Perturbation Methods

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1 Introduction

We present a few problems and analyze them utilizing perturbation techniques including regular perturbation series, Poincare-Linstedt metho, Multiple Scale Expansion, and WKB Approximation.

2 Examples

1. Consider the weakly nonlinear oscillator:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + \epsilon y^5 = 0$$

with $y(0) = 0$ and $y'(0) = A > 0$ and with $0 < \epsilon \ll 1$.

a) Use a regular perturbation expansion and calculate the first two terms.

Proof. First we will suppose that $y(t) = \sum_{n=0}^{\infty} \epsilon^n y_n(t) = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$. Now rewriting the equation in our expanded form gives ,

$$(y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + \dots) + (y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots) + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots)^5 = 0$$

with boundary conditions

$$\begin{aligned}
(y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \dots) &= 0 \\
(y_0'(0) + \epsilon y_1'(0) + \epsilon^2 y_2'(0) + \dots) &= A > 0
\end{aligned}$$

Now collecting orders of ϵ,

$\mathcal{O}(1)$ $y_0'' + y_0 = 0$ with $y_0(0) = 0$, $y_1'(0) = A$
$\mathcal{O}(\epsilon)$ $y_1'' + y_1 + y_0^5 = 0$ with $y_1(0) = 0$, $y_1'(0) = 0$
$\mathcal{O}(\epsilon^2)$ $y_2'' + y_2 + y_1^5 = 0$ with $y_2(0) = 0$, $y_2'(0) = 0$

Our leading order equation has general solution $y_0(t) = c_1 \sin(t) + c_2 \cos(t)$. Enforcing boundary conditions gives

$$y_0(0) = A \sin(t).$$

The first order correction term is found by solving

$$\begin{aligned}
y_1'' + y_1 &= -(A^5 \sin^5(t)) \\
&= -\frac{A^5}{16}(10\sin(t) - 5\sin(3t) + \sin(5t))
\end{aligned}$$

with boundary conditions specified above. This is a 2[nd] order non homogeneous constant coefficient problem with solution,

$$y_1(t) = A^5\big(-\frac{1}{3}\sin(t) - \frac{1}{16}\sin(3t) + \frac{1}{240}\sin(5t) + \frac{1}{2}t\cos(t)\big).$$

Notice that we have a secular growth term, $t\cos(t)$, that shows up because in our first order correction equation we have a sine term which is sitting in the null space of our (self) adjoint equation. ■

b) Determine at what time the approximation in *a*) fails to hold.

Proof. Observe that our our order 1 in ϵ equation is,

$$\epsilon A^5\big(-\frac{1}{3}\sin(t) - \frac{1}{16}\sin(3t) + \frac{1}{240}\sin(5t) + \frac{1}{2}t\cos(t)\big).$$

When $t = \frac{1}{\epsilon}$ then the cosine term is no longer of the order ϵ and will begin to dominate leading order behavior. ■

c) Use a Poincare-Linstedt expansion and determine the first two terms and frequency corrections.

Proof. First we consider a new time scale $\tau = (\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots)t$. This change of variables gives $\frac{\mathrm{d}}{\mathrm{d}t} = \omega\frac{\mathrm{d}}{\mathrm{d}\tau}$. So our equation in τ is

$$\omega^2\frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau} + y(\tau) + \epsilon y(\tau)^5 = 0.$$

Now suppose that we can write our solution as $y(\tau) = \sum_{n=0}^{\infty}\epsilon y_n(\tau)$. Notice that the notation, f', is denoting differentiation with respect to τ. Writing our differential equation with expanded y and ω,

$$(\omega_0 + \epsilon\omega_1^2 + \dots)^2(y_0'' + \epsilon y_1'' + \dots) + (y_0 + \epsilon y_1 + \dots) + \epsilon(y_0 + \epsilon y_1 + \dots) = 0$$

with boundary conditions

$$(y_0(0) + \epsilon y_1(0) + \dots) = 0$$
$$(\omega_0 + \epsilon\omega_1^2 + \dots)(y_0'(0) + \epsilon y_1'(0) + \dots) = A > 0.$$

Now collecting orders of ϵ,

$\mathcal{O}(1)$ $y_0'' + \frac{1}{w_0^2}y_0 = 0$ with $y_0(0) = 0, \omega_0 y_0'(0) = A$

