Any linear combination of normal random variables is again normal.

- $X_1, X_2, \ldots, X_n$ normal

$$\Rightarrow \quad a + \sum_{i=1}^{n} a_i X_i \quad \text{normal}$$

- This includes linear transformations of single normal random variables.

$$X \sim N(\mu, \sigma^2) \quad \Rightarrow \quad Z := \frac{X - \mu}{\sigma}$$

$$Z \sim N(0, 1) \quad \Rightarrow \quad X := \sigma Z + \mu \sim N(0, 1)$$

$$X \sim N(\mu, \sigma^2)$$

$$\Rightarrow \quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$Z \sim N(0, 1)$$

$$\Rightarrow \quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

The cdf can not be written down in closed form.

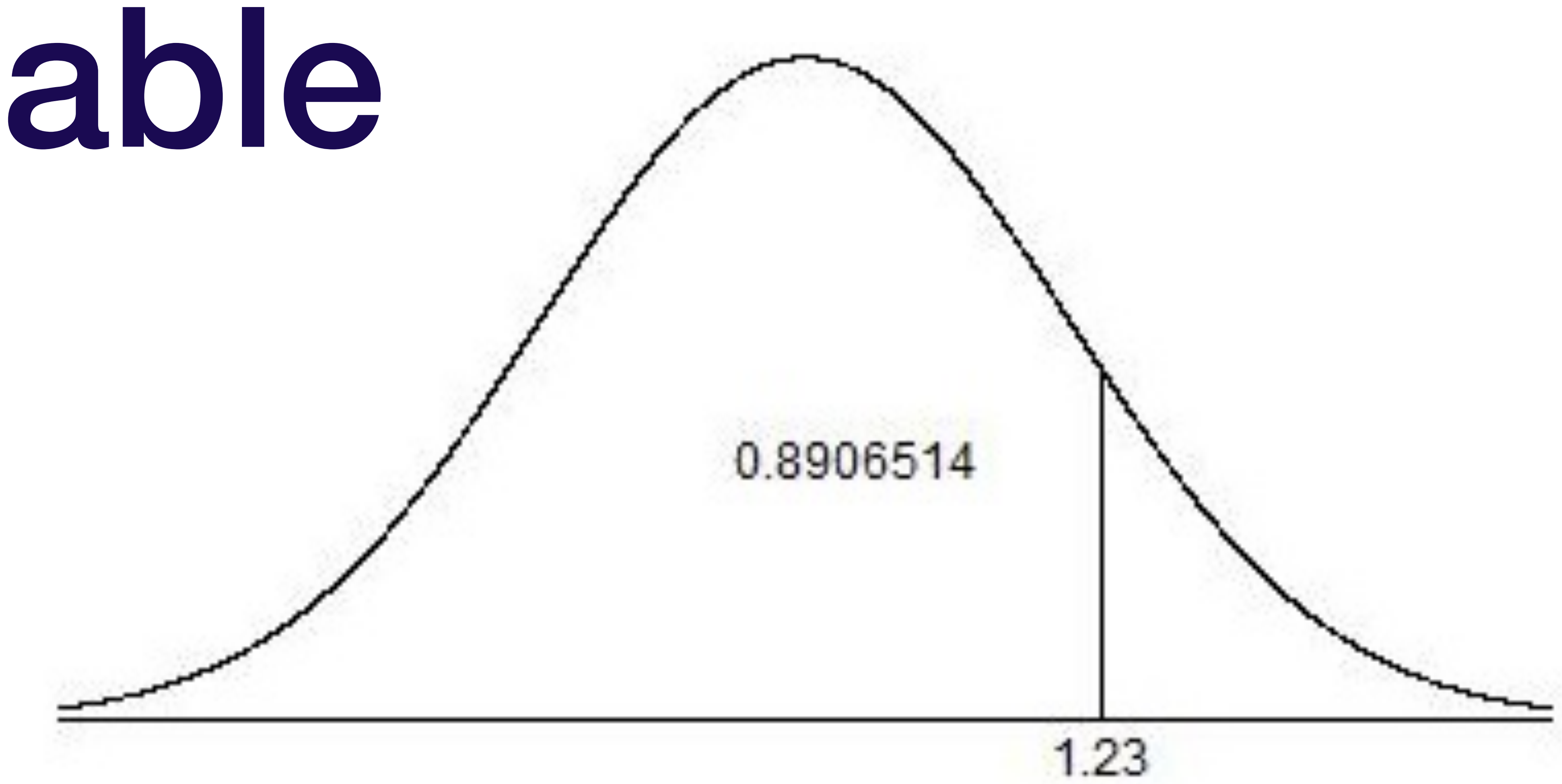
Notation:

$$Z \sim N(0, 1)$$ "standard normal"

$$\Phi(z) = P(Z \leq z)$$

Can be integrated numerically.

