Photon counting statistics—Undergraduate experiment

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substituting Eq. (2.41) into Eq. (2.40), switching the order of integration, and then invoking the delta function formula

$$\delta(x)=(2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\omega x}\, d\omega=2\int_0^{\infty} \cos(2\pi\nu x)d\nu.$$

[18]A rigorous derivation of the widely known formulas (2.48), which is sometimes called the Box–Muller algorithm, may be found in Ref. 10, Sec. 1.8, which also describes a simple procedure for generating values for r_1 and r_2 in Eqs. (2.48a).

[19]It is possible to construct a simulation algorithm for O–U processes that is exact for *both* Y and X, but to do that we need a little more random variable theory than is given in Sec. II B.

[20]A nice account of Einstein's work in Ref. 1 is given in Ref. 9, pp. 2–6.

[21]D. T. Gillespie, "Fluctuation and dissipation in Brownian motion," Am. J. Phys. **61**, 1077–1083 (1993). A more sophisticated argument is given in Ref. 10, Secs. 4.5 and 4.6.

[22]Nyquist's original analysis in Ref. 5 proceeds quite differently from the analysis that we give here; see Ref. 7, p. 592.

[23]J. R. Reitz and F. J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1960), p. 176.

[24]See, e.g., Ref. 7, Sec. 7.5.

[25]A short proof of this "lemma" may be found in Ref. 10, p. 114.

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A photon counting experiment for student physics laboratory is described. It is designed to illustrate the probabilistic nature of the photodetection process itself as well as statistical fluctuations of light. The setup enables the student to measure photon count distributions for both coherent and pseudothermal light sources yielding Poisson and Bose–Einstein distributions, respectively.

I. INTRODUCTION

Photon counting is a technique commonly used to measure extremely low light fluxes. A photomultiplier tube (PMT) of proper design is used to convert light into an electrical signal. (The most important property of photon counting PMT is very high gain at the first dynode. This allows one to distinguish between the pulse[1] resulting from electrons ejected from the photocathode and those coming from the dynodes.) Light impinging on the photocathode of the PMT ejects electrons from it. Assuming that the gain of the PMT itself and that of the following electronics is high enough, one can distinguish individual electrical pulses, each of them corresponding to a single photoelectron. The electrical pulses from the PMT are fed into a discriminator and pulses with amplitudes higher than a given threshold value are counted. Those are usually referred to as photon counts. This way one can count the number of electrical pulses, ideally each of them corresponding to a single photoelectron. Since for each photoelectron created one photon of the light field has to be destroyed, the method is commonly called photon counting. Thus, in this oversimplified picture, one can think of the method as a way to prove the existence of photons. This is not true. Actually there is no need to quantize the electromagnetic field in order to explain all the features of the photoelectric process. All that is necessary is an assumption that the light interacts with matter which is described quantum mechanically. This leads to what is commonly referred to as a semiclassical description,[2] a model in which light is described as a classical electromagnetic wave and the atomic system, the photocathode in our case, quantum mechanically. The question of whether one has to invoke the quantum nature of electromagnetic field at all has been disputed ever since Einstein introduced the notion of the "light quantum" in 1905 in his paper describing the external photoelectric process.[3] Several different models for the photon have been proposed starting with a simple particle model and ending with what is known as Dirac's model,[4] each of them presenting its own difficulties in interpretation. Amazingly enough, the answer to this question has been settled quite recently when experiments on photon antibunching[5] and squeezed states (for a review on the topic see, for example, the paper by Walls[1]) proved that at least in some cases a quantum mechanical description of the light is necessary. Since in the experiments described in this paper the light can be perfectly described in a classical way, the semiclassical picture will be used henceforth.

