

# Emory University

# MATH 212 Differential Equations Learning Notes

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June 18, 2025

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# 1 First Order ODEs

## 1.1 Introduction

**Definition 1.1.1 (Ordinary Differential Equations/ODEs).** An *ordinary differential equation* is an equation that contains one or more derivatives of an unknown function  $y = y(x)$ .

**Definition 1.1.2 (Order of ODEs).** The *order* of an ODE is the maximum order of the derivatives appearing in the equation.

**Definition 1.1.3 (Solution to ODEs).** The *solution* to an ODE is a function  $y$  that satisfies the equation.

**Example 1.1.4** Solve  $y'' = 3x + 1$ .

**Solution 1.**

$$\begin{aligned}y' &= \int 3x + 1 \, dx = \frac{3}{2}x^2 + x + C \\y &= \int y' \, dx = \int \frac{3}{2}x^2 + x + C \, dx = \frac{1}{2}x^3 + \frac{1}{2}x^2 + Cx + D.\end{aligned}$$

□

**Definition 1.1.5 (Linear ODEs/Non-Linear ODEs).** A first order ODE is *linear* if it can be written as

$$y' + p(x)y = f(x).$$

Otherwise, it is *non-linear*.

**Definition 1.1.6 (Homogenous/Non-Homogenous Linear ODEs).** If  $f(x) = 0$ , then the linear ODE is *homogenous*. That is,

$$y' + p(x)y = 0.$$

Otherwise, it is *non-homogenous*.

**Definition 1.1.7 (Trivial/Non-Trivial Solution).**  $y = 0$  is a *trivial solution* to a homogenous ODE. Any other solutions are *non-trivial*.

**Definition 1.1.8 (One-Parameter Family of Solutions).** We call  $C$  a *parameter* and the equation, therefore solution, defines a *one-parameter family* of solutions.

**Example 1.1.9** For the ODE  $y' = 1$ ,  $y_1 = x + C_1$  is a solution to it, and it is a one-parameter family of solutions. Similarly, for  $y' = \frac{1}{x^2}$ , the one-parameter families of solutions are defined by  $y_2 = -\frac{1}{x} + C_2$  on the interval  $(-\infty, 0) \cup (0, \infty)$ .

**Definition 1.1.10 (General Solution).** Given the general form of the linear ODE  $y' + p(x)y = f(x)$  if  $p$  and  $f$  are continuous on some open interval  $(a, b)$  and there is a unique formula  $y = y(x, c)$  and we have the following properties:

- for each fixed  $c$ , the resulting function of  $x$  is a solution of the ODE on  $(a, b)$ , and
- if  $y$  is a solution of the ODE, then  $y$  can be obtained by choosing the value of  $c$  appropriately.

The function  $y = y(x, c)$  is called a *general solution*.

More generally, we can write an ODE as

$$P_0(x)y' + P_1(x)y = F(x).$$

In this case, the ODE has a general solution on any open interval in which  $P_0$ ,  $P_1$ , and  $F$  are continuous and  $P_0 \neq 0$ .

**Definition 1.1.11 (Initial Value Problem (IVP)).** A differential equation with an initial condition.

**Example 1.1.12** Let  $a$  be a constant. Find the general solution of  $y' - ay = 0$  and solve the IVP  $\begin{cases} y' - ay = 0 \\ y(x_0) = y_0. \end{cases}$

**Solution 2.**

Classification: First order, Linear, Homogeneous.

Trivial Solution:  $y = 0$ .

General solution:

$$\begin{aligned} \frac{dy}{dx} &= ay \\ \int \frac{1}{y} dy &= \int a dx \\ \ln |y| &= ax + c \\ y &= e^{ax+c} = Ae^{ax}. \end{aligned}$$

This general solution includes the trivial solution.

IVP: Substitute  $x = x_0$  and  $y = y_0$ :

$$y_0 = Ae^{ax_0} \quad \longrightarrow \quad A = y_0 e^{-ax_0}$$

So,

$$y^{\text{IVP}} = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

This IVP is a “generic initial condition.” We need more information on  $x_0, y_0$  to get a more specific solution.  $\square$

## 1.2 Linear First Order ODEs

### Theorem 1.2.1

If  $p$  is continuous on  $(a, b)$ , then the general solution of the homogeneous equation  $y' + p(x)y = 0$  on  $(a, b)$  is given by

$$y = ce^{-\int p(x) dx}.$$

#### *Proof 1.*

(a). Substitute the solution formula to show that  $y = ce^{-\int p(x) dx}$  is a solution for any choice of  $c$ .

$$y' = c \left( - \int p(x) dx \right)' e^{-\int p(x) dx} = -cp(x)e^{-\int p(x) dx}.$$

Then,

$$y' + p(x)y = -cp(x)e^{-\int p(x) dx} + cp(x)e^{-\int p(x) dx} = 0.$$

So,  $y = ce^{-\int p(x) dx}$  is a solution for any choice of  $c$ .  $\square$

(b). Want to show: any solution of  $y' + p(x)y = 0$  can be written as  $y = ce^{-\int p(x) dx}$ . Note that  $y = 0$  is a trivial solution, so we assume  $y \neq 0$ .

$$\begin{aligned} y' + p(x)y &= 0 \\ y' &= -p(x)y \\ \frac{y'}{y} &= -p(x) \\ \rightsquigarrow \int \frac{1}{y} dy &= \int -p(x) dx \\ \ln |y| &= - \int p(x) dx \\ y &= ce^{-\int p(x) dx}. \end{aligned}$$

Note that when  $c = 0$ ,  $y = 0$  is the trivial solution. So, any solution of  $y' + p(x)y = 0$  can be written as  $y = ce^{-\int p(x) dx}$ .  $\blacksquare$

### Example 1.2.2 Solve the IVP

$$\begin{cases} xy' + y = 0 \\ y(1) = 3. \end{cases}$$

#### *Solution 2.*

Note that  $P_0(x) = x$  and  $P_1(x) = 1$ , which are continuous on  $\mathbb{R}$ . Since we need  $P_0(x) \neq 0$ ,  $x \neq 0$ . So the interval of validity is  $\mathbb{R} \setminus \{0\}$ .

Method 1: Separation of Variables

$$y' = -\frac{y}{x}.$$

Note that  $y = 0$  is a solution. Assume  $y \neq 0$ .

$$\begin{aligned}\frac{y'}{y} = -\frac{1}{x} \quad \rightsquigarrow \quad \int \frac{1}{y} dy &= -\int \frac{1}{x} dx + k \\ \ln|y| &= -\ln|x| + k \\ |y| &= e^k \frac{1}{|x|} \\ y &= \frac{c}{x}\end{aligned}$$

**Method 2: Solution Formula** By Theorem 1.2.1,

$$y = ce^{-\int p(x) dx} = ce^{-\int \frac{1}{x} dx} = ce^{-\ln|x|} = \frac{c}{x}.$$

**Solving the IVP** Substitute  $x = 1$  and  $y = 3$ :

$$3 = \frac{c}{1} \quad \longrightarrow \quad c = 3.$$

$$\text{So, } y^{\text{IVP}} = \frac{3}{x}.$$

□

**Example 1.2.3** Given the equation  $(4 + x^2)y' + 2xy = 4x$ . Classify the equation and find the general solution  $y = y(x, c)$ .

**Solution 3.**

This is a first order, linear, non-homogeneous differential equation.

Note that  $P_0(x) = 4 + x^2$ ,  $P_1(x) = 2x$ ,  $F(x) = 4x$ , and  $P_0 \neq 0 \forall x \in \mathbb{R}$ , so the interval of validity is  $\mathbb{R}$ . Also note that  $\frac{d}{dx}[4 + x^2] = 2x$ , so the equation can be written as

$$(4 + x^2)\frac{dy}{dx} + \frac{d}{dx}[4 + x^2]y = 4x.$$

Using the product rule to re-write the LHS as

$$\begin{aligned}\frac{d}{dx}[(4 + x^2)y] &= 4x \\ \int \frac{d}{dx}[(4 + x^2)y] dx &= \int 4x dx + c \\ (4 + x^2)y &= 2x^2 + c \\ y &= \frac{2x^2 + c}{4 + x^2}.\end{aligned}$$

□

**Example 1.2.4** Given the equation  $y' - 2y = 4 - x$ . Classify the equation and find the general solution  $y = y(x, c)$ .

**Solution 4.**

This is a first order, linear, non-homogeneous differential equation.

Since  $P_0(x) = 1$ ,  $P_1(x) = -2y$ ,  $F(x) = 4 - x$ , and  $P_0(x) \neq 0 \forall x \in \mathbb{R}$ , the interval of validity is  $\mathbb{R}$ . Consider  $\mu = \mu(x) \neq 0$ . Multiply both sides of the equation by  $\mu(x)$ :

$$\mu(x)y' - 2\mu(x)y = \mu(x)(4 - x) \quad (1)$$

To make the LHS a product rule, we need

$$\frac{d}{dx}[\mu(x)y(x)] = \mu'(x)y(x) + \mu(x)y'(x) = \mu(x)y'(x) - 2\mu(x)y.$$

So, we have  $\mu' = -2\mu$ , or  $\mu' + 2\mu = 0$ , a first order, linear, homogeneous ODE. Solving this ODE, we get  $\mu(x) = ce^{-2x}$ . Since we only want one specific  $\mu$  that would work, take  $c = 1$ . So,  $\mu(x) = e^{-2x}$ . Substituting  $\mu(x) = e^{-2x}$  to Eq. (1):

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}(4 - x), \quad \tilde{P}_0 = e^{-2x} \neq 0, \quad \tilde{P}_1 = -2e^{-2x}.$$

Using the product rule:

$$\begin{aligned} \frac{d}{dx}[e^{-2x}y] &= 4e^{-2x} - xe^{-2x} \\ \int \frac{d}{dx}[e^{-2x}y] dx &= \int 4e^{-2x} - xe^{-2x} dx + c \\ e^{-2x}y &= \frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c \\ y &= e^{2x} \left( \frac{1}{2}xe^{-2x} - \frac{7}{4}e^{-2x} + c \right) \\ &= \frac{1}{2}x - \frac{7}{4} + ce^{2x}. \end{aligned}$$

□

**Theorem 1.2.5 Method of Integrating Factor**

Given the first order linear differential equation  $y' + p(x)y = f(x)$ , with  $p$  and  $f$  both continuous on some interval  $(a, b)$ ,

$$y(x) = \frac{1}{\mu(x)} \left[ \int \mu(x)f(x) dx + c \right]$$

is the general solution to the equation, with

$$\mu(x) = e^{\int p(x) dx}.$$

We call  $\mu(x)$  the *integrating factor*.

**Proof 5.** Consider  $\mu = \mu(x) \neq 0$ . Multiplying the both sides of  $y' + p(x)y = f(x)$  by  $\mu$ :

$$\mu y' + p\mu y = \mu f. \quad (2)$$

Impose  $\mu y' + p\mu y = \frac{d}{dx}[\mu y]$  to find  $\mu = \mu(x)$ :

$$\begin{aligned} \mu y' + p\mu y &= \mu' y + \mu y' \\ \mu' - p\mu &= 0, && \text{first order, linear, homogeneous ODE} \\ \mu(x) &= e^{\int p(x) dx}, && \text{the integrating factor} \end{aligned}$$

Substitute  $\mu(x) = e^{\int p(x) dx}$  into Eq. (2):

$$\begin{aligned} \frac{d}{dx}[\mu y] &= \mu f \\ \int \frac{d}{dx}[\mu y] dx &= \int \mu f dx + c \\ \mu y &= \int \mu f dx + c \\ y(x) &= \frac{1}{\mu(x)} \left[ \int \mu(x)f(x) dx + c \right]. \end{aligned}$$

■

**Example 1.2.6** Give the equation  $y' + 2y = x^3e^{-2x}$ . Classify the equation and find the general solution  $y = y(x, c)$ .

**Solution 6.**

It is a first order, linear, non-homogeneous ODE, with  $p = 2$  and  $f = x^3e^{-2x}$ . Let  $\mu(x)$  be the integrating factor. Then,

$$\mu(x) = e^{\int 2 dx} = e^{2x}.$$

So, by the method of integrating factor, we know

$$\begin{aligned}\frac{d}{dx} [\mu(x)y] &= \mu(x)f(x) \\ \int \frac{d}{dx} [e^{2x}y] dx &= \int e^{2x}x^3e^{-2x} dx + c \\ e^{2x}y &= \int x^3 dx + c \\ e^{2x}y &= \frac{1}{4}x^4 + c \\ y &= \frac{1}{4}x^4e^{-2x} + ce^{-2x}.\end{aligned}$$

□

**Remark.** Re-examine the formula we derived from the method of integrating factor:

$$y(x) = \frac{1}{\mu} \int f \mu dx + \boxed{\frac{c}{\mu}}.$$

The part being boxed,  $\frac{c}{\mu}$ , is independent from  $f$  and is exactly  $ce^{-\int p dx}$  if we expand, which is the solution for a homogeneous differential equation.

**Definition 1.2.7 (Complementary Equation).** The *complementary equation* to a first order ODE  $y' + py = f$  is the homogeneous part of it. i.e.,  $y' + py = 0$ .

### Theorem 1.2.8 Method of Variation of Parameters

Given the first order linear differential equation  $y' + p(x)y = f(x)$ , with  $p$  and  $f$  both continuous on some interval  $(a, b)$ ,

$$y(x) = y_1(x) \left[ \int \frac{f(x)}{y_1(x)} dx + c \right]$$

is the general solution to the equation, where  $y_1$  is a solution of the complementary euqation  $y' + py = 0$ .

**Proof 7.** Call  $y_1$  a solution of the complementary equation  $y' + p(x)y = 0$ . Then, we want to find  $y(x) = u(x)y_1(x)$ , the general solution of  $y' + p(x)y = f(x)$ , where  $u$  is an unknown function of  $f$ . Note that, by product rule,  $y'(x) = u'y_1 + uy'_1$ . Then, the equation becomes

$$\begin{aligned}(u'y_1 + uy'_1) + p(x)(uy_1) &= f(x) \\ u'y_1 + uy'_1 + puy_1 &= f \\ y_1u' + \underbrace{(y'_1 + py_1)}_0 u &= f \\ y_1u' = f &\implies u(x) = \int \frac{f(x)}{y_1(x)} dx + c.\end{aligned}$$

Therefore, the formula to find  $y$  is given by

$$y = y_1 u = y_1(x) \left[ \int \frac{f(x)}{y_1(x)} dx + c \right].$$

■

**Remark.** *The method of variation of parameters will be more useful when solving second or higher order differential equations.*

**Example 1.2.9** Give the equation  $y' + 2y = x^3 e^{-2x}$ . Find the general solution  $y = y(x, c)$  using the method of variation of parameters.

**Solution 8.**

It is a first order, linear, non-homogeneous ODE, with  $p = 2$  and  $f = x^3 e^{-2x}$ . Let  $y_1$  be the solution of the complementary equation  $y' + 2y = 0$ . Then,  $y_1(x) = e^{-\int 2 dx} = e^{-2x}$ . By the method of variation of parameters, suppose  $y = uy_1$ , where  $u$  is an unknown function of  $x$ . Then,

$$u(x) = \int \frac{f(x)}{y_1(x)} dx + c = \int \frac{x^3 e^{-2x}}{e^{-2x}} dx + c = \int x^3 dx + c = \frac{1}{4}x^4 + c.$$

So,

$$y = uy_1 = e^{-2x} \left( \frac{1}{4}x^4 + c \right) = \frac{1}{4}x^4 e^{-2x} + ce^{-2x}.$$

□

**Theorem 1.2.10 Existence and Uniqueness Theorem**

Suppose that  $p = p(x)$  and  $f = f(x)$  are continuous on  $(a, b)$ . Then, a general solution of  $y' + p(x)y = f(x)$  on  $(a, b)$  is

$$y(x) = y_1(x) \left[ \int \frac{f(x)}{y_1(x)} dx + c \right],$$

where  $y_1(x)$  is a solution of the complementary equation (i.e.,  $y' + p(x)y = 0$ ).

If  $x_0$  is an arbitrary point in  $(a, b)$  and  $y_0$  is an arbitrary real number, then the initial value problem,

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a unique solution on  $(a, b)$ .

### 1.3 Separable Equations

**Definition 1.3.1 (General Forms).** The general form of a non-linear first order ODE is given by

$$y' = f(x, y(x)).$$

If we take  $M(x, y) = -f(x, y)$  and  $N(x, y) = 1$ , we can also re-write the equation into

$$M(x, y) + N(x, y)y' = 0, \quad \text{or} \quad M(x, y) dx + N(x, y) dy = 0.$$

**Definition 1.3.2 (Separable Equations).** If  $M(x, y) = M(x)$  and  $N(x, y) = N(y)$ , then the ODE is called *separable*.

**Theorem 1.3.3 Separation of Variables (SoV)**

Consider the non-linear first order ODE  $M(x, y) + N(x, y)y' = 0$ , with  $M(x, y) = M(x)$  and  $N(x, y) = N(y)$ . Then we can find an implicit solution of the ODE in the form of

$$F(x, y) = c,$$

where  $F(x, y)$  is a function of  $x$  and  $y$  and

$$F(x, y) = \int M(x) dx + \int N(y) dy.$$

**Proof 1.** Let  $H'_1(x) = M(x)$  and  $H'_2(y) = N(y)$ . Then, the equation becomes

$$\begin{aligned} H'_1(x) + H'_2(y)y' &= 0 \\ \frac{d}{dx} [H_1(x)] + \frac{d}{dy} [H_2(y)] \frac{dy}{dx} &= 0 \end{aligned}$$

By using the chain rule,  $\frac{d}{dy} [H_2(y)] \frac{dy}{dx} = \frac{d}{dx} [H_2(y(x))]$ . So, the equation becomes

$$\begin{aligned} \frac{d}{dx} [H_1(x)] + \frac{d}{dx} [H_2(y(x))] &= 0 \\ \frac{d}{dx} [H_1(x) + H_2(y)] &= 0 \\ H_1(x) + H_2(y) &= c \\ \int M(x) \frac{d}{dx} + \int N(y) dy &= c \\ F(x, y) &= c \end{aligned}$$

■

**Example 1.3.4** Given the equation  $y' = \frac{x^2}{1-y^2}$ . Classify the differential equation and find the general solution.

**Solution 2.**

It is a first order, non-linear ODE. Since  $y' = \frac{x^2}{1-y^2}$ , so we have  $1-y^2 \neq 0$ . That is,

$y^2 \neq 1$ , or  $y \neq \pm 1$ . Using the separation of variables (SoV), we have

$$\begin{aligned}(1 - y^2)y' &= x^2 \\ \int 1 - y^2 \, dy &= \int x^2 \, dx \\ y - \frac{1}{3}y^3 &= \frac{1}{3}x^3 + c \\ y - \frac{1}{3}y^3 - \frac{1}{3}x^3 &= c \\ 3y - y^3 - x^3 &= c\end{aligned}$$

□

**Example 1.3.5** Given the equation  $y' = \frac{(y - 3) \cos x}{1 + 2y^2}$ . Classify the equation and find the general solution.

