# Maximum Probability Shortest Path Problem

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Shortest Path is a known problem and has many applications in "real life".

- Goods transport (industrial and private)
- Food Delivery (Deliveroo - Foodora - UberEats)

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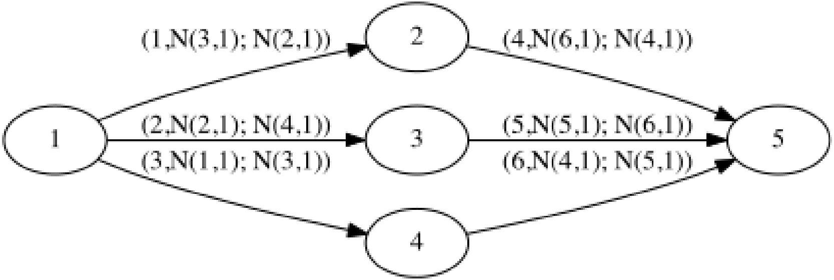
or also with a different optimization problem like utility functions to maximize or cost functions to minimize.

Stochastic resource constrained shortest path problem (SRCSP)



- Graph with weights on arcs, source node $s$ and sink node $t$,
- Stochastic resource consumptions with normal distribution,
- K resources,
- Threshold C of the cost function (maximum allowed weight of the path).

SRCSP can be formulated as this optimization problem :

$$max\ \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k, k = 1..K\}$$

$$\begin{aligned} s.t.\ c^T x &\leq C \\ Mx &= b \\ x &\in \{0,1\}^n \end{aligned}$$

where :

- $x(e) = 1$ if $x(e) \in$ path $P$
- The $a_k$ are multi-variate vectors with mean $\mu_k$ and known covariance matrix $V_k$
- M is the node-arc incidence matrix. $M(i,e) \in \{-1, 0, 1\}$
- b is a vector with 0 everywhere except $b(s) = 1$ and $b(t) = -1$

SRCSP can be reformulated as :

$$
\begin{aligned}
& max\ \ p \\
& s.t.\ \ p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k, k = 1..K\} \\
& \qquad c^T \leq C \\
& \qquad Mx = b \\
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## Branch and bound framework

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With $K = 1$ we have :

$$\begin{aligned} & max\ p \\ s.t.\ & p \leq \mathbf{Pr}\{\tilde{a}_1^T x \leq d_1\} \\ & c^T \leq C \\ & Mx = b \\ & x \in \{0,1\}^n \end{aligned}$$

Using the known multivariate distribution parameters we get :

$$
\begin{aligned}
& max \ p \\
s.t. \ & F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \leq d_1 - \mu_1^T x \\
& c^T \leq C \\
& Mx = b \\
& x \in \{0, 1\}
\end{aligned}
$$

Relaxing the problem we get (SRCSPI) with $p \leq \frac{1}{2}$ :

$$\begin{aligned} & max\ 0 \\ s.t.\ & F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \leq d_1 - \mu_1^T x \\ & c^T \leq C \\ & Mx = b \\ & 0 \leq x \leq 1 \end{aligned}$$

We can solve it using the binary search procedure. We take $p_1 \leq \frac{1}{2}$ a feasible solution, a lower bound of SRCSP. $p_l$ and $p_u$ are lower and upper bounds of SRCSPI. Then we iterate :

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**Start** $p_l = \frac{1}{2}$ and $p_u = 1$. Iteration counter $t = 1$

**Search** Solve SRCSPI with $p = p_t$. If SRCSPI has an optimal solution, set $p_l = p_t$, otherwise $p_u = p_t$.

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**Start** $p_l = \frac{1}{2}$ and $p_u = 1$. Iteration counter $t = 1$

**Search** Solve SRCSPI with $p = p_t$. If SRCSPI has an optimal solution, set $p_l = p_t$, otherwise $p_u = p_t$.

**Stop** Stop when $\frac{p_u - p_l}{2} \leq \epsilon$. Otherwise $t++$ and $p_t = \frac{p_l + p_u}{2}$

We now consider $K > 1$.
We can formulate the joint probabilistic SRCSP as follow :

$$
\begin{aligned}
max \;& p \\
s.t. \;& p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k,\ k = 1, .., K\} \\
SRCSPJ \;& c^T \leq C \\
& Mx = b \\
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\end{aligned}
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If $p$ is greater than a treshold value (theoreme 4.0.1), M(p) is convex.

