1 + (u_1 \cdot r) + (u_1 \cdot r^2) + \dots + (u_1 \cdot r^{n-1})$

Then,  $rS_n = (u_1 \cdot r) + (u_1 \cdot r^2) + \dots + (u_1 \cdot r^n)$ .

Subtract the first expression from the second:

$$rS_n - S_n = u_1 \cdot r^n - u_1 \Rightarrow (r-1)S_n = u_1(r^n - 1)$$

$$\therefore S_n = \frac{u_1(r^n - 1)}{r - 1}$$

- If  $r > 1$ , the sequence is an exponential growth.  
If  $0 < r < 1$ , the sequence has an exponential decay.

- When  $r > 1$ , series approaches  $\infty$ .

When  $-1 < r < 1$ , or  $|r| < 1$ , the series converges:

$$S_{\infty} = \frac{u_1}{1-r}, |r| < 1$$

## 2 Exponents and Logarithms

1.  $a^m \cdot a^n = a^{m+n}$   
 $a^m \div a^n = a^{m-n}$   
 $(a^m)^n = a^{mn}$
2.  $x^0 = 1$  ( $x^0 = x^{1-1} = \frac{x^1}{x^1} = 1$ )  
 $x^{-m} = \frac{1}{x^m}$   
 $x^{\frac{1}{n}} = \sqrt[n]{x}$  ( $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$ )
3. If  $a = b$ , then  $a^n = b^n$   
 If  $m = n$ , then  $a^m = a^n$   
 For  $a^b = 1$ :  $a = 1, b \in \mathbb{R}$ ;  $a \neq 1, b = 0$ ; OR  $a = -1, b = 2n$
4. When solving exponential equations, convert them to the same base.
5. Division Theorem.

### **Theroem: 2.1**

If  $a^x = b^y$  given  $a > 0$  and  $b > 0$ , then  $a = b^{\frac{y}{x}}$ .

### **Proof: 2.1**

$$a^x = b^y$$

$$(a^x)^{\frac{1}{x}} = (b^y)^{\frac{1}{x}} \Rightarrow a = b^{\frac{y}{x}}$$

6.  $a = b^x \Leftrightarrow x = \log_b a$ , where  $a, b \in \mathbb{R}^+$  and  $b \neq 1$ .

7. Logarithmic rules:

- $\log_a x + \log_a y = \log_a(xy)$

### **Proof: 2.2**

Let  $\log_a x = p$ ,  $\log_a y = q$ .  $\Rightarrow a^p = x, a^q = y$ .

Then,  $x \cdot y = a^p \cdot a^q = a^{p+q}$ .

$$\therefore \log_a(xy) = p + q = \log_a x + \log_a y.$$

- $\log_a x - \log_a y = \log_a \left( \frac{x}{y} \right)$

**Proof: 2.3**

Let  $\log_a x = p$ ,  $\log_a y = q$ .  $\Rightarrow a^p = x, a^q = y$ .

Then,  $\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$ .

$$\therefore \log_a \left( \frac{x}{y} \right) = p - q = \log_a x - \log_a y.$$

- $\log_a x^n = n \log_a x$
- $\log_a 1 = 0$
- $\log_a a = 1$
- $-\log_a x = \log_a \frac{1}{x}$
- $\log_a x = \frac{\log_b x}{\log_b a}$
- $s \log_a b = \frac{1}{\log_b a}$

### 3 Proof

1. Direct proof:

**Example: 3.1**

**Show that the sum of two even numbers is always even.**

---

Let  $m$  and  $n$  be two even positive integers.

$m = 2p, n = 2q$ , where  $p$  and  $q \in \mathbb{Z}^+$ .

Then,  $m + n = 2p + 2q = 2(p + q)$ , which is an even number.

**Example: 3.2**

**Show that  $\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 \equiv x^2 + ax$ .**

---

$$\text{LHS} = x^2 + \frac{a^4}{4} + ax - \frac{a^4}{4} = x^2 + ax = \text{RHS}.$$

Equations "=": only true from some values.

Identities " $\equiv$ ": true for all values.

