Introduction to Machine Learning

Arturo Argüelles &
Carlos Galíndez
Quantum Optics Group

Machine Learning lecture

- Overview on the Fourier Analysis
- Inner Products of functions and vectors
- Fourier Series and Fourier Transforms
- Wavelet

Definition 1.1. *Let V be a complex vector space. A Hermitian inner product on V is a function*

$$\langle\ ,\ \rangle \colon V \times V \longrightarrow \mathbb{C},$$

which is

- ***conjugate-symmetric**, that is*

$$\langle u, v\rangle = \overline{\langle v, u\rangle}.$$

- ***sesquilinear**, that is linear in the first factor*

$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle,$$

  for all scalars λ and

$$\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle,$$

  for all vectors u_1, u_2 and v.

- ***positive** that is*

$$\langle v, v\rangle \geq 0.$$

- ***non-degenerate** that is if*

$$\langle u, v\rangle = 0$$

  for every $v \in V$ then $u = 0$.

We say that V is a ***complex inner product space****. The* ***associated quadratic form*** *is the function*

$$Q \colon V \longrightarrow \mathbb{C},$$

defined by

$$Q(v) = \langle v, v \rangle.$$

Since a Hermitian inner product is linear in the first variable and conjugate-symmetric (so that switching factors we get the complex conjugate), we have

$$\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle \qquad \text{and} \qquad \langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle.$$

Note that

$$\langle v, v \rangle,$$

is a real number (it is equal to its complex conjugate, by conjugate-symmetry) and so it makes sense to ask for it to be non-negative. One can define the associated norm, as in the real case, and one can recover the Hermitian inner product from the norm, as in the real case.

The classic example of a Hermitian inner product space is the standard one on $\mathbb{C}^n$,

$$\langle x, y \rangle = \sum x_i \bar{y}_i.$$

For this inner product, we have

$$\langle Au, v\rangle = \langle u, \overline{A}^t v\rangle.$$

Definition 1.2. *Let $A \in M_{n,n}(\mathbb{C})$. We say that A is* ***Hermitian*** *if A is invertible and*

$$A^{-1} = \overline{A}^t.$$

Note that a real orthogonal matrix is Hermitian if and only if it is orthogonal.

$$\langle f(x), g(x)\rangle = \int_a^b f(x)\bar{g}(x)\, dx$$

$$\langle \mathbf{f}, \mathbf{g}\rangle = \mathbf{g}^* \mathbf{f} = \sum_{k=1}^{n} f_k \bar{g}_k = \sum_{k=1}^{n} f(x_k)\bar{g}(x_k).$$

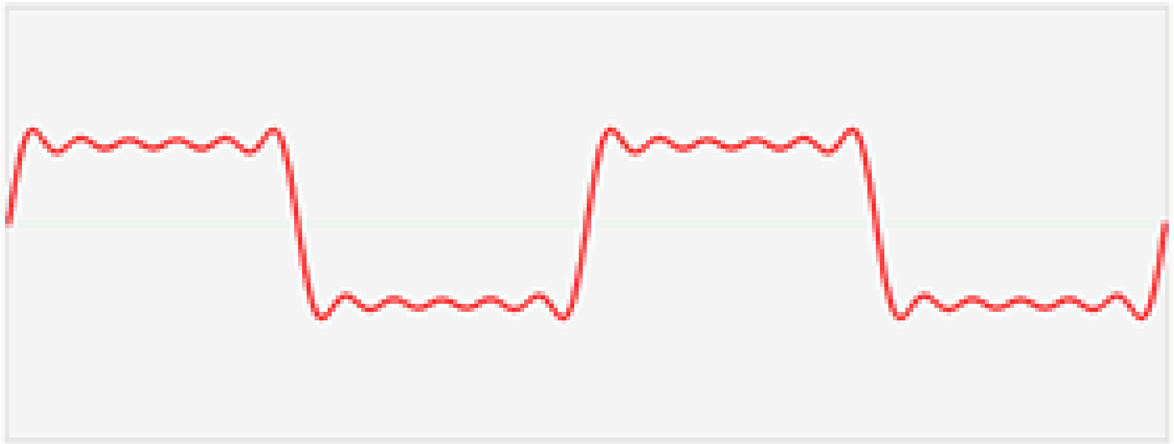
$$\frac{b-a}{n-1}\langle \mathbf{f}, \mathbf{g}\rangle = \sum_{k=1}^{n} f(x_k)\bar{g}(x_k)\Delta x,$$

$$\|f\|_2 = (\langle f, f\rangle)^{1/2} = \sqrt{\langle f, f\rangle}$$

$$= \left(\int_a^b f(x)\bar{f}(x)\,dx\right)^{1/2} = \quad L^2([a,b])$$

Lebesgue Integrable functions

Fourier Series



$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

The coefficients a_k and b_k are given by

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \qquad a_k = \frac{1}{\|\cos(kx)\|^2} \langle f(x), \cos(kx) \rangle$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \qquad b_k = \frac{1}{\|\sin(kx)\|^2} \langle f(x), \sin(kx) \rangle,$$

where $\|\cos(kx)\|^2 = \|\sin(kx)\|^2 = \pi$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi k x}{L}\right) + b_k \sin\left(\frac{2\pi k x}{L}\right) \right)$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi k x}{L}\right) dx$$

$$b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi k x}{L}\right) dx.$$

Fourier Series

$$\boxed{\begin{aligned} e^{ikx} &= \cos(kx) + i\sin(kx) \\ c_k &= \alpha_k + i\beta_k \end{aligned}}$$

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} (\alpha_k + i\beta_k)\,(\cos(kx) + i\sin(kx)) \\ &= (\alpha_0 + i\beta_0) + \sum_{k=1}^{\infty} \left[(\alpha_{-k} + \alpha_k)\cos(kx) + (\beta_{-k} - \beta_k)\sin(kx) \right] \\ &\quad + i\sum_{k=1}^{\infty} \left[(\beta_{-k} + \beta_k)\cos(kx) - (\alpha_{-k} - \alpha_k)\sin(kx) \right]. \end{aligned}$$

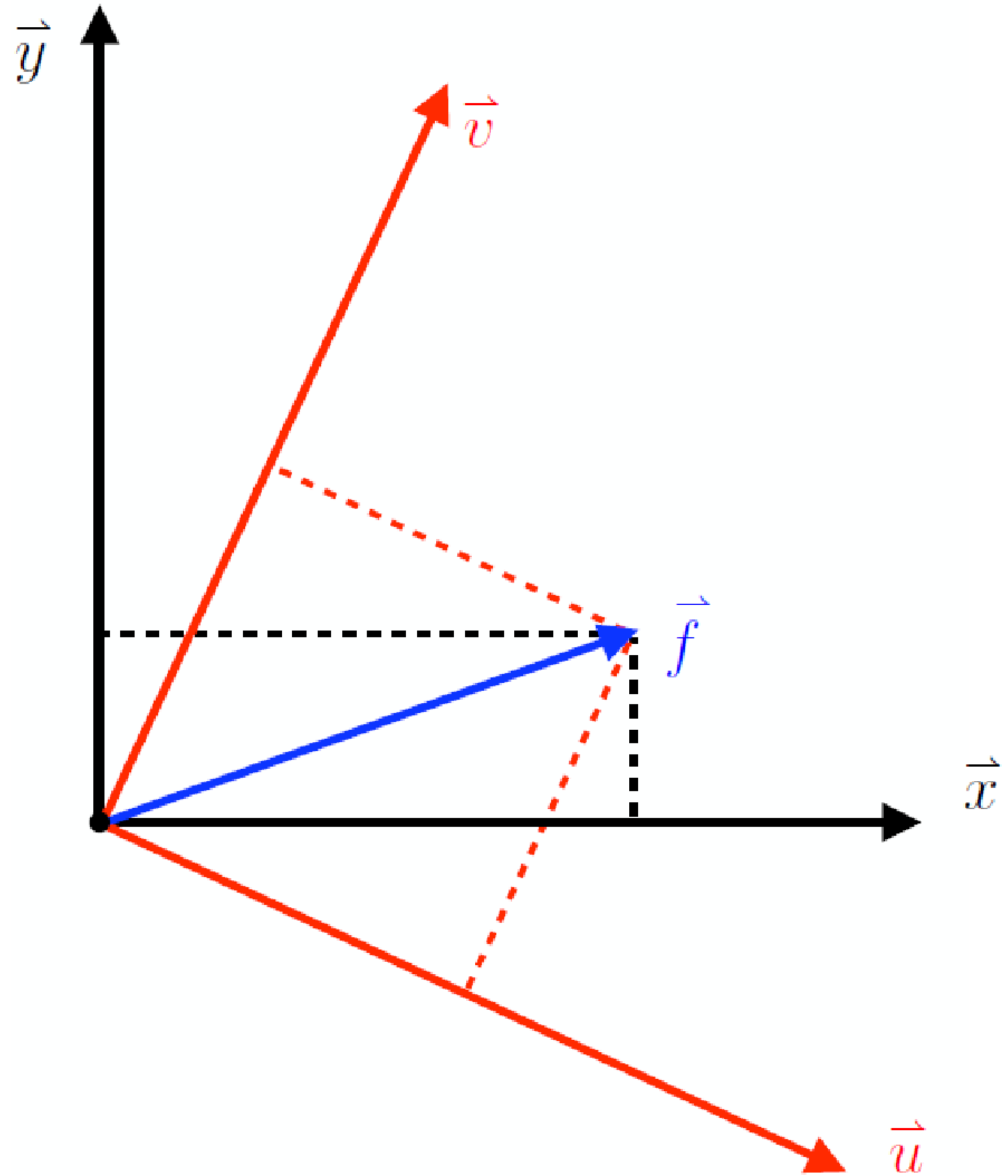
Thus, the functions $\psi_k = e^{ikx}$ for $k \in \mathbb{Z}$ (i.e., for integer k) provide a basis for periodic, complex-valued functions on an interval $[0, 2\pi)$. It is simple to see that these functions are orthogonal:

$$\langle \psi_j, \psi_k \rangle = \int_{-\pi}^{\pi} e^{ijx} e^{-ikx} dx = \int_{-\pi}^{\pi} e^{i(j-k)x} dx = \left[\frac{e^{i(j-k)x}}{i(j-k)} \right]_{-\pi}^{\pi} = \begin{cases} 0 & \text{if } j \neq k \\ 2\pi & \text{if } j = k. \end{cases}$$

So $\langle \psi_j, \psi_k \rangle = 2\pi\delta_{jk}$, where δ is the Kronecker delta function. Similarly, the functions $e^{i2\pi kx/L}$ provide a basis for $L^2([0, L))$, the space of square integrable functions defined on $x \in [0, L)$.

In principle, a Fourier series is just a change of coordinates of a function $f(x)$ into an infinite-dimensional orthogonal function space spanned by sines and cosines (i.e., $\psi_k = e^{ikx} = \cos(kx) + i \sin(kx)$):