Deterministic reformulation of SRCSPJ :

$$
\begin{aligned}
& max\ \ p \\
& s.t.\ F^{-1}(p^{y_k})(x^T V_k x)^{\frac{1}{2}} \leq d_k - \mu_k^T x,\ \ k = 1, .., K \\
& \qquad \sum_{k=1}^{K} y_k = 1,\ \ y_k \geq 0,\ \ k = 1, ..., K \\
& \qquad c^T x \leq C \\
& \qquad Mx = b,\ \ x \in \{0,1\}^n.
\end{aligned}
$$

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- Firstly we approximate $F^{-1}(p_0^k)$ with a piecewise tangent approximation of $y_k$.
  Afterwards, we get an approximation, which is SOCP problem.
- Secondly, we solve the SOCP problem, whose optimal value is an upper bound of SRCSP.
- Finally, we found the optimum with the binary search algorithm.

Finally, we can approximate it with the following convex relaxation :

$$
\begin{aligned}
\min \sum_{k=1}^{K} y_k & \\
s.t.\ & (\tilde{z}_k^T V_k \tilde{z}_k)^{\frac{1}{2}} \leq d_k - \mu_k^T x,\ \ k = 1, \dots, K \\
& \tilde{z}_{ki} \geq a_j x_i + b_j y_{ki},\ \ j = 0, \dots, n,\ \ i = 1, \dots, n \\
& 0 \leq y_k \leq -\log_{p_0}(2),\ \ k = 1, \dots, K \\
& 0 \leq y_{ki} \leq y_k,\ \ y_{ki} \geq y_k + x_i - 1,\ \ i = 1, \dots, n,\ \ k = 1, \dots, K \\
& c^T x \leq C,\ \ Mx = B,\ 0 \leq x_i \leq 1,\ \ i = 1, \dots, n \\
& M\tilde{y}_k = y_k b,\ \ k = 1, \dots, K
\end{aligned}
$$

where $\tilde{y}_k = (y_{k1}, \dots, y_{kn})$

We formulate this problem taking individual probabilities constraints :

$$
\begin{aligned}
& max \;\; p \\
& s.t. \;\; p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k\}, \; \mathbf{k=1,..,K} \\
& \qquad c^T \leq C \\
& \qquad Mx = b \\
& \qquad x \in \{0,1\}^n
\end{aligned}
$$

We can relax the equation as we did before to get (RSRCSPJI) :

$$
\begin{aligned}
& max\ 0 \\
& s.t.\ F^{-1}(p)(x^T V_k x)^{\frac{1}{2}} \leq d_k - \mu_k^T x,\ \ k = 1, .., K \\
& \quad c^T \leq C \\
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& \quad 0 \leq x_i \leq 1,\ \ i = 1, .., n
\end{aligned}
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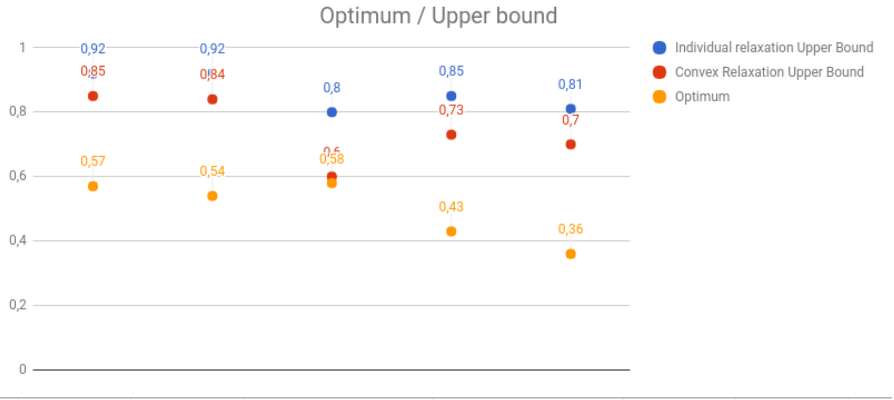
We use the relaxed equation using the Binary Search Procedure again :

