

Maximum Probability Shortest Path Problem

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1 State of the art

2 Problem

- Description
- Hypothesis
- Formulation

3 Resolution

- One resource case
 - Formulation
 - Solution
- Joint probabilities
 - Formulation
- Individual relaxed probabilities
 - Formulation
 - Solution
- Results

Shortest Path is a known problem and has many applications in "real life".

- Goods transport (industrial and private)
- Food Delivery (Deliveroo - Foodora - UberEats)

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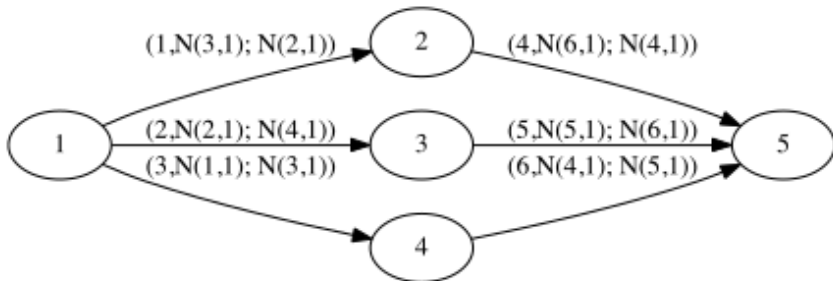
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As this problem is well known, lots of scientists presented their work on related subject, most of the times with different hypotheses :

- Without resource constraints
- With deterministic resource constraints
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or also with a different optimization problem like utility functions to maximize or cost functions to minimize.

Stochastic resource constrained shortest path problem (SRCSP)



- Graph with weights on arcs, source node s and sink node t ,
- Stochastic resource consumptions with normal distribution,
- K resources,
- Threshold C of the cost function (maximum allowed weight of the path).

SRCSP can be formulated as this optimization problem :

$$\max \Pr\{\tilde{a}_k^T x \leq d_k, k = 1..K\}$$

$$s.t. \ c^T x \leq C$$

$$Mx = b$$

$$x \in \{0, 1\}^n$$

where :

- $x(e) = 1$ if $x(e) \in \text{path } P$
- The a_k are multi-variate vectors with mean μ_k and known covariance matrix V_k
- M is the node-arc incidence matrix. $M(i, e) \in \{-1, 0, 1\}$
- b is a vector with 0 everywhere except $b(s) = 1$ and $b(t) = -1$

SRCSP can be reformulated as :

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k, k = 1..K\} \\ & c^T \leq C \\ & Mx = b \\ & x \in \{0, 1\}^n \end{aligned}$$

Branch and bound framework

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- find the upper bound

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- find the optimum, using branch and bound

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- find the upper bound - **relaxation problem**
- find the optimum, using branch and bound

With $K = 1$ we have :

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & p \leq \mathbf{Pr}\{\tilde{a}_1^T x \leq d_1\} \\ & c^T \leq C \\ & Mx = b \\ & x \in \{0, 1\}^n \end{aligned}$$

Using the known multivariate distribution parameters we get :

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \leq d_1 - \mu_1^T x \\ & c^T \leq C \\ & Mx = b \\ & x \in \{0, 1\} \end{aligned}$$

Relaxing the problem we get (SRCSPI) with $p \leq \frac{1}{2}$:

$$\max 0$$

$$s.t. F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \leq d_1 - \mu_1^T x$$

$$c^T \leq C$$

$$Mx = b$$

$$0 \leq x \leq 1$$

We can solve it using the binary search procedure. We take $p_1 \leq \frac{1}{2}$ a feasible solution, a lower bound of SRCSP. p_l and p_u are lower and upper bounds of SRCSPI. Then we iterate :

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Start $p_l = \frac{1}{2}$ and $p_u = 1$. Iteration counter $t = 1$

Search Solve SRCSPI with $p = p_t$. If SRCSPI has an optimal solution, set $p_l = p_t$, otherwise $p_u = p_t$.

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Stop Stop when $\frac{p_u - p_l}{2} \leq \epsilon$. Otherwise $t++$ and $p_t = \frac{p_l + p_u}{2}$

We now consider $K > 1$.

We can formulate the joint probabilistic SRCSP as follow :

$$\begin{aligned} & \max p \\ & \text{s.t. } p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k, \ k = 1, \dots, K\} \\ & \text{SRCSPJ } c^T \leq C \\ & Mx = b \\ & 0 \leq x_i \leq 1, \ i = 1, \dots, n \end{aligned}$$

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If p is greater than a threshold value (theoreme 4.0.1), $M(p)$ is convex.

Deterministic reformulation of SRCSPJ :

$\max p$

$$\text{s.t. } F^{-1}(p^{y_k})(x^T V_k x)^{\frac{1}{2}} \leq d_k - \mu_k^T x, \quad k = 1, \dots, K$$

$$\sum_{k=1}^K y_k = 1, \quad y_k \geq 0, \quad k = 1, \dots, K$$

$$c^T x \leq C$$

$$Mx = b, \quad x \in \{0, 1\}^n.$$

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- Firstly we approximate $F^{-1}(p_0^k)$ with a piecewise tangent approximation of y_k .
Afterwards, we get an approximation, which is SOCP problem.
- Secondly, we solve the SOCP problem, whose optimal value is an upper bound of SRCSP.
- Finally, we found the optimum with branch and bound algorithm.

