IB Mathematics Analysis and Approaches HL

Topic 1 Number and Algebra

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March 5, 2022

Contents

1 Sequences and Series

1. Terms: $u_1,\ u_2,\ u_3 \ldots$
   Position: n
   Sum: S

2. **Arithmetic Sequence**/Arithmetic Progession (AP):

   - Recursive formula: $u_{n+1} = u_n + d$, d is the common difference.
   - Explicit formula: $u_n = u_1 + d(n-1)$
   - Summation: $S_n = \frac{1}{2}[2u_1 + d(n-1)]$

   Proof: 1.1.1

   Let $u_1,\ u_2,\ u_3,\ \ldots,\ u_n$ be an arithmetic sequence with d as common difference.
   Then, $S_n = u_1 + u_2 + u_3 + \ldots + u_n = u_1 + (u_1 + d) + (u_1 + 2d) + \ldots + (u_1 + (n-1)d)$
   Also, $S_n = [u_1 + (n-1)d] + \ldots + (u_1 + d) + u_1$.
   Add two expressions together:

   $$2S_n = [2u_1 + (n-1)d]n$$

   $$\therefore S_n = \frac{n}{2}[2u_1 + (n-1)d].$$

3. **Geometric Sequence**

   - Recursive formula: $u_{n+1} = r \cdot u_n$, r is the common ratio.
   - Explicit formula: $u_n = u_1 \cdot r^{n-1}$
   - $$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \ldots$$
   - Summation: $S_n = \frac{u_1(r^n-1)}{r-1}$

   Proof: 1.1.2

   Let $u_1,\ u_2,\ u_3,\ \ldots,\ u_n$ be a geometric sequence with r as common ratio.
   $S_n = u_1 + u_2 + u_3 + \ldots + u_n = u_1 + (u_1 \cdot r) + (u_1 \cdot r^2) + \ldots + (u_1 \cdot r^{n-1})$
   Then, $rS_n = (u_1 \cdot r) + (u_1 \cdot r^2) + \ldots + (u_1 \cdot r^n)$.
   Substract the first expression from the second:

   $$rS_n - S_n = u_1 \cdot r^n - u_1 \Rightarrow (r-1)S_n = u_1(r^n - 1)$$

   $$\therefore S_n = \frac{u_1(r^n - 1)}{r - 1}$$

   - If $r > 1$, the sequence is an exponential growth.
     If $0 < r < 1$, the sequence has an exponential decay.

- When $r > 1$, series approaches ∞.
  When $-1 < r < 1$, or $|r| < 1$, the series converges:

$$S_\infty = \frac{u_1}{1-r}, |r| < 1$$

2 Exponents and Logarithms

1. $a^m \cdot a^n = a^{m+n}$
   $a^m \div a^n = a^{m-n}$
   $(a^m)^n = a^{mn}$
2. $x^0 = 1$ $(x^0 = x^{1-1} = \frac{x^1}{x^1} = 1)$
   $x^{-m} = \frac{1}{x^m}$
   $x^{\frac{1}{n}} = \sqrt[n]{x}$ $(x^{\frac{m}{n}} = (\sqrt[n]{x})^m)$
3. If $a = b$, then $a^n = b^n$
   If $m = n$, then $a^m = a^n$
   For $a^b = 1$: $a = 1, b \in \mathbb{R}$; $a \neq 1, b = 0$; OR $a = -1, b = 2n$
4. When solving exponential equations, convert them to the same base.
5. Division Theorem.

Theroem: 1.2.1

If $a^x = b^y$ given $a > 0$ and $b > 0$, then $a = b^{\frac{y}{x}}$.