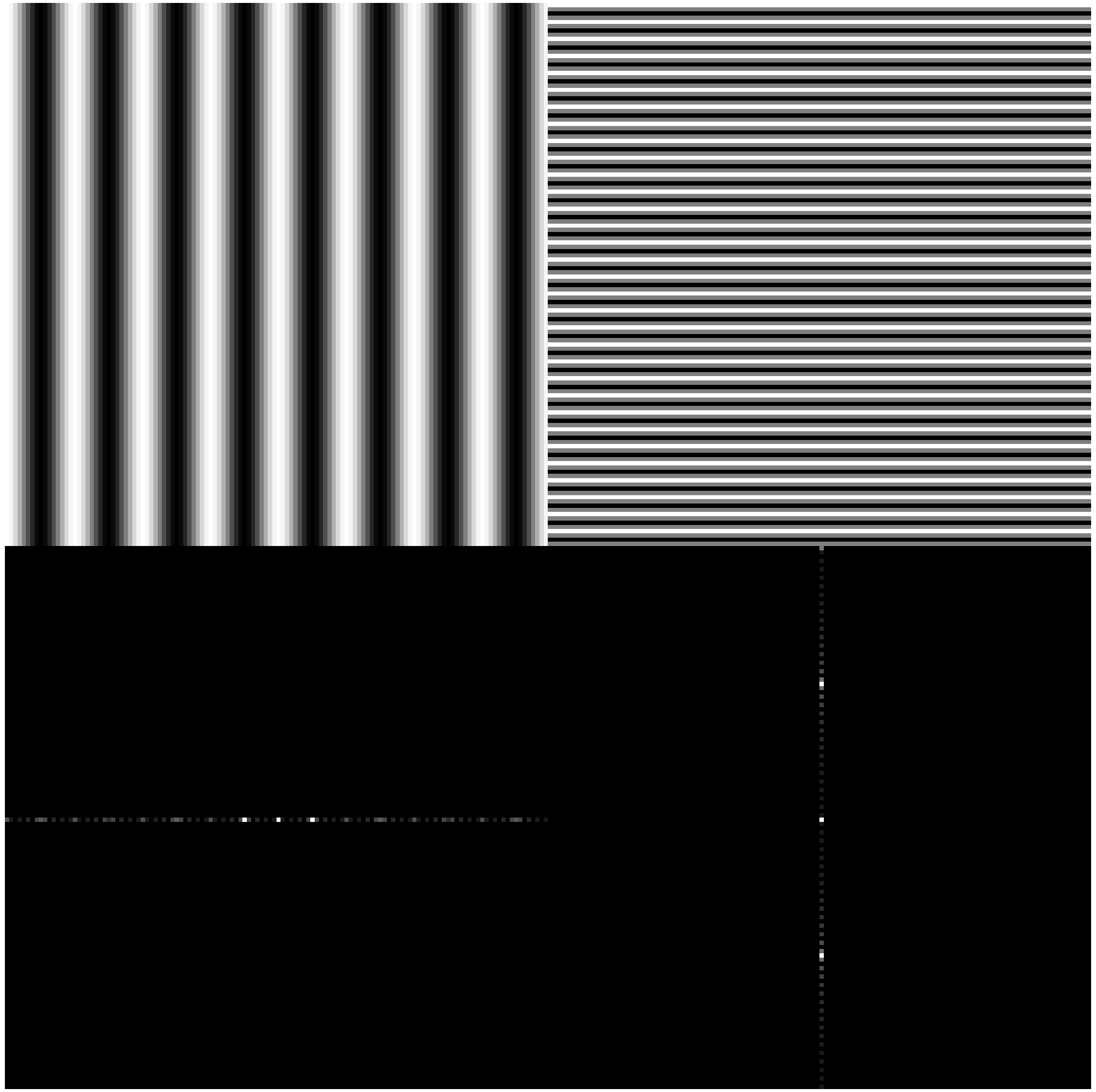
$$\boxed{f(x) = \sum_{k=-\infty}^{\infty} c_k \psi_k(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \langle f(x), \psi_k(x) \rangle \psi_k(x)} \quad c_k = \tfrac{1}{2\pi} \langle f(x), \psi_k(x) \rangle$$

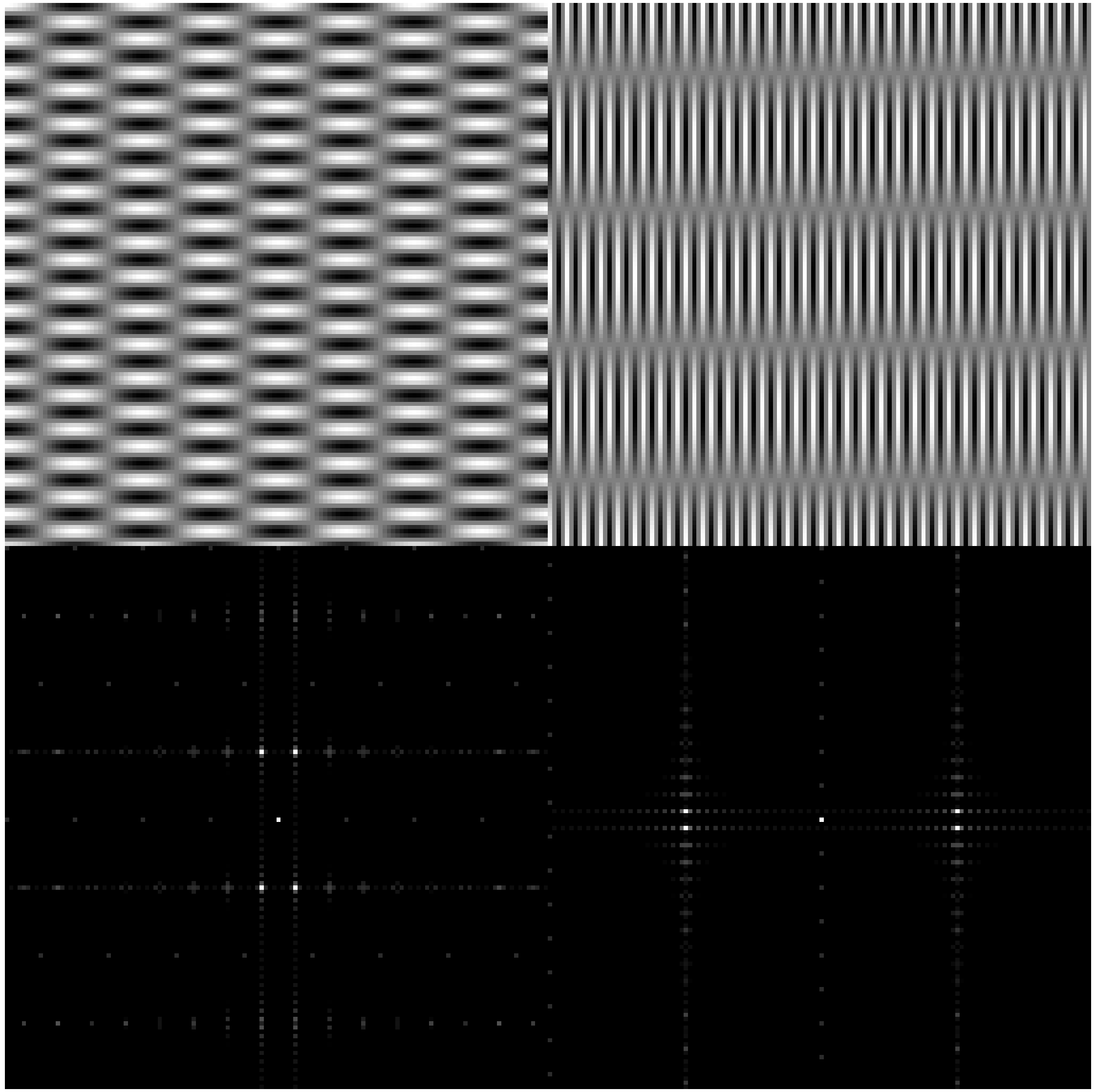


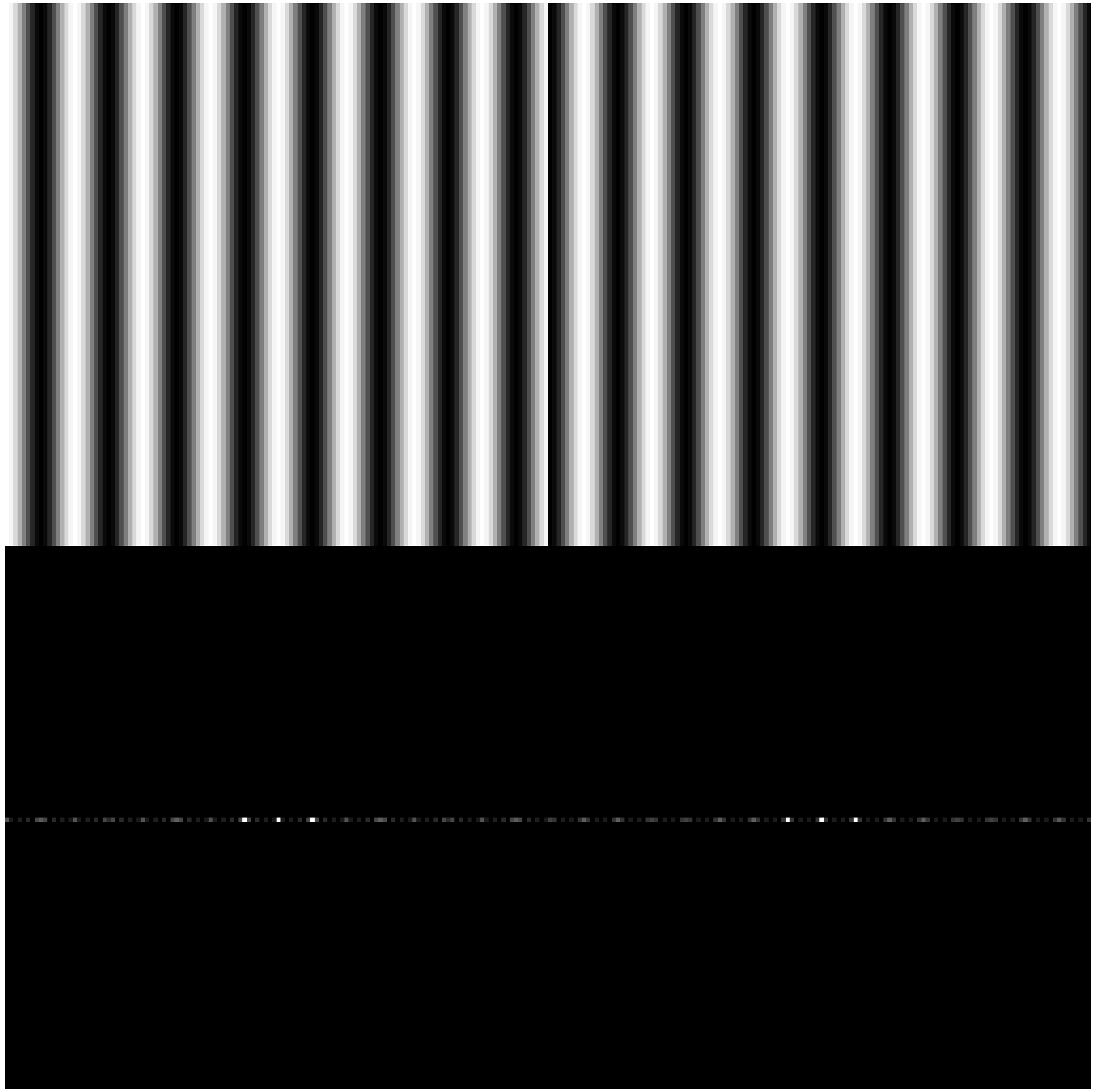
$$\vec{f} = \langle \vec{f}, \vec{x} \rangle \frac{\vec{x}}{\|\vec{x}\|^2} + \langle \vec{f}, \vec{y} \rangle \frac{\vec{y}}{\|\vec{y}\|^2}$$

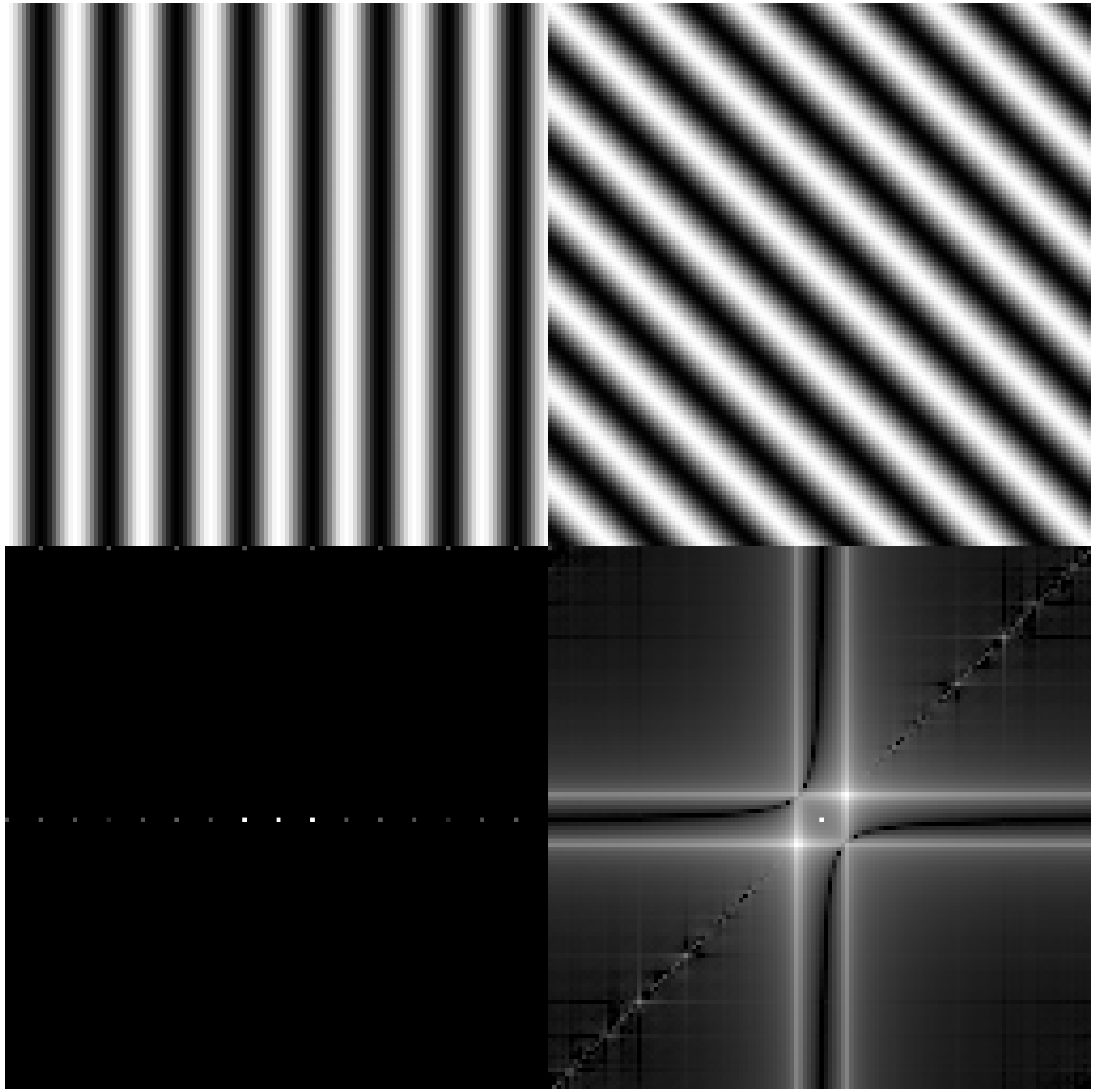
$$= \boxed{\langle \vec{f}, \vec{u} \rangle \frac{\vec{u}}{\|\vec{u}\|^2}} + \langle \vec{f}, \vec{v} \rangle \frac{\vec{v}}{\|\vec{v}\|^2}$$

