Two Populations:

Test

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

- Suppose that $X_{1,1}, X_{1,2}, \ldots, X_{1,n_1}$ is a random sample of size n_1 from the normal distribution with mean μ_1 and variance σ_1^2.

- Suppose that $X_{2,1}, X_{2,2}, \ldots, X_{2,n}$ is a random sample of size n_2 from the normal distribution with mean μ_2 and variance σ_2^2.

- Suppose that σ_1^2 and σ_2^2 are unknown and that the samples are independent.

- Don't assume that σ_1^2 and σ_2^2 are equal!

- There is no known exact test
- This is known as the
  Behrens-Fisher problem.
- The most popular approximate solution is given by Welch's t-test.

Welch says that:

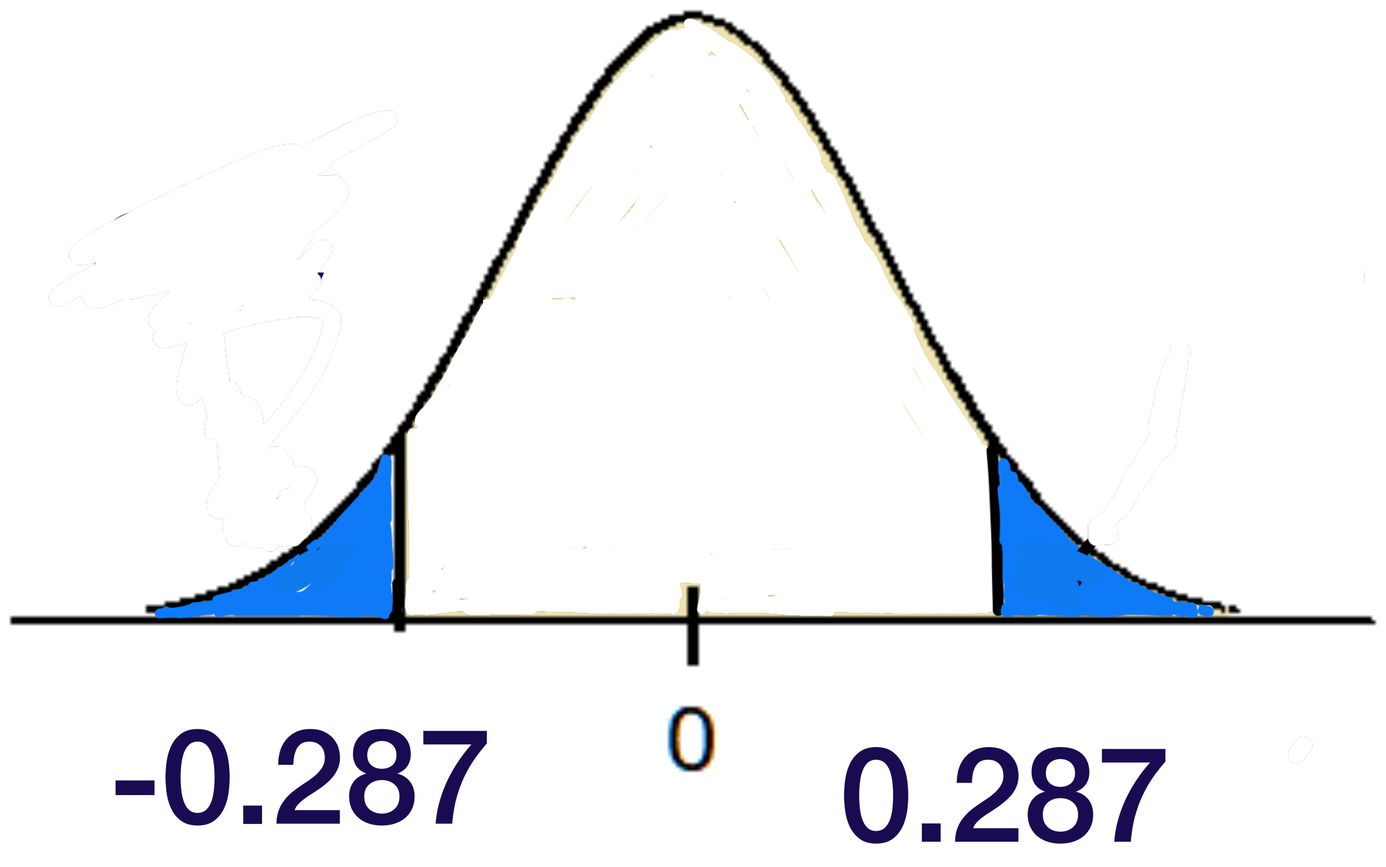
$$\frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

has an approximate t-distribution with r degrees of freedom where

$$r = \frac{S_1^2/n_1 + S_2^2/n_2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

rounded down.

Example: (In R)




```
> x<-c(1.2,3.2,2.7,1.6,2.1)
> y<-c(4.2,0.8,2.2,2.3,1.5,3.0)
> t.test(x,y)

        Welch Two Sample t-test

data:  x and y
t = -0.28741, df = 8.742, p-value = 0.7805
alternative hypothesis: true difference in mean
95 percent confidence interval:
 -1.543768  1.197102
sample estimates:
mean of x mean of y
 2.160000  2.333333
```

2*pt(-0.28741,8.742)
= 0.7804947

Example:

A random sample of 6 students' grades were recorded for Midterm 1 and Midterm 2.

Assuming exam scores are normally distributed, test whether the true (total population of students) average grade on Midterm 2 is greater than Midterm 1.

$$\alpha = 0.05$$

The Data

Student	Midterm 1 Grade	Midterm 2 Grade
1	72	81
2	93	89
3	85	87
4	77	84
5	91	100
6	84	82

The data are “paired”.

