[320] Complexity + Big O

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Outline

Performance and Complexity

What is a step?

Counting Executed Steps

Big O: for functions/curves

Big O: for algorithms

Performance vs. Complexity

Things that affect performance (total time to run):

- ????

Performance vs. Complexity

Things that affect performance (total time to run):

- speed of the computer (CPU, etc)
- speed of Python (quality+efficiency of interpretation)
- **algorithm:** strategy for solving the problem
- **input size:** how much data do we have?

Performance vs. Complexity

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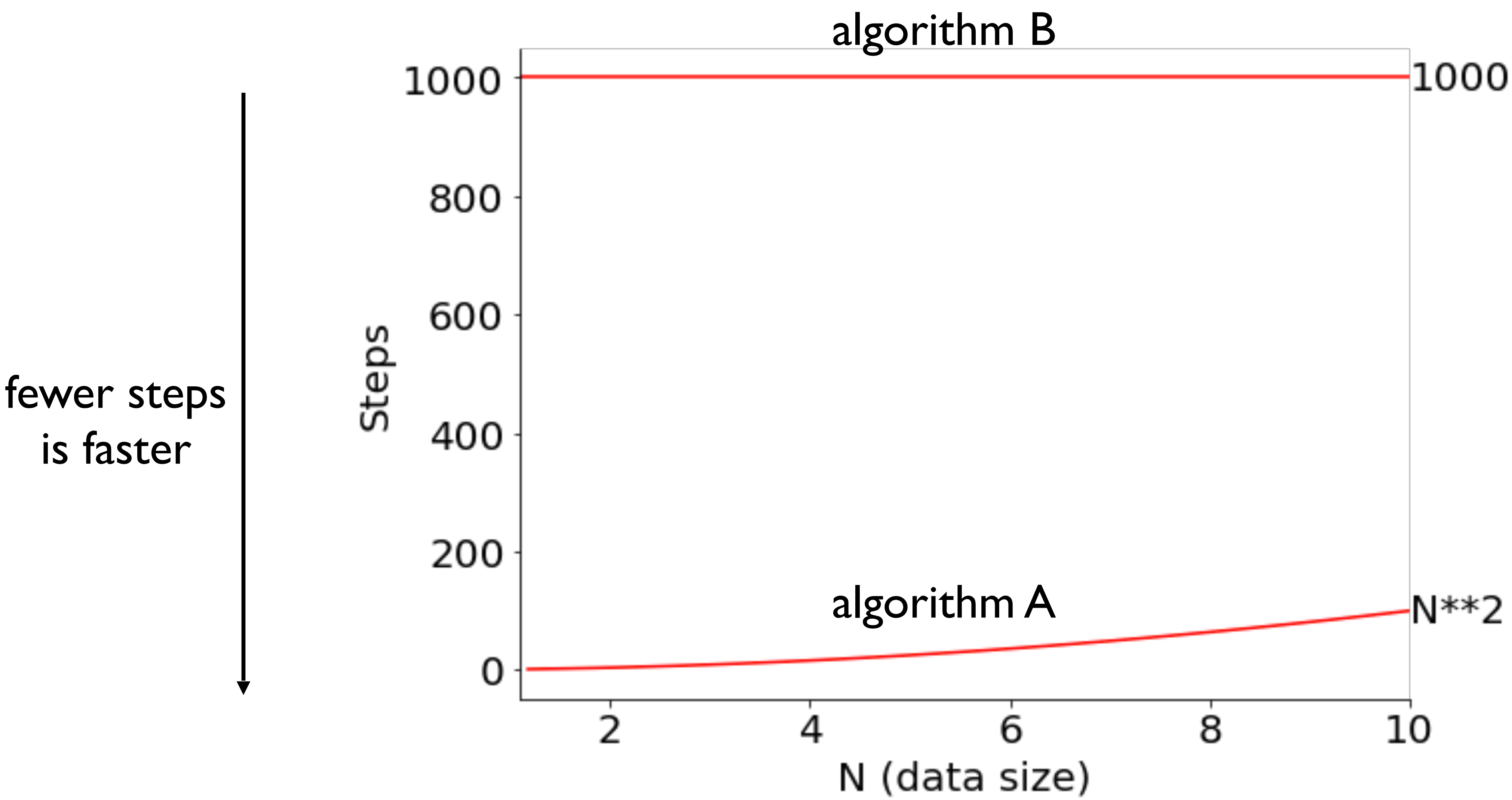
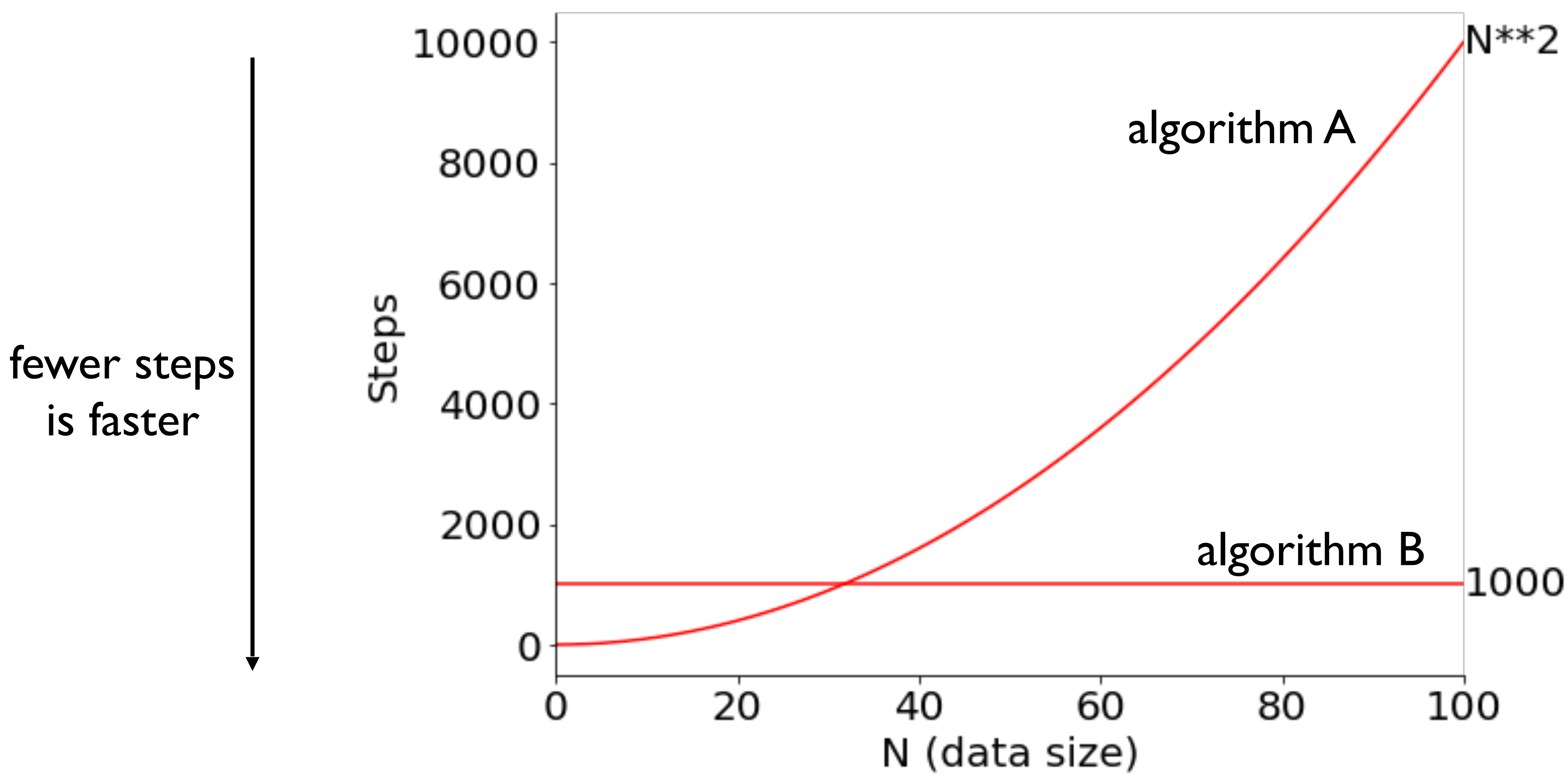
complexity analysis: how many steps must the algorithm perform, as a function of input size?