**Example: 3.3 Question**

**Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.**

### Example: 3.3 Answer

Let  $n$  be a 4-digit number:  $n = 1000a + 100b + 10c + d$ , where  $0 \leq a, b, c, d \leq 9$ , and  $a \neq 0$ .

It is given that  $a + b + c + d = 3k, k \in \mathbb{Z}$ :

$$\begin{aligned} n &= 1000a + 100b + 10c + d + 3k - a - b - c - d \\ &= 999a + 99b + 9c + 3k \\ &= 3(333a + 33b + 3c + k) \end{aligned}$$

Since  $(333a + 33b + 3c + k) \in \mathbb{Z}$ , it implies that  $n$  is divisible by 3.

## 2. Proof by Contradiction:

### Example: 3.4

**Prove the statement: If the integer  $n$  is odd, then  $n^2$  is also odd.**

Let, if possible,  $n^2$  is even and  $n$  is odd.

Then,  $n^2 = 2k, k \in \mathbb{Z} \Rightarrow n \times n = 2k$ , which indicates the product of two odd number is even, and which is not true.

Hence, there is a contradiction.

$\therefore$  Our assumption is wrong, and thus given that  $n$  is odd,  $n^2$  is also odd.

### Example: 3.5

**Show that  $\sqrt{2}$  is irrational.**

Let us assume, if possible, that  $\sqrt{2}$  is rational:

$\sqrt{2} = \frac{p}{q}$ , where  $p, q \in \mathbb{Z}$ , and  $p, q$  have no common factors,  $q \neq 0$ .

$$\therefore 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \quad (1).$$

$\therefore p^2$  is even, and thus  $p$  is also even.

As  $p$  is an even number, we can write:  $p = 2k, k \in \mathbb{Z} \Rightarrow \therefore p^2 = (2k)^2 = 4k^2 \quad (2).$

From (1) and (2):  $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$  is even, and thus  $q$  is also an even number.

But since  $p$  and  $q$  have no common factors, they cannot have "2" as a common factor.

Hence, we have arrived at a contradiction.

$\therefore$  Our assumption is incorrect, and  $\sqrt{2}$  is irrational.

**Definition 1:** A number is **rational** if it can be written as  $\frac{p}{q}$ , where  $p, q \in \mathbb{Z}$ , and  $q \neq 0$ .

### Example: 3.6 Question

**Prove that there is no  $x \in \mathbb{R}$  such that  $\frac{1}{x-2} = 1 - x$**

**Example: 3.6 Answer**

Assume there is a real number  $x$  such that  $\frac{1}{x-2} = 1 - x$ .

$$\therefore (1-x)(x-2) = 1 \Rightarrow x^2 - 3x + 3 = 0$$

Solving the equation, we get  $x = \frac{3 \pm \sqrt{9-12}}{2}$ , which  $\notin \mathbb{R}$

$\therefore$  We arrived at a contradiction, and our assumption is incorrect. There is no  $x \in \mathbb{R}$  such that  $\frac{1}{x-2} = 1 - x$

**3. Proof by Mathematical Induction****Definition 2: Principle of Mathematical Induction (PMI):**

Suppose  $P_n$  is a proposition which is defined for every integer  $n \geq a$ ,  $a \in \mathbb{Z}$ . If  $P_a$  is true, and if  $P_{k+1}$  is true whenever  $P_k$  is true, then  $P_n$  is true  $\forall n \geq a$ .

**Example: 3.7**

**Prove that  $4^n + 2$  is divisible by 3 for  $n \in \mathbb{Z}$ ,  $n \geq 0$ , by using PMI.**

For  $n = 0$ , LHS  $= 4^0 + 2 = 1 + 2 = 3$ , which is divisible by 3.

$\therefore P_0$  (OR denoted as  $P(0)$ ) is true.

Assume that  $P_k$  is true: i.e.,  $4^k + 2$  is divisible by 3.  $\Rightarrow 4^k + 2 = 3A$ ,  $A \in \mathbb{Z}^+ \Rightarrow 4^k = 3A - 2$ .