$$f(x) = \boxed{\frac{1}{2\pi}} \sum_{k=-\infty}^{\infty} \boxed{\langle f(x), \psi_k(x) \rangle} \boxed{\psi_k(x)}$$







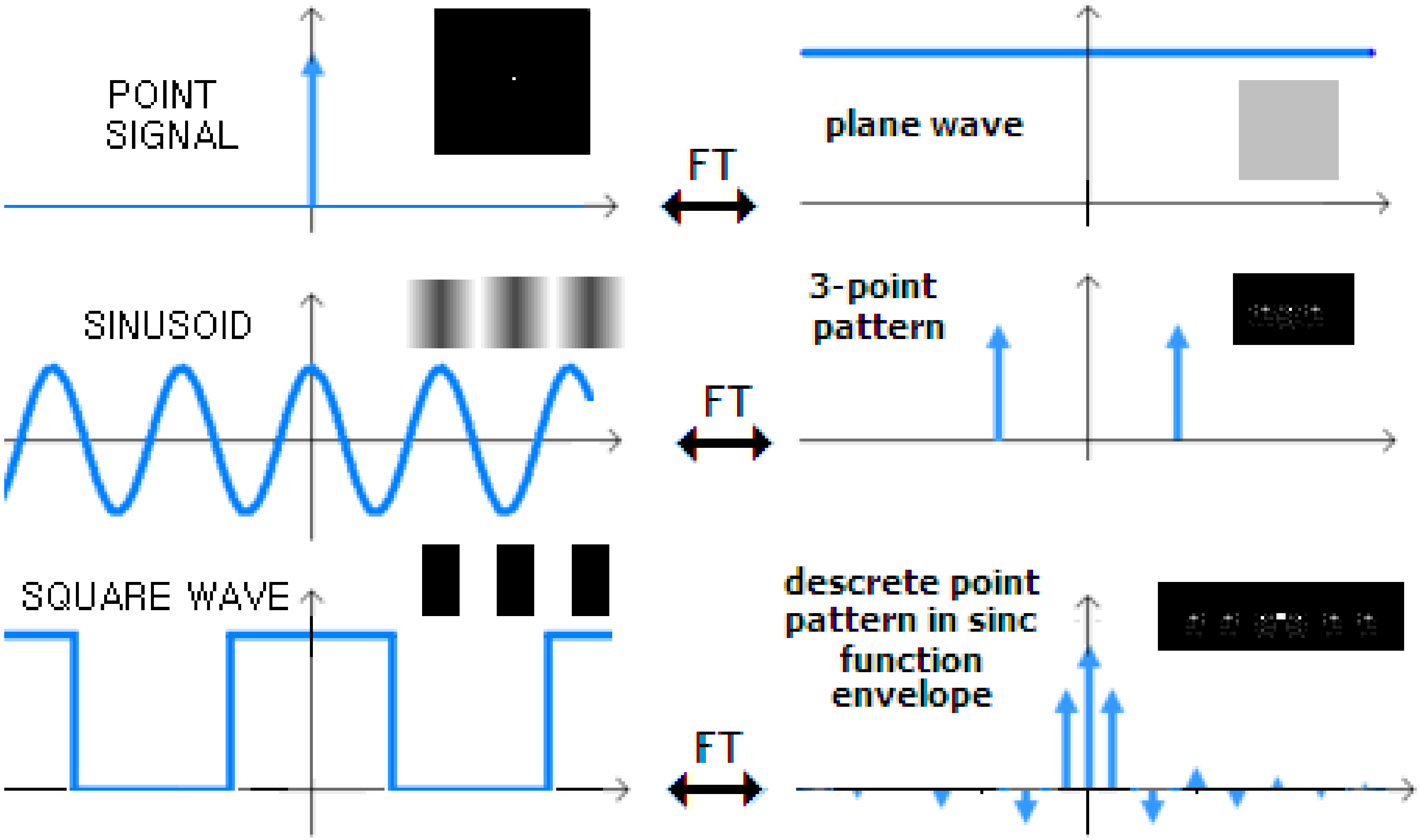


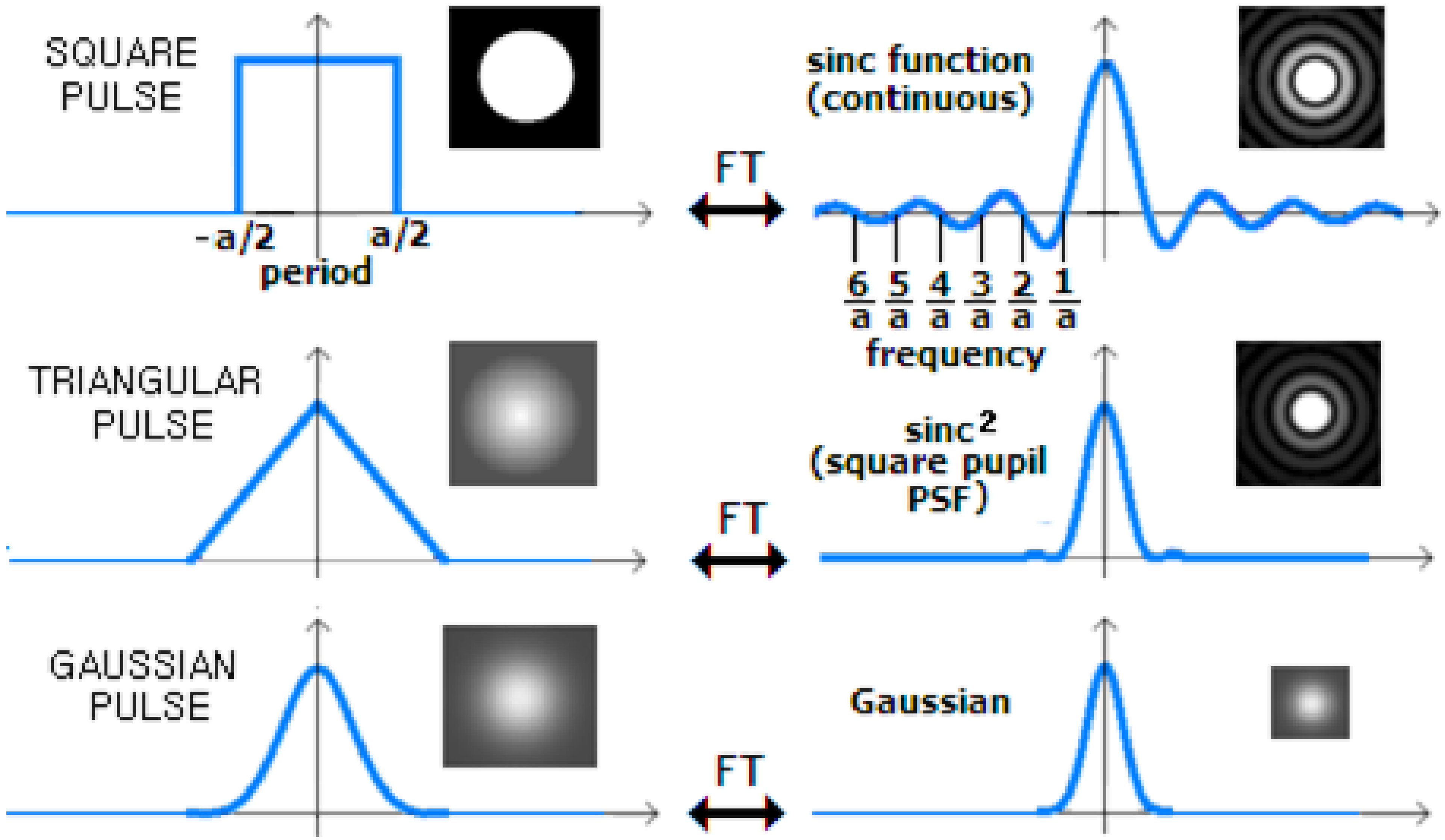
FIELD IN EXIT PUIL

AMPLITUDE AS A FUNCTION
OF SPATIAL PERIOD

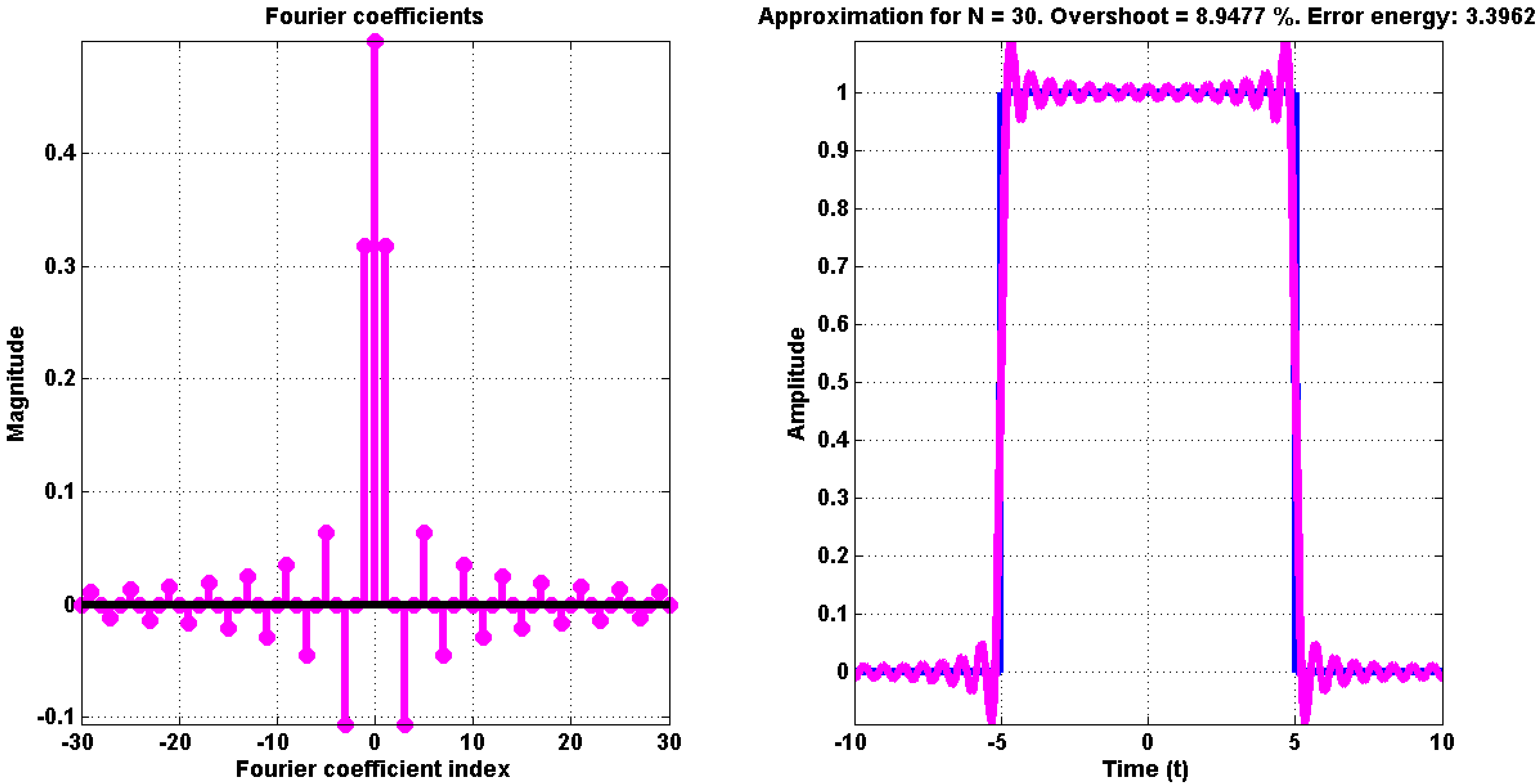
FOURIER TRANSFORM

SPECTRUM OF SINUSOIDAL FREQUENCIES
THE PUPIL FUNCTION DECOMPOSES TO;
FIELD DIFFRACTED TO INFINITY IS
PROPORTIONAL TO THE FREQUENCY DISTRIBUTION



