Iterative Solvers

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Consider the boundary value problem:

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a(x,y)\frac{\partial u}{\partial y}\right)=f(x,y),\ \text{in } (0,1)\times(0,1) \tag{1}$$

$$u(x,0)=u(x,1)=u(0,y)=u(1,y)=0,$$

where $a(x,y)=1+x^2+y^2$ and $f(x,y)=1$. Solve by the Jacobi method, then Gauss Seidel method, then by the SOR method. For each method make a plot of the relative residual nor, $||b-Au^k||/||b||$ versus iteration number k. Try several different values for the parameter ω in SOR, until you find one that works well.

Then try solving system using conjugate gradient method. You can write your own CG code or use the one in Matlab (pcg). First try CG without a preconditioner (ie preconditioner equal to identity) and then try with CG with Incomplete Cholesky decomposition as a preconditioner (ichol). Make a plot of the relative residual norm versus iteration number for the CG method.

Experiment with a few different mesh sizes and comment on how the number of iterations required to reach a fixed level of accuracy seems to vary with h for each method.

Solution:

We use the code from steady2.m to build the matrix. To compute the Conjugate Gradient and Precondition Conjugate Gradient we used MATLAB's PCG function. We can see from Fig. 1 that for the SOR methods we pick the value of $\omega = 1.9$ which converged to a tolerance of 10^{-5} and $h = .01$ in about 600 iterations. In Fig. 2 we plot the residual error vs the number of iterations for the non conjugate gradient iterative methods (on log scale, tolerance is $10^{-5}, h = .01$). We see that the SOR far outperform the Jacobi and Gauss-Siedel methods. In Fig. 3, as we decrease the mesh spacing, h, we see a significant jump in iterations required to converge (tolerance 10^{-5} for the Jacobi and Gauss-Siedel Methods. We see that Jacobi cannot converge in 8000 iterations for $h < .015$ and Gauss-Siedel for $h < .01$. SOR (with this particular ω) was able to converge in under 1000 iterations for $h = .008$. In Fig. 4 we plot the residual error vs the iteration number for the conjugate gradient methods both with and without a preconditioner. We use the preconditioner suggested in the question, incomplete Cholesky decomposition. We see that the preconditioned CG quickly converges (tol $= 10^{-5}$ with $h = .01$) in under 100 iterations while CG takes a bit over 300 iteration. The most impressive contrast is Fig. 5 in which we see the real power of PCG. As we decrease the mesh size PCG doesn't break a sweat to converge quickly while the conjugate gradient method must quickly increase the number of iterations required to converge on finer grids.

