Deriving Finite Differences

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1. Derive divided difference approximations for the following expressions.

   a) Second-order approximation of u_{xy}.

   Proof. We have that $u_x(x,y) + \mathcal{O}(h^2) = \frac{u(x+h,y)-u(x-h,y)}{2h}$. We now want to use a centered difference in y on this. We want to determine coefficients such that,

$$u_{xy}(x,y) + \mathcal{O}(h^2) = au_x(x,y-h) + bu_x(x,y) + cu_x(x,y+h).$$

Taylor expanding each term and bringing along the $\mathcal{O}(h^2)$ term from the inner differencing of u_x gives,

$$\begin{aligned} u_{xy}(x,y) + \mathcal{O}(h^2) &= a(u_x(x,y) - hu_{xy}(x,y) + \frac{h^2}{2!}u_{xyy}(x,y) + \mathcal{O}(h^3)) \\ &+ bu_x(x,y) \\ &+ c(u_x(x,y) + hu_{xy}(x,y) + \frac{h^2}{2!}u_{xyy}(x,y) + \mathcal{O}(h^3)) \\ &+ (a+b+c)\mathcal{O}(h^2). \end{aligned}$$

Now collecting in order of h^n we have,

$$(a+b+c)(u_x(x,y) + \mathcal{O}(h^2)) + (-a+c)hu_{xy}(x,y) + (a+c)\frac{h^2}{2!}u_{xyy}(x,y) + (-a+c)\mathcal{O}(h^3).$$

We now enforce

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1/h \\ 0 \end{bmatrix}.$$

This yields, $a = -1/2h, b = 0, c = 1/2h$. We then have the scheme,

$$u_{xy}(x,y) + \mathcal{O}(h^2) = \frac{1}{(2h)^2}(u(x+h,y+h) - u(x-h,y+h) - u(x+h,y-h) + u(x-h,y-h)).$$

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   b) Second-order approximation of $(a(x)u_x)_x$.

   Proof. We want to determine coefficients such that

$$(a(x)u_x(x))_x + \mathcal{O}(h^2) = A(a(x-h)u_x(x-h)) + Ba(x)u_x(x) + C(a(x+h)u_x(x+h),$$

where we will substitute the centered differencing in for $u_x(x)$. Taylor expanding each term and bringing along the $\mathcal{O}(h^2)$ from the centered differencing of u_x gives,

$$\begin{aligned}(a(x)u_x(x))_x + \mathcal{O}(h^2) &= A([a(x) - ha_x(x) + \frac{h^2}{2!}a_{xx}(x) + \mathcal{O}(h^2)][u_x(x) - hu_x(x) + \frac{h^2}{2!}u_{xx}(x) + \mathcal{O}(h^2)]) \\ &+ Ba(x)u_x(x) \\ &+ C([a(x) + ha_x(x) + \frac{h^2}{2!}a_{xx}(x) + \mathcal{O}(h^2)][u_x(x) + hu_x(x) + \frac{h^2}{2!}u_{xx}(x) + \mathcal{O}(h^2)]) \\ &+ (A+B+C)\mathcal{O}(h^2).\end{aligned}$$

Now collecting in order of h^n we have,

$$\begin{aligned}(a(x)u_x(x))_x + \mathcal{O}(h^2) = (A+B+C)(a(x)u_x(x) + \mathcal{O}(h^2)) + (-A+C)(a(x)u_{xx}(x) \\ + a_x(x)u_x(x)) + (A+C)\mathcal{O}(h^2) + (-A+C)\mathcal{O}(h^3).\end{aligned}$$

Enforcing the same conditions above we arrive at $A = -1/2h, B = 0, C = 1/2h$. Using these coefficients and substituting the centered differencing for u_x we have the scheme,

$$(a(x)u_x(x))_x + \mathcal{O}(h^2) = \frac{1}{(2h)^2}\big(a(x+h)[u(x+2h) - u(x)] - a(x-h)[u(x) + u(x-2h)]\big).$$

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c) Fourth-order approximation of u_x.

Proof. We want

$$u_x + \mathcal{O}(h^4) = au(x-2h) + bu(x-h) + cu(x) + du(x+h)eu(x+2h).$$

Now Taylor expanding about x we have that,

$$\begin{aligned}u_x + \mathcal{O}(h^4) = a[u(x) - 2hu'(x) + \frac{(2h)^2}{2!}u''(x) - \frac{(2h)^3}{3!}u'''(x) + \frac{(2h)^4}{4!}u^{(iv)}(x) + \mathcal{O}(h^5)] \\ + b[u(x) - hu'(x)\frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^{(iv)}(x) + \mathcal{O}(h^5)] \\ + cu(x) \\ + d[u(x) + 2hu'(x) + \frac{(2h)^2}{2!}u''(x) + \frac{(2h)^3}{3!}u'''(x) + \frac{(2h)^4}{4!}u^{(iv)}(x) + \mathcal{O}(h^5)] \\ + e[u(x) + hu'(x)\frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^{(iv)}(x) + \mathcal{O}(h^5)].\end{aligned}$$

Now we collect in order of h^n,

$$\begin{aligned}u_x + \mathcal{O}(h^4) = (a+b+c+d+e)u + (-2a-b+d+2e)hu' + (4a+b+d+4e)\frac{h^2}{2!}u'' \\ + (-8a-b+d+8e)\frac{h^3}{3!}u''' + (16a+b+d+16e)\frac{h^4}{4!}u^{(iv)} + \mathcal{O}(h^5).\end{aligned}$$

We now enforce that,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 1/h \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives $a = \frac{1}{12h}, b = -\frac{2}{3}, c = 0, d = \frac{2}{3h}, e = -\frac{1}{12h}$. We then have,

$$u_x + \mathcal{O}(h^4) = \frac{1}{12h}(u(x-2h) - 8u(x-h) + 8u(x+h) - u(x+2h).$$

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