

# Double Integrator Control: Cascaded P Controller Gains vs Damping Ratio

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Let's say we have a system where we command an acceleration  $\ddot{x}$  with the goal of controlling position  $x$  to a desired value  $u$ .

We use a cascaded proportional controller: an inner one for velocity, with gain  $K_v$  and an outer one with gain  $K_p$ . A cascaded P controller may be used instead of a PD controller instead of a for various reasons, including being able to explicitly constrain the desired velocity.

How does the ratio between  $K_v$  and  $K_p$  relate to system stability / overshoot / undershoot behavior? How can we make tuning these parameters simpler?

Start by writing out the math for a double proportional controller:

$$\ddot{x} = K_v(K_p(u - x) - \dot{x}) \quad (1)$$

where  $\ddot{x}$  is the commanded acceleration,  $u$  is the desired position,  $K_p$  is the position proportional gain,  $K_v$  is the velocity proportional gain, and  $x$  and  $\dot{x}$  are the position and velocity.

The above equation can be written out as

$$\ddot{x} + K_v\dot{x} + K_vK_px = K_vK_pu \quad (2)$$

which means the transfer function is

$$G(s) = \frac{y(s)}{u(s)} = \frac{K_vK_p}{s^2 + K_vs + K_vK_p} \quad (3)$$

Compare this to the canonical  $\omega_n$  (natural frequency) and  $\zeta$  (damping ratio) formulation for the transfer function of a 2nd order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2} \quad (4)$$

By doing some algebra we find

$$K_p = \frac{\omega_n}{2\zeta}, \quad K_v = 2\zeta\omega_n \quad (5)$$

which finally leads us to the ratio of the inner gain to outer gain:

$$\frac{K_v}{K_p} = 4\zeta^2 \quad (6)$$

For a critically damped system,  $\zeta = 1$ , meaning  $K_v/K_p = 4$ . For an underdamped system, e.g.  $\zeta = 0.7$ ,  $K_v/K_p = 1.96$ . In many situations, we can achieve reasonable tuning by assuming that  $\zeta$  should be somewhere between 0.7 and 1, meaning that the ratio  $K_v/K_p$  should be around 2-4.

## NOTES

- Please note that this is NOT a PD controller, which typically has gains such as  $K_p$  and  $K_d$ , and which we are not talking about here.
- There are often additional complexities such as underlying dynamics to  $\ddot{x}$ , saturation, etc, that you should consider in tuning your system.