IB Mathematics Analysis and Approaches HL

Topic 2 Functions

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Contents

1 Foundations of Functions

1. Relations and functions:

> Definition 1: A **relation** R is a set of ordered pairs (x,y) such that $x \in A,\ y \in B$, and sets $A,\ B$ are not empty.

> Definition 2: A **function** f is a relation in which every x-value has a unique y-value.

2. Domain and Range:

> Definition 3: **Domain** is the set of x-values.

> Definition 4: **Range** is the set of y-values.

- Domain and Range should be in inverval notation.
  (a) Using invervals to express the inequalities

  Example: 2.1.2

  $$[\,3,4\,[\ \text{ means } 3 \le x < 4$$

  (b) If the interval will be joint, we use $\cup$ to join the inverval.

  Example: 2.1.2

  $$3 < x < 4 \text{ or } x \ge 5 \ \Rightarrow\ ]\,3,4\,] \cup [5,+\infty\,[$$

  Example: 2.1.3

  Find the interval notation for the domain of $f(x) = \dfrac{1}{x}$.

  $$x \in\ ]-\infty,0\,[\ \cup\ ]\,0,+\infty\,[\ \text{ OR } x \in \mathbb{R} \setminus 0$$

  Note: $\setminus$ means "exclude."

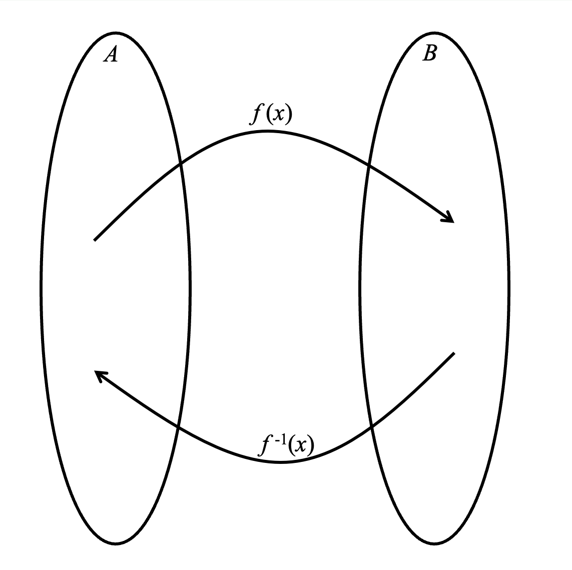
- Since the y-values (outputs) depend on the x-values (inputs), y is the **dependent variable**, and x is the **independent variable**.
- The independent vairbale x is also called the **argument** of the function.

3. Vertical Line test:

- To test whether a relation is a function.
- Since every x has one and only one value of y, there should be only one intersects.

4. Inverse of a function:

Definition 5: $f^{-1}(x)$ is the **inverse function** of $f(x)$.



Example: 2.1.3

$f(1) = 3 \Rightarrow f^{-1}(3) = 1$; $f(x) = x+5 \Rightarrow f^{-1}(x) = x-5$

- In inverse function, the input becomes the output, the output becomes the input.
- In inverse function, the domain becomes the range, the range becomes the domain.

Example: 2.1.4

(a) **Find the inverse function of** $y = \dfrac{x+2}{3}$**.**

$$3y = x+2 \Rightarrow x = 3y-2$$
$$f^{-1}(x) = 3x-2$$

(b) **Find the inverse function of** $f(x) = \dfrac{x}{x+1}$**.**

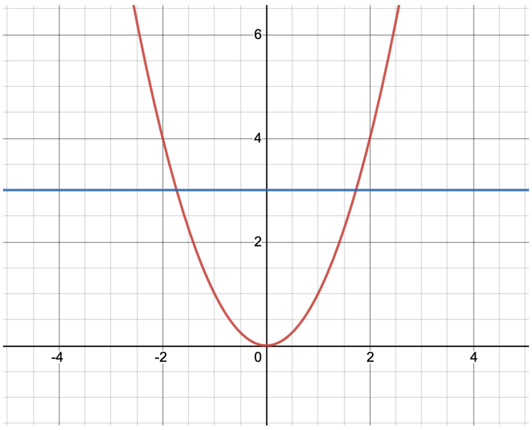
$$y = \frac{x}{x+1} \Rightarrow xy+y = x \Rightarrow xy+x = y$$
$$\therefore y(x-1) = -x \Rightarrow y = -\frac{x}{x-1}$$

