

A Few Continuous Distributions:

- the normal distribution

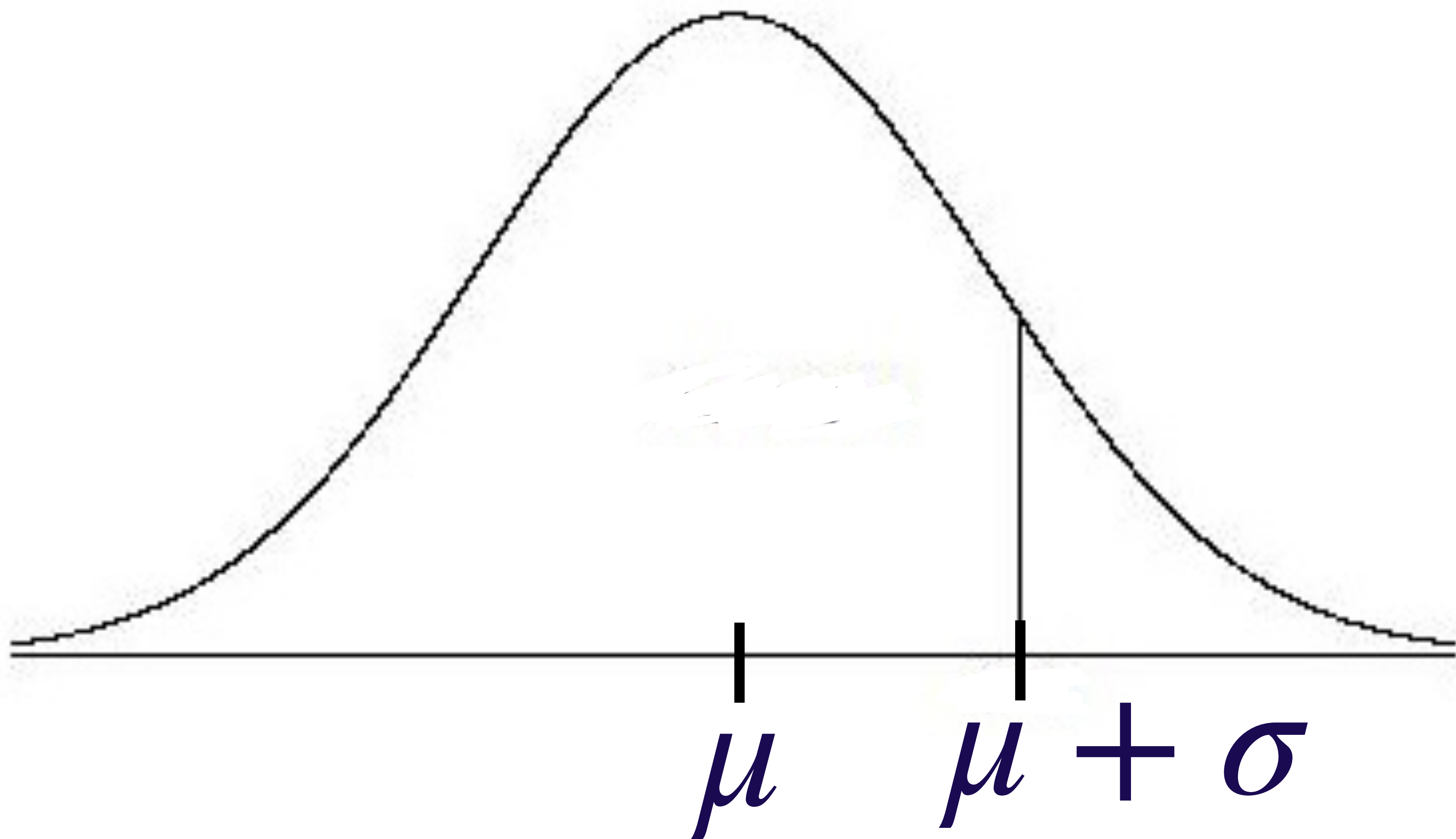
Two Parameters:

- Mean : $-\infty < \mu < \infty$
- Variance: $\sigma^2 > 0$

The Probability Density Function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

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$$X \sim N(\mu, \sigma^2)$$

A Few Continuous Distributions:

- the exponential distribution

One Parameter:

