# A Computational Study of the Gravitational Dynamics in a Multi-planetary System

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The inherent complexity of multi-planetary gravitational interactions poses challenges in predicting and understanding their evolution over time. In this study, we present a computational model that simulates the motion of planets in a solar system influenced solely by their mutual gravitational attractions. Our simulation leverages a fourth-order Runge-Kutta (RK4) method, providing a robust numerical solution for the differential equations governing planetary motion. By integrating the effects of pairwise gravitational forces between planets, the model calculates the cumulative forces acting on each planet, thus updating their positions and velocities over specified time intervals.

Initial conditions for our system were sourced from data files containing parameters such as planet masses and initial positions and velocities. The simulation's output delineates the time-evolution trajectories of these celestial bodies and is visually represented through plots generated via GNUplot, offering a comprehensive view of the planetary paths over time. This methodology can serve as a foundational tool in astrophysical studies, especially in scenarios involving the gravitational dynamics of multi-body systems. Moreover, the adaptability of our computational framework allows for easy incorporation of additional planets or modifications in initial conditions, making it versatile for varied astronomical investigations.

## I. INTRODUCTION

The Solar System contains the most significant collection of planets to human existence (at least for the foreseeable future). In terms of non-terrestrial exploration, the planets that make up our solar system are of the utmost priority. Given recent developments in space fairing technology, the possibility of exploring these planets has become a realistic possibility. This begs the question, "How are we supposed to get to the planets in our solar system without knowing exactly where they are?" This paper describes an elementary model of the movement of the various planets in our solar system. With this information, we can engage in exploration, observation, or decimation of the various planets in the solar system as we see fit.

This simulation provides the framework for simulating any solar system with an arbitrary number of bodies. While the choice of units is optimized for our specific solar system, it is intended for general use.

## II. BACKGROUND [1]

The gravitational force is one of the four fundamental forces in nature and governs the interaction between objects with mass. The concept of gravitational attraction was first systematically studied by Sir Isaac Newton in the 17th century, culminating in his Universal Law of Gravitation.

This model is based on *Newton's Universal Law of Gravitation*, which describes the attractive force that exists between any two masses. This gravitational force, denoted as $F_g$, is directly proportional to the product of the two masses, $M$ and $m$, and inversely proportional to the square of the distance, $r$, between their centers. The mathematical expression for this relationship is given as:

$$F_g = \frac{GMm}{r^2} \tag{1}$$

Where:

- $F_g$ is the gravitational force between the two bodies.
- $M$ and $m$ are the respective masses of the two objects.
- $r$ is the distance between the centers of the two masses.
- $G$ is the gravitational constant, an empirical physical constant involved in the calculation of the gravitational attraction between objects with mass. Its approximate value is $6.67430 \times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$.

Several key points to understand in the context of this law are:

- **Universality:** The law is universal. This means that every object in the universe attracts every other object with a force that is directly proportional to the product of their masses.
- **Directionality:** Gravitational force is a vector quantity. It has both magnitude and direction. The force acts along the line joining the centers of the two masses. For two objects separated in space, each object experiences a force that points directly towards the other object.
- **Dependency on Distance:** As the distance between the two objects increases, the gravitational force between them decreases rapidly, and vice versa. This is because the force is inversely proportional to the square of the distance between them.

- **Invariance with Mass Distribution:** For spherically symmetric bodies, the gravitational force outside the body behaves as if all the object's mass were concentrated at a point at its center. This is why planets, which are roughly spherical, can be treated as point masses when calculating gravitational forces at distances much larger than the planet's radius.

It's worth noting that while Newton's Law of Gravitation provides an excellent approximation for many practical purposes, it doesn't hold in all circumstances. For situations involving very strong gravitational fields, such as those near a black hole, Einstein's theory of General Relativity provides a more accurate description.

### A. Kepler's Laws

Johannes Kepler, a prominent figure of the scientific revolution, formulated three empirical laws in the early 17th century to describe the motion of planets around the Sun. These laws, derived from the careful observational data collected by the astronomer Tycho Brahe, are now known as Kepler's Laws of Planetary Motion. They provide foundational knowledge for understanding the motion of celestial bodies within our solar system and played a critical role in the development of Newtonian physics.

