Johnson-Nyquist Noise: A Look Into the Relationship Between Thermo- and Electro- dynamics

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Basic circuit physics predicts $\Delta V = 0$ for a resistor experiencing zero current flow, stemming from the basic physical relation $V = IR$. Due to advanced physical properties like thermodynamic fluctuations and charge quantization, resistors will instead display a noise voltage $V_J \neq 0$, acting as a built-in emf. This noise voltage will on average be zero, but will experience fluctuations around the average point. In order to quantify these fluctuations, a value called mean square voltage, $< V_J^2 >$ can be defined. This study conducts experimental validation surrounding the value of $< V_J^2 >$ and its dependence on physical factors, following previously established protocols for this type of experiment to determine the fit to theoretical results like those first posited by Nyquist. This result does not add to past scientific knowledge. Despite this, in the past this experiment has served as a crucial stepping stone in the development of electronics and noise theory, and, as part of this study, it will help aid in the development of the experimental capabilities. Noise detection and measurement is crucial when utilizing electrical systems, and the presence of noise can play a significant role in the efficiency of low voltage electronics for communication, power transfer, and other purposes. Understanding the measurement of noise is crucial for an engineer working with electronics. This includes the management of equipment such as a digital multimeter and oscilloscope, in addition to data processing techniques using a software like Excel, Python, or MATLAB.

INTRODUCTION

In 1905, Einstein published that Brownian motion, the then unexplained movement of pollen in water, was due to the atomic nature of matter. Based on this result, Einstein then published the next year that suspended particles would have a mean squared motion of:

$$\bar{x^2} = 2k_BTB\tau.$$

Where B is how free the particle is to move and τ is the period of which the particle is being viewed, k_B is Boltzmann's constant, and T is the temperature of the particle.

In 1927, 20 years later, John Johnson of Bell Labs found noise in his communication systems. Following this discovery, Johnson used band-pass filters and an oscilloscope to show the impact of two factors on the observed fluctuations. He observed that the Δf (bandwidth frequency) the filter allowed and the resistance of the resistor used affected the mean squared voltage observed. Johnson then took his result to fellow Bell Labs researcher, Harry Nyquist, who explained the discovered phenomena as follows:

Say there are two resistors with resistance R with resistanceless transmission cables between them. There is some thermal noise being transferred between the resistors. This thermal noise must be identical. If it were not identical, energy would be transferred from one resistor to the other, which has not been observed. Thus, the amount of noise, which would be proportional to the energy transferred, can only be dependent on temperature, resistance, and frequency. The power given by these standing waves of noise can be defined, given by

$$P = E/t,$$

where t is the length of the wire divided by the velocity of the propagation of the noise and E is the energy of the propagating noise waves. For thermal energy, it is known $E/t = k_BT\Delta f$, where Δf is all available propagating frequencies. Since there are two resistors the answer is multiplied by two. The given expressions can only be satisfied if the mean squared voltage is given by

$$\bar{V^2} = 4k_BTR\Delta f \tag{1}$$

for each resistor.

Given some transformation, this is remarkably similar to Einstein's equation. Since $R = B^{-1}$ (resistance is the inverse of mobility), $\tau = \frac{1}{\Delta f}$, and $i = q/\tau$ (where the particle is electrons so $x = q$). Combining this with Ohm's law, it can be shown get:

$$\bar{x^2} = 2k_BTB\tau$$
$$\bar{V^2} = 4k_BTR\tau^{-1}$$

(note again the factor of two from the two resistors).

In the experiment, the goal was to observe Johnson-Nyquist noise and find that indeed it does follow the original Nyquist equation with a linear relationship between the mean squared voltage and the resistance, temperature, and bandwidth frequency. [1]

It is also noteworthy that Nyquist theorized this noise as being due to thermal fluctuations. Heat energy is observed in particles as little fluctuations. When these fluctuations occur in electrons, they move across resistors that results in a voltage ($V = IR$). However, these fluctuations are random, and therefore their net voltage should be zero. Therefore, to measure the effect of the noise, the voltage squared is needed to quantify the value. Since this is a random fluctuation, it is also important to take the mean for a meaningful interpretation. Therefore, Johnson noise will be measured as mean squared voltage ($< V^2 >$) from this point further.