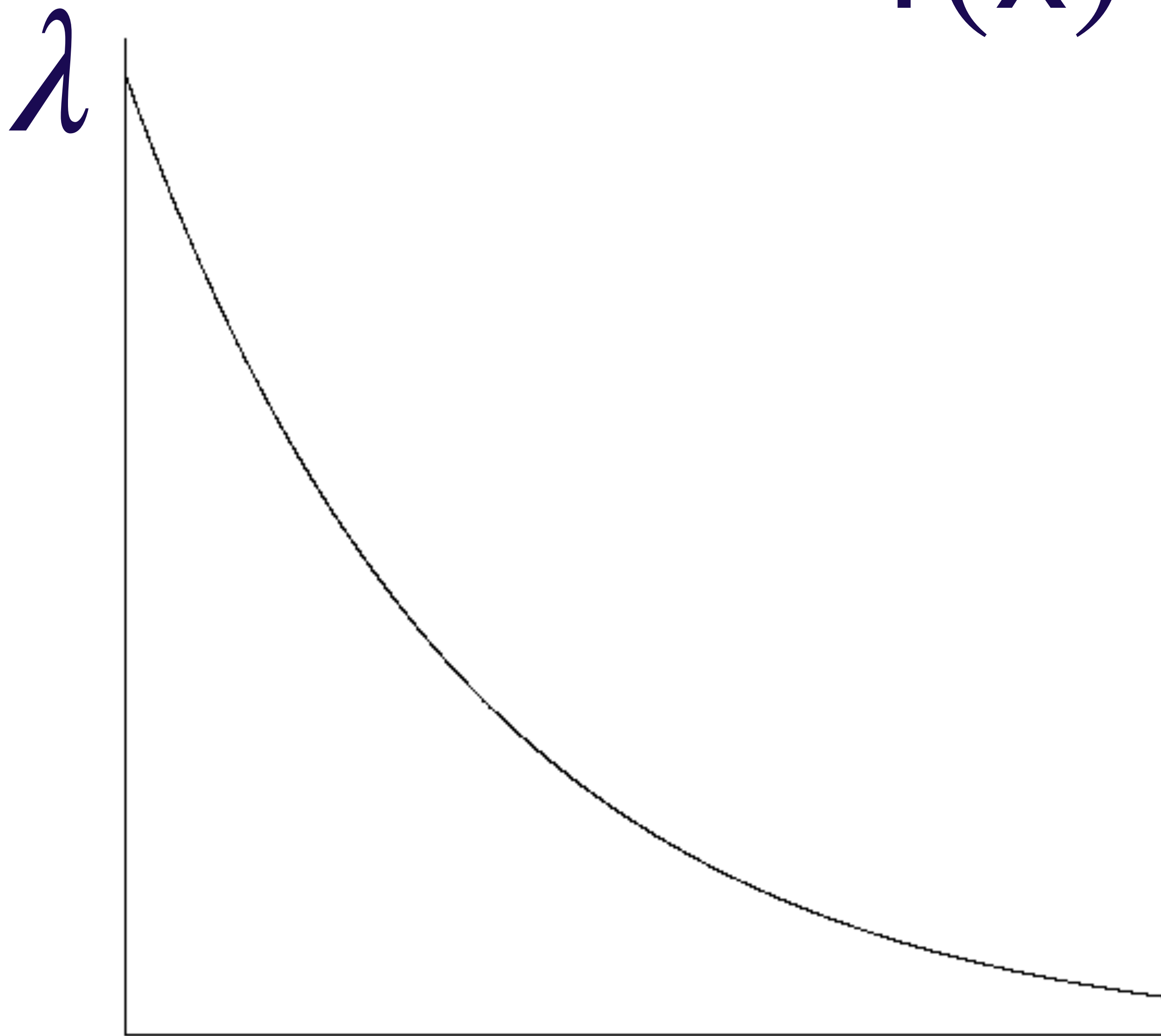
- Rate: $\lambda > 0$

The Probability Density Function:

$$f(x) = \lambda e^{-\lambda x}$$

for $x > 0$

$$f(x) = \lambda e^{-\lambda x}$$



- the exponential distribution

- mean

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \frac{1}{\lambda}$$

- the exponential distribution

- Variance

$$\begin{aligned}\sigma^2 = \text{Var}[X] &= E[(X - \mu)^2] \\ &= E[X^2] - (E[X])^2 \\ &= \dots = \frac{1}{\lambda^2}\end{aligned}$$

- the exponential distribution

$$f(x) = \lambda e^{-\lambda x}$$

If we reparameterize this as

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda}$$

the mean would be λ .

People write $X \sim \exp(\lambda)$.

