Some Elliptic PDE/BVP Theory

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1 Introduction

We show a few problems where we derive weak/variational forms of elliptic PDEs. Once the weak forms are developed one can select a space of test functions (such as $P1$/hat functions) to solve these PDEs with the Finite Element Method. We also show how one typically proves existence and uniqueness of solutions to elliptic PDEs via Lax-Milgram Theorem.

2 Existence and Uniqueness

1. Consider the boundary value problem $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ such that

$$-u_{xx} + e^y u = f, \quad \text{for } (x, y) \in (0, 1)^2,$$
$$u(0, y) = 0, u(1, y) = \cos(y), \quad \text{for } y \in (0, 1).$$

Rewrite this as a variational problem and show that there exists a unique solution. Be sure to define your function spaces carefully and identify where f must lie.

Proof. We seek to rewrite our BVP as a homogeneous problem. To this end, we consider the lift $w = u - x\cos(y)$ so that $u = w - x\cos(y)$. Our BVP then becomes,

$$\begin{cases} -w_{xx} + e^y w = f - e^y x\cos(y) & \text{on } (0,1)^2 \\ w(0, y) = 0, \quad w(1, y) = 0. \end{cases}$$

To formulate the variational problem we take a test function v and integrate. At this point, we see that we can then take $v \in H = \{g \in H^1((0, 1)^2) \mid g(0, y) = g(1, y) = 0\}$.

$$\int_0^1 \int_0^1 -w_{xx} v \; \mathrm{d}x\mathrm{d}y + \int_{(0,1)^2} e^y w \; \mathrm{d}x\mathrm{d}y = \int_{(0,1)^2} (f - e^y x\cos(y))v \; \mathrm{d}x\mathrm{d}y.$$

We now integrate by parts in x on the second derivative term which gives

$$\begin{aligned}\int_0^1 \int_0^1 -w_{xx} v \; \mathrm{d}x\mathrm{d}y &= \int_0^1 \int_0^1 w_x(0, y)v(0, y) - w(1, y)v(1, y) \; \mathrm{d}x \; \mathrm{d}y + \int_0^1 \int_0^1 w_x v_x \; \mathrm{d}x\mathrm{d}y \\ &= \int_{(0,1)^2} w_x v_x \; \mathrm{d}x\mathrm{d}y.\end{aligned}$$

We then have the variational form,

$$B(w, v) = \int_{(0,1)^2} w_x v_x \; \mathrm{d}x\mathrm{d}y + \int_{(0,1)^2} e^y wv \; \mathrm{d}x\mathrm{d}y,$$

$$F(v) = \int_{(0,1)^2} (f - e^y x \cos(y))v \ \mathrm{d}x\mathrm{d}y.$$

To show that there exists a unique solution we need to satisfy the hypothesis of Lax-Milgram. It is clear that B is indeed bilinear and F is indeed linear. It stands to show 1) continuity of B, 2) continuity of F, and 3) coercivity of B.

For the continuity of B, we want to show that $|B(w,v)| \leq c||w||_{H^1}||v||_{H^1}$.

$$\begin{aligned}
|B(w,v)| &\leq \int_{(0,1)^2} |w_x v_x| \ \mathrm{d}x\mathrm{d}y + \int_{(0,1)^2} |e^y w v| \ \mathrm{d}x\mathrm{d}y \\
&\leq \int_{(0,1)^2} |w_x v_x| \ \mathrm{d}x\mathrm{d}y + e\int_{(0,1)^2} |wv| \ \mathrm{d}x\mathrm{d}y \\
&\leq ||w_x||_{L^2}||v_x||_{L^2} + e||w||_{L^2}||v||_{L^2} \\
&\leq e||w||_{H^1}||v||_{H^1}.
\end{aligned}$$

For continuity of F we want to show that $|F(v)| \leq c||v||_{H^1}$.

$$\begin{aligned}
|F(v)| &\leq \int_{(0,1)^2} |fv| \ \mathrm{d}x\mathrm{d}y + \int_{(0,1)^2} |e^y x \cos(y) v| \ \mathrm{d}x\mathrm{d}y \\
&\leq \int_{(0,1)^2} |fv| \ \mathrm{d}x\mathrm{d}y + e\int_{(0,1)^2} |v| \ \mathrm{d}x\mathrm{d}y \\
&\leq ||f||_{L^2}||v||_{L^2} + e||\chi_{(0,1)}||_{L^2}||v||_{L^2} \\
&\leq 2\max(||f||_{L^2}, e)||v||_{H^1}
\end{aligned}$$

At this point, we see that we ought to take $f \in L^2$.