The noise present in photon counting can be separated into two terms. The first is of a technical nature and is caused predominantly by electrical pulses created by amplification of electrons thermally released from the photocathode or the first dynode, which cannot be distinguished from the pulses corresponding to photoelectrons. Those pulses are present even if there is no light falling on the photocathode and for this reason are called dark counts. The dark count rate can be minimized by proper design and cooling of the PMT. Currently, even modestly priced systems have a dark count rate as small as a few counts per second. We will assume throughout this paper that the dark counts can be neglected altogether. The second contribution to the noise in the photon counting experiment and the only one considered in this paper is of a fundamental nature and cannot be eliminated. Again, it can be divided into two parts caused by the stochastic nature of photoelectric process and light intensity fluctuations, respectively. Using the semiclassical model mentioned above one finds that for a constant intensity of light reaching the photocathode the photoelectrons tend to leave the photocathode at random times. Thus the number of photon counts in a given time interval is not constant. Instead it fluctuates, leading to noise which is often referred to as shot noise. It is a fundamental feature of the photoelectric process. On the one hand, one can think that the photocurrent from the photocathode is constant but is formed by a discrete flow of electrons. Then the shot noise in the PMT tube is of the same nature as the shot noise in any vacuum tube. On the other hand one can think of each of the photoelectrons as being ejected as a result of absorption and destruction of one photon from the light beam. In this picture the statistics of photon counts reflects the statistics of photons in the measured light beam. It is important to stress that the shot noise will be present in photon counting experiments whenever the radiation measured can be described classically. However, for squeezed states a subshot noise operation of the photon counting PMT is possible.[6] Turning back to a semiclassical description, in addition to the shot noise, extra fluctuations in the photon count number arise whenever the intensity of the light being measured is not constant. In this case the photon counting statistics depend on the experimental details and reveal both the random nature of the photoelectric process itself and the nature of light fluctuations.

II. PHOTON COUNTING STATISTICS

The elementary experiment in photon counting is one in which photoelectrons are counted during a given time interval T. Since, as we have already mentioned in the Introduction, the photoelectric process is stochastic, one should expect that the number of photon counts will be stochastic, too. This means that the outcome of such an experiment cannot be predicted in advance. The most one can know is the probability of obtaining any given result. The formula for this probability distribution was first derived in the late 1950s by Mandel.[7] Before writing Mandel's formula explicitly, let us define the integrated light intensity W as[8]

$$W=\int\int_A \int_t^{t+T} I(x,y;\xi)d\xi\, dx\, dy, \tag{1}$$

where $I(x,y;\xi)$ is the intensity of the light wave at point (x,y) and time ξ, and A is the illuminated area of the photocathode. As defined by formula (1), W is the energy of the light beam reaching the photocathode during a time interval starting at t and ending at $t+T$. In general, W is a stochastic variable with a probability density function given by $P_W(W)$. The probability of observing K photon counts during time interval T is given by Mandel's formula[7]

$$P(K)=\int_0^{\infty}\frac{(\alpha W)^K}{K!}\exp(-\alpha W)P_W(W)dW, \tag{2}$$

where $\alpha=\eta/h\nu$ and η is the quantum efficiency of the photocathode, h is Planck's constant, and ν is the light frequency. As should be expected, the number of photon counts recorded in the time interval T is proportional to the energy delivered by the light beam to the photocathode during this time interval. Thus, in general, any fluctuations in the light intensity will, in principle, lead to fluctuations in the recorded number of photon counts. However, even for the light beam with perfectly constant intensity the number of the photon counts recorded during time T is not constant. This can be easily seen by performing the integration in formula (2) with an assumption

$$P(W)=\delta(W-\bar{W}),$$

where $\bar{W}$ is the average (constant) integrated intensity. The result of the integration is the Poisson distribution

$$P(K)=\frac{\bar{K}^K}{K!}\exp(-\bar{K}), \tag{3}$$

with the average number of photon counts $\bar{K}=\alpha\bar{W}$. Therefore, even in the photon counting experiment with a constant intensity light source, one observes stochastic variations in the number of counts. The resulting noise is the shot noise.

