IB Mathematics Analysis and Approaches HL

Topic 3 Geometry and Trigonometry

Jiuru Lyu

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Contents

1 Trigonometry

1.1 Radian

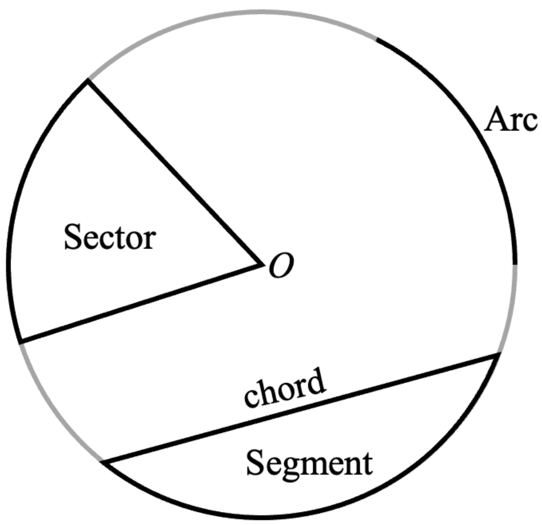
1. Radian as the unit of angle:

   - $$\pi \text{ rad} = 180^\circ$$
   - rad can be omitted. i.e., $\widehat{A} = 1$ means angle A is 1 radian.
   - Unit coversion:

   $$\text{degree} \times \frac{\pi}{180^\circ} = \text{radian; radian} \times \frac{180^\circ}{\pi} = \text{degree.}$$

2. Arc:

   - The **circumference** (perimeter) is $2\pi r$.



   - If the angle of the arc is θ (in radian), the length of arc$(l) = r \cdot \theta$.
   - The area of a sector:

   $$A = \frac{1}{2} r^2 \theta.$$

   - The area of a segment:

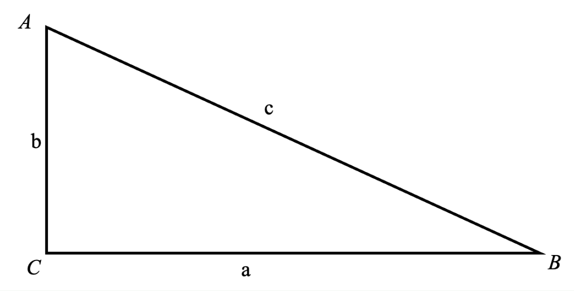
   $$A = \frac{1}{2} r^2 (\theta - \sin\theta).$$

   (Proof: the area of the triangle according to the sine rule is $\frac{1}{2} ab \sin C$)

1.2 Solution of Triangle

1. Define sine, cosine, and tangent:

Definition 1:



$$
\sin A = \frac{a}{c},\ \sin B = \frac{b}{c};
$$
$$
\cos A = \frac{b}{c},\ \cos B = \frac{a}{c};
$$
$$
\tan A = \frac{a}{b},\ \tan B = \frac{b}{a}.
$$

2. The Sine Rule:

Theroem: 3.1.2.1 The Sine Rule

$$
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.
$$

- The bigger the angle, the longer the side.
- Area of a triangle:

$$
A = \frac{1}{2} ab \sin C.
$$

3. The Consine Rule:

Theroem: 3.1.2.2 The Cosine Rule

$$
b^2 + c^2 - a^2 = 2bc \cdot \cos A;
$$
$$
a^2 + c^2 - b^2 = 2ac \cdot \cos B;
$$
$$
a^2 + b^2 - c^2 = 2ab \cdot \cos C.
$$

4. Inverse Trigonometric Functions:

Definition 2:

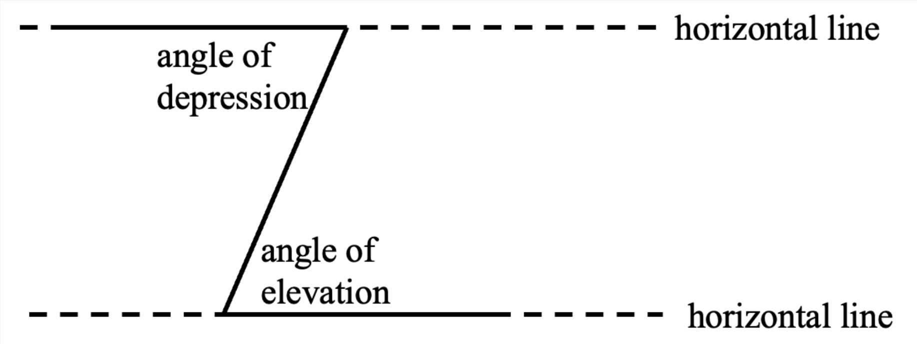
$$
\sin^{-1}\theta = \arcsin\theta;
$$
$$
\cos^{-1}\theta = \arccos\theta;
$$
$$
\tan^{-1}\theta = \arctan\theta.
$$

5. Ambiguity of Sine Rule:

$$\sin\theta = \sin(180^\circ - \theta) \text{ OR } \sin\theta = \sin(\pi - \theta).$$

6. Angle of Elevation and Depression:

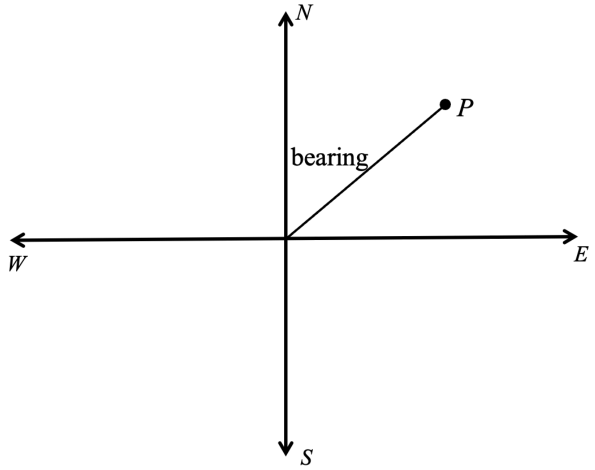
Definition 3:



- **Angle of Elevation** is the angle "up" from horizontal.
- **Angle of Depression** is the angle "down" from horizontal.

7. Bearing:

- Bearing is a way of describing direction.
- All bearings are measured clockwise from the North direction.

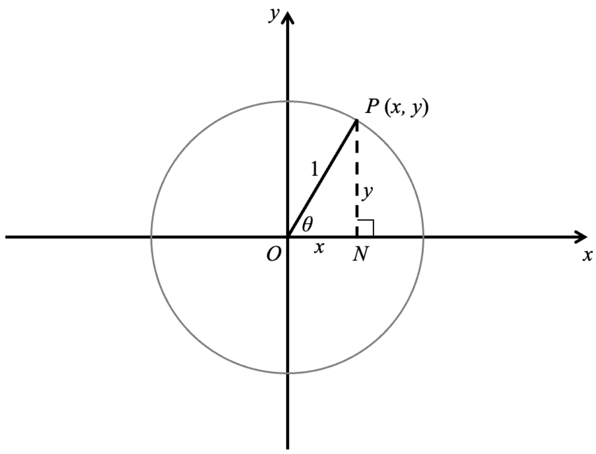


- Bearing of A from B: construct at B.
  N.B.: Bearing of A from B is different from bearing of B from A.

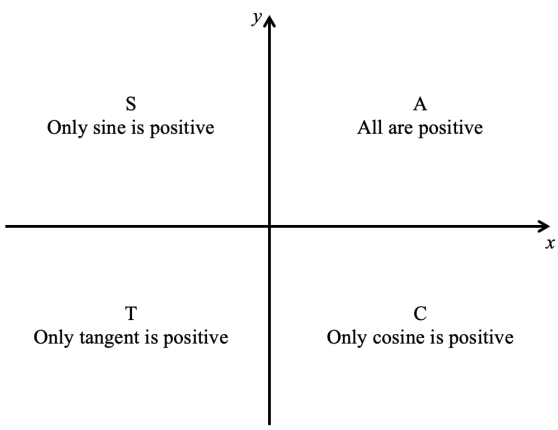
1.3 Definition of Trigonometric Function

1. Unit Circle:

- Center at $(0,0)$ with a radius of 1.