Which algorithm is better?



Do you prefer A or B?

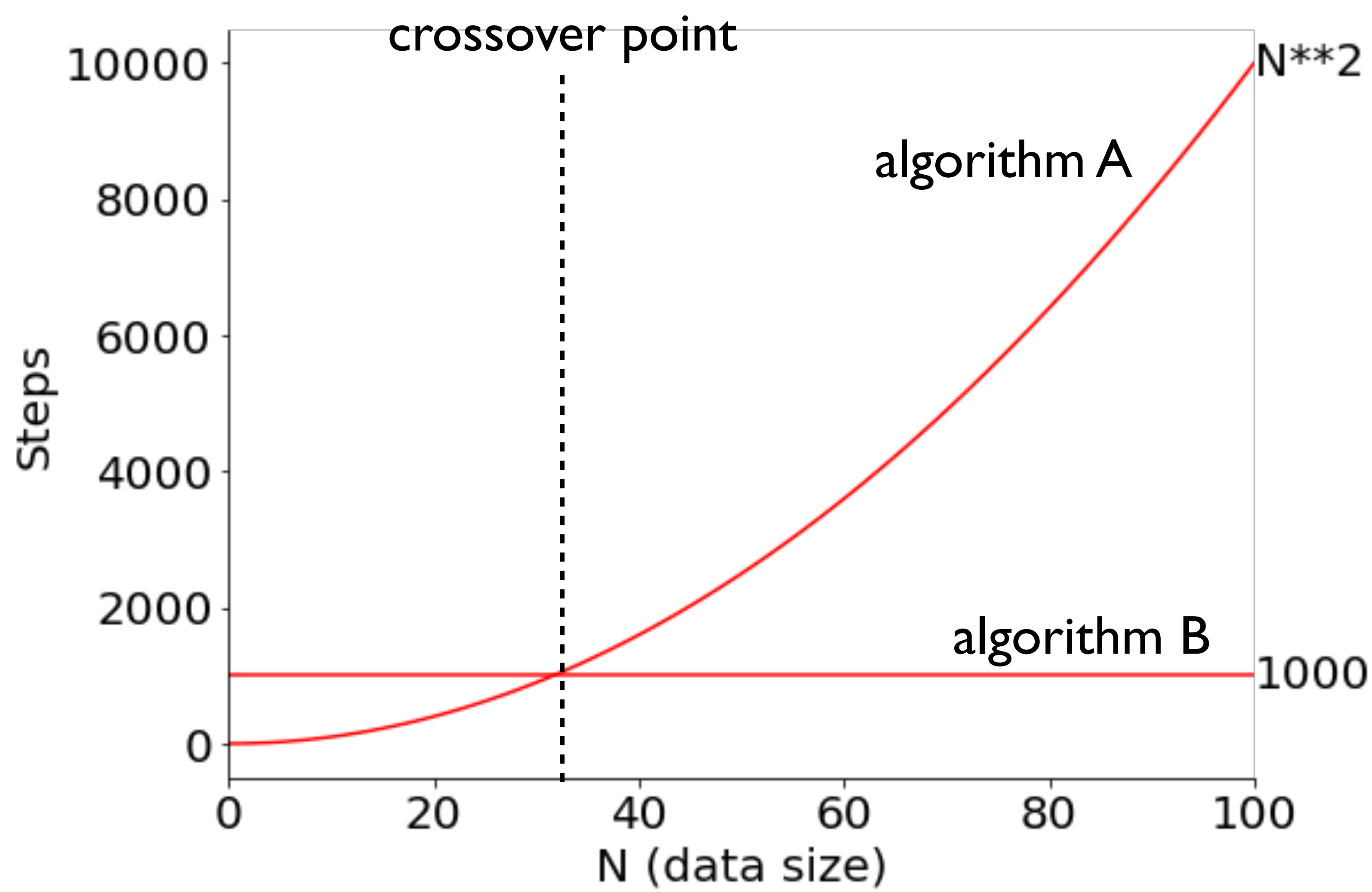
Which algorithm is better?