**Solution 3.**

It is a first order, non-linear ODE. Since  $1 + 2y^2 \neq 0 \quad \forall y \in \mathbb{R}$ . Note that if we take  $y - 3 = 0$ , we get  $y = 3$ , a constant solution to the differential equation. Now, assume  $y \neq 3$ . Then, use SoV:

$$\int \frac{1 + 2y^2}{y - 3} \, dy = \int \cos x \, dx + c = \sin x + c.$$

Set  $t = y - 3$ ,  $dt = dy$ . So,  $y = t + 3$  and  $y^2 = (t + 3)^2$ . Then,

$$\begin{aligned}\int \frac{1 + 2y^2}{y - 3} \, dy &= \int \frac{1 + 2(t + 3)^2}{t} \, dt = \int \frac{1 + 2t^2 + 12t + 18}{t} \, dt \\ &= \int \frac{19}{t} + 12 + 2t \, dt \\ &= 19 \ln |t| + 12t + t^2 \\ &= 19 \ln |y - 3| + 12(y - 3) + (y - 3)^2 \\ &= 19 \ln |y - 3| + 6y + y^2 - 27.\end{aligned}$$

So,

$$19 \ln |y - 3| + y^2 + 6y - 27 - \sin x = c$$

□

**Example 1.3.6** Give the equation  $y' = \frac{1}{2}x(1 - y^2)$ . Classify the equation and find the general solution.

**Solution 4.**

It is a first order, non-linear ODE. Notice that we have the constant solutions when

we take  $1 - y^2 = 0$ , or  $y = \pm 1$ . Now, assume  $y \neq \pm 1$ . Using SoV:

$$\int \frac{2}{1-y^2} dy = \int x dx + c = \frac{1}{2}x^2 + c.$$

Note that  $\frac{2}{1-y^2} = \frac{2}{(1-y)(1+y)}$ . Use partial fractions. Assume

$$\frac{2}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}.$$

Then, we get  $A(1+y) + B(1-y) = 2$ . That is,  $(A+B) + (A-B)y = 2$ . Attempting to solve the system of equations  $\begin{cases} A-B=0 \\ A+B=2 \end{cases}$ , then we get  $A=B=1$ . Therefore,

$$\frac{2}{1-y^2} = \frac{1}{1-y} + \frac{1}{1+y}.$$

Then,

$$\begin{aligned} \int \frac{1}{1-y} + \frac{1}{1+y} dy &= \frac{1}{2}x^2 + c \\ -\ln|1-y| + \ln|1+y| &= \frac{1}{2}x^2 + c \\ \ln|1-y| - \ln|1+y| &= -\frac{1}{2}x^2 + c \\ \ln\left|\frac{1-y}{1+y}\right| &= -\frac{1}{2}x^2 + c \\ \left|\frac{y-1}{y+1}\right| &= e^{-\frac{1}{2}x^2+c} = e^{-\frac{1}{2}x^2}e^c \\ \frac{y-1}{y+1} &= c_2e^{-\frac{1}{2}x^2} \\ y-1 &= (y+1)c_2e^{-\frac{1}{2}x^2} \\ (1-c_2e^{-\frac{1}{2}x^2})y &= 1+c_2e^{-\frac{1}{2}x^2} \\ y &= \frac{1+c_2e^{-\frac{1}{2}x^2}}{1-c_2e^{-\frac{1}{2}x^2}} \end{aligned}$$

The value of  $c_2$  si chosen according to the sign of  $\frac{y-1}{y+1}$ . □

**Example 1.3.7** Given the equation  $y' = \frac{3x^2 + 4x + 2}{2(y-1)}$ , with  $y(0) = 1$ . Classify the equation, find the general solution, and solve the IVP.

### Solution 5.

It is a first order, nonlinear, separable ODE. Note that  $y-1 \neq 0$ , so  $y \neq 1$ . Assume

$y \neq 1$ , use SoV:

$$\begin{aligned} \int 2(y-1) \, dy &= \int 3x^2 + 4x + 2 \, dx + c \\ (y-1)^2 &= x^3 + 2x^2 + 2x + c \\ y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + c \\ y^2 - 2y &= x^3 + 2x^2 + 2x + c. \end{aligned}$$

Substitute  $y = -1$  when  $x = 0$ :

$$1 + 2 = c \implies c = 3.$$

So,

$$\begin{aligned} y^2 - 2y &= x^3 + 2x^2 + 2x + 3, \quad y \neq 1 \\ y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + 4 \\ (y-1)^2 &= x^3 + 2x^2 + 2x + 4 \\ y-1 &= \pm\sqrt{x^3 + 2x^2 + 2x + 4} \\ y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \end{aligned}$$

If  $y = -1$  and  $x = 0$ :  $-1 = 1 \pm \sqrt{4} = 1 \pm 2$ . So, it must be that  $-1 = 1 - 2$ . So,

$$y^{\text{IVP}} = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

Note that now we have another condition for  $x$ :

$$\begin{aligned} x^3 + 2x^2 + 2x + 4 &\geq 0 \\ (x+2)(x^2+2) &\geq 0 \\ x+2 &\geq 0 \\ x &\geq -2 \end{aligned}$$

So,

$$y^{\text{IVP}} = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}, \text{ with } y \neq 1 \text{ and } x \geq -2.$$

□

**Example 1.3.8** Solve the IVP  $\begin{cases} y' = \sqrt[3]{y} = y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$ .

**Solution 6.**

It is a first order, nonlinear, separable ODE. The initial interval of validity:  $x \in \mathbb{R}$  and

$y \in \mathbb{R}$ . Note that if  $y = 0$ , there is a constant solution. Assume  $y \neq 0$ , use SoV:

$$\begin{aligned} \int y^{-\frac{1}{3}} dy &= \int dx + c \\ \frac{3}{2}y^{\frac{2}{3}} &= x + c \\ y^{\frac{2}{3}} &= \frac{2}{3}x + c \\ y &= \pm \left( \frac{2}{3}x + c \right)^{\frac{3}{2}} \end{aligned}$$

Substitute  $y(0) = 0$ :

$$0 = 0 + c \implies c = 0.$$

So,

$$y^{\text{IVP}} = \pm \left( \frac{2}{3}x \right)^{\frac{3}{2}}.$$

□

### Theorem 1.3.9 Existence and Uniqueness of Solutions to Nonlinear ODEs

Consider the IVP

$$y' = f(x, y(x)) \quad \text{with } y(x_0) = y_0.$$

- If  $f$  is continuous on an open rectangle  $R\{a < x < b, c < y < d\}$  that contains  $(x_0, y_0)$ , then the IVP has *at least* one solution on some open subinterval of  $(a, b)$  that contains  $x_0$ .
- If both  $f$  and  $\frac{\partial f}{\partial y}$  are continuous on  $R$ , then the IVP has a *unique* solution on some open subinterval of  $(a, b)$  that contains  $x_0$ .

**Example 1.3.10** In the IVP above (Example 1.3.8),  $f(x, y) = y^{\frac{1}{3}}$ , and so  $\frac{\partial f}{\partial y} = \frac{1}{3}y^{-\frac{2}{3}}$ , which is not continuous at  $y = 0$ . So, the IVP  $\nexists$  a unique solution on the interval given:  $R = \{x \in \mathbb{R}, y \in \mathbb{R}\}$ .

## 1.4 Exact Equations

### Theorem 1.4.1 Multivariable Chain Rule

Given  $F(x, y) = c$ , where  $y = y(x)$ . Then, the total derivative with respect to  $x$  is

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0.$$

**Example 1.4.2** Exact ODEs

$$\begin{aligned} M(x, y) \, dx + N(x, y) \, dy &= 0 \\ M(x, y) + N(x, y) \frac{dy}{dx} &= 0 \quad \text{if } y = y(x) \\ M(x, y) \frac{dx}{dy} + N(x, y) &= 0 \quad \text{if } x = x(y). \end{aligned}$$

**Example 1.4.3** Show that  $x^4y^3 + x^2y^5 + 2xy = c$  is an implicit solution of

$$(4x^3y^3 + 2xy^5 + 2y) \, dx + (3x^4y^2 + 5x^2y^4 + 2x) \, dy = 0.$$

**Solution 1.**

$$\frac{\partial F}{\partial x} = 4x^3y^3 + 2xy^5 + 2y; \quad \frac{\partial F}{\partial y} = 3x^4y^2 + 5x^2y^4 + 2x$$

If  $y = y(x)$ :

$$(4x^3y^3 + 2xy^5 + 2y) + (3x^4y^2 + 5x^2y^4 + 2x) \frac{dy}{dx} = 0.$$

If  $x = x(y)$ :

$$(4x^3y^3 + 2xy^5 + 2y) \frac{dx}{dy} + (3x^4y^2 + 5x^2y^4 + 2x) = 0.$$

So the implicit function is a solution to the differential equation.  $\square$ **Theorem 1.4.4**Given an implicit function  $F(x, y) = c$ . It is a solution to the differential equation if

$$F_x \, dx + F_y \, dy = 0.$$

**Definition 1.4.5 (Exact ODEs).** We say that an ODE of the form  $M(x, y) \, dx + N(x, y) \, dy = 0$  is *exact* if  $\exists F(x, y) = c$ , with  $F_x$  and  $F_y$  continuous, such that

$$M(x, y) = F_x(x, y) \quad \text{and} \quad N(x, y) = F_y(x, y).$$

**Theorem 1.4.6 Characterization of Exact ODEs**

Suppose that  $M$  and  $N$  are continuous on  $R$  and have continuous partial derivatives  $M_y, N_x$  on  $R$ . Then,

$$M(x, y) dx + N(x, y) dy = 0$$

is exact if and only if

$$M_y(x, y) = N_x(x, y)$$

on  $R$ .

**Example 1.4.7** Show this ODE is exact:

$$(4x^3y^3 + 2xy^5 + 2y) dx + (3x^4y^2 + 5x^2y^4 + 2x) dy = 0.$$

**Solution 2.**

$$M(x, y) = 4x^3y^3 + 2xy^5 + 2y; \quad M_y(x, y) = 12x^3y^2 + 10xy^4 + 2$$

$$N(x, y) = 3x^4y^2 + 5x^2y^4 + 2x; \quad N_x(x, y) = 12x^3y^2 + 10xy^4 + 2$$

Since  $M_y(x, y) = N_x(x, y)$ , it is exact. □

**Example 1.4.8** Find the general solution of

$$(4x^3y^3 + 3x^2) dx + (3x^4y^2 + 6y^2) dy = 0$$

**Solution 3.**

Note that  $M(x, y) = 4x^3y^3 + 3x^2$  and  $N(x, y) = 3x^4y^2 + 6y^2$ , so

$$M_y(x, y) = 12x^3y^2; \quad N_x(x, y) = 12x^3y^2.$$

Since  $M_y(x, y) = N_x(x, y)$ , the ODE is exact. Then,

$$\begin{aligned} F(x, y) &= \int M(x, y) dx + \varphi(y) \\ &= \int 4x^3y^3 + 3x^2 dx + \varphi(y) \\ &= x^4y^4 + x^3 + \varphi(y). \end{aligned}$$

Since  $F_y(x, y) = 3x^4y^2 + \varphi'(y) = 3x^4y^2 + 6y^2$ , we have  $\varphi'(y) = 6y^2$ . That is,

$$\varphi(y) = \int 6y^2 dy + c = 2y^3 + c.$$

So,  $F(x, y) = x^4y^3 + x^3 + 2y^3 + c$ . Then, the implicit solution is

$$x^4y^3 + x^3 + 2y^3 = c.$$

Alternatively, we can use  $F(x, y) = \int N(x, y) dy + \psi(x)$  to get the same result.  $\square$

### Theorem 1.4.9 Method of Integrating Factors for Exact ODEs

Given  $M(x, y) dx + N(x, y) dy = 0$  is not exact. Consider  $\mu = \mu(x, y) \neq 0$ . Multiply

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \quad \text{or} \quad \mu(y) = e^{\int \frac{N_x - M_y}{M} dy},$$

we could make the ODE exact and thus solvable.

**Proof 4.** Given  $M(x, y) dx + N(x, y) dy = 0$  is not exact. Consider  $\mu = \mu(x, y) \neq 0$ . Multiply both sides by  $\mu$ :

$$\mu M(x, y) dx + \mu N(x, y) dy = 0 \quad (3)$$

Then, we have

$$\widetilde{M}(x, y) = \mu M; \quad \text{and} \quad \widetilde{N}(x, y) = \mu N.$$

Thus, the condition for Eq. (3) to be exact is  $\widetilde{M}_y = \widetilde{N}_x$ , or  $(\mu M)_y = (\mu N)_x$ . By product rule,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x \quad (4)$$

**Remark.** Eq. (4) is a PDE, which we cannot solve. So, make the assumption that  $\mu$  is a function of only  $x$  or only  $y$ .

- If  $\mu = \mu(x)$ , then  $\mu_y = 0$ . So, we have

$$\begin{aligned} \mu M_y &= \mu_x N + \mu N_x \\ N\mu_x + (N_x - M_y)\mu &= 0 \\ \mu(x) &= e^{\int \frac{M_y - N_x}{N} dx}. \end{aligned}$$

If  $\mu = \mu(y)$ , then  $\mu_x = 0$ . So, we have

$$\begin{aligned} \mu_y M + \mu M_y &= \mu N_x \\ M\mu_y + (M_y - N_x)\mu &= 0 \\ \mu(y) &= e^{\int \frac{N_x - M_y}{M} dy}. \end{aligned}$$



**Remark.** To decide on if we should use  $\mu(x)$  or  $\mu(y)$ , test if  $\frac{M_y - N_x}{N}$  is a function of  $x$  only and  $\inf \frac{N_x - M_y}{M}$  is a function of  $y$  only.

**Example 1.4.10** Given the equation  $(3x + 2y^2) dx + 2xy dy = 0$ . Find the integrating factor and solve the ODE.

**Solution 5.**

Note that  $M(x, y) = 3x + 2y^2$  and  $N(x, y) = 2xy$ . So,  $M_y = 4y$  and  $N_x = 2y$ . So, the ODE is not exact, and we should find an integrating factor. It is easier to divide by  $N$  since there is only 1 term:

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x},$$

is a function of  $x$  only. So, by the Method of Integrating Factors, we know

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x \quad (x \neq 0).$$

Now, multiply both decide of the original ODE by  $x$ :

$$(3x^2 + 2xy^2) dx + 2x^2y dy = 0,$$

where  $\widetilde{M}(x, y) = 3x^2 + 2xy^2$  and  $\widetilde{N}(x, y) = 2x^2y$ . So,

$$\begin{aligned} F(x, y) &= \int \widetilde{N}(x, y) dy + \psi(x) \\ &= \int 2x^2y dy + \psi(x) \\ &= x^2y^2 + \psi(x) \end{aligned}$$

Compute the partial deriviate of  $F$  with respect to  $x$ , we get

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= 2xy^2 + \psi'(x) = 3x^2 + 2xy^2 \\ \psi'(x) &= 3x^2 \\ \psi(x) &= \int 3x^2 dx + c = x^3 + c \end{aligned}$$

So, the solution is

$$x^2y^2 + x^3 = c.$$

□

**Example 1.4.11** Solve the equation  $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) dx + (3x^2y^2 + 4y) dy = 0$ .

**Solution 6.**

- Test if the equation is exact:

$$M(x, y) = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x; \quad M_y = 6xy^2 - 6x^3y^2 - 8xy$$

$$N(x, y) = 3x^2y^2 + 4y; \quad N_x = 6xy^2$$

Since  $M_y \neq N_x$ , the equation is no exact.

- Find an integrating factor. Try:

$$\frac{M_y - N_x}{N} = \frac{6xy^2 - 6x^3y^2 - 8xy - 6xy^2}{3x^2y^2 + 4y} = \frac{-2x(3x^2y^2 + 4y)}{3x^2y^2 + 4y} = -2x,$$

which is a function of  $x$  only. So, the integrating factor  $\mu = \mu(x) \neq 0$  is

$$\mu = e^{\int -2x \, dx} = e^{-x^2}.$$

- Multiply the equation by  $\mu$ :

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \, dx + e^{-x^2}(3x^2y^2 + 4y) \, dy = 0.$$

We can easily show that the equation is exact.

$$\begin{aligned} F(x, y) &= \int e^{-x^2}(3x^2y^2 + 4y) \, dy + \psi(x) \\ &= e^{-x^2}(x^2y^3 + 2y^2) + \psi(x). \end{aligned}$$

So,

$$\begin{aligned} F_x(x, y) &= -(2x)e^{-x^2}(x^2y^3 + 2y^2) + e^{-x^2}(2xy^3) + \psi'(x) \\ &= e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x). \end{aligned}$$

Note that

$$\begin{aligned} e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x) &= e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \\ \psi'(x) &= e^{-x^2}(2x) \\ \psi(x) &= \int (2x)e^{-x^2} \, dx + c = -e^{-x^2} + c. \end{aligned}$$

- So, the solution is

$$e^{-x^2}(x^2y^3 + 2y^2 - 1) = c.$$

□

## 1.5 Autonomous ODEs

**Definition 1.5.1 (Autonomous ODEs).** ODEs in the form  $y' = f(y)$  is called *autonomous ODEs*. In other words, the dependent variable  $x$  does not appear explicitly in the equation.

**Remark.** *Autonomous ODEs are separable.*

### Example 1.5.2 Exponential Growth

Let  $y = \varphi(t)$  be the population of the given species at time  $t$ . Assume that the rate of change of the population is proportional to the current value of  $y$ , and the constant of proportionality is given by  $r$ , called the rate of growth ( $r > 0$ ) or decline ( $r < 0$ ).

A differential equation to model this situation is given by

$$\begin{cases} y' = ry \\ y(0) = y_0 \end{cases}$$

and the solution is

$$y = y_0 e^{rt}.$$

### Example 1.5.3 Logistic Growth

The *logistic growth* is modeled by the differential equation

$$y' = r \left(1 - \frac{y}{K}\right) y,$$

where  $r > 0$  represents the intrinsic growth rate in absence of limiting factors, and  $K > 0$  is the environmental carrying capacity.

In this system, we have  $y = 0$  and  $y = k$  as the constant solutions/equilibria/critical points of the ODE.  $y = 0$  is the *unstable equilibrium solution*, and  $y = k$  is the *asymptotic stable solution*.

This model shows that when  $0 < y < K$ , the population will grow until  $K$  (that is,  $\lim_{t \rightarrow \infty} y = K$ ) and when  $y > K$ , the population will decrease to  $K$  ( $\lim_{t \rightarrow \infty} y = K$ ).

We can model the logistic growth using the following ODE as well

$$y' = -r \left(1 - \frac{y}{T}\right) y,$$

where  $r$  represents the intrinsic growth rate in absence of limiting factors and  $T$  is the threshold level.

In this system, we have  $y = 0$  and  $y = T$  as the equilibrium solutions,  $y = T$  is the unstable solution and  $y = 0$  is the asymptotic stable solution. That is, when  $0 < y < T$ , the population will decrease to 0 ( $\lim_{t \rightarrow \infty} y = 0$ ) and when  $y > T$ , the population will grow forever ( $\lim_{t \rightarrow \infty} y = \infty$ ).

A solution to the IVP

$$y' = -r\left(1 - \frac{y}{T}\right)y; \quad y(0) = y_0$$

is  $y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$ .

The solution also have a vertical asymptote  $t = t^*$  such that  $y \rightarrow \infty$ . To find such an asymptote, we want  $\lim_{t \rightarrow t^*} y^{\text{IVP}} = 0$ . So, we set the denominator to 0, i.e.,  $y_0 + (T - y_0)e^{rt} = 0$ , and we get  $t = \frac{1}{r} \ln\left(\frac{y_0}{y_0 - T}\right)$  with  $y_0 > T$  by solving the equation.

Combining the two models, we give a more accurate model of the population. We call it *Logistic Growth with a Threshold*.

$$y = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}y\right),$$

where  $r > 0$  is the intrinsic growth rate in the absence of limiting factors and  $0 < T < K$ , where  $T$  is the threshold level and  $K$  is the environmental carrying capacity.

In this system,  $y = K$  and  $y = 0$  are asymptotic stable solutions and  $y = T$  is the unstable equilibrium solution. Given the IVP

$$y = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}y\right); \quad y(0) = y_0.$$

The behavior of the IVP solution with respect to  $y_0$  is given as the following:

- If  $0 < y_0 < T$ , then  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ .
- If  $y_0 = T$ , then  $y(t) = T$  for all  $t$ .
- If  $T < y_0 < K$ , then  $y(t) \rightarrow K$  as  $t \rightarrow \infty$ .
- If  $y_0 = K$ , then  $y(t) = K$  for all  $t$ .
- If  $y_0 > K$ , then  $y(t) \rightarrow K$  as  $t \rightarrow \infty$ .