$\mathcal{O}(\epsilon)$ $y_1'' + \frac{1}{\omega_0^2}y_1 = -\frac{1}{\omega_0^2}2\omega_0\omega_1 y_0'' - \frac{1}{\omega_0^2}y_0^5$ with $y_1(0) = 0, \omega_0 y_1'(0) + \omega_1 y_0'(0) = 0$

So our leading order equation has general solution of sines and cosines. Enforcing boundary conditions gives,

$$y_0(\tau) = A\sin(\omega_0\tau).$$

Since we are looking for 2π periodic solutions we have $\omega_0 = 1$. Then we can write our first order correction equation as,

$$\begin{aligned} y_1'' + y_1 &= 2\omega_1 A\sin(\tau) - A^5\sin^5(\tau) \\ &= \sin(\tau)\big(2\omega_1 A - \frac{A^5 5}{8}\big) + \frac{5}{16}\sin(3\tau) + \frac{1}{16}\sin(5\tau) \end{aligned}$$

Notice that we have a $\sin(\tau)$ which is sitting in the null space of our (self) adjoint operator. So we define ω_1 to satisfy the Fredholm Alternative Theorem by killing the $\sin(\tau)$ term. Then the above equation gives,

$$\omega_1 = \frac{A^4 5}{16}.$$

Now to obtain the first order correction we need to solve the inhomogeneous 2nd order constant coefficient differential equation,

$$y_1'' + y_1 = 5\sin(3\tau) + \sin(5\tau).$$

with boundary conditions

$$y_1(0) = 0 \text{ and } y_1'(0) + \frac{5A^5}{16} = 0$$

We have a 2^{nd} order non homogeneous constant coefficient problem which has solution,

$$y(\tau) = -\frac{1}{48}\sin(\tau)(15A^4 + 64\cos(2t) + 4\cos(4t) - 68).$$

So our first order approximation in t is,

$$\begin{aligned} y_{\text{approx}}(t) = {} & A\sin\left((1+\epsilon\frac{5A^4}{16})t\right) \\ & + \epsilon\big[-\frac{1}{48}\sin\left((1+\epsilon\frac{5A^4}{16})t\right)(15A^4 + 64\cos\left(2(1+\epsilon\frac{5A^4}{16})t\right) \\ & + 4\cos\left(4(1+\epsilon\frac{5A^4}{16})t\right) - 68)\big] + \mathcal{O}(\epsilon^2) \end{aligned}$$

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d) For $\epsilon = 0.1$, plot the numerical solution, the regular expansion solutions, and the Poincare-Linstedt solution for $0 \leq t \leq 20$.