- If an angel θ opens in a counterclockwise direction, then θ is positive.
  If an angle θ opens in a clockwise direction, then θ is negative.
- In the diagram, $\theta = \theta + 2k\pi,\ k \in \mathbb{Z}$.
- $$\sin\theta = \frac{PN}{OP} = \frac{y}{1} = y;$$
  $$\cos\theta = \frac{ON}{OP} = \frac{x}{1} = x;$$
  $$\tan\theta = \frac{PN}{ON} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta};$$
- In Q_1 and Q_2, $\sin\theta$ will be positive.
  In Q_1 and Q_4, $\cos\theta$ will be positive.
  In Q_1 and Q_3, $\tan\theta$ will be positive.
  $\Rightarrow$ CAST:



2. Special Angles:

$$\sin 0^\circ = 0 = \cos 90^\circ$$
$$\sin 30^\circ = \frac{1}{2} = \cos 60^\circ$$
$$\sin 45^\circ = \frac{\sqrt{2}}{2} = \cos 45^\circ$$
$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \cos 30^\circ$$
$$\sin 90^\circ = 1 = \cos 0^\circ$$

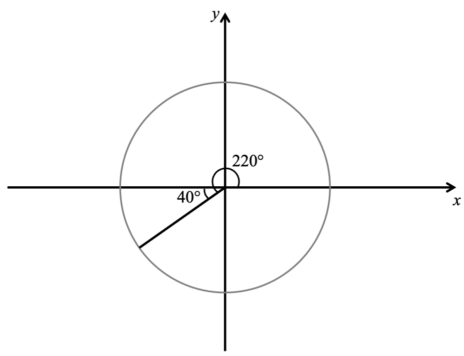
$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = 0$$
$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\sqrt{3}}{3}$$
$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1$$
$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}$$
$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \infty$$

3. Relative Acute Angles (RAA):
   - Acute angle is the angle with x-axis.
   - The absolute value of angles have the same acute angle is the same.

Example: 3.1.3.1

(a) $30^\circ,\ 150^\circ,\ 210^\circ,\ 330^\circ$ have the same acute angle.
$\therefore |\sin 30^\circ| = |\sin 150^\circ| = |\sin 210^\circ| = |\sin 330^\circ|$.

(b)
$$\tan 220^\circ = \tan 40^\circ;\ \cos 215^\circ = -\cos 35^\circ$$



1.4 Trigonometric Identity

1. Pythagorean's Identity:
$$\sin^2\theta + \cos^2\theta \equiv 1.$$

Proof: 3.1.4.1

$$a^2 + b^2 = c^2 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$
$$\Rightarrow \sin^2\theta + \cos^2\theta = 1.$$

2. Definition of Tangent:

   - $$\tan\theta = \frac{\sin\theta}{\cos\theta};$$
   - $$\cot\theta = \frac{1}{\tan\theta};$$
   - $$\sec\theta = \frac{1}{\cos\theta};$$
   - $$\csc\theta = \frac{1}{\sin\theta}.$$

3. Extended Pythagorean's Identity:

$$\tan^2\theta + 1 = \sec^2\theta;$$

$$\cot^2\theta + 1 = \csc^2\theta.$$

Proof: 3.1.4.2

$$\begin{aligned} \sin^2\theta + \cos^2\theta = 1 \Rightarrow & \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \tan^2 + 1 = \sec^2\theta; \\ & \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow \cot^2\theta + 1 = csc^2\theta. \end{aligned}$$

N.B.: a reflex angle is an angle bigger than $180°$, smaller than $360°$.

4. Compound Angle Formula:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B;$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B;$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B;$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B;$$

Example: 3.1.4.1

Find the exact value of $\cos\frac{\pi}{12}$.

$$\begin{aligned} \cos\frac{\pi}{12} &= \cos\frac{\pi}{4} - \frac{\pi}{6} \\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}. \end{aligned}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Proof: 3.1.4.3

$$\begin{aligned}\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} &= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \\ &= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}} \\ &= \frac{\tan A + \tan B}{1 - \tan A\tan B}.\end{aligned}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

5. In the linear function $y = mx + b$, $m = \tan\theta$, where θ is the angle between the line and the positive x-axis.

6. Double Angle Formula:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta;$$

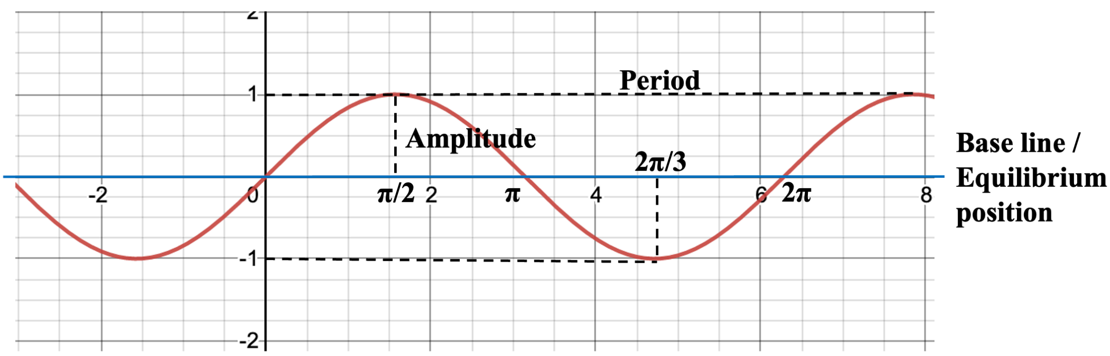
$$\sin(2\theta) = 2\sin\theta\cos\theta;$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

