Finite Element Method

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1 Introduction

We provide a couple of examples for deriving weak forms for the Finite Element Method.

2 Examples

1. Rewrite the following ODE in variational (weak) form,

$$-(a(x)y')' + b(x)y' + c(x)y = d(x), \;\; 0 < x < 1,$$
$$y(0) = 0, y'(1) = 0.$$

Solution: Let $\phi(x)$ be a test function (we will define where our function and test function ought to live later). We now multiply and integrate which gives,

$$-\int_0^1 (a(x)y')'\phi(x)\mathrm{d}x + \int_0^1 b(x)y'\phi(x)\mathrm{d}x + \int_0^1 c(x)y\phi(x)\mathrm{d}x = \int_0^1 d(x)\phi(x)\mathrm{d}x.$$

Using integration by parts, we shift a derivative off of $(a(x)y')'$ to our test function giving,

$$-\phi(x)(a(x)y'(x)\Big|_0^1 + \int_0^1 a(x)y'(x)\phi'(x)\mathrm{d}x + \int_0^1 b(x)y'(x)\phi(x)\mathrm{d}x + \int_0^1 c(x)y(x)\phi(x)\mathrm{d}x = \int_0^1 d(x)\phi(x)\mathrm{d}x.$$

Now let's deal with the boundary terms,

$$-\phi(x)(a(x)y'(x)\Big|_0^1 = -a(1)y'(1)\phi(1) + a(0)y'(0)\phi(0) = a(0)y'(0)\phi(0).$$

Where the last equality comes from the imposed boundary condition $y'(1) = 0$. We either want to choose a space so that the second term vanishes or absorb it into our bilinear form (since it contains a y term). Since we don't know that $y(1) = 0$ we can't choose $H_0^1(0,1)$. Instead, we absorb it into the bilinear form. We now see that we can take $\phi, y \in H^1(0,1)$ and the weak form becomes,

$$\int_0^1 a(x)y'(x)\phi'(x)\mathrm{d}x + a(0)y'(0)\phi(0) + \int_0^1 b(x)y'(x)\phi(x)\mathrm{d}x + \int_0^1 c(x)y(x)\phi(x)\mathrm{d}x = \int_0^1 d(x)\phi(x)\mathrm{d}x.$$

We can rewrite this a bilinear form $A(y,\phi)$ and linear form $f(\phi)$ where,

$$A(y,\phi) = \int_0^1 a(x)y'(x)\phi'(x)\mathrm{d}x + a(0)y'(0)\phi(0) + \int_0^1 b(x)y'(x)\phi(x)\mathrm{d}x + \int_0^1 c(x)y(x)\phi(x)\mathrm{d}x,$$

$$f(\phi) = \int_0^1 d(x)\phi(x)\mathrm{d}x.$$

Remark: You can also choose a subspace of $H^1(0,1)$ such that $\phi(0) = 0$ then we can eliminate the boundary condition to obtain a symmetric form of the bilinear form.

2. Determine the linear system of equations that results from a finite element approximation of the equation below when using standard hat functions,

$$\begin{gathered} -y'' + y = f(x), \ \ 0 < x < 1, \\ y(0) = 0, y'(1) = 0. \end{gathered}$$

Solution: We have the scenario from Example 1 with $a(x) = 1, b(x) = 0, c(x) = 1$, and $d(x) = f(x)$. We have the general weak form,

$$\int_0^1 y'(x)\phi'(x)\mathrm{d}x + a(0)y'(0)\phi(0) + \int_0^1 y(x)\phi(x)\mathrm{d}x = \int_0^1 f(x)\phi(x)\mathrm{d}x,$$

where $\phi(x)$ is a test function. We take our basis to be the hat functions,

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}}, & [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & [x_i . x_{i+1}], \\ 0, & \text{elsewhere} \end{cases}$$

