

Suppose that X_1, X_2, \dots, X_n is a random sample from the exponential distribution with rate $\lambda > 0$.

Derive a **uniformly most powerful** hypothesis test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

(Was $H_1 : \lambda = \lambda_1$ for $\lambda_1 > \lambda_0$)

The **uniformly most powerful** test of size α for testing

$$H_0 : \theta \in \Theta_0 \quad \text{vs.} \quad H_1 : \theta \in \Theta \setminus \Theta_0$$

is a test defined by a rejection region R^* such that

1. It has size α .

$$\text{i.e.} \quad \max_{\theta \in \Theta_0} P(\vec{X} \in R^*; \theta) = \alpha$$

The **uniformly most powerful** test of size α for testing

$$H_0 : \theta \in \Theta_0 \quad \text{vs.} \quad H_1 : \theta \in \Theta \setminus \Theta_0$$

is a test defined by a rejection region R^* such that

2. It has higher power for all $\theta \in \Theta \setminus \Theta_0$

$$\text{i.e. } \gamma_{R^*}(\theta) \geq \gamma_R(\theta) \text{ for all } \theta \in \Theta \setminus \Theta_0$$

$$\text{i.e. } P(\vec{X} \in R^*; \theta) \geq P(\vec{X} \in R; \theta)$$

$$\text{for all } \theta \in \Theta \setminus \Theta_0$$

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Derive a **uniformly most powerful** hypothesis test of size α for

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Step One:

Consider the simple versus simple hypotheses

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

for some fixed $\lambda_1 > \lambda_0$.

Steps Two, Three, and Four:

Find the best test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

for some fixed $\lambda_1 > \lambda_0$.

This test is to reject H_0 , in favor of H_1 if

$$\bar{X} < \frac{\chi^2_{1-\alpha, 2n}}{2n\lambda_0}$$

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Note that this test does not depend on the particular value of λ_1 .

*** It does, however, depend on the fact that $\lambda_1 > \lambda_0$.

“Reject H_0 , in favor of H_1 , if

$$\left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0 - \lambda_1) \sum_{i=1}^n X_i} \leq c$$

⋮

$$-(\lambda_0 - \lambda_1) \sum_{i=1}^n X_i \leq c_1$$

$$\sum_{i=1}^n X_i \leq c_2$$

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$$-(\lambda_0 - \lambda_1) \sum_{i=1}^n X_i \leq c_1$$

$$\sum_{i=1}^n X_i \geq c_2$$

if $\lambda_1 < \lambda_0$

The best (most powerful) test of

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

for $\lambda_1 > \lambda_0$ is to reject H_0 , in favor of H_1 if

$$\bar{X} < \frac{\chi^2_{1-\alpha, 2n}}{2n\lambda_0}$$

Note that this test does not depend on the particular value of λ_1 as long as $\lambda_1 > \lambda_0$.

It is the uniformly most powerful (best) test for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

The “UMP” test for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

is to reject H_0 , in favor of H_1 if

$$\bar{X} < \frac{\chi^2_{1-\alpha, 2n}}{2n\lambda_0}$$

The “UMP” test for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda < \lambda_0$$

is to reject H_0 , in favor of H_1 if

$$\bar{X} > \frac{\chi^2_{\alpha, 2n}}{2n\lambda_0}$$

Does there exist a “UMP” test for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda \neq \lambda_0 \quad ?$$

Answer: No!

For any $\lambda_1 \neq \lambda_0$,

- The best test if $\lambda_1 > \lambda_0$ is to reject H_0 if

$$\bar{X} < \frac{\chi^2_{1-\alpha, 2n}}{2n\lambda_0}$$

- The best test if $\lambda_1 < \lambda_0$ is to reject H_0 if

$$\bar{X} > \frac{\chi^2_{\alpha, 2n}}{2n\lambda_0}$$

There is no one best test that we can use for all $\lambda_1 \neq \lambda_0$!