Общее задание

Найти натуральное уравнение кривой

Кривизна:

$$k(t) = \frac{\left|[\bar{\gamma}', \bar{\gamma}'']\right|}{|\gamma'|^3}$$

Кручение:

$$\Xi(t) = \frac{(\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''')}{\left|[\bar{\gamma}', \bar{\gamma}'']\right|^2}$$

№1

Условие

$$\gamma(t) = (a\cos(t), a\sin(t), bt) \quad t \geq 0$$

Решение

Кривизна:

$$k(t) = \frac{\left|[\bar{\gamma}', \bar{\gamma}'']\right|}{|\gamma'|^3}$$

Кручение:

$$\Xi(t) = \frac{(\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''')}{\left|[\bar{\gamma}', \bar{\gamma}'']\right|^2}$$

$$\gamma'(t) = (-a\sin(t), a\cos(t), b)$$
$$|\gamma'(t)| = \sqrt{a\sin^2(t) + a^2\cos^2(t) + b^2} = \sqrt{a^2 + b^2}$$
$$\gamma''(t) = (-a\cos(t), -a\sin(t), 0)$$
$$\gamma'''(t) = (a\sin(t), -a\cos(t), 0)$$

$$[\gamma', \gamma''] = \left(ab\sin(t), -ab\cos(t), a^2\right)$$
$$|[\gamma', \gamma'']| = a\sqrt{a^2 + b^2}$$
$$k(t) = \frac{a}{a^2 + b^2}$$

$$[\gamma'', \gamma'''] = \left(0, 0, a^2\right)$$
$$(\gamma', \gamma'', \gamma''') = a^2 b$$
$$\Xi(t) = \frac{b}{a^2 + b^2}$$

$$s(t) = \int\limits_1^t |\gamma'(\phi)| d\phi = \sqrt{b^2 + a^2} t$$
$$t = \frac{s}{\sqrt{a^2 + b^2}}$$
$$\gamma(s(t)) = \left(a\cos(\frac{s}{\sqrt{a^2 + b^2}}), a\sin(\frac{s}{\sqrt{a^2 + b^2}}), \frac{bs}{\sqrt{a^2 + b^2}} \right)$$
$$k(s) = \frac{a}{a^2 + b^2}$$
$$\Xi(s) = \frac{b}{a^2 + b^2}$$

№2

Условие

$$\gamma(t) = \left(\frac{t^2}{2}, \frac{2t^3}{3}, \frac{t^4}{2} \right) \quad t \geq 0$$

Решение

$$\gamma'(t) = \left(t, 2t^2, 2t^3 \right)$$
$$|\gamma'(t)| = \sqrt{t^2 + 4t^4 + 4t^6} = 2t^3 + t$$
$$\gamma''(t) = \left(1, 4t, 6t^2 \right)$$
$$[\gamma', \gamma''] = \left(4t^4, -4t^3, 2t^2 \right)$$
$$|[\gamma', \gamma'']| = 2t^2 \left(2t^2 + 1 \right)$$
$$\gamma'''(t) = (0, 4, 12t)$$
$$[\gamma'', \gamma'''] = \left(24t^2, -12t, 4 \right)$$
$$|[\gamma'', \gamma''']| = 4\sqrt{36t^4 + 9t^2 + 1}$$
$$(\gamma', \gamma'', \gamma''') = 8t^3$$

$$k(t) = \frac{2}{t(2t^2 + 1)^2}$$
$$\Xi(t) = \frac{t^3}{2\left(36t^4 + 9t^2 + 1\right)}$$

$$s(t) = \int\limits_0^t |\gamma'(\phi)| d\phi = \int\limits_0^t \left[2\phi^3 + \phi \right] d\phi = \frac{t^4 + t^2}{2}$$
$$t_{1,2} = \pm \frac{\sqrt{\sqrt{8s + 1} - 1}}{\sqrt{2}}$$
$$t_{3,4} = \pm i \frac{\sqrt{\sqrt{8s + 1} + 1}}{\sqrt{2}}$$

Так как $t \geq 0$, то $t = \dfrac{\sqrt{\sqrt{8s+1}-1}}{\sqrt{2}}$

$$k(s) = \frac{2}{\frac{\sqrt{\sqrt{8s+1}-1}}{\sqrt{2}}\left(2\left(\frac{\sqrt{\sqrt{8s+1}-1}}{\sqrt{2}}\right)^2+1\right)^2}$$

$$\Xi(s) = \frac{\left(\frac{\sqrt{\sqrt{8s+1}-1}}{\sqrt{2}}\right)^3}{2\left(36\left(\frac{\sqrt{\sqrt{8s+1}-1}}{\sqrt{2}}\right)^4+9\left(\frac{\sqrt{\sqrt{8s+1}-1}}{\sqrt{2}}\right)^2+1\right)}$$

№3

Условие

$\gamma(t) = (a \cdot \cosh(t), b \cdot \sinh(t), at) \quad a > 0$

Решение

$$\gamma'(t) = (a\sinh(t), b\cosh(t), a)$$
$$\gamma''(t) = (a\cosh(t), b\sinh(t), 0)$$
$$\gamma'''(t) = (a\sinh(t), b\cosh(t), 0)$$

$$k(t) = \frac{a}{\cosh^2(t)\left(a^2+b^2\right)}$$
$$\Xi(t) = \frac{b}{\cosh^2(t)\left(a^2+b^2\right)}$$

$$s(t) = \int\limits_0^t \sqrt{a^2\left(\sinh^2(\phi)+1\right)+b^2\cosh(\phi)}d\phi = \sqrt{a^2+b^2}\sinh(t)$$
$$t = \operatorname{arcsinh}\left(\frac{s}{\sqrt{a^2+b^2}}\right)$$
$$\gamma = \left(\operatorname{arcsinh}\left(\frac{s}{\sqrt{a^2+b^2}}\right), a, b\right)$$

$$\cosh\left(\operatorname{arcsinh} x\right) = \sqrt{1+x^2}$$
$$\cosh^2\left(\operatorname{arcsinh}\left(\tfrac{s}{\sqrt{a^2+b^2}}\right)\right) = 1+\left(\tfrac{s}{\sqrt{a^2+b^2}}\right)^2 = \tfrac{a^2+b^2+s^2}{a^2+b^2}$$

$$k(s) = \frac{a}{a^2+b^2+s^2}$$
$$\Xi(s) = \frac{b}{a^2+b^2+s^2}$$