- standard normal table
- R code: pnorm( )



Example: pnorm(1.23) will give you 0.891.

Example 1:

Suppose that $X \sim N(1, 4)$.

Find $P(X \leq 2)$.

$$P(X \leq 2) = P\left(\frac{X - \mu}{\sigma} \leq \frac{2 - 1}{\sqrt{4}}\right)$$

$$= P(Z \leq 0.5)$$

$$\approx 0.6915$$

R Code: pnorm(0.5)

Example 2:

Suppose that $X_1, X_2, X_3 \sim N(1, 4)$.

Find $P(\overline{X} \leq 2)$.

- $\overline{X}$ has a normal distribution
- $E[\overline{X}] = E[X_1] = 1$
- $\mathrm{Var}[\overline{X}] = \dfrac{\mathrm{Var}[X_1]}{n} = \dfrac{4}{3}$

Example 2, continued:

$$P(\overline{X} \leq 2) = P\left(\frac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} \leq \frac{2 - 1}{2/\sqrt{3}}\right)$$

$$= P\left(Z \leq \sqrt{3}/2\right)$$

$$\approx 0.8068$$

R Code: `pnorm(sqrt(3)/2)`

Convergence in Distribution

Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables where X_n has some cdf $F_n(x) = P(X_n \leq x)$.

Let X be a random variable with cdf $F(x) = P(X \leq x)$.

The sequence **converges in distribution** if

$$\lim_{n \to \infty} F_n(x) = F(x)$$

at all points of continuity of F.

Convergence in Distribution

We write $X_n \xrightarrow{d} X$.

This is a "weak" form of convergence.

The random variables in the sequence are <u>not</u> getting closer to each other.

Their <u>distributions</u> are getting close!