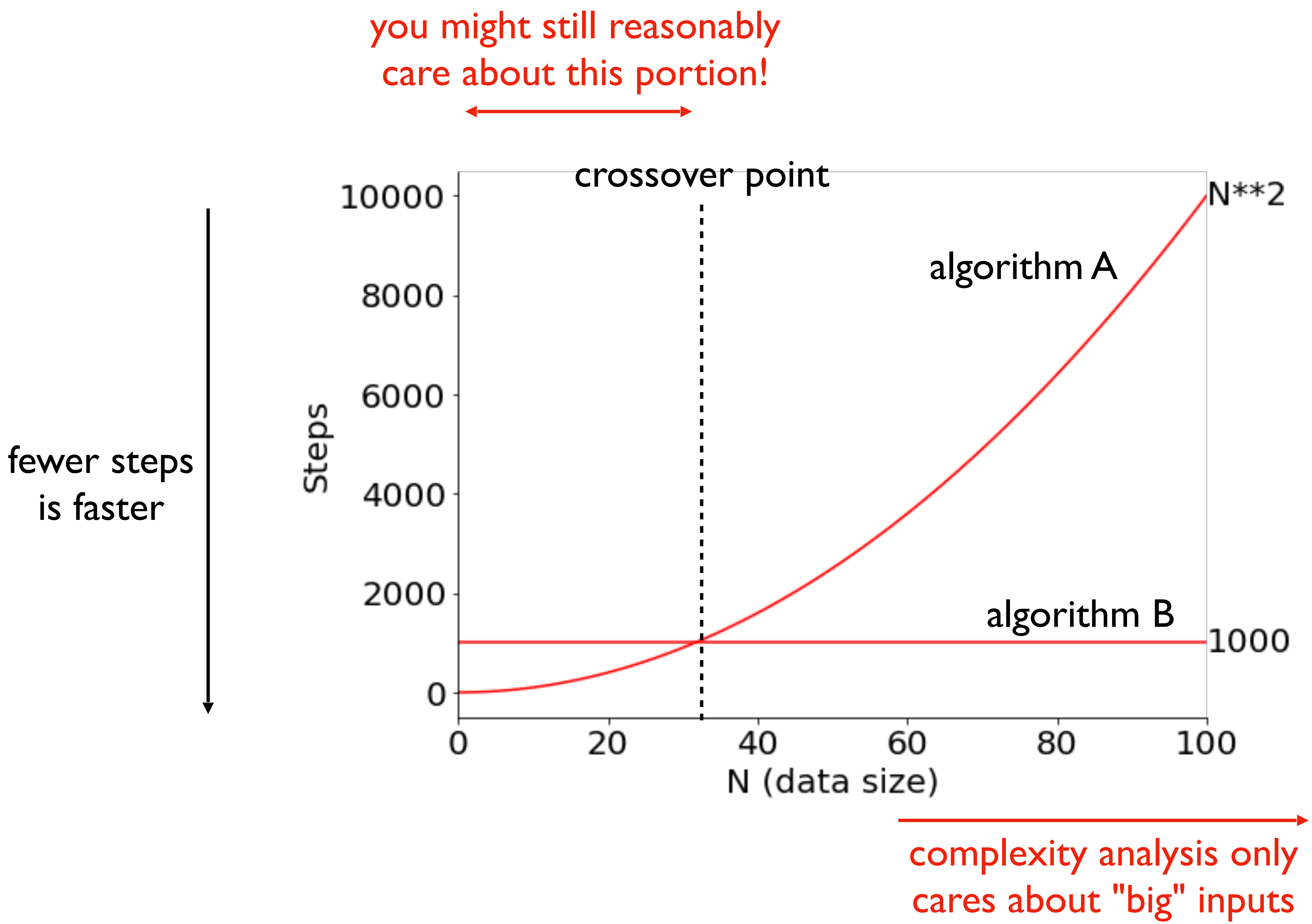
Do you prefer A or B?

Which algorithm is better?

fewer steps
is faster



Which algorithm is better?



What is the asymptotic behavior of the function?

Performance vs. Complexity

Things that affect performance (total time to run):

- speed of the computer (CPU, etc)
- speed of Python (quality+efficiency of interpretation)
- **algorithm**: strategy for solving the problem
- **input size:** how much data do we have?

what is this?

complexity analysis: how many **steps** must the algorithm perform, as a function of input size?

Outline

Performance and Complexity

What is a step?

Counting Executed Steps

Big O: for functions/curves

Big O: for algorithms

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

input size is length of this list

```
input_nums = [2, 3, ...]

STEP   odd_count = 0
STEP   odd_sum = 0
STEP   for num in input_nums:
STEP       if num % 2 == 1:
STEP           odd_count += 1
STEP           odd_sum += num
STEP   odd_avg = odd_sum
STEP   odd_avg /= odd_count
```

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]
```

STEP	`odd_count = 0` `odd_sum = 0`
STEP	`for num in input_nums:`
STEP	`    if num % 2 == 1:`
STEP	`        odd_count += 1` `        odd_sum += num`
STEP	`odd_avg = odd_sum` `odd_avg /= odd_count`

also a valid breakdown into steps

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

	`input_nums = [2, 3, ...]`
STEP	`odd_count = 0` `odd_sum = 0`
STEP	`for num in input_nums:`
STEP	`    if num % 2 == 1:`
STEP	`        odd_count += 1` `        odd_sum += num`
STEP	`odd_avg = odd_sum / odd_count`

One line can do a lot, so no reason to
have lines and steps be equivalent

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

	`input_nums = [2, 3, ...]`
STEP	`odd_count = 0` `odd_sum = 0`
STEP	`for num in input_nums:`
STEP	`    if num % 2 == 1:`
STEP	`        odd_count += 1` `        odd_sum += num`
STEP	`odd_avg = odd_sum / odd_count`

Sometimes a single line is not a single step:

```
found = X in L
```

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]
```

STEP

```
odd_count = 0
odd_sum = 0
```

STEP

```
for num in input_nums:
```

STEP

```
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
```

STEP

```
odd_avg = odd_sum / odd_count
```

???

What is a step?

"bounded" doesn't mean "fixed"

A **step** is any unit of work with bounded execution time (it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]
```

STEP

```
odd_count = 0
odd_sum = 0
```

STEP

```
for num in input_nums:
```

STEP

```
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
```

STEP

```
odd_avg = odd_sum / odd_count
```

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]
```

STEP

```
odd_count = 0
odd_sum = 0
```

STEP
(whole loop execution,
not one pass through)

```
for num in input_nums:
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
```

STEP

```
odd_avg = odd_sum / odd_count
```

???

is this a valid way to identify steps?