Proof: 1.2.1

$$a^x = b^y$$
$$(a^x)^{\frac{1}{x}} = (b^y)^{\frac{1}{x}} \Rightarrow a = b^{\frac{y}{x}}$$

6. $a = b^x \Leftrightarrow x = \log_b a$, where $a, b \in \mathbb{R}^+$ and $b \neq 1$.
7. Logarithmic rules:
   - $\log_a x + \log_a y = \log_a (xy)$

Proof: 1.2.2

Let $\log_a x = p$, $\log_a y = q$. $\Rightarrow a^p = x, a^q = y$.
Then, $x \cdot y = a^p \cdot a^q = a^{p+q}$.

$$\therefore \log_a (xy) = p + q = \log_a x + \log_a y.$$

- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

Proof: 1.2.3

Let $\log_a x = p$, $\log_a y = q$. $\Rightarrow a^p = x, a^q = y$.
Then, $\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$.

$$\therefore \log_a \left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y.$$

- $\log_a x^n = n \log_a x$
- $\log_a 1 = 0$
- $\log_a a = 1$
- $-\log_a x = \log_a \frac{1}{x}$
- $\log_a x = \frac{\log_b x}{\log_b a}$
- $s \log_a b = \frac{1}{\log_b a}$

3 Proof

1. Direct proof:

Example: 1.3.1

Show that the sum of two even numbers is always even.

Let m and n be two even positive integers.
$m = 2p, n = 2q$, where p and $q \in \mathbb{Z}^+$.
Then, $m + n = 2p + 2q = 2(p+q)$, which is an even number.

Example: 1.3.2

Show that $\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 \equiv x^2 + ax$.

$$\text{LHS} = x^2 + \frac{a^4}{4} + ax - \frac{a^4}{4} = x^2 + ax = \text{RHS}.$$

Equations "$=$": only true from some values.
Identities "$\equiv$": true for all values.

Example: 1.3.3 Question

Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.

Example: 1.3.3 Answer

Let n be a 4-digit number: $n = 1000a + 100b + 10c + d$, where $0 \leq a, b, c, d \leq 9$, and $a \neq 0$.
It is given that $a + b + c + d = 3k, k \in \mathbb{Z}$:

$$\begin{aligned} n &= 1000a + 100b + 10c + d + 3k - a - b - c - d \\ &= 999a + 99b + 9c + 3k \\ &= 3(333a + 33b + 3c + k) \end{aligned}$$

Since $(333a + 33b + 3c + k) \in \mathbb{Z}$, it implies that n is divisible by 3.

2. Proof by Contradiction:

Example: 1.3.4

Prove the statement: If the integer n is odd, then n^2 is also odd.

Let, if possible, n^2 is even and n is odd.
Then, $n^2 = 2k,\ k \in \mathbb{Z} \Rightarrow n \times n = 2k$, which indicates the product of two odd number is even, and which is not true.
Hence, there is a contradiction.
$\therefore$ Our assumption is wrong, and thus given that n is odd, n^2 is also odd.

Example: 1.3.5

Show that $\sqrt{2}$ is irrational.

Let us assume, if possible, taht $\sqrt{2}$ is rational:
$\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, and p, q have no common factors, $q \neq 0$.
$\therefore 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \qquad (1)$.
$\therefore p^2$ is even, and thus p is also even.
As p is an even number, we can write: $p = 2k, k \in \mathbb{Z}. \Rightarrow \therefore p^2 = (2k)^2 = 4k^2 \qquad (2)$.
From (1) and (2): $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even, and thus q is also an even number.
But since p and q have no common factors, they cannot have "2" as a common factor.
Hence, we have arrived at a contradiction.
$\therefore$ Our assumption is incorrect, and $\sqrt{(2)}$ is irrational.

Definition 1: A number is **rational** if it can be written as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, and $q \neq 0$.

Example: 1.3.6 Question

Prove that there is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$

Example: 1.3.6 Answer

Assume there is a real number x such that $\frac{1}{x-2} = 1-x$.
$\therefore (1-x)(x-2) = 1 \Rightarrow x^2 - 3x + 3 = 0$
Solving the equation, we get $x = \frac{3 \pm \sqrt{9-12}}{2}$, which $\notin \mathbb{R}$
$\therefore$ We arrived at a contradiction, and our assumption is incorrect. There is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1-x$

3. Proof by Mathematical Induction

Definition 2: **Principle of Mathematical Induction (PMI)**:
Suppose P_n is a proposition which is defined for every integer $n \geq a,\ a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true $\forall n \geq a$.

Example: 1.3.7

Prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z},\ n \geq 0$, by using PMI.