MEASUREMENTS

The apparatus used to measure Johnson noise consisted of a Low Level Electronics system, a High Level Electronics system and a digital multimeter. The observations of the noise stemmed from the Low Level Electronics (LLE), that allowed us to measure these tiny fluctuations. Inside of the LLE are varied resistors for measurements, an initial amplification chain, in addition to a heater and other external settings for noise management. The LLE unit is then attached to the High Level Electronics (HLE) with included settings for adding a low-pass filter, high-pass filter, amplifier, and squarer. Figure 1 displays the electrical setup used during the course of this experiment. Also pictured are the Dewar Temperature Unit and Signal Generator, to be utilized later.

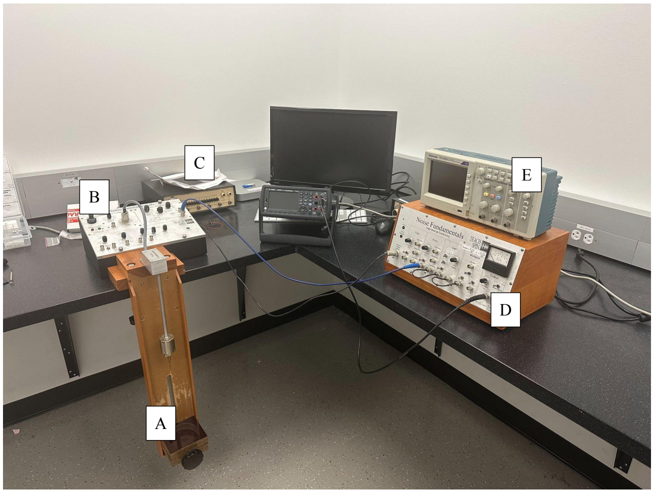

FIG. 1: Image of the electrical setup used during the course of this experiment. (A) Dewar Unit for temperature variation. (B) LLE (C) Signal Generator for bandwidth-signal interaction (D) HLE (E) Oscilloscope

First steps of the experiment involved confirming the readings were indeed stemming from noise. For noise, we expected to see that any peak in the observed signals was not correlated with any other peak. The procedure involved taking a signal from the LLE, amplifying it with the HLE, and then putting it in an oscilloscope for readout. The oscilloscope was set to average outputs around the peaks. It was found that around each peak there were no other correlated peaks, meaning on average around a peak there was no voltage. This showed the unrelated nature of each peak, and that noise was actually being observed. Attached below, Figure 2 displays the results of the oscilloscope, aligning with expectations for a noise reading.

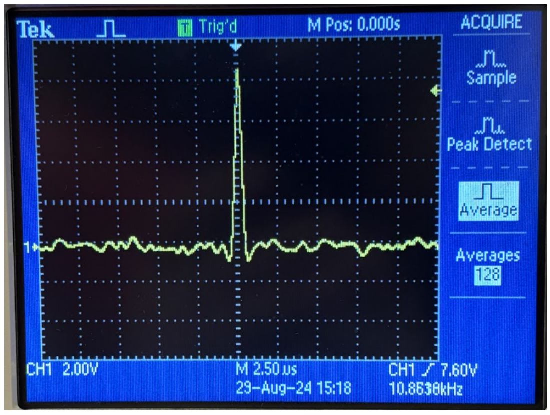

FIG. 2: Oscilloscope capture showing a sharp voltage spike centered at 0 μs with surrounding baseline noise.

Following this confirmation, the Johnson noise was calculated. First, the Johnson noise was checked over the 10k resistor in the LLE and the 10k resistor in an external probe (A in Figure 1). The following settings were used for the HLE processing in this procedure: low-pass filter of 100 kHz, HLE amplification by a factor of 1000, output squaring, and averaging the output over a second time frame. This process allowed us to calculate the mean squared voltage for the noise at the output. In order to turn the mean squared amplified voltage back to the mean squared voltage of the original source, the voltage had to be divided by the total amplification squared. There were two contributions to the gain: 600 from the LLE (G_1) and the aforementioned 1000 from the HLE (G_2). The voltage also had to be multiplied by the value 10 to account for effects imposed by the HLE. The transformation is given below:

$$< V_{in}^2 >= \frac{10* < V_{out}^2 >}{(G_1 * G_2)^2}$$

This procedure was followed with similar steps involving the 10 Ω resistors from the LLE and also the probe (A in Figure 1). The results of all 4 experiments are shown in Table 1, at beginning of the next page.