Consider  $P_{k+1}$ :

$$\begin{aligned} 4^{k+1} + 2 &= 4^k \cdot 4^1 + 2 \\ &= (3A - 2) \cdot 4 + 2 \\ &= 12A - 6 \\ &= 3(4A - 2). \end{aligned}$$

$\therefore 4A - 2$  is an integer as  $A \in \mathbb{Z}^+$ ,  $4^{k+1} + 2$  is divisible by 3 whenever  $4^k + 2$  is divisible by 3.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}$ ,  $n \geq 0$ .

**Example: 3.8**

**A sequence is defined by  $u_{n+1} = 2u_n + 1 \forall n \in \mathbb{Z}^+$ . Prove that  $u_n = 2^n - 1$ .**

For  $n = 1$ ,  $u_1 = 2^1 - 1 = 1 \Rightarrow P_1$  is true.

Let  $P_k$  be true:  $u_k = 2^k - 1$  for some  $k \in \mathbb{Z}^+$ .

Consider  $P_{k+1}$ :

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}^+$ .

**Example: 3.9**

**Prove that**  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}, \forall n \in \mathbb{Z}^+.$

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For  $n = 1$ , LHS =  $1^2 = 1$ , RHS =  $\frac{1(1+1)(2+1)}{6} = 1$

$\therefore$  LHS = RHS  $\Rightarrow P_1$  is true.

Assume that  $P_k$  is true,  $k \in \mathbb{Z}^+ : 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}.$

Consider  $P_{k+1}$ :

$$\begin{aligned}
 \text{LHS} &= 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 \\
 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
 &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
 &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
 &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
 &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = \text{RHS}.
 \end{aligned}$$

Thus,  $P_{k+1}$  is true whenever  $P_k$  is true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}^+.$

**Example: 3.10**

**Prove that if**  $x \neq 1$ , **the**  $\prod_{i=1}^n (1 + x^{2^{i-1}}) = (1+x)(1+x^2)(1+x^4) \dots (1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x}.$

---

For  $n = 1$ , LHS =  $1 + x$ , RHS =  $\frac{1-x^{2^1}}{1-x} = \frac{1-x^2}{1-x} = 1 + x. \Rightarrow \therefore$  LHS = RHS,  $P_1$  is true.

Assume that  $P_k$  is true:  $(1+x)(1+x^2)(1+x^4) \dots (1+x^{2^{k-1}}) = \frac{1-x^{2^k}}{1-x}.$

Conosider  $P_{k+1}$ :

$$\begin{aligned}
 \text{LHS} &= (1+x)(1+x^2)(1+x^4) \dots (1+x^{2^{k-1}})(1+x^{2^k}) \\
 &= \frac{1-x^{2^k}}{1-x} (1+x^{2^k}) \\
 &= \frac{1+x^{2^k} - x^{2^k} + (x^{2^k})^2}{1-x} \\
 &= \frac{1-x^{2^k \cdot 2}}{1-x} \\
 &= \frac{1-x^{2^{k+1}}}{1-x} = \text{RHS}.
 \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true  $\forall n \in \mathbb{Z}^+.$

## 4 Counting and Binomial Theorem

1. Choose  $r$  from  $n$ :  $\binom{n}{r} = {}_nC_r$

- $\binom{n}{m} = \binom{n}{n-m}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Factorial notation:  $n! = n(n-1)(n-2) \cdots 2 \cdot 1$   
e.g.  $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 5 \times 2 = 10$ .

### Example: 4.1

Write  $\frac{(n!)^2}{(n-1)!(n-2)!}$  without using factorial notation.

---

$$(n!)^2 = n! \times n! = n(n-1)! \times n(n-1)(n-2)!$$

$$\therefore \frac{(n!)^2}{(n-1)!(n-2)!} = \frac{n(n-1)! \times n(n-1)(n-2)!}{(n-1)!(n-2)!} = n \cdot n(n-1) = n^3 - n^2.$$

2. The number of ways of arranging  $n$  distinct objects in a row is  $n!$ .
3. The number of permutations of  $r$  objects out of  $n$  distinct objects is given by

$${}_nP_r = \frac{n!}{(n-r)!}.$$

4. In permutations, the order matters.  
In combinations, the order does not matter.
5. The Binomial Theorem:

### Theorem: 4.1 The Binomial Theorem

$$\begin{aligned} (a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n, \quad n \in \mathbb{N} \\ &= \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r \end{aligned}$$

### Example: 4.2

Find  $(2x+3)^4$ .

