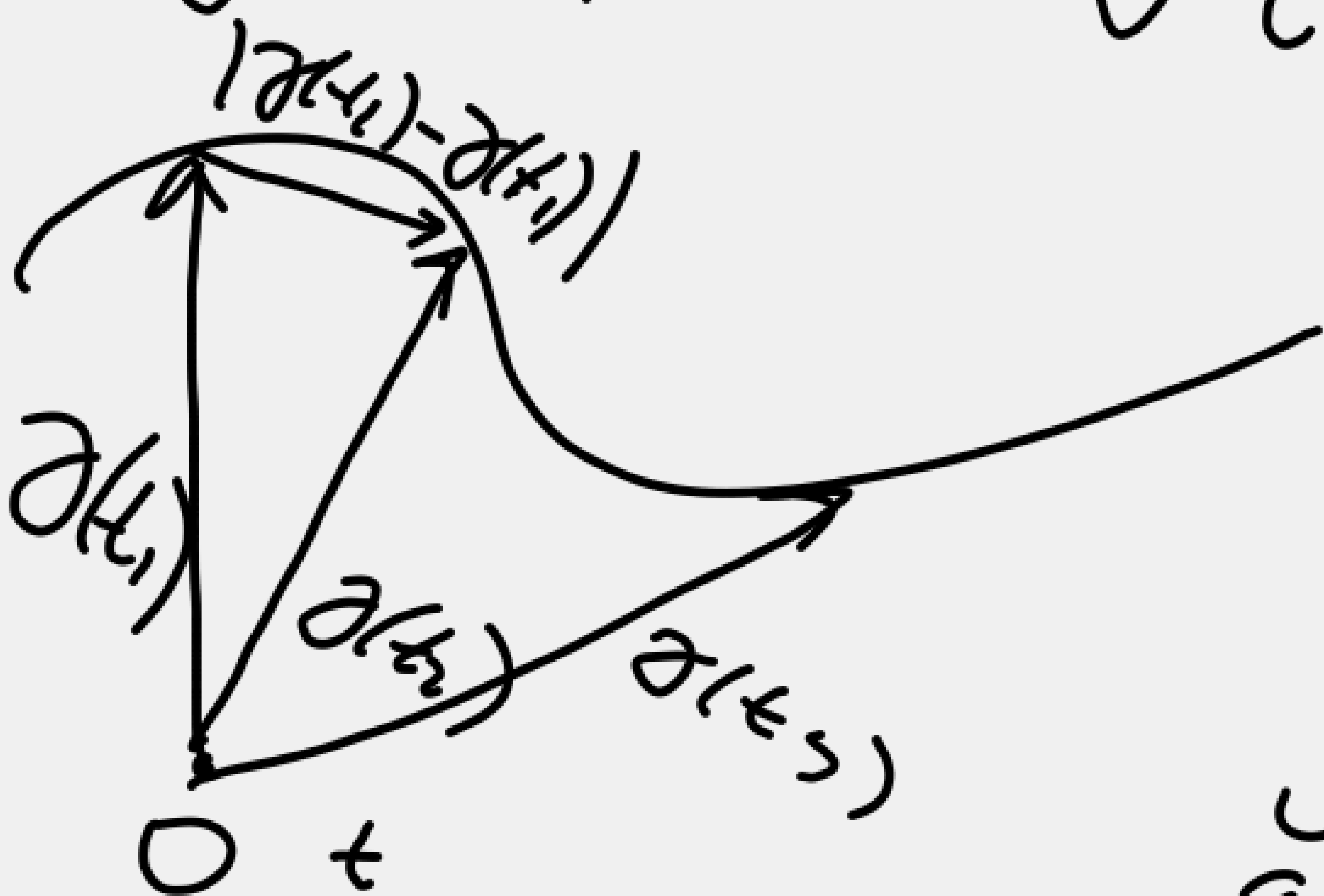
$\gamma(t),\ t \in [a,b],\ \mathbb{R}^2$