Two distinct light sources are of particular interest. The first one is a single-mode laser operating well above the threshold. The light emitted by such a laser is classically described by a constant amplitude perfectly monochromatic wave. Obviously the intensity of such a light beam is constant. Therefore, in this case, one should expect that the photon count distribution will be given by the Poisson distribution (3). It is worth mentioning that the quantum mechanical description of the light from a single-mode laser uses a coherent state mode[9] which converges to the classical description in the high-intensity limit.[10] The second light source of interest is a thermal source, i.e., a discharge lamp, incandescent lamp, etc. It is easy to see why the intensity of light from such a source is not constant. In this case the light field is a superposition of many waves with random amplitudes and phases being emitted by individual atoms or molecules in the discharge. It is very useful to introduce the notion of the coherence time for such field. We will not do it rigorously here; instead, we will define the coherence time heuristically. One can think of an amplitude and phase of the thermal light as changing stochastically but with a finite rate. This means that given the knowledge of the amplitude and phase at any given moment of time, one can with some accuracy predict what these parameters will be some short time later. On the other hand, after a long enough time, the field amplitude and phase are not correlated with their initial values and one cannot make any predictions. Thus one can say that the field "remembers" its previous parameters over some characteristic time. This time is called the coherence time. It is inversely proportional to the spectral width of the light source and for true thermal sources such as an incandescent lamp is very short. The instantaneous intensity probability density function for a linearly polarized thermal field is a negative exponential with zero intensity being the most probable:

$$P(I)=\frac{1}{\bar{I}}\exp\left(-\frac{I}{\bar{I}}\right), \tag{4}$$

where $\bar{I}$ is the average intensity.

For thermal light the integrated intensity probability density $P_W(W)$ depends on the time interval used. Two limiting cases are easy to analyze. For T much shorter than the coherence time τ_c, the integrated intensity W has the same distribution as the instantaneous intensity [formula (4)] and the photon count distribution is given by the Bose–Einstein formula:

$$P(K)=\frac{1}{1+\bar{K}}\left(\frac{\bar{K}}{1+\bar{K}}\right)^K \tag{5}$$

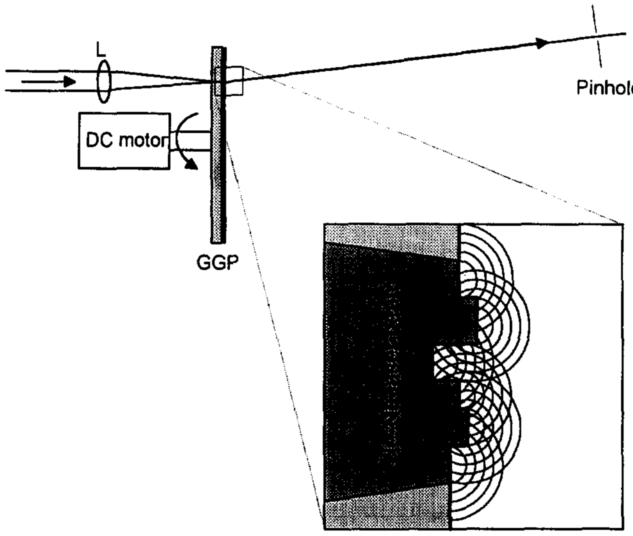

Fig. 1. Principle of pseudothermal light generation. L is a lens and GGP is a ground-glass plate. The laser beam is focused on the ground surface of the plate and the scattered light observed behind the pinhole P. The inset shows schematically a magnified part of the plate with several areas of various thicknesses illuminated by the laser beam. Sets of concentric circles indicate that each of the elementary areas is a source of a spherical wave.