For $n = 0$, LHS $= 4^0 + 2 = 1 + 2 = 3$, which is divisible by 3.
$\therefore P_0$ (OR denoted as $P(0)$) is true.
Assume that P_k is true: i.e., $4^k + 2$ is divisible by 3. $\Rightarrow 4^k + 2 = 3A,\ A \in \mathbb{Z}^+ \Rightarrow 4^k = 3A - 2$.
Consider P_{k+1}:

$$\begin{aligned} 4^{k+1} + 2 &= 4^k \cdot 4^1 + 2 \\ &= (3A-2) \cdot 4 + 2 \\ &= 12A - 6 \\ &= 3(4A-2). \end{aligned}$$

$\therefore 4A - 2$ is an integer as $A \in \mathbb{Z}^+$, $4^{k+1} + 2$ is divisible by 3 whenever $4^k + 2$ is divisible by 3.
Since P_0 is true, and P_{k+1} is true whenever Pk is true, P_n is ture $\forall n \in \mathbb{Z},\ n \geq 0$.

Example: 1.3.8

A sequence is defined by $u_{n+1} = 2u_n + 1\ \forall n \in \mathbb{Z}^+$. Prove that $u_n = 2^n - 1$.

For $n = 1,\ u_1 = 2^1 - 1 = 1 \Rightarrow \therefore P_1$ is ture.
Let P_k be true: $u_k = 2^k - 1$ for some $k \in \mathbb{Z}^+$.
Consider P_{k+1}:

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Since P_1 is ture, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

Example: 1.3.9

Prove that $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6},\ \forall n\in\mathbb{Z}^+$**.**

For $n=1,\ \text{LHS}=1^2=1,\ \text{RHS}=\frac{1(1+1)(2+1)}{6}=1$
$\therefore \text{LHS}=\text{RHS}\Rightarrow P_1$ is true.
Assume that P_k is true, $k\in\mathbb{Z}^+:\ 1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$.
Consider P_{k+1}:

$$\begin{aligned}
\text{LHS} &= 1^2+2^2+3^2+\cdots+k^2+(k+1)^2\\
&= \frac{k(k+1)(2k+1)}{6}+(k+1)^2\\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\
&= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}\\
&= \frac{(k+1)(2k^2+7k+6)}{6}\\
&= \frac{(k+1)(k+2)(2k+3)}{6}\\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}=\text{RHS}.
\end{aligned}$$

Thus, P_{k+1} is true whenever P_k is true.
Since P_1 is true, and P_{k+1} is true whenver P_k is true, P_n is true $\forall n\in\mathbb{Z}^+$.

Example: 1.3.10

Prove that if $x\neq 1$**, the** $\prod_{i=1}^{n}(1+x^{2^{i-1}})=(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})=\frac{1-x^{2^n}}{1-x}$**.**

For $n=1$, $\text{LHS}=1+x,\ \text{RHS}=\frac{1-x^{2^1}}{1-x}=\frac{1-x^2}{1-x}=1+x.\ \Rightarrow\therefore\text{LHS}=\text{RHS},\ P_1$ is true.
Assume that P_k is true: $(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{k-1}})=\frac{1-x^{2^k}}{1-x}$.
Conosider P_{k+1}:

$$\begin{aligned}
\text{LHS} &= (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{k-1}})(1+x^{2^k})\\
&= \frac{1-x^{2^k}}{1-x}(1+x^{2^k})\\
&= \frac{1+x^{2^k}-x^{2^k}+(x^{2^k})^2}{1-x}\\
&= \frac{1-x^{2^k\cdot 2}}{1-x}\\
&= \frac{1-x^{2^{k+1}}}{1-x}=\text{RHS}.
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n\in\mathbb{Z}^+$.