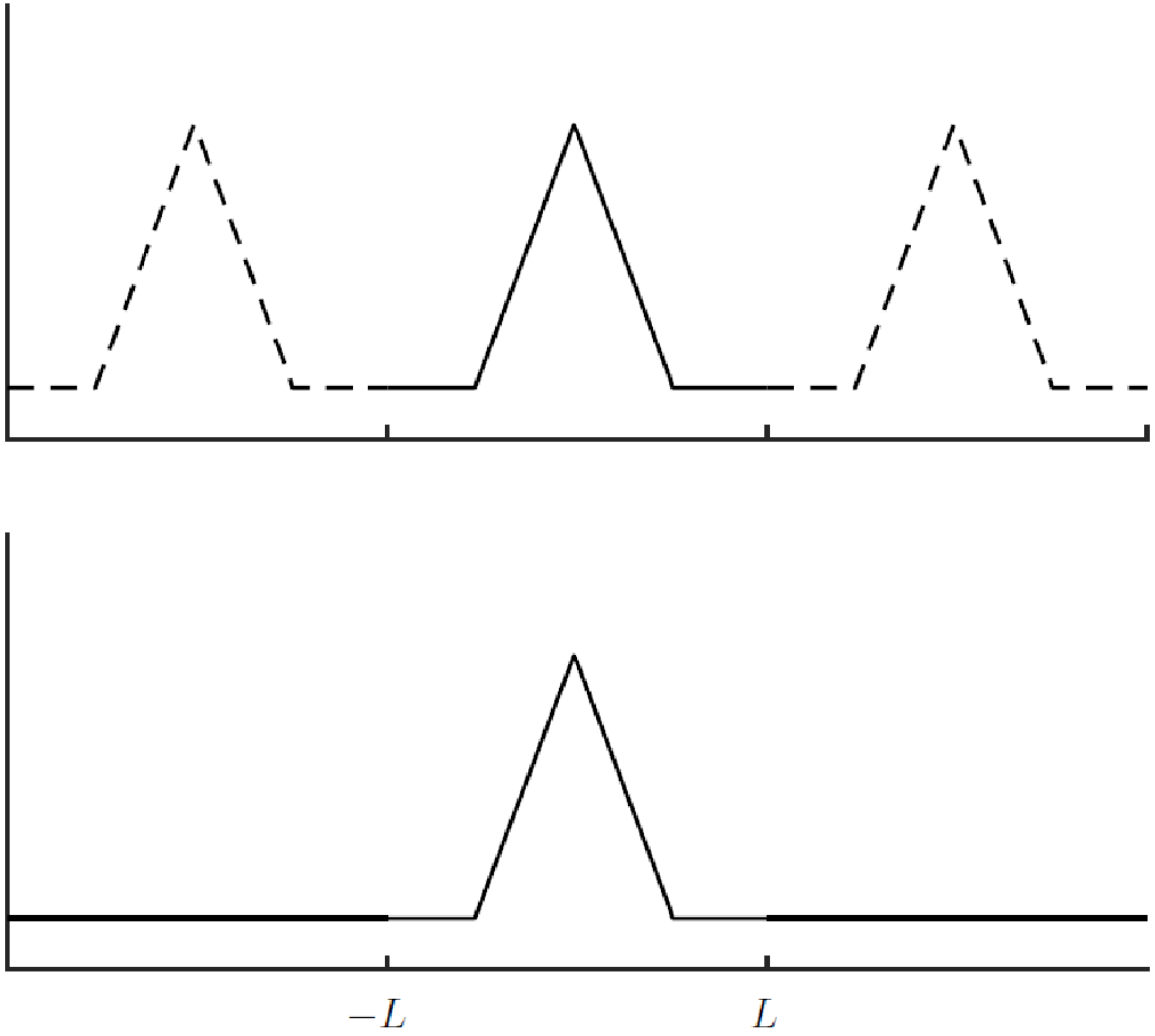


SQUARE PULSE
-a/2
a/2
period
FT
sinc function (continuous)
6/a 5/a 4/a 3/a 2/a 1/a
frequency
TRIANGULAR PULSE
FT
sinc²
(square pupil PSF)
GAUSSIAN PULSE
FT
Gaussian



Fourier coefficients
Magnitude
0.4
0.3
0.2
0.1
0
-0.1
-30
-20
-10
0
10
20
30
Fourier coefficient index
Approximation for N = 30. Overshoot = 8.9477 %. Error energy: 3.3962
Amplitude
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-5
0
5
10
Time (t)

Fourier Transform



$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right] = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/L}$$

$$c_k = \frac{1}{2L} \langle f(x), \psi_k \rangle = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-ik\pi x/L}\, dx$$

$$\omega_k = k\pi/L$$

as $L \to \infty$, these discrete frequencies become a continuous range of frequencies. Define $\omega = k\pi/L$, $\Delta\omega = \pi/L$, and take the limit $L \to \infty$, so that $\Delta\omega \to 0$:

$$f(x) = \lim_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \underbrace{\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-ik\Delta\omega\xi}\, d\xi}_{\langle f(x), \psi_k(x) \rangle} e^{ik\Delta\omega x}.$$

When we take the limit, the expression $\langle f(x), \psi_k(x) \rangle$ will become the Fourier transform of $f(x)$, denoted by $\hat{f}(\omega) \triangleq \mathcal{F}(f(x))$. In addition, the summation with weight $\Delta\omega$ becomes a Riemann integral, resulting in the following:

$$f(x) = \mathcal{F}^{-1}\left(\hat{f}(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\, d\omega$$
$$\hat{f}(\omega) = \mathcal{F}\left(f(x)\right) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx.$$

$$
\begin{aligned}
\mathcal{F}\left(\frac{d}{dx}f(x)\right) &= \int_{-\infty}^{\infty} \overbrace{f'(x)}^{dv}\,\overbrace{e^{-i\omega x}}^{u}\,dx \\
&= \Big[\underbrace{f(x)e^{-i\omega x}}_{uv}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{f(x)}_{v}\Big[\underbrace{-i\omega e^{-i\omega x}}_{du}\Big]\,dx \\
&= i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\
&= i\omega\mathcal{F}(f(x)).
\end{aligned}
$$

$$
\begin{array}{ccc}
u_{tt} = cu_{xx} & \overset{\mathcal{F}}{\Longrightarrow} & \hat{u}_{tt} = -c\omega^2\hat{u}. \\
\text{(PDE)} & & \text{(ODE)}
\end{array}
$$

Discrete Fourier transform –DFT

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}$$

Inverse Discrete Fourier transform –iDFT

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_j e^{i2\pi jk/n}$$

$$\{f_1, f_2, \cdots, f_n\} \quad \xRightarrow{\text{DFT}} \quad \{\hat{f}_1, \hat{f}_2, \cdots \hat{f}_n\}$$

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix}.$$

Fast Fouerier Transform

$$\hat{\mathbf{f}} = \mathbf{F}_{1024}\mathbf{f} = \begin{bmatrix} \mathbf{I}_{512} & -\mathbf{D}_{512} \\ \mathbf{I}_{512} & -\mathbf{D}_{512} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{512} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{512} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{\text{even}} \\ \mathbf{f}_{\text{odd}} \end{bmatrix}$$

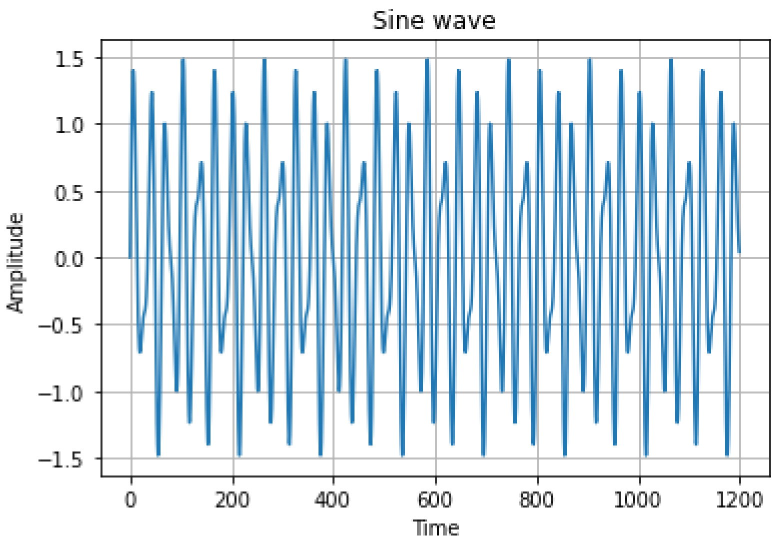
where $\mathbf{f}_{\text{even}}$ are the even index elements of $\mathbf{f}$, $\mathbf{f}_{\text{odd}}$ are the odd index elements of $\mathbf{f}$, $\mathbf{I}_{512}$ is the 512×512 identity matrix, and $\mathbf{D}_{512}$ is given by

$$\mathbf{D}_{512} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{511} \end{bmatrix}$$

```
#Importing the libraries

import numpy as np
import matplotlib.pyplot as plt
from scipy.fft import fft, fftfreq

#Making the sine waves and adding them

#  sample points
N = 1200

# sample spacing
T = 1.0 / 1600.0

x = np.linspace(0.0, N*T, N,
endpoint=False)
sum = np.sin(50.0 * 2.0*np.pi*x) +
0.5*np.sin(80.0 * 2.0*np.pi*x)
```

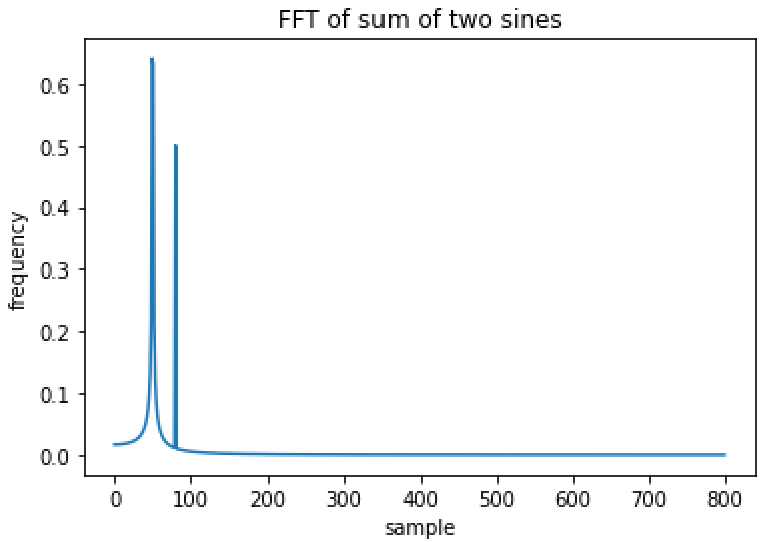
```
plt.plot(sum)
plt.title('Sine wave')
plt.xlabel('Time')
plt.ylabel('Amplitude')
plt.grid(True, which='both')
plt.show()
```

Output:




```
sumf = fft(sum)
xf = fftfreq(N, T)[:N//2]
plt.ylabel('frequency')
plt.xlabel('sample')
plt.title("FFT of sum of two
sines")
plt.plot(xf, 2.0/N *
np.abs(sumf[0:N//2]))
plt.show()
```