The first column contains the observed voltage from the digital multimeter attached to the HLE. In the second column, there is the noise density (S). S is calculated

TABLE I: Johnson Noise Calculations

	$< V_{in}^2 >$	S(10^{-17})	Noise Resistor(10^{-17})
Probe 10K	0.813	19.74	13.60
Probe 10	0.253	6.14	
Internal 10k	0.924	22.44	16.32
Internal 10	0.252	6.12	

with the un-amplified voltage divided by the frequency range, and represents the noise per Hz added to the overall range. Note that for a reading with no high-pass filter and 100k HZ low-pass filter, the frequency range is not 100k Hz, as intuition would suggest. It is instead, 114400 Hz, the mathematics for this will be discussed in *Noise vs Bandwidth*.

For these calculations, it was first assumed that the Johnson noise from the 10 Ω resistors would be effectively 0 due to their small resistance values. This was not the case, and there was still a value for the Johnson noise from these measurements. This noise originated from the additional resistors, further up the chain, in the LLE and HLE. In order to measure the true noise density, this noise had to be subtracted from the observed noise from the 10 kΩ resistor. These steps allowed us to find the noise from the electrons, flowing only over the 10 kΩ resistor itself. These results can be found in the last column of Table 1.

Observing the in-lab thermometer, the temperature was found to be 296K. Using this value, it was determined that $4k_BTR = 1.36 * 10^{-16}$. Looking at the Nyquist equation, $\bar{V}^2 = 4k_BTR\Delta f \rightarrow S = \bar{V}^2/\Delta f = 4k_BTR$. Mentioned earlier, S is the noise density or the amount of Johnson noise that is observed per Hz of allowed frequency. The value of the noise density found here by measuring the noise was remarkably close to the value found later by using the bandwidth. The probe's value of noise density is also close, but a little smaller, as seen in the last column of Table 1. In response to this observation, it was theorized that the probe's resistor has been affected by the many times it has been placed in high and low temperature environments. If this lowered the resistor's resistance, that would explain why less Johnson noise was observed. The reasons why the probe was subjected to these extreme conditions will be discussed later.

NOISE VS RESISTANCE

The first variable contained within the Nyquist equation that will be analyzed is resistance.

Johnson noise vs resistance was plotted and a log-log plot was used to avoid issues of scale. The internal resistors from the LLE and the external resistors from the probe, as well as a few extra resistors from the lab, were

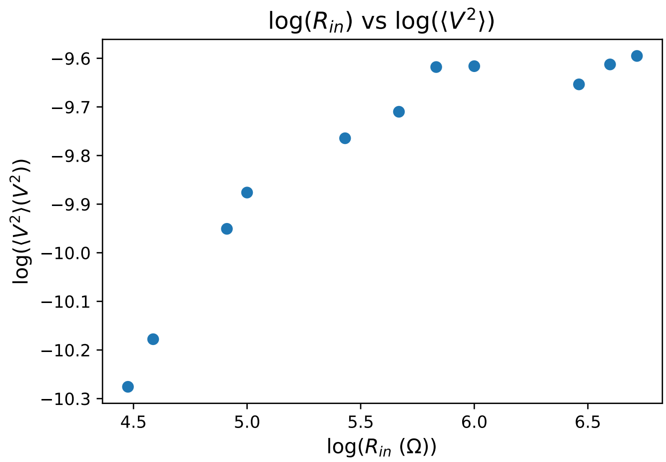

FIG. 3: Log-Log plot of the varied input resistance R_{in} vs the mean-squared voltage $< V_J^2(t) >$ for Johnson Noise. For this plot, log_{10} was used.

used. Figure 3 displays the plot, which deviates from the expected linear behaviour.

After 2 MΩ the Johnson voltage plateaued. This change occurred past $log(R_{in}) \approx 6.3$ on the x-axis of Figure 3, where all three points appear to deviate from the established linear pattern. This error may have been because the amplifier in the LLE had a limited output. If that was the case, the LLE could only give up to a certain voltage to HLE, and so for the last few data points the LLE gave the same outputs to the HLE even though the true relationship is linear.

