Heat Equation with Finite Differences

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1. Let $\Omega \subset \mathbb{R}^2$ be the unit square. Consider the parabolic differential equation,

$$u_t - \Delta u = 0, \quad x \in \Omega, t > 0,$$
$$\text{Periodic boundary conditions for } t > 0$$
$$u(x,0) = u_0(X) \quad x \in \Omega, t = 0.$$

   a) Write down a finite difference scheme that uses second-order centered differencing and Crank-Nicolson in time.

   Proof. We have the centered finite differencing in 2D,

$$\Delta u = \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{(\Delta x)^2} + \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{(\Delta y)^2}.$$

   Crank-Nicolson gives us,

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} = \frac{1}{2}(F(u^{n+1}_{i,j}) + F(u^n_{i,j})).$$

   Here we take $F(u)$ to be then centered differencing. This yields the scheme,

$$\begin{aligned} u^{n+1}_{i,j} = u^n_{i,j} + \frac{\Delta t}{2}\Bigg(&\frac{u^{n+1}_{i+1,j} - 2u^{n+1}_{i,j} + u^{n+1}_{i-1,j}}{(\Delta x)^2} + \frac{u^{n+1}_{i,j+1} - 2u^{n+1}_{i,j} + u^n_{i,j-1}}{(\Delta y)^2} \\ &+ \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{(\Delta x)^2} + \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{(\Delta y)^2}\Bigg). \end{aligned}$$

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   b) Show that your scheme is stable by bounding

$$\max_n ||u_n||^2 \text{ in terms of } ||u_0||,$$

   using Von Neumann analysis.

   Proof. We let $(\xi_1, \xi_2) = (2\pi k_x/J_x, 2\pi k_y/J_y)$ be the transform variable of x, y where J_x, J_y are the x, y number of intervals, respectively. For convenience we take $\Delta x = \Delta y = h$. We then have the Discrete Fourier Transform of the scheme,

$$\begin{aligned} \hat{u}^{n+1}_{i,j} = \hat{u}^n_{i,j} + \frac{\Delta t}{2h^2}\Big(&(e^{i\xi_1}\hat{u}^{n+1}_k - 2\hat{u}^{n+1}_k + e^{-i\xi_1}\hat{u}^{n+1}_k) + (e^{i\xi_2}\hat{u}^{n+1}_k - 2\hat{u}^{n+1}_k + e^{-i\xi_2}\hat{u}^{n+1}_k) \\ &+ (e^{i\xi_1}\hat{u}^n_k - 2\hat{u}^n_k + e^{-i\xi_1}\hat{u}^n_k) + (e^{i\xi_2}\hat{u}^n_k - 2\hat{u}^n_k + e^{-i\xi_2}\hat{u}^n_k)\Big). \end{aligned}$$

Combining and rearranging some things gives us,

$$\left(1 - \frac{\Delta t}{2h^2}\Big[\cos(\xi_1) + \cos(\xi_2) - 4\Big]\right)\hat{u}_k^{n+1} = \left(1 + \frac{\Delta t}{2h^2}\Big[\cos(\xi_1) + \cos(\xi_2) - 4\Big]\right)\hat{u}_k^{n}.$$

We then need to put restrictions on our amplification factor,

$$|A| \leq 1 \implies \left|\frac{\left(1 + \frac{\Delta t}{2h^2}\Big[\cos(\xi_1) + \cos(\xi_2) - 4\Big]\right)}{\left(1 - \frac{\Delta t}{2h^2}\Big[\cos(\xi_1) + \cos(\xi_2) - 4\Big]\right)}\right| < 1.$$

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2. Solve numerically the heat equation problem,

$$u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t < 1$$
$$u(0,t) = u(1,t) = 0, \quad u(x,0) = .5 - |x - .5|$$

by using centered divided differencing in space and forward difference in time with $\Delta t = 0.1, \Delta x = .05, .1$. Comment on the result.

Solution: We have the scheme

$$\frac{u_l^{n+1} - u_j^n}{\Delta t} = \frac{1}{(\Delta x)^2}(u_{j+1}^n - 2u_j + u_{j-1}^n).$$

See Fig. 1 and Fig. 2 for stable step sizes and Fig. 3 for an example of violating the CFL-type condition.