$\gamma(t) = (x(t), y(t)),\ x, y \in C^n$

$|\gamma(t)| \neq 0 \quad \forall t$



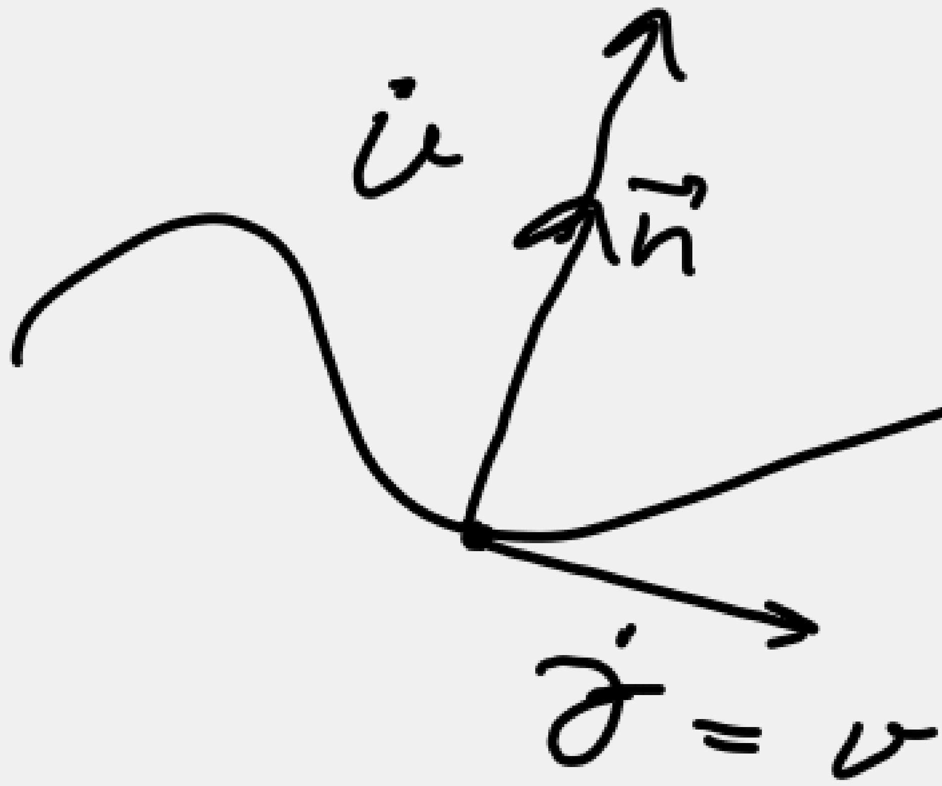
$$l(\gamma) = \sum |\gamma(t_{i+1}) - \gamma(t_i)|$$

$$\int_a^t |d\gamma| = \int_a^b |\gamma'|\,dt$$

длина кривой

$$s = \underbrace{\int_a^t |\gamma'|\,du}_{f(t)} \Rightarrow t = f^{-1}(s),$$

$$\gamma'(s) =: \dot\gamma,\quad |\dot\gamma(s)| = 1$$



$\mathbb{R}^2$

$\ddot\gamma \neq 1$

$|\dot\gamma| = 1 \Rightarrow \sqrt{(v,v)} = 1 \Rightarrow (v,v) = 1 \Rightarrow \frac{d}{ds}(v,v) = 0$

$\Rightarrow (\dot v, v) + (v, \dot v) = 0 \Rightarrow 2(\dot v, v) = 0 \Rightarrow \dot v \perp v$

$v = \dot\gamma$

$n = \frac{\ddot\gamma}{|\ddot\gamma|} = \frac{\dot v}{|\dot v|} \Rightarrow \dot v = \underbrace{\overset{k}{\overset{\|}{|\dot v|}}}_{\text{кривизна кривой в точке } s} n$

