Math Camp: Day 1

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August 22, 2017

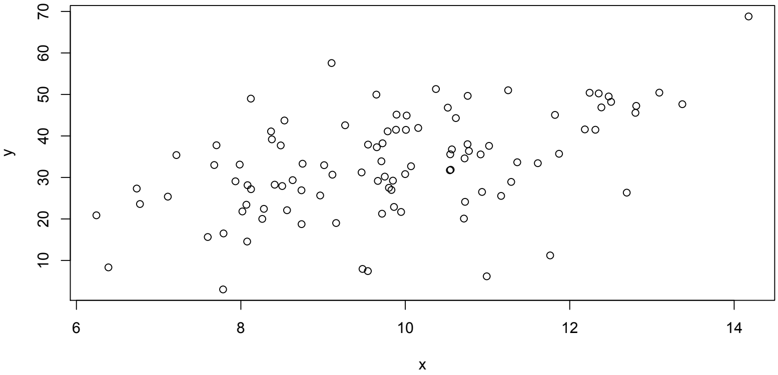
Why do we need math?

- At first thought, many of the world's political phenomenon do not require math:
  - Why do countries go to war?
  - Why do people vote for particular candidates?
  - How does Congress pass legislation?
- If you dig deeper many of these question start to lend themselves to a relationships between factors.

Why do we need math?

- At first thought, many of the world's political phenomenon do not require math:
  - Why do countries go to war?
    - Does GDP contribute to decisions to go to war?
  - Why do people vote for particular candidates?
    - Do women and men vote differently?
  - How does Congress pass legislation?
    - Does party polarization reduce the number of bills passed?
- If you dig deeper many of these question start to lend themselves to a relationships between factors.

How Do We Describe Relationships Between Data?



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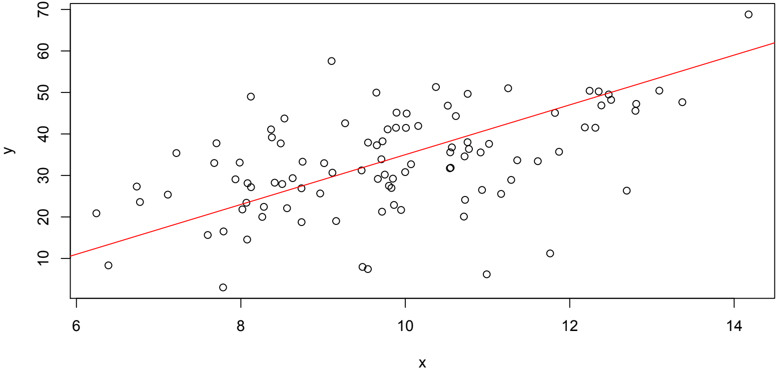

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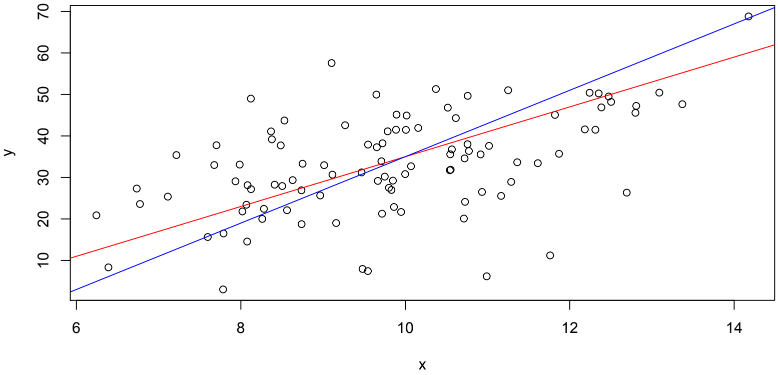

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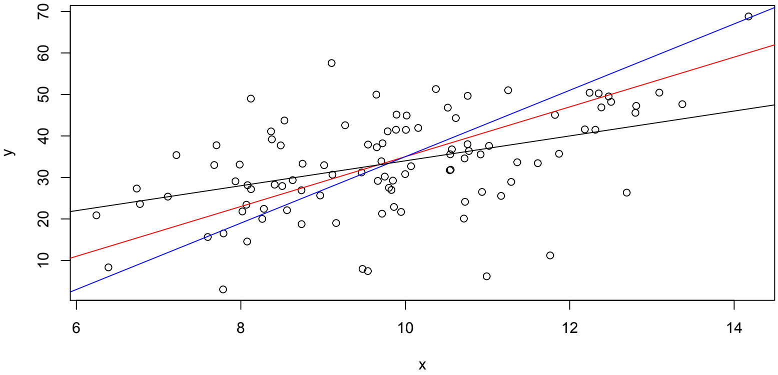