The Data

Student	Midterm 1 Grade	Midterm 2 Grade	Differences: Midterm 2 minus Midterm 1
1	72	81	9
2	93	89	-4
3	85	87	2
4	77	84	7
5	91	100.	9
6	84	82	-2

The Hypotheses:

Let μ be the true average difference for all students.

$$H_0 : \mu = 0$$
$$H_1 : \mu > 0$$

This is simply a one sample t-test on the differences.

Data:

$$9, \ -4, \ 2, \ 7, \ 9, \ -2$$

$$\sum x_i = 23 \qquad \sum x_i^2 = 267 \qquad n = 6$$

This is simply a one sample t-test on the differences.

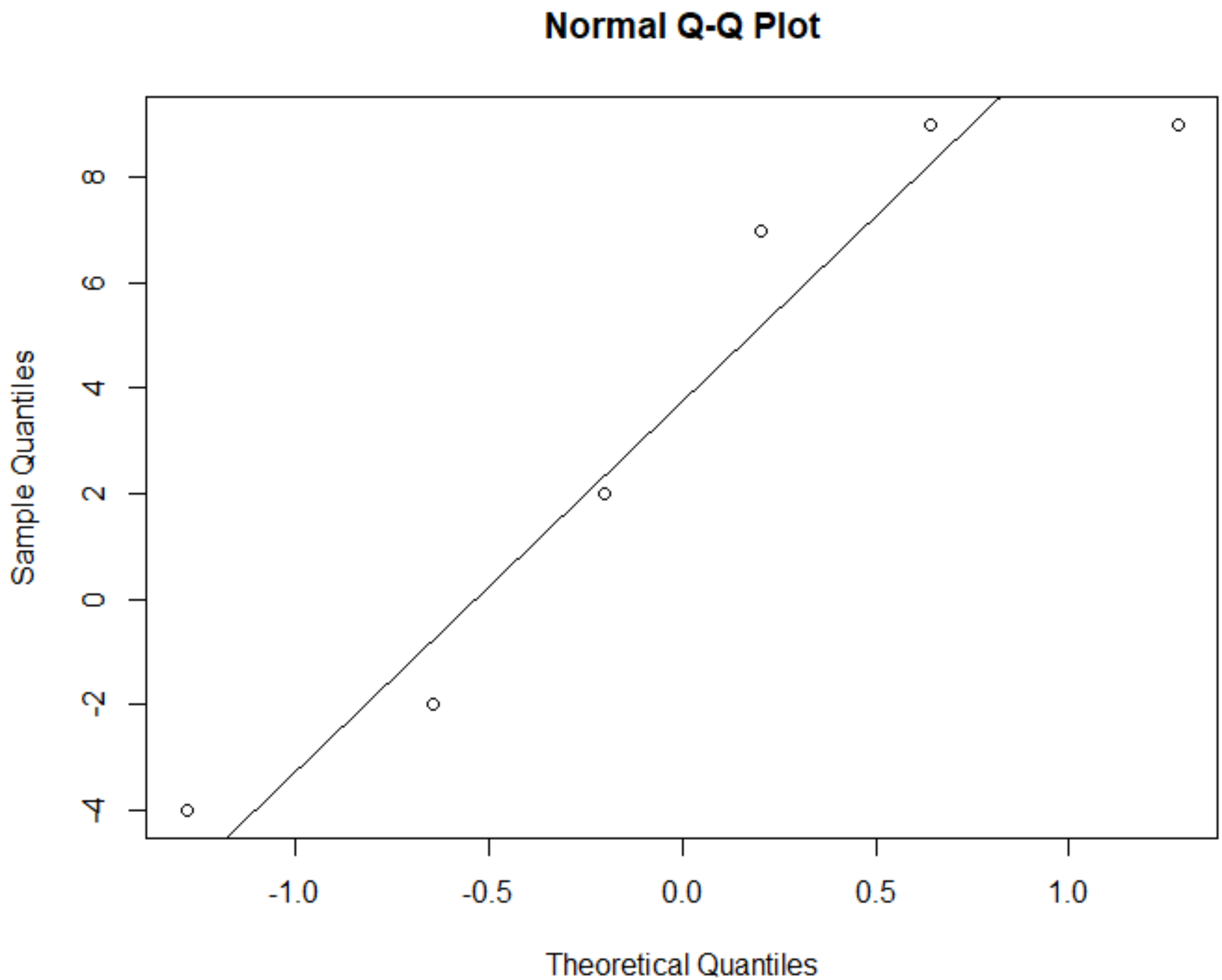
Data: $9, \ -4, \ 2, \ 7, \ 9, \ -2$

$$\sum x_i = 23 \qquad \sum x_i^2 = 267 \qquad n = 6$$

This is simply a one sample t-test on the differences.

$$\bar{x} = 3.5$$

$$s^2 = \frac{\sum x_i^2 - (\sum x_i)^2/n}{n-1} = 32.3$$


Normal Q-Q Plot
Sample Quantiles
-4
-2
0
2
4
6
8
-1.0
-0.5
0.0
0.5
1.0
Theoretical Quantiles

$$H_0 : \mu = 0$$

$$H_1 : \mu > 0$$

$$t_{\alpha,n-1} = t_{0.05,5} = 2.01$$

Reject H_0**, in favor of** H_1**, if**

$$\underset{3.5}{\bar{X}} > \underbrace{\mu_0 + t_{\alpha,n-1}\frac{S}{\sqrt{n}}}_{4.66}$$

Conclusion:

We fail to reject H_0, in favor of H_1, at 0.05 level of significance.

These data do not indicate that Midterm 2 scores are higher than Midterm 1 scores.