Output:



FFT applications

- Noise Filtering
- Spectral derivatives
- Solving partial differential equations
  - Heat Equation

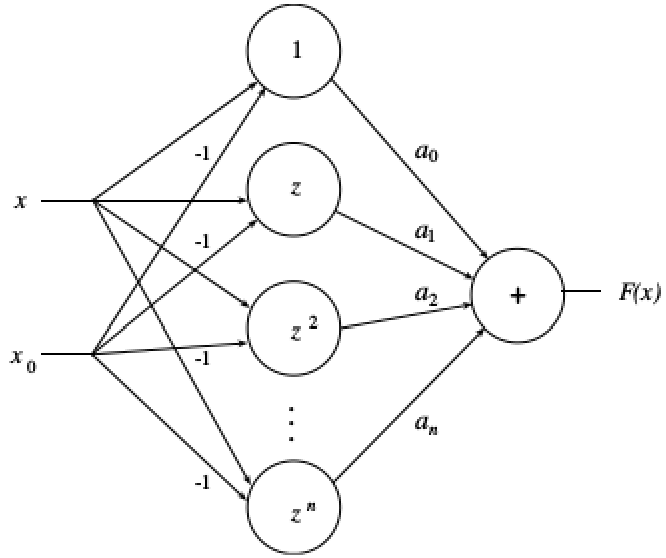


Fig. 1.16. A Taylor network

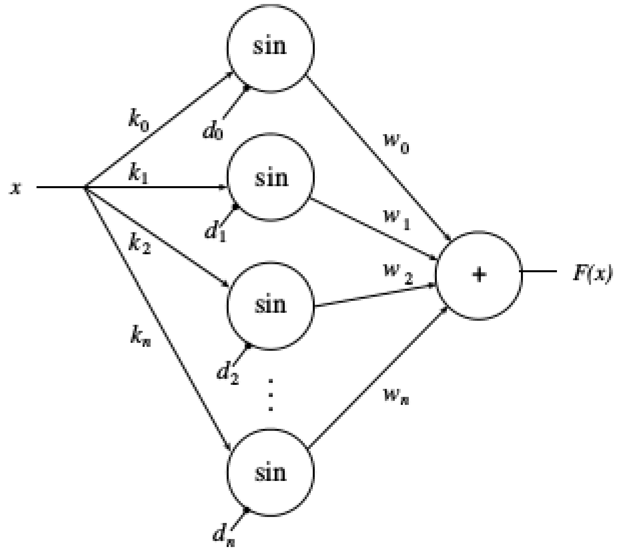
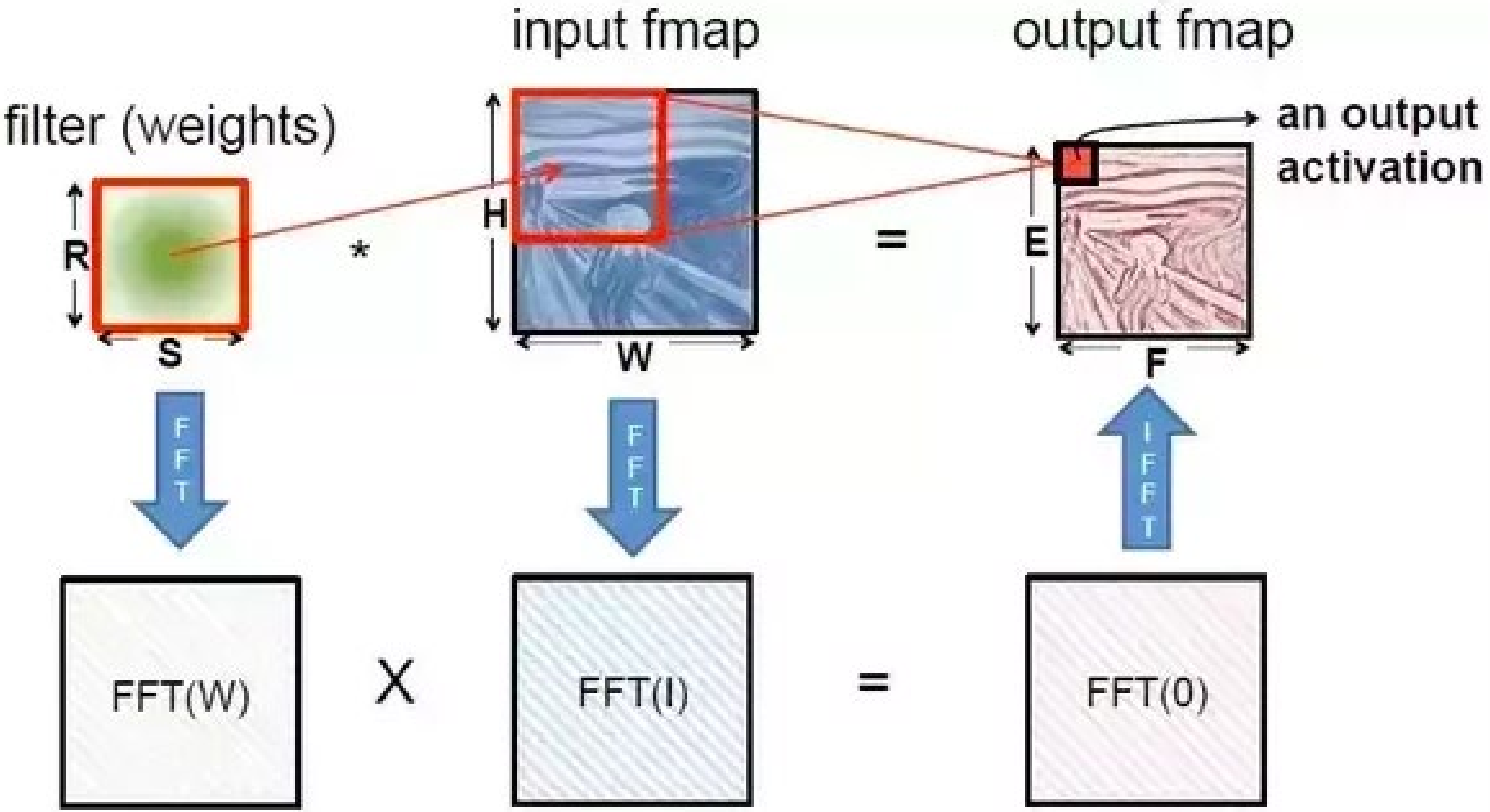
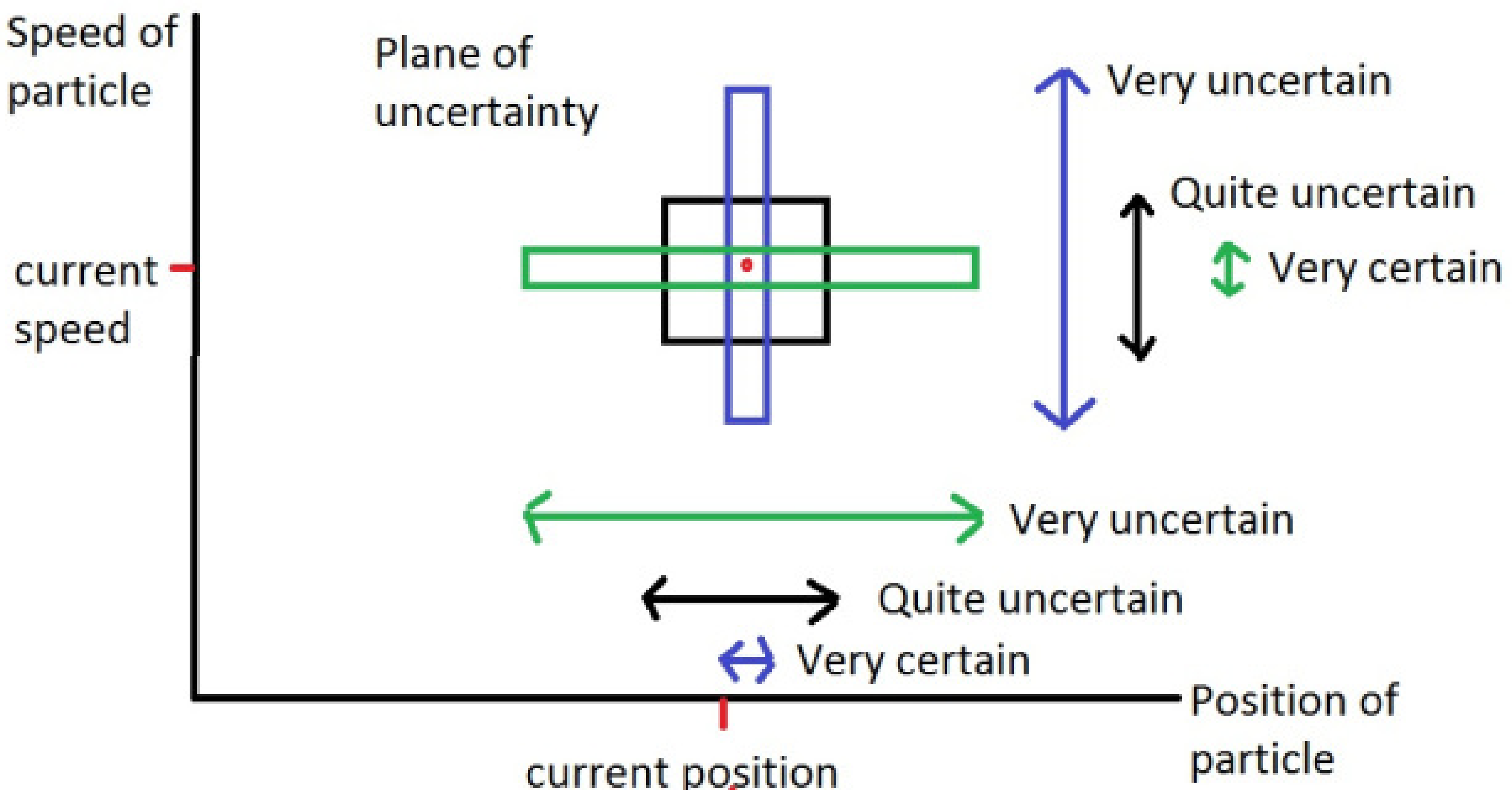


