Finite Element Method for Heat Equation

Jon Staggs

January 10, 2024

Solve numerically the heat equation problem,

$$u_t = u_{xx}, \;\; 0 < x < 1, 0 < t < 1$$
$$u(0,t) = u(1,t) = 0, \;\; u(x,0) = .5 - |x - .5|$$

by using piecewise linear finite elements in space and Crank-Nicholson (trapezoidal rule) in time. Plot $u_h(x,0)$ and $u_h(x,1)$ with $\Delta x = .1, \Delta t = .01$.

Solution:

Since our BVP is zero on the boundary we take our test functions $v \in H_0^1$. Writing in the variational form gives,

$$\int_\Omega \frac{\partial u}{\partial t} v \;\mathrm{d}x = -\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\;\mathrm{d}x.$$

We now consider the descritized versions using hat/P_1 functions, $u_h = \sum_{j=0}^J \alpha(t)\phi_j(x)$ and since our variational form holds for $v \in V_h$ then it also holds for basis functions $\phi_k(x)$. We then write our descritized variational problem as,

$$\sum_{j=0}^J \alpha_j'(t)\langle \phi_j(x), \phi_k(x)\rangle_{L^2} = -\sum_{j=0}^J \alpha_j(t)\langle \frac{\mathrm{d}\phi_j}{\mathrm{d}x}, \frac{\mathrm{d}\phi_k}{\mathrm{d}x}\rangle_{L^2}. \qquad (1)$$

Since we have selected hat functions then we have that $\langle \phi_j, \phi_k\rangle \neq 0$ for $k = j, j \pm 1$ and same for the derivatives of ϕ_j, ϕ_k. This yields a tridiagonal matrix and since there is no first order derivative term (i.e. $b(x) = 0$) then our matrix is also symmetric. We can then rewrite (1) as

$$B\alpha'(t) + A\alpha(t) = 0$$

subject to the initial condition

$$\alpha(t_0) \text{ such that } \sum_{j=0}^J \alpha_j(t_0)\phi_j(x) \approx u(x,0) = .5 - |x - .5|.$$

Computing the integrals with our hat functions yield the matrices,

$$A = \frac{1}{\Delta x}\begin{bmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & -1 \\ \vdots & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$B = \frac{h}{6\Delta x}\begin{bmatrix} 4 & 1 & 0 & 0 & \dots \\ 1 & 4 & 1 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 1 \\ \vdots & 0 & 0 & 1 & 4 \end{bmatrix}$$

Notice that A yields the typical second derivative finite difference matrix.
We know want to use Crank- Nicholson for time stepping. The general Crank- Nicholson schemes is:

$$y_{n+1} - y_n = \frac{\Delta t}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1})).$$

We want to derive a matrix version for our time stepping scheme with $\alpha(t)$. We abstractly have,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) &= F(\alpha(t)) \\ &= B^{-1}A\alpha(t).\end{aligned}$$

Discretizing with Crank- Nicholson gives,

$$\alpha(t_{n+1}) = \alpha(t_n) + \frac{\Delta t}{2}(B^{-1}A\alpha(t_n) + B^{-1}A\alpha(t_{n+1}))$$

$$\implies B\alpha(t_{n+1}) = B\alpha(t_n) + \frac{\Delta t}{2}(A\alpha(t_n) + A\alpha(t_{n+1}))$$

$$\implies (B - \frac{\Delta t}{2}A)\alpha(t_{n+1}) = (B + \frac{\Delta t}{2}A)\alpha(t_n)$$

In Fig. 1 t denotes the time step so the interpreted value in time is $t/\Delta t$.

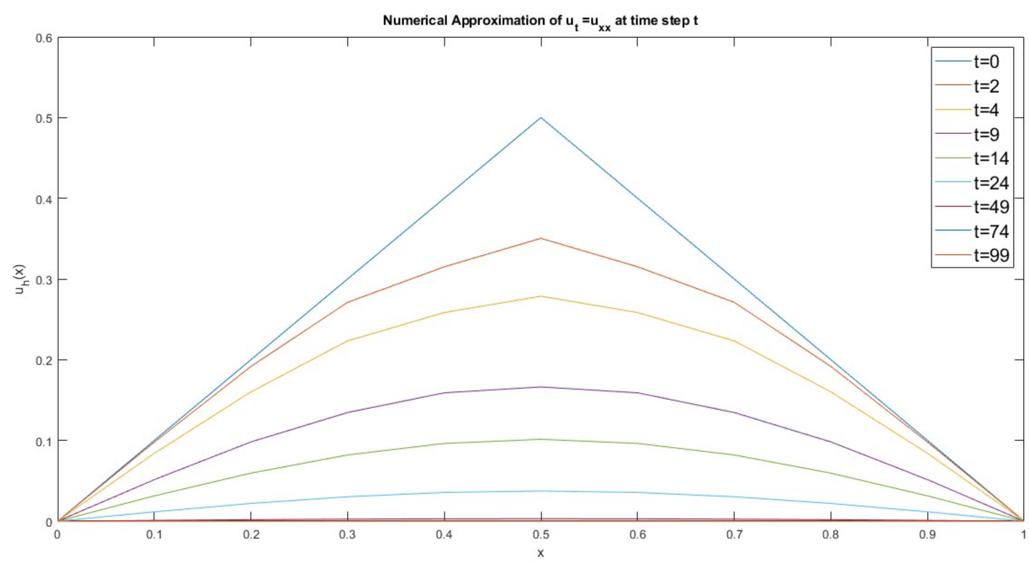

Figure 1: Numerical Approximation of $u_t = u_{xx}$ at various time steps.