7. Proving Identities.

1.5 Trigonometric Functions and Transformation

1. Sine: Odd function: $\sin(-x) = -\sin x$.



$$\text{T(Period)} = 2\pi;$$

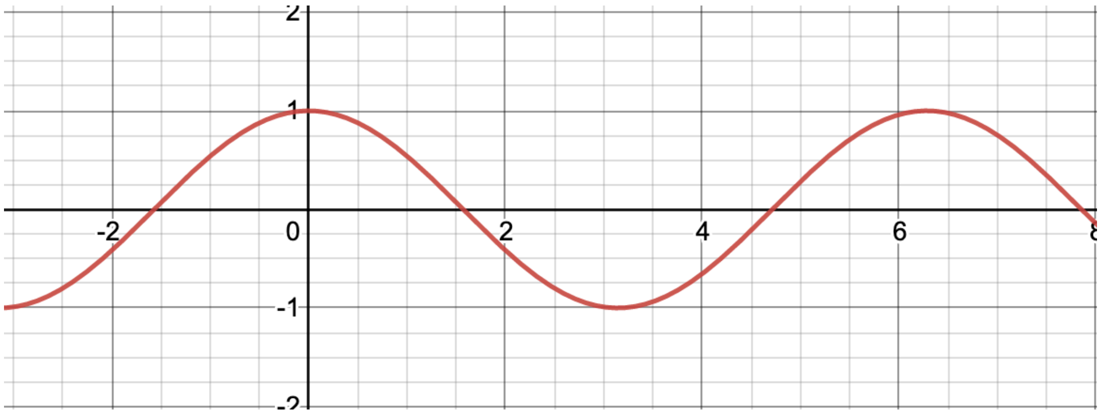
$$\text{Base line} = 0;$$

$$\text{Amplitude} = \left|\frac{y_{\max} - y_{\min}}{2}\right| = 1;$$

$$\text{Range: } \sin x \in [-1,\ 1];$$

$$\text{Domain: } x \in \mathbb{R}.$$

2. Cosine: Even function: $\cos(-x) = \cos x$.



$$\text{T(Period)} = 2\pi;$$

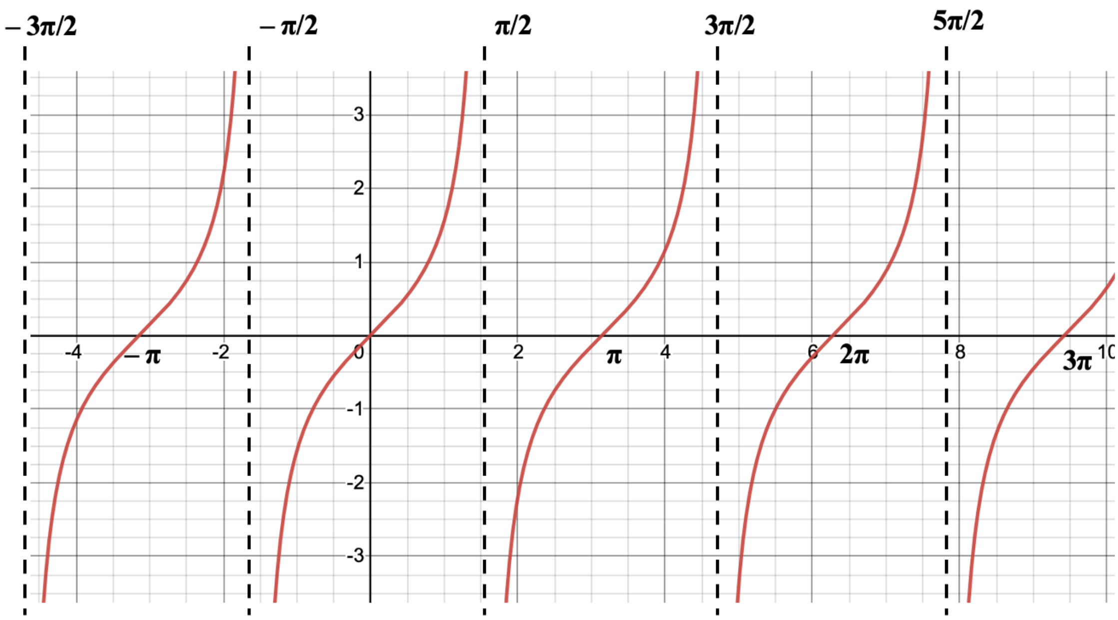
$$\text{Base line} = 0;$$

$$\text{Amplitude} = \left|\frac{y_{\max} - y_{\min}}{2}\right| = 1;$$

$$\text{Range: } \cos x \in [-1,\ 1];$$

$$\text{Domain: } x \in \mathbb{R}.$$

3. Tangent:



$$\text{T(Period)} = \pi;$$

$$\text{No amplitude(A)};$$

$$\text{V.A.: } x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z};$$

$$\text{Range: } \tan x \in \mathbb{R};$$

$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}.$$

4. Transformation of Sine and Cosine:

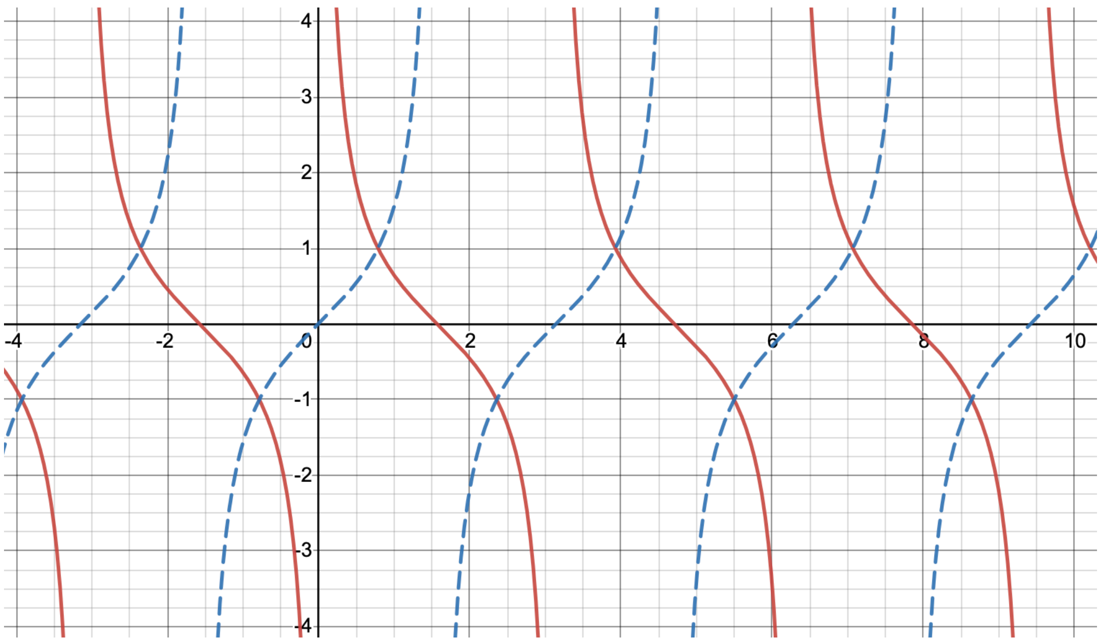
$$y = A\sin(\omega(x-\varphi)) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. $\Rightarrow$ changes $T = \frac{\pi}{\omega}$.
- Horizontal translate to the right φ units. $\Rightarrow$ changes the initial point to $(\varphi, 0)$.
- Vertical stretch with a scale factor of A. $\Rightarrow$ changes the amplitude$= |A|$.
- Vertical translation of h units upwards. $\Rightarrow$ changes the equilibrium position $y = h$.
- Range of $y = A\sin(\omega(x-\varphi)) + h$: $y \in [h-A, h+A]$.

$$y = A\cos(\omega(x-\varphi)) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. $\Rightarrow$ changes $T = \frac{\pi}{\omega}$.
- Horizontal translate to the right φ units. $\Rightarrow$ changes the initial point to $(\varphi, 1)$.
- Vertical stretch with a scale factor of A. $\Rightarrow$ changes the amplitude$= |A|$, initial point (φ, A).
- Vertical translation of h units upwards. $\Rightarrow$ changes the equilibrium position $y = h$, initial point $(\varphi, A+h)$.