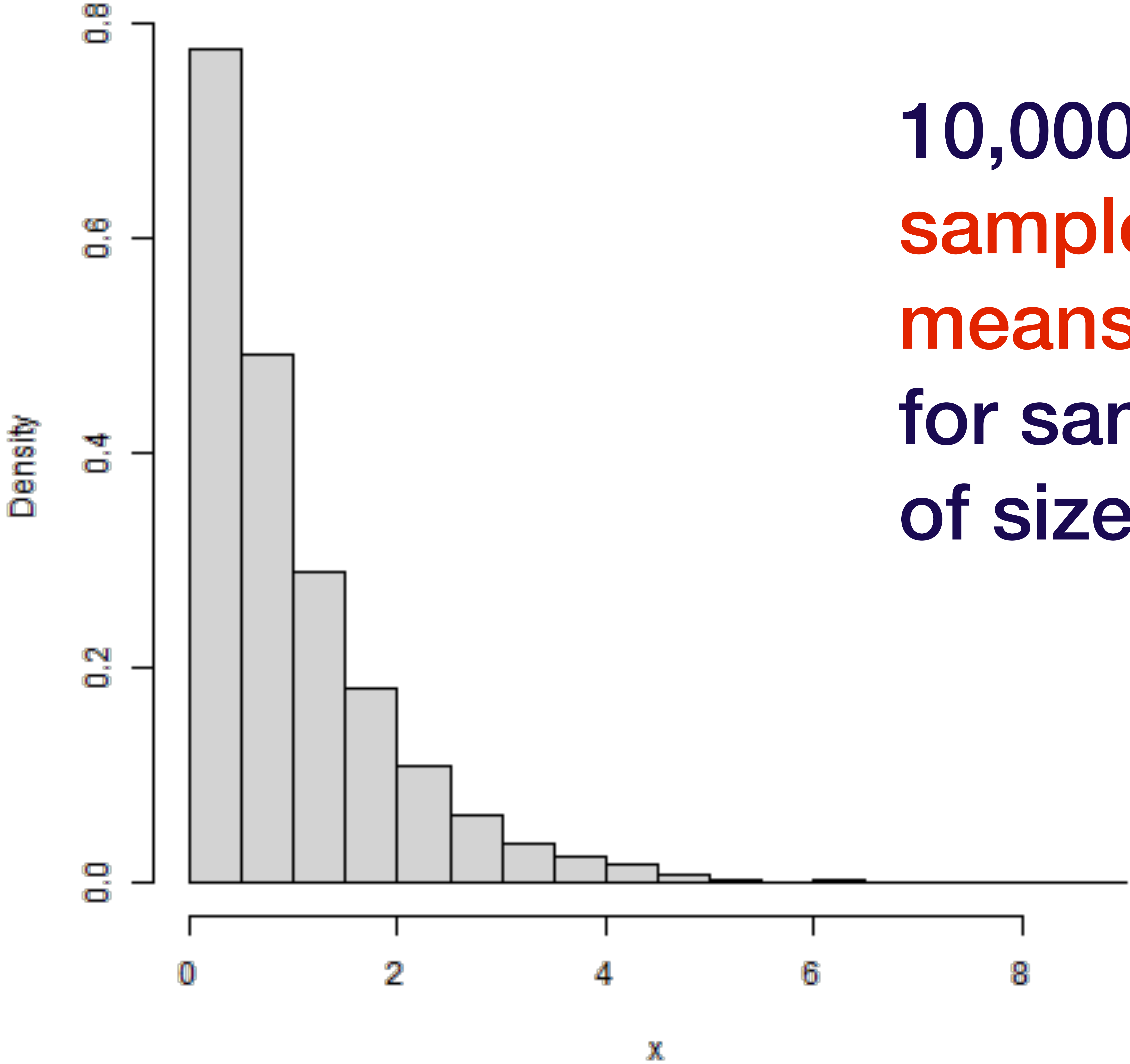
Example: $X_1, X_2, X_3, \ldots$

with $X_n \sim N(1/n, \sigma^2)$

$X_n \xrightarrow{d} Z$

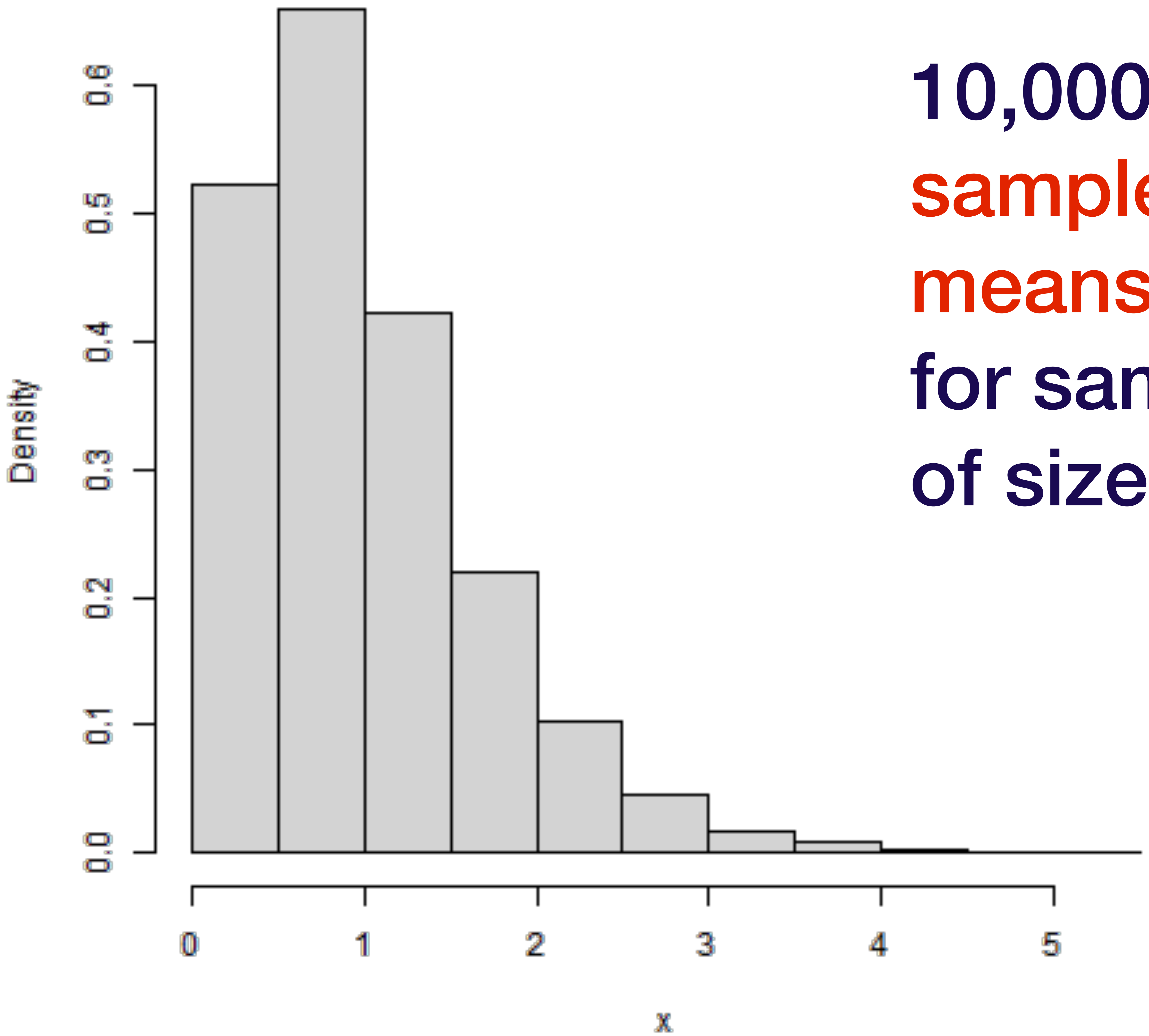
where $Z \sim N(0,1)$.