---

$$\begin{aligned} (2x+3)^4 &= (2x)^4 + \binom{4}{1}(2x)^3(3)^1 + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)(3)^3 + 3^4 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81 \end{aligned}$$



**Example: 4.3**

**Find the term independent of  $x$  in the expansion of  $\left(x - \frac{2}{x^2}\right)^{12}$ .**

General term:  $\binom{12}{r} x^{12-r} \left(-\frac{2}{x^2}\right)^r = \binom{12}{r} x^{12-r-2r} = \binom{12}{r} x^{12-3r}$

When  $12 - 3r = 0$ , the term is independent of  $x$ :  $12 - 3r = 0 \Rightarrow r = 4$ .

$$\therefore \binom{12}{4} x^0 = 7920.$$

1. The independent term should not involve  $x$  in it since the independent term does not vary as  $x$  varies. (constant term)
2. The coefficient should not include  $x$  as well.

**Example: 4.4**

**Find the coefficient of  $x^3 y^2$  in the expansion of  $(2x + y) \left(x + \frac{y}{x}\right)^5$ .**

Assume  $2x \cdot A$  and  $y \cdot B$  will yield the term  $x^3 y^2$ .  $\Rightarrow A = x^2 y^2$ ,  $B = x^3 y$ .

General term:  $\binom{5}{r} x^{5-r} \left(\frac{y}{x}\right)^r = \binom{5}{r} x^{5-2r} y^r$ .

When  $r = 2$ ,  $5 - 2r = 1 \neq 2 \Rightarrow x^2 y^2$  is not possible.

When  $r = 1$ ,  $5 - 2r = 3 \Rightarrow x^3 y$  is possible.

$$\therefore \text{Coefficient} = \binom{5}{1} = 5.$$

**Example: 4.5**

**Find the coefficient of  $x^2$  in the expansion of  $(1 - 2x)(1 - 4x)^7$ .**

Assume  $1 \cdot A = x^2$ ,  $-2x \cdot B = x^2$ .  $\Rightarrow A = x^2$ ,  $B = x$ .

General term:  $\binom{7}{r} (-4x)^{7-r} (1)^r$

When  $7 - r = 2$ ,  $r = 5$ :  $\binom{7}{5} (-4x)^2 (1)^5 = 336x^2$ .  $\Rightarrow 1 \cdot 336x^2 = 336x^2$

When  $7 - r = 1$ ,  $r = 6$ :  $\binom{7}{6} (-4x)^1 (1)^6 = -28x$ .  $\Rightarrow (-2x) \cdot (-28x) = 56x^2$

$$\therefore \text{Coefficient} = 336 + 56 = 392.$$

## 6. AHL - Extension of Binomial Theorem:

**Theorem: 4.2 Binomial Theorem Extended**

$$\begin{aligned} (a+b)^n &= a^n \left(1 + \frac{b}{a}\right)^n \\ &= a^n \left(1 + n \cdot \frac{b}{a} + \frac{n(n-1)}{2!} \left(\frac{b}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{b}{a}\right)^3 + \dots\right), n \in \mathbb{Q}, \left|\frac{b}{a}\right| < 1 \end{aligned}$$

**Example: 4.6**

**Expand  $\sqrt{1+2x}$  ( $|x| < \frac{1}{2}$ ) and  $\frac{2}{1-3x}$  ( $|x| < \frac{1}{3}$ ) upto  $x^3$  term.**

---

$$\begin{aligned}(1+2x)^{\frac{1}{2}} &= 1 + \frac{1}{2}(2x) + \frac{1}{2} \left( \frac{1}{2} - 1 \right) \frac{(2x)^2}{2!} + \frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right) \frac{(2x)^3}{3!} + \dots \\ &= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots\end{aligned}$$

$$\begin{aligned}2(1-3x)^{-1} &= 2(1 - (-3x) - (-1-1) \frac{(-3x)^2}{2!} - (-1-1)(-1-2) \frac{(-3x)^3}{3!} + \dots \\ &= 2(1 + 3x + x^2 + 27x^3 + \dots \\ &= 2 + 6x + 18x^2 + 54x^3 + \dots\end{aligned}$$