We then have that our function can be represented in this bases. We let $\tilde{y}$ be our finite approximation in this basis. That is,

$$\tilde{y}(x) = \sum_{j=1}^{n-1} c_j \phi_j(x).$$

We also have that our test functions $\phi_i \in V_h$. Since for any $\phi \in V_h$ we have $\langle L\overline{u} - f, \phi\rangle = 0$ we have this equality namely for the basis functions ϕ_i,

$$\langle L\tilde{u} - f, \phi_i\rangle = 0.$$

Then for each i we have,

$$\int_0^1 \Big(\sum_{j=1}^{n-1} c_j \frac{\mathrm{d}\phi_j}{\mathrm{d}x}\Big)\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\ \mathrm{d}x + \sum_{j=1}^{n-1} c_j \frac{\mathrm{d}\phi_j}{\mathrm{d}x}(0)\phi_i(0) + \int_0^1 \sum_{j=1}^{n-1} c_j\phi_j\phi_i\ \mathrm{d}x - \int_0^1 f(x)\phi_i\ \mathrm{d}x = 0,$$

$$\implies \sum_{j=1}^{n-1} c_j \int_0^1 \frac{\mathrm{d}\phi_j}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\ \mathrm{d}x + \sum_{j=1}^{n-1} c_j \frac{\mathrm{d}\phi_j}{\mathrm{d}x}(0)\phi_i(0) + \sum_{j=1}^{n-1} c_j \int_0^1 \phi_j\phi_i\ \mathrm{d}x - \int_0^1 f(x)\phi_i\ \mathrm{d}x = 0 \quad \text{for } i = 1, ..., n-1.$$

The derivatives of our hat functions are,

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi_i(x) = \begin{cases} \frac{1}{x_i-x_{i-1}}, & [x_{i-1}, x_i], \\ \frac{-1}{x_{i+1}-x_i}, & [x_i, x_{i+1}], \\ 0, & \text{elsewhere} \end{cases}$$

We can now write our problem as $A\vec{c} = \vec{f}$ where $f_i(x) = \int_0^1 f(x)\phi_i(x)\ \mathrm{d}x$. Next we notice that ϕ_i and ϕ_j are both nonzero only if $j = i, i \pm 1$. Hence, we will have a tridiagonal matrix. We have the diagonal elements,

$$\begin{aligned}
a_{i,i} &= \int_0^1 \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\ \mathrm{d}x + \frac{\mathrm{d}\phi_i}{\mathrm{d}x}(0)\phi_i(0) + \int_0^1 \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_i\ \mathrm{d}x \\
&= \int_{x_{i-1}}^{x_i} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\ \mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_i\ \mathrm{d}x + \frac{\mathrm{d}\phi_i}{\mathrm{d}x}(0)\phi_i(0) + \int_{x_{i-1}}^{x_i} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\ \mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_i\ \mathrm{d}x.
\end{aligned}$$

For the off-diagonal terms $a_{i,i-1}, a_{i,i+1}$ notice that by symmetry we have $a_{i,i-1} = a_{i,i+1}$

$$a_{i,i-1} = a_{i,i+1} = \int_{x_{i-1}}^{x_i} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i+1}}{\mathrm{d}x}\ \mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_{i+1}\ \mathrm{d}x + \frac{\mathrm{d}\phi_i}{\mathrm{d}x}(0)\phi_{i+1}(0) + \int_{x_{i-1}}^{x_i} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i+1}}{\mathrm{d}x}\ \mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_{i+1}\ \mathrm{d}x.$$

The right-hand side vector's elements are,

$$f_i(x) = \int_0^1 f(x)\phi_i(x)\ \mathrm{d}x = \int_{x_{i-1}}^{x_i} f(x)\phi_i(x)\ \mathrm{d}x + \int_{x_i}^{x_{i+1}} f(x)\phi_i(x)\ \mathrm{d}x.$$

Typically, we determine the right-hand side via numerical integration depending on what f is given.