$$X \sim \exp(\text{rate} = \lambda)$$



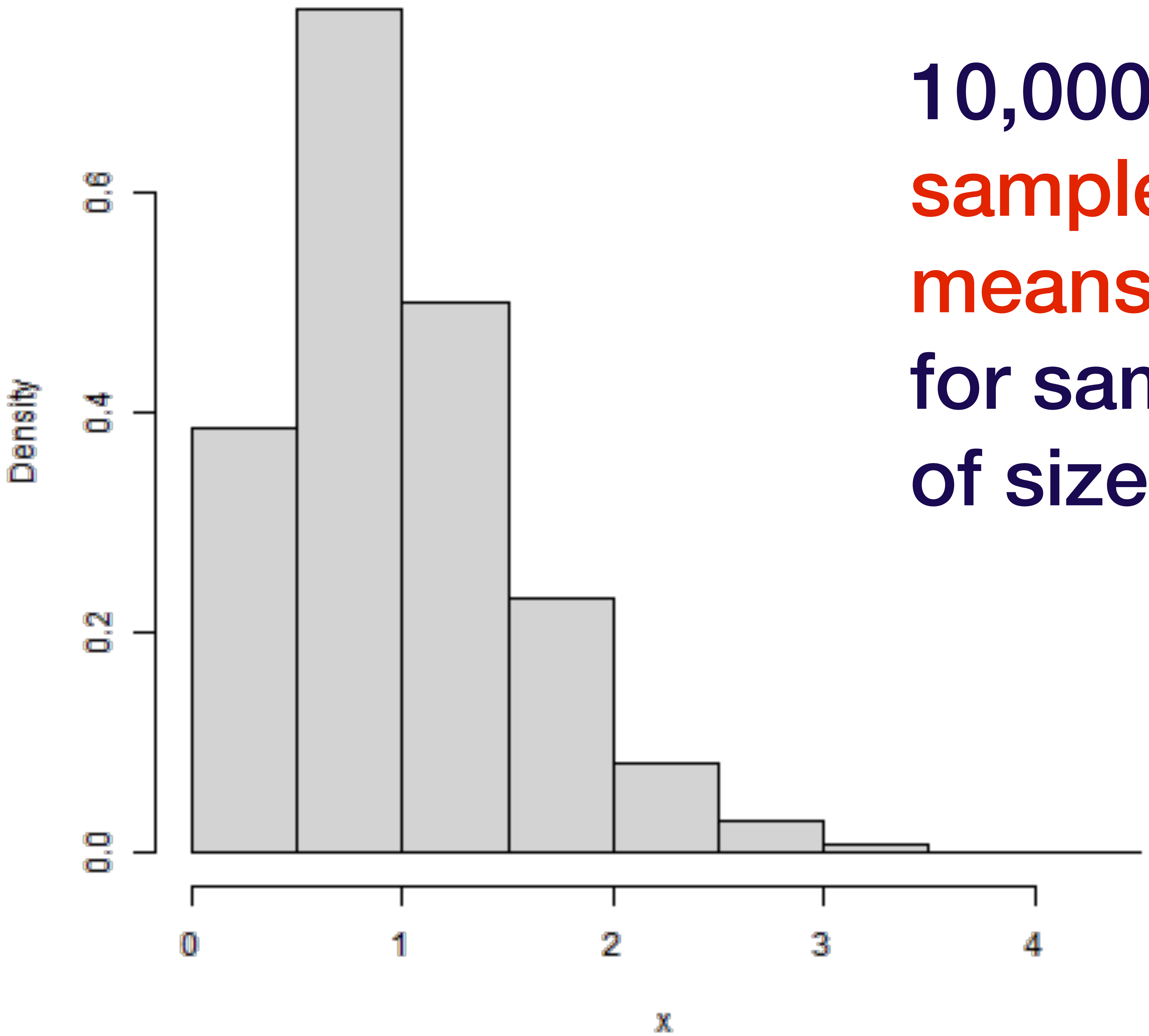
10,000
sample
means
for samples
of size n=1