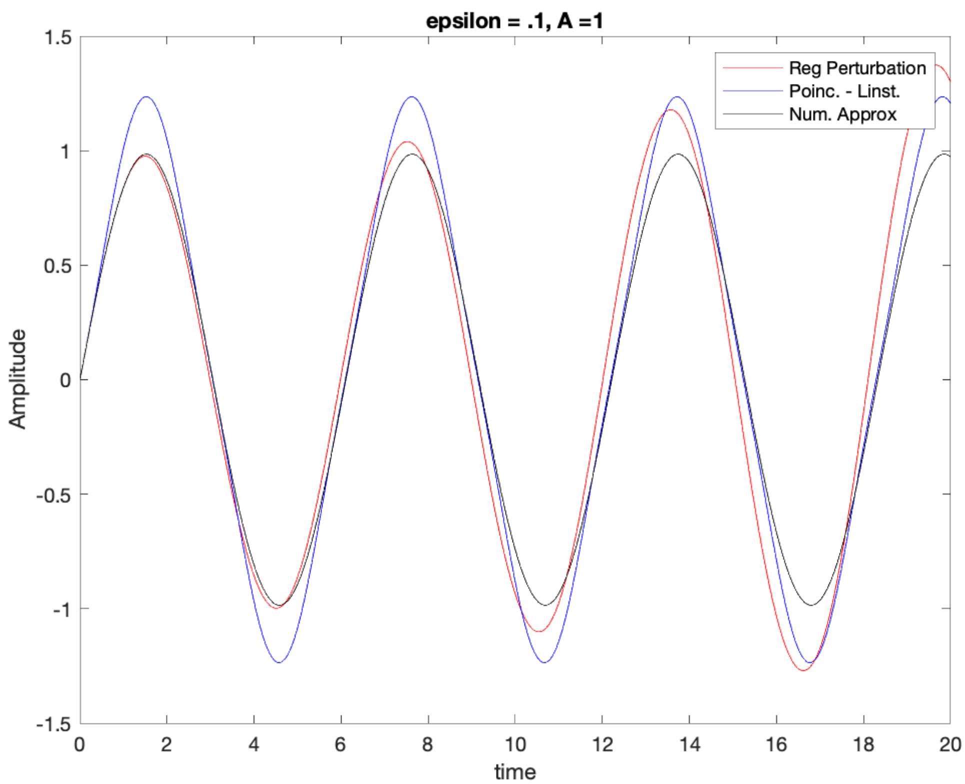

Figure 1: Weak non linear oscillator approximations

2. Consider Rayleigh's equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + \epsilon\Big[-\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{3}\Big(\frac{\mathrm{d}y}{\mathrm{d}t}\Big)\Big] = 0$$

which has only one periodic solution called a "limit cycle" ($0 < \epsilon \ll 1$). Given $y(0) = 0$ and $\frac{\mathrm{d}y(0)}{\mathrm{d}t} = A^*$.

a) Use a multiple scale expansion to calculate the leader order behavior.

Proof. First we define a slow time scale $\tau = \epsilon t$ and we will treat τ and t as independent variables. We also define $y = y(t, \tau)$. Our change of variables yields,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \epsilon y_\tau + y_t$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = y_{tt} + 2\epsilon y_{\tau t} + \epsilon^2 y_{\tau\tau}$$

Writing our equation in terms of $y = y(t, \tau)$ we have,

$$(y_{tt} + 2\epsilon y_{\tau t} + \epsilon^2 y_{\tau\tau}) + y + \epsilon(-(y_t + \epsilon y_\tau) + \frac{1}{3}(y_t + \epsilon y_\tau)^3) = 0.$$

Now suppose that we can write $y(t, \tau) = \sum_{n=0}^{\infty} y_n \epsilon^n$. Rewriting our differential equation in expanded y,

$$\begin{aligned}
&[(y_{0_{tt}} + \epsilon y_{1_{tt}} + \dots) + 2\epsilon(y_{0_{\tau t}} + \epsilon y_{1_{\tau t}} + \dots) + \epsilon^2(y_{0_{\tau\tau}} + \epsilon y_{1_{\tau\tau}} + \dots)] + (y_0 + \epsilon y_1 + \dots) \\
&\quad + \epsilon\Big[-\big((y_{0_t} + \epsilon y_{1_t} + \dots) + \epsilon(y_{0\ \tau} + \epsilon y_\tau)\big) + \frac{1}{3}\big((y_{0_t} + \epsilon y_{1_t} + \dots) + \epsilon(y_{0\ \tau} + \epsilon y_\tau)\big)^3\Big] \\
&\quad = 0
\end{aligned}$$

with boundary conditions

$$y_0(0,0) + \epsilon y_1(0,0) + \dots = 0$$
$$(y_{0_t}(0,0) + \epsilon y_{1_t}(0,0) + \dots) + \epsilon(y_{0_\tau}(0,0) + \epsilon y_{1_\tau}(0,0) + \dots) = A^*$$

Collecting terms in orders of ϵ,

$\mathcal{O}(1)$ $y_{0_{tt}} +\ y_0 = 0$ with $y_0(0,0) = 0, y_{0_t}(0,0) = A^*$

$\mathcal{O}(\epsilon)$ $y_{1_{t\tau}} + y_1 = -2y_{0_{\tau t}} + y_{0_t} - y_{0_t}^3$ with $y_1(0,0) = 0, y_{1_t}(0,0) + y_{0_\tau} = 0$

So our leading order solution, assumed to be in product form,

$$y_0(t, \tau) = A(\tau)\cos(t) + B(\tau)\sin(t)$$

with boundary conditions $A(0) = 0, B(0) = A^*$.