Functions

- Functions map one variable to another $f(x) : X \to Y$
- Functions can also map multiple variables onto a single variable
  - Examples:
    - $f(x) = 4x + 5$
    - $y = x^2 + 2x + 6$
    - $f(x, y) = x^2 + 2xy + 4y^2$
    - $z = y^2 + x^2 + 1$
- Variables on the right side of the equation are called input variables, and the variable on the left side is the output.
- The input variables are also called Independent variables, and the output is called a Dependent variable.

Types of Functions

- Monomial: $f(x) = bx^k$; b is the coefficient and k is the degree.
- Polynomial: sum of monomials; i.e. $y = 3x + 2$; $y = x^3 + 4x^2$. Polynomials should be written starting with the highest degree monomial to the lowest degree. The degree of the polynomial is the highest monomial degree.
  - Linear functions are degree 1.
  - Nonlinear functions is anything not of degree 1.
- Rational function: ratio of polynomials; i.e. $y = \frac{3x^2+4x+6}{12x^4+3x^2}$
- Exponential Function: function where the input variable is an exponent; i.e. $y = 12^x$
- Logarithmic Function: function with a logarithm; i.e. $y = ln(x)$
- Trigonometric Function: function of an angle; i.e. $y = sin(x)$

Linear Functions

- $y = a + bx$
- Slope: $b = \frac{\delta y}{\delta x} = \frac{y_1 - y_0}{x_1 - x_0}$
- Intercept: a The point at which the line crosses 0 on the y-axis.
- Exercise: Find the equation of the line for points (2,5) and (6,3)
  - $b = \frac{3-5}{6-2} = \frac{-2}{4} = -\frac{1}{2}$
  - Now to find the intercept, use one of the points:
    - $3 = -\frac{1}{2}(6) + a$
    - $3 = -3 + a$
    - $a = 6$
  - $y = 6 - \frac{1}{2}x$

Exponents and Logarithms

- Properties of exponents
  - $b^2 = b \times b$
  - $b^{-x} = \frac{1}{b^x}$
  - $b^0 = 1$
- Properties of logarithms
  - $log_b X = Y \leftrightarrow b^Y = X$
  - $b^{log_b x} = x$
  - $log_b(xy) = log_b(x) + log_b(y)$
  - $log_b(\frac{x}{y}) = log_b(x) - log_b(y)$
  - $log_b y^x = x log_b y$
  - $log_b \frac{1}{x} = -log_b(x)$
  - $log_b(1) = 0$
  - $log_b(x) = \frac{log_a(x)}{log_a(b)}$
- Common Logarithms
  - Base 10: $y = log_{10}(x) \leftrightarrow 10^y = x$
  - Base e: $y = log_e(x) \leftrightarrow e^y = x$
    - Also known as the natural logarithm: $log_e(x) = ln(x)$

Summations and Products

- $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$
- $\prod_{i=1}^{n} x_i = x_1 x_2 x_3 \ldots x_n$
  - Examples: x=(5,6,8,10,14)
    - $\sum_{i=1}^{3} x_i = 5 + 6 + 8 = 19$
    - $\prod_{i=1}^{4} x_i = 5 \times 6 \times 8 \times 10 = 2400$

Summations and Products

- Properties:
  - $\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$
  - $\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$
  - $\sum_{i=1}^{n} c = nc$
  - $\prod_{i=1}^{n} cx_i = c^n \prod_{i=1}^{n} x_i$
  - $\prod_{i=1}^{n} (x_i + y_i) = (x_1 + y_1)(x_2 + y_2)\dots(x_n + y_n)$
  - $\prod_{i=1}^{n} c = c^n$

Connection Between Sums, Products, and Logarithms

$$
\begin{aligned}
log(\prod_{i=1}^{n} x_i) &= log(x_1 \times x_2 \times x_3 \ldots {}_n) \\
&= log(x_1) + log(x_2) + log(x_3) + \ldots log(x_n) \\
&= \sum_{i=1}^{n} log(x_i)
\end{aligned}
$$