(c) **Find the inverse of** $\{.(4,2),(0,2),(-2,2)\}$

Inverse: $\{(2,4),(2,0),(2,-2)\}$

- By restricting the domain, we can find $f^{-1}(x)$ of $f(x)$, if the direct inverse of $f(x)$ is not a function.

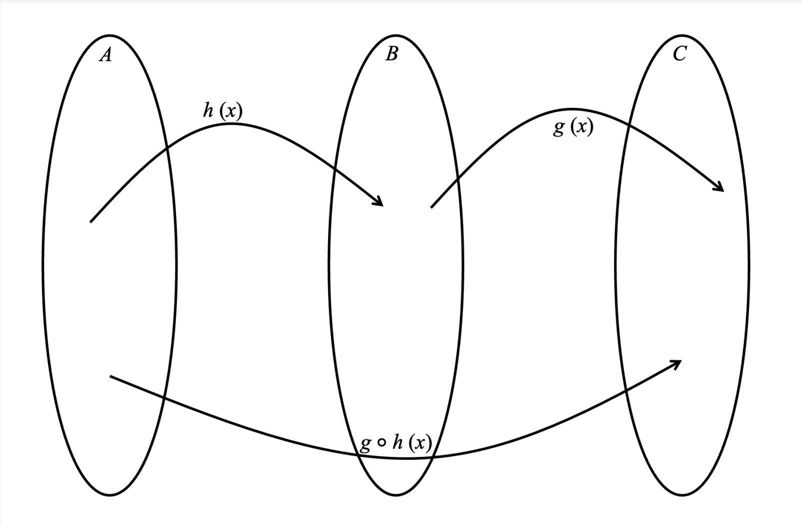
Example: 2.1.5



Horizontal line test: The largest domain we can find $f^{-1}(x)$ is $x \le 0$ or $x > 0$.

5. Composite Functions:

Definition 6: We use $(g \circ h)(x)$ or $g(h(x))$ to represent composite functions.



Example: 2.1.6

Given $f : x \mapsto 3x-6,\ g : x \mapsto \frac{1}{3}x+2$**. Find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$**.**

$$(f \circ g)(x) = f(g(x)) = 3(\frac{1}{3}x+2)-6 = x.$$

$$(g \circ f)(x) = g(f(x)) = \frac{1}{3}(3x-6)+2 = x.$$

When f and g are inverse functions:

$$(f \circ g)(x)(x) = x = (g \circ f)(x).$$

6. $f(x)$ and $f^{-1}(x)$ are symmetrical to $y = x$ since $D_f = R_{f^{-1}}$, $R_f = D_{f^{-1}}$. That is, if $f(x)$ passes through (a,b), $f^{-1}(x)$ passes through (b,a).

2 Quadratic Functions

1. The Standard Form:

$$y = ax^2 + bx + c,$$

where a is the coefficient of x^2, b is the coefficient of x, and c is the constant or y-intercept. $a, b, c \neq 0$.

- Zeros of the function (x-intercepts):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $\Delta = b^2 - 4ac$ is the discriminant of the function.

- Euqation of the line of symmetry & x-coordinate of the vertex

$$x = -\frac{b}{2a}.$$

- Vieta's Formula:

Theroem: 2.2.1: The Vieta's Theorem

Assume x_1, x_2 are two roots for equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then

$$x_1 + x_2 = -\frac{b}{a};$$

$$x_1 \cdot x_2 = \frac{c}{a}.$$

- When $a > 0$, the parabola opens upwards.
  When $a < 0$, the parabola opens downwards.