$\{\vec v, \vec n\}$ — базис Френе

$\gamma(t)$ t — произв. параметр

$\{v, n\}$

$$v = \dot{\gamma} = \frac{d\gamma}{ds} = \frac{d\gamma}{dt} \cdot \frac{dt}{ds} \Rightarrow \frac{\gamma'}{|\gamma'|}$$

$$S = \int_a^t |\gamma'(\varphi)|\, d\varphi \Rightarrow \frac{ds}{dt} = |\gamma'(t)| \Rightarrow \frac{dt}{ds} = \frac{1}{|\gamma'(t)|}$$

$$n = \frac{\dot{v}}{|\dot{v}|} \to \boxed{\dot{v} = \underbrace{|\dot{v}|}_{k} n}$$

$$\begin{array}{ccc} v & k & n \\ \| & \| & \\ \end{array}$$

$$|[\dot{\gamma}, \ddot{\gamma}]| = k$$

$$|[v, \dot{v}]| = k$$

$$\underline{\dot{v}} = \ddot{\gamma} = \frac{d}{ds}\left(\frac{d\gamma}{ds}\right) = \frac{d}{ds}\left(\frac{d\gamma}{dt} \cdot \frac{dt}{ds}\right) =$$

$$= \left(\frac{d}{ds} \cdot \frac{d\gamma}{dt}\right)\frac{dt}{ds} + \frac{d\gamma}{dt}\left(\frac{d}{ds}\,\frac{dt}{ds}\right) =$$

$$= \frac{d^2\gamma}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{d\gamma}{dt} \cdot \frac{d^2t}{ds^2} = \boxed{\frac{\vec{\gamma}''}{|\vec{\gamma}'|^2} + \vec{\gamma}' \cdot \frac{d^2t}{ds^2} = \underline{k \cdot \vec{n}}}$$

$$\left|\left[\frac{\gamma'}{|\gamma'|}\frac{\gamma''}{|\gamma'|^2}+\gamma'\frac{d^2t}{ds^2}\right]\right| = k(t)$$

$$\left|\left[\frac{\gamma'}{|\gamma'|},\frac{\gamma''}{|\gamma'|^2}\right]\right| = \boxed{\frac{|[\gamma',\gamma'']|}{|\gamma'|^3}}$$

$$\dot{n} = a_{21}v$$

$$(v,n)=0 \quad \Big| \frac{d}{ds}$$

$$(\dot{v},n)+(v,\dot{n})=0$$

$$(kn,n)+(v,a_{21}v)=0 \Rightarrow k+a_{21}=0$$

$$a_{21}=-k$$

$$\begin{cases}\dot{v} = \overset{0}{a_{11}}v + a_{12}n = kn \\ \dot{n} = a_{21}v + \overset{0}{a_{22}}n\end{cases} \Leftrightarrow \frac{d}{ds}\begin{pmatrix}v\\n\end{pmatrix} = \begin{pmatrix}0 & k\\ -k & 0\end{pmatrix}\begin{pmatrix}v\\n\end{pmatrix}$$

$$\begin{pmatrix}\dot{v}\\\dot{n}\end{pmatrix}$$

$$(\dot{e},e)=0$$

$$(e,e)=1 \Rightarrow \frac{d}{ds}(e,e) = (\dot{e},e)+(e,\dot{e})=0$$

$$(\dot{e},e)=0 \quad \dot{e}\perp e$$

$r(s) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}\right)$, $\boxed{R = \frac{1}{k}}$

радиус кривизны кривой в точке.

$k(s_0) = |\dot{v}|$

$k = \left\| \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right)' \right\| = \left\| \left(-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right) \right\| = \frac{1}{R}$

$\gamma(s_0) \rightsquigarrow k(s_0)$

Окружность с такой же кривизной назыв. соприкасающ. окружность

$\gamma(s) \rightsquigarrow \vec{R}(s,\varphi) = \gamma(s_0) + \frac{1}{k}\vec{n} + \frac{1}{k}(\cos\varphi, \sin\varphi) =$

$= \gamma(s_0) + \frac{1}{k}\left(n + (\cos\varphi, \sin\varphi)\right)$

$= \gamma(s_0)$

$\gamma(s) \in \mathbb{R}^3$

$v = \dot\gamma$

$n = \dfrac{\dot v}{|\dot v|} \Rightarrow \dot v = kn$

$b = \dfrac{[v,n]}{|[v,n]|} = [v,n] = \left[v, \dfrac{\dot v}{|\dot v|}\right] = \dfrac{[v,\dot v]}{|\dot v|}$

$$\frac{d}{ds}\begin{pmatrix} v \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \varkappa \\ 0 & -\varkappa & 0 \end{pmatrix}\begin{pmatrix} v \\ n \\ b \end{pmatrix}$$

ф. Френе для прост. кривой

$a_{ii} = 0$ т.к. $(\dot e, e) = 0$

$\dot v = kn + a_{13} b$

$\parallel$

$kn \Rightarrow a_{13} = 0,$

$a_{23} = \varkappa$ кручением

его кручение.