3. Repeat 2. with Crank-Nicolson time discretization.

Solution: We use the 1D version of the scheme in question 4. We end up with the system,

$$(I - \alpha F)u_i^{n+1} = (I + \alpha F)u_i^n$$

where F is the finite difference matrix corresponding to the second derivative centered difference and $\alpha = \frac{\Delta t}{2(\Delta x)^2}$. See Fig. 4 and Fig. 5

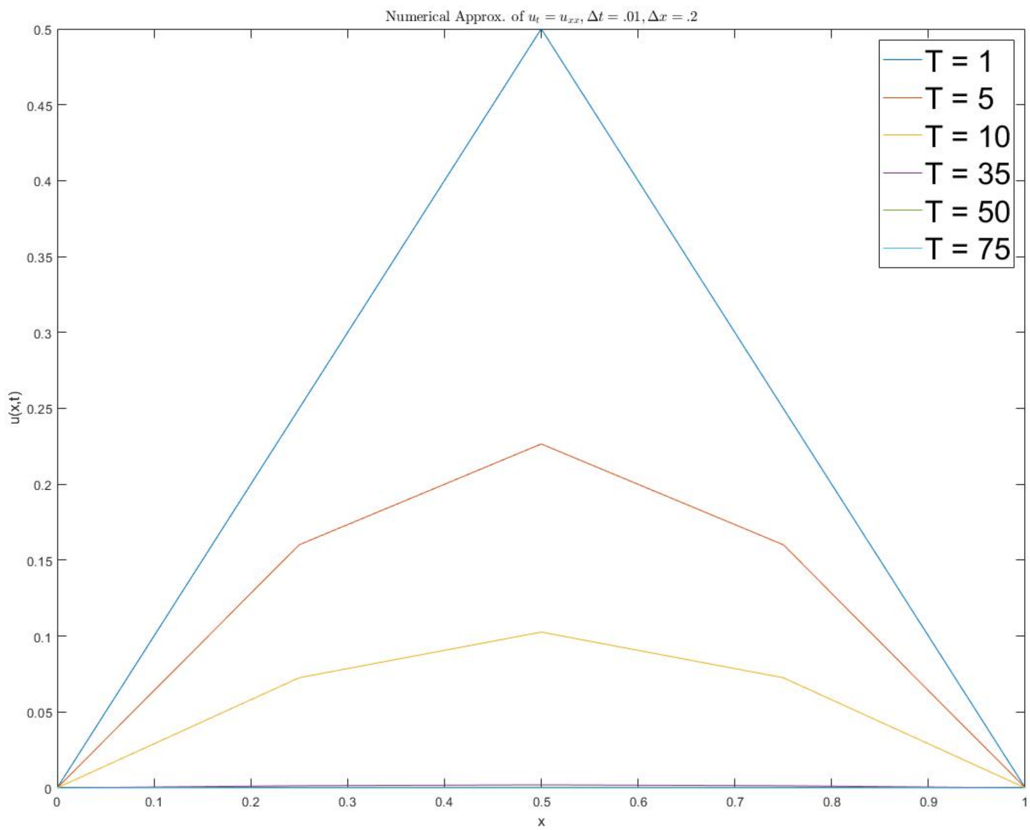

Figure 1: Numerical Approximation of Heat Eqn. with $\Delta t = .01, \Delta x = .2$