2. Completion of square:

$$x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2.$$

3. The Vertex Form:

$$y = a(x-h)^2 + k, \text{ where } (h,k) \text{ is the vertex}.$$

Example: 2.2.1

Given that $f(x) = ax^2 + bx + c$, find the axis of symmetry and vertex.

$$\begin{aligned} f(x) &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}. \end{aligned}$$

$$\begin{aligned} \therefore \text{ axis of symmetry: } x &= -\frac{b}{2a} \\ \text{vertex: } &\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right). \end{aligned}$$

3 Higher Order Polynomial Functions

1. Factor Theorem:

Theroem: 2.3.1: The Factor Theorem

If $(x-a)$ is a factor of a polynomial $P(x)$, then $x = a$ must be a root for $P(x) \Rightarrow P(a) = 0$.

Proof: 2.3.1

Assume the quotient when $P(x)$ is divided by $(x-a)$ is $Q(x)$, then $P(x) = Q(x) \cdot (x-a)$. Then, $P(a) = Q(a) \cdot (a-a) = 0$.

2. Long division: solving polynomial equation.

Example: 2.3.1

For a cubic function, $P(x) = 2x^3 + bx^2 + cx + d,\ P(1) = P(2) = P(3) = 2$. **What is** $P(0)$**?**

Since $P(1) = P(2) = P(3) = 2, Q(1) = Q(2) = Q(3) = 0,$ where $Q(x) = P(x) - 2$.
Thus, $Q(x) = 2(x-1)(x-2)(x-3)$.

$$\therefore P(x) = Q(2) + 2 = 2(x-1)(x-2)(x-3) + 2.$$

$$\therefore P(0) = 2(-1)(-2)(-3) + 2 = -10.$$

3. Remainder Theorem:

Theroem: 2.3.2: The Remainder Theorem

When a polynomial $P(x)$ is divided by $(ax-b)$, the remainder R of this division must be

$$P\left(\frac{b}{a}\right).$$

Proof: 2.3.2

Assume the quotient is $Q(x)$, and the reminder is R:

$$P(x) = (ax-b)Q(x) + R.$$

$$P\left(\frac{b}{a}\right) = 0 \cdot Q(X) + R = R.$$

4. Roots of Cubic Functions:

Theroem: 2.3.3

For a cubic function $f(x) = ax^3 + bx^2 + cx + d$, given the roots of it are α, β, and γ. Then,

$$\begin{cases} \alpha+\beta+\gamma = -\frac{b}{a} \Rightarrow \sum\alpha = -\frac{b}{a} \\ \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} \Rightarrow \sum\alpha\beta = \frac{c}{a} \\ \alpha\beta\gamma = -\frac{d}{a} \Rightarrow \sum\alpha\beta\gamma = -\frac{d}{a} \end{cases}$$

Proof: 2.3.3

Since α, β, γ are roots of $f(x)$,

$$f(x) = a(x-\alpha)(x-\beta)(x-\gamma).$$

$$\text{So } a(x-\alpha)(x-\beta)(x-\gamma) = ax^3 + bx^2 + cx + d,$$

$$\text{i.e., } ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\alpha\gamma+\beta\gamma)x - a\alpha\beta\gamma = ax^3 + bx^2 + cx + d.$$

$$\Rightarrow \ \alpha+\beta+\gamma = -\frac{b}{a}, \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}.$$

Theroem: 2.3.4: An extention to Theorem 2.3.3

$$\sum\alpha = -\frac{b}{a}, \sum\alpha\beta = \frac{c}{a}, \sum\alpha\beta\gamma = -\frac{d}{a}, \sum\alpha\beta\gamma\delta = \frac{e}{a}.$$

4 Rational Functions

1. Reciprocal Functions: $f(x) = \dfrac{1}{x}$.

   - Domain: $x \in \mathbb{R},\ x \neq 0$
   - As x increases, $\dfrac{1}{x}$ decreases $\Rightarrow x \to \infty, \dfrac{1}{x} \to 0$.
   - Range: $y \in \mathbb{R},\ y \neq 0$
   - **Asymptotes**: $x = 0$, $y = 0$.
   - Axis of symmetry: $y = x$, $y = -x$.
   - **Self-inversing function**: have axis of symmetry $y = x$.

