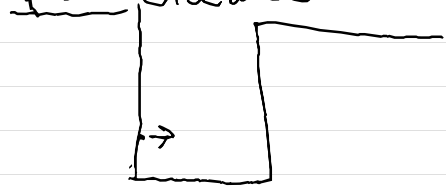
Infinite Potential Well

$$V(x) = \begin{cases} 0 & \forall x \in [0,a] \\ \infty & \text{elsewise} \end{cases}$$

Schrodinger: $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$



$\rightarrow$ In well $(0,a)$: $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = E\psi \rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi$

let $k = \sqrt{2mE}/\hbar$

$\hookrightarrow \psi'' = -k^2 \psi$

general solution: $\psi(x) = A\sin kx + B\cos kx$

BC

$\psi(0) = \psi(a) = 0$

$\psi(0) = 0$: $\psi(0) = A\sin(0) + B\cos(0) = 0$

$\Rightarrow B = 0$

$\psi(a) = 0$: $\psi(a) = A\sin(ka) = 0$

$\sin(ka) = 0$

$ka = n\pi \quad n \in \mathbb{N}$

$k = \frac{n\pi}{a}$

$\psi(x) = A\sin\left(\frac{n\pi x}{a}\right)$

Normalization

$|\psi|^2 = 1$

$\hookrightarrow A = \sqrt{\frac{2}{a}}$

$\psi(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

$k^2 = 2mE/\hbar^2 \rightarrow E = \frac{k^2\hbar^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$

$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$

Inf Well - Variational Guess

$$E_1 \leq \langle \Psi | H | \Psi \rangle \equiv \langle H \rangle$$

$H = V + K$ In well: $V = 0$

$$H = K = \frac{p^2}{2m} \qquad \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

Ansatz choice: $\Psi(x) = Ax^2$

Normalization: $1 = \int_0^a (Ax^2)^* Ax^2 \, dx = A^2 \int_0^a x^4 \, dx$

$$= A^2 \left[\frac{1}{5} x^5 \right]_0^a = \frac{A^2 a^5}{5} = 1$$

$$A^2 = \frac{5}{a^5} \qquad A = \sqrt{\frac{5}{a^5}} \qquad \Psi(x) = \sqrt{\frac{5}{a^5}} \, x$$

$$\langle H \rangle = \langle V \rangle + \langle K \rangle$$

$\hookrightarrow$ In well: $\langle H \rangle = \langle K \rangle = \frac{p^2}{2m} = +\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$

$$\langle H \rangle = \langle \Psi | H | \Psi \rangle = \int_0^a (Ax^2)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)(Ax^2)$$

$$= \frac{\hbar^2 A^2}{2m} \int_0^a x^2 \frac{\partial}{\partial x}[2x] \, dx = \frac{\hbar^2 A^2}{m} \int_0^a x^2 \, dx = \frac{\hbar^2 A^2}{m} \left[\frac{1}{3} x^3 \right]_0^a$$

$$= \frac{\hbar^2 A^2}{m} \cdot \frac{1}{3} a^3 = \frac{\hbar^2 \cdot 5}{m a^5} \cdot \frac{1}{3} a^3 = \frac{5}{3} \cdot \frac{\hbar^2}{m a^2}$$

Harmonic oscillator