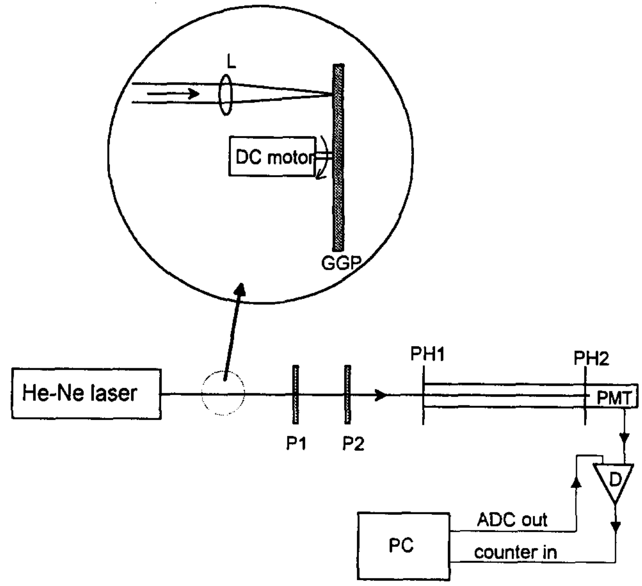

Fig. 2. Experimental setup P1, P2—Polaroid polarizers; PH1 and PH2—pinholes; PMT—photon counting photomultiplier tube; D—discriminator; PC—personal computer. The inset shows a ground-glass plate (GGP) mounted on a dc motor, and a lens (L) which were optionally inserted into the laser beam in order to produce pseudothermal light.

while for $T \gg \tau_c$ the function $P_W(W)$ is constant, the same as for a single-mode laser. Unfortunately, the coherence time for true thermal sources is very short (usually shorter than 1 ps) and for technical reasons one cannot measure $P(K)$ for $T \ll \tau_c$. However, one can easily make a pseudothermal light from the laser light. This can be achieved by scattering laser light from a large set of moving, randomly distributed scattering centers. Two methods for achieving this have been reported as early as the 1960s. The first one relies on scattering the light from a collection of submicron-sized plastic balls suspended in liquid,[11] while the second uses a rotating ground-glass plate.[12] The latter was used in our experiment. The principle of generating pseudothermal light using this method is illustrated in Fig. 1. Each of the elementary areas on the uneven glass surface illuminated by the laser beam forms a source of a spherical wave as illustrated in the inset. The optical field observed in the position of the pinhole is a sum of many waves with amplitudes and phases determined, respectively, by the size and relative positions of the respective scattering areas. Since both size and position of these areas are random, the resulting field is composed of many components with random amplitudes and phases. The resulting light intensity varies dramatically with the position of the observation point. This leads to a well-known phenomenon called speckle. If the glass plate is translated perpendicularly to the beam the speckle pattern changes and the light field observed behind the pinhole fluctuates. It can be shown that for the aperture diameter much smaller than speckle grain size, the intensity of light behind the aperture displays fluctuations described by the same formula as those of the thermal light.[11] The advantage of using this method to produce pseudothermal light is that one can easily control the correlation time of the fluctuating light produced this way. It can be continuously adjusted by changing the speed with which the glass moves across the laser beam. Light scattered from the ground-glass plate at some angle to the beam axis was used in our experiment to imitate a thermal source.

III. EXPERIMENT

The experimental setup is shown in Fig. 2. A 5-mW polarized He–Ne laser was used as a light source throughout the measurements. The laser beam passed through two Polaroid polarizers P1 and P2, and two pinholes, PH1 and PH2, mounted at the ends of a black metal tube of about 0.5-m length before reaching the photon counting PMT. The purpose of the pinholes and the metal tube is to decrease the solid angle viewed from the photocathode of the PMT and thus limit as far as possible the amount of the stray light reaching the photocathode. At the same time by appropriate choice of the pinhole diameters the intensity of the laser light impinging on the photocathode can be brought to a reasonably low level of about few thousand counts per second. The pinholes were made by carefully piercing holes in aluminum foil with a sharp needle. The pinhole diameters used were from about 50 μm to about 200 μm as measured from the diffraction pattern of the laser beam. Final adjustment of the light intensity and thus the photoelectron count rate was made by rotating the second polarizer. To produce pseudothermal light a 10-cm focal length lens and a slowly rotating ground glass disk were inserted into the laser beam as shown schematically in the picture. A dc motor was used to turn the glass. A photodiode with a small active area was temporarily set in the position of the first pinhole and the signal from the photodiode observed on an oscilloscope. This enabled us to adjust the motor speed in such a way that the intensity fluctuations observed on the scope had a characteristic time of about 100 ms. This gave us an order of magnitude estimate for the coherence time τ_c of the light. It should be much longer than the photon counting time interval T (in our case 1 ms) if the Bose–Einstein distribution is to be observed. The PMT used was a Hamamatsu H4730-01 photon counting head consisting of the PMT tube itself as well as an amplifier and a discriminator. The discriminator produced a transistor–transistor logic (TTL) level pulse whenever the output pulse from the PMT tube had an amplitude higher than a given