4 Counting and Binomial Theorem

1. Choose r from n: $\binom{n}{r} =_n C_r$
   - $\binom{n}{m} = \binom{n}{n-m}$
   - $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
   - Fractorial notation: $n! = n(n-1)(n-2)\cdots 2\cdot 1$
     e.g. $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\times4\times3!}{3!\times2} = 5\times 2 = 10.$

Example: 1.4.1

Write $\frac{(n!)^2}{(n-1)!(n-2)!}$ without using fractorial notation.

$$(n!)^2 = n!\times n! = n(n-1)!\times n(n-1)(n-2)!$$

$$\therefore \frac{(n!)^2}{(n-1)!(n-2)!} = \frac{n(n-1)!\times n(n-1)(n-2)!}{(n-1)!(n-2)!} = n\cdot n(n-1) = n^3 - n^2.$$

2. The number of ways of arranging n distinct objects in a row is $n!$.

3. The number of permutations of r objects out of n distinct objects is given by

$$_nP_r = \frac{n!}{(n-r)!}.$$

4. In permutations, the order matters.
   In combinations, the order does not matter.

5. The Binomial Theorem:

Theroem: 1.4.1 The Binomial Theorem

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n,\ n\in\mathbb{N}\\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r\end{aligned}$$

Example: 1.4.2

Find $(2x+3)^4$.

$$\begin{aligned}(2x+3)^4 &= (2x)^4 + \binom{4}{1}(2x)^3(3)^1 + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)(3)^3 + 3^4\\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81\end{aligned}$$

Example: 1.4.3

Find the term independent of x in the expasion of $\left(x-\frac{2}{x^2}\right)^{12}$.

General term: $\binom{12}{r}x^{12-r}\left(-\frac{2}{x^2}\right)^r$
Thus, the general expression for $x: x^{12-r-2r}=x^{12-3r}$
When $12-3r=0$, the term is independent of x: $12-3r=0 \Rightarrow r=4$.

$$\therefore \binom{12}{4}x^{12-4}\left(-\frac{2}{x^2}\right)^4=7920.$$

1. The independent term should not involve x in it since the independent term does not vary as x varies. (constant term)
2. The coefficient should not include x as well.

Example: 1.4.4

Find the coefficient of x^3y^2 in the expansion of $(2x+y)\left(x+\frac{y}{x}\right)^5$.

Assume $2x\cdot A$ and $y\cdot B$ will yield the term x^3y^2. $\Rightarrow A=x^2y^2,\ B=x^3y$.
General term: $\binom{5}{r}x^{5-r}(\frac{y}{x})^r=\binom{5}{r}x^{5-2r}y^r$.
When $r=2,\ 5-2r=1\neq 2 \Rightarrow x^2y^2$ is not possible.
When $r=1,\ 5-2r=3 \Rightarrow x^3y$ is possible.

$$\therefore \text{Coefficient}=\binom{5}{1}=5.$$

Example: 1.4.5

Find the coefficient of x^2 in the expansion of $(1-2x)(1-4x)^7$.

Assume $1\cdot A=x^2,\ -2x\cdot B=x^2. \Rightarrow A=x^2,\ B=x$.
General term: $\binom{7}{r}(-4x)^{7-r}(1)^r$
When $7-r=2,\ r=5:\ \binom{7}{5}(-4x)^2(1)^5=336x^2. \Rightarrow 1\cdot 336x^2=336x^2$
When $7-r=1,\ r=6:\ \binom{7}{6}(-4x)^1(1)^6=-28x. \Rightarrow (-2x)\cdot(-28x)=56x^2$

$$\therefore \text{Coefficient}=336+56=392.$$

6. AHL - Extention of Binomial Theorem:

Theroem: 1.4.2 Binomial Theorem Extended

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$
$$= a^n(1+n\cdot\frac{b}{a}+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\frac{n(n-1)(n-2)}{3!})\left(\frac{b}{a}\right)^3+\cdots),\ n\in\mathbb{Q},\ \left|\frac{b}{a}\right|<1$$

Example: 1.4.6

Expand $\sqrt{1+2x}$ $\left(|x|<\frac{1}{2}\right)$ **and** $\frac{2}{1-3x}$ $\left(|x|<\frac{1}{3}\right)$ **upto** x^3 **term.**

$$(1+2x)^{\frac{1}{2}} = 1+\frac{1}{2}(2x)+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{(2x)^2}{2!}+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{(2x)^3}{3!}+\cdots$$
$$= 1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots.$$

$$2(1-3x)^{-1} = 2(1-(-3x)-(-1-1)\frac{(-3x)^2}{2!}-(-1-1)(-1-2)\frac{(-3x)^3}{3!}+\cdots$$
$$= 2(1+3x+x^2+27x^3+\cdots$$
$$= 2+6x+18x^2+54x^3+\cdots.$$