Finally, we can approximate it with the following convex relaxation :

$$\begin{aligned} \min \quad & \sum_{k=1}^K y_k \\ \text{s.t.} \quad & (\tilde{z}_k^T V_k \tilde{z}_k)^{\frac{1}{2}} \leq d_k - \mu_k^T x, \quad k = 1, \dots, K \\ & \tilde{z}_{ki} \geq a_j x_i + b_j y_{ki}, \quad j = 0, \dots, n, \quad i = 1, \dots, n \\ & 0 \leq y_k \leq -\log_{p_0}(2), \quad k = 1, \dots, K \\ & 0 \leq y_{ki} \leq y_k, \quad y_{ki} \geq y_k + x_i - 1, \quad i = 1, \dots, n, \quad k = 1, \dots, K \\ & c^T x \leq C, \quad Mx = B, \quad 0 \leq x_i \leq 1, \quad i = 1, \dots, n \\ & M\tilde{y}_k = y_k b, \quad k = 1, \dots, K \end{aligned}$$

where $\tilde{y}_k = (y_{k1}, \dots, y_{kn})$

We formulate this problem taking individual probabilities constraints :

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & p \leq \mathbf{Pr}\{\tilde{a}_k^T x \leq d_k\}, \mathbf{k=1,...,K} \\ & c^T \leq C \\ & Mx = b \\ & x \in \{0, 1\}^n \end{aligned}$$

We can relax the equation as we did before to get (RSRCSPJI) :

$$\max 0$$

$$s.t. F^{-1}(p)(x^T V_k x)^{\frac{1}{2}} \leq d_k - \mu_k^T x, \quad k = 1, \dots, K$$

$$c^T \leq C$$

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We use the relaxed equation using the Binary Search Procedure again :

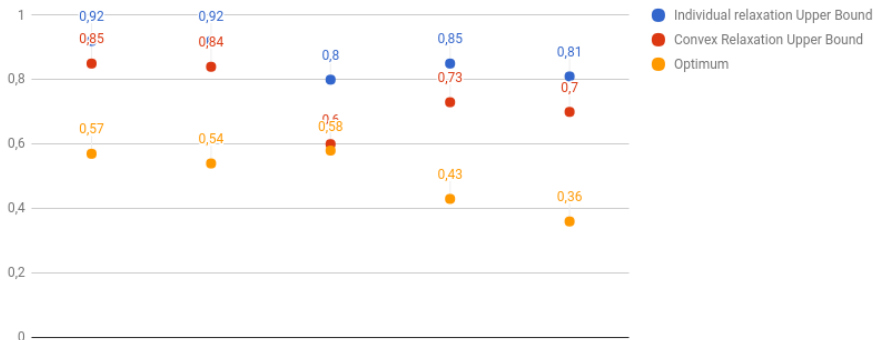
Start $p_l = \frac{1}{2}$ and $p_u = 1$. Iteration counter $t = 1$

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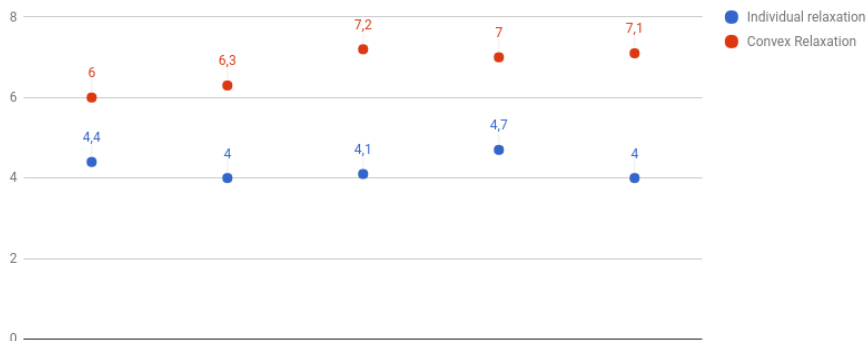
- upper bound is more accurate

Optimum / Upper bound



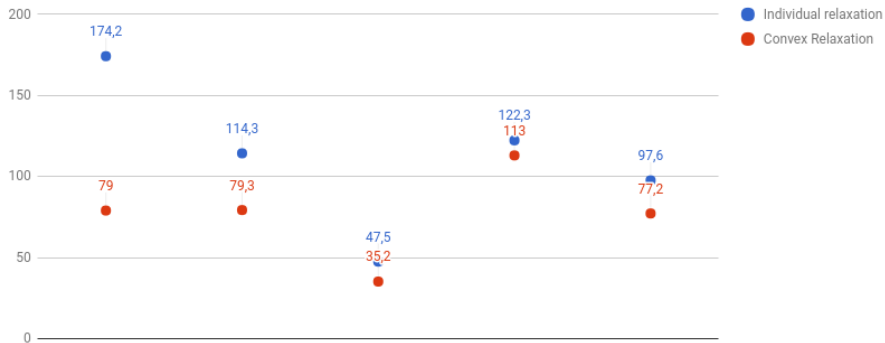
- Individual relaxation upper bound computation is faster

CPU time for upper bound computation



- convex relaxation optimum computation is faster (upper bound is smaller)

CPU time for optimum computation



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Method easily extendable to solve larger size instances.

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- study the influence of some variable (for exemple N , the number of segment in tangent approximation, on the CPU time / upper bound value).

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- it introduces a convex relaxation to find the upper bound.
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Going further

- generalize to dependent random variable
- study the influence of some variable (for exemple N , the number of segment in tangent approximation, on the CPU time / upper bound value).
- find a lower bound to give the branch and bound algorithm with another approximation of $F^{-1}(p_0^k)$.

Questions ?