Truncation Error and Stability Analysis for Hyperbolic Problems

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1. The hyperbolic partial differential equation,

$$u_t + au_x + b(x)u = 0, \quad 0 < x < 1, t > 0, a > 0, b(x) < 0,$$
$$u(x,0) = u_0(x), \quad 0 < x < 1$$
$$\text{Periodic Boundary Conditions}$$

is approximated by the standard upwind finite difference method,

$$u_j^{n+1} = \big(1 - \frac{a\Delta t}{\Delta x} - \Delta t b(x_j)\big)u_j^n + \frac{a\Delta t}{\Delta x}u_{j-1}^n, 0 < j \leq J, n > 0,$$
$$u_0^n = u_J^n, n > 0 \text{ (periodic boundary conditions)}$$

a) Determine the order of the local truncation error for the method.

Proof. First observe that we can rewrite this scheme as,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a\frac{u_j^n - u_{j-1}^n}{\Delta x} + b(x_j)u_j^n = 0.$$

We now see that the time step and spatial step are both forward Euler which is first order. We then have that the scheme is $\mathcal{O}(\Delta t + \Delta x)$.

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b) Prove L^2 stability for the method under suitable conditions on Δx, Δt.

Proof. We use a Von Neumann analysis and take the Discrete Fourier Transform to obtain (*Remark*: We can initially ingore the $b(x)$ term and reintroduce since $\mathcal{O}(\Delta t)$ terms do not affect the stability - Prof. Engquist.),

$$\begin{aligned}
\hat{u}_k^n &= \left(1 - \frac{a\Delta t}{\Delta x}\right)\hat{u}_k^n + \frac{a\Delta t}{\Delta x}e^{-i2\pi k/J}\hat{u}_k^n \\
&= \left(1 - \frac{a\Delta t}{\Delta x} + \frac{a\Delta t}{\Delta x}e^{-i2\pi k/J}\right)\hat{u}_k^n \\
&= \left(1 - \frac{a\Delta t}{\Delta x}(1 + e^{-i2\pi k/J})\right)\hat{u}_k^n
\end{aligned}$$

Since the Fourier Transform is an isometric isomorphism on L^2 we only need to find the value of the amplification factor. We now want to enforce that the amplification factor has norm less than 1.

$$|A| < 1 \implies (1 - \frac{a\Delta t}{\Delta x}(1 + \cos(2\pi k/J)))^2 + (\frac{a\Delta t}{\Delta x}\sin(2\pi k/J))^2 < 1.$$

Doing some algebra we arrive at the requirement,

$$2\frac{a\Delta t}{\Delta x}(\frac{a\Delta t}{\Delta x}-1)(1-\cos(2\pi k/J)) \leq 1 \iff \frac{a\Delta t}{\Delta x} \leq 1.$$

Using this constraint and introducing b back in and assuming that $b \in L^\infty$ and $b^* = ||b||_{L^\infty}$.

$$\begin{aligned}
||\hat{u}^n||_{L^2} &\leq \left|1+\frac{a\Delta t}{\Delta x}(e^{-i\xi}-1)\right| ||\hat{u}^n||_{L^2} + b^*\Delta t||u^n||_{L^2} \\
&\leq (1+b^*\Delta t)||\hat{u}^n||_{L^2} \\
&\leq e^{b^*\Delta t}||\hat{u}^n||_{L^2} \\
&\leq e^{b^*\Delta t n}||u^0||_{L^2} \\
&\leq e^{b^*T}||u^0||_{L^2},
\end{aligned}$$

where $T = n\Delta t$.

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c) Prove a maximum principle for the method, under suitable conditions on Δx, Δt.

Proof. We have that

$$\begin{aligned}
|u_j^{n+1}| &\leq |1-\frac{a\Delta t}{\Delta x}-\Delta t b(x_j)|\,|u_j^n| + \frac{a\Delta t}{\Delta x}|u_{j-1}^n| \\
&\leq \big(|1-\frac{a\Delta t}{\Delta x}-\Delta t b(x_j)| + \frac{a\Delta t}{\Delta x}\big)\max_k |u_k^n|.
\end{aligned}$$

Now we would enforce this amplification factor to be less than or equal to one. (This seems like a big restriction to make on $b(x)$.) We then have that,

$$|u_j^{n+1}| \leq ||u^n||_{L^\infty} \text{ for all j.}$$

Hence we have that,

$$\max_j |u_j^{n+1}| = ||u^{n+1}||_{L^\infty} \leq \max_j ||u^n||_{L^\infty} = ||u^n||_{L^\infty}.$$

$$\implies ||u^n||_{L^\infty} \leq ||u^0||_{L^\infty}.$$

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