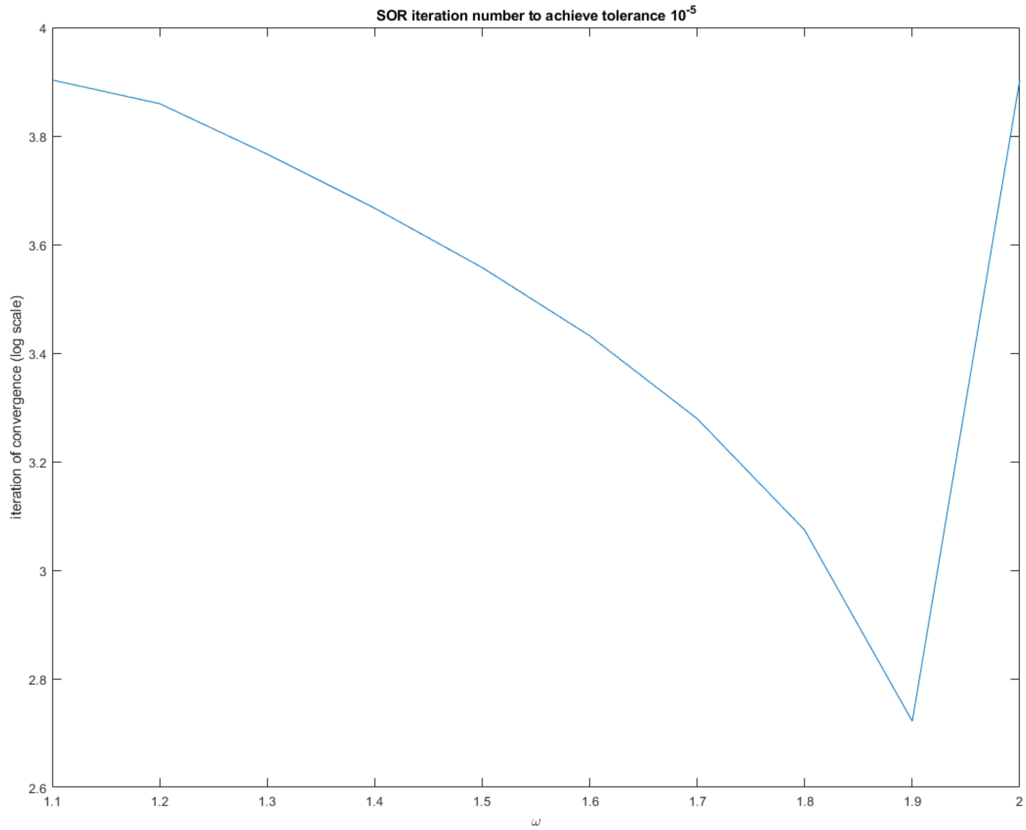

Figure 1: Number of Iterations required to converge for the SOR method for various ω values.

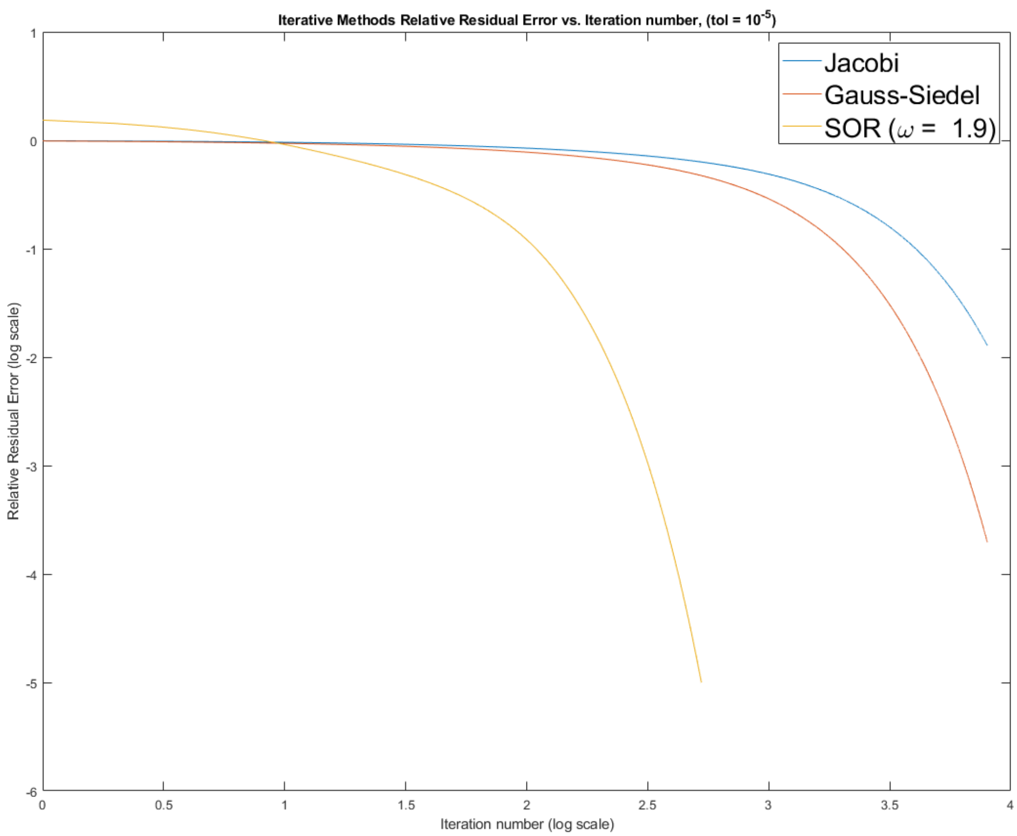

Figure 2: Relative Residual Error for iteration number for Jacobi, Gauss-Siedel, and SOR.