- the exponential distribution

Our notation

$$X \sim \text{exp}(\text{rate} = \lambda)$$

$$f(x) = \lambda e^{-\lambda x}$$

$$X \sim \text{exp}(\text{mean} = \lambda)$$

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda}$$

A Few Continuous Distributions:

- the gamma distribution

Two Parameters:

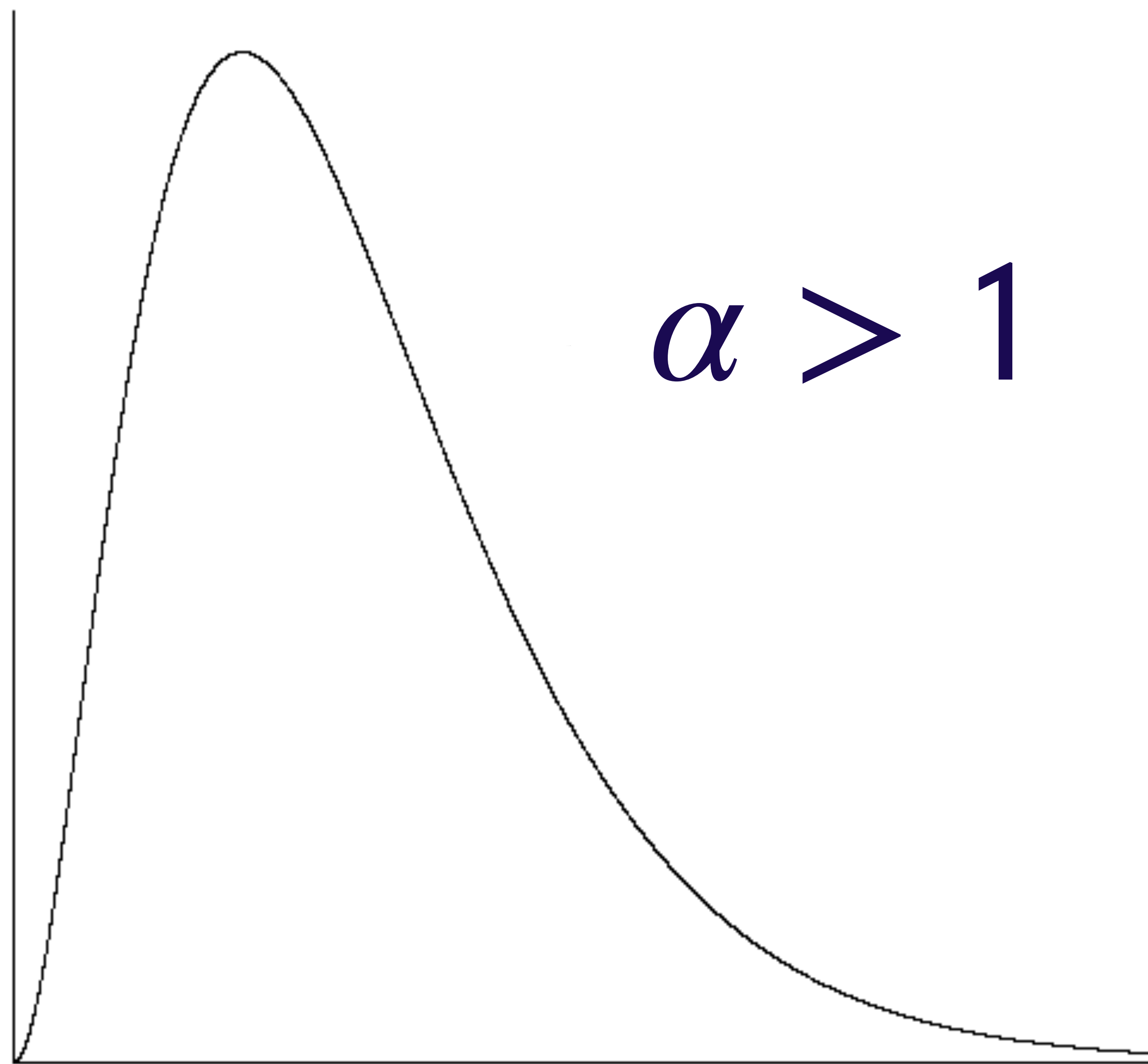
- Shape: $\alpha > 0$
- Inverse Scale: $\beta > 0$

The Probability Density Function:

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x}$$

for $x > 0$

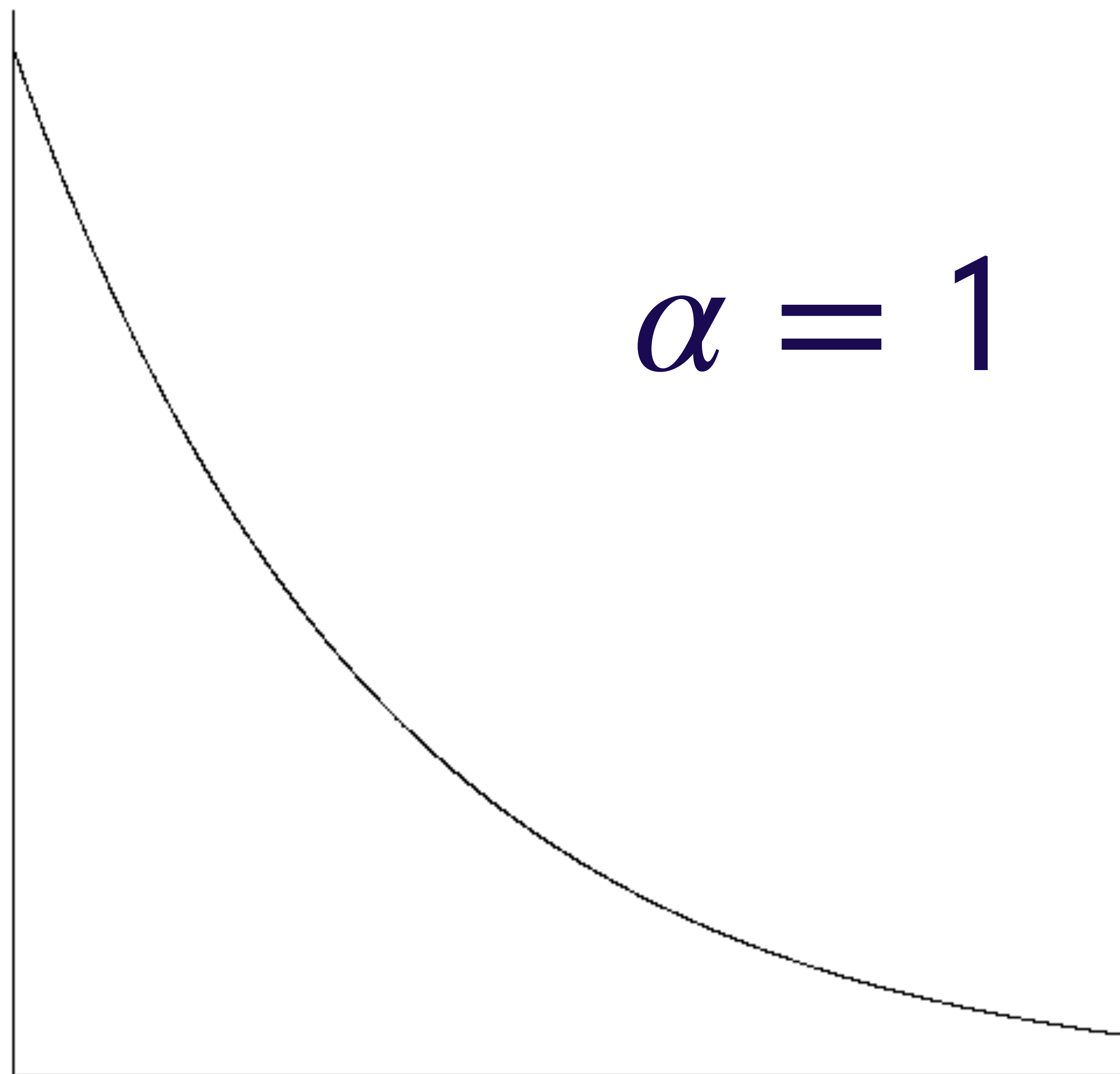
$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \quad \text{for } x > 0$$