   $$f(x) = f^{-1}(x).$$

2. $y = \dfrac{a}{bx+c}$

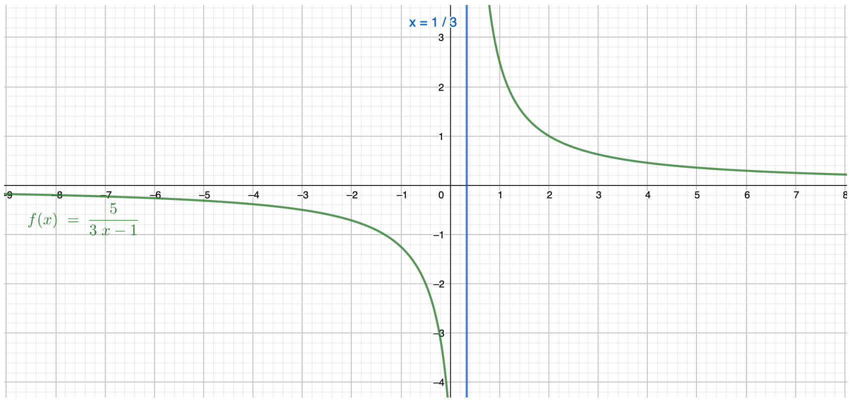
   - Vertical asymptotes (V.A.): $bx + c = 0$
   - Horizontal asymptotes (H.A.): $y = 0$

   Example: 2.4.1

   Draw the diagram of $y = \dfrac{5}{3x-1}$**.**

   ---

   x-intercept: $0 = \frac{5}{3x-1} \Rightarrow$ no solution, no intercept.
   H.A.: $y = 0$
   y-intercept: $y = -5$
   V.A.: $3x - 1 = 0,\ x = \dfrac{1}{3}$

3. $y = \dfrac{ax+b}{cx+d}$

   - V.A.: $cx + d = 0$
   - H.A.: $y = \dfrac{a}{c}$

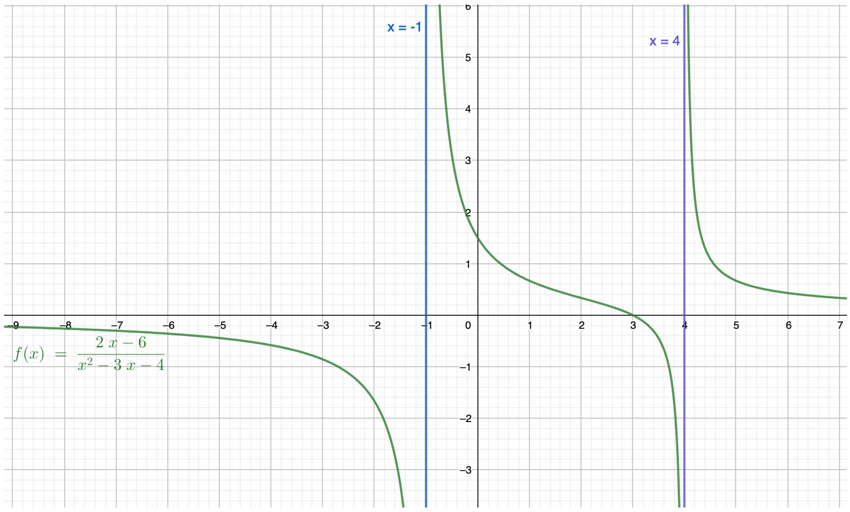
4. $y = \dfrac{ax+b}{cx^2+dx+e}$

- V.A.: $cx^2 + dx + e = 0$
- H.A.: As $x \to \pm\infty$, $\dfrac{ax}{cx^2} \to 0$, $y = 0$
- Intercepts: $\left(0, \dfrac{e}{c}\right)$, $\left(-\dfrac{e}{d}, 0\right)$

Example: 2.4.2

Draw the diagram of $y = \dfrac{2x-6}{x^2-3x-4}$.

