Additional PDE Theory

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1 Introduction

We present some additional ways to solve PDEs and ODEs with the Banach Contraction Mapping Theorem, Implicit Function Theorem, and Green's Functions.

2 Contraction Mapping

1. Consider the first-order differential equation,

$$u'(t) + u(t) = \cos(u(t))$$

posed as an initial-value problem for $t > 0$ with initial condition

$$u(0) = u_0.$$

a) Use the contraction-mapping theorem to show that there is exactly one solution, u, corresponding to any given $u_0 \in \mathbb{R}$.

Proof. First, we rewrite our equation as

$$u(t) = \int_0^t \cos(u(s)) - u(s) \ \mathrm{d}s + u_0 = (Fu)(t).$$

Generally, we might first establish that F maps from a ball to a ball, but we will remark on this after showing F is a contraction. Now establishing that F is a contraction,

$$\begin{aligned} |(Fu)(t) - (Fv)(t)| &\leqslant \int_0^t |\cos(u(s)) - \cos(v(s)) + v(s) - u(s)| \ \mathrm{d}s \\ &= \int_0^t |\sin(\eta)||u(s) - v(s)| \ \mathrm{d}s \int_0^t |v(s) - u(s)| \ \mathrm{d}s \\ &\leqslant t||u - v||_L + t||u - v||_{L^\infty} \\ &\leqslant 2t||u - v||_{L^\infty}. \end{aligned}$$

We then enforce $2t < 1 \implies t < 1/2$ so we can take $t = 1/(2+\epsilon)$ then we can repeat by advancing over intervals of this size.

Coming back to F mapping bounded functions to bounded functions notice that our bound in the contraction constant is does not depend on where u and v are from so we don't require that F is mapping from a ball to a ball.

∎

b) Prove that there is a number ξ such that $\lim_{t\to\infty} u(t) = \xi$ for any solution u, independent of the value of u_0.

Proof. We want $u'(t) = (Gu)(t)$ then if there exists a ξ such that $G(\xi) = 0$ then $u'(t) = 0$ hence there is an equilibrium solution. $(Gu\xi) = 0 \iff \cos(\xi) = \xi$ so we are looking for a fixed point of $\cos(\xi) = F\xi$. Now observe that since $u'(t) = 0$ then $u(t)$ is a constant. Applying the mean value theorem for $x, y \in [0,1]$ we have that $F\xi = \cos(\xi)$ has a unique fixed point. To show that any solution tends to ξ notice that if we start below ξ then Fu is positive (so u will tend up to the fixed point) and if we start above ξ then Fu is negative (so u will tend down to the fixed point). ∎

2. Set up and apply the contraction mapping principle to show that the problem

$$-u_{xx} + u - \epsilon u^2 = f(x), \quad x \in \mathbb{R},$$

has a smooth bounded solution if $\epsilon > 0$ is small enough where $f \in \mathcal{S}(\mathbb{R})$.

Proof. First, we write our equation as,

$$-u_{xx} + u = f + \epsilon u^2 = g.$$

Since we have yet to decide where things live we proceed formally and take the Fourier Transform,

$$\begin{aligned}
&\mathcal{F}(-u_{xx} + u) = \mathcal{F}(g) \\
&\implies (1+\xi)^2 \hat{u} = \hat{g} \\
&\implies u = g * \left(\frac{1}{1+\xi^2}\right)^{\vee} \\
&\implies u = \int \frac{1}{2} e^{-|x-y|}(f(y) + \epsilon u^3(y) \, \mathrm{d}y \\
&\implies u = Gu.
\end{aligned}$$

Since we are looking for smooth bounded solution we look in the space of $C_B(B_R(0))$ where the radius of this ball will be determined. First, we show that $G : B_R(0) \to B_R(0)$.

$$\begin{aligned}
\left| \int_{B_R} \frac{1}{2} e^{-|x-y|}(f(y) + \epsilon u^2(y) \, \mathrm{d}y \right| &\leqslant \frac{1}{2} ||e^{-|x-y|}||_{L^\infty(B_R)} ||f(y) + \epsilon u^2(y)||_{L^\infty(B_R)} \\
&\leqslant ||f||_{L^\infty(B_R)} + \epsilon ||u||^2_{L^\infty(B_R)} \\
&\leqslant ||f||_{L^\infty(B_R)} + CR^2.
\end{aligned}$$

Remark: L^∞ norm on bounded domain is sub-multiplicative hence $||u^2|| \leqslant ||u||^2$.

The above calculation gives us our first constraint that $||f||_{L^\infty(B_R)} + R^2\epsilon < R$.