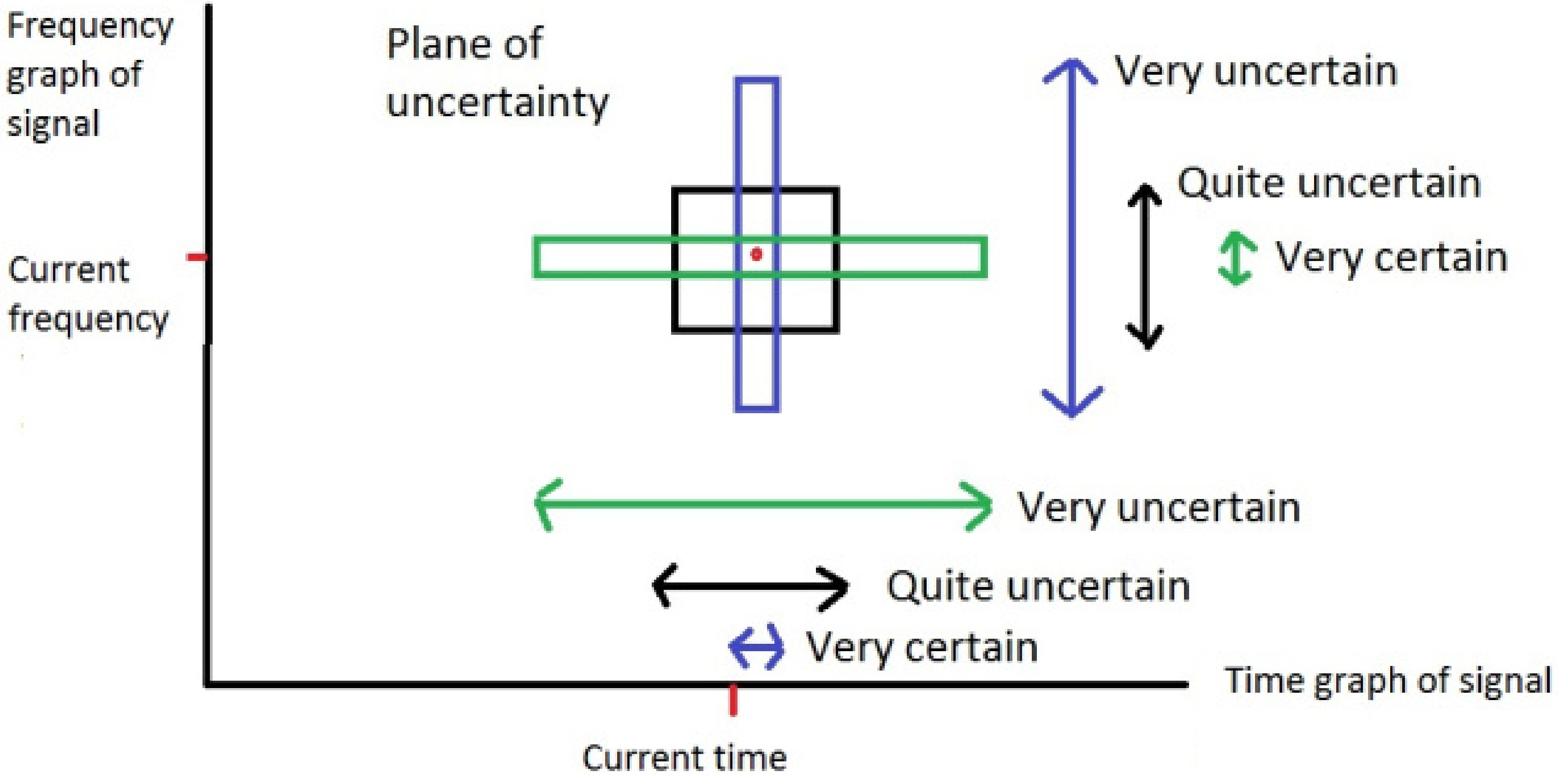
Fig. 1.17. A Fourier network

Convolutional Neural Networks, CNNs



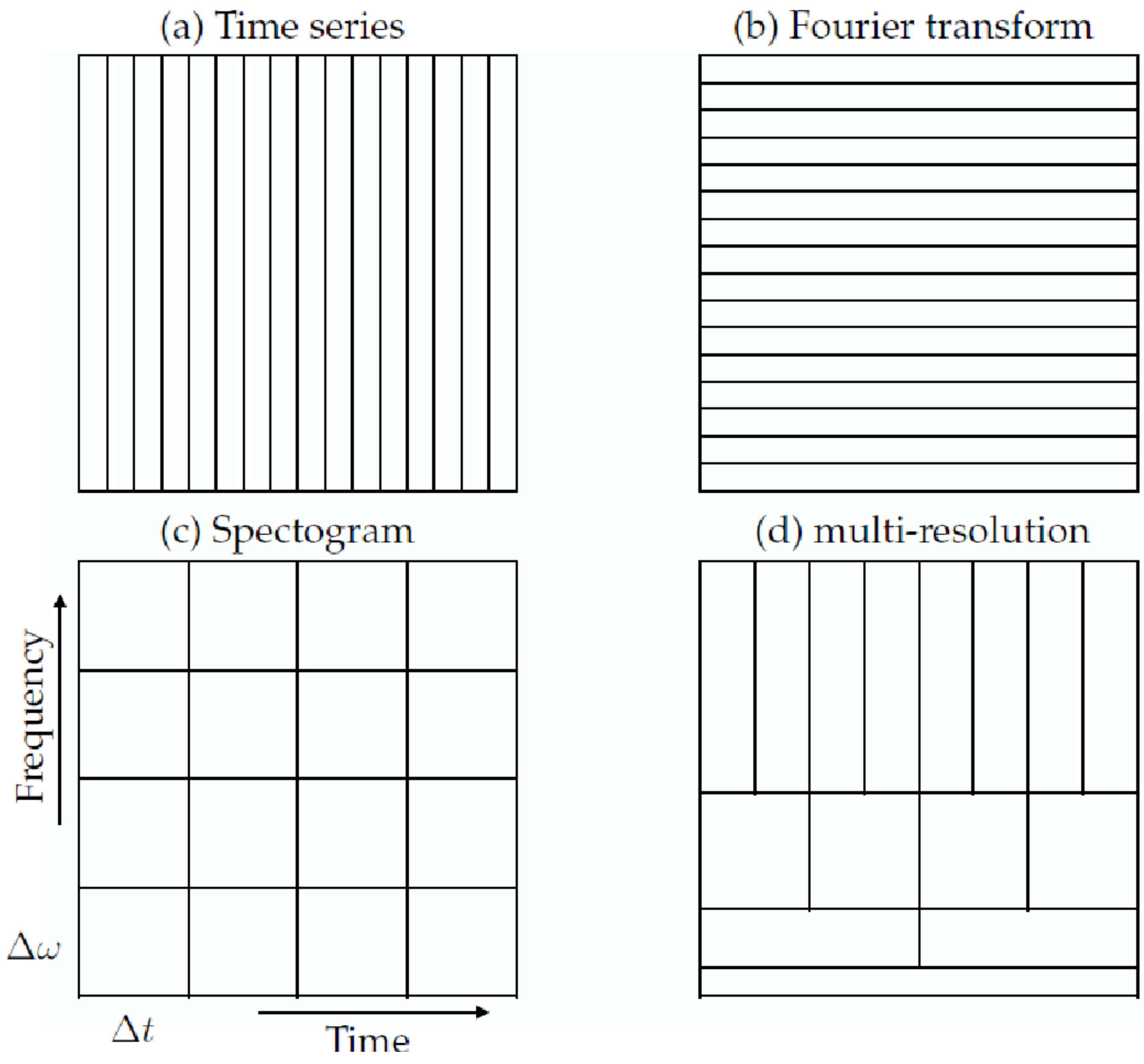


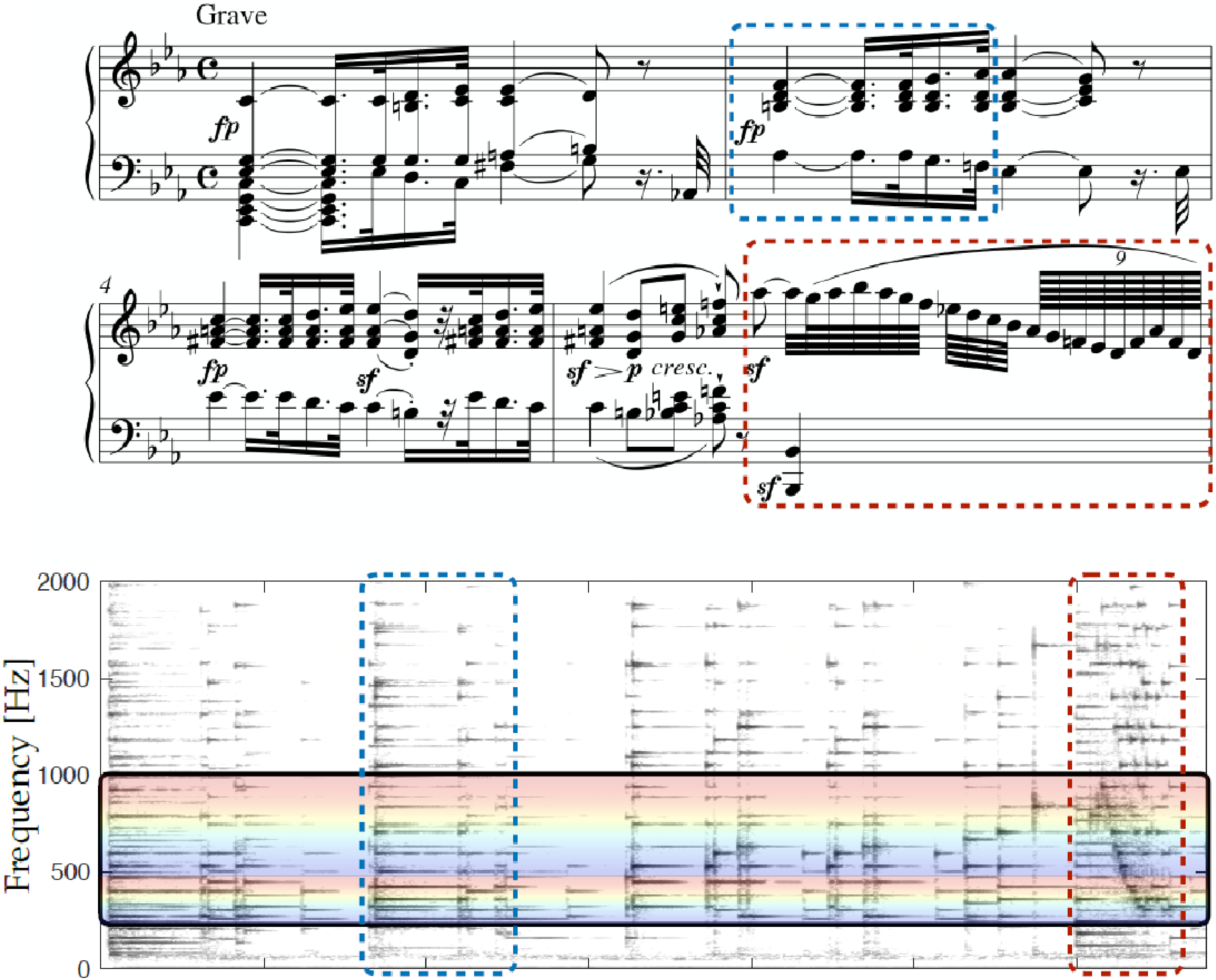
$$\Delta x \Delta p \geq h/4\pi$$



$$\Delta f \Delta T \geq 2\pi$$

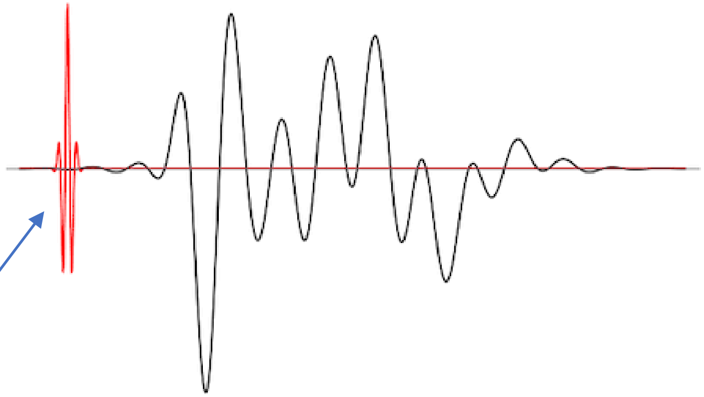
Gabor transform and the spectrogram

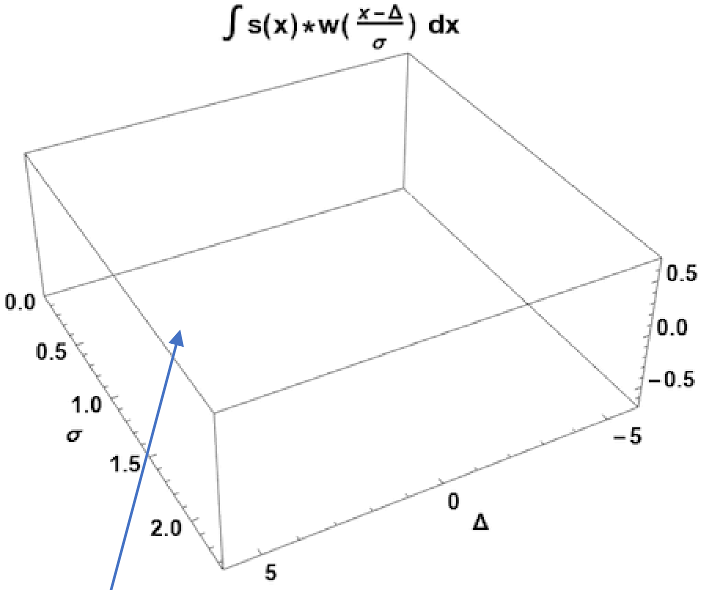


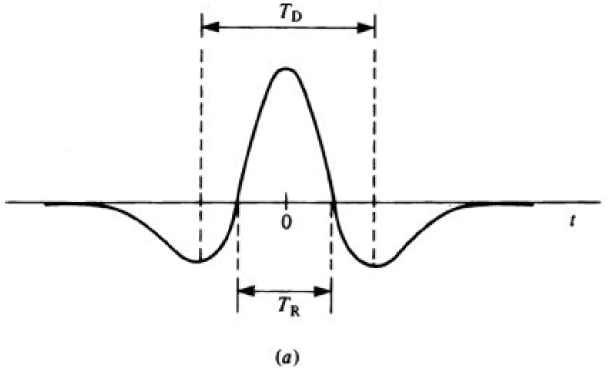


Grave
fp
4
fp
sf
sf
p
cresc.
sf
sf
9
2000
1500
1000
500
0
Frequency [Hz]