For coercivity of B, we want to show that $\gamma||v||^2_{H^1} \leq |B(v,v)|$.

$$\begin{aligned}
|B(v,v)| &= \left| \int_{(0,1)^2} v_x^2 \ \mathrm{d}x\mathrm{d}y + \int_{(0,1)^2} e^y v^2 \ \mathrm{d}x\mathrm{d}y \right| \\
&\geq \left| \int_{(0,1)^2} v_x^2 \ \mathrm{d}x\mathrm{d}y + \int_{(0,1)^2} v^2 \ \mathrm{d}x\mathrm{d}y \right|
\end{aligned}$$

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2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz bonudary, $f \in L^2(\Omega)$, and $\alpha > 0$. Consider the Robin boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega. \end{cases}$$

a) For this problem, formulate a variational principle,

$$B(u, v) = (f, v) \quad \text{for all } v \in H^1(\Omega).$$

b) Show that this problem has a unique weak solution.

Proof. We take $v \in H^1(\Omega)$, multiply, and integrate to obtain,

$$\int_\Omega -\Delta uv \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x$$

$$\iff \int_\Omega \nabla u \nabla v \,\mathrm{d}x - \int_{\partial\Omega} \nabla u \cdot \nu v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x$$

Now using the Robin boundary condition we have the Bilinear and linear forms, respectively,

$$B(u, v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x + \alpha \int_{\partial\Omega} uv \,\mathrm{d}x,$$

$$F(u, v) = \int_\Omega fv \,\mathrm{d}x.$$

To show that there exists a unique weak solution we show that the variational problem satisfies Lax-Milgram. It is clear that B, F are linear. First, we show the continuity of F. We want to show $|F(v)| \leq c||v||_{H^1}$.

$$\begin{aligned} \left| \int_\Omega fv \,\mathrm{d}x \right| &\leq ||f||_{L^2}||v||_{L^2} \\ &\leq ||f||_{L^2}||v||_{H^1} \end{aligned}$$

For continuity of B, we want to show $|B(u, v)| \leq c||u||_{H^1}||v||_{H^1}$.

$$\begin{aligned} |B(u, v)| &= \left| \int_\Omega \nabla uv \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x + \alpha \int_{\partial\Omega} uv \,\mathrm{d}x \right| \\ &\leq \int_\Omega |\nabla u \nabla v| \,\mathrm{d}x + \int_\Omega |uv| \,\mathrm{d}x + \alpha \int_{\partial\Omega} |uv| \,\mathrm{d}x \\ &\leq ||\nabla u||_{L^2}||\nabla v||_{L^2} + ||u||_{L^2}||v||_{L^2} + \alpha||u||_{L^2(\partial\Omega)}||v||_{L^2(\partial\Omega)} \\ &\leq 2||u||_{H^1}||v||_{H^1} + \alpha c^2||u||_{H^1}||v||_{H^1} \\ &= (2 + \alpha c^2)(||u||_{H^1}||v||_{H^1}), \end{aligned}$$

where the last inequality comes from the Trace Theorem.
Now it stands to show coercivity of B, that is $|B(v, v)| \geq \gamma||v||^2_{H^1}$. For sake of contradiction, suppose otherwise so that for each $\gamma_n > 0$ there exists v_n such that

$$|B(v_n, v_n)| \leq \gamma_n||v_n||^2_{H^1}.$$

We take $\gamma_n = 1/n$ and v_n to be normalized so we have that,

$$
\begin{aligned}
&|B(v_n, v_n)| \leq \frac{1}{n} \\
\implies &\left| \int_{\Omega} \nabla v_n^2 \ \mathrm{d}x + \int_{\Omega} v_n^2 \ \mathrm{d}x + \alpha \int_{\partial\Omega} v_n^2 \ \mathrm{d}x \right| \leq \frac{1}{n} \\
\implies &||v_n||_{H^1}^2 + \alpha \left| \int_{\partial\Omega} v_n^2 \ \mathrm{d}x \right| \leq \frac{1}{n} \\
\implies &||v_n||_{H^1}^2 \leq \frac{1}{n} \to 0.
\end{aligned}
$$

This is a contradiction since we assumed that v_n had norm 1.

