lecture 08
Rigid Body Motions

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Modern Robotics Ch. 3.1-3.2

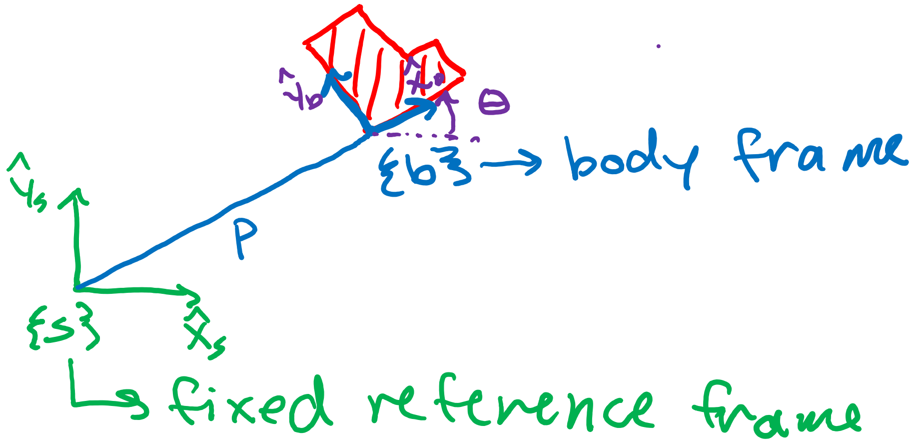
Admin

- HW4 is the PF coding assignment!
- HW5 will be posted soon
- Reflection 1 is due Sunday 9/29 at midnight

Who was Benjamin Olinde Rodrigues?

- A mathematician who turned to finance for employment
  - PhD in math and became a banker
- His dissertation introduced the Rodrigues formula
  - Formerly known as the Ivory–Jacobi formula, as it was independently introduced by Olinde Rodrigues (1816), Sir James Ivory (1824) and **Carl Gustav Jacobi (1827)**

Frames of Reference



body frame origin p can be expressed: $p = p_x \hat{x}_s + p_y \hat{y}_s$

to describe the orientation of {b} we use θ to write:

$$\hat{x}_b = \cos\theta\,\hat{x}_s + \sin\theta\,\hat{y}_s$$

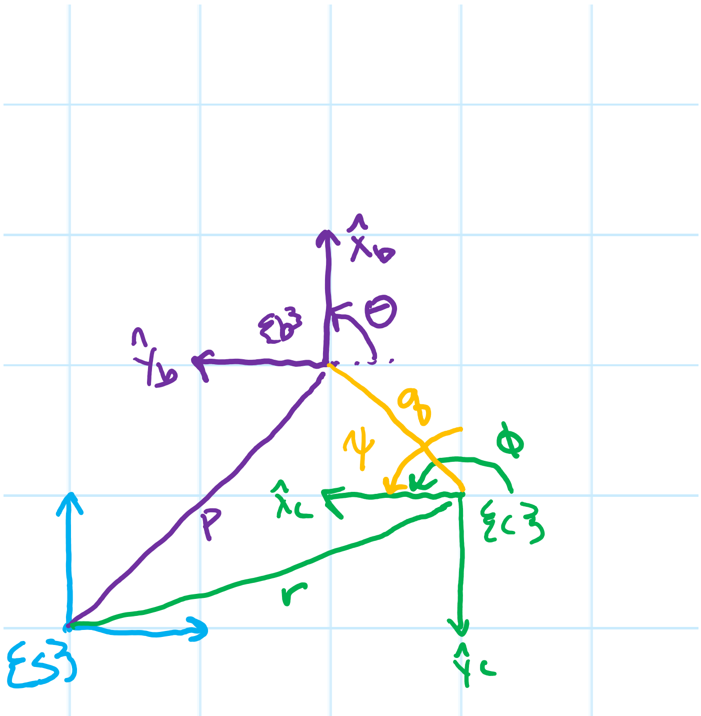
$$\hat{y}_b = -\sin\theta\,\hat{x}_s + \cos\theta\,\hat{y}_s$$

to express everything in terms of {s}, we need

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \quad \& \quad P = \begin{bmatrix} \hat{x}_b & \hat{y}_b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

column vectors

Changing frames



The pair (P, p), parameterized by θ, provide a description of pos + ori of $\{b\}$ relative to $\{s\}$