Limits

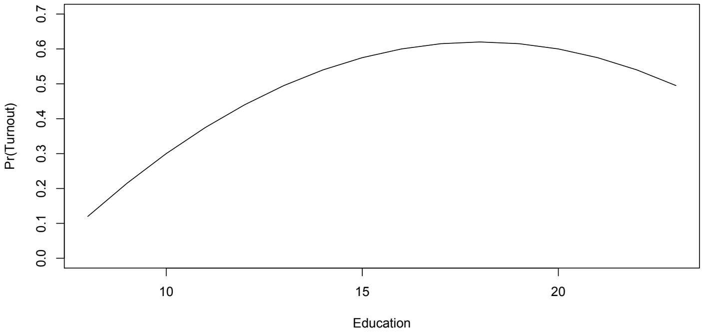
- Definition
  - L is the limit of $f(x)$ as x approaches c when the value of $f(x)$ nears L as x nears c
    - $\lim_{x \to c} f(x) = L$
- Right-hand limits
  - The value of $f(x)$ when approaching c from the right
    - $\lim_{x \to c^+} f(x) = L$
- Left-hand limits
  - The value of $f(x)$ when approaching c from the left
    - $\lim_{x \to c^-} f(x) = L$
- Infinite limits ("Asymptotes")
  - The value of $f(x)$ as the function approaches infinity.
    - $\lim_{x \to \infty} f(x) = L$

Properties of Limits

- Let $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$
  - If $f = g$, then $L_1 = L_2$
  - $\lim_{x\to c} k = k$
  - $\lim_{x\to c} x = c$
  - $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L_1 + L_2$
  - $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x) = kA$
  - $\lim_{x\to c} f(x)g(x) = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)] = AB$
  - $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{A}{B}$ for all $B \neq 0$
  - If $\lim_{x\to L_2} f(x) = L_3$ and $\lim_{x\to c} g(x) = L_2$, then $\lim_{x\to c} f(g(x)) = L_3$

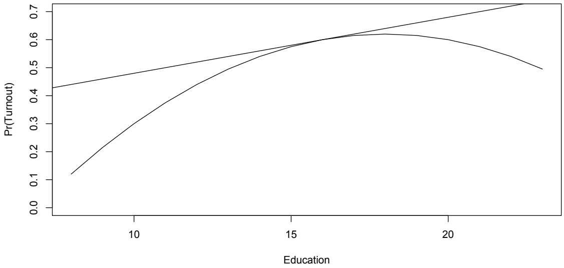
Turnout and Education

- The probability of turning out to vote has a quadratic relationship with education.
- Let's assume the following relationship:
  $Pr(Turnout) = -.005Ed^2 + .18Ed - 1$



Turnout and Education

- What if we want to know how the likelihood of turnout is changing at 16 years (college education)
- We need to know the slope of the function at this point.
- How do we find this?
- We can determine the slope of the tangent line touching the function at that point



Slope of Tangent Line

- So how do we find this slope?
- Let $h > 0$ be some small amount.
  $m = \frac{f(x+h)-f(x)}{(x+h)-x}$
- To find the slope, m, we take the limit of this function as it approaches 0.
  - $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Slope of Tangent Line

- Let's do this with our turnout function:

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{-.005(x+h)^2+.18(x+h)-1-(-.005x^2+.18x-1)}{h} \\
&= \lim_{h\to 0} \frac{-.005(x-h)^2+.18h+.005x^2}{h} \\
&= \lim_{h\to 0} \frac{-.005x^2-.01xh-.005h^2+.18h+.005x^2}{h} \\
&= \lim_{h\to 0} \frac{-.01xh-.005h^2+.18h}{h} \\
&= \lim_{h\to 0} -.01x-.005h+.18 \\
&= \text{- .01x - .005} \lim_{h\to 0} h+.18 \\
&= \text{- .01x + .18}
\end{aligned}
$$

- This equation gives the slope of a tangent line touching the function for any x.
- For $x = 16$, the slope of a tangent line touching the function is .02.
- Thus, the function is changing at a rate of .02 at $x = 16$.