NOISE VS BANDWIDTH

The next variable considered from Nyquist's formulation was the bandwidth. Various filters were used that prevent AC voltages of certain frequencies from getting past. These filters allowed for changes to the Δf that was being observed, limiting what frequencies voltage would be seen. There are three types of filters. Low-pass filters let lower frequencies than the target frequency (f_c) pass. High-pass filters, meanwhile, let only voltages of frequencies higher than the target threshold pass. Band-pass filters are similar to a low-pass and high-pass filter combined; they allow only voltages with frequencies close to the given frequency to pass.

It would be intuitive to think that a low-pass filter of 100 kHz would let any voltage from below 100 kHz through 100% of the time and anything above 100 kHz would never get through. However, these 'brick wall' filters are much less predicable. Instead, filters that trail off were used. To describe these filters, gain is used, formulated as $G = V_{out}/V_{in}$. For the filters gain is given as:

$$G_{Low-pass} = [1 + (f/f_c)^4]^{-1/2} \tag{2}$$

$$G_{High-Pass} = (f/f_c)^2[1 + (f/f_c)^4]^{-1/2} \tag{3}$$

where f_c is the listed filter value or the cornering frequency. To find the effective bandwidth from using these filters use the equation [2]:

$$Bandwidth_{effective} = \int_0^\infty G^2(f)df. \tag{4}$$

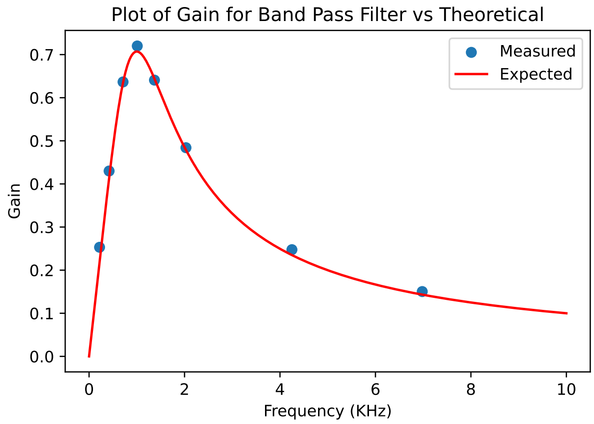

FIG. 4: Gain tracked over different frequency values produced using a function generator for a single band-pass filter. For this plot, the natural logarithm ln was used.

Before use, we wanted to confirm the filters were working as expected. To test this, a signal generator (C in Figure 1) was used to put voltages through the different filters at different frequencies. There are two plots showing the results of this experiment. The first is a plot of a single band-pass filter, and its gain across a chosen frequency range, plotted compared to the theoretical gain (Figure 4). The second result is the gain of one band-pass filter in red and a combination of a low and high band-pass working together (Figure 5). Also presented are the theoretical expectations for these two filters.

It is seen that the band-pass filter fit to the theoretical equation. Therefore it can be assumed that the later filters will follow the expected formulations, and the hardware is working as expected. The high/low-pass filter combination appears to follow the correct shape, but had a higher gain value than expected. The comparison between the two filter types is as expected from literature [2]. The single band-pass filter has a smaller region of optimal gain, but a less severe fall off than the high/low-pass filter combination.

After it was shown the filters worked very close to as expected, the noise could be plotted against the equivalent bandwidth. The Johnson noise was measured for

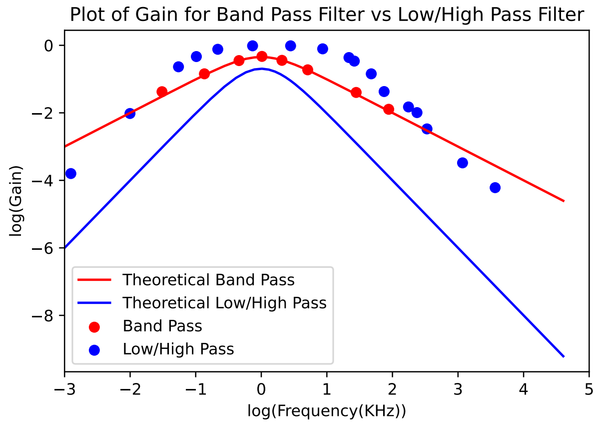

FIG. 5: Gain tracked over different frequency values produced using a function value for both a single band-pass filter and a combination high/low-pass filter

many bandwidth values. Then, those noise values were plotted against the corresponding effective bandwidths. Not shown, the noise was also plotted against the naive bandwidth ($\Delta f = |f_{lowpass} - f_{highpass}|$). This plot did not show a linear relationship. Below (Figure 6) is a log-log plot of the given effective bandwidth and the Johnson noise found. There is a very clear linear relationship revealed between the two.