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3. Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz doomain. Consider the Stokes problem for vector u and scalar p given by

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the first equation holds for each coordinate (i.e. $-\Delta u_j + \frac{\partial p}{\partial x_j} = f_j$ for each $j = 1, ..., d$). This problem is not a minimization problem; it is a *saddle-point problem* in that we minimize some energy subject to the *constraint* $\nabla \cdot u = 0$. However, if we work over the constrained space, we can handle this problem by the ideas of this chapter. Let,

$$H = \{v \in (H_0^1(\Omega))^d \mid \nabla \cdot u = 0\}.$$

a) Verify that H is a Hilbert space.

Proof. We define the inner-product,

$$\langle u, v \rangle_H = \sum_{j=1}^{d} \langle u_j, v_j \rangle_{H_0^1} + \langle \nabla \cdot u, \nabla \cdot v \rangle_{L^2}.$$

For $u, v \in H$, we have that $\nabla \cdot u = \nabla \cdot v = 0$ then

$$\langle u, v \rangle_H = \sum_{j=1}^{d} \langle u_j, v_j \rangle_{H_0^1}.$$

Let $\{u_n\}$ be a Cauchy sequence in H. We then have that

$$\langle u_n - u_m, u_n - u_m \rangle_H = \sum_{j=1}^{d} \langle u_{n_j} - u_{m_j}, u_{n_j} - u_{m_j} \rangle_{H_0^1} \to 0.$$

Since $(H_0^1)^d$ is Hilbert, namely complete, there exists a $u \in (H_0^1)^d$ such that $u_n \to u$ in $(H_0^1)^d$ which is to say,

$$\sum_{j=1}^{d} \langle u_{n_j} - u_j, u_{n_j} - u_j \rangle_{H_0^1} \to 0.$$

Observe that

$$\begin{aligned} \langle \nabla \cdot (u_n - u_m), \nabla \cdot (u_n - u_m) \rangle &= ||\nabla \cdot (u_n - u_m)||_{L^2} \\ &\leq ||u_n - u_m||_H \end{aligned}$$

hence $\{\nabla \cdot u_n\}$ is a Cauchy sequence in L^2 and since L^2 is complete then there exists a $g \in L^2$ such that $\nabla \cdot u_n \to g$ in L^2.
We now need to show that $\nabla \cdot u = g$ and $\nabla \cdot u = 0$. Recall that L^2 convergence implies distributional convergence. Take $\phi \in \mathcal{D}$ then we have,

$$\int_\Omega |\nabla \cdot u_n - g| \phi \, \mathrm{d}x \leq ||\nabla \cdot u_n - g||_{L^2} ||\phi||_L^2 \to 0.$$

We now have that $\nabla \cdot u_n \to g$ distributionally.

Similarly, H_0^1 convergence implies distributional convergence. This gives us,

$$-\int_\Omega u_n \cdot \nabla\phi \ \mathrm{d}x \to -\int_\Omega u \cdot \nabla\phi \ \mathrm{d}x$$
$$\implies \int_\Omega (\nabla \cdot u_n)\phi \ \mathrm{d}x \to \int_\Omega (\nabla \cdot u)\phi \ \mathrm{d}x.$$

This gives us that (in a distributional sense) $\nabla \cdot u_n \to \nabla u$ and since $\nabla \cdot u_n = 0$ then since $\nabla u_n \to g$ we have that $g = \nabla \cdot u = 0$. Then Lebesgue Lemma gives us the desired result for the functions. Hence, $u \in H$ and we conclude H is a Hilbert space.

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b) Determine an appropriate Sobolev space for f, and formulate an appropriate variational problem for the constrained Stokes problem.

Proof. Let $v \in H \subset (H_0^1)^d$ and $f \in (H^{-1})^d$. We then have,

$$\int_\Omega -\Delta u_j v_j \ \mathrm{d}x + \int_\Omega \frac{\partial p}{\partial x_j} v_j \ \mathrm{d}x = f_j(v_j)$$
$$\implies \int_\Omega \nabla u \nabla v \ \mathrm{d}x + \int_{\partial\Omega} v_j(\nabla u_j \nabla v_j) \cdot \nu \ \mathrm{d}x + \int_\Omega \frac{\partial p}{\partial x_j} v_j \ \mathrm{d}x = f_j(v_j).$$

Enforcing the boundary condition gives us the bilinear and linear forms,

$$B(u,v) = \int_\Omega \nabla u_j \nabla v_j \ \mathrm{d}x,$$
$$F(v) = f_j(v_j) - \int_\Omega \frac{\partial p}{\partial x_j} v_j \ \mathrm{d}x.$$

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c) Show that there is a unique solution to the variational problem.