5. Cotangent:

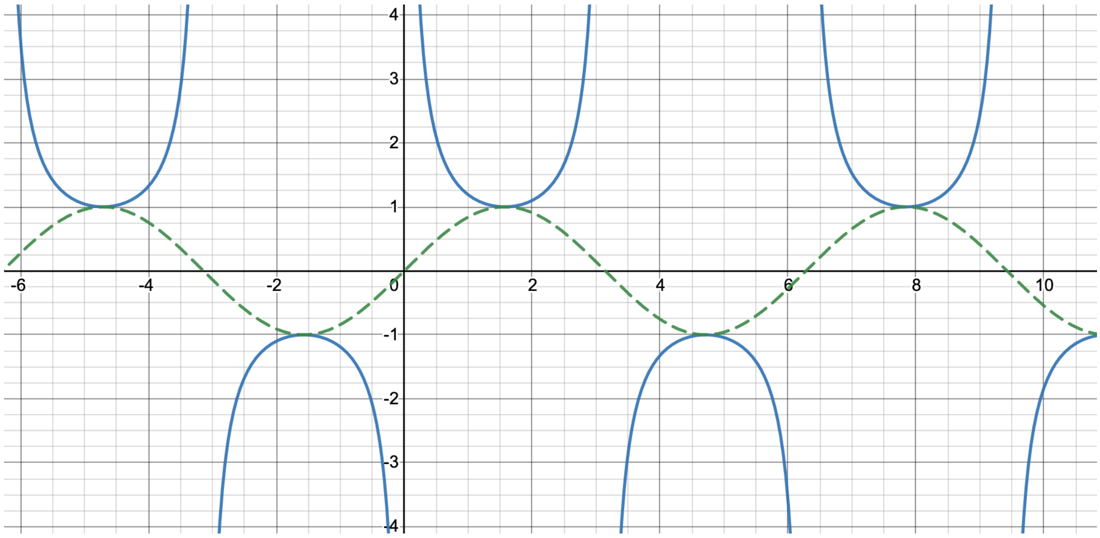


$$\text{V.A.: } x = k\pi$$

$$\text{Period: } \pi$$

$$\text{Pass through } \left(\frac{\pi}{2} + k\pi, 0\right)$$

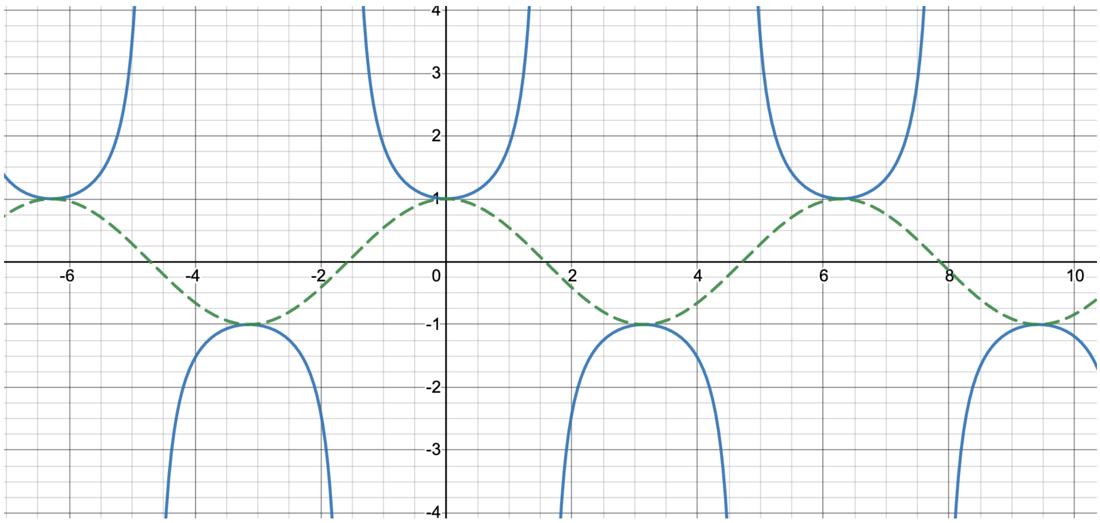
6. Cosecant:



$$\text{Domain: } x \neq k\pi$$

$$\text{Range: } y \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[$$

7. Secant:



$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi$$

$$\text{Range: } y \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[$$

8. When drawing the graph of $secx$ and $\csc x$, draw $cosx$ and $\sin x$ first.

9. Conversion between sine and cosine:

- $$\sin\left(\frac{\pi}{2}-x\right)=\cos x$$
- $$\cos\left(\frac{\pi}{2}-x\right)=\sin x$$
- $$\cos\left(\frac{\pi}{2}+x\right)=\cos\left[\pi-\left(\frac{\pi}{2}-x\right)\right]=-\cos\left(\frac{\pi}{2}-x\right)=-\sin x$$
- $$\sin\left(\frac{\pi}{2}+x\right)=\sin\left[\pi-\left(\frac{\pi}{2}-x\right)\right]=\sin\left(\frac{\pi}{2}-x\right)=\cos x$$

1.6 Solving Trigonometric Functions

1. Solving Trigonometric Functions in Paper 1:
   - Values of sepcial angles
   - From relative acute angles and CAST rule
   - Modification of period
   - Check the solution with domain

Example: 3.1.6.1

Solve for $\cos x=\frac{\sqrt{3}}{2}$ for $0<x<3\pi$.

$$\text{Consider } x\in[0,2\pi]$$

$$x=\frac{\pi}{6},\frac{11\pi}{6}.$$

$$\text{In the domain of } x\in[0,3\pi],$$

$$\text{Another solution is }\frac{13\pi}{6}.$$

2. Transformed Trigonometric Equations:

Example: 3.1.6.2

Solve $6\sin\left(2\left(x-\frac{\pi}{6}\right)\right)-2=1,\ \frac{\pi}{6}<x<2\pi$.

$$\sin\left(2\left(x-\frac{\pi}{6}\right)\right)=\frac{1}{2}.$$

$$\text{Let } t=2\left(x-\frac{\pi}{6}\right):$$

Example: 3.1.6.2 - Continued

$$\because \frac{\pi}{6} < x < 2\pi,$$

$$\therefore 0 < 2\left(x - \frac{\pi}{6}\right) < \frac{11\pi}{3},\ 0 < t < \frac{11\pi}{3}.$$

$$\sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6};$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}.$$

3. Solving Trigonometric Functions in Paper 2:

   - Change mode to RADIAN.
   - Plot the functions.
   - Adjust the window.
   - Calculate the intersects.
   - Repeat step 4 if necessary.

1.7 Inverse Trigonometric Functions

1. Inverse Trigonometric Function:

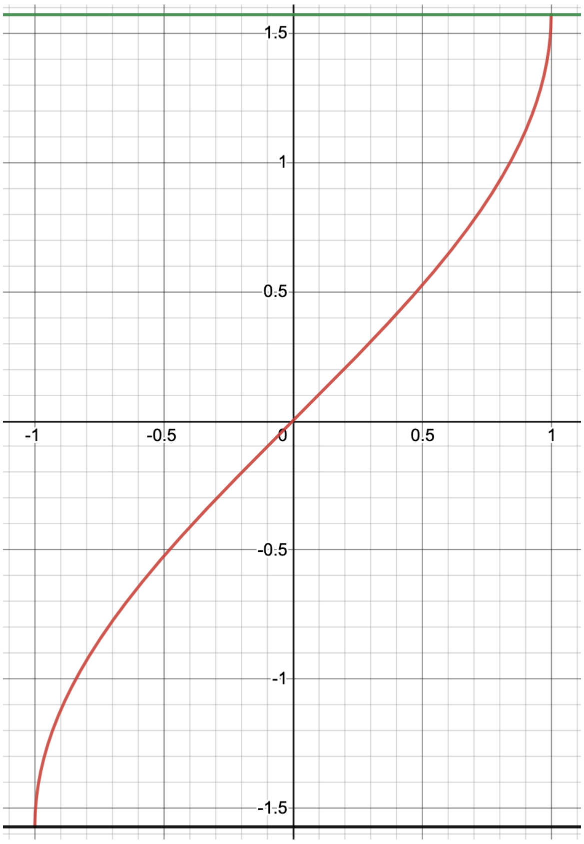
   - $$y = \arcsin x$$
   - $$y = \arccos x$$
   - $$y = \arctan x$$
   - $$\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$$
   - $$\operatorname{arccsc} x = \arcsin\left(\frac{1}{x}\right)$$
   - $$\operatorname{arccot} x = \arctan\left(\frac{1}{x}\right)$$

2. One-to-one Function:

   - In order for functions to have the inverse function, it must be so called **one-to-one** function (bijection).

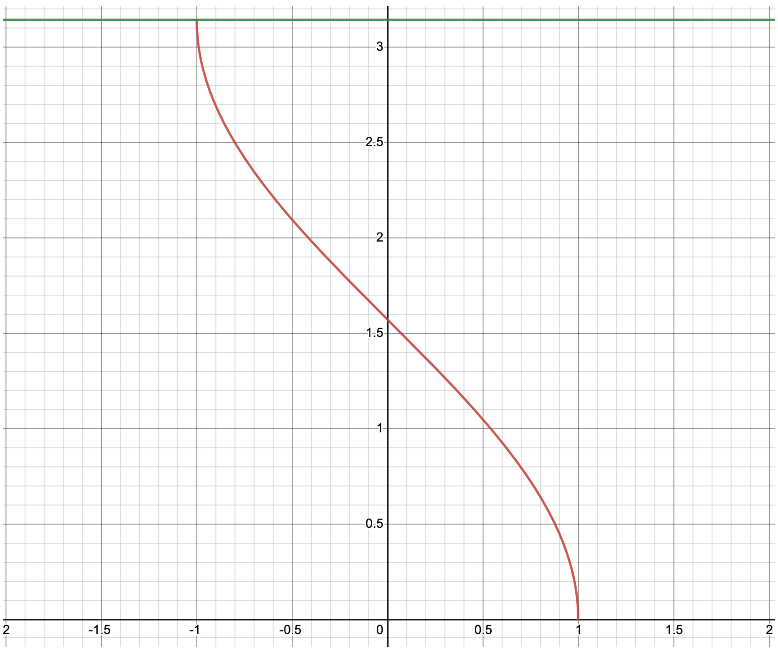
- One x value to one (and only one) y value.
  One y value to one (and only one) x value.