**Example: 4.7**

**Write the first three terms in the expansion of  $(2+x)^{-3}$ .**

---

$$\begin{aligned}(2+x)^{-3} &= 2^{-3} \left( 1 + \frac{x}{2} \right)^{-3} \\ &= \frac{1}{8} \left( 1 + (-3) \frac{x}{2} + (-3)(-3-1) \frac{2^2}{2 \cdot 2!} + \dots \right) \\ &= \frac{1}{8} \left( 1 - \frac{3}{2}x + \frac{12}{4}x^2 + \dots \right) \\ &= \frac{1}{8} - \frac{3}{16}x + \frac{3}{8}x^2 + \dots\end{aligned}$$

**Example: 4.8 Application of Binomial Theorem**

**Find square root of 24 correct to 5 decimal places, using the binomial theorem.**

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$$\begin{aligned}24^{\frac{1}{2}} &= (25-1)^{\frac{1}{2}} = 25^{\frac{1}{2}} \left( 1 - \frac{1}{25} \right)^{\frac{1}{2}} \\ &= 5 \left( 1 + \left( \frac{1}{2} \right) \left( -\frac{1}{25} \right) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \left( -\frac{1}{25} \right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \left( -\frac{1}{25} \right)^3 + \dots \right) \\ &= 5 \left( 1 - \frac{1}{50} - \frac{1}{5000} - \frac{1}{250000} + \dots \right) \\ &= 5(1 - 0.02 - 0.0002 - 0.000004) \\ &= 4.89898 \quad (5 \text{ d.p.}).\end{aligned}$$

## 5 Partial Fraction - AHL

1. Proper fractions: The degree of the numerator is less than the degree of the denominator.
2. Partial fraction: A method to separate one complex fraction into two or more simpler fractions.

### Example: 5.1

**Find the partial fraction of**  $\frac{3x}{(x-1)(x+2)}$ .

---

$$\text{Let } \frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

$$\therefore 3x \equiv A(x+2) + B(x-1).$$

$$\text{When } x = 1, 3 = 3A \Rightarrow A = 1.$$

$$\text{When } x = -2, -6 = -3B \Rightarrow B = 2.$$

$$\therefore \frac{3x}{(x-1)(x+2)} \equiv \frac{1}{x-1} + \frac{2}{x+2}.$$

### Example: 5.2

**Find the partial fraction of**  $\frac{2x+5}{(x-2)(x+1)}$ .

---

$$\text{Let } \frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

$$\therefore 2x+5 \equiv A(x+1) + B(x-2).$$

$$\text{When } x = 2, 9 = 3A \Rightarrow A = 3.$$

$$\text{When } x = -1, 3 = -3B \Rightarrow B = -1.$$

$$\therefore \frac{2x+5}{(x-2)(x+1)} \equiv \frac{3}{x-2} - \frac{1}{x+1}.$$

### Example: 5.3

**Find the partial fraction of**  $\frac{34-12x}{3x^2-10x-8}$ .

---

$$\text{As } \frac{34-12x}{3x^2-10x-8} = \frac{34-12x}{(3x+2)(x-4)}, \text{ let } \frac{34-12x}{(3x+2)(x-4)} = \frac{A}{3x+2} + \frac{B}{x-4}.$$

$$\therefore 34 - 12x \equiv A(x-4) + B(3x+2).$$

$$\text{When } x = 4, -14 = 14A \Rightarrow A = -1.$$

$$\text{When } x = -\frac{2}{3}, 42 = -\frac{14}{3}A \Rightarrow A = -9.$$

$$\therefore \frac{34-12x}{(3x+2)(x-4)} \equiv -\frac{9}{3x+2} - \frac{1}{x-4}.$$

## 6 Complex Number - AHL

### 6.1 Introduction

#### 1. Complex Number:

Definition 3:

**Complex Numbers** are numbers in the form of  $a + bi$ , where  $i^2 = -1$ .

-  $a$  is called the **real part**, denoted as  $\text{Re}(a + bi) = a$ .