Now our first order correction can be written,

$$\begin{aligned}
y_{1_{tt}} + y_1 &= \cos(t)(-2B_\tau + B - BA^2) + \sin(t)(2A_\tau - A + AB^2) \\
&\quad + \cos^3(t)(BA^2 - \frac{B^3}{3}) + \sin^3(t)(\frac{B^3}{3} - AB^2) \\
&= \cos(t)(B - 2B_\tau - \frac{1}{4}BA^2 - \frac{1}{4}B^3) + \sin(t)(2A_\tau - A + \frac{1}{4}AB^2 + \frac{1}{4}B^3) \\
&\quad + \frac{1}{4}\cos(3t)(BA^2 - \frac{B^3}{3}) - \frac{1}{4}\sin(3t)(\frac{A^3}{3} - AB^2).
\end{aligned}$$

Now we need to enforce solvability of the (self) adjoint problem. This requires that the sines and cosines term drop out since they are sitting in the null space of the adjoint. This requires,

$$2A_\tau - A + \frac{1}{4}AB^2 + \frac{1}{4}A^3 = 0$$
$$2B_\tau - B + \frac{1}{4}BA^2 + \frac{1}{4}B^3 = 0.$$

Following the example in lecture we multiply the top by A and bottom by B and add them together to obtain,

$$2AA_\tau + 2BB_\tau - (A^2 + B^2) + \frac{1}{4}(A^4 + 2(AB)^2 + B^4).$$

Now make the substitution $\alpha(\tau) = A(\tau)^2 + B(\tau)^2$. This results in a separable differential equation,

$$\alpha_\tau - \alpha + \frac{1}{4}\alpha^2 = 0.$$

Solving gives,

$$\frac{\alpha}{1 - \frac{\alpha}{4}} = \frac{A^2 + B^2}{1 - \frac{1}{4}(A^2 + B^2)} = Ce^\tau.$$

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Using initial condition $A(0)^2 + B(0)^2 = {A^*}^2$ we have that $C = \frac{{A^*}^2}{1 - \frac{1}{4}{A^*}^2}$.

b) Use Poincare-Linstedt expansion and find an expansion of $A = A_0 + \epsilon A_1 + \dots$ to calculate the leading-order solution and the first non-trivial frequency shift for the limit cycle.

Proof. Consider the stretched time scale $\tau = \big(\sum_{n=0}^\infty w_n \epsilon^n\big)t$. Then our equation in τ is,

$$\omega^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + y + \epsilon\big[-\omega\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{1}{3}\Big(\frac{\mathrm{d}y}{\mathrm{d}\tau}\Big)^3\big].$$

Now suppose that $y(\tau) = \sum_{n=0}^\infty y_n(\tau)\epsilon^n$ and that $A = A_0 + \epsilon A_1 + \dots$. Then expanding our equation in ω and τ gives,

$$(\omega_0 + \epsilon\omega_1 + \dots)^2(y_0'' + \epsilon y_1'' + \dots) + (y_0 + \epsilon y_1 + \dots)$$
$$+ \epsilon\big[-(\omega_0 + \epsilon\omega_1 + \dots)(y_0' + \epsilon y_1' + \dots) + \frac{1}{3}\big((\omega_0 + \epsilon\omega_1 + \dots)(y_0' + \epsilon y_1' + \dots)\big)^3\big]$$

with boundary conditions

$$y_0(0) + \epsilon y_1(0) + \dots = 0$$
$$(\omega_0 + \epsilon\omega_1 + \dots)(y_0'(0) + \epsilon y_1'(0) + \dots) = A_0 + \epsilon A_1 + \dots.$$

Collecting terms of order ϵ,

$\mathcal{O}(1)$ $\omega_0^2 y_0'' + y_0 = 0$ with $y_0(0) = 0, \omega_0 y_0'(0) = A_0$

$\mathcal{O}(\epsilon)$ $\omega_0^2 y_1'' + y_1 = \omega_0 y_0' - \frac{1}{3}(y_0')^3 - 2\omega_0\omega_1 y_0''$ with $y_1(0) = 0, \omega_1 y_0'(0) + y_1'(0) = A_1$

$\mathcal{O}(\epsilon^2)$ $\omega_0^2 y_2'' + y_2 = -y_0(2\omega_0\omega_2 + \omega_1) - y_1(2\omega_0\omega_1) + \omega_1 y_0' + \omega_0 y_1' + \omega_0^3 y_0'^2 y_1' + \omega_0^2\omega_1 y_0'^3$ with $y_2(0) = 0, \omega_0\epsilon^2 y_2'(0) + \omega_1 y_1'(0) + \omega_2 y_0'(0) = A_2$