3. Domain and range for $\arcsin x$:



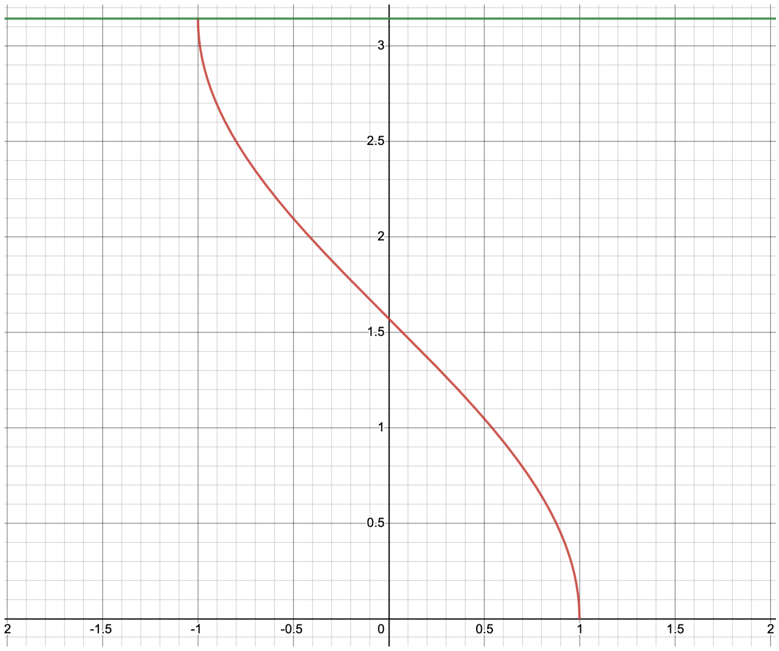
- Domain: $x \in [-1, 1]$ (Range $\sin x \in [-1, 1]$).
- Range: $\arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Domain $\sin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

4. Domain and range for $\arccos x$:



- Domain: $x \in [-1, 1]$.
- Range: $\arccos x \in [0, \pi]$.

5. Domain and range for $\arctan x$:



- Domain: $x \in \mathbb{R}$
- Range: $y \in \,]-\frac{\pi}{2}, \frac{\pi}{2}[$

2 Vectors

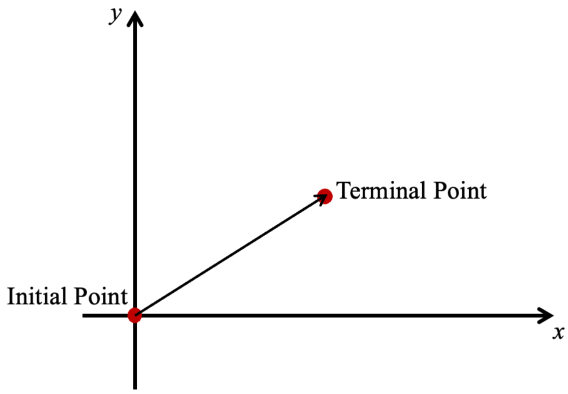
2.1 Introduction to Vectors

1. Vector:

   Definition 4:
   A **vector** is a quantity with a direction and magnitude. It is noted as $\vec{a}$.

2. Components of a vector:

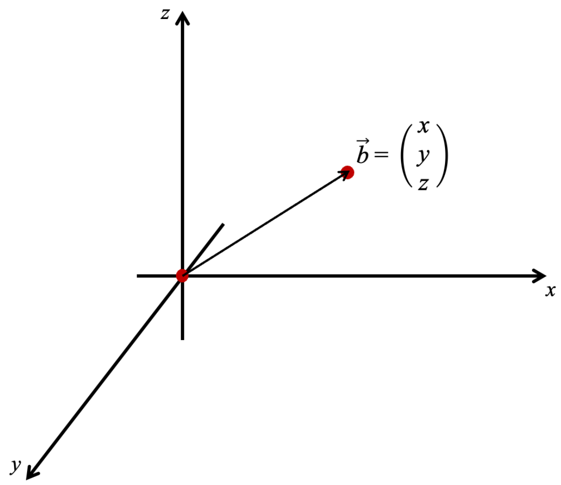
   - 2-D:



Example: 3.2.1.1

The vector $\vec{a} = \binom{3}{2}$ menas 3 units in the horizontal direction and 2 units in the vertical direction.

   - 3D:



3. **Magnitude/Modulus** of vector:

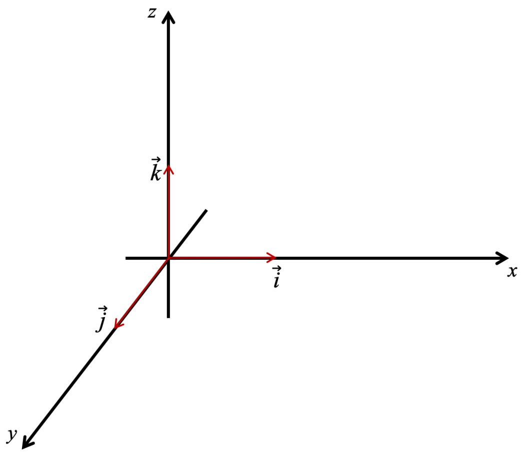
- 2D:

$$\text{For } \vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \; |\vec{a}| = \sqrt{x^2+y^2}.$$

- 3D:

$$\text{For } \vec{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; \left|\vec{b}\right| = \sqrt{x^2+y^2+z^2}.$$

4. **Unit Vector**: A vector of length 1:
    - $\vec{i}$: unit vector on the x-axis.
    - $\vec{j}$: unit vector on the y-axis.
    - $\vec{k}$: unit vector on the z-axis.

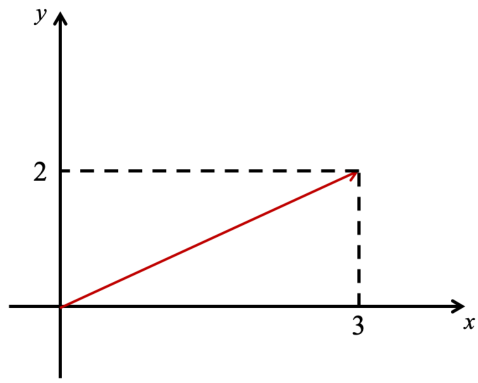


5. Sum of vectors:
    - **Position vector**: A vector that has an initial point at the origin.

Example: 3.2.1.2

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\vec{a} = 3\vec{i} + 2\vec{j}.$$



- Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$

$$\vec{a} + \vec{b} = \begin{pmatrix} x+m \\ y+n \end{pmatrix}.$$

6. Multiplication of vectors by a scalar:

   Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and n be a scalar:

$$n\vec{a} = n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} nx \\ ny \end{pmatrix}.$$

   $n\vec{a}$ and $\vec{a}$ are in the same direction $\Rightarrow$ parallel.

7. Subtracting a vector:

   Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$.

$$\vec{a} - \vec{b} = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

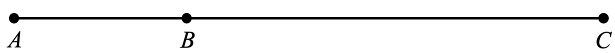
Proof: 3.2.1.1

$$-\vec{b} = (-1)\vec{b} = \begin{pmatrix} -m \\ -n \end{pmatrix}$$

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right) = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

8. **Zero vector**: $\vec{0}$.

9. **Collinear points**: three points, A, B, and C, are said to be collinear if $\vec{AB} = t\vec{AC}$.