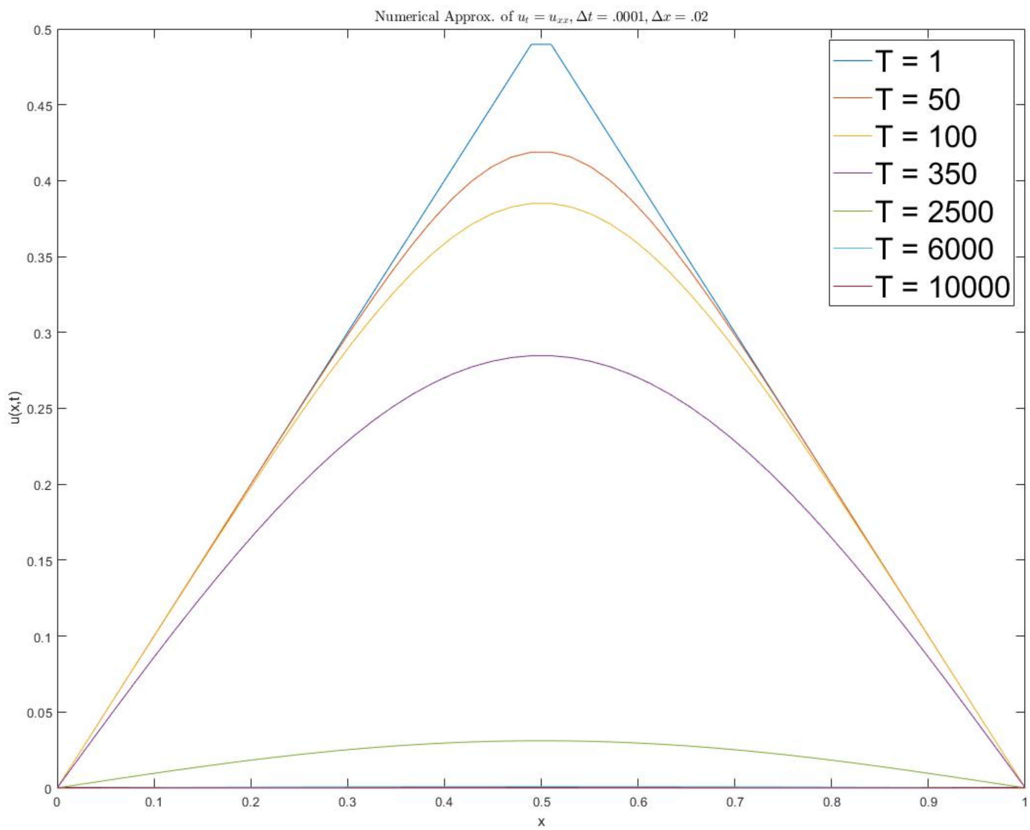

Figure 2: Numerical Approximation of Heat Eqn. with fine grid $\Delta t = .0001, \Delta x = .02$

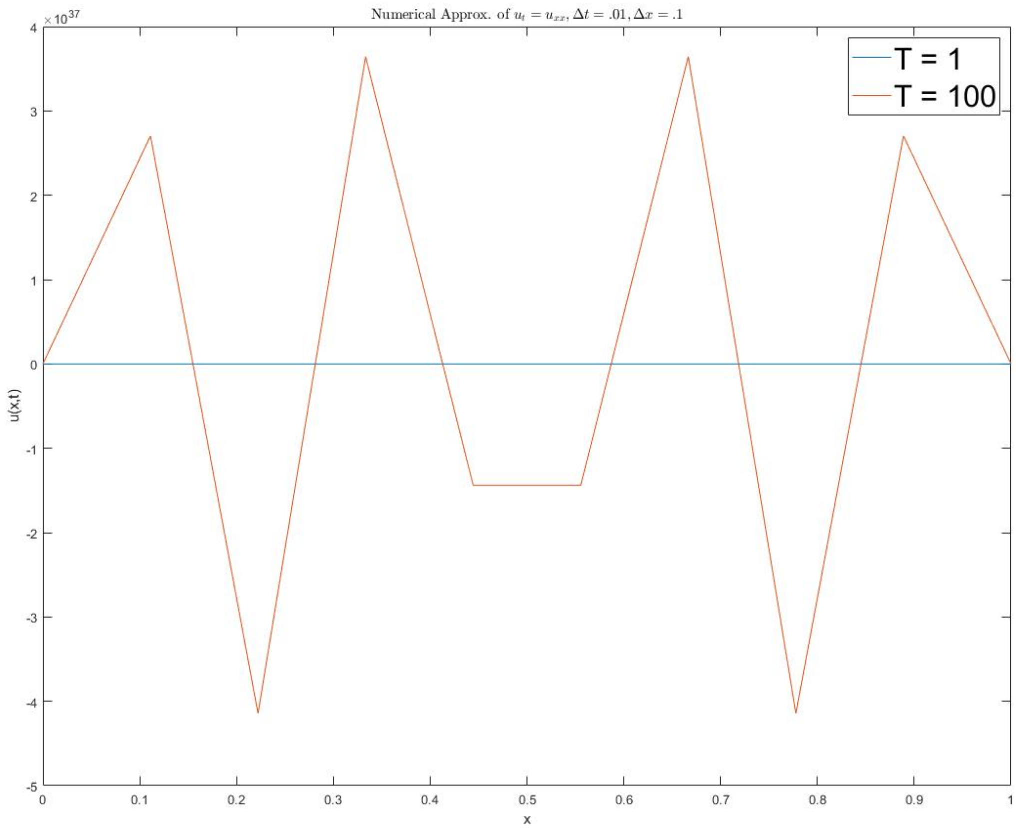

Figure 3: Numerical Approximation of Heat Eqn. with fine grid $\Delta t = .0001, \Delta x = .02$