-  $b$  is called the **imaginary part**, denoted as  $\text{Im}(a + bi) = b$ .

$a + bi$  is called the Cartesian form of complex number.

#### 2. Basic Calculations of Complex Number:

- Define  $z_1 = a + bi$  and  $z_2 = c + di$ :

$$z_1 \pm z_2 = (a \pm c) + (b \pm d)i.$$

- Define  $z_1 = a + bi$  and  $z_2 = c + di$ :

$$z_1 z_2 = (ac - bd) + (ad + bc)i.$$

##### Proof: 6.1.1

$$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) \\ &= ac + (ad + bc)i + bdi^2 \quad [i^2 = -1] \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

- Conjugate complex number:

Definition 4:

We call  $a - bi$  as the **conjugate** of  $z = a + bi$ , denoted as  $z^* = a - bi$ .

##### Theorem: 6.1.1

Define  $z_1 = a + bi$ , and  $z^*$  is the conjugate of  $z_1$ . Then,

$$z_1 z^* = a^2 + b^2.$$

##### Proof: Theorem 6.1.1

By definition,  $z^* = a - bi$ . Thus,

$$\begin{aligned} z_1 z^* &= (a + bi)(a - bi) \\ &= a^2 - (bi)^2 \\ &= a^2 + b^2. \end{aligned}$$

- Define  $z_1 = a + bi$  and  $z_2 = c + di$ :

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i.$$

### Proof: 6.1.2

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{(ac + bd) - (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i. \end{aligned}$$

### Example: 6.1.1

Find  $z \in \mathbb{C}$  that satisfies the equation  $\frac{z+2}{1-i} = \frac{z-3i}{2+i}$ .

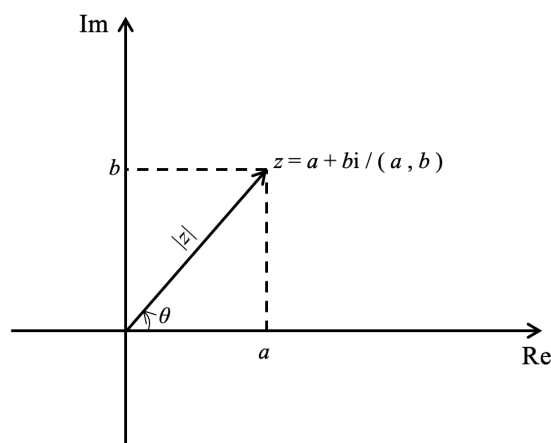
$$\begin{aligned} (z+2)(2+i) &= (z-3i)(1-i) \\ z(2+i) + 4 + 2i &= z(1-i) - 3i + (3i)^2 \\ z(2+i-1+i) &= -3i - 3 - 4 - 2i \\ z(1+2i) &= -7 - 5i \\ z &= \frac{-7-5i}{1+2i} = -\frac{17}{5} + \frac{9}{5}i. \end{aligned}$$

3. If  $s = a + bi$  and  $t = c + di$ , then:

$$\operatorname{Re}(s) + \operatorname{Re}(t) = \operatorname{Re}(s+t); \text{ and } \operatorname{Im}(i \cdot s) = \operatorname{Re}(s).$$

## 6.2 Argand Diagram

1. The Complex Plane:



$z = a + bi$  can be represented on a complex plane with real coordinate  $a$  and imaginary coordinate  $b$ . It can also be denoted as  $z(a, b)$ .

- Modulus of a complex number:

$$|z| = \sqrt{a^2 + b^2}.$$

- Argument of a complex number:

$$\text{Arg}(z) = \arctan\left(\frac{b}{a}\right) (+k\pi) \rightarrow \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

\*When determine a complex number, first draw it on the plane to show which quadrant it is in.

