Finite Element Method for a Second Order ODE

Jon Staggs

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1. (finite elements) Use the Galerkin finite element method with continuous piecewise linear basis functions to solve the problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x), \quad 0 \le x \le 1$$
$$u(0) = 0, u(1) = 0.$$

a) Derive the matrix equation you will need to solve this problem.

Proof. We use the basis of continuous piecewise linear basis functions which are hat functions of the form,

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}}, & [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & [x_i, x_{i+1}] \\ 0, & \text{elsewhere} \end{cases}. \tag{1}$$

We also observe that if u satisfies the above differential equation then it also satisfies

$$\langle Lu, \phi \rangle = \langle f, \phi \rangle \tag{2}$$

where $L[\cdot] = -\frac{\mathrm{d}}{\mathrm{d}x}\big((1+x^2)\frac{\mathrm{d}[\cdot]}{\mathrm{d}x}\big)$, $\langle \cdot, \cdot \rangle$ is the L_2 inner product, and ϕ can be written as a linear combination of our basis function. We can write eqn. (2) as,

$$\int_0^1 -\frac{\mathrm{d}}{\mathrm{d}x}\big((1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\big)\mathrm{d}x = \int_0^1 f(x)\phi(x)\mathrm{d}x$$
$$\implies -(1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\phi(x)|_0^1 + \int_0^1 (1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\mathrm{d}x = \int_0^1 f(x)\phi(x)\mathrm{d}x. \tag{3}$$

We take our approximation of u to be $\hat{u} = \sum_{i=1}^{n-1} c_j\phi_j$. Since inner product is linear and ϕ is a linear combination we can write enq. (refeq:innerprod-weak-form) as $\langle L\hat{u}-f, \phi\rangle = \langle L\hat{u}-f, \sum_{i=1}^{n-1} d_i\phi_i\rangle = \sum_{i=1}^{n-1}\langle L\hat{u} - f, \phi_i\rangle$. For each i we have,

$$-(1+x^2)\frac{\mathrm{d}\hat{u}}{\mathrm{d}x}\phi_i(x)|_0^1 + \int_0^1 (1+x^2)\Big(\sum_{j=1}^{n-1} c_j\frac{\mathrm{d}\phi_j}{\mathrm{d}x}\Big)\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x = \int_0^1 f(x)\phi_i(x)\mathrm{d}x, \;\; i = 1, ..., n-1. \tag{4}$$

Observe that our basis is defined so that $\phi(0) = \phi(1) = 0$ so it boundary term vanishes and since we have a finite sum we can safely pull it out from the integral and obtain,

$$\sum_{j=1}^{n-1} c_j \int_0^1 (1+x^2)\frac{\mathrm{d}\phi_j}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x = \int_0^1 f(x)\phi_i(x)\mathrm{d}x, \;\; i = 1, .., n-1. \tag{5}$$

From eqn. (1) we have

$$\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x} = \begin{cases} \frac{1}{x_i - x_{i-1}}, & [x_{i-1}, x_i] \\ \frac{-1}{x_{i+1}-x_i}, & [x_i, x_{i+1}] \\ 0, & \text{elsewhere} \end{cases} . \tag{6}$$

We can then write our problem as $A\vec{c} = \vec{f}$ where $\vec{c} = (c_1, ...c_{n-1})$ and $f_i(x) = \int_0^1 f(x)\phi_i(x)\mathrm{d}x$. Now observe ϕ_i and ϕ_j are both nonzero only if $j = i, i \pm 1$. This means we only have elements on the upper, lower, and main diagonals. The entries of our matrix can then be written as,

$$a_{i,j} = \int_0^1 \frac{\mathrm{d}\phi_j}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x.$$

Our diagonal elements are then,

$$a_{i,i} = \int_0^1 (1+x^2)\left(\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x = \int_{x_{i-1}}^{x_i} (1+x^2)\left(\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x + \int_{x_i}^{x_{i+1}} (1+x^2)\left(\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{7}$$

and the upper and lower diagonal elements are,

$$a_{i,i+1} = a_{i+1,i} = \int_0^1 (1+x^2)\frac{\mathrm{d}\phi_{i+1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x = \int_{x_i}^{x_{i+1}} (1+x^2)\frac{\mathrm{d}\phi_{i+1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x. \tag{8}$$

The elements of the right hand side vector $\vec{f}$ can be determined from numerical integration. We could in this case compute the integral. The computation would be,

$$\int_0^1 f(x)\phi_i(x)\mathrm{d}x = \int_{x_{i-1}}^{x_i} f(x)\frac{x - x_{i-1}}{x_i - x_{i-1}}\mathrm{d}x + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1} - x}{x_{i+1} - x_i}\mathrm{d}x. \tag{9}$$

$\blacksquare$

b) Try $u(x) = x(1-x)$. Then $f(x) = 2(3x^2 - x + 1)$.