#### Example 1.5.4 Newton's Law of Cooling

Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). This law is summarized using the following ODE

$$T' = -k(T - T_a),$$

where  $T$  represents the temperature of an object,  $T_a$  is the ambient temperature of the surrounding environment, and  $k > 0$  is the cooling constant.

The equilibrium solution  $T = T_a$  is asymptotic stable. Using either integrating factor or SoV, we find the general solution as  $T = Ae^{-kt} + T_a$ .

## 2 System of ODEs

### 2.1 Linear Algebra

**Definition 2.1.1 (System of Linear Ordinary Differential Equations).** A *system of ODEs* is in general given by

$$\begin{cases} y'_1 = f_1(t, y_1, y_2, \dots, y_n) \\ y'_2 = f_2(t, y_1, y_2, \dots, y_n) \\ \vdots \\ y'_n = f_n(t, y_1, y_2, \dots, y_n). \end{cases}$$

A system of ODEs is called *linear* if it can be written in the following form

$$y' = \mathbf{A}(t)y + f(t),$$

where

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

The system can be

- *Homogeneous*, if  $f(t) = \vec{0}$
- *Non-homogeneous*, if  $f(t) \neq \vec{0}$

**Definition 2.1.2 (Matrix Equality).** Suppose  $\mathbf{A}$  and  $\mathbf{B}$  are matrices of the same size. We say  $\mathbf{A} = \mathbf{B}$  if  $A_{ij} = b_{ij}$  for all  $i$  and  $j$ . In other words,  $\mathbf{A} = \mathbf{B}$  if they are equal componentwise.

**Definition 2.1.3 (Null Matrix/Zero).** The symbol  $\mathbf{0}$  will be used to denote the matrix (or vector) each of whose elements is zero.

**Definition 2.1.4 (Transpose).**  $\mathbf{A}^T$  is the *transpose* of  $\mathbf{A}$  and it is obtained by interchanging the rows and columns of  $\mathbf{A}$ . If  $\mathbf{A} = [A_{ij}]$ , then  $\mathbf{A}^T = [A_{ji}]$ .

**Definition 2.1.5 (Matrix Addition).** Suppose  $\mathbf{A}$  and  $\mathbf{B}$  are matrices of the same size. Then,  $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$  for all  $i$  and  $j$ .

**Definition 2.1.6 (Product of a Matrix and a Scalar).** Suppose  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $c \in \mathbb{R}$ , then  $c\mathbf{A} = [ca_{ij}]$ .

**Definition 2.1.7 (Dot Product/Scalar Product).** Suppose  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{2 \times 1}$  and  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^{2 \times 1}$ . Then, the *dot product* of  $\mathbf{x}$  and  $\mathbf{y}$  is defined to be

$$\mathbf{x} \cdot \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2.$$

**Definition 2.1.8 (Matrix Product).** Suppose  $\tilde{\mathbf{A}} \in \mathbb{R}^{m \times n}$  and  $\tilde{\mathbf{B}} \in \mathbb{R}^{n \times p}$ . Then, each  $c_{ij}$  of

$\tilde{\mathbf{C}} \in \mathbb{R}^{m \times p}$  such that  $\tilde{\mathbf{C}} = \tilde{\mathbf{A}}\tilde{\mathbf{B}}$  is the dot product between row  $i$  of  $\mathbf{A}$  and column  $j$  of  $\mathbf{B}$ . That is,

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

**Definition 2.1.9 (Identity Matrix).** The  $n \times n$  multiplicative identity is the *identity matrix*

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

From the definition of matrix multiplication, we have  $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ .

**Definition 2.1.10 (Determinant).** Given a matrix  $\mathbf{A}$ , we can always find the *determinant* of  $\mathbf{A}$ , which is a number associated to  $\mathbf{A}$ , denoted as  $\det(\mathbf{A})$  or  $|\mathbf{A}|$ . Suppose  $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ , then

$$\det(\mathbf{A}) = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

**Definition 2.1.11 (Inverse).** For  $\mathbf{A} \in \mathbb{R}^{n \times n}$ , if  $\mathbf{A}$  is invertible, then  $\exists \mathbf{B} \in \mathbb{R}^{n \times m}$  such that  $\mathbf{AB} = \mathbf{I}$ . We denote  $\mathbf{B}$  as  $\mathbf{A}^{-1}$ , the *inverse* of  $\mathbf{A}$ .

### Theorem 2.1.12 Determinant and Invertibility

If  $\det(\mathbf{A}) \neq 0$ , then the matrix  $\mathbf{A}$  is nonsingular and invertible. Conversely, if  $\det(\mathbf{A}) = 0$ , then the matrix is singular and not invertible.

If  $\det(\mathbf{A}) \neq 0$ , then the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

### Theorem 2.1.13 Determinants are Multiplicable

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det(\mathbf{A}) \det(\mathbf{B}).$$

**Definition 2.1.14 (Linear Systems of Algebraic Equations).** Given an  $n \times n$  matrix  $\mathbf{A}$ , an  $n \times 1$  vector  $b$ , and a vector of unknowns  $x$ ,  $\mathbf{Ax} = b$  is called *linear system of equations*. If  $b = 0$ , the system is called *homogeneous*. If  $b \neq 0$ , then the linear system is called *non-homogeneous*.

**Theorem 2.1.15 Singularity of Systems**

If the system is homogenous, it is nonsingular if it exists a unique solution (which is the null vector); it is singular if it has infinitely many solutions.

If the system is non-homogenous, then it is nonsingular if it has a unique solution ( $x = \mathbf{A}^{-1}\mathbf{b}$ ), and it is singular if it has either infinitely many solutions or no solutions.

**Definition 2.1.16 (Eigenvector, Eigenvalue, Eigenpairs).** Suppose that  $\mathbf{A}$  is an  $n \times n$  matrix. A nonzero vector  $v$  that satisfies the equation

$$\mathbf{A}v = \lambda v$$

is called an *eigenvector* of matrix  $\mathbf{A}$ . The scalar  $\lambda$  is the *eigenvalue* associated with this eigenvector. An *eigenpair* is the pair  $\{\lambda_i, v_i\}$  with  $i = 1, \dots, n$ .

**Theorem 2.1.17 Finding Eigenpairs**

To find an eigenpair  $\{\lambda_i, v_i\}$  with  $i = 1, \dots, n$ , we need to find the roots of the characteristics polynomial of  $\mathbf{A}$ . That is, we need to solve the problem

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

Once we have the eigenvalues, we need to solve the linear systems

$$(\mathbf{A} - \lambda_i \mathbf{I})v_i = 0 \quad \text{with } i = 1, \dots, n.$$

**Proof 1.**

$$\begin{aligned} \mathbf{A}v = \lambda v &\implies \mathbf{A}v - \lambda v = 0 \\ &\implies (\mathbf{A} - \lambda \mathbf{I})v = 0 \end{aligned}$$

To solve this homogeneous linear system, we cannot have  $v = 0$  since  $v$  is defined to be nonzero. Therefore, the system has infinite many solutions. That is,  $\mathbf{A} - \lambda \mathbf{I}$  is singular, or  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ . ■

**Remark.** The matrix  $\mathbf{A} - \lambda \mathbf{I}$  is singular, meaning the system has infinitely many solutions. Finding the eigenvector consists of identifying a direction (invariant under the action of  $\mathbf{A}$ ) and choose a representative of it.

**Definition 2.1.18 (Similar Matrices).** Two  $n \times n$  matrices  $\mathbf{A}$  and  $\mathbf{B}$  are *similar* if there exists an invertible matrix  $\mathbf{T}$  such that

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad \text{and} \quad \mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1}.$$

**Definition 2.1.19 (Diagonalizable Matrix).** A matrix  $\mathbf{A}$  is said to be *diagonalizable* if it is similar to a diagonal matrix  $\mathbf{D}$ . In this case, the entries of the diagonal matrix  $\mathbf{D}$  are the eigenvalues of  $\mathbf{A}$ .

**Theorem 2.1.20 Characterization of Diagonalizability**

A matrix is diagonalizable if and only if it has  $n$  linearly independent eigenvectors.

**Proof 2.** Consider the eigenpairs  $\{\lambda_i, v_i\}$  with  $i = 1, \dots, n$  of an  $n \times n$  matrix  $\mathbf{A}$ . Let  $\mathbf{V}$  be the matrix of the  $n$  linearly independent eigenvectors, namely  $\mathbf{V} = [v_1 \ v_2 \ \dots \ v_n]$ , then

$$\mathbf{D} = \mathbf{V}^{-1} \mathbf{A} \mathbf{V}.$$

■

## 2.2 Basic Theorems on System of ODEs

**Example 2.2.1** Solve the following system of ODEs:

$$\begin{cases} y'_1 = y_1 - y_2 & \textcircled{1} \\ y'_2 = 2y_2 & \textcircled{2} \end{cases}$$

**Solution 1.**

From  $\textcircled{2}$ :  $y'_2 - 2y_2 = 0$  is linear homogeneous ODE, so  $y_2 = c_2 e^{2x}$ . Substitute  $y_2 = c_2 e^{2x}$  into  $\textcircled{1}$ : we have  $y'_1 - y_1 = -c_2 e^{2x}$ . Using the method of integrating factor, we get

$$y_1 = c_1 e^x - c_2 e^{2x}.$$

Therefore,

$$y(x) = \begin{bmatrix} c_1 e^x - c_2 e^{2x} \\ c_2 e^{2x} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^x + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2x}.$$

□

**Corollary 2.2.2:** Forms of Solutions When solving systems, we are aiming to find solutions in the following form:

$$y(t) = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t},$$

where  $c_1, c_2$  are constants,  $x_1, x_2$  are eigenvectors, and  $\lambda_1, \lambda_2$  are eigenvalues.

**Definition 2.2.3 (Continuous).** A matrix is *continuous* if and only if all its components are continuous.

**Theorem 2.2.4 Basic theory of Linear Systems (Homogeneous)**

If  $y' = \mathbf{A}(t)y$  with  $\mathbf{A}(t)$  continuous on  $(a, b)$ , then  $y = 0$  is the trivial solution of the system. Any other solutions will be nontrivial.

**Definition 2.2.5 (Linear Combination of Solutions).** If  $y_1, y_2$  are vector functions defined on  $(a, b)$  and  $c_1, c_2$  are parameters, then  $c_1 y_1 + c_2 y_2$  is a *linear combination* of  $y_1, y_2$ .

**Theorem 2.2.6 Principle of superposition**

If  $y_1, y_2$  are solutions of  $y' = \mathbf{A}(t)y$  on  $(a, b)$ , then  $c_1y_1 + c_2y_2$  is also a solution of  $y' = \mathbf{A}(t)y$  on  $(a, b)$ .

**Definition 2.2.7 (Fundamental set of Solutions).** We say that the set  $\{y_1, y_2\}$  is a *fundamental set of solutions* of  $y' = \mathbf{A}(t)y$  on  $(a, b)$  if every solution of  $y' = \mathbf{A}(t)y$  on  $(a, b)$  can be written as a linear combination of  $y_1$  and  $y_2$ . In this case,  $y(t) = c_1y_1 + c_2y_2$  is a *general solution* of the system on  $(a, b)$ .

**Definition 2.2.8 (Linear Independence).** We say that  $y_1, y_2$  are *linearly independent* on  $(a, b)$  if  $c_1y_1 + c_2y_2 = 0$  if and only if  $c_1 = c_2 = 0$ .

**Theorem 2.2.9 Fundamental Set of Solutions and Linear Independence**

Suppose that  $\mathbf{A}(t)$  is continuous on  $(a, b)$ . The set  $\{y_1, y_2\}$  is a fundamental set of solutions if  $y_1$  and  $y_2$  are linearly independent.

**Definition 2.2.10 (Fundamental Matrix and Wronskian).** Suppose  $y_1$  and  $y_2$  are solutions of  $y' = \mathbf{A}(t)y$ . Suppose the initial condition is  $y(t_0) = y_0$ , then the *fundamental matrix* is

$$\mathbf{Y} = \begin{bmatrix} | & | \\ y_1(t_0) & y_2(t_0) \\ | & | \end{bmatrix} = \begin{bmatrix} y_1^{(1)}(t_0) & y_2^{(1)}(t_0) \\ y_1^{(2)}(t_0) & y_2^{(2)}(t_0) \end{bmatrix},$$

and the *Wronskian* of  $y_1$  and  $y_2$  is

$$\mathbf{W}\left(\begin{bmatrix} y_1 & y_2 \end{bmatrix}\right) = \det(\mathbf{Y}(t_0)).$$

**Remark.** *Intuitions of Fundamental Matrices and Wronskians*

Consider  $c_1y_1 + c_2y_2 = 0$  if and only if  $c_1 = c_2 = 0$  in vector form:

$$c_1 \begin{bmatrix} y_1^{(1)} \\ y_1^{(2)} \end{bmatrix} + c_2 \begin{bmatrix} y_2^{(1)} \\ y_2^{(2)} \end{bmatrix} = 0.$$

Given initial condition:  $y(t_0) = y_0$ :

$$\begin{aligned} \begin{bmatrix} y^{(1)}(t_0) \\ y^{(2)}(t_0) \end{bmatrix} &= \begin{bmatrix} y_0^{(1)} \\ y_0^{(2)} \end{bmatrix} \\ c_1 \begin{bmatrix} y_1^{(1)}(t_0) \\ y_1^{(2)}(t_0) \end{bmatrix} + c_2 \begin{bmatrix} y_2^{(1)}(t_0) \\ y_2^{(2)}(t_0) \end{bmatrix} &= \begin{bmatrix} y_0^{(1)} \\ y_0^{(2)} \end{bmatrix} \\ \begin{bmatrix} y_1^{(1)}(t_0) & y_2^{(1)}(t_0) \\ y_1^{(2)}(t_0) & y_2^{(2)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} y_0^{(1)} \\ y_0^{(2)} \end{bmatrix} \end{aligned}$$

To ensure we have a unique solution  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ , we have that

$$\mathbf{Y}(t_0) = \begin{bmatrix} y_1^{(1)}(t_0) & y_2^{(1)}(t_0) \\ y_1^{(2)}(t_0) & y_2^{(2)}(t_0) \end{bmatrix}$$

should be nonsingular. That is,  $\det(\mathbf{Y}(t_0)) \neq 0$ .

**Definition 2.2.11 (Matrix Differential Equations).** Let  $\mathbf{Y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$ . Then,

$$\mathbf{Y}' = \begin{bmatrix} y'_1 & y'_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}y_1 & \mathbf{A}y_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \mathbf{AY}.$$

The form  $\mathbf{Y}' = \mathbf{AY}$  is called the *matrix differential equations*.

**Example 2.2.12** Write the following system of equations in matrix form

$$\begin{cases} y'_1 = y_1 - y_2 & \textcircled{1} \\ y'_2 = 2y_2 & \textcircled{2} \end{cases}$$

**Solution 2.**

$$y' = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

□

### Theorem 2.2.13 Existence and Uniqueness

If  $\mathbf{A}(t)$  and  $f(t)$  are continuous on  $(a, b)$ ,  $t_0 \in (a, b)$ , and  $k$  is an arbitrary vector, then

the IVP of the system  $\begin{cases} y' = \mathbf{A}(t)y + f(t) \\ y(t_0) = k \end{cases}$  has a unique solution on  $(a, b)$ .

**Example 2.2.14** Given the following system of ODEs

$$\begin{cases} y'_1 = y_1 + 2y_2 + 2e^{4t} \\ y'_2 = 2y_1 + y_2 + e^{4t}. \end{cases}$$

Write the system in the matrix form and conclude that every initial value problem will have a unique solution on  $\mathbb{R}$ . Then, verify that

$$y = \frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

is a solution of the system for all values of  $c_1$  and  $c_2$ .

**Solution 3.**

$$y'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} y(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}. \quad (5)$$

Note that both  $\mathbf{A}$  and  $f$  are continuous on  $\mathbb{R}$ , so every IVP will have a unique solution. Now, consider the solution given. We want to show the solution satisfies Eq. (5).

$$\begin{aligned} \text{LHS} = y' &= \frac{4}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + 3c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \\ &= \begin{bmatrix} 32/5e^{4t} + 3c_1e^{3t} - c_2e^{-t} \\ 28/5e^{4t} + 3c_1e^{3t} + c_2e^{-t} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{RHS: } y &= \frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \\ &= \begin{bmatrix} 8/5e^{4t} + c_1e^{3t} + c_2e^{-t} \\ 7/5e^{4t} + c_1e^{3t} - c_2e^{-t} \end{bmatrix} \\ &\quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 8/5e^{4t} + c_1e^{3t} + c_2e^{-t} \\ 7/5e^{4t} + c_1e^{3t} - c_2e^{-t} \end{bmatrix} + \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix} \\ &= \begin{bmatrix} 8/5e^{4t} + c_1e^{3t} + c_2e^{-t} + 14/5e^{4t} + 2c_1e^{3t} - 2c_2e^{-t} + 2e^{4t} \\ 16/5e^{4t} + 2c_1e^{3t} + 2c_2e^{-t} + 7/5e^{4t} + c_1e^{3t} - c_2e^{-t} + e^{4t} \end{bmatrix} \\ &= \begin{bmatrix} 32/5e^{4t} + 3c_1e^{4t} - c_2e^{-t} \\ 28/5e^{4t} + 3c_1e^{3t} + c_2e^{-t} \end{bmatrix} \end{aligned}$$

So, LHS = RSH. □

**Theorem 2.2.15 Solutions of a System, Fundamental Set of Solutions, Linear independence, and Wronskian**

Suppose the matrix  $\mathbf{A} = \mathbf{A}(t) \in \mathbb{R}^{n \times n}$  is continuous on  $(a, b)$ ; let  $y_1, \dots, y_n$  be solutions of  $y' = \mathbf{A}(t)y$  on  $(a, b)$ . Then, the following statements are equivalent:

- The general solution of  $y' = \mathbf{A}(t)y$  on  $(a, b)$  is  $y = c_1y_1 + \dots + c_n y_n$ , where  $c_1, \dots, c_n$  are arbitrary constants.
- $\{y_1, \dots, y_n\}$  is a fundamental set of solutions of  $y' = \mathbf{A}(t)y$  on  $(a, b)$ .
- $\{y_1, \dots, y_n\}$  is linearly independent on  $(a, b)$ .
- The Wronskian of  $\{y_1, \dots, y_n\}$  is nonzero at some point of  $(a, b)$ .
- The Wronskian of  $\{y_1, \dots, y_n\}$  is nonzero at any point of  $(a, b)$ .

We call  $\mathbf{Y}$  a fundamental matrix if any of the statement is true.

**Example 2.2.16** Given  $\mathbf{A} = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix}$ ,  $y_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix}$ , and  $y_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$ . Compute the Wronskian of the solutions toward the system  $y' = \mathbf{A}y$ . Compute the general solution.

**Solution 4.**

$$\mathbf{W}\left(\begin{bmatrix} y_1 & y_2 \end{bmatrix}\right) = \det \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix} = -e^{2t}e^{-t} - (-e^{-t})(2e^{2t}) = e^t \neq 0 \quad \forall t \in \mathbb{R}.$$

Since  $\mathbf{W}\left(\begin{bmatrix} y_1 & y_2 \end{bmatrix}\right) \neq 0$ ,  $y_1$  and  $y_2$  are linear independent. Then, the general solution is

$$y = c_1y_1 + c_2y_2 = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}.$$

□

**Example 2.2.17** Given the system  $y' = \mathbf{A}y$ , with  $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ . find the general solution.

**Solution 5.**

Solving the eigenvalue problem:  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \implies \lambda_1 = -2$  and  $\lambda_2 = 6$ .