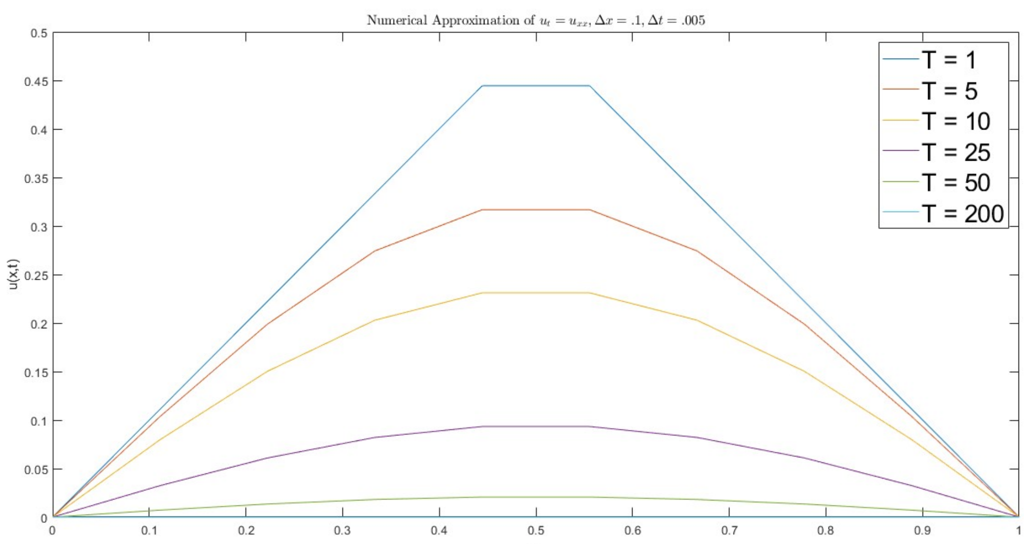

Figure 4: Numerical Approximation of Heat Eqn. (Crank-Nicolson) with $\Delta t = .01, \Delta x = .2$

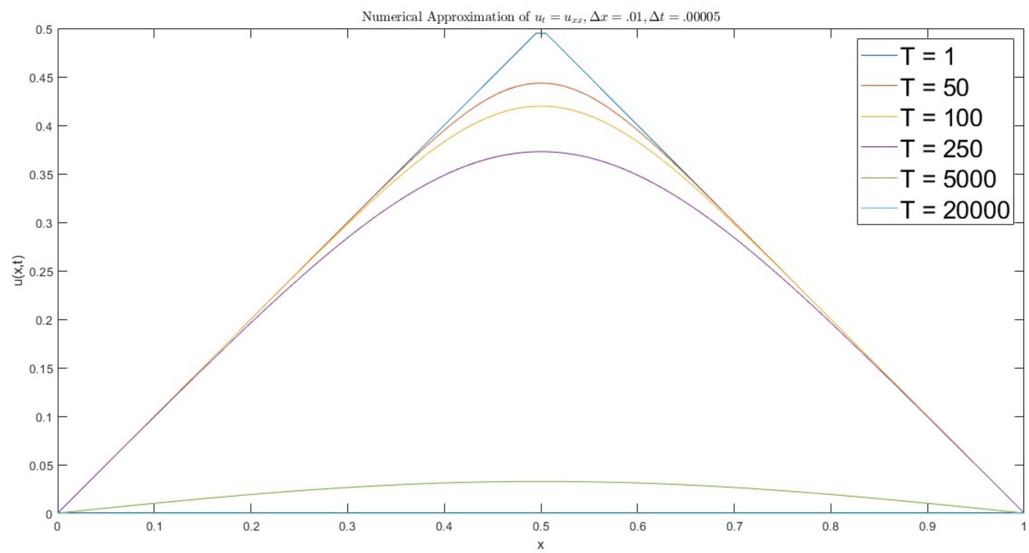

Figure 5: Numerical Approximation of Heat Eqn. (Crank-Nicolson) with fine grid $\Delta t = .0005, \Delta x = .01$