#### *1. Kepler's First Law (The Law of Ellipses)*

The path (orbit) of a planet around the Sun is an ellipse with the Sun at one of the two foci.

Mathematically, the elliptical shape of the orbit is defined by its semi-major axis $a$ and its eccentricity $e$. The distance from the center of the ellipse to a focus is $ae$.

#### *2. Kepler's Second Law (The Law of Equal Areas)*

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. This means that the planet moves faster when it is closer to the Sun (at perihelion) and slower when it is farther from the Sun (at aphelion). The implication of this law is that the angular momentum of the planet about the Sun is conserved.

#### *3. Kepler's Third Law (The Law of Harmonies)*

The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit. This relationship can be expressed as:

$$T^2 \propto a^3 \tag{2}$$

Where:

- $T$ is the period of the planet's orbit (the time it takes for one complete revolution around the Sun).
- $a$ is the semi-major axis of the planet's elliptical orbit.

For the planets orbiting the Sun, the proportionality becomes an equality with the same constant of proportionality for all planets:

$$\frac{T^2}{a^3} = \text{constant} \tag{3}$$

## III. DERIVATION OF INITIAL POSITIONS AND VELOCITIES USING KEPLER'S LAWS

To derive the initial positions and velocities of celestial bodies using Kepler's laws, one would typically follow these steps:

1. From Kepler's First Law, determine the shape and orientation of the ellipse. This will give you the initial position of the celestial body along its orbit.
2. From Kepler's Second Law, find the area swept by the line segment joining the celestial body and the Sun over a given interval of time. This can be used to determine the body's velocity at different positions along its orbit, ensuring that the area swept out in equal times remains constant.
3. With the determined position and velocity, and by employing Newton's Law of Gravitation, one can then calculate the gravitational forces acting on the body and thus predict its future positions and velocities.

Kepler's laws, combined with Newton's Universal Law of Gravitation, allow for a comprehensive understanding and simulation of the motion of celestial bodies in gravitational fields.

## IV. DISCUSSION

### A. Limitations and Assumptions

For the simulation of our solar system, we make the following assumptions and simplifications:

1. The planets move only in two dimensions.
2. The planets all start on the x-axis.
3. Bodies outside of the solar system do not affect the orbits of the planets within the solar system.
4. The primary solar mass of the system remains fixed at the origin.

### B. Various Programs in directory

In the directory of this simulation, there are various programs that aid the user in their different applications. The two main programs are *solarsystem.cpp* and *invelfind.cpp*.

#### 1. *invelfind.cpp*

This program can be utilized by the user to determine the appropriate initial conditions for the celestial bodies in a solar system based on their aphelion, perihelion, eccentricity, and expected period. The program uses the following expressions derived from Kepler's laws and Newton's universal law of gravitation to Calculate the initial parameters For a given planet in the system:

$$a = \frac{A+P}{2} \tag{4}$$

$$v = \sqrt{\frac{GM\left(\frac{1+e}{1-e}\right)}{a}} \tag{5}$$

$$v_y = v\sin(\theta) \tag{6}$$

$$v_x = v\cos(\theta) \tag{7}$$

In the equations, $A$ represents the aphelion distance, $P$ denotes the perihelion distance, $e$ stands for the eccentricity, and $T$ signifies the target period of the celestial body. **It is important to note that these simulations are in spacial units of Au and time units of Years**

#### 2. *solarsystem.cpp*

This is the primary code for the simulation. This code reads in two .dat files. The first, *ssinitial.dat*, contains the initial $x, y, v_x$, and $v_y$ parameters for each planet. The simulation is built to be generalizable to any number of planets with the first row of the file designated for the solar body of the system. The second, *mass.dat*, contains the mass information for the sun and each of the planets in the system. By changing these two files, one could approximate the orbits of any solar system consisting of only planets and one solar body.