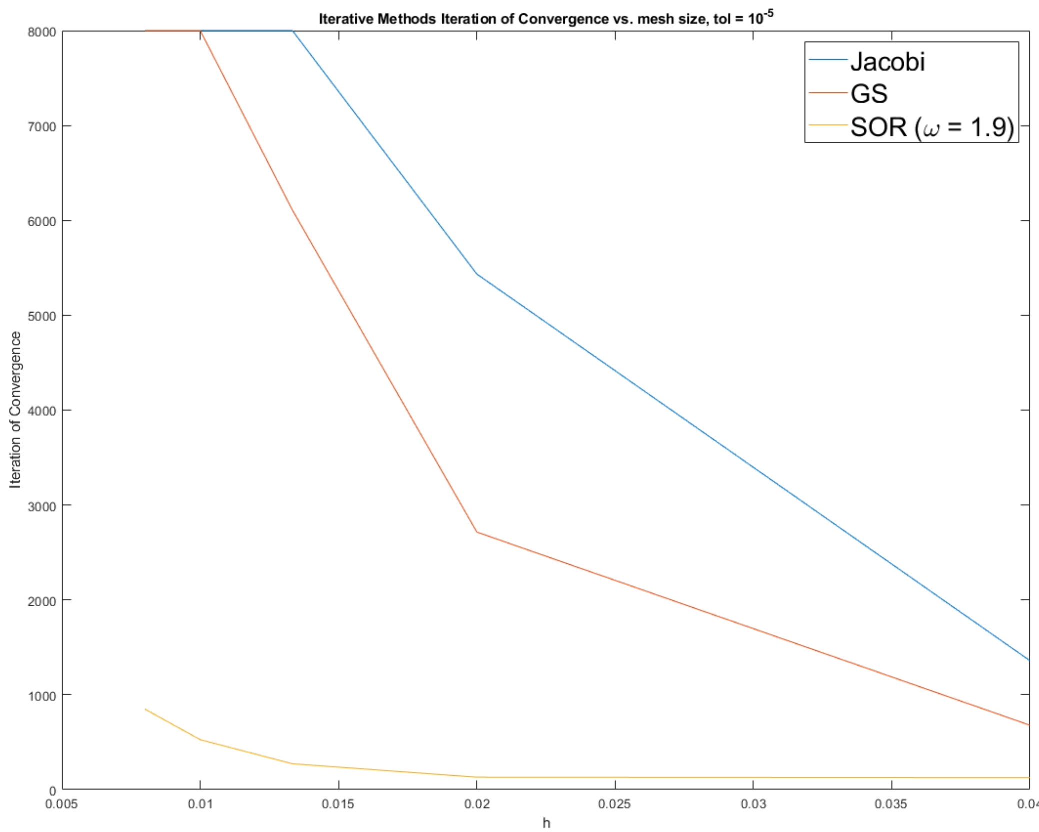

Figure 3: Number of iterations required to converge depending on grid spacing, h, for Jacobi, Gauss-Siedel, and SOR.