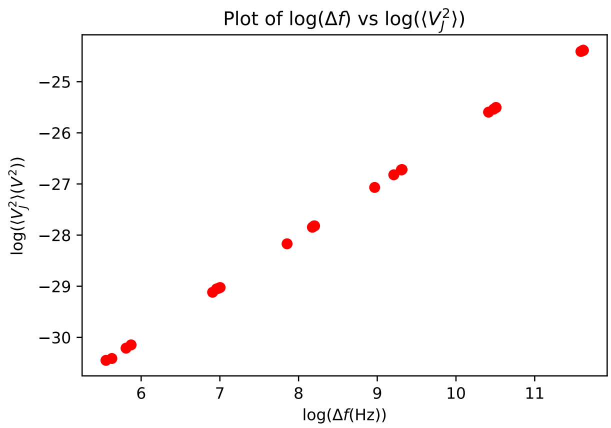

FIG. 6: Log-Log plot of the varied bandwidth Δf vs the mean-squared voltage $< V_J^2(t) >$ for Johnson Noise. For this plot, the natural logarithm ln was used.

Johnson noise was remarkably linear in effective bandwidth across many chosen values of both the high-pass and low-pass filter. From this result, we can attempt to derive Boltzmann's constant (k_B), confirming the validity of our results. Looking at the Nyquist equation and taking the log of both sides:

$$ln(\hat{V^2}) = ln(4k_bTR\Delta f) = ln(4k_bTR) + ln(\Delta f).$$

Because this is a log-log plot, the intercept should be $ln(4k_BTR)$ or $ln(S)$ (because $S = \hat{V^2}/\Delta f = 4k_BTR$). Remember, S is the noise density that was discussed earlier, effectively, it is how much noise comes from a section of white noise. Using Python, the intercept was found and a corresponding S value of $2.23 * 10^{-16} \pm 5 * 10^{-18}$. Using this and the fact that $R = 10000\Omega \pm 10\Omega$, $T = 296.65K \pm .5K$, it was shown that $S/4RT = 1.87 * 10^{-23}J/K \pm 5.64 * 10^{-24} = k_B$. This puts the accepted value of of $k_B = 1.38 * 10^{-23}$ within the error range. The comparison to k_B is an attempt to validate the observations seen for noise, taken within the lab. Our value of k_B being close to the accepted signifies that our readings were mostly correct, with a additional small source of error causing the value to increase past what is expected.

This source of error could have come from the amplifiers' inherent Johnson noise. As discussed earlier, the resistors and components of the HLE may have contributed to the total Johnson noise observed. This effect was not subtracted from observed values of noise. This neglect could have shifted the line systematically. Earlier, we took into account the baseline noise measurement at 10 Ω. Despite this, the baseline does not consider all effects of noise caused by the HLE. This may have been part of the reason why the value of k_B is higher than the accepted value, as this effect could have contributed to our observations. In addition, there was a discrepancy between the gain for the theoretical and observed high/low-pass filter combination. This difference could have also systematically shifted the line, causing our calculated value to change.

NOISE VS TEMP

The final relationship that was investigated was that between temperature and Johnson noise. There are a couple of issues with this.

The first one is that temperature affects both the Johnson noise and the resistor. Most resistors change resistance due to temperature. Thus, there would be two variables changing and there would not be a linear relationship. However, there is a cure for this. If Johnson noise is looked at with no band-pass filters, it would be expected that $\Delta f = \infty$ and therefore there should be infinite Johnson noise. However, a full spectrum Johnson noise measurement was already taken to confirm the readings were indeed noise, and there was not infinite energy.