Derivative

- The function for a slope of a tangent line touching the function at any point is also called the derivative.
- Notationally, a derivative is written as $f'(x) = \frac{dy}{dx} = \frac{d}{dx}y = Df(x) = Df$
- Derivatives can be taken of derivatives: $f''(x)$ or $f'''(x)$, $\frac{d^2y}{dx^2}$
- What can the derivative tell us?
  - Can tell us whether the function is increasing or decreasing at a particular point.
  - Can help us find the minima, maxima, and inflection points of the function.

Derivatives

- Not all functions have derivatives for all values.
  - Example: $|x| atx = 0$
  - The value of a derivative does not exist wherever the function is non-continuous
- The derivative of a scaler is 0.
- For derivatives of polynomials with degree 1, the derivative is a scalar.
- For polynomials of degree 2 or higher, the derivative will depend on the value of x.

Derivative Shortcuts

- Power Rule
  - If $f(x) = \sum_{k=0}^{n} a_k x^k$, then
  - $f'(x) = \sum_{k=0}^{n} k a_k x^{k-1}$
- Constant Rule: $\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$
- Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$
- Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} g(x)$
- Quotient Rule:
  $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2} = \frac{\text{Low}\,d\text{High} - \text{High}\,d\text{Low}}{\text{Denominator}^2}$
- Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Higher Order Derivatives

- A second derivative is the derivative of the first derivative.
- A third derivative is the derivative of the second derivative.
- And so on....

Second Derivatives

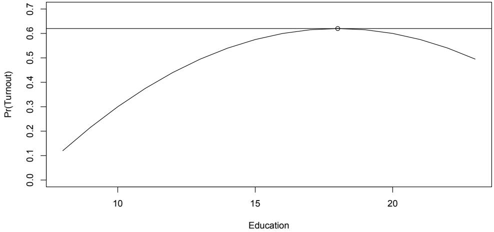
- The first derivative tells us whether the formula is increasing or decreasing.
- The second derivative tells us whether the derivative is increasing or decreasing.
- This also tells us the concavity of the function
- This helps us determine the maxima and minima of the function.

Critical Points

- What are critical points?
  - Maximums
  - Minimums
  - Inflection Points (change in concavity)
- How to find critical points?
  - The derivative
- How do we know what the critical points are?
  - The second derivative

Critical Points

- How do we find the critical points?
  - The derivative tells us the slope of a tangent line.
  - The slopes of tangents at maxima or minima are 0



Critical Points

- Let's consider this with our turnout function: $Pr(Turnout) = -.005Ed^2 + .18Ed - 1$
- The derivative is $-.01Ed + .18$
- Set the derivative equal to 0
  - $-.01Ed + .18 = 0$
  - $Ed = 18$
- So there is a critical point at $Ed = 18$
- How do we know if it is a maximum or a minimum (without looking at the plot)?
  - The second derivative
  - The second derivative is -.01
  - If a second derivative at the value of the critical point is negative, the concavity is negative (concave down), so the point is a maximum. If it is positive, the function is concave up, so the point is a minimum.
  - If the second derivative is a scalar, the function is always either concave up or concave down, so the critical point is a global maximum or minimum
  - $Ed = 18$ is a global maximum

Inflection Points

- When a function changes concavity, this is called an inflection point.
- Inflection points occur where the $f''(x) = 0$.
- When $f''(x) = 0$, there is no information on the concavity.
- Example: $f(x) = x^3$
  - $f'(x) = 3x^2$
  - $3x^2 = 0$; $x = 0$ 0 is a critical point. Is it a maximum or minimum?
  - $f''(x) = 6x$
  - $f''(0) = 0$
  - Since $f''(0) = 0$, $x = 0$ is an inflection point.

Helpful Derivatives

- $f(x) = e^x$; $f'(x) = e^x$
- $f(x) = c^x$; $f'(x) = ln(c)c^x$
- $f(x) = ln(x)$; $f'(x) = \frac{1}{x}$
- $f(x) = log_n(x)$; $f'(x) = \frac{1}{xln(n)}$
- $f(x) = sin(x)$; $f'(x) = cos(x)$
- $f(x) = cos(x)$; $f'(x) = -sin(x)$
- $f(x) = tan(x)$; $f'(x) = sec^2(x)$

Multiple Variables

- What happens when a function has more than one variable?
- $Pr(Vote) = Age + Education + Gender + Income$
- How do we find the derivative?
  - Partial differentiation