### C. Figures With Explanations

This section designates some interesting figures produced when plugging in different initial parameters. While none of these figures can be said to be correspondent to physical systems we see in the real world, one can extrapolate from them some interesting behaviors about real systems.

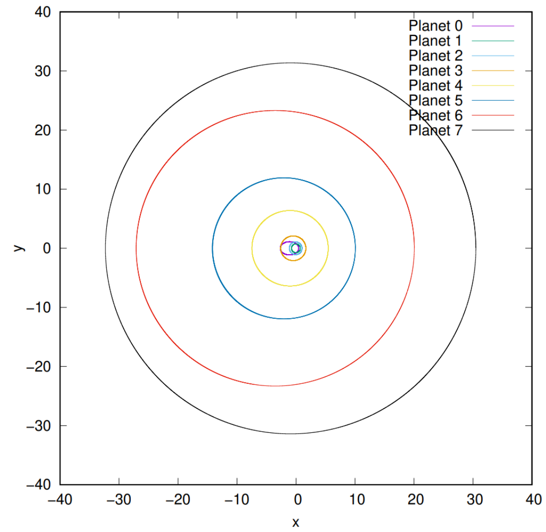


FIG. 1: Our Solar System as Circular Orbits (200 years)

Figure 1 represents a fictitious simulation of our solar system where each planet if modeled on its own would have a perfectly circular orbit [2]. The exact initial conditions can be found in the files fig1inparam.dat and fig1mass.dat in the directory of this project. From this simulation, we can see that the effects of each planet on each other are relatively negligible on this scale when compared to the effect of the sun.

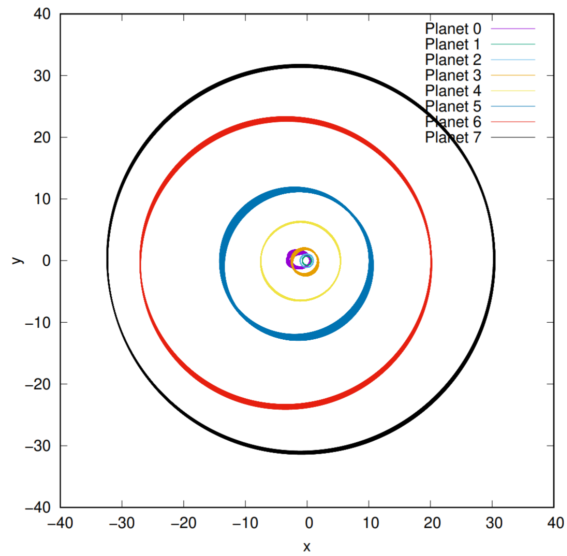


FIG. 2: Our Solar System as Circular Orbits (20,000 years)

Allowing this simulation to run for 20,000 years (Figure 2), however, begins to show signs of deviation from the initial circular orbits. While this is not a perfectly accurate model of the solar system, the forces acting on the planets are on the right scale for us to reasonably deduce that our solar system will not break down within the next 20,000 years.

This then begs the question, what would a system look like that has planets with masses close to the mass of the sun they orbit? To answer this question, consider the following figure.

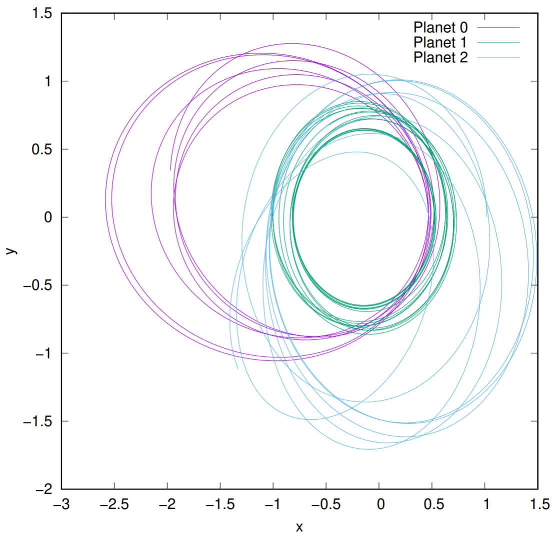


FIG. 3: Three Large Planets Orbit our Sun (10 years)

Figure 3 shows how the orbits become unstable and seemingly random when the masses of the planets are close to the mass of the celestial body. The initial parameters and masses can be found in 3planets.dat and 3planetsmass.dat.