$\dot n = -kv + \varkappa b$

$\dot b = a_{31} v + a_{32} n$

$(v,b) = 0 \Rightarrow (\dot v, b) + (v, \dot b) = 0$

$\Rightarrow (kn, b) + (v, a_{31} v + a_{32} n) = 0 \Rightarrow a_{31} \underbrace{(v,v)}_{1} = 0$

$(n,b) = 0$

$(\dot n, b) + (n, \dot b) = 0$

$\Rightarrow (-kv + \varkappa b, b) + (n, a_{32} n) = 0 \Rightarrow a_{32} = -\varkappa$

$\gamma(t)$, t — произв. парам.

$k(t)$, $\varkappa(t)$

$$k(s) = |\dot v| \Rightarrow \frac{|[\gamma', \gamma'']|}{|\gamma'|^3} = k(t)$$

$$\varkappa(s) =$$

$$\ddot v = \frac{d}{ds}\dot v = \frac{d}{ds}(kn) = \dot k n + k \dot n$$

$$\Rightarrow \dot k n + k(-kv + \varkappa b) = \ddot v \quad \Big| (b, \cdot)$$

$$k\varkappa = (b, \ddot v) \Longrightarrow \varkappa = \frac{(b, \ddot v)}{k}, \quad b = [v, n],$$

$$\varkappa = \frac{([v,n], \ddot v)}{k} = \frac{([v, \frac{\dot v}{k}], \ddot v)}{k} = ([v, \dot v], \ddot v)/k^2 = \frac{(v, \dot v, \ddot v)}{k^2}$$

$$\ddot v = \frac{d}{ds}\left(\gamma'' \frac{1}{|\gamma'|^2} + \gamma' \frac{d^2 t}{ds^2}\right)$$

$$= \frac{d\gamma''}{dt}\frac{dt}{ds}\frac{1}{|\gamma'|^2} + \gamma''\frac{d}{ds}\frac{1}{|\gamma'|^2} + \frac{d\gamma'}{dt}\frac{dt}{ds}\frac{d^2t}{ds^2} + \gamma'\frac{d^3 t}{ds^3}$$

$$\varkappa(t) =$$

$$v = \frac{\gamma'}{|\gamma'|}, \quad \dot v = \ddot\gamma = \frac{d}{ds}\dot\gamma = \frac{d}{ds}\left(\frac{d\gamma}{dt}\frac{dt}{ds}\right) = \frac{d^2\gamma}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{d\gamma}{dt}\frac{d^2t}{ds^2} = \gamma''\frac{1}{|\gamma'|^2} + \gamma'\frac{d^2t}{ds^2}$$

$$) =$$

$$\ddot{v} = \frac{\gamma'''}{|\gamma'|^3} + \gamma'' \frac{d}{ds} \frac{1}{|\gamma'|^2} + \gamma'' \frac{1}{|\gamma'|} \frac{d^2 t}{ds^2} + \gamma' \frac{d^3 t}{ds^3}$$

$$(*)$$

$$\varkappa = \frac{([v, \dot{v}], \ddot{v})}{k^2} =$$

$$[v, \dot{v}] = \left[\frac{\gamma'}{|\gamma'|}, \gamma'' \frac{1}{|\gamma'|^2} + \cancel{\gamma' \frac{d^2 t}{ds^2}}\right] =$$

$$= \frac{1}{|\gamma'|^3} [\gamma', \gamma'']$$

$$\left(\frac{1}{|\gamma'|^3} [\gamma', \gamma''], (*)\right) = \frac{2}{|\gamma'|^3} \left([\gamma', \gamma''], \gamma'''\right)$$

$$\varkappa = \frac{(\gamma', \gamma'', \gamma''')}{|[\gamma', \gamma'']|^2}$$