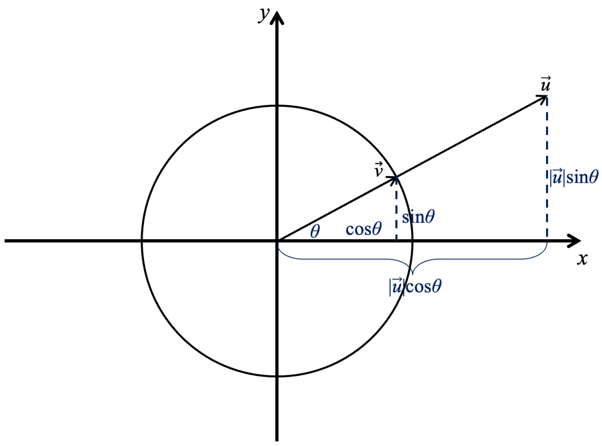


10. Find a unit vector parralel to $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$.

    - Find the value $|\vec{u}|$.
    - Then, the unit vector parralel to $\vec{u}$ is

$$\vec{v} = \frac{\vec{u}}{|\vec{u}|}.$$

11. Vectors and unit circle:



θ is the angle with the horizontal axis. The unit vector $\vec{v}$, in the same direction as $\vec{u}$ is:

$$\vec{v} = \cos\theta \cdot \vec{i} + \sin\theta \cdot \vec{j}$$

$$\begin{aligned} \vec{v} = \frac{1}{|\vec{u}|} \cdot \vec{u} \;\Rightarrow\; \vec{u} = |\vec{u}| \cdot \vec{v} &= |\vec{u}| \cos\theta \cdot \vec{i} + |\vec{u}| \sin\theta \cdot \vec{j} \\ &= |\vec{u}| \left(\cos\theta \cdot \vec{i} + \sin\theta \cdot \vec{j}\right). \end{aligned}$$

2.2 Scalar Product and Its Properties

1. The **scalar product** of two vectors is a real number (scalar).

   - The algebraic definition:
     For $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

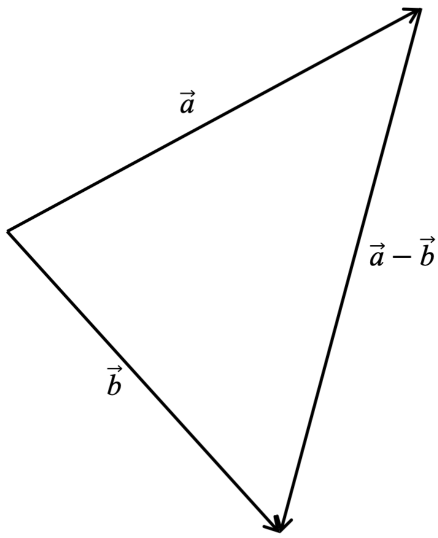
     $$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2.$$

     The scalar product is also called the dot product.

   - The geometric definition:
     For $\vec{a}$ and $\vec{b}$,

     $$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta, \;\; \theta \text{is the angle between the two vectors.}$$

Proof: 3.2.2.1



By cosine rule:

$$\left|\vec{b}-\vec{a}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2\,|\vec{a}|\,\left|\vec{b}\right|\cos\theta$$
$$\left|\vec{b}\right|^2 - 2\vec{a}\vec{b} + |\vec{a}|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2\,|\vec{a}|\,\left|\vec{b}\right|\cos\theta$$
$$\therefore \vec{a}\cdot\vec{b} = |\vec{a}|\,\left|\vec{b}\right|\cos\theta.$$

- Combining the two definitions:

$$\cos\theta = \frac{a_1b_1 + a_2b_2}{\sqrt{\left(a_1^2+a_2^2\right)\left(b_1^2+b_2^2\right)}}.$$

2. 3-D vectors: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

   - $$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$
   - $$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)}}.$$

3. Properties of scalar product:

- If $\vec{a}\cdot\vec{b}=0 \Rightarrow \begin{cases} \vec{a}=0 \\ \vec{b}=0 \\ \vec{a} \text{ and } \vec{b} \text{ are perpendicular (orthogonal)} \Rightarrow \theta=\frac{\pi}{2} \end{cases}$.

- If $\vec{a}$ and $\vec{b}$ are colinear,

$$\vec{a}\cdot\vec{b}=\pm|\vec{a}|\left|\vec{b}\right|.$$

Proof: 3.2.2.2

Angel between $\vec{a}$ and $\vec{b}$ is $0°$.
$\cos 0° = 1 \Rightarrow \vec{a}\cdot\vec{b}=|\vec{a}|\left|\vec{b}\right|$ for $\vec{a}$, $\vec{b}$ at the same direction.
OR $\vec{a}\cdot\vec{b}=-|\vec{a}|\left|\vec{b}\right|$ for $\vec{a}$ and $\vec{b}$ at opposite directions.

- $$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}.$$

- $$\vec{a}\cdot\vec{a}=|\vec{a}|^2$$

Proof: 3.2.2.3

$$\vec{a}\cdot\vec{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}\begin{pmatrix}a_1\\b_1\end{pmatrix}=a_1^2+a_2^2=|\vec{a}|^2.$$

- $$\vec{a}\cdot\left(\vec{b}+\vec{c}\right)=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}.$$

- $$\lambda\left(\vec{a}\cdot\vec{b}\right)=(\lambda\vec{a})\cdot\vec{b}=\vec{a}\cdot\left(\lambda\vec{b}\right).$$

2.3 Vector Equation of a Line

1. There is only one line that passes through two distinct points.

Theroem: 3.2.3.1

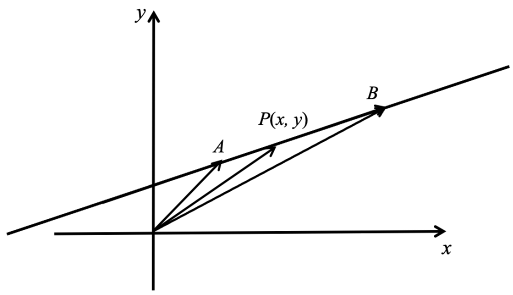
In the coordinate plane, the equation can be found as:
For $A(x_1,y_1)$ and $B(x_2,y_2)$, the line passes through A, B is given by

$$y=\frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1.$$

2. **Slope, y-intercept form**: $y=mx+k$, where m is the slope, and k is the y-intercept.
   It can be rearranged to $ax+by=c$; $a,b,c\in\mathbb{R}$, where a and b cannot be equal to 0 at the same time.

3. **Vector form** of a line:

- For every point $P(x,y)$ that lies on the line AB, the vector $\overrightarrow{AP}$ must be collinear or parallel to $\overrightarrow{AB}$: $\overrightarrow{AP} = k\overrightarrow{AB},\ k \in \mathbb{R}$.



(a) The vector $\overrightarrow{AB}$ is called a **direction vector** of the line.
All the vectors that are parralel to $\overrightarrow{AB}$ can also define the same line.

(b) Assume $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{AB}$ is the direction vector $\vec{d}$. Then, $\overrightarrow{AP} = \vec{p} - \vec{a} = k\overrightarrow{AB} = k\vec{d}$

$$\vec{p} = \vec{a} + k\vec{d},\ k \in \mathbb{R}.$$

- Vector equation of a line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} + k \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},\ k \in \mathbb{R}.$$

- Parametric form:

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases},\ k \in \mathbb{R}.$$

- Cartesian form:

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}.$$

Proof: 3.2.3.1

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases} \Rightarrow \begin{cases} k = \frac{x - x_1}{d_1} \\ k = \frac{y - y_1}{d_2} \end{cases}.$$

(a) Cartesian form can be further rearranged to slope-intercept form

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}$$
$$\frac{d_2}{d_1}(x - x_1) = y - y_1$$
$$y = \frac{d_2}{d_1}(x - x_1) + y_1,$$

where $\frac{d_2}{d_1}$ is the slope.

