Non-Linear Boundary Value Problems

Jon Staggs

January 10, 2024

1. (nonlinear pendulum)

   a) We are solving the boundary value problem

   $$\theta''(t) = -\sin(\theta(t)), \;\; 0 < t < T$$

   $$\theta(0) = \alpha, \theta(T) = \beta.$$

   Our Newton iteration is of the form

   $$\theta^{k+1} = \theta^k - (J_G(\theta^k))^{-1} G(\theta^k)$$

   where G is the nonlinear discretization,

   $$G(\theta_i) = \frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i) = 0,$$

   and $J_G(\theta^k)$ is the Jacobian of G at θ^k

   $$J_{i,j}(\theta) = \begin{cases} \frac{1}{h^2}, & j = i-1 \text{ or } j = i+1 \\ \frac{-2}{h^2} + \cos(\theta_i), & j = i \\ 0, & \text{else} \end{cases}.$$

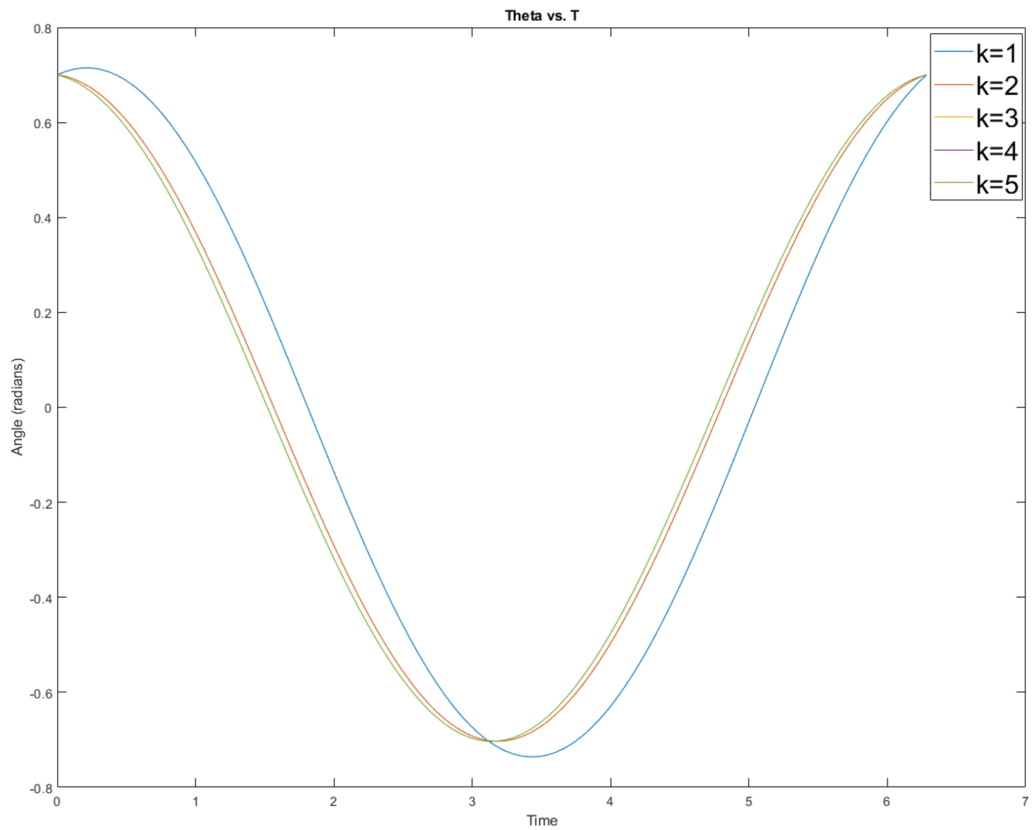


Figure 1: initial guess: $\theta = .7\cos(\theta) + .5\sin(\theta)$ plotted with Newton Iterations, k

Fig. 2 shows an approximate solution to the nonlinear pendulum problem with initial guess $\theta = (\alpha - 1) + \cos(x)$. Note that many solutions can exhibit this general behavior by enforcing the boundaries in the generic way of $c_0 \sin(x)$ where c_0 would be a symmetric boundary value. Essentially we exploit where sin or cos are 0 or 1.