We are looking for a limit cycle with period 2π so $\omega_0 = 1$. Our leading order equation has solution,

$$y_0(\tau) = A_0 \sin(\tau).$$

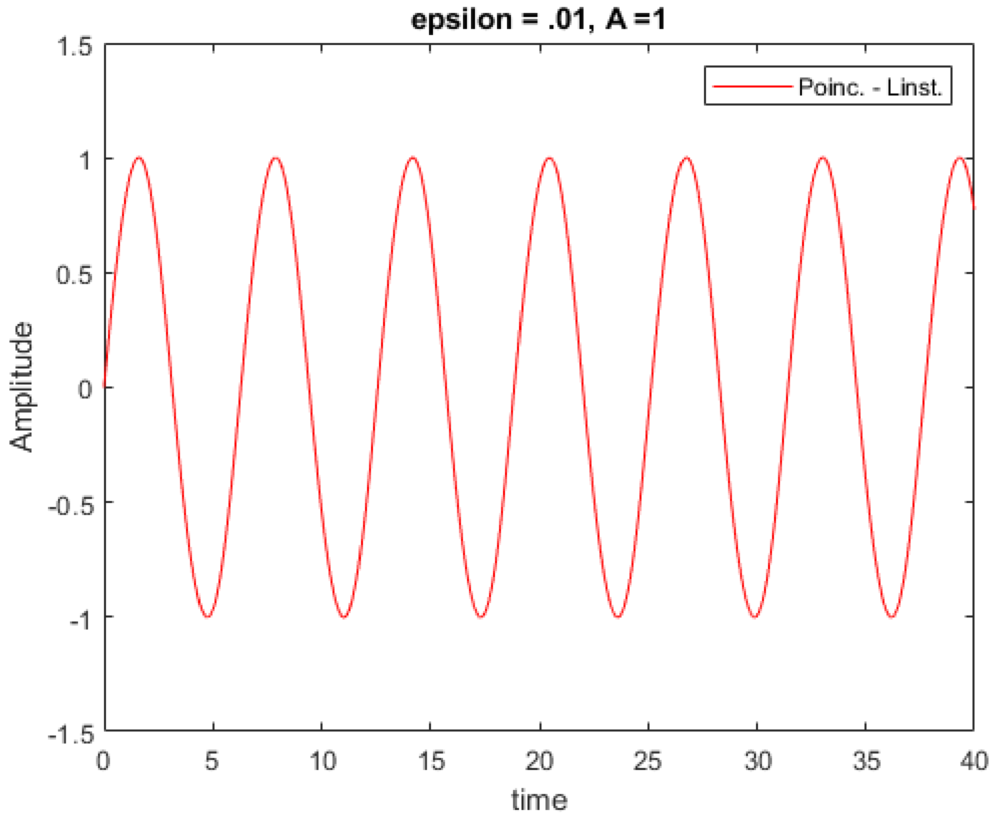

Figure 2: Limit Cycle found from Poincare - Linstedt Approx

Then our first order correction equation,

$$\begin{aligned} y_1'' + y_1 &= \cos(\tau) - \frac{1}{3}(8\cos^3(\tau)) - 2\omega_1 A_0 \sin(\tau) \\ &= \cos(\tau)(A_0 - \frac{A_0^3}{4}) - \frac{A_0^3}{12}\cos(3\tau) - 2\omega_1 \sin(\tau). \end{aligned}$$

In order for solvability to be satisfied we require that $A_0 = \pm 2$ and $\omega_1 = 0$ since $\cos(\tau), \sin(\tau)$ are in the null space of the (self) adjoint operator.
The non homogeneous equation for the first order correction has solution,

$$y_1(\tau) = -\frac{1}{6}\sin(\tau)(\sin(2\tau) - 6A_1)$$

To find the next order correction for the frequency shift we need to enforce solvability on the $\mathcal{O}(\epsilon^2)$ equation,

$$y_2'' + y_2 = \frac{1}{12}(48A_1\cos(\tau) + 12A_1\cos(3\tau) + (24\omega_2 - 1)\sin(t) - 8\sin(3t) - 3\sin(5t))$$

So we require that $A_1 = 0$ and $\omega_2 = -\frac{1}{24}$. ■

c) For $\epsilon = .01, .1, .2,$ and $.3$, plot the numerical solution and the multiple scale expansion for $0 \leq t \leq 40$ and for various values of A for your multiple scale solution. Also plot the limit cycle solution calculated from part (b).

