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STAT 340: Computer Simulation of Complex Systems

Assignment 3 (due on 2015-03-27)

Exercise 18 (Generating random variates) (3+3=6 points)
Provide an algorithm for generating random variates from the

a) distribution function $F(x) = 1 - (1 - x^2)(\frac{e-e^x}{e-1})$, $x \in [0, 1]$;

b) probability mass function $f(k) = (\frac{1}{2})^{k+1} + \frac{2^{k-1}}{2 \cdot 3^k}$, $k \in \mathbb{N}$.

Exercise 19 (The (non-parametric) bootstrap) (3+3=6 points)
Consider the random sample $\{-1, 2, 5\}$ from a distribution function F. By considering all n possible samples precisely once, estimate the *lower endpoint* $\theta = \min\{x : F(x) > 0\}$ of F by its bootstrap estimator $\hat{\theta}_n$ and estimate the standard deviation of $\hat{\theta}_n$.

Exercise 20 (Monte Carlo integration) (4+4=8 points)
In each case below:

i) Determine the exact mathematical answer;

ii) Use Monte Carlo integration to estimate μ (sample size 1000; set the seed to 271) and provide an approximate 95% confidence interval for the true value μ. Include the source code and output in your solution.

a) $\mu = \int_{-4}^{2} \log(x + 5)\, dx$

b) $\mu = \int_{-\infty}^{\infty} \exp(-x^2/2)\, dx$

Exercise 21 (Monte Carlo integration) (1+1+1=3 points)
Let $U \sim \mathrm{U}[0, 1]$. Use simulation (sample size 1e6; set the seed to 271) to approximate

a) $\mathrm{Cor}[U, \sqrt{1 - U^2}]$;

b) $\mathrm{Cor}[U^2, \sqrt{1 - U^2}]$.

Compare the two results.

Exercise 22 (Variance reduction) (3+3+3+3+3=15 points)
In each case below:

i) Use Monte Carlo integration (sample size 1000; set the seed to 271) to estimate the value of the integral;

ii) Use the given variance reduction technique to estimate the value of the integral; and

iii) Compute the variance reduction factor (efficiency) of the given technique in comparison to Monte Carlo integration.

a) $\mu = \int_0^1 \exp(-x^2)\, dx$ using antithetic variates;

b) $\mu = \int_0^1 \log(1 + x)\, dx$ using control variates with the helper function $h(x) = \sqrt{x}$.

c) $\mu = \int_0^1 \log(1 + x)\, dx$ using an optimal control variate and helper function $h(x) = \sqrt{x}$.

d) $\mu = \int_0^1 g(x)\,dx$ where $g(x) = \begin{cases} 8x/3, & \text{if } x \in [0, 3/4], \\ -8x+8, & \text{if } x \in [3/4, 1], \\ 0, & \text{otherwise} \end{cases}$ using stratified sampling.

e) $\mu = \int_0^\infty \sqrt{\frac{2}{\pi}} \exp(-\frac{x^2}{2})\,dx$ using importance sampling with the helper function $h(x) = \exp(-x)$.

Note. Include the source code and output in your solution.

Exercise 23 (2+2+2=6 points)

If X is such that $\mathbb{P}(X \in [0, a]) = 1$, $a > 0$, show that

a) $\mathbb{E}[X^2] \leq a\mathbb{E}[X]$;

b) $\mathrm{Var}[X] \leq \mathbb{E}[X](a - \mathbb{E}[X])$; and

c) $\mathrm{Var}[X] \leq a^2/4$.