derivation_of_expectation_of_power

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1 Expectation of 2D Power Spectrum (and its actual covariance)

In this notebook, we aim to derive from first principles the expectation of the 2D power spectrum of foregrounds, specifically in the context of point-source foregrounds (we hope that this is intuitively generalisable to continuous foregrounds). In the process, we find that we require the covariance of the power spectrum, so that in effect we cover all required elements of the distribution.

What differentiates this from previous work (in the CHIPS paper and Murray et al. 2017) is that we properly retain the structure of the (u, v)-gridding, and therefore retain the covariance between baselines.

1.1 General Derivation

1.1.1 Single-source visibility

Consider a baseline $\vec{u}$ at frequency ν, and a single (point) source with flux density $S(\nu)$ and position $\vec{l}$. The visibility is given by

$$V(\nu, u) = S(\nu)B(\vec{l}, \nu)e^{-2\pi i\vec{u}\cdot\vec{l}} \tag{1}$$

where B is the primary beam. Now imagine that we have a collection of $N_{\rm bl}$ baselines, and to determine the visibility at a particular grid-point $\vec{u}$, we apply a window function W_ν, where $\int W_\nu = 1$. Thus the visibility at arbitrary gridpoint $\vec{u}$ is

$$V(\nu, u) = \sum_{i=1}^{N_{\rm bl}} W_\nu(\vec{u} - f\vec{u}_0^i)S(\nu)B(\vec{l}, \nu)e^{-2\pi i f\vec{u}_0^i\cdot\vec{l}}, \tag{2}$$

where $f = \nu/\nu_0$.

The 2D Power spectrum is $P(\eta, u) = \tilde{V}\tilde{V}^*$, where $\tilde{V}$ is the FT (over frequency) of the ν-space visibility:

$$\tilde{V}(\eta, u) = \sum_{i=1}^{N_{\rm bl}} \int d\nu e^{-2\pi i\nu\eta} W_\nu(\vec{u} - \vec{u}_i)S(\nu)B(\vec{l}, \nu)e^{-2\pi i\vec{u}_i\cdot\vec{l}}. \tag{3}$$

1.1.2 Visibility from all sources

We could use the visibility so far to determine the power spectrum, given a suitable B and W. However, we'll leave that for later and continue in a more general route. Let's assume we have multiple sources across the sky. Since all operations thus far have been linear, we may simply add over each source (we must do this before transforming to the power spectrum itself, since that is nonlinear):

$$\tilde{V}(\eta, u) = \sum_{j=1}^{N_s} \sum_{i=1}^{N_{\rm bl}} \int d\nu e^{-2\pi i\nu\eta} W_\nu(\vec{u} - \vec{u}_i) S_j(\nu) B(\vec{l}_j, \nu) e^{-2\pi i \vec{u}_i \cdot \vec{l}_j}. \tag{4}$$

While this is a useful representation, another useful way to write it down is via the source count model:

$$\tilde{V}(\eta, u) = \sum_{i=1}^{N_{\rm bl}} \int d\nu e^{-2\pi i\nu\eta} W_\nu(\vec{u} - \vec{u}_i) f^{-\gamma} \int dS \int d^2\vec{l} S \frac{dN}{dS}(S, \vec{l}) B(\vec{l}, \nu) e^{-2\pi i \vec{u}_i \cdot \vec{l}}. \tag{5}$$

Here for simplicity we've made the assumption that the SED is universal and a power-law ($f^{-\gamma}$). This can be modified at a later point.

1.1.3 Expectation of the Power Spectrum

Recall the identity

$$\langle XX^* \rangle = \langle X \rangle \langle X^* \rangle + \mathrm{Var}(X). \tag{6}$$

Thus to calculate the expectation of the power, we require both the expectation of $\tilde{V}$ and its variance.

Expectation of Fourier-Visibility Throughout this section, we assume a uniform distribution of sources across the sky (uniform in l, m). The second equation of S1.1.2 is the easier to use here. Since the entire formula is a sum, the expectation can go right through to the only stochastic factor – dN/dS – which merely turns into its non-stochastic expectation. Thus we end up with

$$\langle \tilde{V}(\eta, u) \rangle = \sum_{i=1}^{N_{\rm bl}} \int d\nu e^{-2\pi i\nu\eta} W_\nu(\vec{u} - f\vec{u}_0^i) f^{-\gamma} \mu_1 \hat{B}(fu_0^i, \nu). \tag{7}$$