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]
```

STEP

```
odd_count = 0
odd_sum = 0
```

not a "step", because exec time depends on input size

STEP
(whole loop execution, not one pass through)

```
for num in input_nums:
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
```

STEP

```
odd_avg = odd_sum / odd_count
```

What is a step?

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]
```

STEP

```
odd_count = 0
odd_sum = 0
```

not a "step", because exec time depends on input size

STEP (whole loop execution, not one pass through)

```
for num in input_nums:
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
```

STEP

```
odd_avg = odd_sum / odd_count
```

Note! A loop that iterates a bounded number of times (not proportional to input size) COULD be a single step.

Outline

Performance and Complexity

What is a step?

Counting Executed Steps

Big O: for functions/curves

Big O: for algorithms

Counting Executed Steps

A **step** is any unit of work with bounded execution time (it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]

odd_count = 0
odd_sum = 0
for num in input_nums:
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
odd_avg = odd_sum / odd_count
```

STEP (odd_count = 0, odd_sum = 0)

STEP (for num in input_nums:)

STEP (if num % 2 == 1: ... odd_sum += num)

STEP (odd_avg = odd_sum / odd_count)

How many total steps will **execute** if `len(input_nums) == 10`?

Counting Executed Steps

A **step** is any unit of work with bounded execution time (it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]

odd_count = 0
odd_sum = 0
for num in input_nums:
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
odd_avg = odd_sum / odd_count
```

1 STEP

+ 11 STEP

+ 10 STEP

+ 1 STEP

= 23 steps

For **N** elements, there will be **2*N+3** steps

Counting Executed Steps

A **step** is any unit of work with bounded execution time (it doesn't keep getting slower with growing input size)

```
            input_nums = [2, 3, ...]

?   STEP    odd_count = 0
?   STEP    odd_sum = 0
?   STEP    for num in input_nums:
?   STEP        if num % 2 == 1:
?   STEP            odd_count += 1
?   STEP            odd_sum += num
?   STEP    odd_avg = odd_sum
?   STEP    odd_avg /= odd_count
```

How many total steps will **execute** if `len(input_nums) == 10`?

Counting Executed Steps

A **step** is any unit of work with bounded execution time (it doesn't keep getting slower with growing input size)

```
                      input_nums = [2, 3, ...]

1        STEP  odd_count = 0
1        STEP  odd_sum = 0
11       STEP  for num in input_nums:
10       STEP      if num % 2 == 1:
0 to 10  STEP          odd_count += 1
0 to 10  STEP          odd_sum += num
1        STEP  odd_avg = odd_sum
1        STEP  odd_avg /= odd_count
```

How many total steps will **execute** if `len(input_nums) == 10`?

Counting Executed Steps

A **step** is any unit of work with bounded execution time (it doesn't keep getting slower with growing input size)

```
input_nums = [2, 3, ...]

      1      STEP  odd_count = 0
    + 1      STEP  odd_sum = 0
    + 11     STEP  for num in input_nums:
    + 10     STEP      if num % 2 == 1:
+ 0 to 10    STEP          odd_count += 1
+ 0 to 10    STEP          odd_sum += num
    + 1      STEP  odd_avg = odd_sum
    + 1      STEP  odd_avg /= odd_count
```

For **N** elements, there will be between **2*N+5** and **4*N+5** steps

Counting Executed Steps

A **step** is any unit of work with bounded execution time
(it doesn't keep getting slower with growing input size)

```python
input_nums = [2, 3, ...]
```

1	STEP	`odd_count = 0`
+ 1	STEP	`odd_sum = 0`
+ 11	STEP	`for num in input_nums:`
+ 10	STEP	`    if num % 2 == 1:`
+ 0 to 10	STEP	`        odd_count += 1`
+ 0 to 10	STEP	`        odd_sum += num`
+ 1	STEP	`odd_avg = odd_sum`
+ 1	STEP	`odd_avg /= odd_count`

For **N** elements, there will be between ~~2*N+5~~ and **4*N+5** steps

usually we care about the worst case

Counting Executed Steps

A **step** is any unit of work with bounded execution time (it doesn't keep getting slower with growing input size)

2*N+3 answer 1 **OR** **4*N+5** answer 2

Answer 2 is never bigger than 2 times answer 1.
Answer 1 is never bigger than answer 2.

Important: we might not identify steps the same, but our execution counts can at most differ by a <u>constant</u> factor! → **can we broadly (but rigorously) categorize based on this?**

Outline

Performance and Complexity

What is a step?

Counting Executed Steps

Big O: for functions/curves

Big O: for algorithms

How fast?