$$\alpha > 1$$

$$X \sim \Gamma(\alpha, \beta)$$

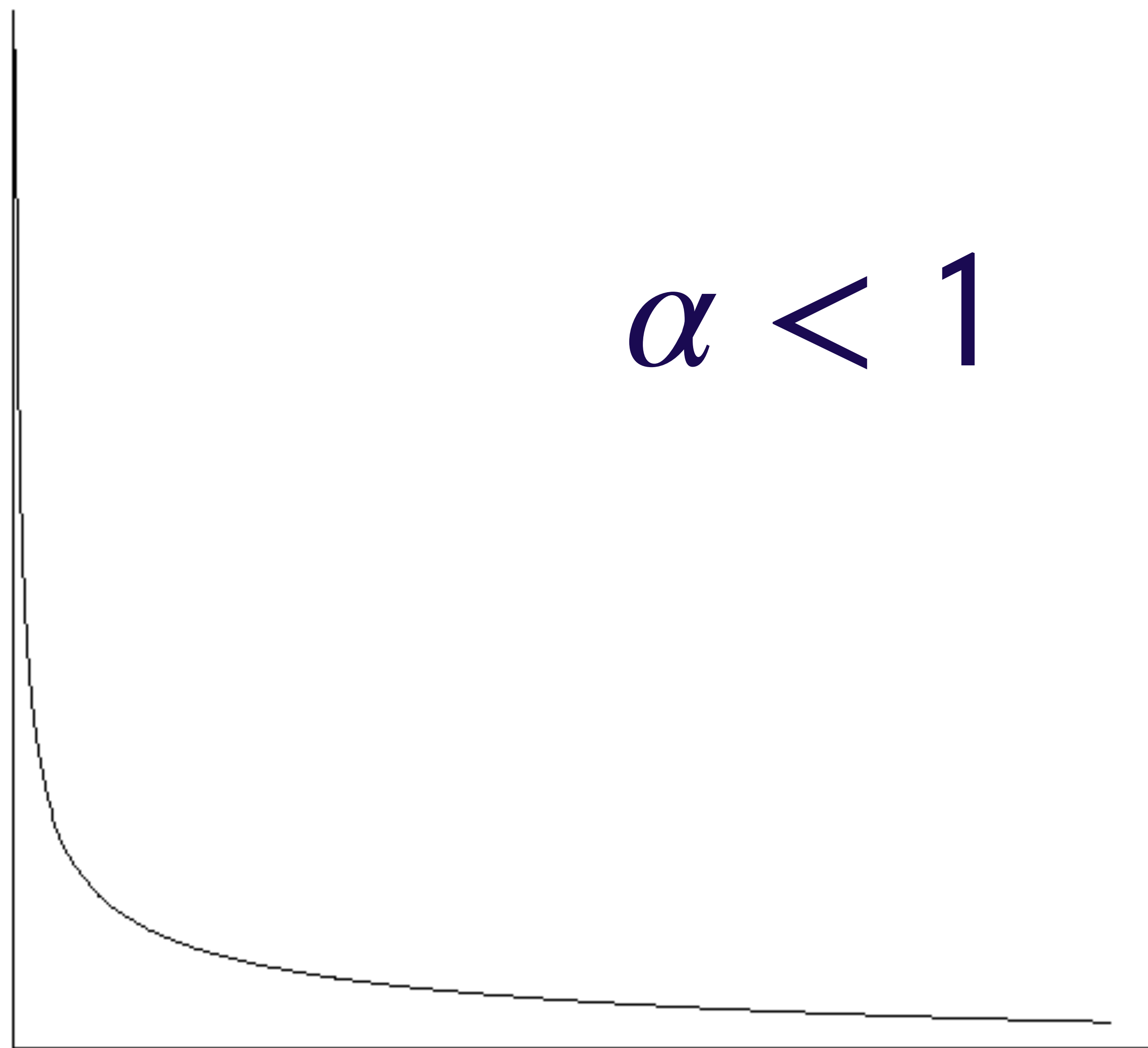
$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \quad \text{for } x > 0$$



$$\alpha = 1$$

$$X \sim \Gamma(\alpha, \beta)$$

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \quad \text{for } x > 0$$



$$X \sim \Gamma(\alpha, \beta)$$

The Gamma Function:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

Properties:

- $\Gamma(1) = 1$
- $\Gamma(\alpha) = (\alpha - 1) \Gamma(\alpha - 1)$ for $\alpha > 1$
- $\Gamma(n) = (n - 1)!$ for an integer
 $n \geq 1$

$$X \sim \Gamma(\alpha, \beta)$$

- mean

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{\alpha}{\beta}$$

$$\int_0^{\infty} x \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} \beta^{\alpha} \underbrace{x^{\alpha} e^{-\beta x}}_{\text{looks like}} dx$$

looks like
 $\Gamma(\alpha + 1, \beta)$ pdf

$$= \frac{1}{\beta} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \int_0^{\infty} \underbrace{\frac{1}{\Gamma(\alpha + 1)} \beta^{\alpha+1} x^{\alpha} e^{-\beta x}}_{\text{is a}} dx$$

is a
 $\Gamma(\alpha + 1, \beta)$ pdf

$$\mu = E[X] = \int_0^{\infty} x \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{1}{\beta} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} = \frac{1}{\beta} \frac{\alpha \Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \frac{\alpha}{\beta}$$



$$X \sim \Gamma(\alpha, \beta)$$

- **Variance**

$$\begin{aligned}\sigma^2 = \text{Var}[X] &= \text{E}[(X - \mu)^2] \\ &= \text{E}[X^2] - (\text{E}[X])^2 \\ &= \dots = \frac{\alpha}{\beta^2}\end{aligned}$$

A Few Continuous Distributions:

- the chi-squared distribution

One Parameter:

- degrees of freedom: $n \geq 1$
(n is an integer)

$$X \sim \chi^2(n)$$

is defined as $\Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

$$X \sim \chi^2(n)$$

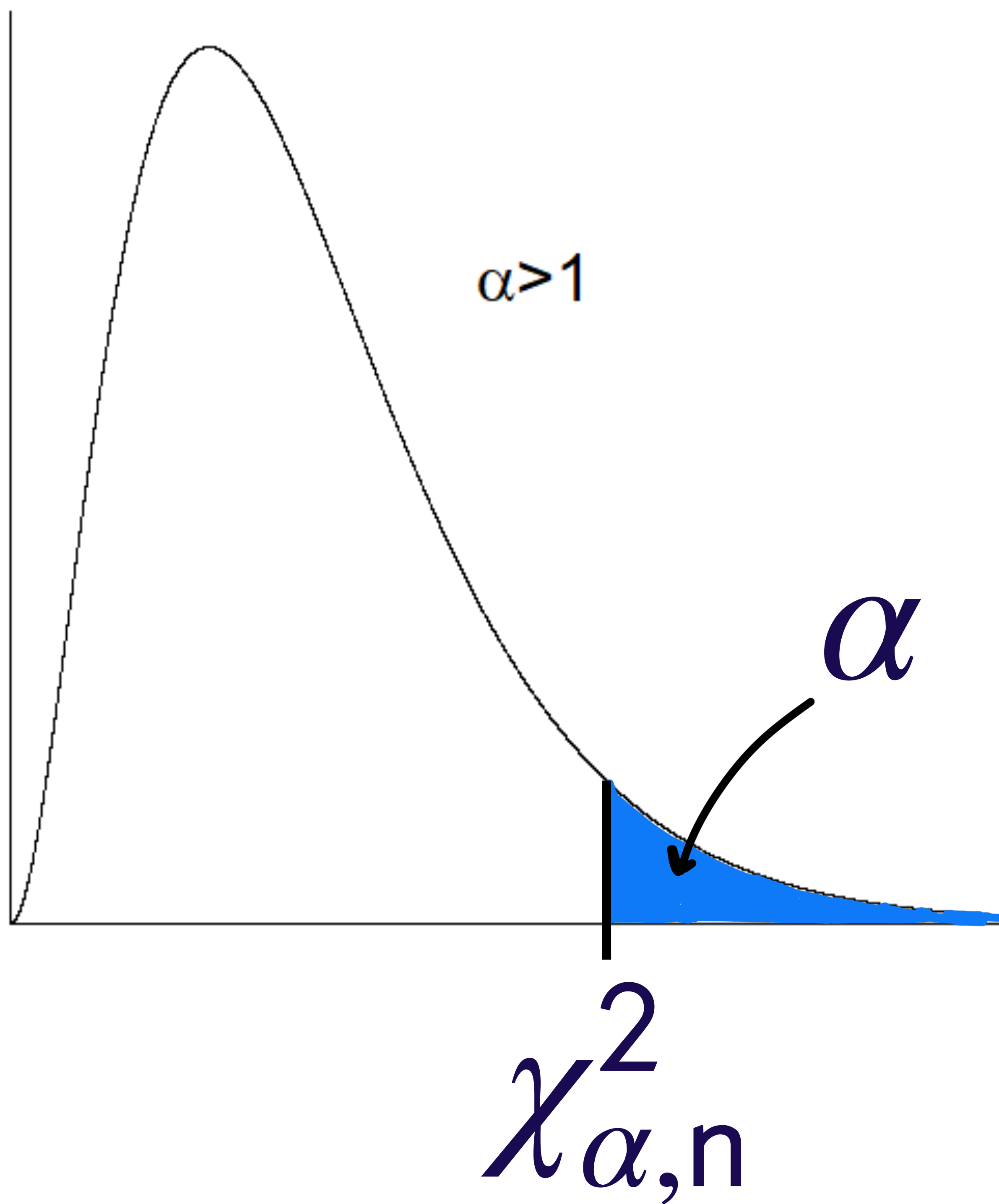
- mean

$$\mu = E[X] = n$$

- variance

$$\sigma^2 = \text{Var}[X] = 2n$$

$$X \sim \chi^2(n)$$



A Few Continuous Distributions:

- the t-distribution

Let $Z \sim N(0, 1)$ and $W \sim \chi^2(n)$ be independent random variables.

Define

$$T = \frac{Z}{\sqrt{W/n}}$$

then T has pdf...