One last interesting case (Figure 4) I chose to illustrate using this model is that of a planet that is not initially part of a solar system but passes by in such a way that it gets sucked into the 3 planet system from the above example. This small planet can be thought to just be passing by when it is sucked in by the gravitational force of the system. The initial conditions and masses for this simulation can be found in 3+1planets.dat and 3+1planetsmass.dat

Interestingly, simulating the system further reveals the unpredictable nature of interactions between massive celestial bodies. Figure 5 shows how all of the initial planets diverge away from the system, while the little planet that wasn't initially part of the system stays trapped within the gravitational field of the sun.

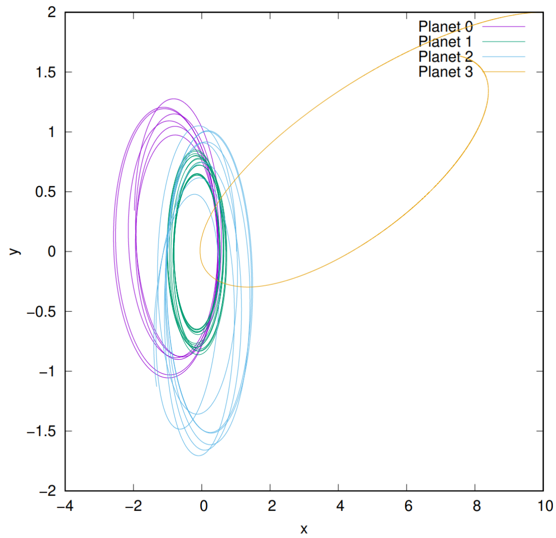


FIG. 4: Three Large Planets Orbit our Sun as a 4th Small Planet Comes Flying In (10 years)

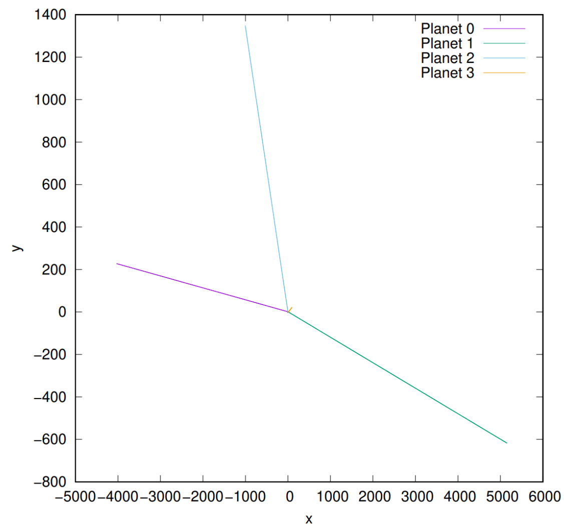


FIG. 5: Three Large Planets Orbit our Sun as a 4th Small Planet Comes Flying In (100 years)

## V. CONCLUSION

While this toy model is not state-of-the-art, it provides the user with the ability to explore a wide variety of multi-planetary solar systems easily. Its use of the rk4 algorithm makes it computationally light to run on most computers. Future improvements can be made to invelfind.cpp to make it better at determining the initial conditions of real-life systems. Additionally, one could improve the life-likeness of the solarsystem.cpp by making the primary solar mass of the system non-stationary.

[1] M. J. Benacquista and J. D. Romano, *Classical Mechanics*, Undergraduate Lecture Notes in Physics (Springer International Publishing, Cham, 2018), ISBN 9783319687797, 1st ed. 2018, URL `https://www.springer.com/gp/book/9783319687797`.

[2] P. Moore, *Firefly Atlas of the Universe* (Firefly Books, 2003), ISBN 978-1552978191.