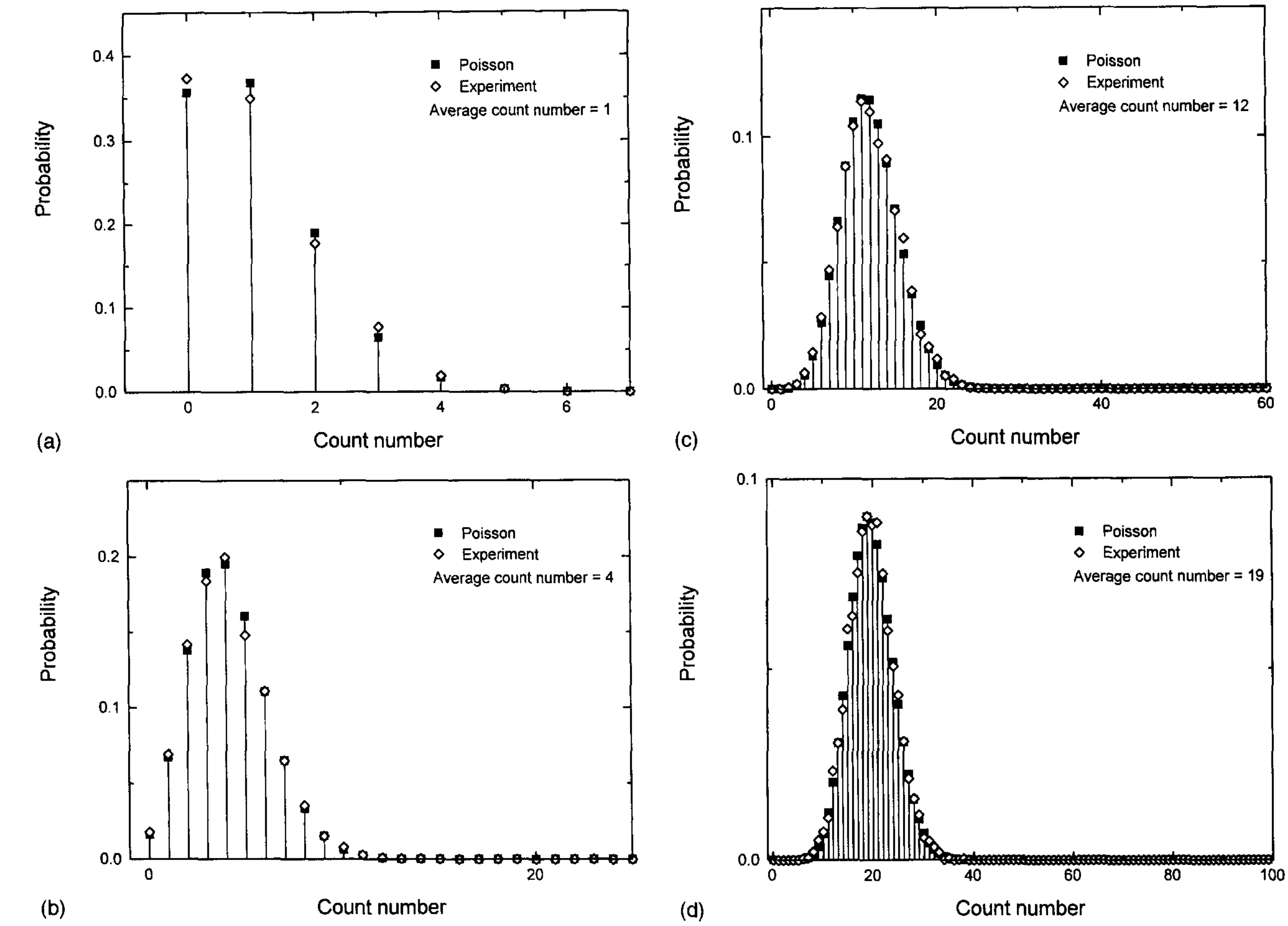

Fig. 3. Experimental photon count distributions (diamonds) measured for the He–Ne laser light and Poisson distributions (squares) with corresponding average values. (a)–(d) Results for count numbers 1, 4, 12, and 19, respectively.

threshold value. The TTL pulses were fed into a pulse counter board (Advantech model PCL-720) in a IBM compatible PC computer. Another plug-in board in the computer was used to provide an analog output voltage which determined the discriminator threshold. A rather straightforward computer code was written to control the DA converter and the counter boards. The code enabled the following tasks to be performed by the computer: setting the discriminator threshold, repetitive photon counting over a given time interval T, calculating average count number $\bar{K}$ and probability distribution.

First, the characteristics of the photon counting system itself were measured. With the PMT supply voltage set at $+1000$ V, a distribution function of the PMT pulse amplitude $K(V_d)$ was measured by varying the discriminator threshold between -0.5 and -2.5 V. Numerical differentiation of $K(V_d)$ yielded the PMT pulse amplitude probability density function. This function showed a rather weakly pronounced single-photon peak centered around -1.5 V. The result confirmed the manufacturer's recommendation for a -1.0 V discriminator threshold and this value has been used in all the following measurements. In order to measure the dark count rate, the input of the PMT was blocked and an average of 100 measurements each lasting 1 s was taken for different values of PMT supply voltage. The results obtained