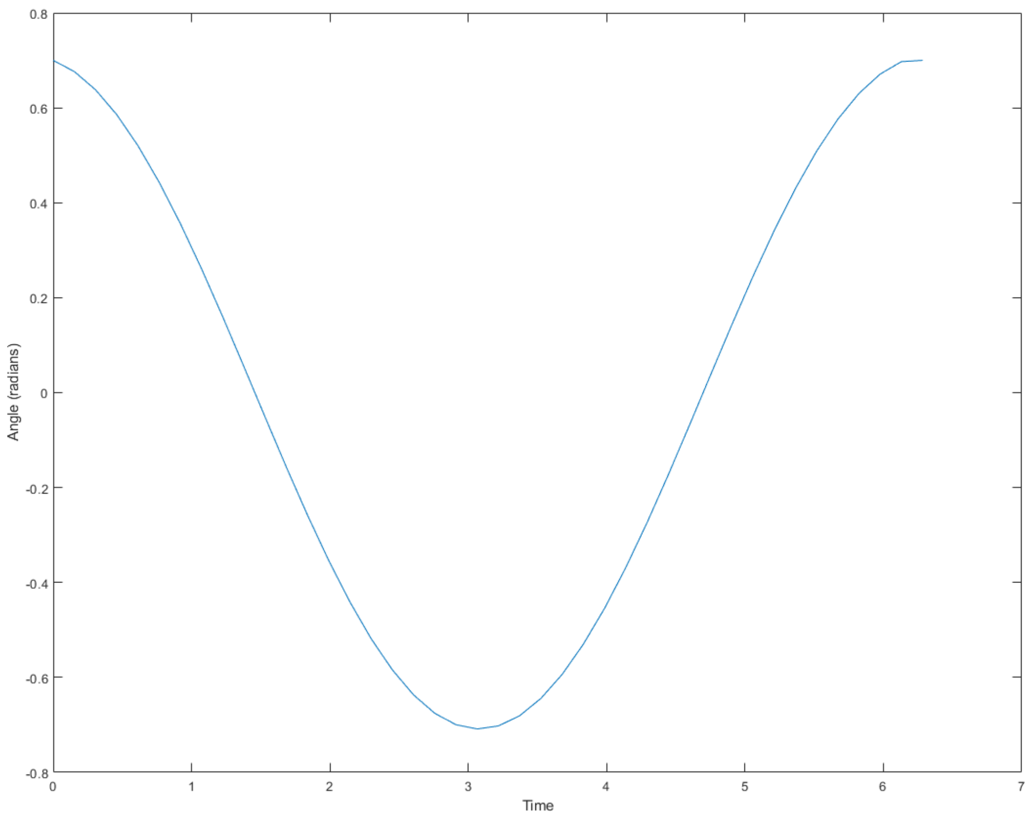
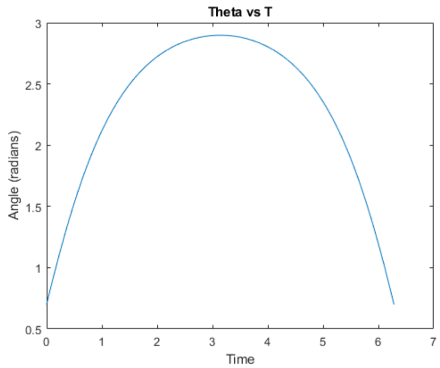
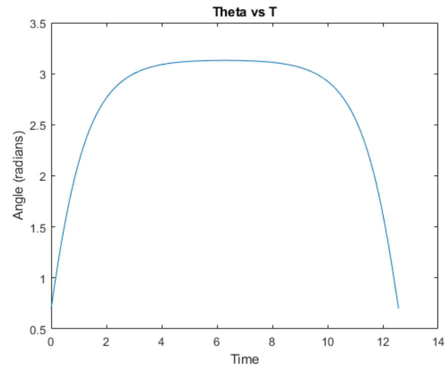


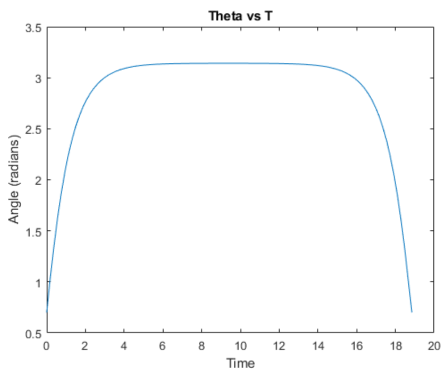
Figure 2: initial guess: $\theta = (.7 - 1) + \cos(x)$ plotted with Newton Iterations, k



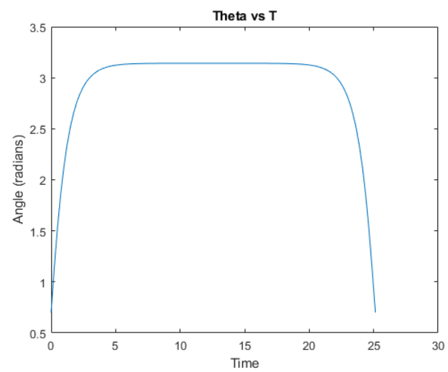
(a) $T = 2\pi$



(b) $T = 4\pi$



(c) $T = 6\pi$



(d) $T = 8\pi$

Figure 3: Initial guess: $\theta = .7 + \sin(t/2)$

b) Find a numerical solution to this BVP with the same general behavior as seen in Figure 2.5 for the case of longer time interval, say $T = 20$, again with $\alpha = \beta = .7$. Try larger values of T. What does $\max_i \theta_i$ approach as T is increased? Note that for large T this solution exhibits "boundary layers".