3. Consider the singular equation:

$$\epsilon y'' + (1+x)^2 y' + y = 0$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$.

a) Obtain a leading order uniform solution using the WKB method.

Proof. We assume a WKB solution form with distinguished limit $\delta = \epsilon$,

$$y(x) = e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty} S_n(x)\epsilon^n}.$$

Plugging in our ansatz into the governing equations gives,

$$\epsilon\Big(\frac{S_{0_x}^2}{\epsilon^2} + \frac{2S_{0_x}S_{1_x}}{\epsilon} + \dots + \frac{S_{0_{xx}}}{\epsilon} + \dots\Big) + (1+x)^2\Big(\frac{S_{0_x}}{\epsilon} + S_{1_x} + \dots\Big) + 1 = 0.$$

Collecting in orders of ϵ,

$\mathcal{O}(\frac{1}{\epsilon}):\ S_{0_x} + (1+x)^2 S_{0_x} = 0.$

$\mathcal{O}(\epsilon):\ S_{0_{xx}} + 2S_{0_x}S_{1_x} + (1+x)^2 S_{1_x} + 1 = 0.$

Notice the leading order equation is solved by two solutions,

1) $S_{0_x} = 0$
2) $S_{0_x} = -(1+x)^2$.

Continuing with case 1) we have that $S_0 = c_1$ a constant. Then the first order correction equation is given by

$$(1+x)^2 S_{1_x} + 1 = 0.$$

The solution, S_1, is given by integrating which gives,

$$S_1 = -\frac{x}{1+x}.$$

The WKB solution for this case is given by,

$$y(x) \approx e^{\frac{1}{\epsilon}(S_0 + \epsilon S_1)} = c_2 e^{-\frac{x}{x+1}}.$$

Now for case 2), $S_{0_x} = -(1+x)^2$, which is found by integrating gives the leading order solution,

$$S_0 = -(x + x^2 + \frac{x^3}{3}).$$

Then our first order correction is given by,

$$\begin{aligned}
&-2(1+x) - 2(1+x)^2 S_{1_x} + (1+x)^2 S_{1_x} + 1 = 0\\
&\implies (1+x)^2 S_{1_x} + 2(1+x) = 1\\
&\implies S_{1_x} = \frac{1}{(1+x)^2} - \frac{2(1+x)}{(1+x)^2}\\
&\implies S_1 = \int_0^x \frac{1}{(1+t)^2}\mathrm{d}t - \ln\big((1+x)^2\big)\\
&= \frac{x}{1+x} - \ln\big((1+x)^2\big)
\end{aligned}$$

Then the WKB solution up to order ϵ, $e^{\frac{1}{\epsilon}(S_0+\epsilon S_1)}$, is given by

$$y(x) \approx \frac{c_3}{(1+x)^2} e^{-\frac{1}{\epsilon}(x+x^2+\frac{x^3}{3})} e^{\frac{x}{1+x}}.$$

Then the total solution is given by,

$$y_{\mathrm{WKB}}(x) = c_2 e^{-\frac{x}{x+1}} + \frac{c_3}{(1+x)^2} e^{-\frac{1}{\epsilon}(x+x^2+\frac{x^3}{3})} e^{\frac{x}{1+x}}.$$

To solve for the constants we use the boundary conditions which give us 2 equations and 2 unknowns. The boundary conditions give

$$\begin{aligned} y(0) &= c_2 + c_3 = 1 \\ y(1) &= c_2 e^{-\frac{1}{2}} + \frac{c_3}{4} e^{-\frac{1}{\epsilon}(\frac{11}{3})} = 1 \end{aligned}$$

Solving for c_2, c_3 gives

$$\begin{aligned} c_3 &= \frac{e^{-1/2} - 1}{e^{-1/2} + \frac{1}{4} e^{-1/\epsilon(11/3)}} \\ c_2 &= 1 - \frac{e^{-1/2} - 1}{e^{-1/2} + \frac{1}{4} e^{-1/\epsilon(11/3)}} \end{aligned}$$

■

b) Plot the uniform solution for $\epsilon = .01, .05, .1, .2$.
Notice that since $b(x) = (1+x)^2 > 0$ on our domain from boundary layer theory we expect to see "fast physics" near $x = 0$.