Proof. Since we are in a Hilbert space H, we need to satisfy the conditions of Lax-Milgram. By our choice of F we only need to put a condition on p. If we take $p \in H^1$ then $F(v)$ is continuous. Now we show continuity of B.

$$\begin{aligned}
|B(u,v)| &= \left| \int_\Omega \nabla u_j \nabla v_j \ \mathrm{d}x \right. \\
&\le \int_\Omega |\nabla u \nabla v| \ \mathrm{d}x \\
&\le ||\nabla u||_{L^2} ||\nabla v||_{L^2} \\
&\le ||u||_H ||v||_H.
\end{aligned}$$

It remains to show coercivity of B. Take $v \in H$, which means $\nabla \cdot v = 0$, and compute,

$$\begin{aligned}
|B(v,v)| &= \int_\Omega (\nabla v)^2 \ \mathrm{d}x \\
&= ||\nabla v||_{L^2}^2 \\
&\ge c^2 ||v||_{H_0^1}^2 \\
&= c^2 ||v||_{H_0^1}^2 + \langle \nabla v, \nabla v \rangle_{L^2} \\
&= c^2 ||v||_H.
\end{aligned}$$

The first inequality comes from Poincaré's Inequality.

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3 Finite Element Method

4. (finite elements) Use the Galerkin finite element method with continuous piecewise linear basis functions to solve the problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x), \quad 0 \le x \le 1$$
$$u(0)=0, u(1)=0.$$

a) Derive the matrix equation you will need to solve this problem.

Proof. We use the basis of continuous piecewise linear basis functions which are hat functions of the form,

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}}, & [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & [x_i, x_{i+1}] \\ 0, & \text{elsewhere} \end{cases}. \tag{1}$$

We also observe that if u satisfies the above differential equation then it also satisfies

$$\langle Lu, \phi \rangle = \langle f, \phi \rangle \tag{2}$$

where $L[\cdot] = -\frac{\mathrm{d}}{\mathrm{d}x}\big((1+x^2)\frac{\mathrm{d}[\cdot]}{\mathrm{d}x}\big)$, $\langle \cdot, \cdot \rangle$ is the L_2 inner product, and ϕ can be written as a linear combination of our basis function. We can write eqn. (2) as,

$$\begin{aligned} &\int_0^1 -\frac{\mathrm{d}}{\mathrm{d}x}\big((1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\big)\mathrm{d}x = \int_0^1 f(x)\phi(x)\mathrm{d}x \\ \implies &-(1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\phi(x)|_0^1 + \int_0^1 (1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\mathrm{d}x = \int_0^1 f(x)\phi(x)\mathrm{d}x. \end{aligned} \tag{3}$$

We take our approximation of u to be $\hat{u} = \sum_{i=1}^{n-1} c_j\phi_j$. Since inner product is linear and ϕ is a linear combination we can write enq. (refeq:innerprod-weak-form) as $\langle L\hat{u}-f, \phi \rangle = \langle L\hat{u}-f, \sum_{i=1}^{n-1} d_i\phi_i \rangle = \sum_{i=1}^{n-1}\langle L\hat{u}-f, \phi_i \rangle$. For each i we have,

$$-(1+x^2)\frac{\mathrm{d}\hat{u}}{\mathrm{d}x}\phi_i(x)|_0^1 + \int_0^1 (1+x^2)\Big(\sum_{j=1}^{n-1} c_j\frac{\mathrm{d}\phi_j}{\mathrm{d}x}\Big)\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x = \int_0^1 f(x)\phi_i(x)\mathrm{d}x, \quad i = 1, ..., n-1. \tag{4}$$

Observe that our basis is defined so that $\phi(0) = \phi(1) = 0$ so it boundary term vanishes and since we have a finite sum we can safely pull it out from the integral and obtain,

$$\sum_{j=1}^{n-1} c_j \int_0^1 (1+x^2)\frac{\mathrm{d}\phi_j}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x = \int_0^1 f(x)\phi_i(x)\mathrm{d}x, \quad i = 1, .., n-1. \tag{5}$$