$X \sim \exp(\text{rate} = \lambda)$



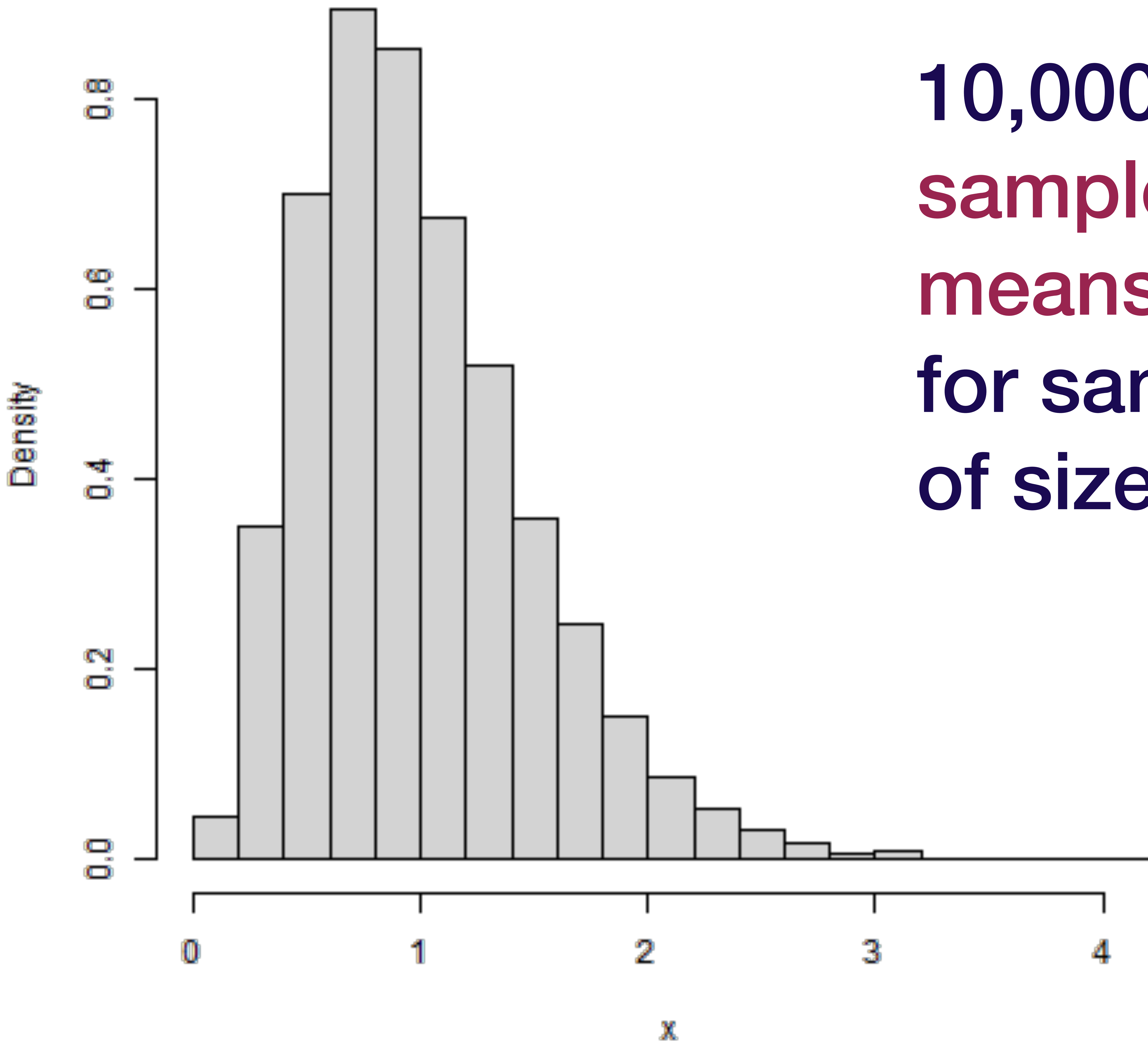
10,000
sample
means
for samples
of size n=2