When  $\lambda_1 = -2$ ,  $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ; when  $\lambda_2 = 6$ ,  $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . So,

$$y_1 = x_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

$$y_2 = x_2 e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}.$$

$$\mathbf{W} \left( \begin{bmatrix} y_1 & y_2 \end{bmatrix} \right) = \det \begin{bmatrix} e^{-2t} & e^{6t} \\ -e^{-2t} & e^{6t} \end{bmatrix} = e^{-2t}(e^{6t}) - (e^{6t})(-e^{-2t}) = 2e^{4t} \neq 0 \quad \forall t \in \mathbb{R}.$$

Then,  $y_1$  and  $y_2$  are linearly independent, and the general solution is

$$y_1 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}.$$

□

**Example 2.2.18** Solve the following system of ODEs:

$$y' = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} y.$$

**Solution 6.**

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\frac{1}{2} - \lambda & 1 \\ 0 & -\frac{1}{2} - \lambda \end{vmatrix} = \left( -\frac{1}{2} - \lambda \right)^2 = 0 \implies \lambda_1 = \lambda_2 = -\frac{1}{2}.$$

Here we have repeated eigenvalues, so we cannot solve the system via solving the eigenvalue problem. However, we are dealing with an upper-triangular system, which has nice properties when solving:

$$\begin{cases} y^{(1)\prime} = -\frac{1}{2}y^{(1)} + y^{(2)} & \textcircled{1} \\ y^{(2)\prime} = -\frac{1}{2}y^{(2)} & \textcircled{2} \end{cases}$$

From  $\textcircled{2}$ ,  $y^{(2)} = c_2 e^{-\frac{1}{2}t}$ . So,  $y^{(1)\prime} + \frac{1}{2}y^{(1)} = c_2 e^{-\frac{1}{2}t}$ . So, we have

$$\begin{cases} y^{(1)} = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t} \\ y^{(2)} = c_2 e^{-\frac{1}{2}t}. \end{cases}$$

So, the general solution is

$$y = \underbrace{c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-\frac{1}{2}t}}_{\text{from eigenvalue problem}} + \underbrace{c_2 \begin{bmatrix} t \\ 1 \end{bmatrix} e^{-\frac{1}{2}t}}_{\text{in the form } (w+tx)e^{\lambda t}}.$$

□

### Theorem 2.2.19

Given  $y' = \mathbf{A}y$ . If  $\mathbf{A}$  has repeated real eigenvalues and eigenvectors, then the solution is given by

$$y = c_1 xe^{\lambda t} + c_2(tx + w)e^{\lambda t},$$

where  $\lambda$  and  $x$  are such that  $\mathbf{A}x = \lambda x$ , and  $w$  is the *generalized eigenvector* such that

$$(\mathbf{A} - \lambda \mathbf{I})w = x.$$

**Proof 7.** Note that  $y = (tx + w)e^{\lambda t}$  is a solution to  $y' = \mathbf{A}y$ . So,

$$\begin{aligned} \text{LHS} &= y' = xe^{\lambda t} + \lambda(tx + w)e^{\lambda t} \\ &= (x + t\lambda x + \lambda w)e^{\lambda t} \end{aligned}$$

and

$$\text{RHS} = \mathbf{A}y = \mathbf{A}(tx + w)e^{\lambda t} = (t\mathbf{A}x + \mathbf{A}w)e^{\lambda t}.$$

By LHS = RHS, we have

$$\begin{aligned} (x + t\lambda x + \lambda w)e^{\lambda t} &= (t\mathbf{A}x + \mathbf{A}w)e^{\lambda t} \\ x + t\lambda x + \lambda w &= t \underbrace{\mathbf{A}x}_{=\lambda x} + \mathbf{A}w \\ x + \lambda w &= \mathbf{A}w \\ \mathbf{A}w - \lambda w &= x \implies (\mathbf{A} - \lambda \mathbf{I})w = x. \end{aligned}$$

■

### Theorem 2.2.20

If  $\lambda$  and  $\bar{\lambda}$  are complex eigenvalues of  $\mathbf{A}$ , and  $x$  is the eigenvector of  $\mathbf{A}$  corresponding to  $\lambda$ , then  $\mathbf{A}\bar{x} = \bar{\lambda}\bar{x}$ . That is,  $\bar{x}$  is the eigenvector of  $\mathbf{A}$  corresponding to  $\bar{\lambda}$ .

**Proof 8.** We know  $\mathbf{A}x = \lambda x$ . So,  $\overline{\mathbf{A}x} = \bar{\lambda}\bar{x}$ . Note that  $\overline{\mathbf{A}x} = \mathbf{A}\bar{x}$  and  $\overline{\lambda x} = \bar{\lambda}\bar{x}$ . So, we have  $\mathbf{A}\bar{x} = \bar{\lambda}\bar{x}$ . ■

**Theorem 2.2.21**

Suppose  $z = a + ib \in \mathbb{C}$ , then  $\bar{z} = a - ib$ ,  $a = \operatorname{Re}(z) \in \mathbb{R}$ , and  $b = \operatorname{Im}(z) \in \mathbb{R}$ . Then,

$$\frac{z + \bar{z}}{2} = \frac{(a + ib) + (a - ib)}{2} = \frac{2a}{2} = a = \operatorname{Re}(z)$$

$$\frac{z - \bar{z}}{2i} = \frac{(a + ib) - (a - ib)}{2i} = \frac{2ib}{2i} = b = \operatorname{Im}(z)$$

**Definition 2.2.22 (Euler's Formula).**

$$e^{i\theta} = \cos \theta + i \sin \theta$$

**Theorem 2.2.23**

Given  $y' = \mathbf{A}y$ , where  $\mathbf{A}$  has complex conjugate eigenvalues and eigenvectors such that  $\lambda = \alpha + i\beta$ ,  $\bar{\lambda} = \alpha - i\beta$ ,  $x = u + iv$ , and  $\bar{x} = u - iv$ , with  $\alpha, \beta \in \mathbb{R}$  and  $u, v \in \mathbb{R}^2$ . Then, a complex solution to the system will be

$$y^{\mathbb{C}} = xe^{\alpha t} = (u + iv)e^{(\alpha+i\beta)t}, \quad \bar{y}^{\mathbb{C}} = \bar{x}e^{\bar{\alpha}t} = (u - iv)e^{\alpha-i\beta}t.$$

Note that, by the Euler's Formula,

$$\begin{aligned} y^{\mathbb{C}} &= (u + iv)e^{\alpha t} e^{i\beta t} = e^{\alpha t}(u + iv)(\cos(\beta t) + i \sin(\beta t)) \\ &= e^{\alpha t}(u \cos(\beta t) - v \sin(\beta t)) + ie^{\alpha t}(u \sin(\beta t) + v \cos(\beta t)) \end{aligned}$$

So, we have

$$y_1 = \operatorname{Re}(y^{\mathbb{C}}) = e^{\alpha t}(u \cos(\beta t) - v \sin(\beta t)); \quad y_2 = \operatorname{Im}(y^{\mathbb{C}}) = e^{\alpha t}(u \sin(\beta t) + v \cos(\beta t))$$

**Remark.** Note that  $y_1$  and  $y_2$  are linearly independent if and only if  $u$  and  $v$  are linearly independent.

**Example 2.2.24** Solve the system of ODEs:  $y' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix}y$ .

**Solution 9.**

Find eigenvalues and eigenvectors:  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \implies \lambda_{1,2} = 1 \pm 4i$ . Suppose  $\lambda = 1 + 4i$ , then

$$\begin{bmatrix} 3 - 4i & -5 \\ 5 & -3 - 4i \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \frac{3 - 4i}{5} x_1^{(1)} = x_1^{(2)} \implies x_1 = \begin{bmatrix} 5 \\ 3 - 4i \end{bmatrix}.$$

Therefore, when  $\bar{\lambda} = 1 - 4i$ , we have  $x_2 = \bar{x} = \begin{bmatrix} 5 \\ 3 + 4i \end{bmatrix}$ . So, the complex solutions are

$$y^C = xe^{\lambda t} = \begin{bmatrix} 5 \\ 3 - 4i \end{bmatrix} e^{(1+4i)t}; \quad \bar{y}^C = \bar{x}e^{\bar{\lambda}t} = \begin{bmatrix} 5 \\ 3 + 4i \end{bmatrix} e^{(1-4i)t}.$$

**Remark.** By Principle of Superposition, any linear combination of  $y_1$  and  $y_2$  is a solution to the system. We want to combine  $y^C$  and  $\bar{y}^C$  in order to get  $\operatorname{Re}(y^C)$  and  $\operatorname{Im}(y^C)$ .

By Euler's Formula:

$$\begin{aligned} y^C &= \begin{bmatrix} 5 \\ 3 - 4i \end{bmatrix} e^{(1+4i)t} = \begin{bmatrix} 5 \\ 3 - 4i \end{bmatrix} e^t e^{4it} = e^t \begin{bmatrix} 5 \\ 3 - 4i \end{bmatrix} (\cos(4t) + i \sin(4t)) \\ &= e^t \begin{bmatrix} 5 \cos(4t) + i 5 \sin(4t) \\ 3 \cos(4t) + i 3 \sin(4t) - i 4 \cos(4t) + 4 \sin(4t) \end{bmatrix} \\ &= e^t \left( \begin{bmatrix} 5 \cos(4t) \\ 3 \cos(4t) + 4 \sin(4t) \end{bmatrix} + i \begin{bmatrix} 5 \sin(4t) \\ 3 \sin(4t) - 4 \cos(4t) \end{bmatrix} \right). \end{aligned}$$

So,

$$y_1 = \operatorname{Re}(y^C) = e^t \begin{bmatrix} 5 \cos(4t) \\ 3 \cos(4t) + 4 \sin(4t) \end{bmatrix} \quad \text{and} \quad y_2 = \operatorname{Im}(y^C) = e^t \begin{bmatrix} 5 \sin(4t) \\ 3 \sin(4t) - 4 \cos(4t) \end{bmatrix}.$$

We found two real-valued solutions to the system. In order to write the general solution, we need to verify that  $\{y_1, y_2\}$  is a fundamental set of solutions (FSS):

$$W[y_1, y_2] = \det \begin{bmatrix} e^t 5 \cos(4t) & e^t 5 \sin(4t) \\ e^t (3 \cos(4t) + 4 \sin(4t)) & e^t (3 \sin(4t) - 4 \cos(4t)) \end{bmatrix}$$

Choose  $t = 0$ , we have  $W[y_1, y_2] = \begin{vmatrix} 5 & 0 \\ 3 & -4 \end{vmatrix} = -20 \neq 0$ . Then,  $\{y_1, y_2\}$  is a FSS, and  $y = c_1 y_1 + c_2 y_2$  is a general solution to the system. □

**Theorem 2.2.25 Summary: Solve Homogeneous System of Linear ODE**

Given  $y' = \mathbf{A}y$ ,  $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ . Then,  $y = xe^{\lambda t}$ , where  $\lambda$  is an eigenvalue of  $\mathbf{A}$  and  $x$  is the corresponding eigenvector. When finding  $\lambda$ , we need to solve  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ . That is,  $\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$ , or in general,  $a\lambda^2 + b\lambda + c = 0$ . Then, we have

$$\lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac$$

- If  $\Delta > 0$ ,  $\lambda_1 \neq \lambda_2$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$ . Then,

$$y_1 = x_1 e^{\lambda_1 t}, \quad y_2 = x_2 e^{\lambda_2 t},$$

and  $y = c_1 y_1 + c_2 y_2$  is a general solution.

- If  $\Delta = 0$ ,  $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ . Then,

$$y_1 = x e^{\lambda t}, \quad y_2 = (tx + w) e^{\lambda t},$$

where  $x$  is such that  $(\mathbf{A} - \lambda\mathbf{I})x = 0$  and  $w$  is such that  $(\mathbf{A} - \lambda\mathbf{I})w = x$ . Meanwhile,  $y = c_1 y_1 + c_2 y_2$  is a general solution.

- If  $\Delta < 0$ , we have  $\lambda_1 = \overline{\lambda_2}$ , and  $\lambda_1, \lambda_2 \in \mathbb{C}$ . Suppose  $y^{\mathbb{C}} = xe^{\lambda t}$ , then  $y_1 = \operatorname{Re}(y^{\mathbb{C}})$  and  $y_2 = \operatorname{Im}(y^{\mathbb{C}})$ , and  $y = c_1 y_1 + c_2 y_2$  is a general solution.

**Example 2.2.26**

- $y' = \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} y$ .

Complex eigenvalues:  $y_1 = \begin{bmatrix} \cos(t/2) \\ -\sin(t/2) \end{bmatrix} e^t$  and  $y_2 = \begin{bmatrix} \sin(t/2) \\ \cos(t/2) \end{bmatrix} e^t$ .

- $y' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} y$ .

Complex eigenvalues:  $y_1 = \begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} e^t$  and  $y_2 = \begin{bmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{bmatrix} e^t$ .

- $y' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} y$ .

Repeated eigenvalues:  $y_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$  and  $y_2 = \left( t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t = \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$ .

- $y' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} y$ .

Complex eigenvalues:  $y_1 = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}$  and  $y_2 = \begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$ .

## 2.3 Phase Portraits

- Setting up:  $\begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ .

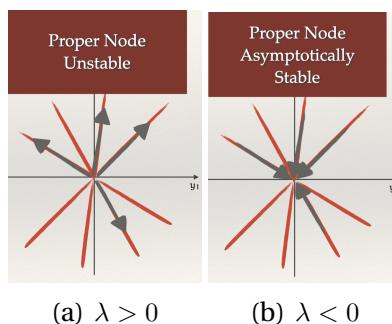
If  $y$  is a constant solution,  $y' = 0$  and  $\mathbf{A}y = 0$ . So, if  $\det(\mathbf{A}) \neq 0$ , we have  $y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

If  $\det(\mathbf{A}) = 0$ , we have infinitely many constant solutions (a line). But we are not interested in constant solutions. If  $y$  is a non-constant, then  $(y_1(t), y_2(t))$  moves along a curve  $C$ , where  $C$  is the trajectory of infinitely many solutions.

- Two trajectories cannot intersect.
  - A trajectory of a nontrivial solution cannot obtain  $(0, 0)$ .
  - $(0, 0)$  is the trajectory of the trivial solution  $y = 0$ .
  - Generally, if  $y = [k_1, k_2]^T$  is a constant solution, its trajectory will be the point  $(k_1, k_2)$ .
- Case of Real Eigenvalues (Non-defective)
- Suppose repeated eigenvalues:  $\lambda_1 = \lambda_2 = \lambda$ :

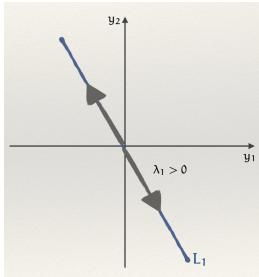
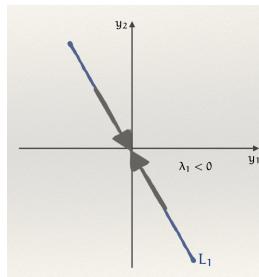
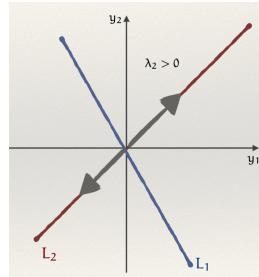
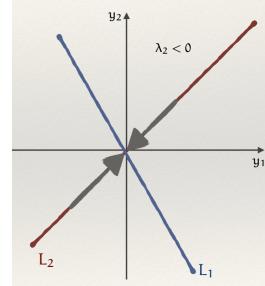
$$y = c_1x_1e^{\lambda t} + c_2x_2e^{\lambda t} = (c_1x_1 + c_2x_2)e^{\lambda t} = xe^{\lambda t}.$$

The trajectories of nontrivial solutions of the system are half-lines through the origin. If  $\lambda > 0$ :  $\lim_{t \rightarrow +\infty} xe^{\lambda t} = +\infty$  and  $\lim_{t \rightarrow -\infty} xe^{\lambda t} = 0$ . So the direction of the trajectories are moving away from the origin. On the other hand, if  $\lambda < 0$ :  $\lim_{t \rightarrow +\infty} xe^{\lambda t} = 0$  and  $\lim_{t \rightarrow -\infty} xe^{\lambda t} = -\infty$ . So the direction of the trajectories are moving towards the origin.



- Suppose  $\lambda_2 > \lambda_1 \neq 0$ . Then,  $y = c_1x_1e^{\lambda_1 t} + c_2x_2e^{\lambda_2 t}$ . Let  $L_1$  be the line parallel to  $x_1$  and  $L_2$  be the line parallel to  $x_2$ .

- (a) If  $c_2 = 0$ , then  $y = c_1 x_2 e^{\lambda_1 t}$ . If  $c_1 \neq 0$ , the trajectory is a half-line of  $L_1$ . If  $\lambda_1 > 0$ , motion is away from the origin. If  $\lambda_1 < 0$ , motion is towards the origin.

(c)  $c_2 = 0; \lambda_1 > 0$ (d)  $c_2 = 0; \lambda_1 < 0$ (e)  $c_1 = 0; \lambda_2 > 0$ (f)  $c_1 = 0; \lambda_2 < 0$ 

- (b) If  $c_1 = 0$ , then  $y = c_2 x_2 e^{\lambda_2 t}$ . If  $c_2 \neq 0$ , the trajectory is a half-line of  $L_2$ . If  $\lambda_2 > 0$ , motion moves away from origin. If  $\lambda_2 < 0$ , motion moves towards origin.

- (c) If  $c_1 \neq 0$  and  $c_2 \neq 0$ , then  $y = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}$ . The trajectory cannot intersect  $L_1$  or  $L_2$ . The trajectories lie entirely in one of the open sectors bounded by  $L_1$  and  $L_2$ . The direction is the same as

- Note that  $e^{-\lambda_2 t} y(t) = c_1 x_1 e^{-(\lambda_2 - \lambda_1)t} + c_2 x_2$ . Therefore as  $t \rightarrow \infty$ , we have  $e^{-\lambda_2 t} y(t) \rightarrow c_2 x_2 \implies$  trajectory is asymptotically parallel to  $L_2$ .
  - Note that  $e^{-\lambda_1 t} y(t) = c_1 x_1 + c_2 x_2 e^{(\lambda_2 - \lambda_1)t}$ . Therefore as  $t \rightarrow -\infty$ , we have  $e^{-\lambda_1 t} y(t) \rightarrow c_1 x_1 \implies$  trajectory is asymptotically parallel to  $L_1$ .
- i. If  $\lambda_2 > \lambda_1 > 0$ .

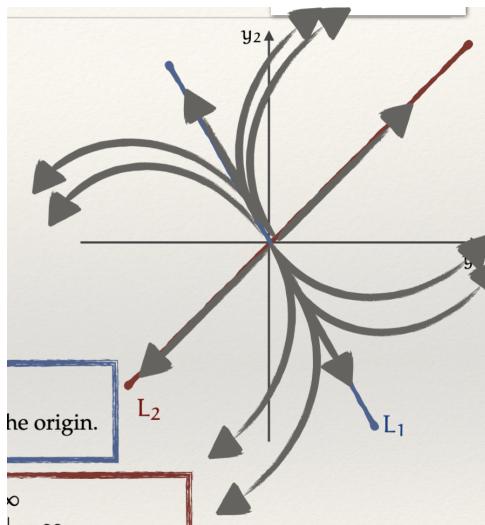


Figure 1: Origin: Unstable Node (Source)

Note that  $\lim_{t \rightarrow -\infty} \|y(t)\| = \lim_{t \rightarrow -\infty} \|c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}\| = 0$ . Therefore, the trajectory is asymptotically tangent to  $L_1$  at origin.

Further note  $\lim_{t \rightarrow +\infty} \|y(t)\| = \lim_{t \rightarrow +\infty} \|c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}\| = \infty$ , and we know

that  $\lim_{t \rightarrow \infty} \|y(t) - c_2 x_2 e^{\lambda_2 t}\| = \lim_{t \rightarrow \infty} \|c_1 x_2 e^{\lambda_1 t}\| = \infty$ . Then, the trajectory is asymptotically parallel to  $L_2$ , but not tangent to  $L_2$ .