Documentation

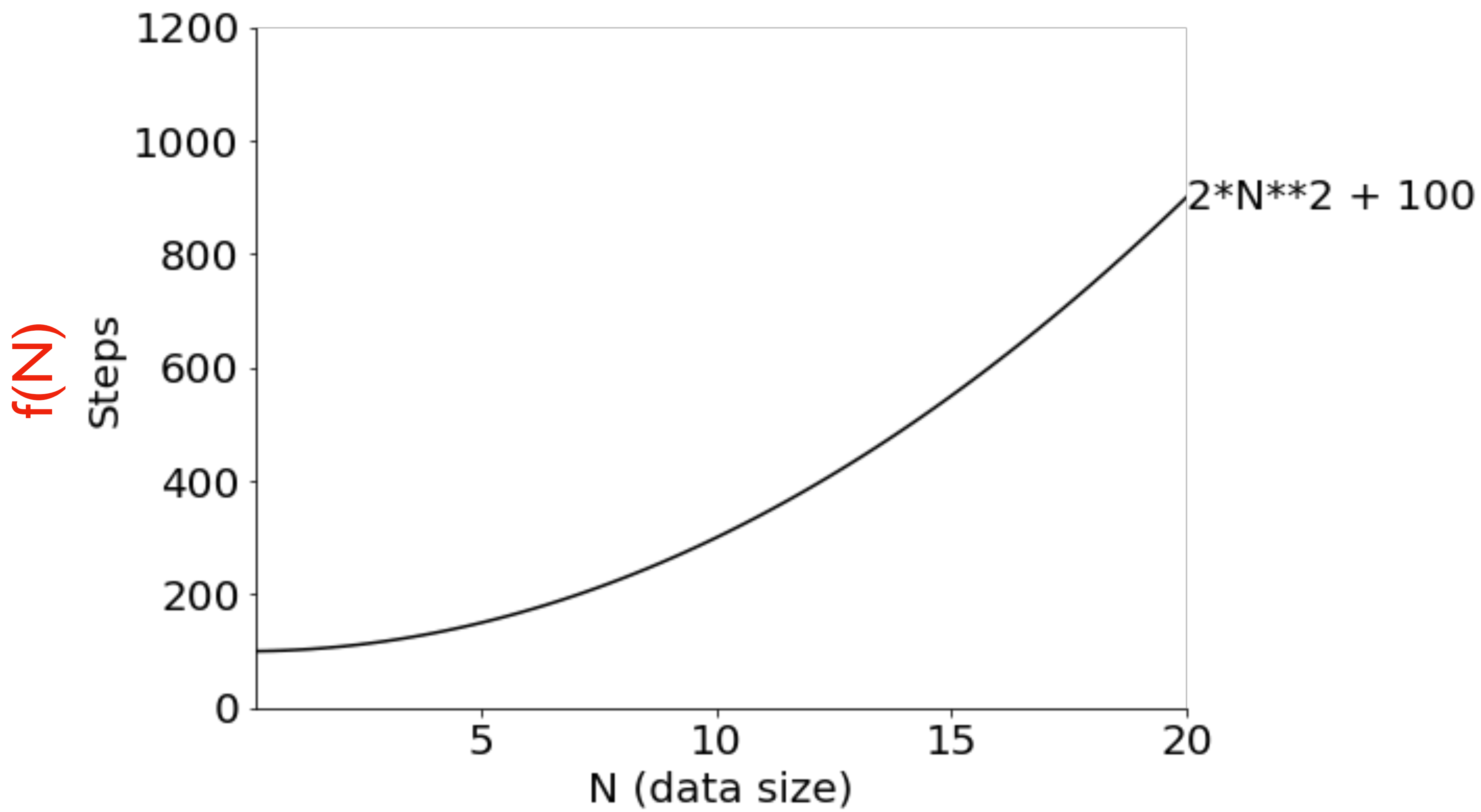
- https://scikit-learn.org/stable/modules/linear_model.html#ordinary-least-squares-complexity

- https://scikit-learn.org/stable/modules/tree.html#complexity

Big O Notation ("O" is for "order of growth")

Goal: categorize functions (and algorithms) by how fast they **grow**

- **do not care** about scale
- **do not care** about small inputs
- **care** about shape of the curve
- **strategy:** find some multiple of a general function that is an upper bound



$f(N) == 2N^2 + 100$

is an $O(N^2)$ function

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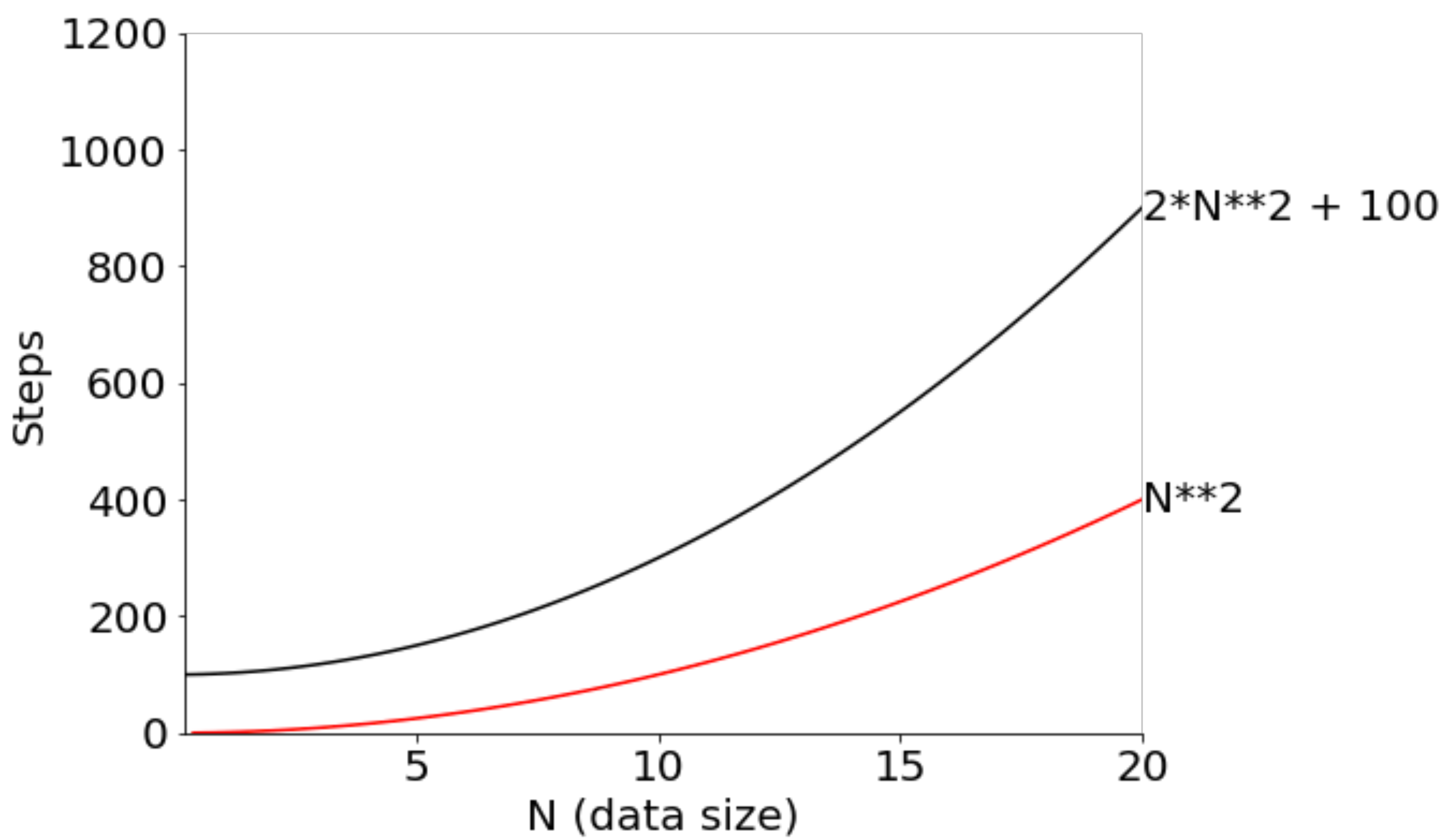