$$X \sim \exp(\text{rate} = \lambda)$$



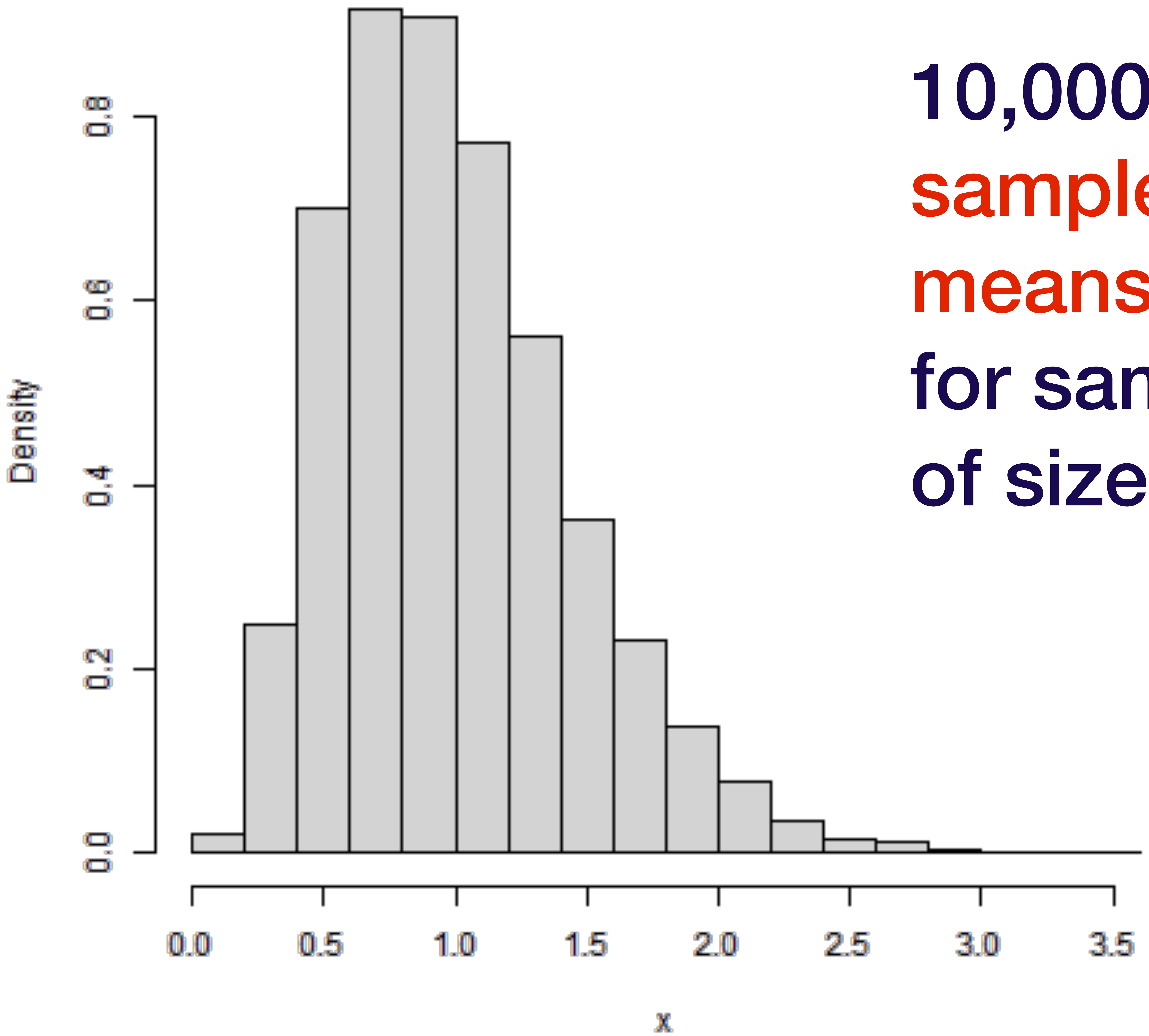
10,000 sample means for samples of size n=3