Variance of Fourier-Visibility Here we go slightly further than what is done in Murray et al. (2017). There, the covariance *between frequencies* for the *same baseline* is determined. Here, we desire the variance of a single *bin* in (η, u)-space. The variance of a sum is merely the sum of covariance of all pairs. Using Eq. 5, the only stochastic part is dN/dS, and so we may take the covariance operator through to there, bringing everything else out. The source counts are independent for different bins of S and l, but are not independent over ν or baselines. Ultimately, we end up with

$$\mathrm{Var}\left(\tilde{V}(\eta, u)\right) = \sum_{i_1=1}^{N_{\rm bl}} \sum_{i_2=1}^{N_{\rm bl}} \int d\nu' e^{-2\pi i\nu'\eta} \int d\nu'' e^{+2\pi i\nu''\eta} W'(\vec{u} - f\vec{u}_0^{i_1}) W''(\vec{u} - f\vec{u}_0^{i_2}) (f'f'')^{-\gamma} \tag{8}$$

$$\times \mu_2 \int d^2\vec{l} B' B'' e^{-2\pi i \vec{l} \cdot (f'\vec{u}_0^{i_1} - f''\vec{u}_0^{i_2})}. \tag{9}$$

This is a fairly involved expression, and clearly it is numerically expensive – ultimately it is 6-dimensional sum for each η and u. Furthermore, it is only applicable to spatially uncorrelated sky models. For the rest of this notebook we'll try looking at simplifications/special cases which may make it easier to deal with.

1.1.4 Covariance of the Power

There are two ways to think about the covariance of the power spectrum. The easiest way forward is to assume that the complex visibilities are themselves drawn from a Complex Normal distribution. In this case, we have the simple expression

$$\mathrm{Cov}(P) = 2\left|\mathrm{Var}(\tilde{V})\right|^2. \tag{10}$$

This assumption is of course not necessarily correct, and a more general approach would be to perform a double integral over Eq. 5 squared (note: this means that the dimensionality of the problem is multiplied by 4 initially). Some of the integrals are contracted due to the independence of the source counts, but the final integral will still be extremely formidable.

1.2 Simplifications and Special Cases

There are several obvious special cases and simplifications that can be made, and some of them may be combined. We list here some of those we will pursue, and some we will also explicitly combine:

- Frequency-dependent Gaussian beam
- Dirac-delta window function + Ideal UV coverage
- Rectangular windows with $\Delta u >> \Delta f u_0$ and ~1 baseline per cell.
- Fourier-beam kernel window function
- Uniform-density UV coverage.
- Non-stochastic sky model
- Single-source model
- FFT-based covariance for the ν integral.

We will start with the simplest and most general.

Preliminary on Gaussian Beam (FDGB) For a start, we note that for a FDGB:

$$B(l,\nu) = e^{-l^2 f^2/2\sigma_0^2}, \tag{11}$$

the Fourier-space beam is

$$\hat{B}(u,\nu) = \frac{2\pi\sigma_0^2}{f^2} e^{-2\pi^2\sigma_0^2 u^2/f^2}. \tag{12}$$

Interestingly, the beam at fu retains no frequency-dependence in the exponent:

$$\hat{B}(fu,\nu) = \frac{2\pi\sigma_0^2}{f^2} e^{-2\pi^2\sigma_0^2 u^2}. \tag{13}$$

1.2.1 Non-Stochastic Sky

In this case, the expectation of the power-spectrum only involves the expectation of the visibility (Eq. 7). We'll go through and derive this term under sum combinations of simplifications

DDWF + IdealUV Assuming we have a baseline at every infinitessimal location, but infinitely small UV bins, the window function becomes a delta function at the bin.

$$\langle \tilde{V}(\eta, u) \rangle = \mu_1 \int d\nu e^{-2\pi i \nu \eta} f^{-(\gamma+2)} \hat{B}_\nu(u). \tag{14}$$

The power spectrum is consequently

$$\langle P(\eta, u) \rangle = \nu_0^2 \mu_1^2 \left| \mathcal{F}\left[f^{-\gamma} \hat{B}_\nu(u) \right] \right|^2. \tag{15}$$

WIth FDGB Letting $y^2 = 4\pi^2\sigma_0^2$, the power spectrum for a FDGB is thus

$$P(\eta, u) = \nu_0^2 y^2 \sigma_0^2 \mu_1^2 \left| \mathcal{F}\left[f^{-(\gamma+2)} \exp\left(-\frac{y^2 u^2}{f^2} \right) \right] \right|^2. \tag{16}$$

This *may* exhibit a wedge, since there is a combination of f and u in the exponential. I'll try it out later...