Example: 1.4.7

Write the first three terms in the expasion of $(2+x)^{-3}$**.**

$$(2+x)^{-3} = 2^{-3}\left(1+\frac{x}{2}\right)^{-3}$$
$$= \frac{1}{8}\left(1+(-3)\frac{x}{2}+(-3)(-3-1)\frac{2^2}{2\cdot 2!}+\cdots\right)$$
$$= \frac{1}{8}\left(1-\frac{3}{2}x+\frac{12}{4}x^2+\cdots\right)$$
$$= \frac{1}{8}-\frac{3}{16}x+\frac{3}{8}x^2+\cdots.$$

Example: 1.4.8 Application of Bionomial Theorem

Find square root of 24 correct to 5 decimal places, using the binomial theorem.

$$
\begin{aligned}
24^{\frac{1}{2}} = (25-1)^{\frac{1}{2}} = 25^{\frac{1}{2}}\left(1-\frac{1}{25}\right)^{\frac{1}{2}} \\
&= 5\left(1+\left(\frac{1}{2}\right)\left(-\frac{1}{25}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{25}\right)^2+\frac{\frac{1}{2(\frac{1}{2}-1)(\frac{1}{2}-2)}}{3!}\left(-\frac{1}{25}\right)^3+\cdots\right) \\
&= 5\left(1-\frac{1}{50}-\frac{1}{5000}-\frac{1}{250000}+\cdots\right) \\
&= 5(1-0.02-0.0002-0.000004) \\
&= 4.89898 \quad (5\ d.p.).
\end{aligned}
$$

5 Partial Fraction - AHL

1. Proper fractions: The degree of the numerator is less than the degree of the denominator.

2. Partial fraction: A method to separate one complex fraction into two or more simpler fractions.

Example: 1.5.1

Find the partial fraction of $\frac{3x}{(x-1)(x+2)}$.

Let $\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$.

$$\therefore 3x \equiv A(x+2) + B(x-1).$$

When $x = 1$, $3 = 3A \Rightarrow A = 1$.
When $x = -2$, $-6 = -3B \Rightarrow B = 2$.

$$\therefore \frac{3x}{(x-1)(x+2)} \equiv \frac{1}{x-1} + \frac{2}{x+2}.$$

Example: 1.5.2

Find the partial fraction of $\frac{2x+5}{(x-2)(x+1)}$.

Let $\frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$.

$$\therefore 2x+5 \equiv A(x+1) + B(x-2).$$

When $x = 2$, $9 = 3A \Rightarrow A = 3$.
When $x = -1$, $3 = -3B \Rightarrow B = -1$.

$$\therefore \frac{2x+5}{(x-2)(x+1)} \equiv \frac{3}{x-2} - \frac{1}{x+1}.$$

Example: 1.5.3

Find the partial fraction of $\frac{34-12x}{3x^2-10x-8}$.

As $\frac{34-12x}{3x^2-10x-8} = \frac{34-12x}{(3x+2)(x-4)}$, let $\frac{34-12x}{(3x+2)(x-4)} = \frac{A}{3x+2} + \frac{B}{x-4}$.

$$\therefore 34 - 12x \equiv A(x-4) + B(3x+2).$$

When $x = 4$, $-14 = 14A \Rightarrow B = -1$.
When $x = -\frac{2}{3}$, $42 = -\frac{14}{3}A \Rightarrow A = -9$.

$$\therefore \frac{34-12x}{(3x+2)(x-4)} \equiv -\frac{9}{3x+2} - \frac{1}{x-4}.$$

6 Complex Number - AHL

6.1 Introduction

1. Complex Number:

> Definition 3:
> **Complex Numbers** are numbers in the form of $a+bi$, where $i^2 = -1$.
> - a is called the **real part**, denoted as $\text{Re}(a+bi) = a$.
> - b is called the **imaginary part**, denoted as $\text{Im}(a+bi) = b$.
> $a+bi$ is called the Cartesian form of complex number.