showed that the dark count rate increases monotonically with the PMT high voltage in the range of $+800$ to $+1200$ V. For a PMT high voltage of $+1000$ V, the dark count rate was about 60 counts/s which is slightly lower than specified for this instrument. All the following measurements were carried out in such a way as to make the contribution of the dark counts insignificant. For example, if the average count number $\bar{K}$ desired was 5, then the counting time T was chosen to be 1 ms and thus average dark count rate during time T was 0.06 which is negligible compared to $\bar{K}$. For higher values of $\bar{K}$ the counting time T could be appropriately extended while keeping the dark count contribution at the negligible level.

Photon count distributions were measured for both He–Ne laser light and pseudothermal light produced by a rotating disk of ground glass as described before. In each of the measurements the procedure was as follows. First, the average count rate was adjusted to the desired value by changing the light intensity with the polarizer P2. Then a series of 10 000 measurements was taken and the data stored in computer memory. Finally, the probability distribution was calculated as the ratio of the number of measurements with a given number of counts to the total number of measurements. The time interval T used varied from 1 to 10 ms, but all the measurements for the pseudothermal light were taken with $T=1$ ms in order to fulfill the condition $T \ll \tau_c$ described earlier.

IV. RESULTS AND DISCUSSION

Figure 3 shows the photon count distributions for the laser light. Four distributions with average count numbers of 1, 4,