RW + 1-BL-per-cell In this case, we expect that each UV cell has about 1 baseline, and that typically that baseline stays within the cell over the bandwidth. For simplicity, we will assume that at ν_0, the baselines exactly represent the UV grid. This turns out to give a very similar result to the DDWF with IdealUV, however the beam's dependence on f in the exponent vanishes, leaving

$$P(\eta, u) = \nu_0^2 y^2 \sigma_0^2 \mu_1^2 \exp\left(-y^2 u^2\right) \left| \mathcal{F}\left[f^{-(\gamma+2)} \right] \right|^2. \tag{17}$$

Radially distributed baselines Let the baselines be infinitessimally separated, with density distribution $\rho(u)$. Then we have

$$\langle \tilde{V}(\eta, u) \rangle = \mu_1 \int d\nu e^{-2\pi i \nu \eta} f^{-\gamma} \int d^2\vec{u}' \rho(fu') W_\nu(\vec{u} - f\vec{u}') \hat{B}(fu', \nu). \tag{18}$$

We make the substitution $x = fu'$.

So long as W is also a radially-symmetric function, the final integral can be reduced to one dimension:

$$\langle \tilde{V}(\eta, u) \rangle = \mu_1 \int d\nu e^{-2\pi i \nu \eta} f^{-(\gamma+2)} \int dl l J_0(lu) \mathcal{F}\left[\rho(u) \hat{B}_\nu(u) \right] \hat{W}_\nu(l). \tag{19}$$

Uniform Distribution If the baselines are distributed uniformly, then the Fourier transform in brackets simplifies significantly, to

$$\langle \tilde{V}(\eta, u) \rangle = \mu_1 \int d\nu e^{-2\pi i \nu \eta} f^{-(\gamma+2)} \int dl l J_0(lu) B_\nu(l) \hat{W}_\nu(l). \tag{20}$$

UD + FourierBeam Window The correct window to use is the Fourier-Beam. Under the assumption of a uniform distribution of baselines, we then have

$$\langle\tilde{V}(\eta,u)\rangle=\mu_1\int d\nu e^{-2\pi i\nu\eta}f^{-(\gamma+2)}\int dll J_0(lu)B_\nu^2(l). \tag{21}$$

Adding in a FDGB, we get

$$\langle\tilde{V}(\eta,u)\rangle=\frac{\mu_1}{2}\int d\nu e^{-2\pi i\nu\eta}f^{-(\gamma+4)}\sigma_0^2\exp(-\pi^2u^2\sigma_0^2/f^2). \tag{22}$$

GaussianDistributedBaselines + FourierBeam Window If the baselines are distributed roughly as a gaussian distribution, we get a very similar result, except that there is a distn width, σ_u, in the exponent as well:

$$\langle\tilde{V}(\eta,u)\rangle=\mu_1\int d\nu e^{-2\pi i\nu\eta}f^{-(\gamma+2)}\int dll J_0(lu)\exp(-l^2/2(\sigma_\nu^2+1/4\pi^2\sigma_u^2)). \tag{23}$$

which leaves

$$\langle\tilde{V}(\eta,u)\rangle=\mu_1\int d\nu e^{-2\pi i\nu\eta}f^{-(\gamma+2)}\Sigma_\nu^2\exp(-2\pi^2u^2\Sigma_\nu^2), \tag{24}$$

with

$$\Sigma_\nu^2=\frac{(\sigma_\nu^2+1/4\pi^2\sigma_u^2)}{2}. \tag{25}$$

1.2.2 Single Source

Recall that for a single source in the sky, the visibility is explicitly

$$\tilde{V}(\eta,u)=\sum_{i=1}^{N_{\rm bl}}\int d\nu e^{-2\pi i\nu\eta}W_\nu(\vec{u}-f\vec{u}_0^i)S(\nu)B(\vec{l},\nu)e^{-2\pi i f\vec{u}_0^i\cdot\vec{l}}. \tag{26}$$

We can calculate all the same simplifications for this kind of sky.

DDWF + IdealUV Here we get

$$P(\eta,u)=S^2(\nu)e^{-4\pi iul\cos\theta}\left|\mathcal{F}\left[B(l,\nu)\right]\right|^2. \tag{27}$$

For a FDGB:

$$P(\eta,u)=\frac{y^2\sigma_0^2S^2(\nu)}{l^4}e^{-4\pi iul\cos\theta}\exp(-y^2u^2/l^2). \tag{28}$$

RW + 1-BL-per-cell Here we get the same again, but keeping the f in the exponent:

$$P(\eta,u)=\frac{y^2\sigma_0^2S^2(\nu)}{l^4}e^{-4\pi iful\cos\theta}\exp(-y^2u^2/l^2). \tag{29}$$

Radially distributed baselines This case is similar to the expectation of the multi-source visibility, except that the fourier beam does not enter:

$$\langle \tilde{V}(\eta, u) \rangle = \mu_1 \int d\nu e^{-2\pi i \nu \eta} f^{-(\gamma+2)} B_\nu(l_0) \int dl l J_0(lu) \hat{\rho}(l) \hat{W}_\nu(l). \tag{30}$$

Uniform Distribution + FourierBeam Window

$$P(\eta, u) = \mu_1^2 \left| \mathcal{F}\left[f^{-(\gamma+2)} B_\nu(l_0) \hat{B}_\nu(u) \right] \right|^2. \tag{31}$$

GDB + FBW Don't even need to do this, it's so similar to the expected.