**Direction: Away from the origin.**

- ii. If  $0 > \lambda_2 > \lambda_1$ .

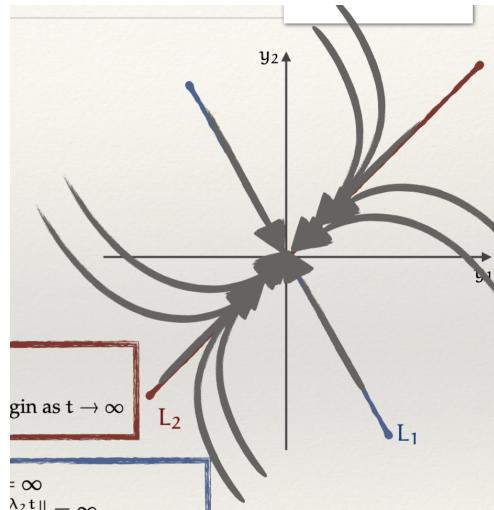


Figure 2: Origin: Asymptotically Stable Node (Sink)

We have  $\lim_{t \rightarrow \infty} \|y(t)\| = \lim_{t \rightarrow \infty} \|c_1 x_2 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}\| = 0$ , so the trajectory is asymptotically to  $L_2$ .

Further we have  $\lim_{t \rightarrow -\infty} \|y(t)\| = \lim_{t \rightarrow -\infty} \|c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}\| = \infty$ , and we have that  $\lim_{t \rightarrow -\infty} \|y(t) - c_1 x_1 e^{\lambda_1 t}\| = \lim_{t \rightarrow -\infty} \|c_2 x_2 e^{\lambda_2 t}\| = \infty$ . So, the trajectory is asymptotically parallel to  $L_1$ .

**Direction: Toward the origin.**

- iii. If  $\lambda_2 > 0 > \lambda_1$

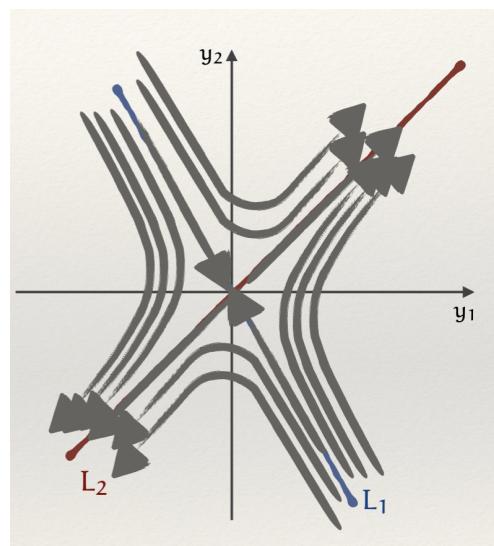


Figure 3: Origin: Unstable Node (Saddle Point)

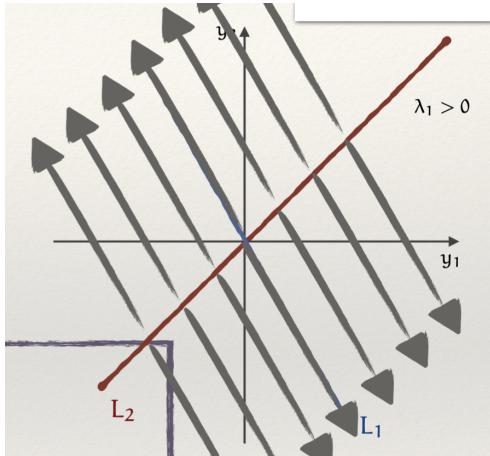
Note that  $\lim_{t \rightarrow \infty} \|y(t)\| = \lim_{t \rightarrow \infty} \|c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}\| = \infty$ , and we know that

$\lim_{t \rightarrow \infty} \|y(t) - c_2 x_2 e^{\lambda_2 t}\| = \lim_{t \rightarrow \infty} \|c_1 x_1 e^{\lambda_1 t}\| = 0$ . Therefore, the trajectory is asymptotically tangent to  $L_2$ .

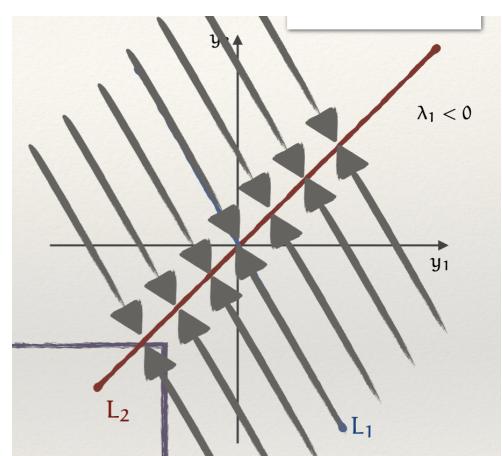
Further we have that  $\lim_{t \rightarrow -\infty} \|y(t)\| = \lim_{t \rightarrow -\infty} \|c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}\| = \infty$ , and we know  $\lim_{t \rightarrow -\infty} \|y(t) - c_1 x_1 e^{\lambda_1 t}\| = \lim_{t \rightarrow -\infty} \|c_2 x_2 e^{\lambda_2 t}\| = 0$ . Therefore, the trajectory is asymptotically tangent to  $L_1$ .

- iv. If  $\lambda_2 = 0, \lambda_1 \neq 0$ . Then,  $y = c_1 x_1 e^{\lambda_1 t} + c_2 x_2$ .

$y = c_2 x_2$  is a constant solution (a line).



(a)  $\lambda_1 > 0$



(b)  $\lambda_1 < 0$

If  $\lambda_1 > 0$ , we have  $\lim_{t \rightarrow \infty} \|y(t)\| = \lim_{t \rightarrow \infty} \|c_1 x_1 e^{\lambda_1 t} + c_2 x_2\| = \infty$ .

If  $\lambda_1 < 0$ , we have  $\lim_{t \rightarrow \infty} \|y(t)\| = \lim_{t \rightarrow \infty} \|c_1 x_1 e^{\lambda_1 t} + c_2 x_2\| = 0$ .

Also, we know  $\lim_{t \rightarrow \infty} \|y(t) - c_1 x_1 e^{\lambda_1 t}\| = \lim_{t \rightarrow \infty} \|c_2 x_2\| = c_2 x_2$ .

We get parallel half-lines because we know all the solutions must be parallel to  $L_1$ .

- Case of Repeated Eigenvalues (Defective):

$$y = c_1 x e^{\lambda t} + c_2 e^{\lambda t} (w + tx).$$

1. Assume  $\lambda \neq 0$ . Call  $L$  the line through origin parallel to  $x$ . If  $c_2 \neq 0$ , the solution is a parametric equation of the half-line of  $L$ .

- In the direction of  $x$  if  $c_1 > 0$ .
- In the direction of  $-x$  if  $c_1 < 0$ .

2. Assume  $c_2 \neq 0$ .

The trajectory cannot intersect  $L$  since every point on  $L$  is on a trajectory obtained by setting  $c_2 = 0$ . The trajectory must lie entirely in one of the open half-planes bounded by  $L$ , but does not obtain any point on  $L$ . Since the initial point  $(y_1(0), y_2(0))$  defined by  $y(0) = c_1 x + c_2 w$  is on the trajectory, we can determine which half-plane contains the trajectory from the sign of  $c_2$ .

We call the half-plane where  $c_2 > 0$  the positive half-plane and  $c_2 < 0$  the negative half-plane.

Multiply the solution by  $e^{-\lambda t}$ , we have

$$e^{-\lambda t}y(t) = c_1x + c_2w + c_2tx.$$

When  $|t|$  is large, the last term is dominant; therefore,

- (a) Along the trajectories in the positive half-plane:

The direction of  $y(t)$  approaches  $x$  as  $t \rightarrow +\infty$ ; the direction of  $y(t)$  approaches  $-x$  as  $t \rightarrow -\infty$ .

- (b) Along the trajectories in the negative half-plane:

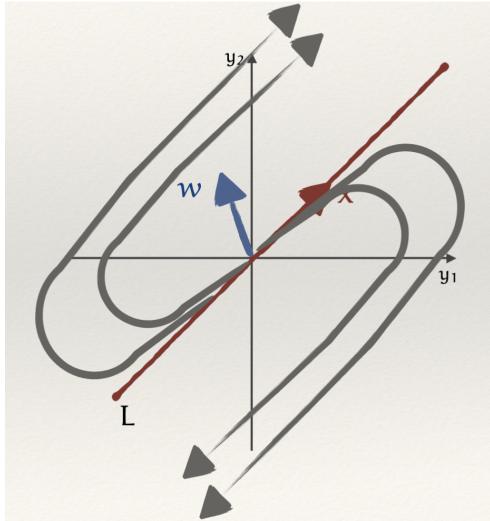
The direction of  $y(t)$  approaches  $-x$  as  $t \rightarrow +\infty$ ; the direction of  $y(t)$  approaches  $x$  as  $t \rightarrow -\infty$ .

3. If  $\lambda > 0$ :

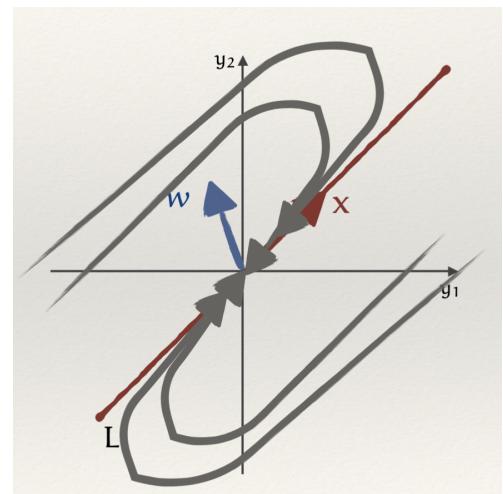
$$\lim_{t \rightarrow \infty} \|axe^{\lambda t} + c_2e^{\lambda t}(w + tx)\| = \infty$$

and

$$\lim_{t \rightarrow -\infty} \|c_1xe^{\lambda t} + c_2e^{\lambda t}(w + tx)\| = 0.$$



(c)  $\lambda > 0$ : Origin: Improper Node (Unstable)



(d)  $\lambda < 0$ : Origin: Improper Node (Asymptotically Stable)

4. If  $\lambda < 0$ :

$$\lim_{t \rightarrow +\infty} \|c_1xe^{\lambda t} + c_2e^{\lambda t}(w + tx)\| = 0$$

and

$$\lim_{t \rightarrow -\infty} \|c_1xe^{\lambda t} + c_2e^{\lambda t}(w + tx)\| = \infty$$

- Case of Complex Eigenvalues:

$$y = e^{\alpha t}[c_1(\cos(\beta t)u - \sin(\beta t)v) + c_2(\cos(\beta t)v + \sin(\beta t)u)],$$

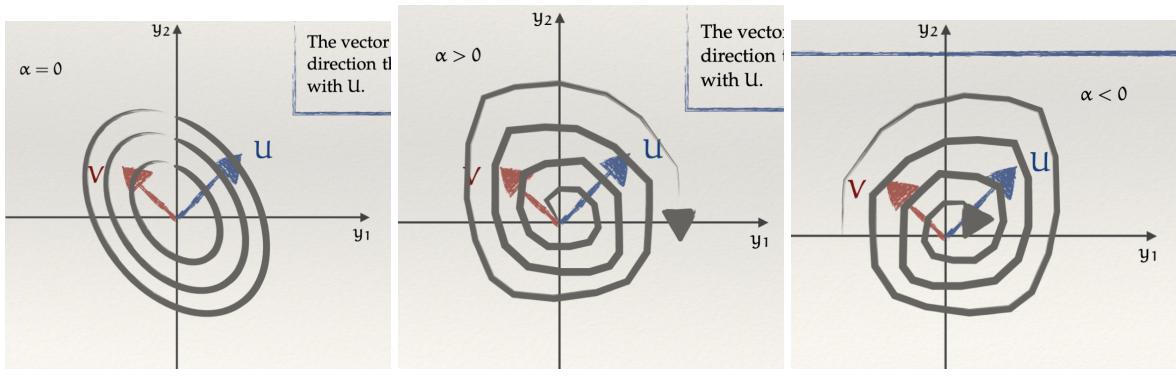
which can be written as

$$y = e^{\lambda t} \{ [c_1 \cos(\beta t) + c_2 \sin(\beta t)]u + [-c_1 \sin(\beta t) + c_2 \cos(\beta t)]v \}.$$

Call  $U = \frac{u}{\|u\|}$  and  $V = \frac{v}{\|v\|}$ , and multiply the solution by  $e^{-\lambda t}$ , and call the curve traversed by  $e^{-\alpha t}y$  a *shadow trajectory*, then

$$e^{-\alpha t}y = z_1U + z_2V,$$

where  $z_1 = \|u\| [c_1 \cos(\beta t) + c_2 \sin(\beta t)]$  and  $z_2 = \|v\| [-c_1 \sin(\beta t) + c_2 \cos(\beta t)]$ . We can verify that  $\frac{z_1^2}{\|u\|^2} + \frac{z_2^2}{\|v\|^2} = c_1^2 + c_2^2$ , meaning the shadow trajectories are ellipses centered at the origin with axes parallel to  $U$  and  $V$ .



(e)  $\alpha = 0$ : Origin: Center (Stable)  
(f)  $\alpha > 0$ : Origin: Spiral Point (Unstable)  
(g)  $\alpha < 0$ : Origin: Spiral Point (Unstable)

## 2.4 Autonomous System of Differential Equations

### 2.4.1 The Predator-Prey Model

#### 1. Setting-ups and Assumptions:

- (a) Rabbits:  $R(t)$  = the number of rabbits.

The rabbits have ample food supplies.

In the absence of predators, the food supply would support exponential growth of the prey:

$$\frac{dR}{dt} = rR, \quad r > 0.$$

- (b) Wolves:  $W(t)$  = the number of wolves.

The wolves feed on the rabbits. In the absence of prey, the predators would decline at a rate which is proportional the actual population:

$$\frac{dW}{dt} = -kW, \quad k > 0.$$

**Theorem 2.4.1 Principle of Mass Action**

Given two populations, the rate of contact between the populations is proportional to the product of their sizes.

2. In the model, we need to consider: (a) the cause of death for the prey are the predators; and (b) the birth rate of the predators depends on the prey. So, we have the following system of ODEs:

$$\begin{cases} \frac{dR}{dt} = rR - aRW \\ \frac{dW}{dt} = -kW + bRW \end{cases}, \quad k, r, a, b > 0.$$

3. **Definition 2.4.2 (Autonomous Systems of ODEs).**

$$\begin{cases} x'(t) = f(x, y), & x = x(t), \text{ population of } x \text{ at time } t \\ y'(t) = g(x, y), & y = y(t), \text{ population of } y \text{ at time } t \end{cases}$$

4. **Definition 2.4.3 (Nullclines).** Given  $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$ , the *x-nullclines* are the curves in the  $xy$ -plane that satisfy  $f(x, y) = 0$ . Along these curves, we have  $x' = 0$ . The *y-nullclines* are the curves in the  $xy$ -plane that satisfy  $g(x, y) = 0$ . Along these curves, we have  $y' = 0$ .

5. **Definition 2.4.4 (Equilibrium Point).** Any point at which an *x*-nullcline intersects a *y*-nullcline is an *equilibrium point*, where both *x* and *y* are not changing.

**Theorem 2.4.5 Phase-Plane Analysis**

1. Set  $f(x, y)$  and  $g(x, y)$  to 0 to find the nullclines and the equilibrium points.
2. Draw nullclines and equilibria on the  $xy$ -plane.
3. Study the signs of functions  $f$  and  $g$ , and draw the combined arrows in each different region of the plane. *At which values of *x* and *y* are functions *f* and *g* increasing or decreasing?*
4. Describe the equilibrium points (extinction of both *x* and *y*, extinction of *x* or *y*, coexistence of *x* and *y*, etc.)
5. Estimate the behavior of  $x(t)$  and  $y(t)$  as  $t \rightarrow \infty$ , and make conclusion about the long term behavior of the populations.

**Example 2.4.6** Consider

$$\begin{cases} R' = rR - aRW \\ W' = -kW + bRW \end{cases}, \quad k, r, a, b > 0.$$

1. Find  $R$ -nullclines:  $R' = rR - aRW = R(r - aW) = 0 \implies R = 0 \text{ or } W = \frac{r}{a}$
2. Find  $W$ -nullclines:  $W' = -kW + bRW = W(bR - k) = 0 \implies W = 0 \text{ or } R = \frac{k}{b}$
3. Find equilibrium points:  $E_1(0, 0)$  and  $E_2\left(\frac{k}{b}, \frac{r}{a}\right)$ .
4. Study sign of  $R'$ :  $R' < 0 \implies rR - aRW > 0 \implies R > 0 \text{ and } W < \frac{r}{a}$ .
5. Study sign of  $W'$ :  $W' > 0 \implies -kW + bRW > 0 \implies W > 0 \text{ and } R > \frac{k}{b}$ .

### 2.4.2 Competing Species

Let  $x, y$  be populations of two species  $X, Y$ . we model  $x, y$  using logistic growth:

$$x' = rx\left(1 - \frac{x}{K}\right), \quad y' = sy\left(1 - \frac{y}{L}\right), \quad r, K, s, L > 0$$

**Example 2.4.7 Model I**

$$x' = 0.05x\left(1 - \frac{x}{20}\right) - 0.002xy \quad y' = 0.09\left(1 - \frac{y}{15}\right) - 0.15xy$$

**Example 2.4.8 Model II**

$$x' = x(0.2 - 0.05x - 0.02y) \quad y' = y(0.1 - 0.01x - 0.02y)$$

**Example 2.4.9 Model III**

$$x' = 6x - (2/3)x^2 - xy \quad y' = 10y - y^2 - 2xy$$

### 2.4.3 SIR Model

**Example 2.4.10** The SIR Model

Let  $S$  be the number of the *susceptible*, the people who are not yet sick but who could become sick,  $I$  the number of the *infected*, the people who are currently sick, and  $R$  the number of *recovered*, the people who have been sick and can no longer infect others or be reinfected.

Since  $S + I + R = n$ , the total population, we only need the populations of  $S$  and  $I$ , and then we can compute  $R$ . Therefore, we form the following system of equations:

$$\frac{dS}{dt} = -aSI \quad \frac{dI}{dt} = aSI - bI$$

Using the phase plane analysis, we can find a threshold value  $S_T$  such that it will maximize  $I$ . Considering the multi-variable chain rule:

$$\frac{dI}{dS} = \frac{dI}{dt} \frac{dt}{dS} = \frac{aSI - bI}{-aSI} = -1 + \frac{b}{aS}$$

Then, we can solve for  $I$  as a function of  $S$  by separation of variables:

$$I(S) = -S + \frac{b}{a} \ln(S) + C.$$

Given some initial conditions, we can easily find  $C$ . By substituting different values of  $S$  we can find the value of  $I$  at any desired time.

## 3 Second Order ODEs

### 3.1 Introduction to Linear Second Order ODEs

**Definition 3.1.1 (Linear Second Order ODEs).** Generally, we can write *linear second order ODEs* in the following form:

$$y'' = f(x, y, y')$$

Echoing what we have did in first order ODEs, we can further write it as

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \quad (6)$$

**Definition 3.1.2 (Homogeneous and Non-homogeneous Second Order ODEs).** In the form of Eq. (6), if  $F(x) = 0$ , then the ODE is *homogeneous*. If  $F(x) \neq 0$ , then the ODEe is *non-homogeneous*.

**Remark.** If  $P_0(x) \neq 0$ , we can rewrite Eq. (6) as

$$y'' + p(x)y' + q(x)y = f(x),$$

where  $p(x) = \frac{P_1}{P_0}$ ,  $q(x) = \frac{P_2}{P_0}$ , and  $f = \frac{F}{P_0}$ .