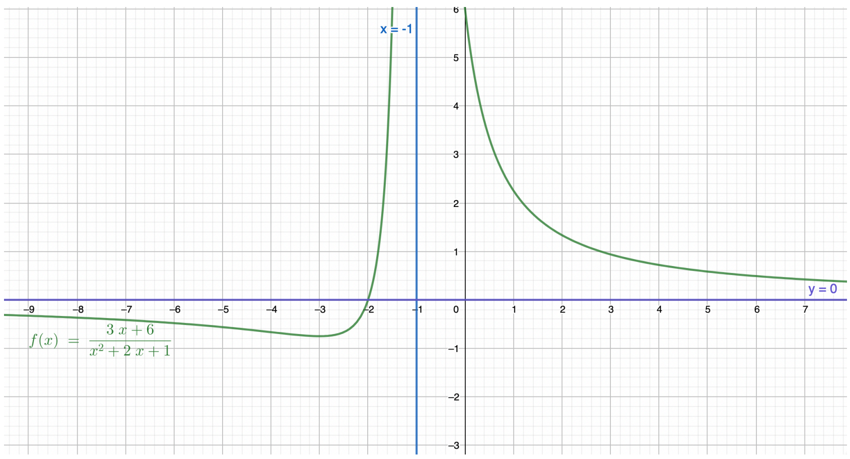
Intercept: $\left(0, \dfrac{3}{2}\right)$, $(3, 0)$
H.A.: $y = 0$
V.A.: $x = -1$, $x = 4$



Example: 2.4.3

Draw the diagram of $y = \dfrac{3x+6}{x^2+2x+1}$.

Intercept: $(0, 6)$, $(-2, 0)$
H.A.: $y = 0$
V.A.: $x = -1$

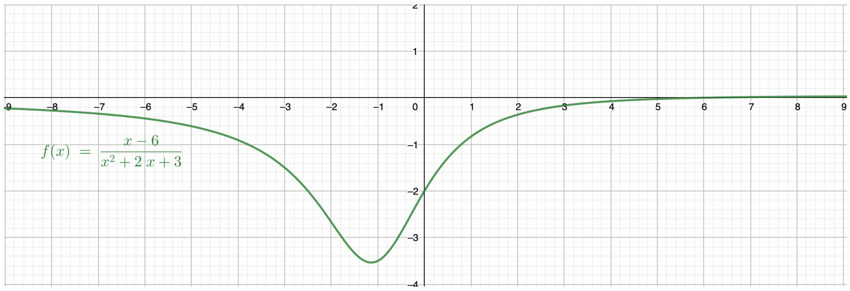


Example: 2.4.4

Draw the diagram of $y = \dfrac{x-6}{x^2+2x+3}$.

Intercept: $(6,0)$, $(0,-2)$

When $x \to \infty$, $f(x)$ is positive. When $x \to -\infty$, $f(x)$ is negative.



5. $y = \dfrac{ax^2+bx+c}{dx+e}$

- V.A.: $dx+e=0$
- **Oblique Asymptote**: Quotient of (ax^2+bx+c) dividied by $(dx+e)$.
- Intercepts: $\left(0, \dfrac{c}{e}\right)$, $ax^2+bx+c=0$