$$X \sim \exp(\text{rate} = \lambda)$$



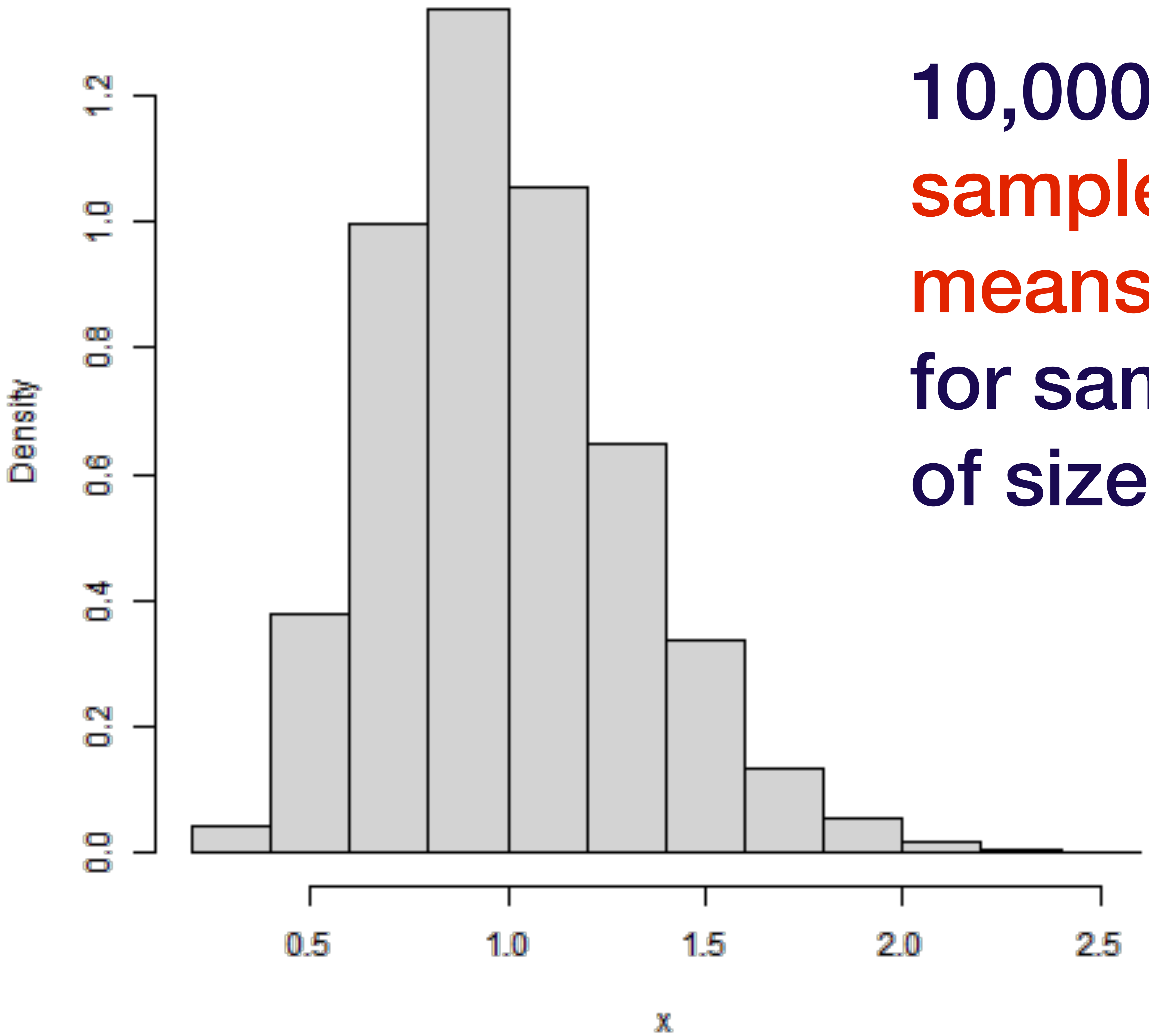
10,000
sample
means
for samples
of size n=4