for $\{c\}$ relative to $\{s\}$, we write (R, r):

$$r = \begin{bmatrix} r_x \\ r_y \end{bmatrix}, \quad R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

for $\{c\}$ in $\{b\}$ coordinates:

$$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix}, \quad Q = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

if we know (Q, q) + (P, p), we can compute $\{c\}$ relative to $\{s\}$:

$$R = PQ, \quad r = Pq + p$$

Rigid Body Motions

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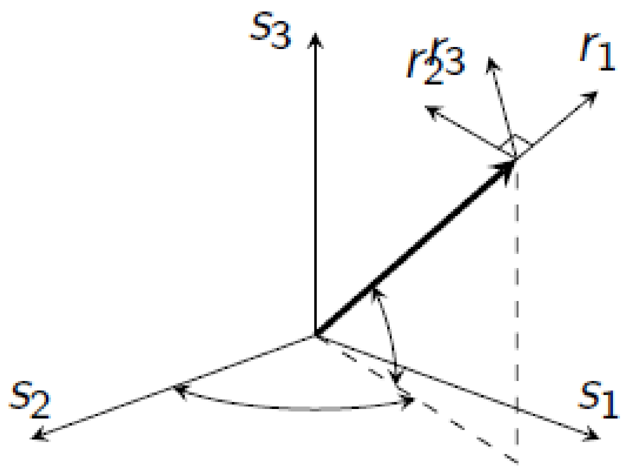


Special Orthogonal Groups

$\boldsymbol{SO(3)}$, the group that represents all spatial rotation matrices, is the set of all 3×3 real matrices R that satisfy

1. $R^{\top} R = I$
2. $\det R = 1$

- Note that (2) implies that both (1) holds and that we obey the righthand rule

Similarly, $\boldsymbol{SO(2)}$, the group describes all planar rotation matrices, is the set of all 2×2 real matrices R, satisfying the same conditions.

Rotation Matrices

- We use the convention R_{ab} for the matrix whose **columns are coordinates of the frame $\boldsymbol{b}$ expressed in the frame $\boldsymbol{a}$**
- We have the following laws:

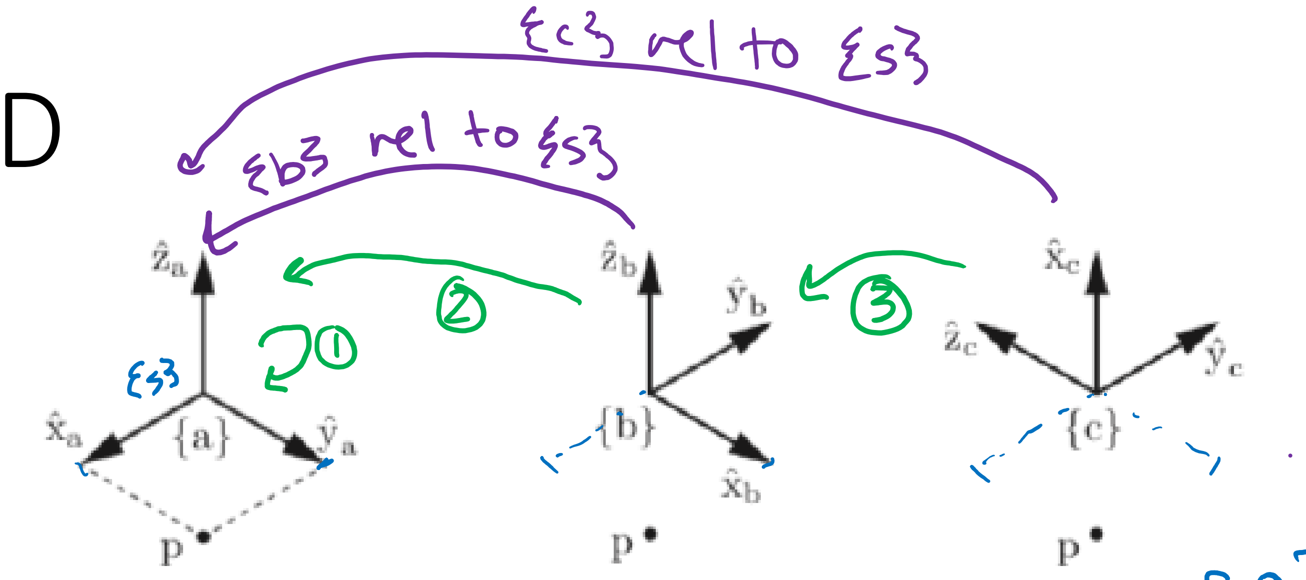
$$R_{ab}R_{bc} = R_{ac} \text{ and } R_{ab}^{-1} = R_{ba}$$