### 3.2 Homogeneous ODEs

**Definition 3.2.1 (Initial Value Problem/IVP).** We need two initial conditions for an IVP:

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(x_0) = k_0, \quad y'(x_0) = k_1 \end{cases}. \quad (7)$$

#### Theorem 3.2.2 Existence and Uniqueness

Given the IVP  $\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(x_0) = k_0, \quad y'(x_0) = k_1 \end{cases}$  as in Eq. (7). If

- $p(x)$  and  $q(x)$  are continuous on  $(a, b)$ ,
- $x_0 \in (a, b)$ , and
- $k_0, k_1 \in \mathbb{R}$ ,

Then, the solution to the IVP exists and it is unique on  $(a, b)$ .

**Definition 3.2.3 (Trivial Solution).** For a homogeneous second order ODE, we always have this *trivial solution*  $y = 0$ .

**Example 3.2.4** Given the equation  $y'' - y = 0$ . Verify  $y_1 = e^x$  and  $y_2 = e^{-x}$  are solutions of the problem on  $(-\infty, +\infty)$ . Further, show that if  $c_1$  and  $c_2$  are two arbitrary constants, then  $y = c_1 y_1 + c_2 y_2$  is a solution of the equation on  $(-\infty, +\infty)$ . Finally solve the IVP with  $y(0) = 1, y'(0) = 3$ .

**Solution 1.**

$$y'_1 = e^x, \quad y''_1 = e^x \implies y'' - y = e^x - e^x = 0$$

and

$$y'_2 = -e^{-x}, \quad y''_2 = e^{-x} \implies y'' - y = e^{-x} - e^{-x} = 0.$$

Since  $p(x) = 0$  and  $q(x) = -1$  are constants so they are continuous on  $\mathbb{R}$ . Then,  $y_1$  and  $y_2$  are solutions on  $\mathbb{R}$ . Since

$$y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{-x},$$

we have

$$y' = c_1 e^x - c_2 e^{-x}, \quad y'' = c_1 e^x + c_2 e^{-x}.$$

Then,

$$y'' - y = c_1 e^x + c_2 e^{-x} - (c_1 e^x + c_2 e^{-x}) = 0 \quad \forall c_1, c_2 \in \mathbb{R}.$$

To solve the IVP we solve the following system

$$\begin{cases} y(0) = c_1 e^0 + c_2 e^0 = 1 \\ y'(0) = c_1 e^0 - c_2 e^0 = 3 \end{cases} \implies \begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 3 \end{cases} \implies \begin{cases} c_1 = 2 \\ c_2 = -1 \end{cases}$$

□

### Theorem 3.2.5 Principle of Superposition

Given  $y'' + p(x)y' + q(x)y = 0$ . If  $y_1$  and  $y_2$  are two solutions on  $(a, b)$ , then any linear combination of  $y_1$  and  $y_2$ , that is,  $c_1 y_1 + c_2 y_2$ , is a solution of the equation on  $(a, b)$ .

**Proof 2.**

$$y = c_1 y_1 + c_2 y_2 : \quad y' = c_1 y'_1 + c_2 y'_2, \quad y'' = c_1 y''_1 + c_2 y''_2.$$

Then,

$$\begin{aligned} y'' + p(x)y' + q(x)y &= c_1 y''_1 + c_2 y''_2 + p(x)(c_1 y'_1 + c_2 y'_2) + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 y''_1 + c_1 p(x)y'_1 + c_1 q(x)y_1 + c_2 y''_2 + c_2 p(x)y'_2 + c_2 q(x)y_2 \\ &= c_1 \underbrace{(y''_1 + p(x)y'_1 + q(x)y_1)}_{y_1 \text{ is solution} \implies =0} + c_2 \underbrace{(y''_2 + p(x)y'_2 + q(x)y_2)}_{y_2 \text{ is solution} \implies =0} \\ &= 0 \end{aligned}$$

Therefore,  $y = c_1y_1 + c_2y_2$  is a solution. ■

**Definition 3.2.6 (Second Order Linear Constant-Coefficient Homogeneous ODEs).** The following second order linear homogenous ODE is called to have *constant coefficients*:

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}.$$

**Example 3.2.7** Find a solution to  $ay'' + by' + cy = 0$ ,  $a, b, c \in \mathbb{R}$ . Note that not all  $y$ 's is going to work. For example, if  $y = x^2$ ,  $y' = 2x$ , and  $y'' = 2$ . Then,  $2a + 2bx + cx^2 = 0$ . This can never happen for  $x \in \mathbb{R}$ . Inspired by Example 3.2.4, we guess  $y = e^{rx}$  is a solution. Then we have  $y' = re^{rx}$  and  $y'' = r^2e^{rx}$ . Then,  $ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$ . That is, we need

$$(ar^2 + br + c)e^{rx} = 0$$

Since  $e^{rx}$  can never be zero, we need  $p(r) = ar^2 + br + c = 0$ . We call this  $p(r)$  the *characteristic polynomial* of the solution.

**Theorem 3.2.8 Solving  $ay'' + by' + cy = 0$**

To solve  $ay'' + by' + cy = 0$ , we solve the characteristic polynomial  $p(r) = ar^2 + br + c = 0$ .

**Example 3.2.9** Suppose  $p(r) = 0$ , and we have  $r_1, r_2 \in \mathbb{R}$  with  $r_1 \neq r_2$ . (That is, we have two distinct real roots). Then,  $y_1 = e^{r_1 x}$  and  $y_2 = e^{r_2 x}$ . So,  $y = c_1y_1 + c_2y_2 = c_1e^{r_1 x} + c_2e^{r_2 x}$ .

Given the IVP  $\begin{cases} ay'' + by' + cy = 0 \\ y(x_0) = k_0, \quad y'(x_0) = k_1 \end{cases}$ , we have

$$\begin{cases} y(x_0) = c_1y_1(x_0) + c_2y_2(x_0) \\ y'(x_0) + c_1y'_1(x_0) + c_2y'_2(x_0) \end{cases},$$

a system of linear equations with unknowns  $c_1$  and  $c_2$ . We can rewrite the system into the matrix form as follows:

$$\underbrace{\begin{bmatrix} y(x_0) \\ y'(x_0) \end{bmatrix}}_y = \underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_c$$

This system can be solved when  $\det \mathbf{A} \neq 0$ . That is,

$$\det \mathbf{A} = y_1(x_0)y'_2(x_0) - y'_1(x_0)y_2(x_0) \neq 0 \quad (8)$$

We call the condition Eq. (8) the Wronskian of  $y_1$  and  $y_2$ .

**Theorem 3.2.10**

Assume that  $y_1$  and  $y_2$  are two solutions of  $y'' + p(x)y' + q(x)y = 0$  with initial conditions  $y(x_0) = k_0$  and  $y'(x_0) = k_1$ . Then, it is always possible to choose constants  $c_1$  and  $c_2$  such that  $y = c_1y_1 + c_2y_2$  satisfy the IVP if and only if

$$\mathbf{W}[y_1, y_2] \Big|_{x_0} \neq 0$$

**Theorem 3.2.11**

Suppose that  $y_1$  and  $y_2$  are two solutions of  $y'' + p(X)y' + q(x)y = 0$ . Then,  $y = c_1y_1 + c_2y_2$  is the general solution of the equation (contains all possible solutions) if and only if  $\mathbf{W}[y_1, y_2] \Big|_{x_0} \neq 0$  at some  $x_0$ .

**Definition 3.2.12 (Fundamental Set of Solutions).** If the general solution of  $y'' + p(x)y' + q(x)y = 0$  can be written as  $y = c_1y_1 + c_2y_2$  on  $(a, b)$ , then the set  $\{y_1, y_2\}$  is called the *fundamental set of solutions*.

**Theorem 3.2.13 Abel's Formula**

Suppose that  $p, q$  are continuous on  $(a, b)$ . Let  $y_1$  and  $y_2$  be two solutions of  $y'' + p(x)y' + q(x)y = 0$  on  $(a, b)$ . Define  $\mathbf{W} = y_1y'_2 - y'_1y_2$ . Let  $x_0 \in (a, b)$ . Then,

$$\mathbf{W}(x) = \mathbf{W}(x_0)e^{-\int_{x_0}^x p(t) dt} \quad (9)$$

Therefore, either  $\mathbf{W}$  has no zeros on  $(a, b)$  or  $\mathbf{W} = 0$  on  $(a, b)$ .

**Proof 3.** Eq. (9) looks like a solution to first order homogenous ODE. So, we will prove the formula by constructing a differential equation.

$$\begin{aligned} \mathbf{W} &= y_1y'_2 - y'_1y_2 \\ \mathbf{W}' &= y'_1y'_2 + y_1y''_2 - y''_1y_2 - y'_1y'_2 \\ &= y_1y''_2 - y''_1y_2 \\ &= y_1(-py'_2 - qy_2) - (-py'_1 - qy_1)y_2 & y''_1 + py'_1 + qy_1 = 0 \\ &= -py_1y'_2 - qy_1y_2 + py'_1y_2 + qy_1y_2 \\ &= -p(y_1y'_2 - y'_1y_2) \\ &= -p\mathbf{W} \end{aligned}$$

When solving  $\mathbf{W}' = -p\mathbf{W}$ , we will have Eq. (9). ■

**Corollary 3.2.14:** If  $\mathbf{W}[y_1, y_2] \neq 0$  at one point of  $x_0$ ,  $\mathbf{W}[y_1, y_2] \neq 0$  for all  $x$ 's in the interval.

**Proof 4.** From Theorem 3.2.13,  $\mathbf{W}(x) = \mathbf{W}(x_0)e^{-\int_{x_0}^x p(t) dt}$ . Since  $e^{-\int_{x_0}^x p(t) dt} > 0$ , we know  $\mathbf{W}(x) = 0 \iff \mathbf{W}(x_0) = 0$ . ■

**Theorem 3.2.15**

Suppose that  $p$  and  $q$  are continuous on  $(a, b)$ , and let  $y_1$  and  $y_2$  be solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad \text{on } (a, b) \quad (10)$$

Then, the following statements are equivalent:

1. The general solution of Eq. (10) on  $(a, b)$  is  $y = c_1y_1 + c_2y_2$ .
2.  $\{y_1, y_2\}$  is a fundamental set of solutions (FSS) for Eq. (10) on  $(a, b)$ .
3.  $y_1$  and  $y_2$  are linearly independent on  $(a, b)$ .
4. The Wronskian of  $\{y_1, y_2\}$  is nonzero at some point in  $(a, b)$ .
5. The Wronskian of  $\{y_1, y_2\}$  is nonzero at all points in  $(a, b)$ .

**Example 3.2.16** Find the solution of the equation  $y'' + 6y' + 5y = 0$  given the initial conditions  $y(0) = 3, y'(0) = -1$ .

**Solution 5.**

The characteristic polynomial is  $r^2 + 6r + 5r = 0 \implies (r+1)(r+5) = 0$ . So,  $r_1 = -1$  and  $r_2 = -5$ . Therefore, we have  $y_1 = e^{-x}$  and  $y_2 = e^{-5x}$ . Then,

$$y = c_1y_1 + c_2y_2 = c_1e^{-x} + c_2e^{-5x}$$

is still a solution, and  $y' = -c_1e^{-x} - 5c_2e^{-5x}$ . Then, by the initial condition:

$$\begin{cases} y(0) = c_1e^0 + c_2e^0 = c_1 + c_2 = 3 \\ y'(0) = -c_1e^0 - 5c_2e^0 = -c_1 - 5c_2 = -1 \end{cases} \implies \begin{cases} c_1 = 7/2 \\ c_2 = -1/2 \end{cases}$$

So,

$$y^{\text{IVP}} = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$$

□

**Example 3.2.17** Find the solution of the equation  $y'' + 5y' + 6y = 0$  given the initial conditions  $y(0) = 2, y'(0) = 3$ .

**Solution 6.**

The characteristic polynomial:  $r^2 + 5r + 6 = 0 \implies (r+2)(r+3) = 0$ . So,  $r_1 = -2$  and  $r_2 = -3$ . So,  $y_1 = e^{-2x}$  and  $y_2 = e^{-3x}$ . Therefore,

$$y = c_1y_1 + c_2y_2 = c_1e^{-2x} + c_2e^{-3x},$$

and then  $y' = -2c_1e^{-2x} - 3c_2e^{-3x}$ . By the initial condition, we have

$$\begin{cases} y(0) = c_1e^0 + c_2e^0 = c_1 + c_2 = 2 \\ y'(0) = -2c_1e^0 - 3c_2e^0 = -2c_1 - 3c_2 = 3 \end{cases} \implies \begin{cases} c_1 = 9 \\ c_2 = -7 \end{cases}$$

So,

$$y^{\text{IVP}} = 9e^x - 7e^{-6x}.$$

□

**Example 3.2.18** Find the general solution of the equation  $y'' + 6y' + 9y = 0$  with initial conditions  $y(0) = 3, y'(0) = -1$ .

**Solution 7.**

The characteristic polynomial:  $r^2 + 6r + 9 = 0 \implies (r + 3)^2 = 0 \implies r_1 = r_2 = -3$ . In this case, we have repeated roots. Suppose  $y = ue^{-3x}$ , where  $u$  is a function of  $x$ .

$$\begin{aligned} y' &= u'e^{-3x} - 3ue^{-3x} = (u' - 3u)e^{-3x} \\ y'' &= (u'' - 3u')e^{-3x} - 3(u' - 3u)e^{-3x} \\ &= (u'' - 6u' + 9u)e^{-3x} \end{aligned}$$

So,

$$\begin{aligned} y'' + 6y' + 9y &= (u'' - 6u' + 9u)e^{-3x} + 6[(u' - 3u)e^{-3x}] + 9ue^{-3x} \\ &= (u'' - 6u' + 9u + 6u' - 18u + 9u)e^{-3x} \\ &= u''e^{-3x} \\ &= 0 \end{aligned}$$

Since  $e^{-3x} > 0 \quad \forall x \in \mathbb{R}$ , it must be  $u'' = 0 \implies u' = c_2 \implies u = c_2x + c_1$ . So,

$$y = ue^{-3x} = (c_2x + c_1)e^{-3x} = c_1e^{-3x} + c_2xe^{-3x}.$$

Let  $y_1 = e^{-3x}$  and  $y_2 = xe^{-3x}$ . Then,

$$\begin{aligned} \mathbf{W}[y_1, y_2] &= \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{vmatrix} \\ &= e^{-3x}(e^{-3x} - 3xe^{-3x}) - (-3e^{-3x})(xe^{-3x}) \\ &= e^{-6x} - 3xe^{-6x} + 3xe^{-6x} \\ &= e^{-6x} \neq 0 \quad \forall x \in \mathbb{R}. \end{aligned}$$

So,  $y = c_1e^{-3x} + c_2xe^{-3x}$  is the general solution, and  $y' = -3c_1e^{-3x} + c_2(e^{-3x} - 3xe^{-3x})$ .

Therefore, by the initial condition, we have

$$\begin{cases} y(0) = c_1 = 3 \\ y'(0) = -3c_1 + c_2 = -1 \end{cases} \implies \begin{cases} c_1 = 3 \\ c_2 = 8 \end{cases}$$

Therefore, solution to the IVP is given by

$$y^{\text{IVP}} = 3e^{-3x} + 8xe^{-3x} = (3 + 8x)e^{-3x}.$$

□

**Example 3.2.19** Find the general solution of the equation  $y'' + 4y' + 13y = 0$ .

**Solution 8.**

The characteristic polynomial:  $r^2 + 4r + 13 = 0 \implies r^2 + 4r + 4 + 9 = 0$  So we have  $(r + 2)^2 = -9 \implies r + 2 = \pm -3i \implies r_{1,2} = -2 \pm 3i$ . In this case, we have complex roots:  $y_1^C = e^{(-2+3i)x}$  and  $y_2^C = e^{(-2-3i)x}$ . Key: any linear combination of  $y_1^C$  and  $y_2^C$  is a solution. So, we can choose  $\frac{y_1^C + y_2^C}{2} = \text{Re}(y_1^C)$  and  $\frac{y_1^C - y_2^C}{2i} = \text{Im}(y_1^C)$ .

$$\begin{aligned} y_1^C &= e^{(-2+3i)x} = e^{-2x}[\cos(3x) + i\sin(3x)] \\ &= e^{-2x}\cos(3x) + ie^{-2x}\sin(3x) \end{aligned}$$

Therefore,

$$y_1 = \text{Re}(y_1^C) = e^{-2x}\cos(3x) \quad \text{and} \quad y_2 = \text{Im}(y_1^C) = e^{-2x}\sin(3x)$$

Compute the Wronskian:

$$\begin{aligned} \mathbf{W}[y_1, y_2] &= \begin{vmatrix} e^{-2x}\cos(3x) & e^{-2x}\sin(3x) \\ -2e^{-2x}\cos(3x) - 3e^{-2x}\sin(3x) & -2e^{-2x}\sin(3x) + 3e^{-2x}\cos(3x) \end{vmatrix} \\ &= e^{-2x}\cos(3x)[-2e^{-2x}\sin(3x) + 3e^{-2x}\cos(3x)] + \\ &\quad - e^{-2x}\sin(3x)[-2e^{-2x}\sin(3x) - 3e^{-2x}\cos(3x)] \\ &= e^{-4x}[-2\cos(3x)\sin(3x) + 3\cos^2(3x)] + \\ &\quad - e^{-4x}[-2\cos(3x)\sin(3x) - 3\sin^2(3x)] \\ &= e^{-4x}(-2\cos(3x)\sin(3x) + 3\cos^2(3x) + 2\cos(3x)\sin(3x) + 3\sin^2(3x)) \\ &= 3e^{-4x}(\sin^2(3x) + \cos^2(3x)) \\ &= 3e^{-4x} \neq 0 \quad \forall x \in \mathbb{R}. \end{aligned}$$

Therefore,  $y_1$  and  $y_2$  are fundamental set of solutions, and the general solution is given by  $y = c_1y_1 + c_2y_2 = c_2e^{-2x}\cos(3x) + c_2e^{-2x}\sin(3x)$ . □

**Theorem 3.2.20 Summary I: Solving Constant-Coefficient Second Order Homogeneous ODE**

Let  $p(r) = ar^2 + br + c$  be the characteristic polynomial associated to the equation  $ay'' + by' + cy = 0$ . Then,

- If  $p(r) = 0$  has two real distinct roots  $r_1, r_2$ , then the general solution of the equation is  $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ .

**Proof 9.** Verify that  $y_1$  and  $y_2$  are linearly independent.

$$\begin{aligned}\mathbf{W}[y_1, y_2] &= \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x} \\ &= (r_2 - r_1) e^{(r_1+r_2)x}\end{aligned}$$

Since  $e^{(r_1+r_2)x} > 0 \quad \forall x \in \mathbb{R}$ ,  $\mathbf{W} \neq 0 \iff r_2 - r_1 \neq 0$ . As, by assumption,  $r_1 \neq r_2$ ,  $r_2 - r_1 \neq 0$ , and thus  $\mathbf{W} \neq 0$ . So,  $y_1$  and  $y_2$  are linearly independent. ■

- If  $p(r) = 0$  has two repeated real roots  $r_1 = r_2 = r$ , then the general solution of the equation is  $y = c_1 e^{rx} + c_2 x e^{rx}$ .
- If  $p(r) = 0$  has two complex conjugated roots  $r_1 = \bar{r}_2 = \alpha + i\beta$ , then the general solution of the equation is  $y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$ .

### 3.3 Method of Undetermined Coefficients

**Theorem 3.3.1 Existence of Solutions**

Suppose that  $p, q, f$  are continuous on  $(a, b)$ . Let  $x_0 \in (a, b)$ , and let  $k_0, k_1$  be arbitrary real numbers. Then, the initial value problem

$$\begin{cases} y'' + p(x)y' + q(x)y = f(x) \\ y(x_0) = k_0, \quad y'(x_0) = k_1 \end{cases}$$

has a unique solution on  $(a, b)$ .