A Few Continuous Distributions:

- the t-distribution

Write $X \sim t(n)$

One Parameter:

- degrees of freedom: $n \geq 1$

(n is an integer)

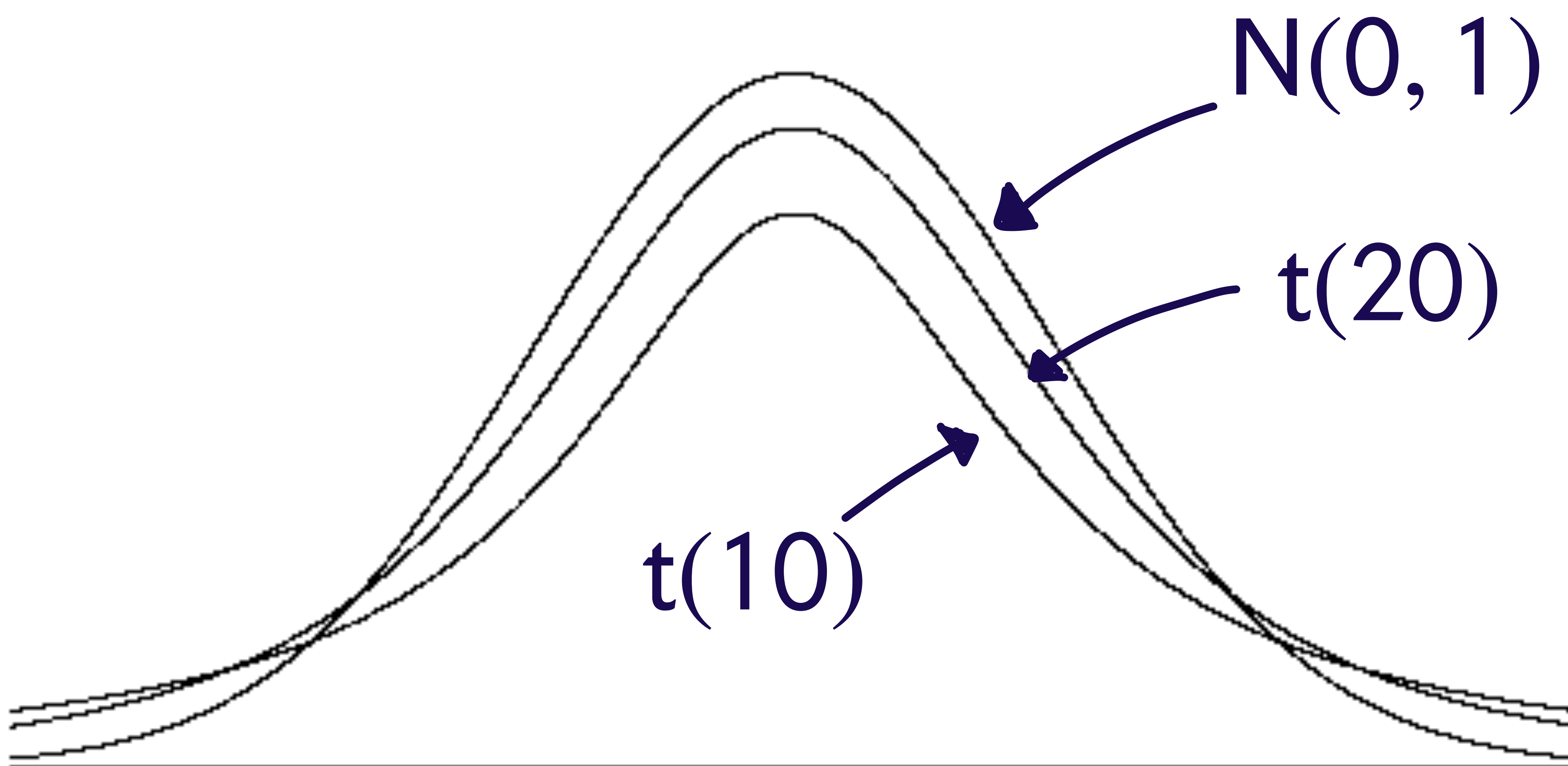
The pdf:

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n} \Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$$