- If the vector p_a is the vector p expressed in frame a, then

$$R_{ba}p_a = p_b$$

- If we rotate a vector v around the $\widehat{\omega}$ axis by angle θ, we say $\mathrm{Rot}(\widehat{\omega}, \theta)v$
  - For a rotation about the x-axis: $\mathrm{Rot}(\widehat{\omega}, \theta)v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} v$
  - Rotations about the y and z axis follow

Frames in 3D



$$P_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad P_b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad P_c = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

① no rot

$$R_a = I$$

② rot about $\hat{z}_a$ by $90°$

$$R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ rot $\hat{y}_b$ by $-90°$

$$R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

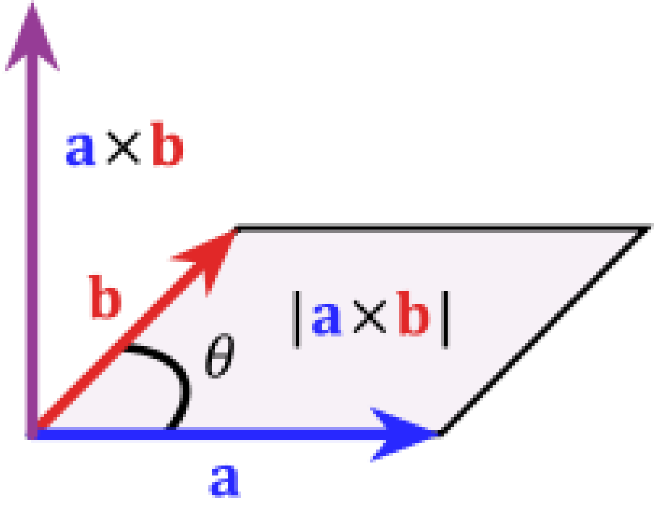
Recall the cross product

- The cross-product is defined as

$$a \times b = \|a\|\|b\| \sin\theta \; N$$

- In coordinates:

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$



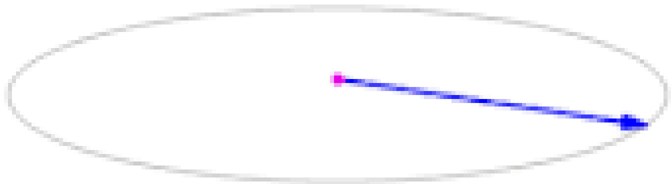
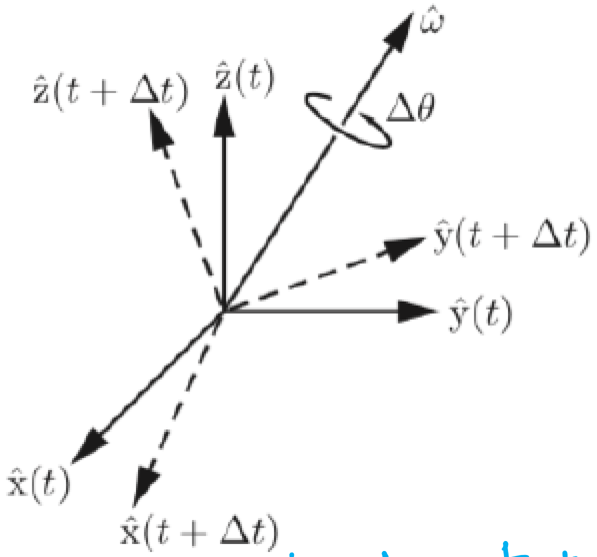
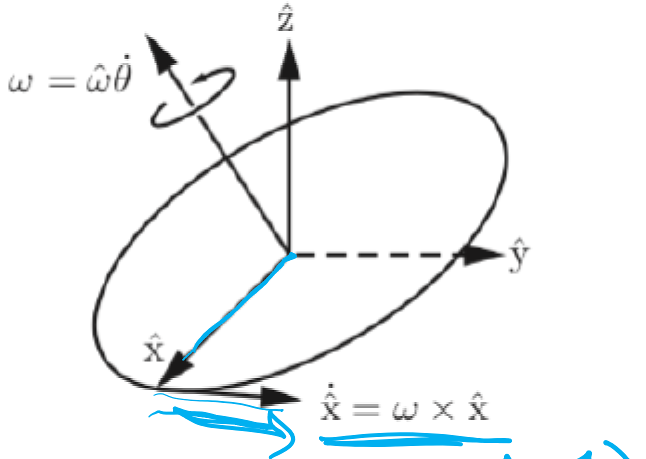