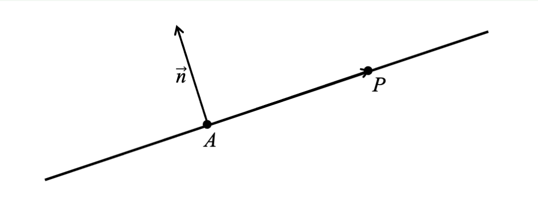
(b) Another way of interpretation:

$$\begin{aligned}
\overrightarrow{AP} = k\overrightarrow{AB} \Rightarrow \vec{p}-\vec{a} &= k\left(\vec{b}-\vec{a}\right) \\
\vec{p} &= (1-k)\vec{a}+k\vec{b},\ k\in\mathbb{R}. \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= (1-k)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k\begin{pmatrix} x_2 \\ y_2 \end{pmatrix},\ k\in\mathbb{R}. \\
\Rightarrow & \begin{cases} x = (1-k)x_1 + kx_2 = x_1 + k(x_2-x_1) \\ y = (1-k)y_1 + ky_2 = y_1 + k(y_2-y_1) \end{cases},\ k\in\mathbb{R}. \\
\Rightarrow & \begin{cases} k = \frac{x-x_1}{x_2-x_1} \\ y = y_1 + k(y_2-y_1) \end{cases} \\
\Rightarrow\ y &= y_1 + \frac{x-x_1}{x_2-x_1}(y_2-y_1) \\
&= \frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1.
\end{aligned}$$

4. Orthogonal / Perpendicular vector of a line.

- There is one and only one line in the plane that is perpendicular to a given line at a particular point on that line.
- Normal Vector:

Definition 5:



A **normal vector** is perpendicular or **orthogonal** to any vector on the lines.

$$\text{i.e., } \vec{n}\cdot\overrightarrow{AP} = 0.$$

Theroem: 3.2.3.2

$$\vec{n}\cdot(\vec{p}-\vec{a}) = 0 \ \Rightarrow\ \vec{n}\cdot\vec{p} = \vec{n}\cdot\vec{a}.$$

- If the direction vector $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, then one possible normal vector would be $\vec{n} = \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$ or any other vectors parallel to it.

- The vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$$

$$\Rightarrow xd_2 - yd_1 = x_1 d_2 - y_1 d_1$$

$$(x - x_1)d_2 = yd_1 - y_1 d_1$$

$$\therefore y = \frac{d_2}{d_1}(x - x_1) + y_1.$$

5. Direction vectors:

   - **Parallel lines** have **collinear** direction vectors.
   - **Perpendicular lines** have **orthogonal** direction vectors, such that the scalar product is equal to 0.

6. Vector equation of lines in 3-D spaces:

   - $$\vec{r} = \vec{a} + \lambda \vec{d}, \ \lambda \in \mathbb{R}.$$
     $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$
   - The parametric form:
     $$\begin{cases} x = a_1 + \lambda d_1 \\ y = a_2 + \lambda d_2 \\ z = a_3 + \lambda d_3 \end{cases}, \ \lambda \in \mathbb{R}.$$
   - The cartesian form:
     $$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$

7. Two lines:

   - 2-D spaces: two distinctive lines can either be parallel or they can intersect.
   - 3-D spaces:
     (a) Lines are parallel.
     (b) Lines intersect at one common points.
     (c) Lines are **skewed** (do not intersect and they are not parallel).

2.4 Vector Product and Properties

1. The vector product is an operation that takes two vectors and results in another vector.

   - Definition

Definition 6:

Given the two vectors and their components, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then the **vector product** is given by:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

- The vector product of two vectors is another vector that is perpendicular to both vectors.
- Magnitude of the vector product:

Theroem: 3.2.4.1

The magnitude of the vector product is given by the formula

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \sin\theta,$$

where θ is the angle between those two vectors. If $\vec{a} \times \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are parallel/colinear.

- The geometrical definition of cross product (vector product):

Theroem: 3.2.4.2

Given two vectors $\vec{a}$ and $\vec{b}$, then the vector product is given by

$$\vec{a} \times \vec{b} = \left(|\vec{a}| \left|\vec{b}\right| \sin\theta\right) \widehat{n},$$

where $\widehat{n}$ is the unit vector whose direction is given by the right-hand screw rule to both $\vec{a}$ and $\vec{b}$ and the vectors $\vec{a}$, $\vec{b}$, and $\widehat{n}$ follows the right-hand rule.

- Geometrical meaning of the magnitude of the vector product:
  It is equal to the area of the parallelogram enclosed by those two vectors.

2. Properties of the vector product:

- $$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

- $$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \vec{a} \times \left(\vec{b} \times \vec{c}\right)$$

- $$\lambda\left(\vec{a} \times \vec{b}\right) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times \left(\lambda\vec{b}\right), \ \lambda \in \mathbb{R}$$

- $$\left(\vec{a} + \vec{b}\right) \times \vec{c} = (\vec{a} \times \vec{c}) + \left(\vec{b} \times \vec{c}\right).$$

3. Mixed product:

- An operation with three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ combining both the vector and scalar product is called a **mixed product**:

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}.$$

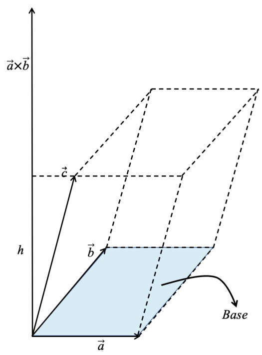
- Given $\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$, $\vec{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, and $\vec{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$, the mixed product is given by:

$$\begin{aligned}\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}&=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}\cdot\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\\&=a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1\end{aligned}$$

- Geometric meaning of mixed products:
  The volume of a parallelepiped fromed by three non-coplanar vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by:

$$V=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|.$$

Proof: 3.2.4.1



$$V=\text{Base}\times h$$

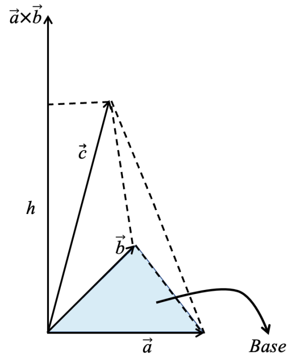
Base=magnitude of cross product of $\vec{a}$ and $\vec{b}$.
$=$ perpendicular projection of $\vec{c}$ to $\vec{a}\times\vec{b}$.

$$\therefore V=\text{Base}\times h=\left|\vec{a}\times\vec{b}\right|\cdot|\vec{c}|\cdot|\cos\theta|=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$$

- Three or more vectors are said to be coplanar if they lie in the same plane.
- Using mixed product to find the volume of a triangular pyramid:

$$V=\frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|.$$

Proof: 3.2.4.2



Since the base is not a parallelogram but a triangle, that is half an area of the parallelogram, we multiply $\frac{1}{2}$ in front of the expression of the cross product.

$$\text{Base} = \frac{1}{2}\left|\vec{a}\times\vec{b}\right|.$$

The volume of a pyramid is $\frac{1}{3}$ of the product of the base and the height.

$$\therefore V = \frac{1}{3}\text{Base}\cdot h = \frac{1}{3}\cdot\frac{1}{2}\left|\vec{a}\times\vec{b}\right| |\vec{c}| \, |\cos\theta| = \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$$

4. Proving vector product using matrix.

Proof: 3.2.4.3

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Convert into a 3×3 matrix: $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$.

Find the determinant: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$

$$\Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

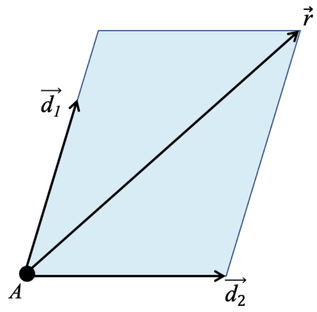
2.5 Vector Equation of a Plane

1. A plane is uniquely determined by three points (or a line and a point outside the line).
   $\rightarrow$ A plane can also be determined by two intersecting lines and a point outside the lines.

2. Vector equation of a plane:

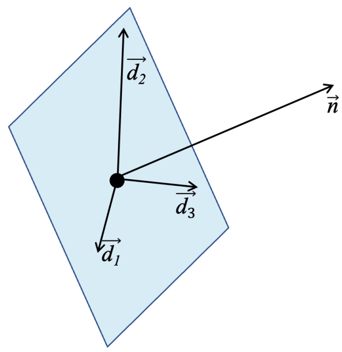
$$\vec{r} = \vec{a} + \lambda\vec{d}_1 + \mu\vec{d}_2, \ \lambda, \mu \in \mathbb{R}.$$

where $\vec{d}_1$ and $\vec{d}_2$ are direction vectors, and $\vec{a}$ is the position vector.



3. The scalar product form:

   - **Normal vector** is a vector that is perpendicular to every line in the plane.