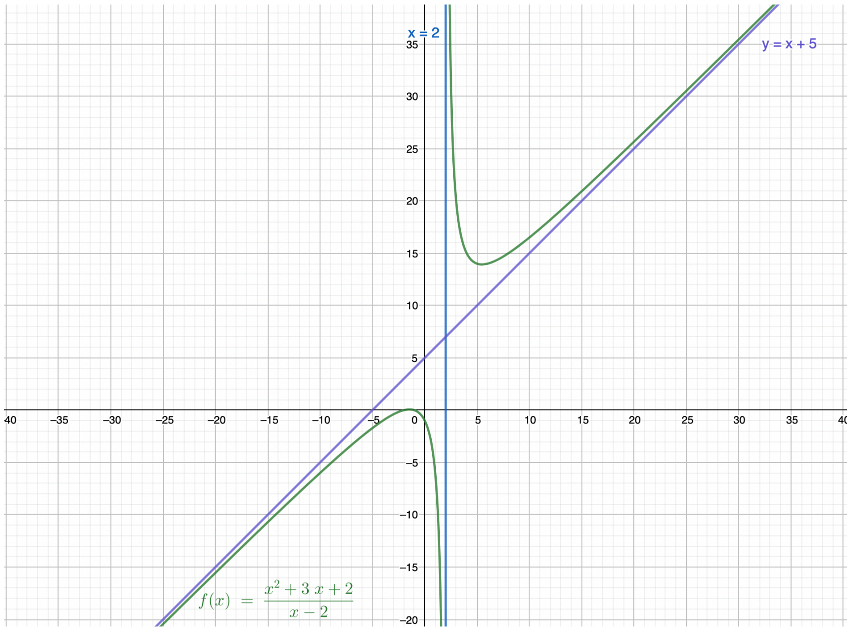
Example: 2.4.5

Draw the diagram of $y = \dfrac{x^2+3x+2}{x-2}$.

Intercept: $(0,-1)$, $(-1,0)$, $(-2,0)$

V.A.: $x=2$

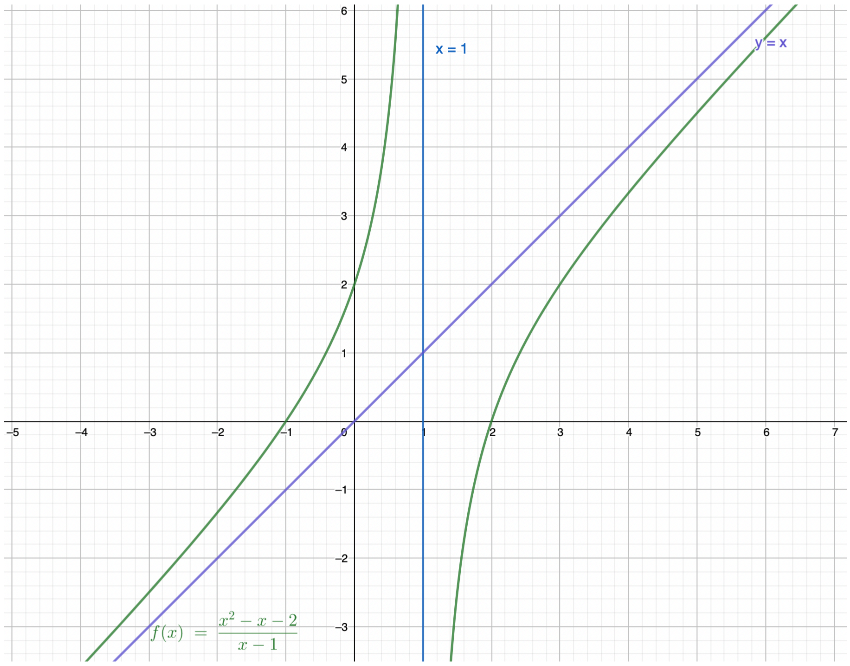
O.A.: $y=x+5$ (Use long division)



Example: 2.4.6

Draw the diagram of $y = \dfrac{x^2 - x - 2}{x - 1}$**.**

Intercept: $(0,2)$, $(2,0)$, $(-1,0)$
V.A.: $x = 1$
O.A.: $y = x$ (Use long division)



6. When the function has asymptotes:

   - Denominator$= 0$;
   - $\log_a 0$ (argument of a logarithm is 0)

5 Transformation of Functions

1. **Translation**:

   - $f(x+n)$ means translate $f(x)$ n units to the left.
   - $f(x-n)$ means translate $f(x)$ n units to the right.
   - $f(x)+n$ means translate $f(x)$ n units upwards.
   - $f(x)-n$ means translate $f(x)$ n units downwards.