Continuous wavelet transform







Ricker Function

$$\psi_{a,b}(\lambda) = \frac{1}{\sqrt{a}}\psi\left(\frac{\lambda - b}{a}\right) \quad (1)$$

$$W_f(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(\lambda)\psi_{a,b}(\lambda)d\lambda \ (2)$$

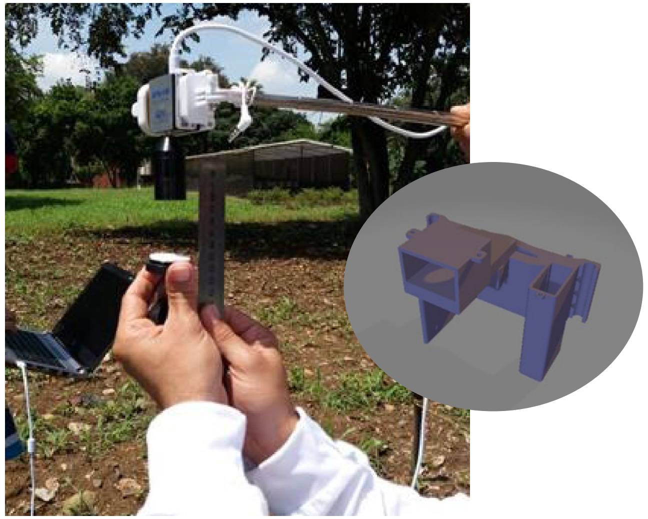
Fourier Transform

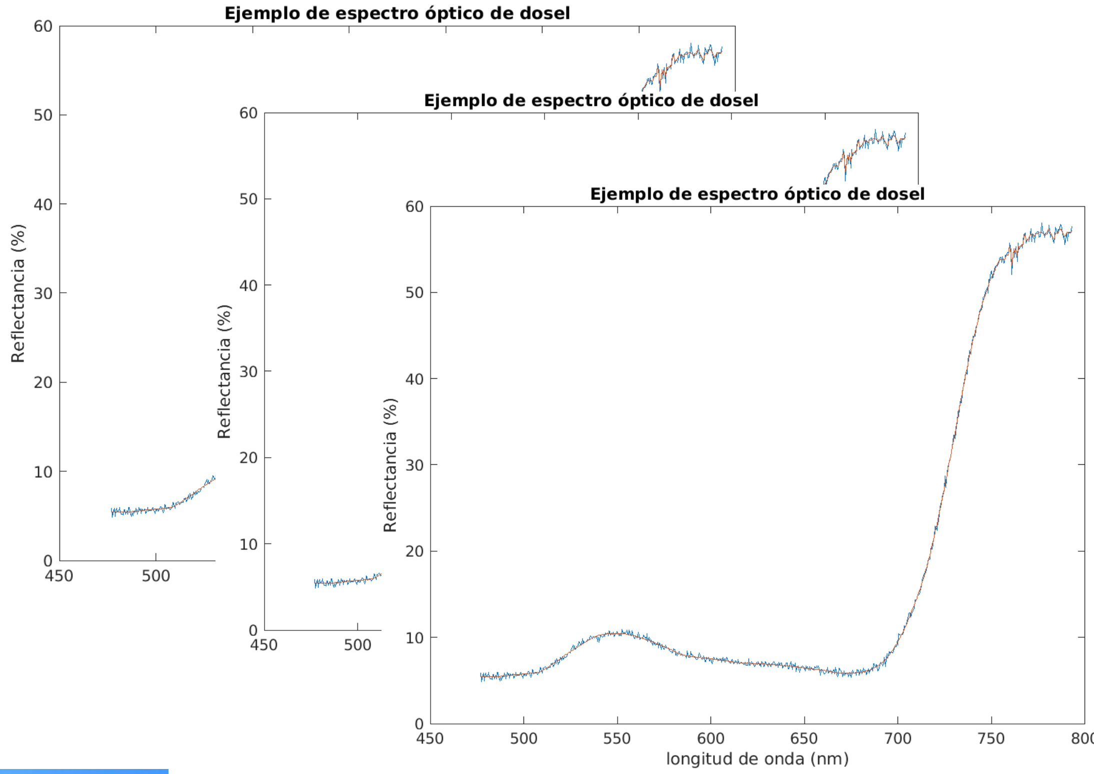
$$X(F) = \int_{-\infty}^{\infty} x(t)\ \boldsymbol{e^{-j2\pi Ft}}\, dt$$

"analyzing function"

Wavelet Transform

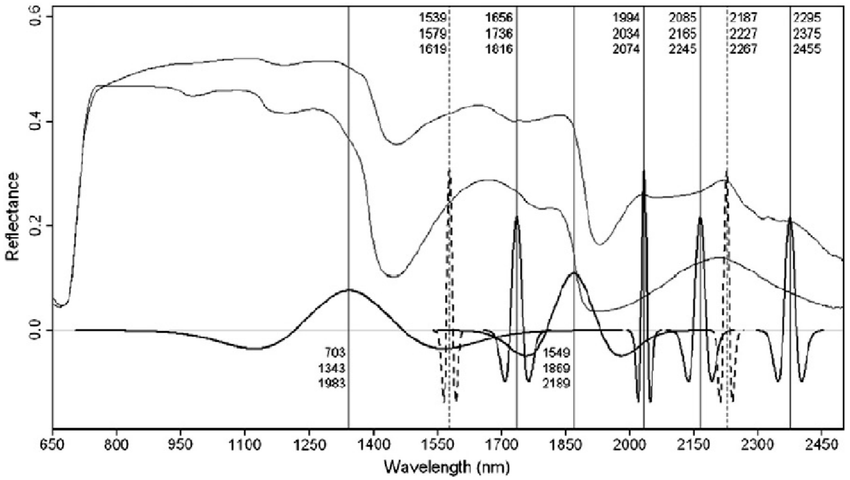
$$X(a,b) = \int_{-\infty}^{\infty} x(t)\ \boldsymbol{\psi^{*}_{a,b}(t)}\, dt$$



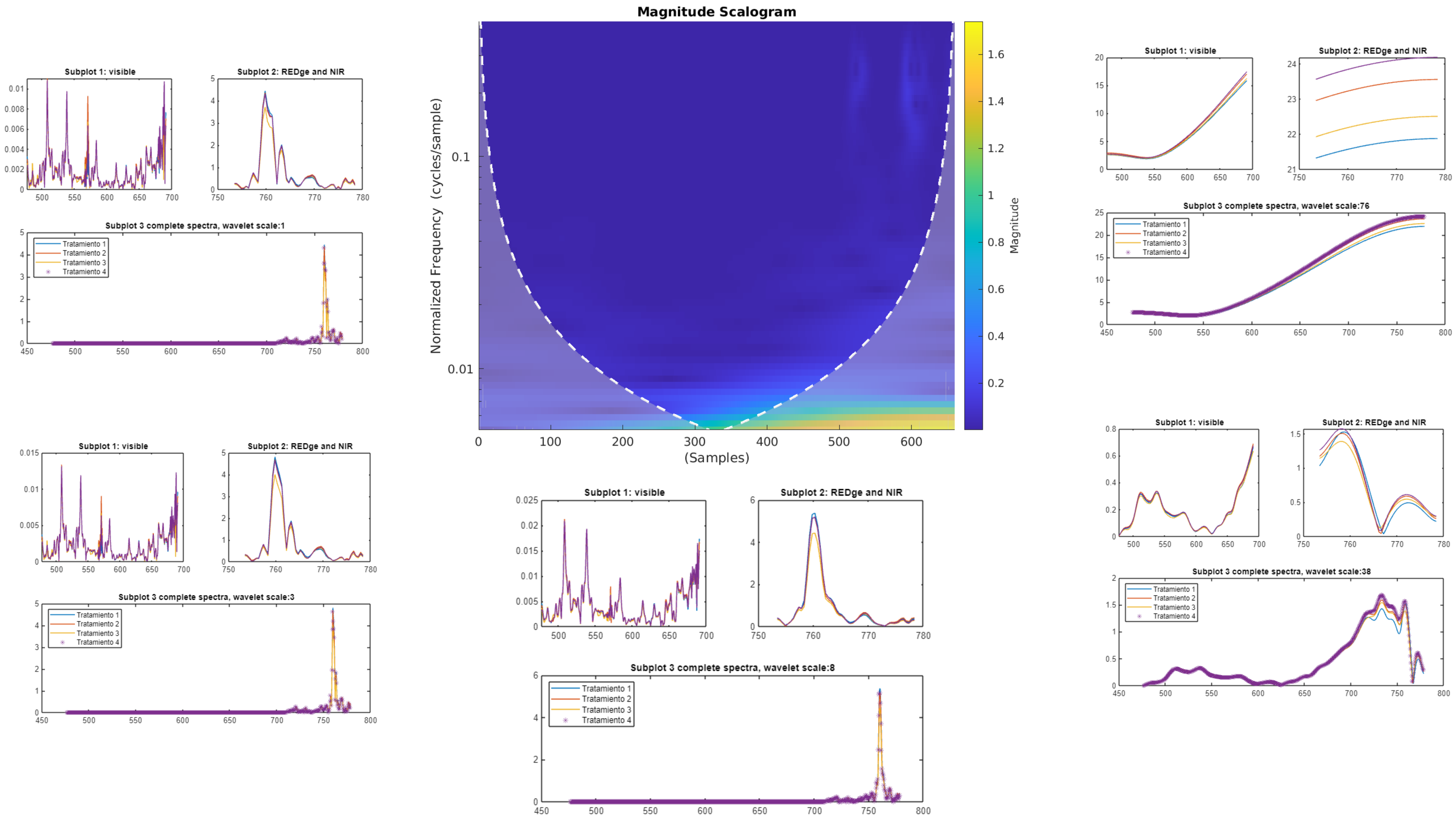


Ejemplo de espectro óptico de dosel
Ejemplo de espectro óptico de dosel
Ejemplo de espectro óptico de dosel
Reflectancia (%)
Reflectancia (%)
Reflectancia (%)
longitud de onda (nm)

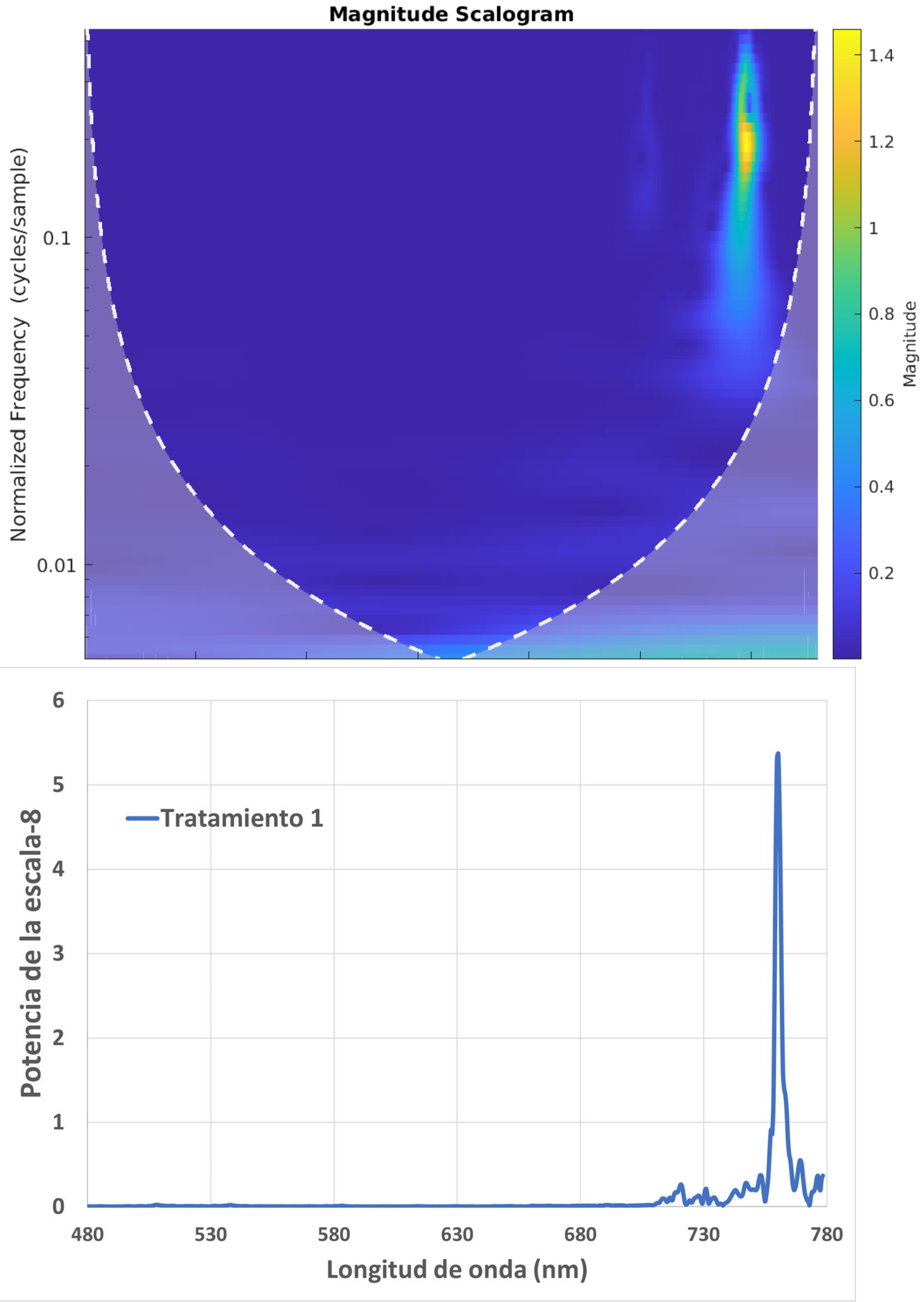
Ocean Insight
STS-NIR
Miniature Spectrometer