Our goal now is to show that G is a contraction.

$$\begin{aligned}
|Gu(x) - Gv(x)| &= \left| \frac{\epsilon}{2} \int_{B_R} e^{|x-y|}(u^2 - v^2) \, \mathrm{d}y \right| \\
&\leqslant \frac{\epsilon}{2} \int_{B_R} |e^{-|x-y|}(u+v)(u-v)| \, \mathrm{d}y \\
&\leqslant \frac{\epsilon}{2} ||u-v||_{L^\infty(B_R)} ||u+v||_{L^\infty(B_R)} ||e^{-|x-y|}||_{L^1(B_R)} \\
&\leqslant \frac{\epsilon}{2} 4R ||u-v||_{L^\infty(B_R)} \\
&= 2R\epsilon ||u-v||_{L^\infty(B_R)}.
\end{aligned}$$

This gives us our second constraint that $2R\epsilon < 1$.

Combining our constraints we have require that $R = 1/(4\epsilon) \implies ||f||_{L^\infty(B_R)} + \frac{1}{16\epsilon} \leqslant \frac{1}{4\epsilon} \implies \epsilon \leqslant \frac{3}{16}\frac{1}{||f||_{L^\infty}}$. ∎

3. Consider the PDE

$$\frac{\partial u}{\partial t} + \frac{\partial^2}{\partial t \partial x^2} u - \epsilon u^3 = f, \quad -\infty < x < \infty, t > 0,$$
$$u(x,0) = g(x).$$

Use the Fourier transform and a contraction mapping argument to show that there exists a solution for small enough ϵ, at least up to some time $T < \infty$. In what spaces should f and g lie?

Proof. First, we rewrite our PDE as

$$\begin{aligned} &\frac{\partial}{\partial t}\big(u - u_{xx}\big) - \epsilon u^3 = f \\ \iff &\frac{\partial}{\partial t}\big(u - u_{xx}\big) = f + \epsilon u^3 = h(u(x,t), f(x,t), x, t). \end{aligned}$$

We proceed formally by taking the Fourier transform in space to obtain,

$$\begin{aligned} &\frac{\partial}{\partial t}(1+\xi^2)\hat{u} = \hat{h} \\ \iff &\frac{\partial \hat{u}}{\partial t} = \frac{\hat{h}}{1+\xi^2}. \end{aligned}$$

We can now write this as,

$$\hat{u}(\xi, t) = \hat{u}(\xi, t) + \frac{1}{1+\xi^2}\int_0^t \hat{h}\ \mathrm{d}s.$$

Inverting our transform now gives,

$$\begin{aligned} u(x,t) &= g(x) + \int_0^t h(s,x) * \frac{1}{2}e^{-|x|}\ \mathrm{d}s \\ &= g(x) + \frac{1}{2}\int_{\mathbb{R}}\int_0^t h(s,x)e^{-|x-y|}\ \mathrm{d}y\mathrm{d}s \\ &= g(x) + \frac{1}{2}\int_{\mathbb{R}}\int_0^t (f(y) + \epsilon u^3(y,s))e^{-|x-y|}\ \mathrm{d}y\mathrm{d}s = Fu. \end{aligned}$$

We now want to show that F is a contraction.

$$\begin{aligned} |Fu - Fv| &= \frac{\epsilon}{2}\int_0^t \int_{\mathbb{R}} e^{-|x-y|}(u^3(y,s) - v^3(y,s)\ \mathrm{d}y\mathrm{d}s \\ &\leqslant \frac{\epsilon}{2}t||e^{-|x-y|}||_{L^1}||u^3 - v^3||_{L^\infty} \\ &= \frac{\epsilon}{2}t||e^{-|x-y|}||_{L^1}||u-v||_{L^\infty}||u^2 + uv + v||_{L^\infty} \\ &\leqslant \epsilon t 3R^2||u-v||_{L^\infty}. \end{aligned}$$

Since the contraction constant depends on R we want to show that F maps from the ball, $B_R(0)$ to the ball. This will give us restrictions on what the radius should be.

$$\begin{aligned} ||(Fu)(x,t)||_{L^\infty} &= \left|\left| \int_0^t (f + \epsilon u(s,x)^3) * \frac{1}{2} e^{-|x|} \ \mathrm{d}s + g(x) \right|\right| \\ &\leqslant ||f||_{L^\infty} \frac{1}{2} ||e^{-|x|}|| + \epsilon ||u^3||_{L^\infty} + ||g||_{L^\infty} \\ &\leqslant ||f||_{L^\infty} + ||g||_{L^\infty} + \epsilon R^3. \end{aligned}$$