It was assumed that Johnson noise was white (uniform across every frequency). However, in any situation with a multimeter, there will be a little added capacitance in the system. This creates a RC circuit which inherently acts as a band-pass filter with a gain function of $G(f) = [1 + f/f_c]^{-1/2}$, where $f_c = (2\pi RC)^-1$. Since the capacitance is very small, the frequencies are all $\ll 1/(RC)$. Thus, in the past this result could be ignored, but in this scenario it has a very useful effect. Plugging this into Eq (4):

$$\hat{V^2} = \int_0^\infty 4k_BTR\frac{1}{1+(f/f_c)^2}df = 4k_BTR\pi/2f_c = k_BT/C.$$

Therefore, the full spectrum Johnson noise is actually linear in T because the effective bandwidth is inversely related to R. This allowed us to take full bandwidth measures of Johnson noise as a function of temperature [2].

The second issue was measuring the temperature itself. The probe was used for this part of the experiment (A in Figure 1). A setup like this was required to take measurements all the way down to 77K, far too cold for a standard thermometer. Thankfully, for a given resistor and diode there is a relationship between current voltage and temperature given by:

$$ln\ \mathrm{i}(\Delta V, T) = \frac{e\Delta V}{k_BT} + ln(D) - \frac{E_g - \alpha T}{k_BT} + (3+\gamma/2)lnT.$$

In this formulation, ΔV is the voltage drop over the resistor and all other non-T variable are constants. Rearranging this equation for a constant current and variable voltage:

$$\Delta V = \frac{E_g}{e} + \frac{k_BT}{e}[constant - (3+\gamma/2)lnT]$$

or put simpler

$$\Delta V = d_1 - d_2T - d_3TlnT.$$

Where d_i are constants that can be found experimentally. Since there are three unknowns, there needs to be 3 data points to solve out for all values. First, the company that made the probe gave us that $d_3 = 0.405mV/K$ [2]. For a second point, a reading was taken at room temperature (298.8K), with an observed $V = 408.3mV$. To cool the probe, liquid nitrogen was used (LN^2), which has a boiling point of 77K. LN_2 was stored inside the Dewar unit, labeled as A in Figure 1. When the probe was put in LN^2, and not heated at all, the probe will cool to 77K. This is our last data point, taken at 77K. We observed a $V = 993.3mV$. Solving for d_1, d_2, we get $d_1 = 1140.18mV/K$ and $d_2 = 0.138mV/K$.

Using these constants, the voltage at a given temperature was able to be discerned. The in-built heater was used to adjust the temperature (and inversely, the voltage) to the desired range from either 77K for LN_2 or $\sim$ 278K for room temperature. The heater then had to be fine tuned to produce an equilibrium probe voltage. Once this was reached,the Johnson noise was measured through the HLE.

TABLE II: Johnson Noise vs Temperature

mV	T(K)	$V_{raw}(V)$E-10	$< V_J^2 > (V^2)$E-10	$V_R(V)$E-10
408.3	298.8	1.55	1.20	0.505
993.3	77	1.18	0.91	0.220
927.6	104.6	1.24	0.96	0.266
863	130.6	1.28	0.99	0.297
790	159.1	1.33	1.03	0.336
729.5	182	1.38	1.06	0.374
667.5	205.1	1.42	1.10	0.405
607.3	227.1	1.45	1.12	0.428
543.6	250.2	1.49	1.15	0.459
484.7	271.2	1.53	1.18	0.490
334.9	323.5	1.53	1.18	0.490
326.4	326.5	1.535	1.18	0.494
317.6	329.5	1.54	1.19	0.498
280	342.4	1.56	1.20	0.513
214.4	364.8	1.6	1.23	0.544

To achieve an equilibrium, values had to be slightly shifted from the desired intervals. This explains the non-uniformity of the spacing of the voltage readings. Table 2, presented above, captures the data taken during this experiment. 10 μA were applied for the probe current. In the table, the observed voltage readings of the heater and the Johnson noise reading are provided. These values can be used to determine V_R, or the noise voltage contained with the resistor.