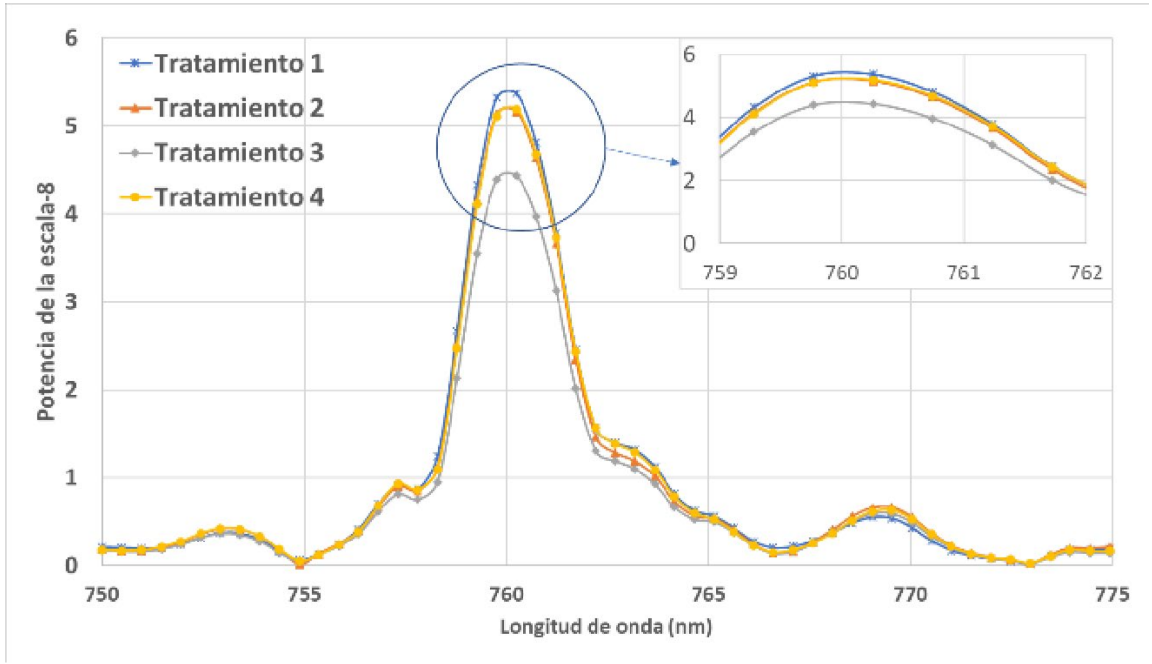
1539 1579 1619
1656 1736 1816
1994 2034 2074
2085 2165 2245
2187 2227 2267
2295 2375 2455
703 1343 1983
1549 1869 2189
Reflectance
Wavelength (nm)



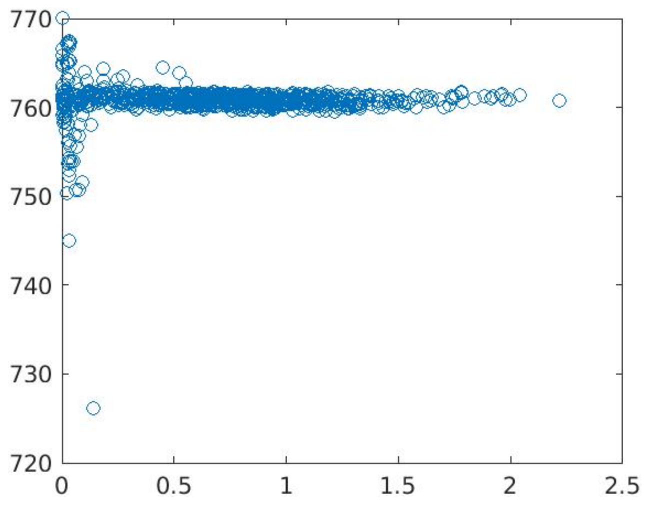
Magnitude Scalogram
Normalized Frequency (cycles/sample)
(Samples)
Magnitude
Subplot 1: visible
Subplot 2: REDge and NIR
Subplot 3 complete spectra, wavelet scale:1
Subplot 3 complete spectra, wavelet scale:3
Subplot 3 complete spectra, wavelet scale:8
Subplot 3 complete spectra, wavelet scale:76
Subplot 3 complete spectra, wavelet scale:38
Tratamiento 1
Tratamiento 2
Tratamiento 3
Tratamiento 4



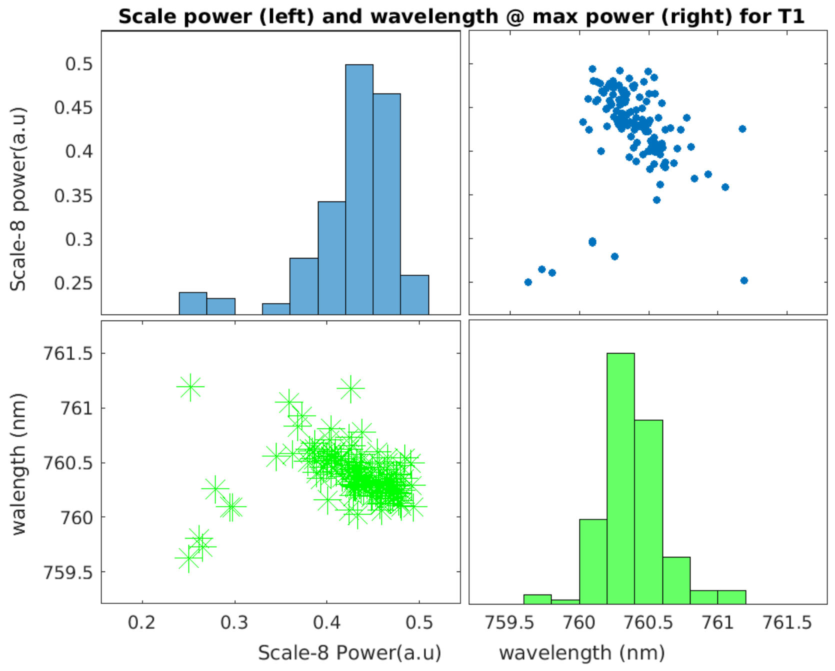
Magnitude Scalogram
Normalized Frequency (cycles/sample)
0.1
0.01
Magnitude
1.4
1.2
1
0.8
0.6
0.4
0.2
Potencia de la escala-8
Tratamiento 1
Longitud de onda (nm)
480
530
580
630
680
730
780



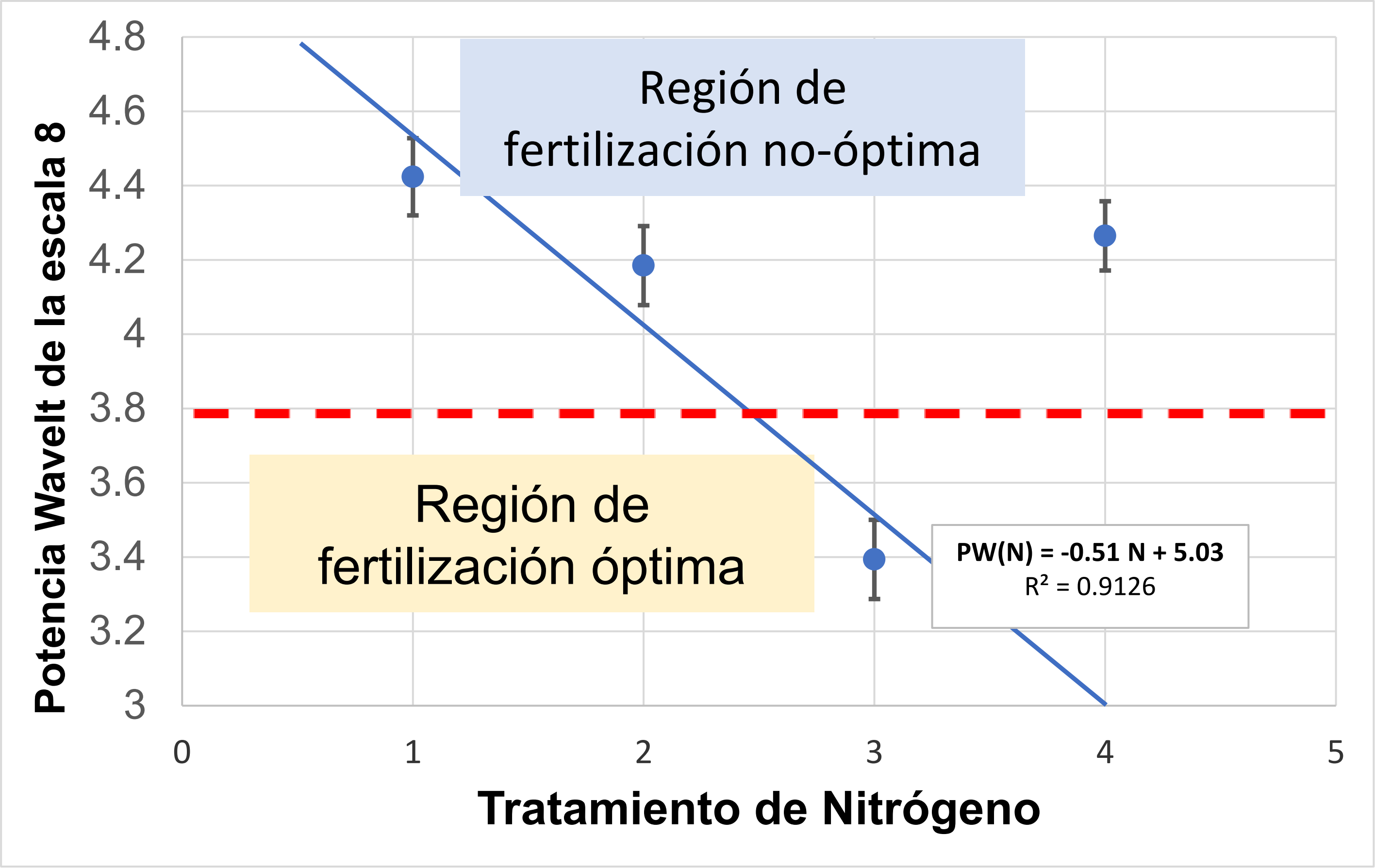
Tratamiento 1
Tratamiento 2
Tratamiento 3
Tratamiento 4
Potencia de la escala-8
Longitud de onda (nm)
750
755
760
765
770
775
759
760
761
762



770
760
750
740
730
720
0
0.5
1
1.5
2
2.5



Scale power (left) and wavelength @ max power (right) for T1
Scale-8 power(a.u)
0.5
0.45
0.4
0.35
0.3
0.25
walength (nm)
761.5
761
760.5
760
759.5
0.2
0.3
0.4
0.5
Scale-8 Power(a.u)
759.5
760
760.5
761
761.5
wavelength (nm)



Potencia Wavelt de la escala 8
4.8
4.6
4.4
4.2
4
3.8
3.6
3.4
3.2
3
0
1
2
3
4
5
Tratamiento de Nitrógeno
Región de fertilización no-óptima
Región de fertilización óptima
PW(N) = -0.51 N + 5.03
R² = 0.9126