2. Basic Calculations of Complex Number:

   - Define $z_1 = a+bi$ and $z_2 = c+di$:

$$z_1 \pm z_2 = (a \pm c) + (b \pm d)i.$$

- Define $z_1 = a + bi$ and $z_2 = c + di$:

$$z_1 z_2 = (ac - bd) + (ad + bc)\mathrm{i}.$$

Proof: 1.6.1.1

$$\begin{aligned} z_1 z_2 &= (a + b\mathrm{i})(c + d\mathrm{i}) \\ &= ac + (ad + bc)\mathrm{i} + bd\mathrm{i}^2 \qquad [\mathrm{i}^2 = -1] \\ &= (ac - bd) + (ad + bc)\mathrm{i}. \end{aligned}$$

- Conjugate complex number:

Definition 4:
We call $a - bi$ as the **conjugate** of $z = a + bi$, denoted as $z^* = a - bi$.

Theroem: 1.6.1.1

Define $z_1 = a + bi$, and z^* is the conjugate of z_1. Then,

$$z_1 z^* = a^2 + b^2.$$

Proof: Theorem 6.1.1

By definition, $z^* = a - b\mathrm{i}$. Thus,

$$\begin{aligned} z_1 z^* &= (a + b\mathrm{i})(a - b\mathrm{i}) \\ &= a^2 - (b\mathrm{i})^2 \\ &= a^2 + b^2. \end{aligned}$$

- Define $z_1 = a + bi$ and $z_2 = c + di$:

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}\mathrm{i}.$$

Proof: 1.6.1.2

$$\begin{aligned} \frac{z_1}{z_2} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}} &= \frac{(a + b\mathrm{i})(c - d\mathrm{i})}{(c + d\mathrm{i})(c - d\mathrm{i})} \\ &= \frac{(ac + bd) - (bc - ad)\mathrm{i}}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}\mathrm{i}. \end{aligned}$$

Example: 1.6.1.1

Find $z \in \mathbb{C}$ that satisfies the equation $\frac{z+2}{1-\mathrm{i}} = \frac{z-3\mathrm{i}}{2+\mathrm{i}}$.

$$\begin{aligned}(z+2)(2+\mathrm{i}) &= (z-3\mathrm{i})(1-\mathrm{i}) \\ z(2+\mathrm{i})+4+2\mathrm{i} &= z(1-\mathrm{i})-3\mathrm{i}+(3\mathrm{i})^2 \\ z(2+\mathrm{i}-1+\mathrm{i}) &= -3\mathrm{i}-3-4-2\mathrm{i} \\ z(1+2\mathrm{i}) &= -7-5\mathrm{i} \\ z &= \frac{-7-5\mathrm{i}}{1+2\mathrm{i}} = -\frac{17}{5}+\frac{9}{5}\mathrm{i}.\end{aligned}$$

3. If $s = a+b\mathrm{i}$ and $t = c+d\mathrm{i}$, then:

$$\mathrm{Re}(s)+\mathrm{Re}(t) = \mathrm{Re}(s+t); \text{ and } \mathrm{Im}(\mathrm{i}\cdot s) = \mathrm{Re}(s).$$

6.2 Argand Diagram

1. The Complex Plane:

$z = a+b\mathrm{i}$ can be represented on a complex plane with real coordinate a and imaginary coordinate b. It can also be denoted as $z(a,b)$.

- Modulus of a complex number:

$$|z| = \sqrt{a^2+b^2}.$$

- Argument of a complex number:

$$\mathrm{Arg}(z) = \arctan\left(\frac{b}{a}\right)(+k\pi) \rightarrow \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

*When determine a complex number, first draw it on the plane to show which quadrant it is in.
The range of arugment is $[0,2\pi]$ or $[-\pi,\pi]$.