Partial Differentiation

- Example: $\frac{\partial f}{\partial x} x^3 + 4x^2 + 12y^2 + 3y + 8$
  - Take the derivative with respect to x.
  - $\frac{\partial f}{\partial x}(x^3 + 4x^2) + \frac{\partial f}{\partial x}(12y^2 + 3y + 8)$
  - The first part we already know: $\frac{\partial f}{\partial x}(x^3 + 4x^2) = 3x^2 + 8x$
  - What about the second part?
    - Take derivative with respect to x
    - Go back to definition: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
    - $\lim_{h \to 0} \frac{(12y^2+3y+8) - (12y^2+3y+8)}{h} = 0$
    - So the derivative is $\frac{\partial f}{\partial x} x^3 + 4x^2 + 12y^2 + 3y + 8 = 3x^2 + 8x$

Partial Differentiation

- Higher order derivatives.
- Perform derivation based on order in the denominator
- Example:
  - $f = x^3 + 4x^2 + 4x^2y + 12y^2 + 3xy + 12y + 8$
  - $\frac{\partial f}{\partial x} = 3x^2 + 8x + 8xy + 3$
  - $\frac{\partial f}{\partial y} = 4x^2 + 24y + 3x + 12$
  - $\frac{\partial^2 f}{\partial x^2} = 6x + 8 + 8y$
  - $\frac{\partial^2 f}{\partial y^2} = 24$
  - $\frac{\partial^2 f}{\partial x \partial y} = 8x$
- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ so long as the second partial is continuous

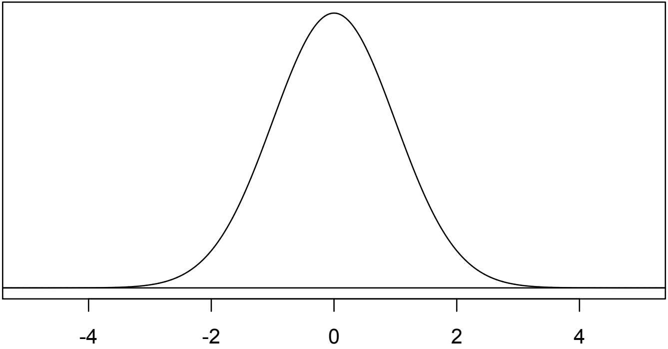
Partial Differentiation

- This becomes helpful when considering effects in statistical models
- Consider this model from Rees & Schultz (1970):
$$\begin{aligned} Income = \; & \beta_0 + \beta_1(Seniority) + \beta_2(Education) \\ & + \beta_3(Experience) + \beta_4(Training) + \beta_5(Commute) \\ & + \beta_6(Age) + \beta_7(Age^2) \end{aligned}$$
- To determine how income is changing based on how age is changing, take derivative with respect to age:

$$\frac{\partial}{\partial Age} Income = \beta_6 + 2\beta_7(Age) \tag{1}$$

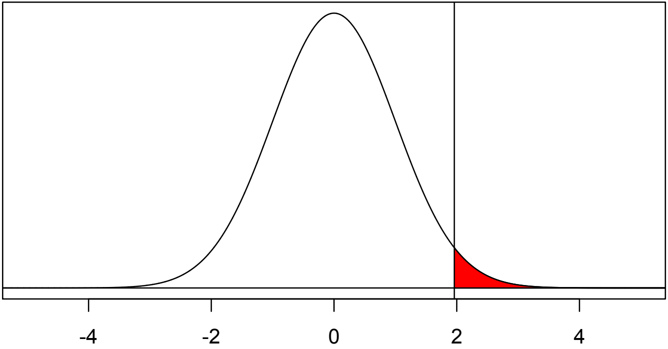
Area Under Curve

Take the standard normal curve: $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



Area Under Curve

What if we want to know the area below the curve when $x \geq 1.96$



Integrals

- The way to find the area under a curve is to take the integral
- What is an integral?
  - The area under the curve $f(x)$ for some range of $x = (a, b)$ is defined as "the definite integral for f from a to b".
  - $\int_a^b f(x)dx$

Integrals and Derivatives

The integral is the inverse function of the derivative:

$$F(x) = \int_a^b f(x)dx$$

$$f(x) = \frac{d}{dx}F(x)$$

So, for any x,

$$F(x) = \int_a^b f(x)dx = \int_a^b \frac{d}{dx}F(x) = F(x)$$

and

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^b f(x)dx = f(x)$$

F(x) is the *antiderivative* of $f(x)$. Then,

$$\frac{d}{dx}\int_a^b f(x)dx = F(b) - F(a)$$

The Antiderivative

$$F(x) = \int f(x)dx$$

Examples:

- $\int 3x^2 dx = x^3 + c$
- $\int sin(x)dx = -cos(x) + c$
- $\int e^x dx = e^x + c$
- $\int (x^2 - 4)dx = \frac{1}{3}x^3 - 4x + c$

Indefinite Integral

- The antiderivative, also called the indefinite integral, is not unique.
- There are multiple antiderivates for each function (for each c).
- This shifts the curve up or down the y-axis
- With information, such as a point the function passes through, you can solve for c.

Definite Integral

- The definite integral, again, is the defined by a specific range:
  - The area under the curve $f(x)$ for some range of $x = (a, b)$ is defined as the definite integral from a to b.
- Since this is determined by taking $F(b) - F(a)$, the constant will be cancelled out in the calculation.

Rules of Integration

- $\int af(x)dx = a\int f(x)dx$
- $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
- $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$
- $\int e^x dx = e^x + c$
- $\int \frac{1}{x}dx = \ln x + c$
- $\int e^{f(x)}f'(x)dx = e^{f(x)} + c$
- $\int [f(x)]^n f'(x)dx = \frac{1}{n+1}[f(x)]^{n+1} + c$
- $\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$

Example

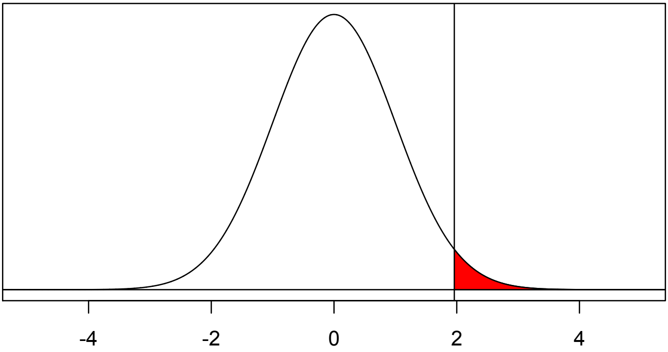
$$\begin{aligned}
\int_1^3 5x^4 + 4x \quad &= 5(\tfrac{1}{5})x^5 + 4(\tfrac{1}{2}x^2)|_1^3 \\
&= ((3)^5 + 2(3)^2) - ((1)^5 + 2(1)^2) \\
&= 261 - 3 \\
&= 258
\end{aligned}$$

Properties of Definite Integrals

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Integral of Normal Distribution

Back to our normal distribution that started this off:



Integral of Normal Distribution

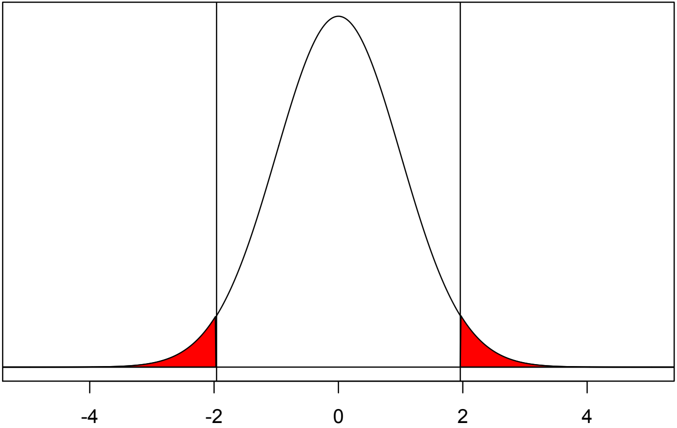
To find the area, need to integrate the function:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\int_{1.96}^{\infty} f(x) = \int_{1.96}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 0.025$$

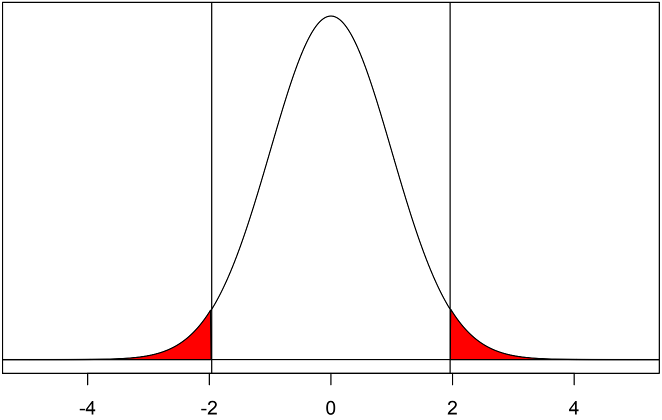
Integral of Normal Distribution

The area under the curve when $x \geq 1.96$ is 0.025. The function is symmetric, so the area when $x \leq -1.96$ is also 0.025. Together it is 0.05



So What?