**Start** $p_l = \frac{1}{2}$ and $p_u = 1$. Iteration counter $t = 1$

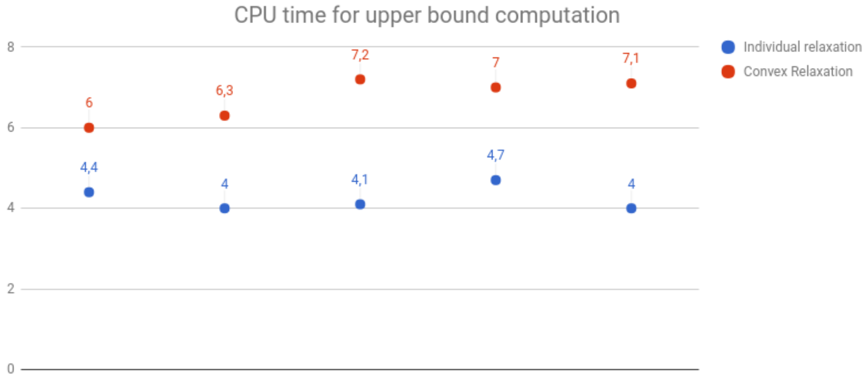
**Search** Solve RSRCSPJI with $p = p_t$. If RSRCSPJI has an optimal solution, set $p_l = p_t$, otherwise $p_u = p_t$.

**Stop** Stop when $\frac{p_u - p_l}{2} \leq \epsilon$. Otherwise $t++$ and $p_t = \frac{p_l + p_u}{2}$

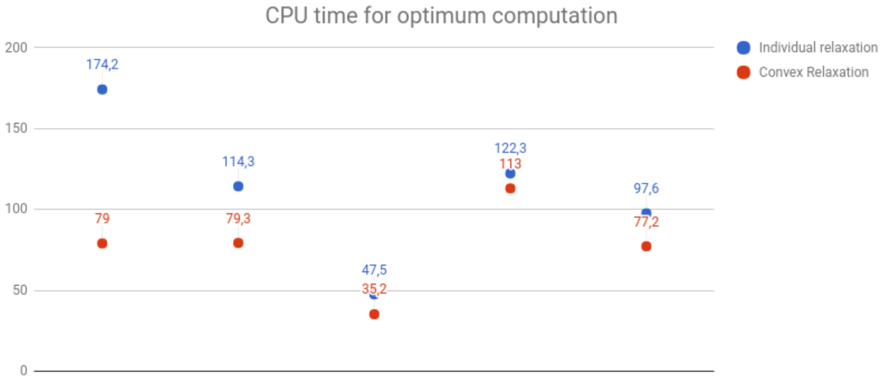
- upper bound is more accurate



- Individual relaxation upper bound computation is faster



- convex relaxation optimum computation is faster (upper bound is smaller)



Graphs kind

- $(n, m) = (23, 40)$ **(chart)**.

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Method easily extendable to solve larger size instances.

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## Going further

- generalize to dependent random variable

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- it introduce a convex relaxation to found the upper bound.
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## Going further

- generalize to dependent random variable
- study the inflence of some variable (for exemple $N$, the number of segment in tangent approximation, on the CPU time / upper bound value.

# Questions ?