**Definition 3.3.2 (Complementary Equation).** The *complementary equation* of  $y'' + p(x)y' + q(x)y = f(x)$  is given by  $y'' + p(x)y' + q(x)y = 0$ .

**Theorem 3.3.3 Solution to Non-Homogeneous Constant Coefficient Second Order Equations**

Suppose that  $p, q$ , and  $f$  are continuous on  $(a, b)$ . Let  $y_p$  be a *particular solution* of  $y'' + p(x)y' + q(x)y = f(x)$ . Let  $\{y_1, y_2\}$  be a fundamental set of solutions of the complementary equation  $y'' + p(x)y' + q(x)y = 0$  on  $(a, b)$ . Then,  $y$  is a solution if and only if  $y = y_p + c_1 y_1 + c_2 y_2$ .

**Proof 1.** To prove this theorem, we need to prove the following two things:

- Show that  $y_p + c_1y_1 + c_2y_2$  satisfies the equation.
- Show that  $y - y_p$  satisfies the complementary equation.

■

**Example 3.3.4** Find the general solution of  $y'' - 2y' + y = -3 - x + x^2$ .

**Solution 2.**

By the method of undetermined coefficients, suppose

$$y_p = A + Bx + Cx^2.$$

So,  $y'_p = B + 2Cx$  and  $y''_p = 2C$ . Therefore, we have

$$\begin{aligned} y'' - 2y' + y &= 2C - 2(B + 2Cx) + A + Bx + Cx^2 \\ &= 2C - 2B + 4Cx + A + Bx + Cx^2 \\ &= (A - 2B + 2C) + (B - 4C)x + Cx^2 \\ &= -3 - x + x^2 \end{aligned}$$

So, we have the following system:

$$\begin{cases} A - 2B + 2C = -3 \\ B - 4C = -1 \\ C = 1 \end{cases} \implies \begin{cases} A = 1 \\ B = 3 \\ C = 1 \end{cases}$$

Thus,

$$y_p = 1 + 3x + x^2.$$

Solve the complementary equation:  $y'' - 2y' + y = 0$ . Consider the characteristic polynomial:  $r^2 - 2r + 1 = 0 \implies (r - 1)^2 = 0 \implies r_1 = r_2 = 1$ . So,  $y_1 = e^x$  and  $y_2 = xe^x$ . Therefore, the general solution is

$$y = c_1y_1 + c_2y_2 + y_p = c_1e^x + c_2xe^x + 1 + 3x + x^2.$$

□

**Example 3.3.5** Given the equation  $y'' - 7y' + 12y = 4e^{2x}$ . Find a particular solution and then find the general solution.

**Solution 3.**

By the method of undetermined coefficients, suppose  $y_p = Ae^{2x}$ , and then we have

$y'_p = 2Ae^{2x}$  and  $y''_p = 4Ae^{2x}$ . So,

$$\begin{aligned} 4Ae^{2x} - 7(2Ae^{2x}) + 12Ae^{2x} &= 4e^{2x} \\ 4A - 14A + 12A &= 4 \\ 2A = 4 &\implies A = 2. \end{aligned}$$

So,  $y_p = 2e^{2x}$ . Now, consider the complementary equation:  $y'' - 7y' + 12y = 0$ . Then, we have the characteristic polynomial  $r^2 - 7r + 12 = 0 \implies (r - 3)(r - 4) = 0$ . Then, we have  $r_1 = 3$  and  $r_2 = 4$ . So,  $y_1 = e^{3x}$  and  $y_2 = e^{4x}$ . Check linearly independence:

$$\mathbf{W}[y_1, y_2] = \begin{vmatrix} e^{3x} & e^{4x} \\ 3e^{3x} & 4e^{4x} \end{vmatrix} = 4e^{3x}e^{4x} - 3e^{3x}e^{4x} = e^{7x} \neq 0.$$

So,  $y_1$  and  $y_2$  are linearly independent, and the general solution is

$$y = c_1y_1 + c_2y_2 + y_p = c_1e^{3x} + c_2e^{4x} + 2e^{2x}.$$

□

**Example 3.3.6** Given the equation  $y'' - 7y' + 12y = 5e^{4x}$ . Find a particular solution and then the general solution.

**Solution 4.**

Suppose  $y_p = Ae^{4x}$ . Then,  $y'_p = 4Ae^{4x}$  and  $y''_p = 16Ae^{4x}$ . So,

$$\begin{aligned} 16Ae^{4x} - 7(4Ae^{4x}) + 12Ae^{4x} &= 5e^{4x} \\ 16A - 28A + 12A &= 5 \\ 0 \cdot A &= 5 \end{aligned}$$

This conclusion indicate that  $e^{4x}$  is a solution in the fundamental solution set. So, we need to try something else:  $y_p = ue^{4x}$ , where  $u$  is a function of  $x$ . Then,

$$\begin{aligned} y'_p &= u'e^{4x} + 4ue^{4x} = e^{4x}(u' + 4u) \\ y''_p &= u''e^{4x} + 4u'e^{4x} + 4u'e^{4x} + 16ue^{4x} \\ &= e^{4x}(u'' + 8u' + 16u) \end{aligned}$$

So,

$$\begin{aligned} e^{4x}(u'' + 8u' + 16u) - 7e^{4x}(u' + 4u) + 12ue^{4x} &= 5e^{4x} \\ u'' + 8u' + 16u - 7u' - 28u + 12u = 5 &\implies u'' + u' = 5 \end{aligned}$$

Guess  $u'_p = 5, u''_p = 0 \implies 0 + u'_p = 5$  So,  $u_p = 5x$ . Therefore,  $y_p = ue^{4x} = 5xe^{4x}$ .

The complementary equation and the characteristic polynomial are the same from the previous question. So,

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 e^{3x} + c_2 e^{4x} + 5xe^{4x}.$$

□

**Example 3.3.7** Given the equation  $y'' - 8y' + 16y = 2e^{4x}$ . Find a particular solution and the general solution.

**Solution 5.**

Complementary equation:  $y'' - 8y' + 16y = 0$ . Consider the characteristic polynomial

$$r^2 - 8r + 16 = 0 \implies (r - 4)^2 = 0 \implies r_1 = r_2 = 4.$$

So, FSS =  $\{e^{4x}, xe^{4x}\}$ . Now, guess  $y_p = ue^{4x}$ . Then, we have

$$\begin{aligned} y'_p &= u'e^{4x} + 4ue^{4x} = e^{4x}(u' + 4u) \\ y''_p &= u''e^{4x} + 4u'e^{4x} + 4u'e^{4x} + 16ue^{4x} \\ &= e^{4x}(u'' + 8u' + 16u) \end{aligned}$$

So, we have

$$\begin{aligned} y'' - 8y' + 16y &= e^{4x}(u'' + 8u' + 16u) - 8e^{4x}(u' + 4u) + 16ue^{4x} = 2e^{4x} \\ u'' + 8u' + 16u - 8u' - 32u + 16u &= 2 \\ u'' &= 2. \end{aligned}$$

So,  $u'' = 2 \implies u'_p = 2x \implies u_p = x^2$ . Therefore,  $y_p = u_p e^{4x} = x^2 e^{4x}$ . Thus, the general solution is

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 e^{3x} + c_2 e^{4x} + x^2 e^{4x}.$$

**Remark.** If we find the general solution to  $u'' = 2$ :

$$u' = 2x + c_2 \implies u = x^2 + c_2 x + c_1$$

Then we have

$$y = ue^{4x} = (x^2 + c_2 x + c_1)e^{4x},$$

exactly the general solution.

□

**Theorem 3.3.8 Summary II: Method of Undetermined Coefficients**

If  $f(x) = e^{\alpha x}$ , when finding the particular solution to

$$y'' + p(x)y' + q(x)y = f(x),$$

we have the following situations when apply the method of undetermined coefficients:

- If  $e^{\alpha x}$  is not in the FSS, then  $y_p = Ae^{\alpha x}$ .
- If  $e^{\alpha x} \in \text{FSS}$ , but  $xe^{\alpha x} \notin \text{FSS}$ , then  $y_p = Axe^{\alpha x}$ .
- If  $e^{\alpha x}, xe^{\alpha x} \in \text{FSS}$ , but  $x^2e^{\alpha x} \notin \text{FSS}$ , then  $y_p = Ax^2e^{\alpha x}$ .

If we want to solve  $ay'' + by' + cy = f(x)$  with the method of undetermined coefficient, in general, we suppose  $f(x) = P(x)e^x$ . With  $\deg(P) = \deg(Q)$ , we have

- $y_p = Q(x)e^{\alpha x}$  if  $e^{\alpha x} \notin \text{FSS}$
- $y_p = xQ(x)e^{\alpha x}$  if  $e^{\alpha x} \in \text{FSS}$  and  $xe^{\alpha x} \notin \text{FSS}$ .
- $y_p = x^2Q(x)e^{\alpha x}$  if  $e^{\alpha x}, xe^{\alpha x} \in \text{FSS}$ .

**Theorem 3.3.9 Superposition**

Suppose  $y_{p_1}$  is a particular solution of  $y'' + py' + qy = f_1$  and  $y_{p_2}$  is a particular solution of  $y'' + py' + qy = f_2$ . Then,  $y_{p_1} + y_{p_2}$  is a particular solution of  $y'' + py' + qy = f_1 + f_2$ .

**Example 3.3.10** Find a particular solution of  $y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}$ .

**Solution 6.**

$2e^{2x}$  is a particular solution of  $y'' - 7y' + 12y = 4e^{2x}$ , and  $5xe^{4x}$  is a particular solution of  $y'' - 7y' + 12y = 5e^{4x}$ . Then,  $y_p = 2e^{2x} + 5xe^{4x}$  is a particular solution of  $y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}$ .  $\square$

**Example 3.3.11** Find the general solution of  $y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2)$ .

**Solution 7.**

Suppose  $y_p = e^{3x}(-A + Bx + Cx^2) \implies y_p = ue^{3x}$ , where  $u$  is a function of  $x$ . So,

$$\begin{aligned} y'_p &= 3ue^{3x} + u'e^{3x} \\ y''_p &= 9ue^{3x} + 3u'e^{3x} + u''e^{3x} + 3u'e^{3x} \\ &= u''e^{3x} + 6u'e^{3x} + 9ue^{3x} \end{aligned}$$

So, we have

$$\begin{aligned} u''e^{3x} + 6u'e^{3x} + 9ue^{3x} - 3(3ue^{3x} + u'e^{3x}) + 2ue^{3x} &= e^{3x}(-1 + 2x + x^2) \\ u'' + 6u' + 9u - 9u - 3u' + 2u &= -1 + 2x + x^2 \\ u'' + 3u' + 2u &= -1 + 2x + x^2. \end{aligned}$$

We don't have to find the general solution. Instead, we just need to find a particular solution.

Let  $u_p = A + Bx + Cx^2 \implies u'_p = B + 2Cx \implies u''_p = 2C$ . Then,

$$\begin{aligned} 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) &= -1 + 2x + x^2 \\ 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2 &= -1 + 2x + x^2 \\ (2A + 3B + 2C) + (2B + 6C)x + 2Cx^2 &= -1 + 2x + x^2 \end{aligned}$$

So, we have the following system of equations:

$$\left\{ \begin{array}{l} 2A + 3B + 2C = -12 \\ B + 6C = 2 \\ 2C = 1 \end{array} \right. \implies \left\{ \begin{array}{l} A = -1/4 \\ B = -1/2 \\ C = 1/2 \end{array} \right.$$

Therefore,  $u_p = -\frac{1}{4} - \frac{1}{2}x + \frac{1}{2}x^2$ , and then

$$y_p = e^{3x} \left( -\frac{1}{4} - \frac{1}{2}x + \frac{1}{2}x^2 \right)$$

The complementary equation is  $y'' - 3y' + 2y = 0$ , and the characteristic polynomial is  $r^2 - 3r + 2 = 0 \implies (r - 1)(r - 2) = 0 \implies r_1 = 1, r_2 = 2$ . So, the general solution is

$$y = c_1y_1 + c_2y_2 + y_p = c_1e^x + c_2e^{2x} + e^{3x} \left( -\frac{1}{4} - \frac{1}{2}x + \frac{1}{2}x^2 \right).$$

□

**Example 3.3.12** Find the general solution of  $y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2)$ .

**Solution 8.**

The complementary equation is  $y'' - 4y' + 3y = 0$ , and thus the characteristic polynomial is  $r^2 - 4r + 3r = 0 \implies (r - 1)(r - 3) = 0 \implies r_1 = 1, r_2 = 3$ . So, FSS =  $\{e^x, e^{3x}\}$ . Suppose  $y_p = ue^{3x}$ . Then, we can find the first and second derivatives of  $y_p$

to be  $y'_p = 3ue^{3x} + u'e^{3x} = e^{3x}(3u + u')$  and

$$\begin{aligned} y''_p &= 9ue^{3x} + 3u'e^{3x} + u''e^{3x} + 3u'e^{3x} \\ &= u''e^{3x} + 6u'e^{3x} + 9ue^{3x} \\ &= e^{3x}(u'' + 6u' + 9u). \end{aligned}$$

So,

$$\begin{aligned} e^{3x}(u'' + 6u' + 9u) - 4e^{3x}(3u + u') + 3ue^{3x} &= e^{3x}(6 + 8x + 12x^2) \\ u'' + 6u' + 9u - 12u - 4u' + 3u &= 6 + 8x + 12x^2 \\ u'' + 2u' &= 6 + 8x + 12x^2. \end{aligned}$$

Suppose  $u'_p = A + Bx + Cx^2$ , then  $u''_p = B + 2Cx$ . So,

$$\begin{aligned} (B + 2Cx) + 2(A + Bx + Cx^2) &= 6 + 8x + 12x^2 \\ B + 2Cx + 2A + 2Bx + 2Cx^2 &= 6 + 8x + 12x^2 \\ (2A + B) + (2B + 2C)x + 2Cx^2 &= 6 + 8x + 12x^2 \end{aligned}$$

So, we have the following system:

$$\begin{cases} 2A + B = 6 \\ 2B + 2C = 8 \\ 2C = 12 \end{cases} \implies \begin{cases} A = 4 \\ B = -2 \\ C = 6 \end{cases}$$

So,  $u'_p = 4 - 2x + 6x^2 \implies u_p = 4x - x^2 + 2x^3$ . Then,  $y_p = u_p e^{3x} = (4x - x^2 + 2x^3)e^{3x}$ . So, the general solution is

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 e^x + c_2 e^{3x} + (4x - x^2 + 2x^3)e^{3x}.$$

□

**Example 3.3.13** Find the general solution of  $y'' - 2y' + y = 5 \cos(2x) + 10 \sin(2x)$ .

**Solution 9.**

The complementary equation is  $y'' - 2y' + y = 0$ , and so the characteristic polynomial is  $r^2 - 2r + 1 = 0 \implies (r - 1)^2 = 0 \implies r_1 = r_2 = 1$ . So, FSS =  $\{e^x, xe^x\}$ . So, suppose  $y_p = A \cos(2x) + B \sin(2x)$ . Then,

$$\begin{aligned} y'_p &= -2A \sin(2x) + 2B \cos(2x) \\ y''_p &= -4A \cos(2x) - 4B \sin(2x) \end{aligned}$$

So,

$$\begin{aligned}
 y_p'' - 2y_p' + y_p &= [-4A \cos(2x) - 4B \sin(2x)] - 2[-2A \sin(2x) + 2B \cos(2x)] \\
 &\quad + [A \cos(2x) + B \sin(2x)] \\
 &= (-4A - 4B + A) \cos(2x) + (-4B + 4A + B) \sin(2x) \\
 &= (-3A - 4B) \cos(2x) + (4A - 3B) \sin(2x) \\
 &= 5 \cos(2x) + 10 \sin(2x)
 \end{aligned}$$

So, we have the following system:

$$\begin{cases} -3A - 4B = 5 \\ 4A - 3B = 10 \end{cases} \implies \begin{cases} A = 1 \\ B = -2 \end{cases} \implies y_p = \cos(2x) - 2s \sin(2x)$$

So, the general solution is

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 e^x + c_2 x e^x + \cos(2x) - 2 \sin(2x).$$

□

**Example 3.3.14** Find the general solution of  $y'' + 4y = 8 \cos(2x) + 12 \sin(2x)$ .

**Solution 10.**

The complementary equation is  $y'' + 4y = 0$ , and so the characteristic polynomial is  $p(r) = r^2 + 4 = 0 \implies r^2 = -4 \implies r = \pm 2i$ . Since  $e^{2xi} = \cos(2x) + i \sin(2x)$ , we know the FSS is  $\{\cos(2x), \sin(2x)\}$ . Suppose  $y_p = x(A \cos(2x) + B \sin(2x))$ . Then,

$$\begin{aligned}
 y_p' &= A \cos(2x) + B \sin(2x) + x(-2A \sin(2x) + 2B \cos(2x)) \\
 &= (A + 2Bx) \cos(2x) + (B - 2Ax) \sin(2x) \\
 y_p'' &= 2B \cos(2x) - 2(A + 2Bx) \sin(2x) - 2A \sin(2x) + 2(B - 2Ax) \cos(2x) \\
 &= (2B - 4Ax + 2B) \cos(2x) + (-2A - 4Bx - 2A) \sin(2x) \\
 &= (4B - 4Ax) \cos(2x) + (-4A - 4Bx) \sin(2x).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 y'' + 4y &= (4B - 4Ax) \cos(2x) + (-4A - 4Bx) \sin(2x) \\
 &\quad + 4x(A \cos(2x) + B \sin(2x)) \\
 &= (4B - 4Ax + 4Ax) \cos(2x) + (-4A - 4Bx + 4Bx) \sin(2x) \\
 &= 4B \cos(2x) - 4A \sin(2x) \\
 &= 8 \cos(2x) + 12 \sin(2x)
 \end{aligned}$$

So, we have the following system

$$\begin{cases} 4B = 8 \\ -4A = 12 \end{cases} \implies \begin{cases} A = -3 \\ B = 2 \end{cases} \implies y_p = x[-3\cos(2x) + 2\sin(2x)].$$

So, the general solution is

$$\begin{aligned} y = c_1y_1 + c_2y_2 + y_p &= c_1\cos(2x) + c_2\sin(2x) + x[-3\cos(2x) + 2\sin(2x)] \\ &= c_1\cos(2x) + c_2\sin(2x) - 3x\cos(2x) + 2x\sin(2x). \end{aligned}$$

□

**Example 3.3.15** Find the general solution of  $y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x$ .

**Solution 11.**

The complementary equation is  $y'' + 3y' + 2y = 0$ , and the characteristic polynomial is  $p(r) = r^2 + 3r + 2 = 0 \implies (r+1)(r+2) = 0 \implies r_1 = -1, r_2 = -2$ . So, we know that the FSS =  $\{e^{-x}, e^{-2x}\}$ .

**Remark.** We might consider to set our initial guess as  $y_p = (A + Bx)\cos x + C\sin x$ . However, it will not work since we will end up with 3 parameters with 4 conditions. Therefore, the rule is treat  $\cos x$  and  $\sin x$  in the same way. So, the true guess should be  $y_p = (Ax + B)\cos x + (Cx + D)\sin x$ .