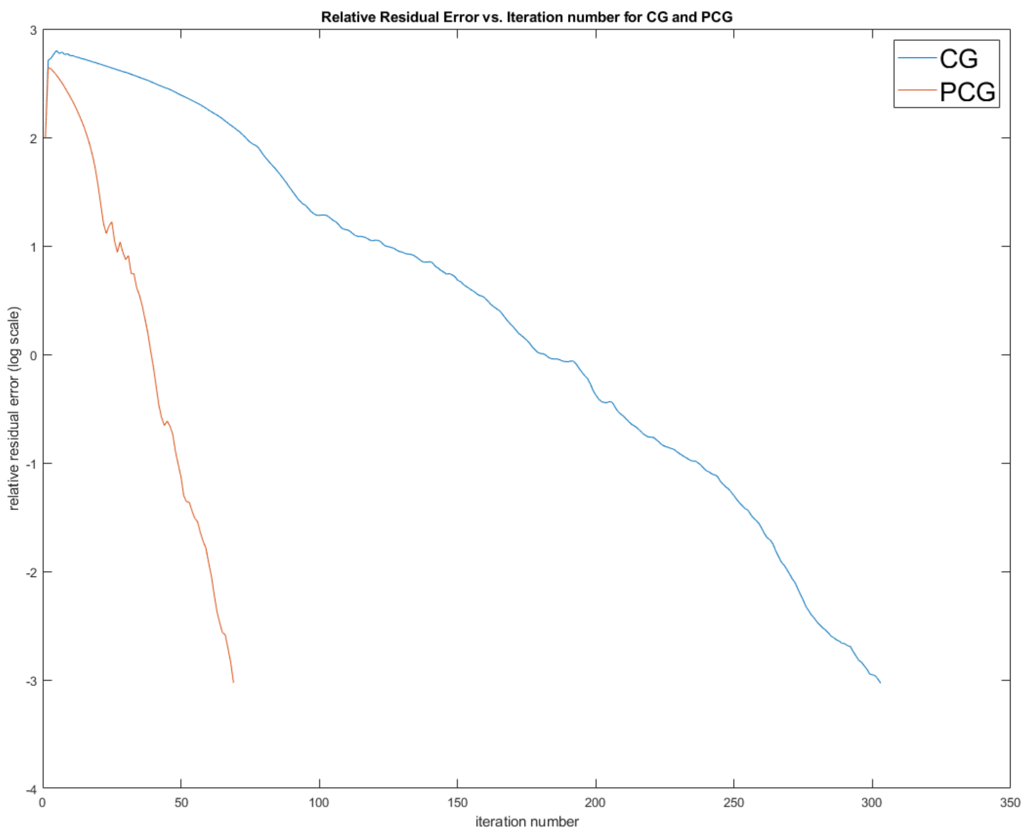

Figure 4: Residual Errors at each iteration for the Non and Pre conditioned Conjugate Gradient Methods.

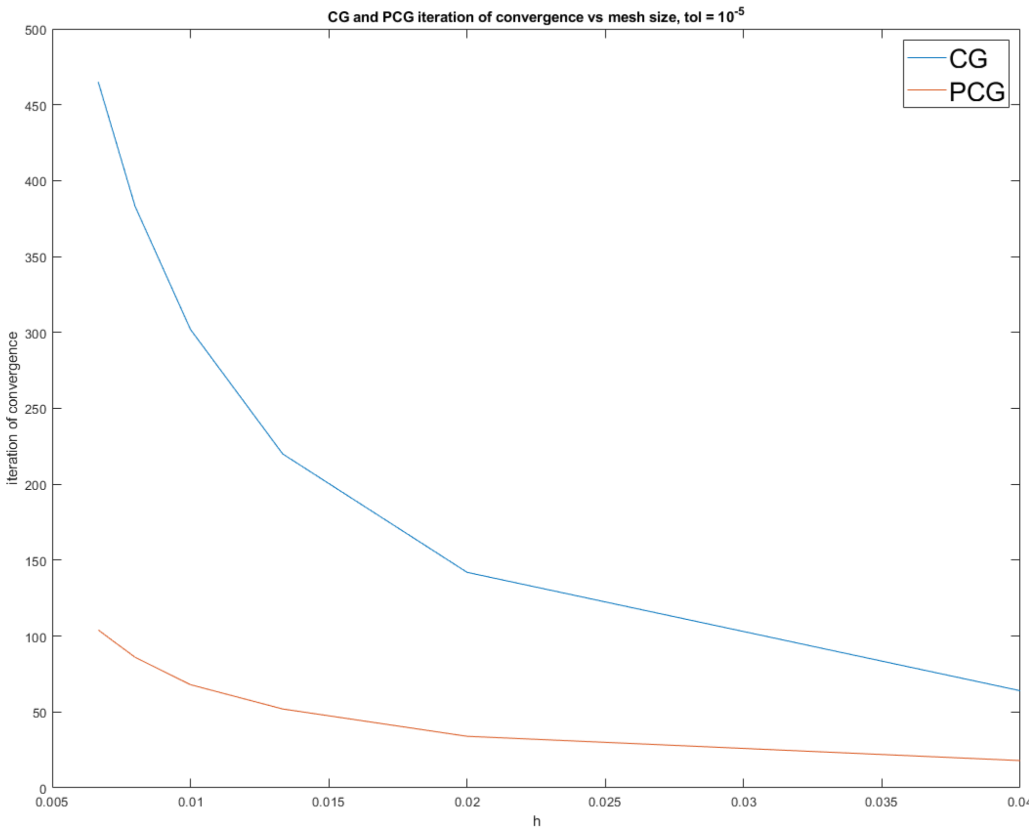
CG and PCG iteration of convergence vs mesh size, tol = 10^{-5}
iteration of convergence
h
CG
PCG
500
450
400
350
300
250
200
150
100
50
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04

Figure 5