Maximum Probability Shortest Path Problem

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Shortest Path is a known problem and has many applications in "real life".

- Goods transport (industrial and private)
- Food Delivery (Deliveroo - Foodora - UberEats)

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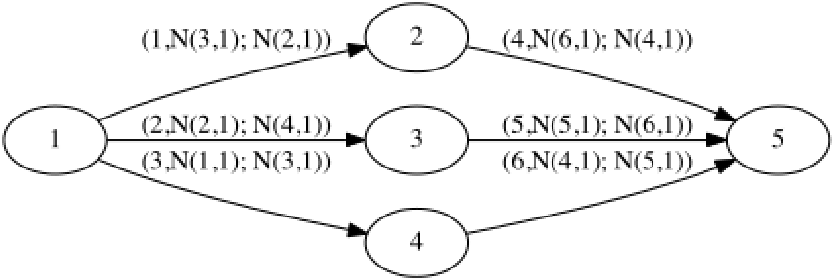
or also with a different optimization problem like utility functions to maximize or cost functions to minimize.

Stochastic resource constrained shortest path problem (SRCSP)



- Graph with weights on arcs, source node s and sink node t,
- Stochastic resource consumptions with normal distribution,
- K resources,
- Threshold C of the cost function (maximum allowed weight of the path).

SRCSP can be formulated as this optimization problem :

$$max\ \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k, k = 1..K\}$$

$$\begin{aligned} s.t.\ c^T x &\leq C \\ Mx &= b \\ x &\in \{0,1\}^n \end{aligned}$$

where :

- $x(e) = 1$ if $x(e) \in$ path P
- The a_k are multi-variate vectors with mean μ_k and known covariance matrix V_k
- M is the node-arc incidence matrix. $M(i,e) \in \{-1,0,1\}$
- b is a vector with 0 everywhere except $b(s) = 1$ and $b(t) = -1$

SRCSP can be reformulated as :

$$
\begin{aligned}
& max \;\; p \\
s.t. \;\; & p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k, k = 1..K\} \\
& c^T \leq C \\
& Mx = b \\
& x \in \{0,1\}^n
\end{aligned}
$$

With $K = 1$ we have :

$$
\begin{aligned}
&max\ p \\
&s.t.\ p \leq \mathbf{Pr}\{\tilde{a}_1^T x \leq d_1\} \\
&\qquad c^T \leq C \\
&\qquad Mx = b \\
&\qquad x \in \{0,1\}^n
\end{aligned}
$$

Using the known multivariate distribution parameters we get :

$$
\begin{aligned}
max \ & p \\
s.t. \ & F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \leq d_1 - \mu_1^T x \\
& c^T \leq C \\
& Mx = b \\
& x \in \{0, 1\}
\end{aligned}
$$

Relaxing the problem we get (SRCSPI) with $p \leq \frac{1}{2}$:

$$
\begin{aligned}
max\ & 0 \\
s.t.\ & F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \leq d_1 - \mu_1^T x \\
& c^T \leq C \\
& Mx = b \\
& 0 \leq x \leq 1
\end{aligned}
$$

We can solve it using the binary search procedure. We take $p_1 \leq \frac{1}{2}$ a feasible solution, a lower bound of SRCSP. p_l and p_u are lower and upper bounds of SRCSPI. Then we iterate :

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Start $p_l = \frac{1}{2}$ and $p_u = 1$. Iteration counter $t = 1$

Search Solve SRCSPI with $p = p_t$. If SRCSPI has an optimal solution, set $p_l = p_t$, otherwise $p_u = p_t$.

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Stop Stop when $\frac{p_u - p_l}{2} \leq \epsilon$. Otherwise $t++$ and $p_t = \frac{p_l + p_u}{2}$

Put the formulation Explain why we are looking for convexity, using the lectures Explain how we get the approximation (Theorem 4.1.2), saying that we used this method in classes

We formulate this problem taking individual probabilities constraints :

$$
\begin{aligned}
& max \;\; p \\
& s.t. \;\; p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k\}, \mathbf{k=1,..,K} \\
& \qquad c^T \leq C \\
& \qquad Mx = b \\
& \qquad x \in \{0,1\}^n
\end{aligned}
$$