The range of argument is  $[0, 2\pi]$  or  $[-\pi, \pi]$ .

- Use modulus and argument to express a complex number:

$$a = |z| \cdot \cos \theta;$$

$$b = |z| \cdot \sin \theta.$$

2. If  $z = a + bi$  and  $|z| = 1$ , then  $z^* = z^{-1}$ .

#### Proof: 6.2.1

$$\because |z| = 1$$

$$\therefore \sqrt{a^2 + b^2} = 1$$

$$\therefore a^2 + b^2 = 1$$

##### Method 1

$$\begin{aligned} \text{RHS} = z^{-1} &= \frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} \\ &= \frac{a - bi}{a^2 + b^2} = a - bi \\ &= z^* = \text{LHS}. \end{aligned}$$

##### Method 2

$$\begin{aligned} z \cdot z^* &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 = 1 \\ \therefore z^* &= z^{-1} \end{aligned}$$

3. When  $|z| \neq 1$ ,  $z^* = \frac{|z|^2}{z}$ , and  $z^{-1} = \frac{z^*}{|z|^2}$ .

4. Properties of modulus and arguments:

For complex number  $s$  and  $t \in \mathbb{C}$ :

•

$$|st| = |s||t|$$

•

$$\left| \frac{s}{t} \right| = \frac{|s|}{|t|}$$

•

$$\text{Arg}(st) = \text{Arg}(s) + \text{Arg}(t) + 2k\pi$$

•

$$\text{Arg}\left(\frac{s}{t}\right) = \text{Arg}(s) - \text{Arg}(t) + 2k\pi$$

## 6.3 Complex Number in Other Forms

1. The Polar Form (Modulus-Argument Form):

•

$$z = r(\cos \theta + i \sin \theta) = r \text{cis} \theta$$

### Proof: 6.3.1

According to the Argand Diagram:

$$z = x + yi = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta).$$

•

$$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$$

•

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$$

2. de Moivre's Theorem:

- By Maclaurin Series:

$$e^{i\theta} = \text{cis} \theta = \cos \theta + i \sin \theta.$$

- Exponential form of complex number:

$$z = re^{i\theta} = r \text{cis} \theta.$$

3. Cartesian Form: Addition and Subtraction  
Modulus-Argument Form: Multiply and Division  
Exponential Form: Exponents and Roots

4. Since  $\text{cis} \theta = \text{cis}(\theta + 2k\pi)$ ,

$$re^{i\theta} = re^{i(\theta + 2k\pi)}.$$

### Example: 6.3.1

Find  $e^{i\frac{17\pi}{12}}$  in the form of Cartesian.

$$\begin{aligned} e^{i\frac{17\pi}{12}} &= e^{i(\frac{7\pi}{6} + \frac{\pi}{4})} = e^{i\frac{7\pi}{6}} \cdot e^{i\frac{\pi}{4}} \\ &= \text{cis}\left(\frac{7\pi}{6}\right) \cdot \text{cis}\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2} - \sqrt{6}}{4} - \frac{\sqrt{2} + \sqrt{6}}{4}i. \end{aligned}$$

## 6.4 Power of Complex Number

1. For a complex number  $z = re^{i\theta}$ ,

$$z^n = r^n e^{in\theta}.$$

### Example: 6.4.1

**Find**  $\left(3 \cos \frac{2\pi}{3} - 3i \sin \frac{\pi}{3}\right)^3$

$$\begin{aligned} \left(3 \cos \frac{2\pi}{3} - 3i \sin \frac{\pi}{3}\right)^3 &= \left(-3 \cos \frac{\pi}{3} - 3i \sin \frac{\pi}{3}\right)^3 \\ &= \left(-3 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)\right)^3 \\ &= (-3)^3 (e^{i\frac{\pi}{3}})^3 \\ &= -27e^{i\pi} \\ &= -27(-1) = 27. \end{aligned}$$

Key learnings from Example 6.4.1:

1.  $z = 3$  is only the fundamental root of equation  $z^3 = 27$ . In  $\mathbb{C}$ , there are other two complex roots that satisfy the equation.
2. In  $\mathbb{C}$ ,  $\sqrt{4} = \pm 2 = 2 + 0 \cdot i$  or  $-2 + 0 \cdot i$ .