Proof. If we have a uniform grid then we can clean up eqns. (7) and (8) by noting that $x_i - x_{i-1} = x_{i+1} - x_i = h$ and $\phi_i' = \frac{1}{h}$ on $[x_{i-1}, x_i]$ and $\phi_i' = -\frac{1}{h}$ on $[x_i, x_{i+1}]$ so for our diagonal elements we have,

$$a_{i,i} = \int_{x_{i-1}}^{x_i} (1+x^2)\left(\frac{1}{h}\right)^2 \mathrm{d}x + \int_{x_i}^{x_{i+1}} (1+x^2)\left(-\frac{1}{h}\right)^2 \mathrm{d}x = \frac{2}{h}\left(\frac{x_{i+1}^3}{3} - \frac{x_{i-1}^3}{3}\right). \tag{10}$$

Our sub and super diagonals are,

$$a_{i,i+1} = a_{i+1,i} = \int_{x_i}^{x_{i+1}} (1+x^2)\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)\mathrm{d}x = -\frac{1}{h}\left(\frac{x_{i+1}^3}{3} - \frac{x_i^3}{3}\right). \tag{11}$$

For the right hand side vector, see part d) and replace instances of $x_i - x_{i-1}$ and $x_{i+1} - x_i$ with h.
We use the midpoint rule to compute the integral of $f\phi_i$ and recall ϕ_i will be non zero in the open interval (x_{i-1}, x_{i+1}). We have,

$$\begin{aligned}\int_{x_{i-1}}^{x_{i+1}} f(x)\phi_i(x)\mathrm{d}x &= f(x_{i-1/2})(x_{i-1/2} - x_i)\frac{x_i - x_{i-1}}{(x_i - x_{i-1})} + f(x_{i+1/2})(x_{i+1} - x_{i+1/2})\frac{x_{i+1} - x_i}{(x_{i+1} - x_i)} \\ &= f(x_{i-1/2})(x_{i-1/2} - x_i) + f(x_{i+1/2})(x_{i+1/2} - x_i).\end{aligned}$$

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c) Try several different values for the mesh size h. Based on your results, what would say is the order of accuracy of the Galerkin method with continuous piece wise linear basis functions?
The table below shows the infinity and 2 norm for various mesh sizes of h.

h	L_∞	L_2
.1	7.812×10^{-4}	5.6648×10^{-4}
.01	7.8686×10^{-6}	5.6627×10^{-6}
.001	7.8686×10^{-8}	5.6626×10^{-8}
.0001	7.8552×10^{-10}	5.6569×10^{-10}

Based on the above table the accuracy appears to be second order.

d) Now try a nonuniform mesh spacing, say, $x_i = (i/(m+1))^2$. $i = 0, 1, ..., m+1$. Do you see the same order of accuracy, if h is defined as the maximum mesh spacing, $\max_i(x_{i+1} - x_i)$?

Proof. Since the mesh is non uniform we need to keep the more general form as in eqns. (7) and (11). Our diagonal elements are,

$$a_{i,i} = \left(x_i + \frac{x_i^3}{3} - x_{i-1} - \frac{x_{i-1}^3}{3}\right)\left(\frac{1}{x_i - x_{i-1}}\right)^2 + \left(x_{i+1} + \frac{x_{i+1}^3}{3} - x_i - \frac{x_i^3}{3}\right)\left(\frac{-1}{x_{i+1} - x_i}\right)^2. \quad (12)$$

The super and sub diagonals are,

$$a_{i,i+1} = a_{i+1,i} = \left(x_{i+1} + \frac{x_{i+1}^3}{3} - x_i - \frac{x_i^3}{3}\right)\left(\frac{-1}{(x_{i+1} - x_i)^2}\right). \quad (13)$$

Our right hand side vector is,

$$\int_{x_{i-1}}^{x_i} 2(3x^2 - x + 1)\frac{x - x_i}{x_i - x_{i-1}}\mathrm{d}x + \int_{x_i}^{x_{i+1}} 2(3x^2 - x + 1)\frac{x_{i+1} - x}{x_{i+1} - x_i}\mathrm{d}x.$$

$$= \left.\frac{2(\frac{3}{4}x^4 - \frac{x^3}{3} + \frac{x^2}{2}) - 2(\frac{x^3}{3} - \frac{x^2}{2} + x)x_i}{x_i - x_{i-1}}\right|_{x_{i-1}}^{x_i} + \left.\frac{2(\frac{x^3}{3} - \frac{x^2}{2} + x)x_{i+1} - 2(\frac{3}{4}x^4 - \frac{x^3}{3} + \frac{x^2}{2})}{x_{i+1} - x_i}\right|_{x_i}^{x_{i+1}}. \quad (14)$$

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The table below shows the infinity and 2 norm errors along with the corresponding largest mesh distance in meshes of different coarseness.