What is the significance of 0.05? P-value!



P-values are integrals under the normal curve.

Integration by Substitution

Say you are presented with a somewhat complicated integral:

$$\int 3x^2\sqrt{x+3}dx$$

If it were $\sqrt{x}$, this would be easy. By substituting, we can make the two easy to multiply. How?

- Set u equal to $x+3$. Then $x = u-3$, $du = dx$
- $\int (3*(u-3)^2)\sqrt{u}du$
- Why is this better? Multiply, create polynomial and solve for antiderivative.
- $\int (3u^2 - 18u + 27)u^{\frac{1}{2}}du$
- $\int (3u^{\frac{5}{2}} - 18u^{\frac{3}{2}} + 27u^{\frac{1}{2}})du$
- $3*\frac{2}{7}u^{\frac{7}{2}} - 18*\frac{2}{5}u^{\frac{5}{2}} + 27*\frac{2}{3}u^{\frac{3}{2}} + c$
- $\frac{6}{7}u^{\frac{7}{2}} - \frac{36}{5}u^{\frac{5}{2}} + 18u^{\frac{3}{2}} + c$
- $\frac{6}{7}(x+3)^{\frac{7}{2}} - \frac{36}{5}(x+3)^{\frac{5}{2}} + 18(x+3)^{\frac{3}{2}} + c$

Integration by Substitution

Say you are presented with a somewhat complicated integral:

$$\int 3x^2\sqrt{x+3}dx$$

Alternatively:

- Set u equal to $\sqrt{x+3}$. Then $x = u^2 - 3$, $dx = 2udu$
- $\int (3 * (u^2 - 3)^2) u2udu$
- $\int (3u^4 - 18u^2 + 27) 2u^2 du$
- $\int (6u^6 - 36u^4 + 54u^2) du$
- $\frac{6}{7}u^7 - \frac{36}{5}u^5 + 18u^3 + c$
- $\frac{6}{7}\sqrt{x+3}^7 - \frac{36}{5}\sqrt{x+3}^5 + 18\sqrt{x+3}^3 + c$
- $\frac{6}{7}(x+3)^{\frac{7}{2}} - \frac{36}{5}(x+3)^{\frac{5}{2}} + 18(x+3)^{\frac{3}{2}} + c$

Integration by Parts

- The reverse of the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- If we integrate,

$$\int uvdx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

- If we rearrange

$$\int u\frac{dv}{dx}dx = uv + \int v\frac{du}{dx}dx$$

- To integrate, find values for u and dv that make it easy to integrate.

Integration by Parts

Take $\int xe^{5x}dx$

- Let $u = x$ and $dv = e^{5x}dx$
- So $du = dx$ and $v = \frac{1}{5}e^{5x}$
- If we plug everything into the integration by parts formula, we get.

$$\begin{aligned}
\int u\frac{dv}{dx}dx &= uv + \int v\frac{du}{dx}dx \\
\int x^2e^{5x}dx &= \frac{1}{5}xe^{5x} + \int \frac{1}{5}e^{5x}dx \\
&= \frac{1}{5}xe^{5x} + \frac{1}{25}e^{5x} + c
\end{aligned}$$

Multiple Integrals

- What happens if you want to know the area in multiple dimensions?
- Multiple integrals.

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy \right] dx$$

Multiple Integral Example

Consider

$$\begin{aligned}\int_3^6 \int_0^4 4x^3 y dy dx &= \int_3^6 2x^3 [y^2|_3^6] dx \\ &= \int_3^6 2x^3 [(6)^2 - (3)^2] dx \\ &= \int_3^6 54x^3 dx \\ &= \frac{54}{4} x^4 |_0^4 \\ &= \frac{54}{4} [(4)^4 - (0)^4] \\ &= 3,456\end{aligned}$$