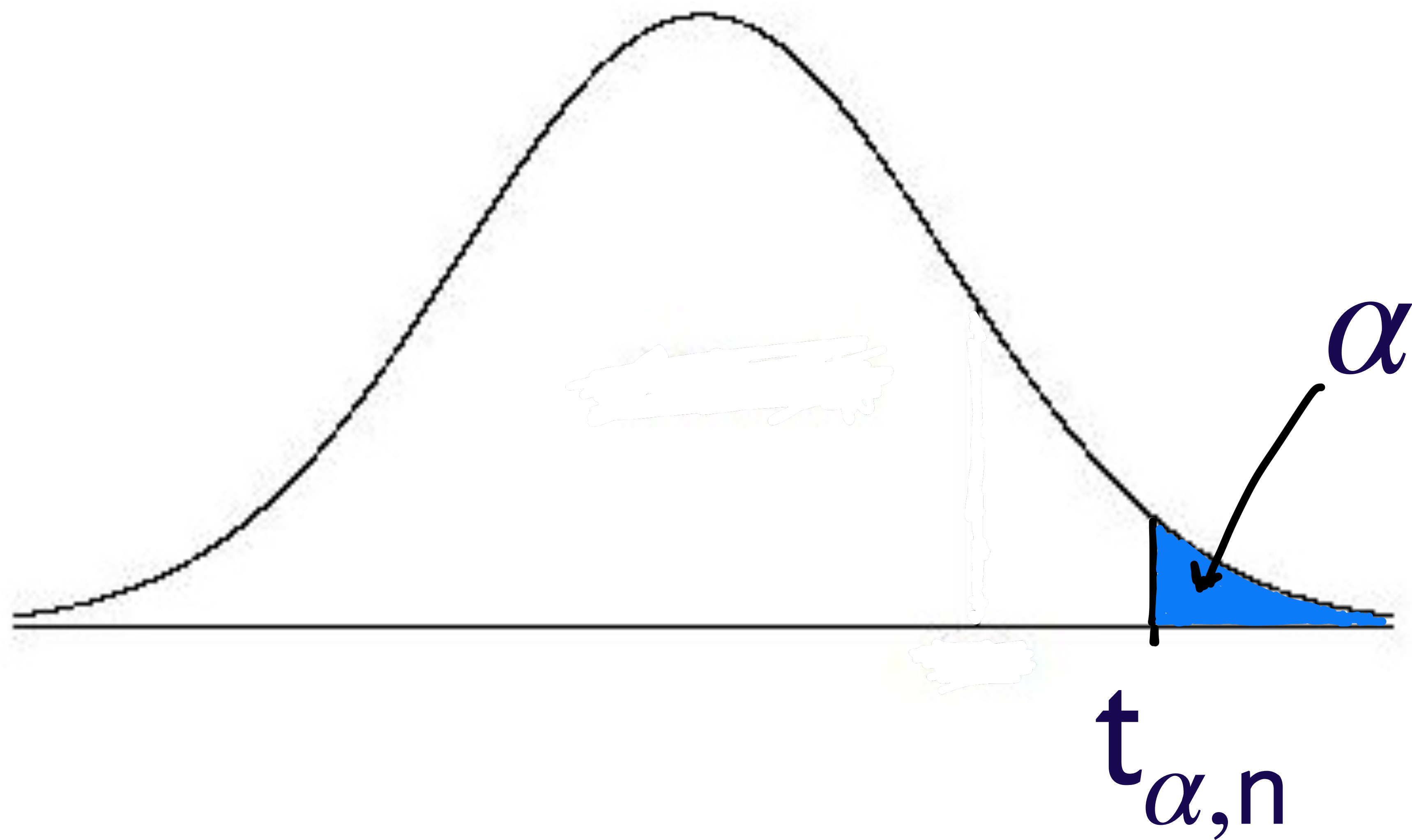
$$-\infty < x < \infty$$

Recall that

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n$$



$$X \sim t(n)$$



Fun Facts!

- $Z \sim N(0, 1) \Rightarrow Z^2 \sim \chi^2(1)$
- X_1, X_2, \dots, X_k independent with $X_i \sim \chi^2(n_i)$

$$\sum_{i=1}^k X_i \sim \chi^2(n_1 + n_2 + \dots + n_k)$$

In particular, $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \chi^2(1)$

$$\Rightarrow \sum_{i=1}^n X_i \sim \chi^2(n)$$

Fun Facts!

- $X \sim \Gamma(\alpha, \beta)$ and $c > 0$
 $\Rightarrow cX \sim \Gamma(\alpha, \beta/c)$

- $X \sim \Gamma(\alpha, \beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x}$$

$$\alpha = 1 \quad \Rightarrow \quad X \sim \exp(\text{rate} = \beta)$$

Fun Facts!

- $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{exp}(\text{rate} = \lambda)$

$$\Rightarrow \sum_{i=1}^n X_i \sim \Gamma(n, \lambda)$$

- $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \Gamma(\alpha, \beta)$

$$\Rightarrow \sum_{i=1}^n X_i \sim \Gamma(n\alpha, \beta)$$

Things we now know...

- $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{exp}(\text{rate} = \lambda)$

$$\Rightarrow \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \sim \Gamma(n, n\lambda)$$

- $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \Gamma(\alpha, \beta)$

$$\Rightarrow \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \sim \Gamma(n\alpha, n\beta)$$

Things we now know...

- $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{exp}(\text{rate} = \lambda)$

$$\Rightarrow \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \sim \Gamma(n, n\lambda)$$

$$\begin{aligned} \Rightarrow 2n\lambda \bar{X} &\sim \Gamma\left(n, \frac{1}{2}\right) \\ &= \Gamma\left(\frac{2n}{2}, \frac{1}{2}\right) = \chi^2(2n) \end{aligned}$$