Suppose  $y_p = (Ax + B)\cos x + (Cx + D)\sin x$ . Then, we have

$$\begin{aligned} y'_p &= A\cos x - (Ax + B)\sin x + C\sin x + (Cx + D)\cos x \\ &= (Cx + A + D)\cos x + (-Ax - B + C)\sin x \\ y''_p &= C\cos x - (Cx + A + D)\sin x + (-A)\sin x + (-Ax - B + C)\cos x \\ &= (-Ax - B + C + C)\cos x + (-Cx - A - D - A)\sin x \\ &= (-Ax - B + 2C)\cos x + (-Cx - 2A - D)\sin x \end{aligned}$$

So,

$$\begin{aligned} y''_p + 3y'_p + 2y_p &= (-Ax - B + 2C)\cos x + (-Cx - 2A - D)\sin x \\ &\quad + 3((Cx + A + D)\cos x + (-Ax - B + C)\sin x) \\ &\quad + 2((Ax + B)\cos x + (Cx + D)\sin x) \\ &= (-Ax - B + 2C + 3Cx + 3A + 3D + 2Ax + 2B)\cos x \\ &\quad + (-Cx - 2A - D - 3Ax - 3B + 3C + 2Cx + 2D)\sin x \\ &= ((A + 3C)x + 3A + B + 2C + 3D)\cos x \\ &\quad + ((C - 3A)x - 2A - 3B + 3C + D)\sin x \end{aligned}$$

So, we have the following system:

$$\begin{cases} A + 3C = 20 \\ 3A + B + 2C + 3D = 16 \\ C - 3A = 0 \\ -2A - 3B + 3C + D = 10 \end{cases} \implies \begin{cases} A = 2 \\ B = 1 \\ C = 6 \\ D = -1 \end{cases}$$

Therefore, the general solution is

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 e^{-x} + c_2 e^{-2x} + (2x + 1) \cos x + (6x - 1) \sin x.$$

□

## 3.4 Variable Coeff Nonhomo Second Order ODE

### 3.4.1 Method of Reduction of Order (RoO)

#### Theorem 3.4.1 Method of Reduction of Order

Given the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x), \quad (11)$$

and a solution  $y_1 = y_1(x)$  of its *complementary equation* ( $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ ). Then the general solution of Eq. (11) has the form  $y = uy_1$ .

**Proof 1.** Given  $y = uy_1$ , we have  $y' = u'y_1 + uy'_1$  and

$$y'' = u''y_1 + u'y'_1 + u'y'_1 + uy''_1 = u''y_1 + 2u'y'_1 + uy''_1.$$

So, Eq. (11) becomes

$$\begin{aligned} P_0(u''y_1 + 2u'y'_1 + uy''_1) + P_1(u'y_1 + uy'_1) + P_2(uy_1) &= F \\ (P_0y_1)u'' + (2P_0y'_1 + P_1y_1)u' + \underbrace{(P_0y''_1 + P_1y'_1 + P_2y_1)}_{=0} u &= F \\ (P_0y_1)u'' + (2P_0y'_1 + P_1y_1)u' &= F \end{aligned}$$

Let  $Q_0 := P_0y_1$  and  $Q_1 := 2P_0y'_1 + P_1y_1$ , so we want to solve  $Q_0u'' + Q_1u' = F$ . Set  $z = u'$  and  $z' = u''$ , we have  $Q_0z' + Q_1z = F \implies z' + \frac{Q_1}{Q_0}z = \frac{F}{Q_0}$ . By the method of integrating factor,

$$\mu(x) = e^{\int Q_1/Q_0 dx}.$$

So,

$$z(x) = e^{-\int \frac{Q_1}{Q_0} dx} \left[ \int \frac{F}{Q_0} e^{\int \frac{Q_1}{Q_0} dx} dx + c_2 \right] = u'(x)$$

Therefore,  $u(x) = \int z(x) dx + c_1$ . So, the general solution is

$$y = u(x)y_1(x) = y_1(x) \int z(x) dx + c_1 y_1(x).$$

■

**Example 3.4.2** Given the equation

$$xy'' - (2x + 1)y' + (x + 1)y = x^2$$

verify that  $y_1 = e^x$  is a solution of the complementary equation and find the general solution of the equation.

**Solution 2.**

- Verify  $y_1 = e^x$  is a solution of the complementary equation. We have  $y_1 = e^x$ ,  $y'_1 = e^x$ , and  $y''_1 = e^x$ . As the complementary equation is  $xy'' - (2x + 1)y' + (x + 1)y = 0$ , consider

$$\begin{aligned} xy''_1 - (2x + 1)y'_1 + (x + 1)y_1 &= xe^x - (2x + 1)e^x + (x + 1)e^x \\ &= 2xe^x - 2xe^x - e^x + e^x \\ &= 0. \end{aligned}$$

- Compute a  $u(x)$  such that  $y = uy_1 = ue^x$ . So, we have  $y' = u'e^x + ue^x$  and

$$y'' = u''e^x + u'e^x + u'e^x + ue^x = u''e^x + 2u'e^x + ue^x.$$

So, we have

$$\begin{aligned} x(u''e^x + 2u'e^x + ue^x) - (2x + 1)(u'e^x + ue^x) + (x + 1)(ue^x) &= x^2 \\ xe^x u'' + (2xe^x - 2xe^x - e^x)u' + \underbrace{(xe^x - 2xe^x - e^x + e^x)}_0 u &= x^2 \\ e^x(xu'' - u') &= x^2 \\ xu'' - u' &= x^2 e^{-x}. \end{aligned}$$

From the original equation, we know that  $x \neq 0$ . So,  $u'' - \frac{1}{x}u' = xe^{-x}$ . Set  $z = u'$  and so  $z' = u''$ . Then,

$$z' - \frac{1}{x}z = xe^{-x} \implies \mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln(x)} = \frac{1}{x}.$$

So,

$$u(x) = \int -xe^{-x} + c_2 x dx = e^{-x}(x + 1) + \frac{1}{2}c_2 x^2 + c_1 = e^{-x}(x + 1) + c_2 x^2 + c_1.$$

Then, the general solution is

$$y = u(x)y_1(x) = e^x(e^{-x}(x+1) + c_2x^2 + c_1) = c_1e^x + c_2x^2e^x + (x+1).$$

- From the general solution, we know FSS =  $\{e^x, x^2e^x\}$  and a particular solution  $y_p = x + 1$ . We can verify these as well.

□

**Example 3.4.3** Given the equation

$$x^2y'' + xy' - y = x^2 + 1.$$

Verify that  $y_1 = x$  is a solution of the complementary equation and find the general solution of the equation.

**Solution 3.**

Note that  $y_1 = x$ ,  $y'_1 = 1$ ,  $y''_1 = 0$ . So,  $x^2y''_1 + xy'_1 - y_1 = x^2(0) + x - x = 0$ , and thus it is a solution of the complementary equation.

Find  $y = ux$ , where  $u$  is a function of  $x$ . Then,  $y' = u + u'x$  and

$$y'' = u' + u''x + u' = 2u' + u''x.$$

So, the original equation becomes

$$\begin{aligned} x^2y'' + xy' - y &= x^2(2u' + u''x) + x(u + u'x) - ux \\ &= x^3u'' + (2x^2 + x^2)u' \\ &= x^3u'' + 3x^2u' = x^2 + 1. \end{aligned}$$

Set  $z = u'$  and  $z' = u''$ . So, we have

$$x^3z' + 3x^2z = x^2 + 1 \implies z' + \frac{3}{x}z = \frac{x^2 + 1}{x^3} \implies \mu(x) = e^{\int \frac{3}{x} dx} = x^3.$$

So,

$$\begin{aligned} z\mu(x) &= \int \mu(x) \cot \frac{x^2 + 1}{x^3} dx = \int x^2 + 1 dx + c_2 \\ z \cdot x^3 &= \frac{1}{3}x^3 + x + c_2 \\ z(x) &= \frac{1}{3} + x^{-2} + c_2x^{-3}. \end{aligned}$$

So,

$$u(x) = \int z(x) dx = \frac{1}{3}x - x^{-1} - c_2e^{-2} + c_1.$$

Therefore, the general solution is

$$y(x) = u(x)y_1(x) = x \left( \frac{1}{3}x - x^{-1} - c_2x^{-2} + c_1 \right) = c_1x - \frac{c_2}{x} + \left( \frac{1}{3}x^2 - 1 \right).$$

□

### 3.4.2 Method of Variation of Parameters (VoP)

#### Theorem 3.4.4 Method of Variation of Parameters

Given the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y_1 = F(x),$$

and two solutions  $y_1 = y_1(x)$  and  $y_2 = y_2(x)$  of its complementary equation, then a particular solution to the equation has the form  $y = u_1y_1 + u_2y_2$ .

**Proof 4.** Since  $y = u_1y_1 + u_2y_2$ , we have  $y' = u'_1y_1 + u_2y'_1 + u'_2y_2 + u_2y'_2$ . Set  $\boxed{u'_1y_1 + u'_2y_2 = 0}$ .

Therefore,

$$y' = u_1y'_1 + u_2y'_2 \implies y'' = u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2.$$

So, the original equation becomes

$$\begin{aligned} P_0(u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2) + P_1(u_1y'_1 + u_2y'_2) + P_2(u_1y_1 + u_2y_2) &= F \\ P_0(u'_1y'_1 + u'_2y'_2) + P_0(u_1y''_1 + u_2y''_2) + P_1(u_1y'_1 + u_2y'_2) + P_2(u_1y_1 + u_2y_2) &= F \\ P_0(u'_1y'_1 + u'_2y'_2) + u_1 \underbrace{(P_0y''_1 + P_1y'_1 + P_2y_1)}_0 + u_2 \underbrace{(P_0y''_2 + P_1y'_2 + P_2y_2)}_0 &= F \\ P_0(u'_1y'_1 + u'_2y'_2) &= F \\ u'_1y'_1 + u'_2y'_2 &= \frac{F}{P_0} \end{aligned}$$

Hence, solving the original differential equation becomes solving the following system:

$$\begin{cases} u'_1y_1 + u'_2y_2 = 0 & \textcircled{1} \\ u'_1y'_1 + u'_2y'_2 = F/P_0 & \textcircled{2} \end{cases}$$

Consider  $\textcircled{1} \cdot y'_2$  :  $u'_1y_1y'_2 + u'_2y_2y'_2 = 0$   $\textcircled{3}$  and  $\textcircled{2} \times y_2$  :  $u'_1y'_1y_2 + u'_2y_2y'_2 = F/P_0 \cdot y_2$   $\textcircled{4}$

Then, we know that ③ – ④ :  $u'_1 y_1 y'_2 - u'_1 y'_1 y_2 = -\frac{F}{P_0} y_2$

$$u'_1 \left[ \underbrace{y_1 y'_2 - y'_1 y_2}_{= \mathbf{W}[y_1, y_2] \neq 0 \text{ as } y_1, y_2 \in \text{FSS}} \right] = -\frac{F}{P_0} y_2$$

$$u'_1 = -\frac{F}{P_0 \cdot \mathbf{W}} y_2 \implies \boxed{u_1(x) = - \int \frac{F(x)}{P_0(x) \mathbf{W}[y_1, y_2]} y_2(x) dx}$$

Similarly, we can find

$$u'_2 = \frac{F y_1}{P_0(y_1 y'_2 - y'_1 y_2)} \implies \boxed{u_2(x) = \int \frac{F(x)}{P_0(x) \mathbf{W}[y_1, y_2]} y_1(x) dx}.$$

So, the general solution is

$$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2.$$

■

**Example 3.4.5** Given the equation

$$x^2 y'' - 2xy' + 2y = x^{9/2}, \quad x \neq 0$$

and two solutions  $y_1 = x$  and  $y_2 = x^2$  of the complementary equation, find the general solution.

**Solution 5.**

The general solution is

$$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2 = c_1 x + c_2 x^2 + u_1 x + u_2 x^2.$$

The Wronskian is

$$\mathbf{W}[x, x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0 \quad \forall x \in \mathbb{R} \setminus \{0\}.$$

So,

$$u_1(x) = - \int \frac{x^{9/2}}{x^2 \cdot x^2} x^2 dx = - \int x^{5/2} dx = -\frac{2}{7} x^{7/2}.$$

$$u_2(x) = \int \frac{x^{9/2}}{x^2 \cdot x^2} \cdot x dx = \int x^{3/2} dx = \frac{2}{5} x^{5/2}.$$

So, the particular solution is

$$\begin{aligned} y_p &= u_1 x + u_2 x^2 = -\frac{2}{7}x^{7/2} \cdot x + \frac{2}{5}x^{5/2}x^2 \\ &= -\frac{2}{7}x^{9/2} + \frac{2}{5}x^{9/2} \\ &\quad + \left(\frac{2}{5} - \frac{2}{7}\right)x^{9/2} = \frac{4}{35}x^{9/2}. \end{aligned}$$

Then, the general solution is given by

$$y = c_1 x + c_2 x^2 + \frac{4}{35}x^{9/2}.$$

□

**Remark.** When given two solutions from the FSS, we can choose to use either RoO or VoP. With constant coefficient equations, RoO and VoP can also be used in addition to method of undetermined coefficients.

## 3.5 Application of Second Order ODEs

### 3.5.1 Vibrating Springs

**Example 3.5.1** Assume no Friction.

We can have the following equation

$$F = -kx, \quad k > 0.$$

Since we know  $F = ma = -kx$  and  $a = x''$ , then

$$mx'' + kx = 0,$$

which is a second order, linear, constant coefficient, homogeneous ODE.

Attempt to solve this ODE, we consider the characteristic polynomial.

$$p(r) = mr^2 + k = 0 \implies r^2 = -k/m \implies r_{1,2} = \pm i\sqrt{\frac{k}{m}}.$$

Let  $\omega := \sqrt{\frac{k}{m}}$ , we have

$$x^C = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t).$$

So,

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

This is called the *simple harmonic motion*.

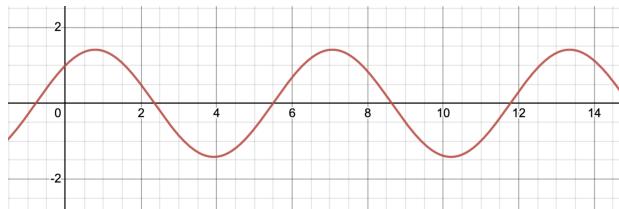


Figure 4: The Simple Harmonic Motion

**Example 3.5.2** Take Friction into Account.

If we take friction into account, then we should have the following equation

$$F = -kx - cv \quad \xrightarrow{v=x'} \quad F = -kx - cx'$$

Similarly, since  $F = ma = -kx - cx'$ , we have

$$mx'' + cx' + kx = 0.$$

To solve this ODE, we consider the characteristic polynomial.

$$p(r) = mr^2 + cr + k = 0 \implies r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \quad \xrightarrow{\Delta := c^2 - 4mk} \quad r_{1,2} = \frac{-c \pm \sqrt{\Delta}}{2m}.$$

**Case 1** Overdamping.  $\Delta > 0$ , then  $r_1 = \frac{-c + \sqrt{\Delta}}{2m}$  and  $r_2 = \frac{-c - \sqrt{\Delta}}{2m}$ . So,

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

If  $r_1$  or  $r_2 > 0$ , then  $x \rightarrow \infty$  when  $t \rightarrow \infty$ . That is impossible. So, we need  $r_1, r_2 < 0$ . Since  $-c - \sqrt{\Delta} < 0$  for sure, we need

$$\sqrt{\Delta} = \sqrt{c^2 - 4mk} < \sqrt{c^2} = c \implies -c + \sqrt{\Delta} < 0.$$

So, as  $t \rightarrow \infty$ ,  $x(t) \rightarrow 0$ .

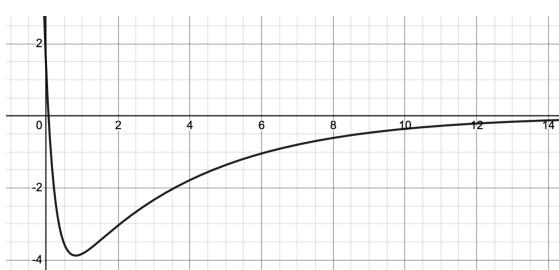
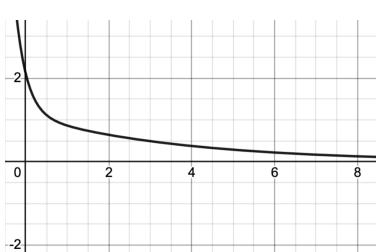


Figure 5: Overdamping

**Case 2** Critical Damping.  $\Delta = 0$ , then  $r_1 = r_2 = r = \frac{-c}{2m}$ . So,

$$x(t) = c_1 e^{rt} + c_2 x e^{rt} = (c_1 + c_2 x) e^{-\frac{c}{2m} t}$$

**Case 3** Underdamping.  $\Delta < 0$ , then  $r_1 = \frac{-c + i\sqrt{-\Delta}}{2m}$  and  $r_2 = \frac{-c - i\sqrt{-\Delta}}{2m}$ . Let  $\omega := \frac{\sqrt{-\Delta}}{2m}$ , then

$$r_1 = \frac{-c + i\sqrt{-\Delta}}{2m} = \frac{-c}{2m} + i\frac{-\Delta}{2m} = \frac{-c}{2m} + i\omega.$$

So,

$$x^C(t) = e^{r_1 t} = e^{(-\frac{c}{2m} + i\omega)t} = e^{-\frac{c}{2m}t} [\cos(\omega t) + i \sin(\omega t)].$$

Therefore,

$$x(t)[c_1 \cos(\omega t) + c_2 \sin(\omega t)] e^{-\frac{c}{2m}t}.$$

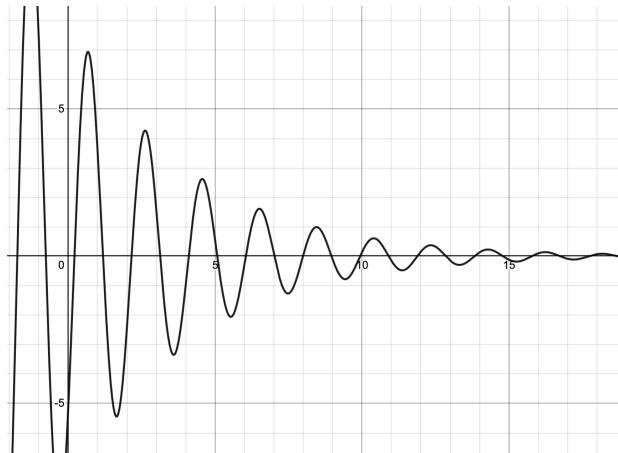


Figure 6: Underdamping

**Example 3.5.3** If we have an external force, then we will have the following non-homogeneous system.

$$mx'' + cx' + kx = F(t).$$

**Example 3.5.4** A spring with a mass of 2kg has natural length 0.5m. A force of 25.6N is required to maintain it stretched to a length of 0.7m. If the spring is stretched to a length of 0.7m and then released with initial velocity 0, find the position of the mass at any time  $t$ .

**Solution 1.**

We attempt to solve

$$mx'' + kx = 0.$$

From the question, we know

$$\begin{aligned} F &= -kx \quad k > 0 \\ -25.6 &= -k(0.7 - 0.5) \\ k &= \frac{25.6}{0.2} = 128. \end{aligned}$$

Given:  $m = 2, \quad k = 128, \quad x(0) = 0.7 - 0.5 = 0.2, \quad x'(0) = 0.$

$$2x'' + 128x = 0 \implies x'' + 64x = 0.$$

The characteristic polynomial:  $p(r) = 2r^2 + 128 = 0 \implies r_{1,2} = \pm 8i.$  So,

$$x^C = r^{r_1 t} = e^{8it} = \cos(8t) + i \sin(8t)$$

That is,

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t).$$

So,

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

We solve the following system:

$$\begin{cases} x(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 0.2 \\ x'(0) = -8c_1 \sin(0) + 8c_2 \cos(0) = 8c_2 = 0 \end{cases} \implies \begin{cases} c_1 = 0.2 \\ c_2 = 0 \end{cases}$$

Then,

$$x^{\text{IVP}}(t) = 0.2 \cos(8t).$$

□

### 3.5.2 Circuit with Resistor, Inductors, and Capacitors

The general equation is

$$LI' + RI = \frac{Q}{C} = E(t).$$

Since  $I = Q'$ , we can also write the equation to be

$$LQ'' + RQ' + \frac{Q}{C} = E(t).$$

Therefore, depending on the question (what are given), we could have a first order or a second order equation.