$\max_i(x_{i+1} - x_i)$	L_2	L_∞
.1736	.0082	.0025
.0197	9.8020×10^{-5}	1.0181×10^{-5}
.002	3.3122×10^{-8}	9.9800×10^{-7}
1.9997×10^{-4}	1.0502×10^{-10}	9.9980×10^{-9}

From this table, it appears the scheme is second order and agrees with part b).

e) Suppose the boundary conditions were $u(0) = a$, $u(1) = b$. Show how you would represent the approximate solution $\hat{u}$ as a linear combination of hat functions and how the matrix equation in part (a) would change.

Proof. We now need our linear combination of basis functions to satisfy $\sum c_j\phi_j(0) = a$ and $\sum c_j\phi_j(1) = b$. Since all of our basis hat functions are zero at the end points we can add a $\phi_0(x)$ and $\phi_m(x)$ that satisfy the boundary conditions. The most natural first guess would be "half" hat functions (i.e. lines) that start at the boundary value and are zero at the next neighboring point in the domain.

$$\phi_0(x) = \begin{cases} \frac{x_1 - x}{x}, & [0, x_1] \\ 0, & \text{elsewhere} \end{cases} \tag{15}$$

$$\phi_m(x) = \begin{cases} \frac{x - x_{m-1}}{x_m - x_{m-1}}, & [x_{m-1}, x_m] \\ 0, & \text{elsewhere} \end{cases}. \tag{16}$$

Now we need to update our matrix and note that we have two new coefficient values $c_0 = a$ and $c_m = b$. Recall that the weak form, eqn. (**??**), and we now take our approximation $\hat{u} = \sum_{j=0}^{m} c_j\phi_j(x)$ so the weak form can be written,

$$-(1+x^2)\sum_{j=0}^{m} c_j \frac{\mathrm{d}\phi_j(x)}{\mathrm{d}x}\phi_i(x)\Big|_0^1 - \sum_{j=0}^{n} c_j \int_0^1 (1+x^2)\frac{\mathrm{d}\phi_j(x)}{\mathrm{d}x}\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\mathrm{d}x. \tag{17}$$

Rather than constructing a larger matrix and $\vec{c}, \vec{f}$ we can instead use the same idea from making finite difference matrices and subtract the boundary term in the right hand side vector, $\vec{f}$. The equation that comes from the first row in our matrix is

$$c_1\int_0^1 (1+x^2)\frac{\mathrm{d}\phi_1}{\mathrm{d}x}\frac{\mathrm{d}\phi_1}{\mathrm{d}x}\mathrm{d}x + c_2\int_0^1 (1+x^2)\frac{\mathrm{d}\phi_2}{\mathrm{d}x}\frac{\mathrm{d}\phi_1}{\mathrm{d}x}\mathrm{d}x =$$

$$\int_0^1 f(x)\phi_1(x)\mathrm{d}x + a(1+x^2)\frac{\mathrm{d}\phi_0}{\mathrm{d}x}\phi_1\Big|_0^1 - a\int_0^1 (1+x^2)\frac{\mathrm{d}\phi_0}{\mathrm{d}x}\frac{\mathrm{d}\phi_1}{\mathrm{d}x}\mathrm{d}x.$$

Observe that the $\phi_1(x)$ term in the right hand side that we subtracted will vanish as we defined ϕ_1 so that it's zero on the boundaries.
Similarly in the last row of our matrix we obtain the equation,

$$c_{m-2}\int_0^1 (1+x^2)\frac{\mathrm{d}\phi_{m-2}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{m-1}}{\mathrm{d}x}\mathrm{d}x + c_{m-1}\int_0^1 (1+x^2)\frac{\mathrm{d}\phi_{m-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{m-1}}{\mathrm{d}x}\mathrm{d}x =$$

$$\int_0^1 f(x)\phi_{m-1}(x)\mathrm{d}x + b(1+x^2)\frac{\mathrm{d}\phi_m}{\mathrm{d}x}\phi_{m-1}\Big|_0^1 - b\int_0^1 (1+x^2)\frac{\mathrm{d}\phi_m}{\mathrm{d}x}\frac{\mathrm{d}\phi_{m-1}}{\mathrm{d}x}\mathrm{d}x.$$

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