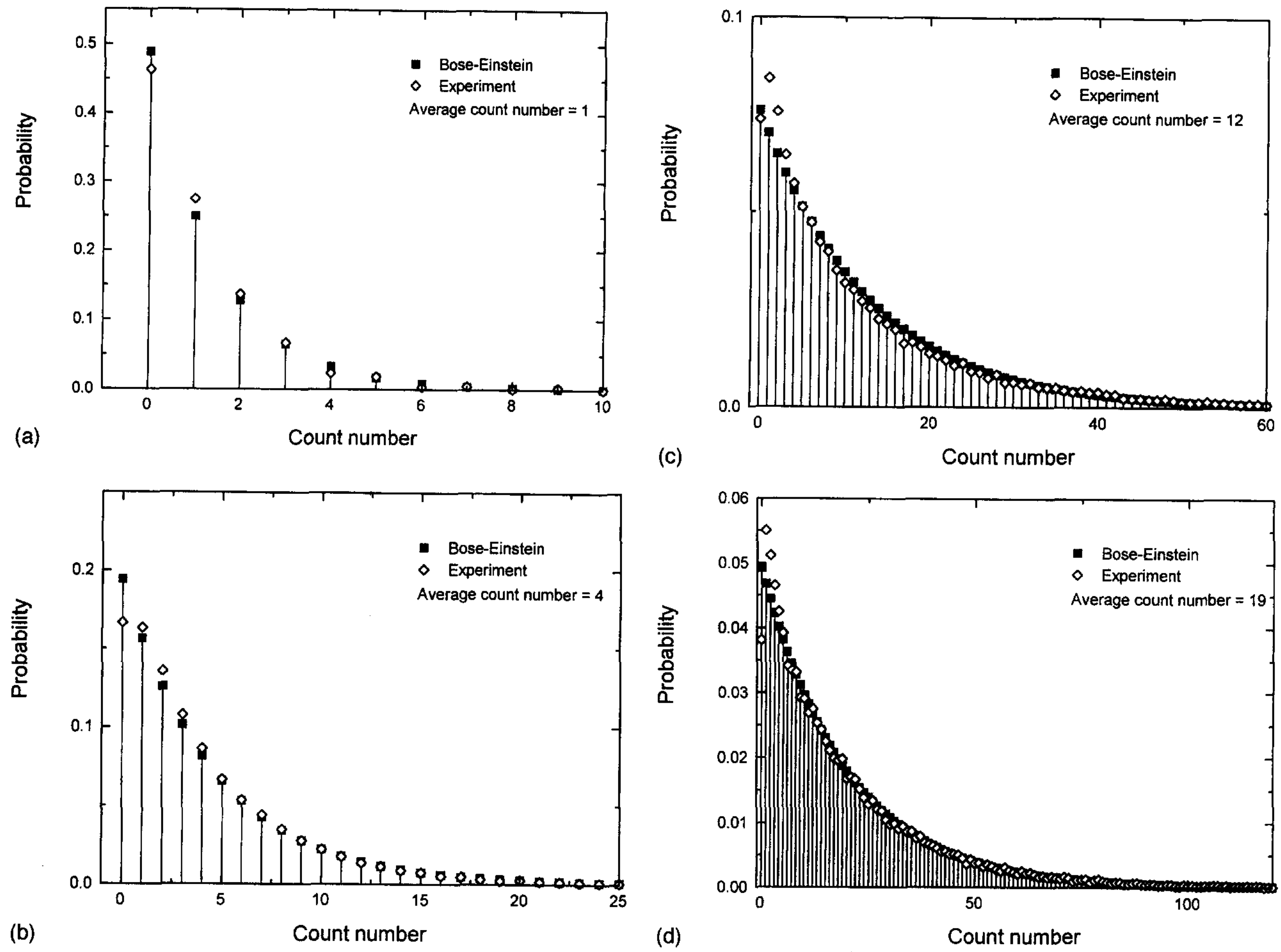

Fig. 4. Experimental photon count distributions (diamonds) for the pseudothermal light and Bose–Einstein distributions (squares) with corresponding average values. (a)–(d) Results for count numbers 1, 4, 12, and 19, respectively.

12, and 19 are shown. Also shown in the picture are Poisson distributions with $\bar{K}$ equal to 1, 4, 12, and 19. As can be seen in the picture, the distributions measured for the He–Ne laser light are almost identical to the Poisson distributions with corresponding average count numbers. This might seem a bit surprising since the laser used in the experiment was not a single longitudinal mode laser and did not emit completely coherent light of constant intensity. Still the photon count distributions are exactly the same as those expected for coherent light. It can be explained as follows. Since the laser operates on a few longitudinal modes simultaneously, the instantaneous intensity varies in time because of mode beating. However the characteristic time of the intensity fluctuations (coherence time τ_c) is relatively short (of the order of 1 ns). Since the shortest time interval T used in the experiment was 1 ms, the relation $T \gg \tau_c$ holds very well, the integrated intensity is constant except for technical noise caused for example by fluctuations of the discharge current in the laser tube, etc., and the experiment yields a Poisson distribution. This shows that one does not need an expensive single-mode He–Ne laser in order to demonstrate the photon count distribution of the coherent light. Such a demonstration can be achieved with a multimode laser at the cost of a somewhat more complex theoretical description. The data in Fig. 3 show the importance of shot noise in the measurements of very low light fluxes. The variance of the photon count number for the Poisson distribution varies as $\sqrt{\bar{K}}$, and thus the signal to noise ratio scales as