1.2.3 Stochastic Whole Sky

The total power spectrum, in the case of a stochastic sky, is a sum of the non-stochastic solutions above, with the solution to the variance of a visibility:

$$\mathrm{Var}\left(\tilde{V}(\eta, u)\right) = \sum_{i_1=1}^{N_{\rm bl}} \sum_{i_2=1}^{N_{\rm bl}} \int d\nu' e^{-2\pi i \nu' \eta} \int d\nu'' e^{+2\pi i \nu'' \eta} W'(\vec{u} - f \vec{u}_0^{i_1}) W''(\vec{u} - f \vec{u}_0^{i_2}) (f' f'')^{-\gamma} \tag{32}$$

$$\times \mu_2 \int d^2\vec{l} B' B'' e^{-2\pi i \vec{l} \cdot (f' \vec{u}_0^{i_1} - f'' \vec{u}_0^{i_2})}. \tag{33}$$

In this section, we apply the same kinds of simplifications as in the previous sections to this equation to see if a useful form will come out.

DDWF + IdealUV In this case, we get

$$\mathrm{Var}\left(\tilde{V}(\eta, u)\right) = 2\mu_2 \sigma_0^2 \int d\nu' \int d\nu'' e^{-2\pi i (\nu' - \nu'')} \frac{(f' f'')^{-\gamma}}{f'^2 + f''^2}. \tag{34}$$

RW + 1-BL-per-cell Ini this case, the initial baselines must be the same to be contributing to the same cell, as per our assumptions, but the frequencies could be different.

$$\mathrm{Var}\left(\tilde{V}(\eta, u)\right) = \mu_2 \int d\nu' \int d\nu'' e^{-2\pi i (\nu' - \nu'') \eta} (f' f'')^{-\gamma} \int d^2\vec{l} B' B'' e^{-2\pi i (f' - f'') \vec{l} \cdot \vec{u})}. \tag{35}$$

This is precisely the same as the default CHIPS case:

$$\mathrm{Var}\left(\tilde{V}(\eta, u)\right) = 2\pi (f_0' f_0'')^{-\gamma} \mu_2 \int d\nu' \int d\nu'' e^{-2\pi i (\nu' - \nu'') \eta} \Sigma_\nu^2 \exp(-2\pi^2 u^2 f_\nu^2 \Sigma_\nu^2), \tag{36}$$

where

$$\Sigma_\nu^2 = \frac{\sigma''^2 \sigma'^2}{\sigma''^2 + \sigma'^2} \quad [\mathrm{sr}]. \tag{37}$$

However, it is probably the poorest assumption for binning.

Radially Distributed Baselines

$$\mathrm{Var}\left(\tilde{V}(\eta,u)\right) = \mu_2 \int d\nu' \int d\nu'' e^{-2\pi i(\nu'-\nu'')\eta} (f'f'')^{-\gamma} \int d^2\vec{l} B'B'' \tag{38}$$

$$\int d^2\vec{u}_1 \int d^2\vec{u}_2 \rho(f'u_1)\rho(f''u_2) W'(\vec{u}-f'\vec{u}_1) W''(\vec{u}-f''\vec{u}_2) e^{-2\pi i\vec{l}\cdot(f'\vec{u}_1 - f''\vec{u}_2)}. \tag{39}$$

Make the substitution $x' = f'u$:

$$\mathrm{Var}\left(\tilde{V}(\eta,u)\right) = \mu_2 \int d\nu' \int d\nu'' e^{-2\pi i(\nu'-\nu'')\eta} (f'f'')^{-(\gamma+2)} \int d^2\vec{l} B'B'' \tag{40}$$

$$\int d^2\vec{x}_1 \int d^2\vec{x}_2 \rho(x_1)\rho(x_2) W'(\vec{u}-\vec{x}_1) W''(\vec{u}-\vec{x}_2) e^{-2\pi i\vec{l}\cdot(\vec{x}_1 - \vec{x}_2)}. \tag{41}$$

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