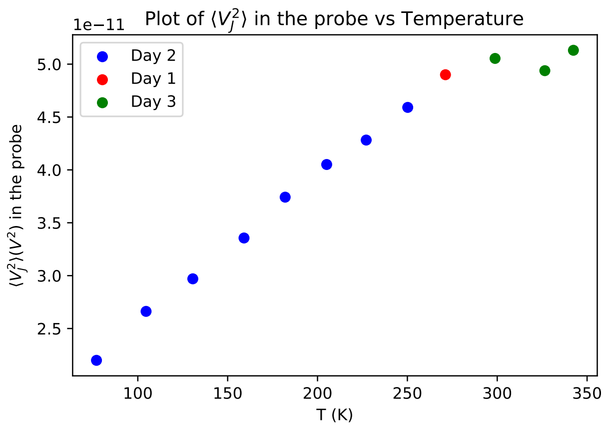

FIG. 7: Gain tracked over different frequency values produced using a function value for both a single band-pass filter and a combination high/low-pass filter

It is apparent that indeed that there was a linear relationship between the temperature and the mean squared Johnson voltage, aligning with expectation. Figure 7 displays the results of this visualization.

Data points were derived on 3 separate days due to time constraints. The days are marked with different colors. It appears that the pattern starts to shift for the Day 3 readings, and this is a possible explanation for the breakdown of linear behaviour at high temperature. Despite this change, a linear pattern clearly formed between temperature and the found Johnson noise on any given day.

CONCLUSION

In this paper, the goal was to show that the Johnson noise in resistors follows the Nyquist equation. Three independent variables, resistance, bandwidth, and temperature, were observed and experiments were done to show they all had a linear relationship with the mean squared voltage. This shows that the fundamental assumptions made by Nyquist when deriving his formula hold, at least within the frequencies, resistances, and temperatures measured.

We observed error, both in deviations from k_B and in differences in the expected and observed behaviour for high/low-pass filters. A lot of this error comes from the nature of Johnson noise. Because any resistor can have Johnson noise interactions, any part of the setup can experience Johnson noise. A way of dealing with this issue was shown in section devoted to calibration, but these values can change, especially if a component heats up due to repeated use. More general hardware issues may also be factor, as this system has seen repeated use for over 10 years now.

The results, however, did reaffirm the Nyquist equation.

First, when calibrating, Johnson noise was observed over one 10 kΩ resistor at room temperature with a low-pass filter of 100 kHz. The readings of noise, when the noise from other sources was accounted for, was remarkably similar to the expected value from the Nyquist equation.

Second, in the Noise v Resistance section, we saw that there was a general linear relationship between resistance and Johnson noise as expected. After about 2 $M\Omega$, the linear relationship broke down, but this result could be explained by the nature/state of the amplifier in the LLE. If the amplifier had a maximum output, like most amplifiers do, then that would explain why plateauing occurred in the observed Johnson noise.

Next, in the Noise vs Bandwidth section, the linear relationship between effective bandwidth and Johnson noise was observed. It was confirmed that we observed that there is a linear relationship between the effective bandwidth and the noise outputted by that bandwidth. To check that this relationship was not only linear but exactly what the Nyquist equation dictates, the implied value of k_B was found and compared to the accepted the value. The found value was within the error bounds of the accepted value, signifying a small but acceptable error within our calculations. This systematic error could have stemmed from other sources of noise that would move the observed value closer to the accepted one.

Finally, in the Noise vs Temperature section we confirmed that the relationship between temperature and noise was linear. There was some issues, as we took data on multiple days leading to pattern inconsistencies. In addition, keeping the probe at a constant temperature, or at equilibrium, led to some issues in taking data points at uniform distances. Despite these issues, overall we found that there was a linear relationship backing the thermal nature of Johnson noise.

The experiment set out to confirm the Nyquist equation for Johnson noise. In it, it was shown that all the independent variables are indeed linearly related to the Johnson noise. In some situations, it was shown that the corresponding linear factor was correct. It is remarkable to note that Nyquist theorized this noise as being between two far apart resistors, and yet the effect was still visible when measuring over a variety of different resistors in both the probe and LLE. It displays the universality of Nyquist's findings. It is important to see that these theoretical noise equations apply to real world resistors and it is important to consider noisy results like this especially in low voltage or high temperature situations. Escaping the theoretical is vital to determining the real-world effects and implications of dealing with noise in electrical systems.

ACKNOWLEDGEMENTS

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[1] G. Dörfel (2012), [Accessed 19-09-2024].

[2] D. V. Baak, *Noise Fundamentals NF1-A Instructors Manual* (Teach Spin Inc., 2010).