Spherical Bessel function and Spherical Associated Legendre function.

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Introduction

- I encounter these eqs. when I was solving the Helmholtz eq for EM wave of radiation.
- Suppose the E wave and B wave in the following forms:
- Propagate in radial direction.

$$\vec{E} = E_0(r,\theta,\phi)e^{ikt}\hat{r} \qquad \vec{B} = B_0(r,\theta,\phi)e^{ikt}\hat{r}$$

- By Maxwell’s eqs.: $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} \to (\nabla^2 + k^2)\vec{E} = 0 \qquad \nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2} \to (\nabla^2 + k^2)\vec{B} = 0$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- So that’s the Helmholtz eq.

Solve Helmholtz eq.

- Change rectangular coordinate to spherical coordinate:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(r^2 \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

- By separate variable, $E(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$
- Then Helmholtz eq. becomes

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta r^2 \sin\theta}\frac{d}{d\theta}\left(r^2 \frac{d}{d\theta}\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{d^2\Phi}{d\phi^2} + k^2 = 0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d}{\sin\theta d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + k^2 r^2 = 0$$

Solution for θ and ϕ part

- The first two for Φ component are independent of phi, so

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2 \rightarrow \Phi(\phi) = e^{-im\phi}$$

$$\left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2\right] + \left[\frac{1}{\Theta}\frac{d}{\sin\theta d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\right] = 0$$

- Then solve θ component

$$\frac{1}{\Theta}\frac{d}{\sin\theta d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -n(n+1)$$

- Since we’ve known the solution of this part is associated Legendre, so that’ our assumption.

Solution for θ and ϕ part

- Let $x = \cos\theta$, $y = \Theta$, then the ODE becomes

$$(1-x^2)y''-2xy'+[n(n+1)-\frac{m^2}{1-x^2}]=0$$

- Where $0 \le \theta \le \pi$, $-1 \le y \le 1$, $n \in Z^+$, $-n \le m \le n$.
- Its solution is $P_n^m(\cos\theta)$, m is the order, n is n^{th} solution.
- The normalized form is

$$\tilde{P}_n^m(\cos\theta)=\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)$$

Solution for radial part

- Finally, for r component, $\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2r^2-n(n+1)=0$

- This is spherical Bessel's function, there are 2 solution:

$$j_n(r)=\sqrt{\frac{\pi}{2r}}J_{n+\frac{1}{2}}(r) \qquad y_n(r)=\sqrt{\frac{\pi}{2r}}Y_{n+\frac{1}{2}}(r)$$

- *J* and *Y* are ordinary Bessel's and Neumann's function, where *j* is real number, *y* is imaginary number, so the solution is

$$h_n(r)=j_n(r)\pm iy_n(r)$$

- *h* is Spherical Hankle function.

Solution

- Finally, substitute the solution to original E, we get

$$E(r,\theta,\phi) = h_n(r)\tilde{P}_n^m(\cos\theta)e^{\pm im\phi} = \frac{h_n(r)}{\sqrt{2\pi}}\tilde{Y}_n^m(\cos\theta) \qquad \tilde{Y}_n^m(\cos\theta) = \tilde{P}_n^m(\cos\theta)e^{\pm im\phi}$$

- Here $Y_n^m(\cos\theta)$ is called "Spherical Harmonic function", and its normalized form is

$$\tilde{Y}_n^m(\cos\theta) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}Y_n^m(\cos\theta)$$

- A little different from P because of ϕ component.

Bessel and Neumann function

- Here we construct j_n and y_n by their 1st and 2nd order function, then use recurrence relation for higher order n $(2 \leq n)$

$$j_0 = \frac{\sin x}{x} \qquad y_0 = -\frac{\cos x}{x}$$

$$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad y_1 = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

- recurrence relation: $f_{n+1}(z) = \frac{2n+1}{z} f_n(z) - f_{n-1}(z)$, f_n correspond to j_n or y_n

Ref: http://people.math.sfu.ca/~cbm/aands/page_361.htm

Derivative of Bessel and Neumann function

- Their derivative can be represent by ordinary form, too. In reference paper eq 9.1.27, there are 4 formula, here we choose the last one because it uses *n* and *n*+1 order, *n* won't be negative.

$$f'_n(z) = -f_{n+1}(z) + \frac{n}{z} f_n(z)$$

Ref: http://people.math.sfu.ca/~cbm/aands/page_361.htm

Associated Legendre function

- This function is more complex than previous, because there are 2 index *m* and *n*, but we still use the *m* = 0 and *m* = 1 order to construct the higher order.
- For simplify, 1st we calculate *m* = *n*

$$\left.\begin{aligned} P_n^m(x) &= \sqrt{(1-x^2)^m}\,\frac{d^m}{dx^m}P_n(x) \\ P_n(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \end{aligned}\right\} \quad P_m^m(x) = \sqrt{\frac{2n+1}{2(2m)!}}(2m-1)!!\sqrt{(1-x^2)^m}$$

Associated Legendre function

- Next we calculate $m \le n$.

$$(n-m)P_n^m(x) = x(2n-1)P_{n-1}^m(x) - (n+m-1)P_{n-2}^m(x)$$

$$P_n^m(x) = x\frac{2n-1}{n-m}P_{n-1}^m(x) - \frac{n+m-1}{n-m}P_{n-2}^m(x)$$

- So that we get the $0 \le m$ Associated Legendre function.
- For m is negative, there is also a recurrence relation:

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!}P_n^m(x)$$

Derivative Associated Legendre function

- Like Bessel and Neumann function, there is also a recurrence relation of its derivative.

$$\frac{d}{dx}P_n^m(x) = \frac{-x(n+1)}{x^2-1}P_n^m(x) + \frac{(n-m+1)}{x^2-1}P_{n+1}^m(x)$$

- Much information in the ref.

Ref: numerical recipes in c p252