$f(N) == 2N^2 + 100$
is an $O(N^2)$ function

not because N^2
is an upper bound

Big O Notation ("O" is for "order of growth")

Goal: categorize functions (and algorithms) by how fast they **grow**

- **do not care** about scale
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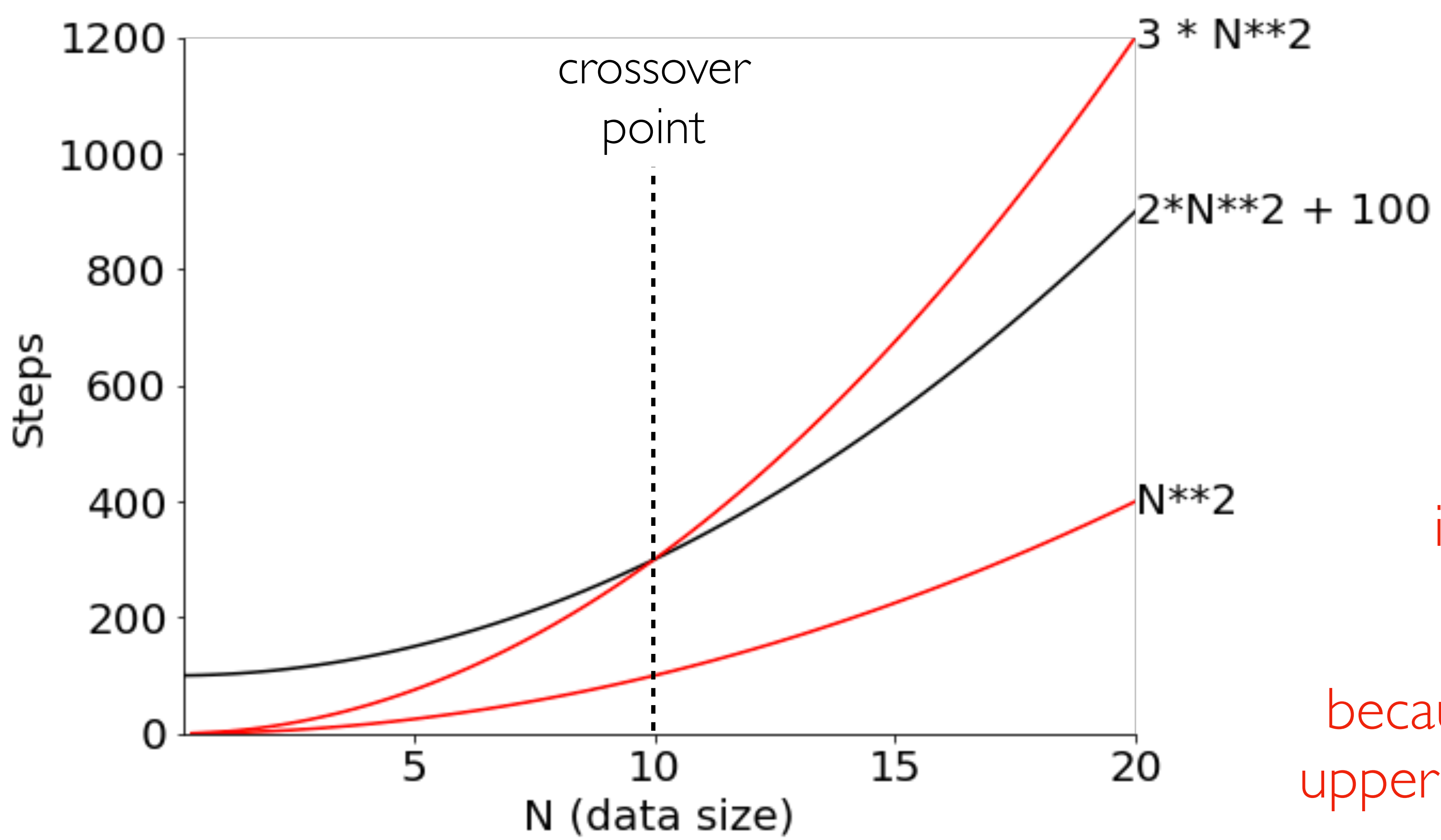


$f(N) == 2N^2 + 100$
is an **$O(N^2)$** function

not because **N^2**
is an upper bound

because **some multiple** is an
upper bound after **some point**

Defining Big O

care about shape of the curve

do not care about small inputs

do not care about scale

If $f(N) \leq C * g(N)$ *for large N values and some fixed* *<u>constant</u>* *C*