We can visually see that the max θ is approaching θ. Physically as we increase T we seem to be stabilizing (at least locally in time) the pendulum at the inverted position then it falls back down to the boundary value. For our initial guess we solve the problem in part a) and take that solution as our initial guess. We do this because we are still trying to obtain a solution that is in the same family/exhibits the half period behavior as part a).

T	max(θ)
2π	2.8973
4π	3.1311
6π	3.1411
8π	3.1416

2. The problem that we are approximating is,

$$\epsilon u'' + u(u' - 1) = 0, \;\; a < x < b$$
$$u(a) = \alpha, u(b) = \beta.$$

We have the vector valued function $G(u_i)$,

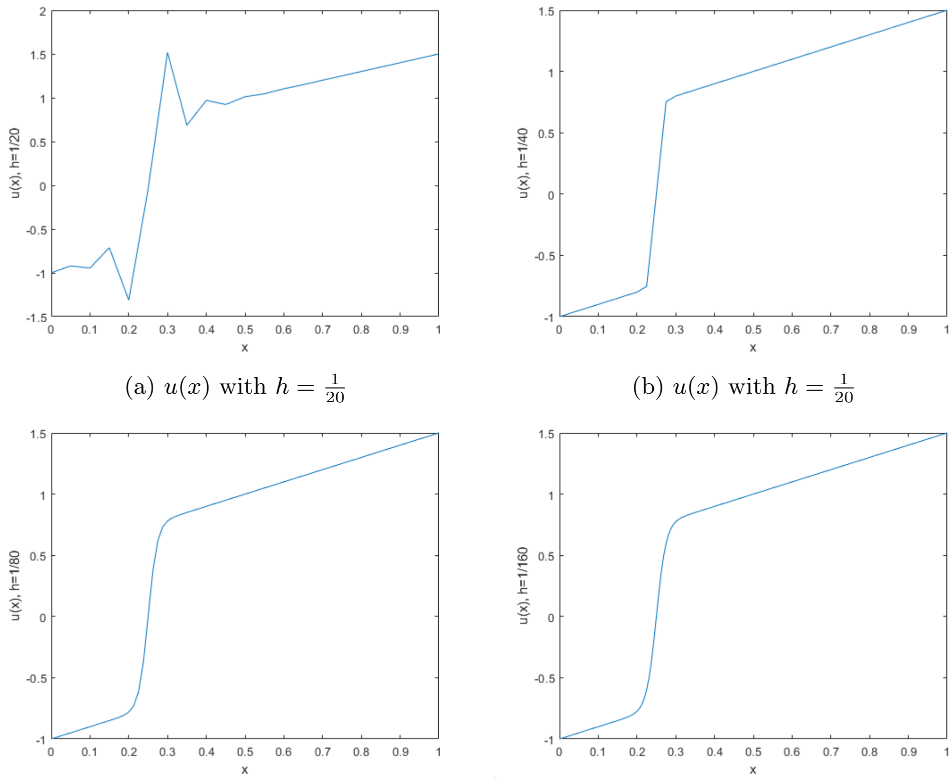
$$G(u_i) = \epsilon\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}\right) + u_i\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) = 0.$$

Computing our Jacobian notice that $\frac{\partial G_i}{\partial u_{i+t}} = 0$ for $|t| = 2, 3, ..$ meaning that anything off the upper, lower, and main diagonal is 0. Our Jacobian is,

$$J_{G_{i,j}}(u) = \begin{cases} -\frac{2\epsilon}{h^2} + \left(\frac{u_{i+1} - u_{i-1}}{2h} - 1\right), & j = i \\ \frac{\epsilon}{h^2} + \frac{u_i}{2h}, & j = i+1 \\ \frac{\epsilon}{h^2} - \frac{u_i}{2h}, & j = i-1 \\ 0, & \text{else} \end{cases}.$$

The book gives us a good initial guess,

$$u^0 = x - \bar{x} + w_0 \tanh(w_0(x - \bar{x})/2\epsilon), \;\; \bar{x} = \frac{1}{2}(a + b - \alpha - \beta), w_0 = \frac{1}{2}(a - b + \beta - \alpha).$$



(a) $u(x)$ with $h = \frac{1}{20}$

(b) $u(x)$ with $h = \frac{1}{20}$

(c) $u(x)$ with $h = \frac{1}{80}$

(d) $u(x)$ with $h = \frac{1}{160}$

Figure 4: Numerical Approximation of $u(x)$ with various h.