We can relax the equation as we did before to get (RSRCSPJI) :

$$
\begin{aligned}
& max\ 0 \\
& s.t.\ F^{-1}(p)(x^T V_k x)^{\frac{1}{2}} \leq d_k - \mu_k^T x,\ \ k = 1, .., K \\
& \qquad c^T \leq C \\
& \qquad Mx = b \\
& \qquad 0 \leq x_i \leq 1,\ \ i = 1, .., n
\end{aligned}
$$

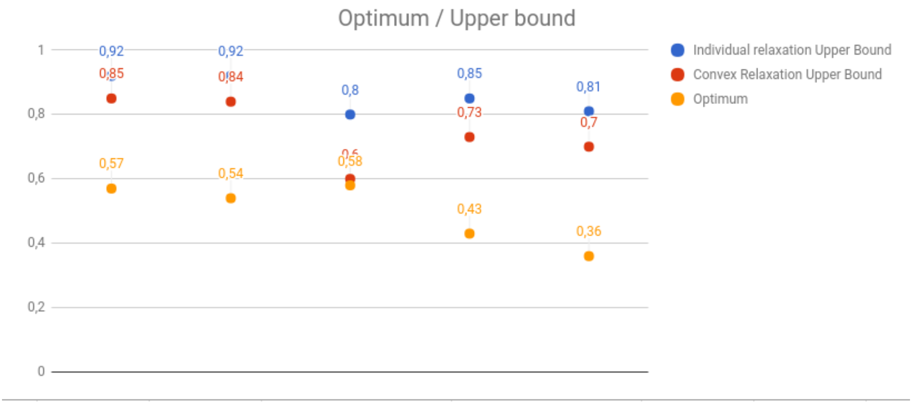
We use the relaxed equation using the Binary Search Procedure again :

Start $p_l = \frac{1}{2}$ and $p_u = 1$. Iteration counter $t = 1$

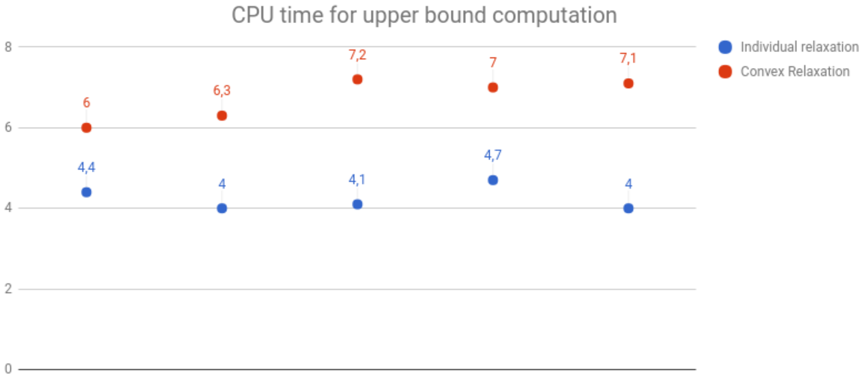
Search Solve RSRCSPJI with $p = p_t$. If RSRCSPJI has an optimal solution, set $p_l = p_t$, otherwise $p_u = p_t$.

Stop Stop when $\frac{p_u - p_l}{2} \leq \epsilon$. Otherwise $t++$ and $p_t = \frac{p_l + p_u}{2}$

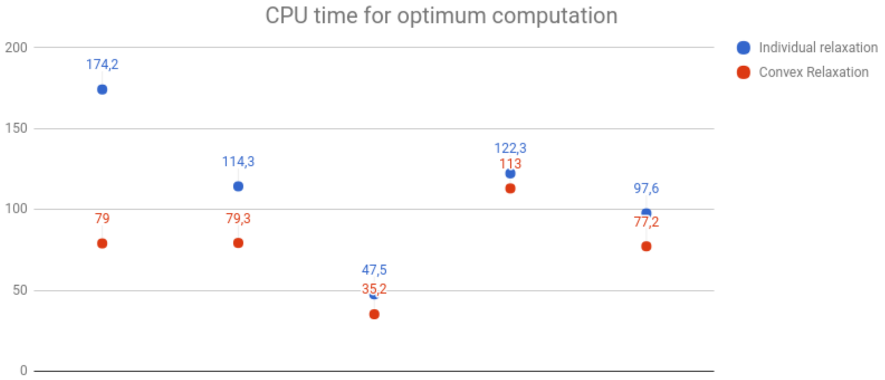
- upper bound is more accurate



- Individual relaxation upper bound computation is faster



- convex relaxation optimum computation is faster (upper bound is smaller)



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Method easily extendable to solve larger size instances.