2. Use translation vector to represent translation:

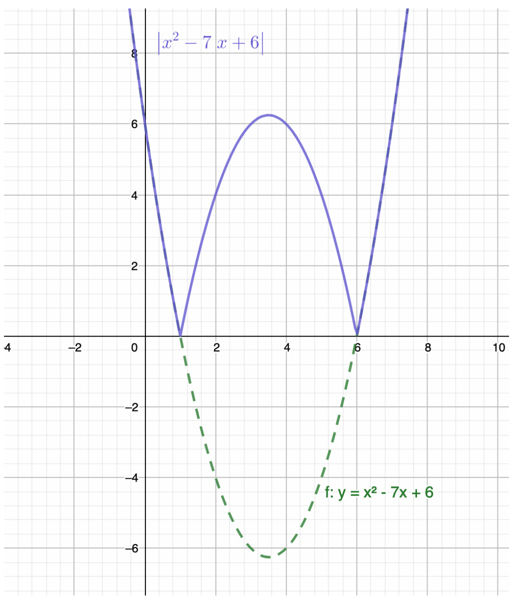
   A vector $\begin{pmatrix} a \\ b \end{pmatrix}$ means a units in the horizontal axis and b units in the vertical axis.

Example: 2.5.1

A translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ means $f(x+2)+3$, 2 units to the left and 3 units upwards.

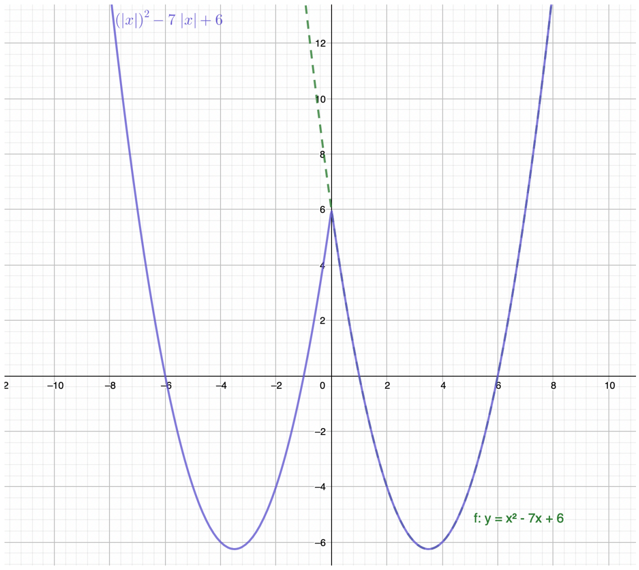
3. **Reflections**:
   - $f(-x)$ reflects in the y-axis.
   - $-f(x)$ reflects in the x-axis.
   - $f^{-1}(x)$ reflects in the $y = x$.
   - $-f(-x)$ reflects in the origin.
4. **Stretches**:
   - $f(qx)$ is a horizontal stretch of a scale factor of $\frac{1}{q}$.
   - $pf(x)$ is a vertical stretch of a scale factor of p.
5. When a graph is transforming, the points shift but the connection remains.
6. Sequence of transformation:
   - Do the horizontal translation before the horizontal stretch.
   - The vertical translation is always after the vertical stretch.
   - Vertical stretch $\rightarrow$ Reflection $\rightarrow$ Horizontal translation $\rightarrow$ Horizontal stretch $\rightarrow$ Vertical translation
7. **Modulus Function**
   - $|f(x)|$: Fold everything below x-axis above x-axis.

Example: 2.5.2



- $f(|x|)$: Reflect everthing on the right of y-axis to the left. Since $|x|$ must be positive, $|x| = |-x| \;\Rightarrow\; f(-x) = f(x)$, which is an even function.

Example: 2.5.3



8. **Reciprocal of $f(x)$**

- Table of Summary:

$f(x)$	$g(x) = \frac{1}{x}$
$f(a) = 0$	Line $x = a$ is vertical asymptote
Line $x = a$ is vertical asymptote	$g(a) = 0$
$f(x) \to \infty$	$g(x) \to 0$
$f(x) \to 0$	$g(x) \to \infty$
Line $y = b$ is horizontal asymptote	Line $y = \frac{1}{b}$ is horizontal asymptote
$f(x) = a$	$g(x) = \frac{1}{a}$

- When $f(x)$ increases, $g(x)$ decreases.