   - All planes with the same normal vector are parallel to each other.
   - If R is any other point on the plane, then $\overrightarrow{AR}$ lies in the plane, and it is perpendicular to the normal vector $\vec{n}$.

Theroem: 3.2.5.1

$$\overrightarrow{AR} \cdot \vec{n} = 0 \Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\therefore \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

where $\vec{a}$ is the position vector, and $\vec{n}$ is the normal vector.

4. The Cartesian equation of a plane:

$$n_1x + n_2y + n_3z = d, \text{ where } n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \ d = \vec{a} \cdot \vec{n}.$$

Proof: 3.2.5.1

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \; d = \vec{a} \cdot \vec{n}, \; \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The scalar product form converts to:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \vec{a} \cdot \vec{n}$$

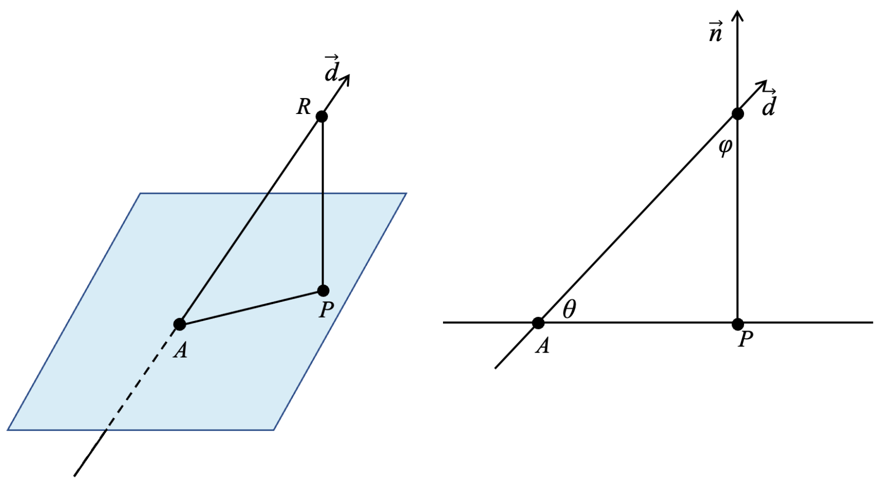
$$\Rightarrow n_1 x + n_2 y + n_3 z = d.$$

5. A plane with the vector equation $\vec{r} = \vec{a} + \lambda \vec{d}_1 + \mu \vec{d}_2$ has a normal vector $\vec{n} = \vec{d}_1 \times \vec{d}_2$.

2.6 Lines, Planes, and Angles

1. Angles and intersections between lines and planes:

   - When a line intersects a plane, the angle between them is defined as the smallest possible angle that the line makes with any of the lines in the plane.



   (a) $\overrightarrow{AR}$: the direction vector of the line, $\vec{d}$.
   (b) Point P is the projection of point R onto the plane. $\overrightarrow{AP}$ is the shadow of $\overrightarrow{AR}$ on the plane.
   (c) $\overrightarrow{PR}$ is in the direction of $\vec{n}$ since it is perpendicular to the plane.
   (d) φ is the angle between $\vec{n}$ and $\vec{d}$.
   (e)

$$\theta = 90^\circ - \varphi, \; \cos\varphi = \frac{\left|\vec{n} \cdot \vec{d}\right|}{|\vec{n}| \left|\vec{d}\right|}.$$

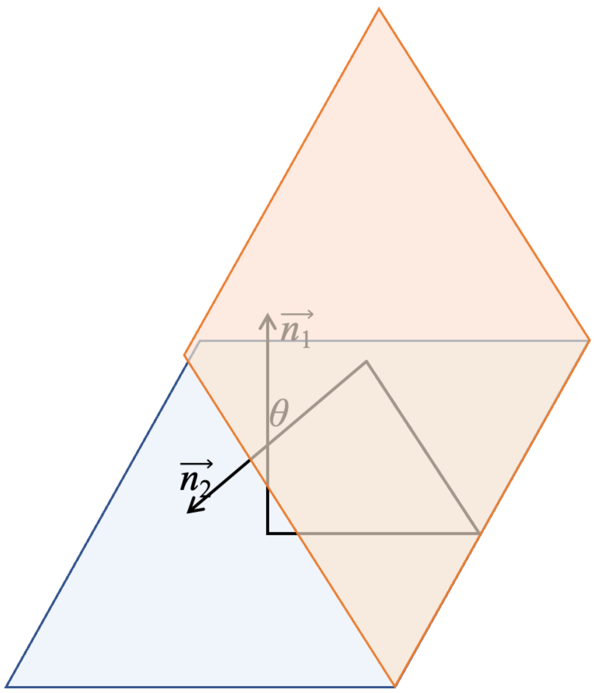
- A line that is not parallel to a plane intersects a plane at one point. The coordinates of this point of intersection satisfies both the equation of the line and the equation of the plane.

2. Relationship of two planes:

   - Two planes can either intersect at a line or they can be parallel.
   - When two planes are parallel, their normal vectors are colinear; otherwise they intersect at a line.

3. Angles between two planes:

   - The angle between two planes is **the angle between their normal vectors**.
   - $$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}.$$



4. Two non-parallel planes intersect along a line. The equation of this line is formed by treating the Cartesian equation of two planes as simultaneous equations and finding the general solution.

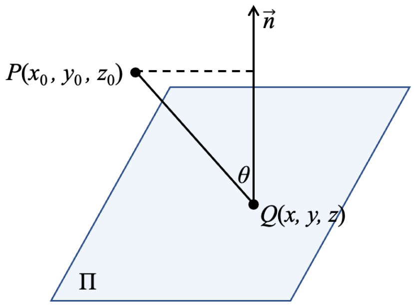
5. Distance between a point and a plane.

   - The distance, d, between a point $P(x_0, y_0, z_0)$, and a plane with equation $Ax + By + Cz = D$ where $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$, is given by:

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

- Proof:

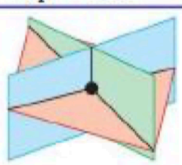
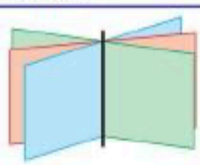
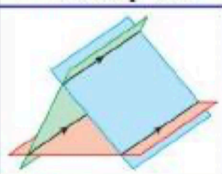
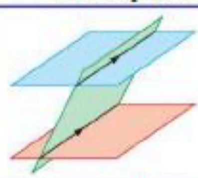
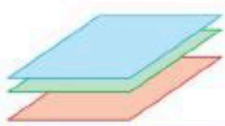
Proof: 3.2.6.1



Let $Q(x,y,z)$ be any point on the plane Π.
The distance, d, is the projection of the distance of point P to the plane on the normal vector, $\vec{n}$.

$$
\begin{aligned}
d = \left|\overrightarrow{QP}\right| \cdot |\cos\theta| &= \left|\overrightarrow{QP}\right| \cdot \frac{\overrightarrow{QP}\cdot\vec{n}}{\left|\overrightarrow{QP}\right| \cdot |\vec{n}|} \\
&= \frac{\overrightarrow{QP}\cdot\vec{n}}{|\vec{n}|} = \frac{|\langle A,B,C\rangle \cdot \langle (x_0-x),(y_0-y),(z_0-z)\rangle|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|Ax_0+By_0+Cz_0-(Ax+By+Cz)|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{Ax_0+Bx_0+Cz_0-D}{\sqrt{A^2+B^2+C^2}}
\end{aligned}
$$

6. Intersection of three points:

Unique solution	Infinitely many solutions	No solutions (inconsistent system)		
		No normals parallel	Two normals parallel	Three normals parallel
				
Three planes intersect at a point	Three planes intersect along a line	Three planes form a prism	One plane cutting two parallel planes	Three parallel planes

- The plane intersect:
  (a) At a point: the system of equations will have a unique solution.
  (b) Alone a line: the system of equations will have infinitely many solutions
- The systems of equations have no solutions:
  (a) No normals are parallel (the planes from a prism)

(b) 2 normals are parallel or three normal are parallel (the planes are parallel)