$$S/N = \sqrt{\bar{K}}. \tag{6}$$

Thus the shot noise is the major problem whenever the light intensity and the count number are low. This is where squeezed states with possible sub-Poissonian distributions are most relevant. For very high light intensities and thus high values of $\bar{K}$, one can usually neglect shot noise altogether. This corresponds to the limit in which the coherent state describes the classical monochromatic wave. The results of measurements for pseudothermal light are shown in Fig. 4. The average count numbers shown are again 1, 4, 12, and 19. Also shown in the picture are theoretical distributions (Bose–Einstein distributions) with corresponding average count numbers. The overall agreement is very good although a slight discrepancy can be seen for low count numbers indicating that the light scattered by rotating ground-glass plate used in our experiment is not perfect pseudothermal light. The comparison of Figs. 3 and 4 shows a dramatic difference between the two distributions. The most striking property of the Bose–Einstein distribution, as compared to the Poisson distribution, is that its variance is larger than the average value and the signal to noise ratio given by the formula

$$S/N = \sqrt{\frac{\bar{K}}{1+\bar{K}}} \tag{7}$$

is always smaller than 1. In this case the noise observed in the photon counting experiment is, for larger values of $\bar{K}$, totally dominated by the fluctuations of the light intensity, while the stochastic nature of the photoelectron process itself manifests itself only for very small values of $\bar{K}$.

In conclusion, we have described a photon counting experiment for the advanced student physics laboratory that demonstrates both shot noise for a constant intensity source and intensity noise for thermal radiation.

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Photon states made easy: A computational approach to quantum radiation theory

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Students first meet the wave-particle paradox through the photon and wave descriptions of light. Yet, in basic courses on quantum mechanics, they study matter particles only, because the mathematics of the quantized radiation field is usually considered too advanced. An oscillating electromagnetic field is formally similar to a harmonic oscillator, whose energy eigenstates can represent states of well-defined photon number. Using a computer program from the CUPS project, an approach will be described which demonstrates the action of the annihilation operator on these states, constructs coherent states which behave like classical electromagnetic fields, and shows how such states can be squeezed. All of these have practical relevance in modern optics. This is just one example of the computer making a hitherto unapproachable subject accessible to ordinary undergraduates. Computers have already changed how much of quantum mechanics is taught. As more such possibilities are realized, the teaching of the whole subject must surely change radically. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

In the 70 or so years since the original formulation of quantum mechanics, the way the subject is taught has become highly standardized. Almost without exception, all post-introductory textbooks arrange their material in the same way. In particular, they start with one dimensional topics and introduce as many concepts as they can until it becomes too difficult, mathematically, to go further.

In recent years, much software has been written for use in standard courses,[1] to relieve the dependence on extensive analytical expertise which so many students do not have. Nowadays it is easy to generate solutions of problems like bound states in different potential wells, plane waves incident on various barriers, and the motion of wave packets. As a result, instructors able to push the study of one-dimensional quantum mechanics much further than they once could.

Another feature of the standard approach is the prominence given to the idea that the same physical system can be described by different formalisms. The harmonic oscillator, for example, is often treated in three different ways: in Schrödinger representation, in matrix mechanics, and by operator methods. The idea that *different* physical systems can be modeled by the *same mathematics* is equally valuable but seldom exploited. A good example occurs in the quantum theory of radiation.

When most students are introduced to modern physics, the