We now have the constraints $\epsilon t 3R^2 < 1$ and $||f||_{L^\infty} + ||g||_{L^\infty} + \epsilon R^3 < R$. We then need to select an R and ϵ such that

$$||f||_{L^\infty} + ||g||_{L^\infty} < R - \epsilon R^3.$$

We then have a solution up to time $t < \frac{1}{3R^2\epsilon}$. ■

4. Suppose $f \in C^0([0,1])$ and that we want to solve

$$\frac{1}{1+\epsilon u^2} u' = f(x) \quad x \in (0,1) \qquad \text{and } u(0) = 0.$$

Use the Banach Contraction Mapping Theorem to show that there is a unique continuous solution in a closed ball about u_0 in an appropriate Banach space for ϵ sufficiently small. Note that there is a unique solution $u_0(x)$ when $\epsilon = 0$.

Proof. First, we want to check that $G : C(B_R(0)) \to C(B_R(0))$ with R to be determined.

$$\begin{aligned} ||Gu||_{L^\infty} &= \sup_t \left| \int_0^t f(s)(1 + \epsilon u^2(s)) \ \mathrm{d}s \right| \\ &\leqslant ||f||_{L^\infty}(1 + \epsilon R^2). \end{aligned}$$

We then enforce $||f||_{L^\infty}(1 + \epsilon R^2) < R$. Now we check that G is indeed a contraction.

$$\begin{aligned} |G(u) - G(v)| &\leqslant \sup_t \int_0^t |f(s)\epsilon(u^2 - v^2)| \ \mathrm{d}s \\ &= \sup_t \int_0^t |f(s)\epsilon(u-v)(u+v)| \ \mathrm{d}s \\ &\leqslant ||f||_{L^\infty} \epsilon 2R ||u - v||_{L^\infty}. \end{aligned}$$

This provides an additional constraint $||f||_{L^\infty} \epsilon 2R < 1$. ■

3 Implicit Function Theroem

5. Suppose $f \in C^0([0,1])$ and that we want to solve

$$\frac{1}{1+\epsilon u^2} u' = f(x) \quad x \in (0,1) \quad \text{ and } u(0) = 0.$$

Use the Implicit Function Theorem to show that there is a continuously differentiable solution for ϵ small enough. Note that there is a unique solution $u_0(x)$ when $\epsilon = 0$.

Proof. The Implicit Function Theorem gives us the unique existence of a function satisfying $F(z, g(z,y)) = y$ and we want that F to be our u, the solution to the above ODE. To this end, we define F to be the left-hand side of the ODE by $F(\epsilon, u(\epsilon, x)) = f$ with $F : \mathbb{R} \times X \to Y$ where $X = \{u \in C^1([0,1]) \mid u(0) = 0\}, Y = \{u \in C^0([0,1])$, which are closed subspaces of Banach spaces, hence, themselves Banach spaces. To satisfy the assumptions of the Implicit Function Theorem we need F to be $C^1(\mathbb{R} \times X)$ in a neighborhood of $(\epsilon = 0, u_0(x))$ and $D_x F(\epsilon = 0, u_0(x)) \in GL(X,Y)$. We now check the C^1 condition which is the Frechet Derivative exists and is continuous. We use Proposition of 9.3 which says that if partials exists and are continuous then the Fréchet Derivative exists.

$$D_\epsilon F(0, u_0(x)) = u' u^2 (1 + \epsilon u^2)^{-2}.$$

which is continuous given a constraint on ϵ. Now checking the partial in u we need to be a little more careful and apply to an $h \in C^1(X)$

$$D_u F(0, u_0(x))(h) = (1+\epsilon u^2)^{-1} h' - (1+\epsilon u^2)^{-2} 2\epsilon u u' h$$

which is continuous. Now to check that the partial derivative with respect to u at $\epsilon, u_0(x)$ is an element of $GL(X,Y)$. We have $D_u F(0, u_0(x))(h) = h'$. We need to check that this map 1) has trivial kernel, 2) surjective, 3) bounded, and 4) Linear. First, assume $h \in X$ then $D_u F(0, u_0(x))(h) = 0$ then h is a constant, but since $\in X$ then $h = 0$. For surjectivity (also injectivity actually) we can take any $f \in Y$ we want a u such that $u(0) = 0$ and $D_u F(0, u)) = f$ then since we are given there is a solution $u_0(x)$ we are done. For boundedness (with respect to C^1 norm) we have that $||h'||_{C_0} \leqslant ||h||_{C^1}$.

Hence by the Implicit Function Theorem there exists a unique $u(\epsilon, x)$ for sufficiently small ϵ solving the ODE. ∎

4 Green's Functions

Under Construction.