6 Exponential and Logarithmic Functions

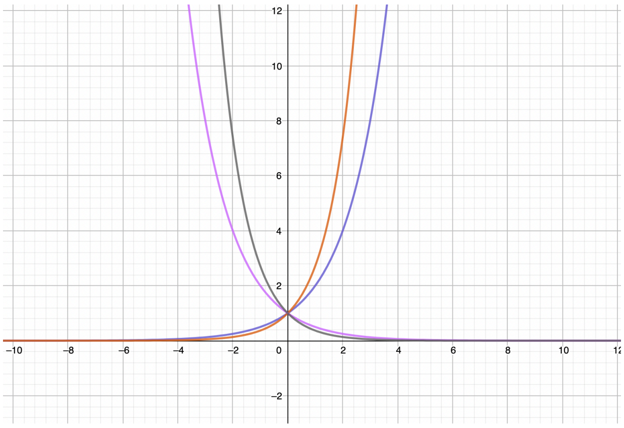
1. Exponential functions:

- $f(x) = a^x$, $a > 1$ (increasing) and $0 < a < 1$ (decreasing).
- $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$ are symmetric to the y-axis.

Proof: 2.6.1

$$g(x) = \left(\frac{1}{a}\right)^x = (a^-1)^x = a^{-x} = f(-x).$$

- Domain: $x \in \mathbb{R}$, Range: $y > 0$
- Common point: $(0,1)$; common H.A.: $y = 0$
- Graph:



2. Logarithmic functions:

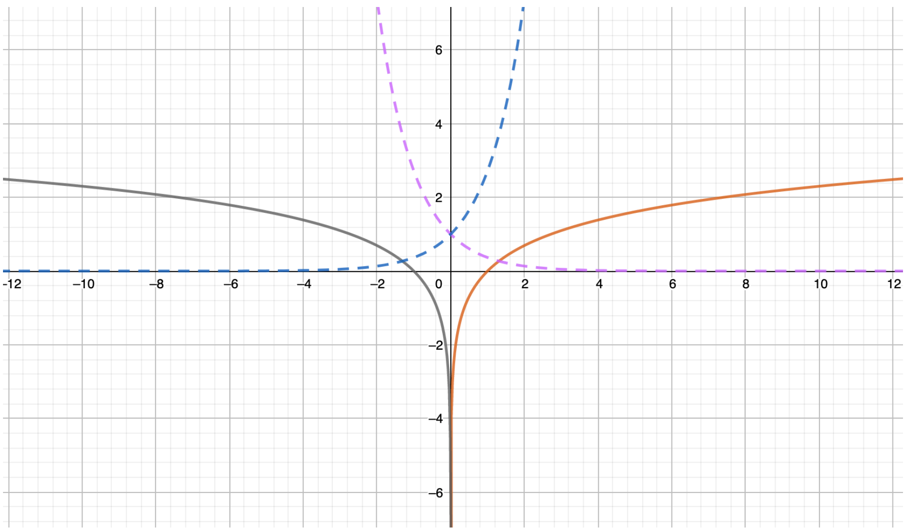
- $f(x) = \log_a x = g^{-1}(x), g(x) = a^x$.
- Common point: $(1,0)$; common V.A.: $x = 0$.
- $f(x) = \log_a x$ and $g(x) = \log_{\frac{1}{a}} x$ are symmetric to the x-axis.

Proof: 2.6.2

$$\log_{\frac{1}{a}} x = \frac{\log_a x}{\log_{\frac{1}{a}} a} = \frac{\log_a x}{-1} = -\log_a x,$$

$$\therefore g(x) = \log_{\frac{1}{a}} x = -\log_a x = -f(x).$$

- When $a > 1$, increasing function; when $0 < a < 1$, decreasing function.
- Domain: $x > 0$, Range: $y \in \mathbb{R}$
- Graph:



3. Solving logarithmic equations.

4. Solving exponential equations: take logarithm on both sides.