Fibonacci number

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Introduction

- Fibonacci number F_n is defined by Fibonacci sequence $F_n = F_{n\text{-}2}+F_{n\text{-}1}$
- $F_0 = 0$, $F_1 = 1$, $F_2 = 1$
- It applies in many region, like mathematics, computer science, etc.
- The sequence can be expressed as a matrix form

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \rightarrow F_{k+1} = AF_k$$

- The eigenvalues A are $\lambda_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$
- The corresponding eigenvectors are $u_{1,2} = \begin{bmatrix} \dfrac{1 \pm \sqrt{5}}{2} \\ 1 \end{bmatrix}$

Derive the general form

- So that A is diagonalizable and can be written as
  $A = PDP^{-1} \to F_1 = AF_0 \to F_{n+1} = AF_n = PD^n P^{-1} F_0$

$$P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \quad P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

Derive the general form

$$D^n = \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \qquad F_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Derive the general form

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n} \end{bmatrix} = \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n} & \left(\frac{1-\sqrt{5}}{2}\right)^{n} \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n} & \left(\frac{1-\sqrt{5}}{2}\right)^{n} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} & * \\ * & * \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Derive the general form

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \Rightarrow F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

- Computing Fibonacci number is a good example for starter practicing to the call function itself, just needs a recursive, but the easiest way needs the most time consuming. So the applications use the iterative or dynamic programming.

C code recursive

- int Fibonacci(int n)
- {

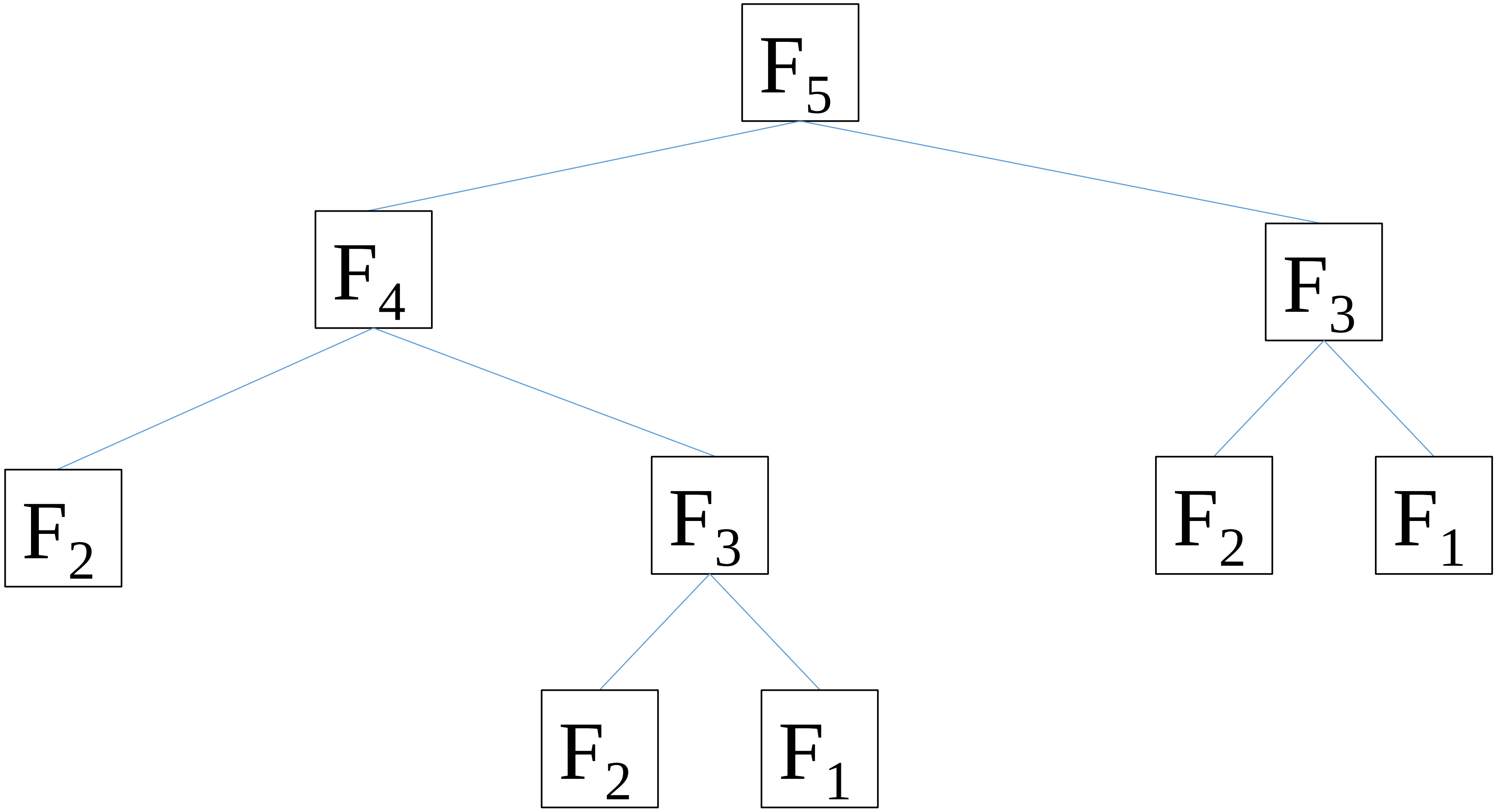
```
if(n == 0) return n;
else if(n == 1 || n == 2)
{
      int k = 1;
      return k;
}
else return F(n-1) + F(n-2);
```

- }

Run time analyze recursive

In analysis, we just consider $n \geq 3$, and $n = 3$ is the 1st term.
Suppose $n = 5$, the we need to know F_4 and F_3, by computing F_4, we need to know F_3 and F_2 and so on.
The process ends when the last term is 2 or 1.

Run time analyze recursive



The summation is like the general form of F_n, so the run-time is

$$\left(\frac{1+\sqrt{5}}{2}\right)^n$$

C code iterative

```
int Fibonacci(int n, int F[])
{
        int i;
        F[0] = 1;F[1] = 1;F[2] = 2;
        for(i=3;i<n;i++)
        {
                F[i] = F[i-1] + F[i-2];
        }
}
```

Run time analyze iterative

The iterative method just uses a for loop, so the run-time is *n*