- Use modulus and argument to express a complex number:

$$a = |z|\cdot\cos\theta;$$

$$b = |z|\cdot\sin\theta.$$

2. If $z = a+b\mathrm{i}$ and $|z| = 1$, then $z^* = z^{-1}$.

Proof: 1.6.2.1

$$\because |z| = 1$$
$$\therefore \sqrt{a^2+b^2} = 1$$
$$\therefore a^2+b^2 = 1$$

$\boxed{\text{Method 1}}$

$$\begin{aligned} \text{RHS} = z^{-1} &= \frac{1}{a+b\mathrm{i}} = \frac{a-b\mathrm{i}}{(a+b\mathrm{i})(a-b\mathrm{i})} \\ &= \frac{a-b\mathrm{i}}{a^2+b^2} = a-b\mathrm{i} \\ &= z^* = \text{LHS}. \end{aligned}$$

$\boxed{\text{Method 2}}$

$$\begin{aligned} z \cdot z^* &= (a+b\mathrm{i})(a-b\mathrm{i}) \\ &= a^2+b^2 \\ &= |z|^2 = 1 \\ \therefore\ z^* &= z^{-1} \end{aligned}$$

3. When $|z| \neq 1$, $z^* = \frac{|z|^2}{z}$, and $z^{-1} = \frac{z^*}{|z|^2}$.
4. Properties of modulus and arguments:
   For complex number s and $t \in \mathbb{C}$:
   - $$|st| = |s||t|$$
   - $$\left|\frac{s}{t}\right| = \frac{|s|}{|t|}$$
   - $$\text{Arg}(st) = \text{Arg}(s) + \text{Arg}(t) + 2k\pi$$
   - $$\text{Arg}\left(\frac{s}{t}\right) = \text{Arg}(s) - \text{Arg}(t) + 2k\pi$$

6.3 Complex Number in Other Forms

1. The Polar Form (Modulus-Argument Form):
   - $$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\text{cis}\theta$$

   Proof: 1.6.3.1

   According to the Argand Diagram:

   $$z = x + y\mathrm{i} = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta).$$

   - $$z_1z_2 = r_1r_2\text{cis}(\theta_1+\theta_2)$$
   - $$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1-\theta_2)$$

2. de Movrie's Theorem:

   - By Maclaurin Series:
     $$e^{\mathrm{i}\theta} = \operatorname{cis}\theta = \cos\theta + \mathrm{i}\sin\theta.$$

   - Exponential form of complex number:
     $$z = re^{\mathrm{i}\theta} = r\operatorname{cis}\theta.$$

3. Cartesian Form: Addition and Substraction
   Modulus-Argument Form: Multiply and Division
   Exponential Form: Exponents and Roots

4. Since $\operatorname{cis}\theta = \operatorname{cis}(\theta + 2k\pi)$,
   $$re^{\mathrm{i}\theta} = re^{\mathrm{i}(\theta+2k\pi)}.$$

Example: 1.6.3.1

Find $e^{\mathrm{i}\frac{17\pi}{12}}$ in the form of Cartesian.

$$\begin{aligned} e^{\mathrm{i}\frac{17\pi}{12}} = e^{\mathrm{i}\left(\frac{7\pi}{6}+\frac{\pi}{4}\right)} = e^{\mathrm{i}\frac{7\pi}{6}} \cdot e^{\frac{\pi}{4}} &= \operatorname{cis}\left(\frac{7\pi}{6}\right) \cdot \operatorname{cis}\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2}-\sqrt{6}}{4} - \frac{\sqrt{2}+\sqrt{6}}{4}\mathrm{i}. \end{aligned}$$

6.4 Power of Complex Number

1. For a complex number $z = re^{\mathrm{i}\theta}$,
   $$z^n = r^n e^{\mathrm{i}n\theta}.$$

Example: 1.6.4.1

Find $\left(3\cos\frac{2\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3$

$$\begin{aligned}\left(3\cos\frac{2\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3 &= \left(-3\cos\frac{\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3 \\ &= \left(-3\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)\right)^3 \\ &= (-3)^3 (e^{\mathrm{i}\frac{\pi}{3}})^3 \\ &= -27e^{\mathrm{i}\pi} \\ &= -27(-1) = 27.\end{aligned}$$

Key learnings from Example 6.4.1:
1. $z = 3$ is only the fundemental root of equation $z^3 = 27$. In $\mathbb{C}$, there are other two complex roots that satisfy the euqation.
2. In $\mathbb{C}$, $\sqrt{4} = \pm 2 = 2 + 0 \cdot \mathrm{i}$ or $-2 + 0 \cdot \mathrm{i}$.