Then $f(N) \in O(g(N))$

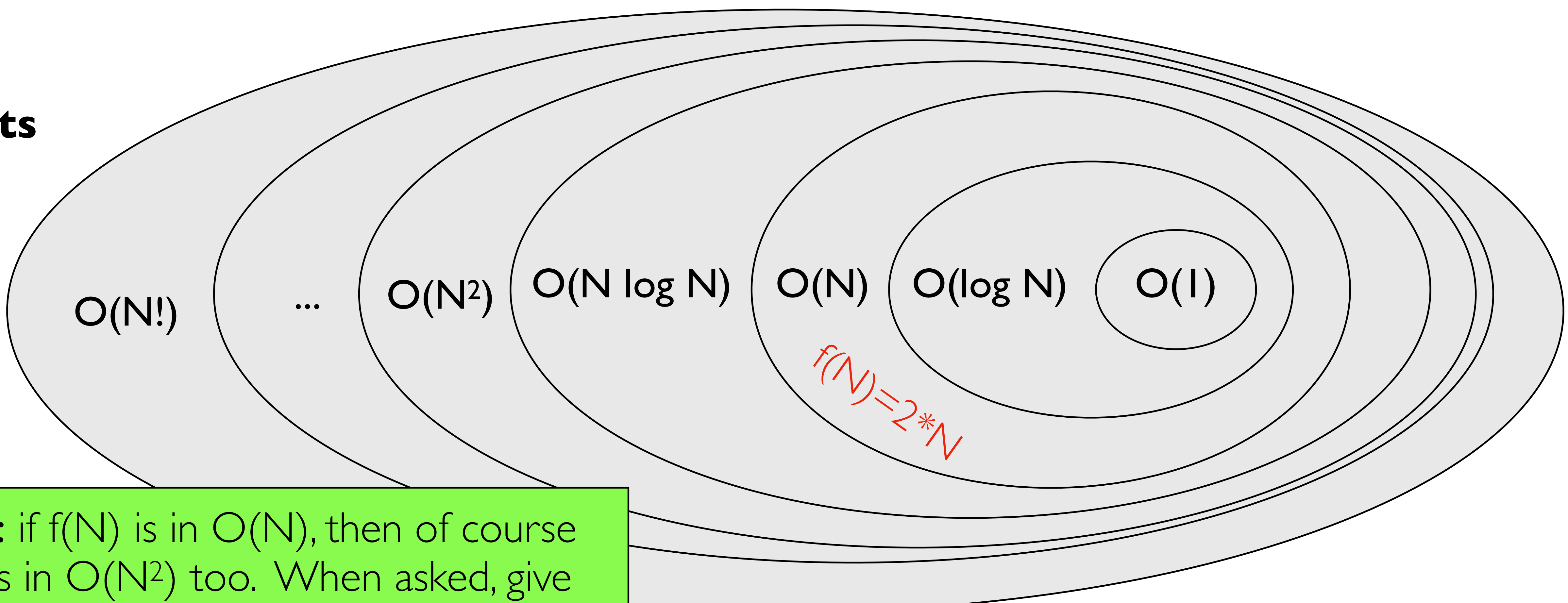
Defining Big O

care about shape of the curve | do not care about small inputs | do not care about scale

If $f(N) \leq C * g(N)$ *for large N values and some fixed* *<u>constant</u>* *C*

Then $f(N) \in O(g(N))$

Sets



Note: if f(N) is in O(N), then of course f(N) is in $O(N^2)$ too. When asked, give the most informative answer.

Defining Big O

If $f(N) \le C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

---

which ones
are true?

$f(N) = 2N \in O(N)$

$f(N) = 100N \in O(N^2)$

$f(N) = N^2 \in O(1000000N)$

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = 2N \in O(N)$

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = 2N \in O(N)$

$f(N) = 2N$ and $g(N) = N$

Defining Big O

If $f(N) \le C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = 2N \in O(N)$

$f(N) = 2N$ and $g(N) = N$ $2N \le C*N$, for some constant C

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = 2N \in O(N)$

$f(N) = 2N$ and $g(N) = N$

$2N \leq C*N$, for some constant C

Since N is a positive number, $2N \leq 3*N$, that is, $C = 3$.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = 2N \in O(N)$

$f(N) = 2N$ and $g(N) = N$
$2N \leq C*N$, for some constant C
Since N is a positive number, $2N \leq 3*N$, that is, $C = 3$.
Therefore, $f(N) = 2N \in O(N)$

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed* *constant* *C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Defining Big O

If $f(N) \le C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$ Option 1. Choose constant $C = 100$ that satisfies the inequality for $N \ge 1$.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed* *constant* *C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Option 1. Choose constant $C = 100$ that satisfies the inequality for $N \geq 1$.

$100\ N \leq 100\ N^2$.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Option 1. Choose constant $C = 100$ that satisfies the inequality for $N \geq 1$.

$100\ N \leq 100\ N^2$. Therefore, $(N) = 100N \in O(N^2)$.

Defining Big O

If $f(N) \le C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Option 2. Choose constant $C = 1$, but choose $N \ge 100$ to satisfy the inequality
$100\ N \le 1 * N^2$

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed* <u>*constant*</u> *C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Option 2. Choose constant $C = 1$, but choose $N \geq 100$ to satisfy the inequality $100\ N \leq 1 * N^2$.

That is, $100 \leq N$, by cancelling N on both sides.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Option 2. Choose constant $C = 1$, but choose $N \geq 100$ to satisfy the inequality $100\ N \leq 1 * N^2$.

That is, $100 \leq N$, by cancelling N on both sides.

Therefore, $100\ N \leq 1 * N^2$ for $N \geq 100$.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Option 2. Choose constant $C = 1$, but choose $N \geq 100$ to satisfy the inequality $100\ N \leq 1 * N^2$.

That is, $100 \leq N$, by cancelling N on both sides.

Therefore, $100\ N \leq 1 * N^2$ for $N \geq 100$.

Hence, $f(N) = 100N \in O(N^2)$.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = 100N \in O(N^2)$

$f(N) = 100\ N$ and $g(N) = C*N^2$

Option 1. Choose constant $C = 100$ that satisfies the inequality for $N \geq 1$.

$100\ N \leq 100\ N^2$. Therefore, $(N) = 100N \in O(N^2)$.

Option 2. Choose constant $C = 1$, but choose $N \geq 100$ to satisfy the inequality $100\ N \leq 1 * N^2$.

That is, $100 \leq N$, by cancelling N on both sides.
Therefore, $100\ N \leq 1 * N^2$ for $N \geq 100$.

Hence, $f(N) = 100N \in O(N^2)$.

Defining Big O

If $f(N) \le C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = N^2 \in O(1000000N)$

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = N^2 \in O(1000000N)$

Suppose $N^2 \in O(1000000)$.

Defining Big O

If $f(N) \le C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = N^2 \in O(1000000N)$

Suppose $N^2 \in O(1000000)$.

Therefore, there exists a constant C, such that

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = N^2 \in O(1000000N)$

Suppose $N^2 \in O(1000000)$.

Therefore, there exists a constant C, such that

$N^2 \leq C*1000000$ for large N values.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed* *<u>constant</u>* *C*

Then $f(N) \in O(g(N))$

$f(N) = N^2 \in O(1000000N)$

Suppose $N^2 \in O(1000000)$.

Therefore, there exists a constant C, such that

$N^2 \leq C*1000000$ for large N values.

Thus $N^2 \leq C*1000000$ implies N is a fixed number.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

$f(N) = N^2 \in O(1000000N)$

Suppose $N^2 \in O(1000000)$.

Therefore, there exists a constant C, such that

$N^2 \leq C*1000000$ for large N values.

Thus $N^2 \leq C*1000000$ implies N is a fixed number.

However, N is a natural number, there we arrived at a contradiction.