From eqn. (1) we have

$$\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x} = \begin{cases} \frac{1}{x_i-x_{i-1}}, & [x_{i-1}, x_i] \\ \frac{-1}{x_{i+1}-x_i}, & [x_i, x_{i+1}] \\ 0, & \text{elsewhere} \end{cases}. \tag{6}$$

We can then write our problem as $A\vec{c} = \vec{f}$ where $\vec{c} = (c_1, ...c_{n-1})$ and $f_i(x) = \int_0^1 f(x)\phi_i(x)\mathrm{d}x$. Now observe ϕ_i and ϕ_j are both nonzero only if $j = i, i \pm 1$. This means we only have elements on the upper, lower, and main diagonals. The entries of our matrix can then be written as,

$$a_{i,j} = \int_0^1 \frac{\mathrm{d}\phi_j}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x.$$

Our diagonal elements are then,

$$a_{i,i} = \int_0^1 (1+x^2)\left(\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x = \int_{x_{i-1}}^{x_i} (1+x^2)\left(\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x + \int_{x_i}^{x_{i+1}} (1+x^2)\left(\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x \quad (7)$$

and the upper and lower diagonal elements are,

$$a_{i,i+1} = a_{i+1,i} = \int_0^1 (1+x^2)\frac{\mathrm{d}\phi_{i+1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x = \int_{x_i}^{x_{i+1}} (1+x^2)\frac{\mathrm{d}\phi_{i+1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x. \quad (8)$$

The elements of the right hand side vector $\vec{f}$ can be determined from numerical integration. We could in this case compute the integral. The computation would be,

$$\int_0^1 f(x)\phi_i(x)\mathrm{d}x = \int_{x_{i-1}}^{x_i} f(x)\frac{x - x_{i-1}}{x_i - x_{i-1}}\mathrm{d}x + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1} - x}{x_{i+1} - x_i}\mathrm{d}x. \quad (9)$$

∎

5. Consider the 1D Poisson Equation,

$$\begin{cases} -u'' = f & \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

a) Modify the method to account for non-homogeneous Neumann conditions.

Proof. Let $u'(0) = a, u'(1) = b$. We try to write the variational problem and discover what issues we need to fix. Take $v \in H_0^1(\Omega)$, multiply, and integrate by parts to obtain

$$\begin{aligned} -u'v\Big|_0^1 + \int_0^1 u'v' \ \mathrm{d}x &= \int_0^1 fv \ \mathrm{d}x \\ \iff -av(0) + bv(1) + \int_0^1 u'v' \ \mathrm{d}x &= \int_0^1 fv \ \mathrm{d}x. \end{aligned}$$

We then have the Bilinear and linear forms,

$$\begin{aligned} B(u,v) &= \int_0^1 u'v' \ \mathrm{d}x, \\ F(v) &= \int_0^1 fv \ \mathrm{d}x - av(0) + bv(1). \end{aligned}$$

Since we have no first order term and Neumann boundary conditions our solution u would not be unique (since we could add any constant and satisfy the same equation) so we instead need to take $u, v \in \tilde{H} = \{v \in H^1 \mid \int_\Omega v = 0\} \cong H^1/Z$ where $Z = \{v \in H^1(\Omega) \mid v \text{ constant a.e. in } \Omega\}$. In other words, u has zero average on Ω. However, this space does not admit a nice basis like P_1/hat functions so we really want $u \in \tilde{H}$ and $v \in H^1$. We have that $B(u, v + \alpha) = B(u, v)$ and since $B(u, v+\alpha) = F(v) + F(\alpha)$ so we want constant functions to be in the kernel of F which is to say that we need,

$$\int_a^b f(x) \ \mathrm{d}x = a - b.$$

We can conclude the continuity of F from Hölder's and the Trace theorem.

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b) Modify the method to account for non-homogeneous Dirichlet conditions.

Proof. Consider the Dirichlet lift $\tilde{u} = -xb - (1-x)a + u$. We then have that $\tilde{u}'' = u$. We now have the homogeneous problem,

$$\begin{cases} -\tilde{u}'' = f & \text{on } (0,1) \\ \tilde{u}(0) = \tilde{u}(1) = 0 \end{cases}.$$

We then proceed in the usual manner. ∎