Example: 1.6.4.2

Given a complex number $\omega \neq 1$ is one of the solutions of $z^3 = 1$.
a. Prove $\omega^2 + \omega + 1 = 0$;
b. Calculate $\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022}$.

(a) $\boxed{\text{Approach A}}$

$$\begin{aligned}&\because \omega^3 = 1 \\ &\therefore \omega^3 - 1 = 0 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0 \\ &\because \omega \neq 1 \\ &\therefore \omega^2 + \omega + 1 = 0.\end{aligned}$$

$\boxed{\text{Approach B}}$ $\omega^2 + \omega + 1 = 0$ is a geometric sequence, $u_1 = 1,\ r = \omega$:

$$S_3 = \frac{u_1(1 - r^3)}{1 - r} = \frac{1 - \omega^3}{1 - \omega} = \frac{0}{1 - \omega} = 0.$$

(b)

$$\begin{aligned}\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022} &= \omega^{2019} \times (1 + \omega + \omega^2 + \omega^3) \\ &= \omega^{2019}(0 + 1) = \omega^{2019} \\ &= \left(\omega^3\right)^{673} = 1.\end{aligned}$$

Example: 1.6.4.3

Find:
a. 1^{i};
b. $\ln(-1)$;
c. $\ln(-c)$, **where** c **is a constant.**

(a)

$$1 = e^{\mathrm{i}2\pi} \Rightarrow 1^{\mathrm{i}} = \left(e^{\mathrm{i}2\pi}\right)^{\mathrm{i}} = e^{-2\pi}. \quad (1^{\mathrm{i}} = e^{-2k\pi}, k \in \mathbb{Z})$$

(b)

$$-1 = e^{\mathrm{i}\pi} \Rightarrow \ln(-1) = \ln\left(e^{\mathrm{i}\pi}\right) = \mathrm{i}\pi.$$

(c)

$$\ln(-c) = \ln[(-1)\cdot c] = \ln(-1) + \ln(c) = \ln(c) + \mathrm{i}\pi.$$

6.5 Polynomial Function with Complex Roots

1. Conjugate Pair Theorem:

Theroem: 1.6.5.1 Conjugate Pair Theorem

If z is a complex root of $P(x)$, then the conjugate of $z(z^*)$ is also a complex root of $P(x)$. ($P(x)$ should be a polynomial with rational coefficients.)

2. Properties of Conjugate.

- $$(s \pm t)^* = s^* \pm t^*$$
- $$(st)^* = s^* t^*$$
- $$\left(\frac{s}{t}\right)^* = \frac{s^*}{t^*}$$

6.6 Root of Complex Numbers

1. The Root of Unity:

Theroem: 1.6.6.1 The Root of Unity

For any complex equation $\omega^n = 1$, there are n distinct roots:

$$1 = e^{\mathrm{i}(0+2k\pi)} = \omega^n,\ k \in \mathbb{Z} \quad \Rightarrow \quad \omega = e^{\mathrm{i}\frac{2k\pi}{n}},\ k \in \mathbb{Z}.$$

Example: 1.6.6.1

Solve $z^3 = 8$.

$$
\begin{aligned}
z^3 = 8 \cdot 1 = 8e^{i(0+2k\pi)} \ \Rightarrow \ z = 2e^{i\frac{2k\pi}{3}},\ k \in \mathbb{Z} \\
k = 0:\ z &= 2 \\
k = 1:\ z &= 2e^{i\frac{2\pi}{3}} = 2\operatorname{cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}\mathrm{i} \\
k = 2:\ z &= 2e^{i\frac{4\pi}{3}} = 2\operatorname{cis}\left(\frac{4\pi}{3}\right) = -1 - \sqrt{3}\mathrm{i}
\end{aligned}
$$

2. Property of $\operatorname{cis}\theta$:

$$\operatorname{cis}(-\theta) = \cos\theta - \mathrm{i}\sin\theta$$

Proof: 1.6.6.1

$$
\begin{aligned}
\cos\theta - \mathrm{i}\sin\theta &= \cos(-\theta) + \mathrm{i}\sin(-\theta) \\
&= \operatorname{cis}(-\theta).
\end{aligned}
$$