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

$f(N) = N^2 \in O(1000000N)$

Suppose $N^2 \in O(1000000)$.

Therefore, there exists a constant C, such that

$N^2 \leq C*1000000$ for large N values.

Thus $N^2 \leq C*1000000$ implies N is a fixed number.

However, N is a natural number, there we arrived at a contradiction.

Hence, our supposition is wrong, that is, $N^2 \in O(1000000)$ is not true.

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If $f(N) \leq C * g(N)$ *for large N values and some fixed constant C*

Then $f(N) \in O(g(N))$

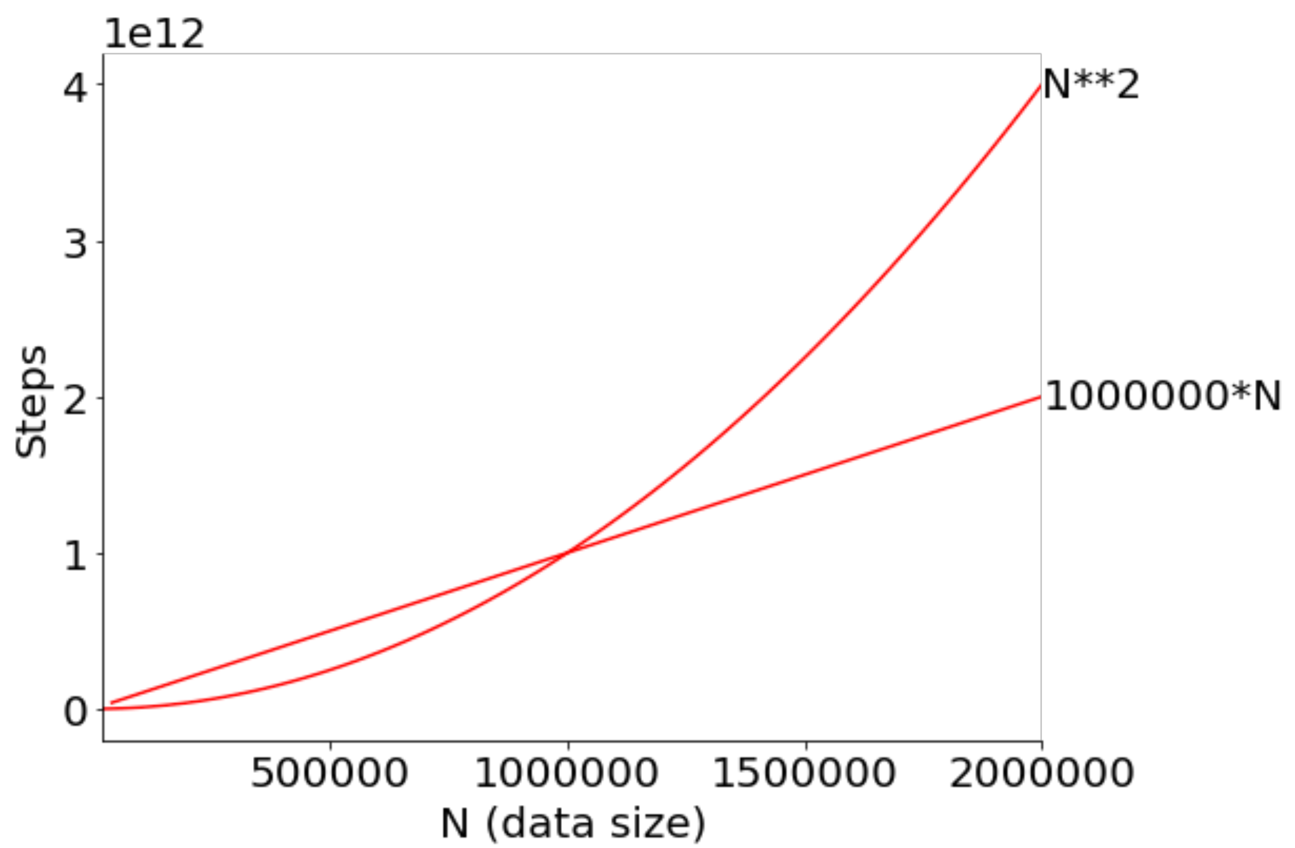
---

which ones are true?

$f(N) = 2N \in O(N)$

$f(N) = 100N \in O(N^2)$

$\cancel{f(N) = N^2 \in O(1000000N)}$



Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

---

shortcuts

$O(3N^5 + N^4 + 8N^3 + 3N^2 + N + 5)$

keep leading term
(if finite number)

$O(3N^5)$

drop coefficients

$O(N^5)$

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What is a step?

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Big O: for functions/curves

Big O: for algorithms

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

We'll let $f(N)$ be the number of steps that some **Algorithm A** needs to perform for input size **N**.

When we say **Algorithm A** $\in$ $O(g(N))$, we mean that $f(N) \in O(g(N))$

Defining Big O

If $f(N) \leq C * g(N)$ *for large N values and some fixed <u>constant</u> C*

Then $f(N) \in O(g(N))$

STEP
STEP
STEP
STEP

```
odd_count = 0
odd_sum = 0
for num in input_nums:
    if num % 2 == 1:
        odd_count += 1
        odd_sum += num
odd_avg = odd_sum / odd_count
```

For N elements, there will be 2*N+3 steps

$2*N+3 \leq 3 * N$

[for big N values]

therefore

this code is O(N)

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Then $f(N) \in O(g(N))$

```
STEP   odd_count = 0
STEP   odd_sum = 0
STEP   for num in input_nums:
STEP       if num % 2 == 1:
STEP           odd_count += 1
STEP           odd_sum += num
STEP   odd_avg = odd_sum
STEP   odd_avg /= odd_count
```

For N elements, there will be between 2*N+5 and 4*N+5 steps

$4*N+5 \leq 5 * N$

[for big N values]

therefore

this code is O(N)

Analysis of Algorithms: Key Ideas

complexity: relationship between input size and steps executed

step: an operation of bounded cost (doesn't scale with input size)

asymptotic analysis: we only care about very large N values for complexity (for example, assume a big list)

worst-case: we'll usually assume the worst arrangement of data because it's harder to do an average case analysis (for example, assume search target at the end of a list)

big O: if $f(N) \leq C * g(N)$ for large N values and some fixed constant **C**,
then $f(N) \in O(g(N))$