### Example: 6.4.2

**Given a complex number  $\omega \neq 1$  is one of the solutions of  $z^3 = 1$ .**

**a. Prove  $\omega^2 + \omega + 1 = 0$ ;**

**b. Calculate  $\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022}$ .**

(a) Approach A

$$\begin{aligned} \because \omega^3 &= 1 \\ \therefore \omega^3 - 1 &= 0 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0 \\ \because \omega &\neq 1 \\ \therefore \omega^2 + \omega + 1 &= 0. \end{aligned}$$

Approach B  $\omega^2 + \omega + 1 = 0$  is a geometric sequence,  $u_1 = 1$ ,  $r = \omega$ :

$$S_3 = \frac{u_1(1-r^3)}{1-r} = \frac{1-\omega^3}{1-\omega} = \frac{0}{1-\omega} = 0.$$

(b)

$$\begin{aligned} \omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022} &= \omega^{2019} \times (1 + \omega + \omega^2 + \omega^3) \\ &= \omega^{2019}(0 + 1) = \omega^{2019} \\ &= (\omega^3)^{673} = 1. \end{aligned}$$



### Example: 6.4.3

**Find:**

- a.  $1^i$ ;
  - b.  $\ln(-1)$ ;
  - c.  $\ln(-c)$ , where  $c$  is a constant.
- 

(a)

$$1 = e^{i2\pi} \Rightarrow 1^i = \left(e^{i2\pi}\right)^i = e^{-2\pi}. \quad (1^i = e^{-2k\pi}, k \in \mathbb{Z})$$

(b)

$$-1 = e^{i\pi} \Rightarrow \ln(-1) = \ln(e^{i\pi}) = i\pi.$$

(c)

$$\ln(-c) = \ln[(-1) \cdot c] = \ln(-1) + \ln(c) = \ln(c) + i\pi.$$

## 6.5 Polynomial Function with Complex Roots

1. Conjugate Pair Theorem:

### Theorem: 6.5.1 Conjugate Pair Theorem

If  $z$  is a complex root of  $P(x)$ , then the conjugate of  $z(z^*)$  is also a complex root of  $P(x)$ . ( $P(x)$  should be a polynomial with **rational** coefficients.)

2. Properties of Conjugate.

•

$$(s \pm t)^* = s^* \pm t^*$$

•

$$(st)^* = s^* t^*$$

•

$$\left(\frac{s}{t}\right)^* = \frac{s^*}{t^*}$$

## 6.6 Root of Complex Numbers

1. The Root of Unity:

### Theorem: 6.6.1 The Root of Unity

For any complex equation  $\omega^n = 1$ , there are  $n$  distinct roots:

$$1 = e^{i(0+2k\pi)} = \omega^n, k \in \mathbb{Z} \Rightarrow \omega = e^{i\frac{2k\pi}{n}}, k \in \mathbb{Z}.$$

**Example: 6.6.1**

Solve  $z^3 = 8$ .

---

$$z^3 = 8 \cdot 1 = 8e^{i(0+2k\pi)} \Rightarrow z = 2e^{i\frac{2k\pi}{3}}, k \in \mathbb{Z}$$

$$k = 0 : z = 2$$

$$k = 1 : z = 2e^{i\frac{2\pi}{3}} = 2\text{cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$$

$$k = 2 : z = 2e^{i\frac{4\pi}{3}} = 2\text{cis}\left(\frac{4\pi}{3}\right) = -1 - \sqrt{3}i$$

2. Property of  $\text{cis}\theta$ :

$$\text{cis}(-\theta) = \cos \theta - i \sin \theta$$

**Proof: 6.6.1**

$$\begin{aligned}\cos \theta - i \sin \theta &= \cos(-\theta) + i \sin(-\theta) \\ &= \text{cis}(-\theta).\end{aligned}$$