Image Credit: Wikipedia

Angular Velocities





for a small rotation $\Delta\theta$, from t to $t+\Delta t$, of frame $\{\hat{x}, \hat{y}, \hat{z}\}$ about vector $\hat{\omega}$. we observe:

$$\hat{x}(t+\Delta t) \simeq x(t) + \underline{\hat{\omega}\Delta\theta \times \hat{x}}$$

as $\Delta t \to 0$, $\dot{\hat{x}} = w \times \hat{x}$, where $w = \underline{\hat{\omega}\dot{\theta}}$

$$\dot{\hat{y}} = w \times \hat{y}$$

$$\dot{\hat{z}} = w \times \hat{z}$$

Angular Velocities in Reference Frame

In fixed frame $\{s\}$, $R(t)$ is the rotation matrix that describes the orientation of the body w.r.t. $\{s\}$ at t

let $\hat{x} = r_1(t)$, $\hat{y} = r_2(t)$, $\hat{z} = r_3(t)$

let $\omega_s = w$ ← angular velocity in $\{s\}$

$\dot{r}_i = \omega_s \times r_i$, $i = 1, 2, 3$ → $\dot{R} = [\omega_s \times r_1 \quad \omega_s \times r_2 \quad \omega_s \times r_3]$

$= \omega_s \times R$

let's simplify by eliminating cross-product → use new matrix!

for each w, define a unique skew-symmetric matrix*

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} := [w]$$

now write:

$\dot{R} = w \times R$ ← R_{sb}

$\updownarrow$

$\dot{R} = [w] R$

$\Updownarrow$

$[w] = \dot{R} R^{-1} = \dot{R} R^T$

* $A = -A^T$, denoted $so(3)$

Some useful properties and relations

Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following holds: $R[\omega]R^\top = [R\omega]$

Now recall: $[\omega_s] = \dot{R}R^\top$

If R is R_{sb}, we have that $\omega_s = R\omega_b \Leftrightarrow \omega_b = R^\top \omega_s$

$$[\omega_b] = [R^\top \omega_s] = R^\top \left(\dot{R}R^\top\right)R = R^\top \dot{R} = R^{-1}\dot{R}$$

This gives us: $[\omega_s] = \dot{R}R^\top$ and $[\omega_b] = R^\top \dot{R}$

Exponential Coordinate Representation

- Instead representing orientation as a rotation matrix, we introduce a three parameter representation: **exponential coordinates.**
- Recall: $\hat{\omega}$ rotation axis, θ angle of rotation
- Then $\hat{\omega}\theta \in \mathbb{R}^3$gives the exponential coordinate representation

- A new interpretation for a frame coincident with $\{s\}$:
  - A rotation for 1 second around $\hat{\omega}$ at angular velocity θ, then the resulting frame is R
  - A rotation for θ seconds around $\hat{\omega}$ at angular velocity 1, then the resulting frame is R

Review: Linear ODEs and the Matrix Exponential

$\dot{x} = A\,x(t)$, $x(0) = x_0$ $\longrightarrow$ soln: $x(t) = \underline{\underline{e^{At}}}\,x_0$

$\dot{x} \leftarrow \mathbb{R}^n$, $A \leftarrow \mathbb{R}^{n\times n}$

matrix exponential / exponentia function expansion:

$$e^{At} = I + At + \tfrac{1}{2}A^2t^2 + \tfrac{1}{3!}A^3t^3 + \cdots$$

Properties:

