⑤

$$D^4 f(x_j) \cong \frac{f(x_{j+2}) - 4f(x_j + 1) + 6f(x_j) - 4f(x_{j-1}) + f(x_{j-2})}{h^4}$$

$x_j = x_0 + jh$

$$(1)\ f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(x)$$

$$(2)\ f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(x)$$

Podemos saltarnos a la segunda derivada con la suma de la expresión (1) y (2)

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12}f^{(4)}(x) + \frac{h^6}{120}f^{(6)}(x)$$

$$\frac{f(x+h) + f(x-h) - 2f(x) - \frac{h^4}{12}f^{(4)}(x) - \frac{h^6}{120}f^{(6)}(x)}{h^2} = f''(x)$$

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} - \frac{h^2}{12}f^{(4)}(x) - \underbrace{\frac{h^4}{120}f^{(6)}(x)}_{O(h^4)} = f''(x)$$

Como necesito considerar más términos para poder despejar $f^{(4)}(x_j)$ y $f(x_{j+1})$ y $f(x_j - 1)$ ya no son suficientes para obtener una forma funcional en términos de la función únicamente, hago la expansión de:

$$(3)\ f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4h^3}{3}f'''(x) + \frac{2h^4}{3}f^{(4)}(x) + \frac{4h^5}{15}f^{(5)}(x) + \frac{4h^6}{45}f^{(6)}(x)$$

$$(4)\ f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3}h^3 f'''(x) + \frac{2h^4}{3}f^{(4)}(x) - \frac{4h^5}{15}f^{(5)}(x) + \frac{4h^6}{45}f^{(6)}(x)$$

Al sumar las dos expresiones anteriores obtenemos:

$$f(x+2h) + f(x-2h) = 2f(x) + 4h^2 f''(x) + \frac{4h^4}{3}f^{(4)}(x) + \underbrace{\frac{8h^6}{45}f^{(6)}(x)}_{O(h^6)}$$

$$f(x+2h) + f(x-2h) - 2f(x) - 4h^2 f''(x) - \frac{8h^6}{45} f^{(6)}(x) = \frac{4h^4}{3} f^{(4)}(x)$$

$$\frac{3f(x+2h)}{4h^4} + \frac{3f(x-2h)}{4h^4} - \frac{3f(x)}{2h^4} - \frac{3}{h^2} f''(x) - \underbrace{\frac{2h^2}{15} f^{(6)}(x)}_{O(h^2)} = f^{(4)}(x)$$

Reemplazando $f''(x)$ en $f^{(4)}(x)$, obtenemos:

$$\frac{3f(x+2h)}{4h^4} + \frac{3f(x-2h)}{4h^4} - \frac{3f(x)}{2h^4} - \frac{3}{h^2}\left(\frac{f(x+h)+f(x-h)-2f(x)}{h^2} - \frac{h^2}{12} f^{(4)}(x) - O(h^4)\right) - O(h^2) = f^{(4)}(x)$$

$$\frac{3f(x+2h)}{4h^4} + \frac{3f(x-2h)}{4h^4} - \frac{3f(x)}{2h^4} - \frac{3f(x+h)}{h^4} - \frac{3f(x-h)}{h^4} + \frac{6f(x)}{h^4} + \frac{3h^2}{12h^2} f^{(4)}(x) + \frac{3}{h^2} O(h^4) - O(h^2) = f^{(4)}(x)$$

$$\frac{3f(x+2h)}{4h^4} + \frac{3f(x-2h)}{4h^4} + \frac{9f(x)}{2h^4} - \frac{3f(x+h)}{h^4} - \frac{3f(x-h)}{h^4} + \frac{3}{h^2} O(h^2) = f^{(4)}(x) - \frac{3}{12} f^{(4)}(x)$$

$$\frac{f(x+2h)}{h^4} + \frac{f(x-2h)}{h^4} + \frac{6f(x)}{h^4} - \frac{4f(x+h)}{h^4} - \frac{4f(x-h)}{h^4} + O(h^2) = D_f^{(4)}(x)$$

$$\frac{f(x_{j+2}) + f(x_{j-2}) + 6f(x_j) - 4f(x_{j+1}) - 4f(x_{j-1})}{h^4} = D_f^{(4)}(x)$$