① $\frac{d}{dt}\left(e^{At}\right) = A\,e^{At} = e^{At}A$

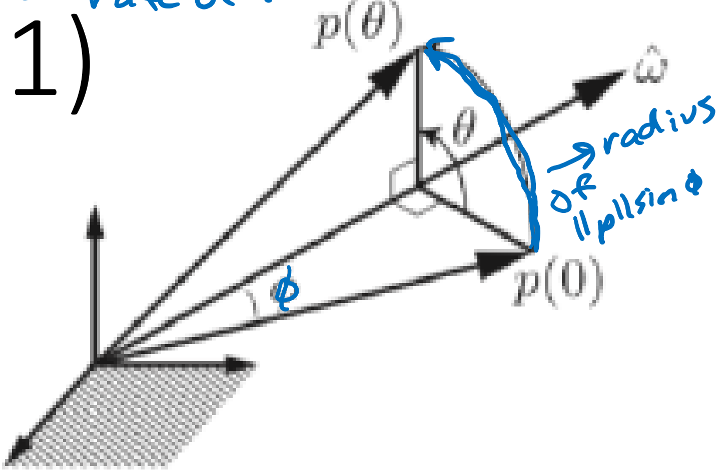
② If $A = PDP^{-1}$ (D diag), then $e^{At} = P\,e^{Dt}P^{-1}$

③ If $AB = BA$, then $e^{A}e^{B} = e^{B}e^{A} = e^{A+B}$

④ always invertible: $\left(e^{At}\right)^{-1} = e^{-At}$

Exponential Coordinates of Rotations (1)

imagining a rate of 1 rad/s



vector $p(0)$ is rotated around $\hat{\omega}$ to $p(\theta)$

Recall: $\dot{p} = \hat{\omega} \times p \longrightarrow \dot{p} = [\hat{\omega}]p$

$\downarrow$ diff eq soln

$p(t) = e^{[\hat{\omega}]t} p(0)$

note a few tricks:

$[\hat{\omega}]$ is skew-symmetric $\longrightarrow [\hat{\omega}]^3 = -[\hat{\omega}]$ and $[\hat{\omega}]^4 = -[\hat{\omega}]^2$

$\sin x = x - \frac{1}{3!}x^3 + \dots$ and $\cos x = 1 - \frac{1}{2}x^2 + \dots$

$$\hookrightarrow e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\theta + \frac{1}{2}[\hat{\omega}]^2\theta^2 + \frac{1}{3!}[\hat{\omega}]^3\theta^3 + \dots$$

$$= I + \underbrace{\left(\theta - \theta^3/3! + \dots\right)}_{\sin\theta}[\hat{\omega}] + \underbrace{\left(\frac{\theta^2}{2} - \frac{\theta^4}{4!} + \dots\right)}_{1-\cos\theta}[\hat{\omega}]^2$$

Rodrigues Formula $\mathrm{Rot}(\hat{\omega}, \theta) = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$

$\longrightarrow R = e^{[\hat{\omega}]\theta}$

Matrix Logarithm

- Given rotation matrix R, we need to take the *logarithm* to find the exponential coordinates:

exp: $[\hat{\omega}]\theta \in so(3) \quad \rightarrow R \in SO(3)$

log: $R \in SO(3) \quad \rightarrow [\hat{\omega}]\theta \in so(3)$

- If we expand Rodrigues formula:

$$\mathrm{Rot}(\hat{\omega}, \theta) = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

Matrix Logarithm Method

- If we take the trace of the matrix, we can solve for θ:
  - $tr(R) \coloneqq r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$
- If we compute $R^\top - R$, we get:
  - $r_{32} - r_{23} = 2\hat{\omega}_1 \sin\theta$
  - $r_{13} - r_{31} = 2\hat{\omega}_2 \sin\theta$
  - $r_{21} - r_{12} = 2\hat{\omega}_3 \sin\theta$

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R^\top - R)$$

- But what if $\sin\theta = 0$? (called singularities)
  - If $\theta = 2k\pi$, we have rotated by 360 degrees
  - If $\theta = (2k+1)\pi$, then Rodrigues formula is $R = I + 2[\hat{\omega}]^2$, which gives:
    $\hat{\omega}_i = \pm\sqrt{\frac{r_{ii}+1}{2}}$ and $2\hat{\omega}_i\hat{\omega}_j = r_{ij}$ for $i \neq j$