$$X \sim \exp(\text{rate} = \lambda)$$



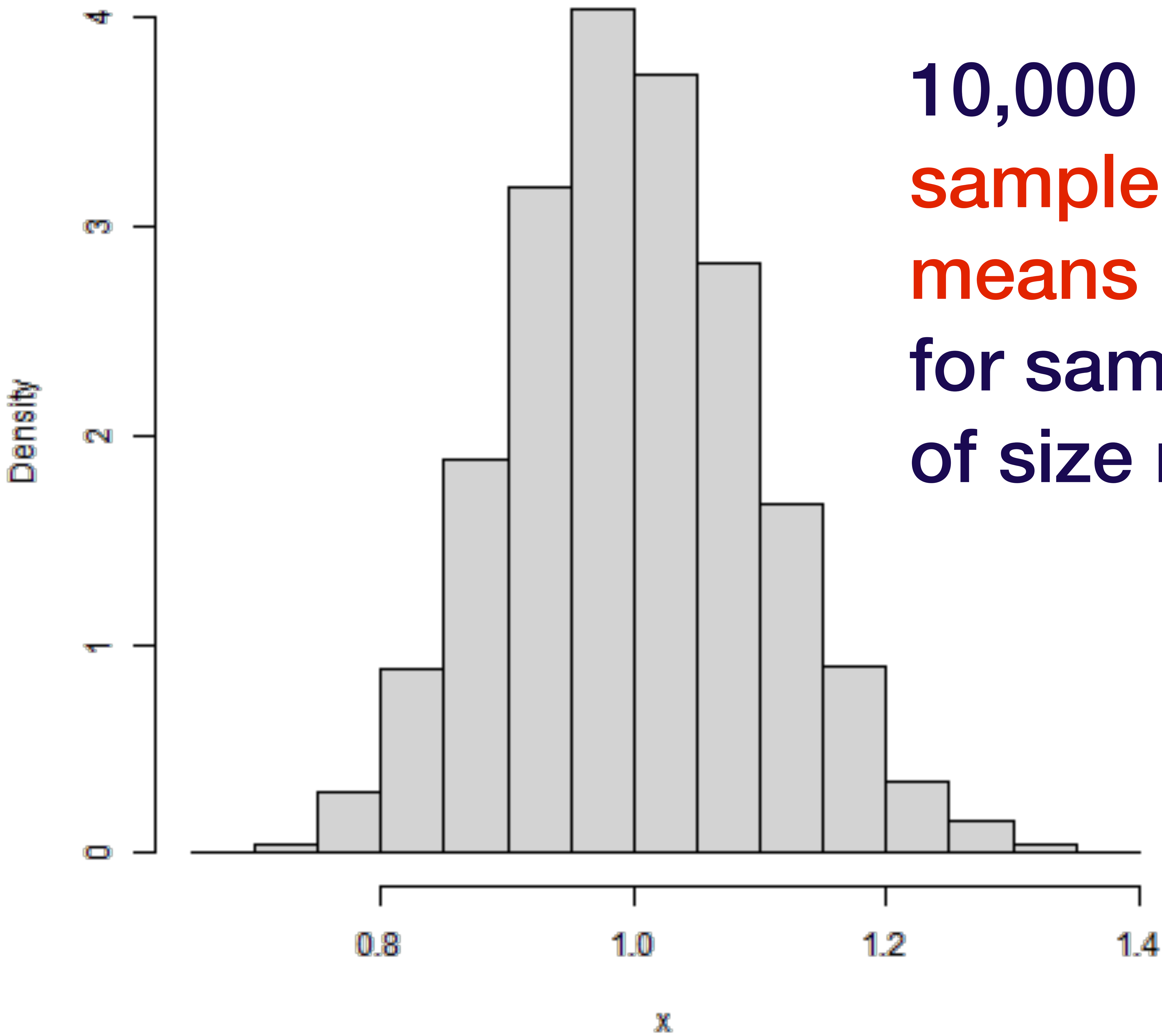
10,000 **sample means** for samples of size n=5

$$X \sim \exp(\text{rate} = \lambda)$$



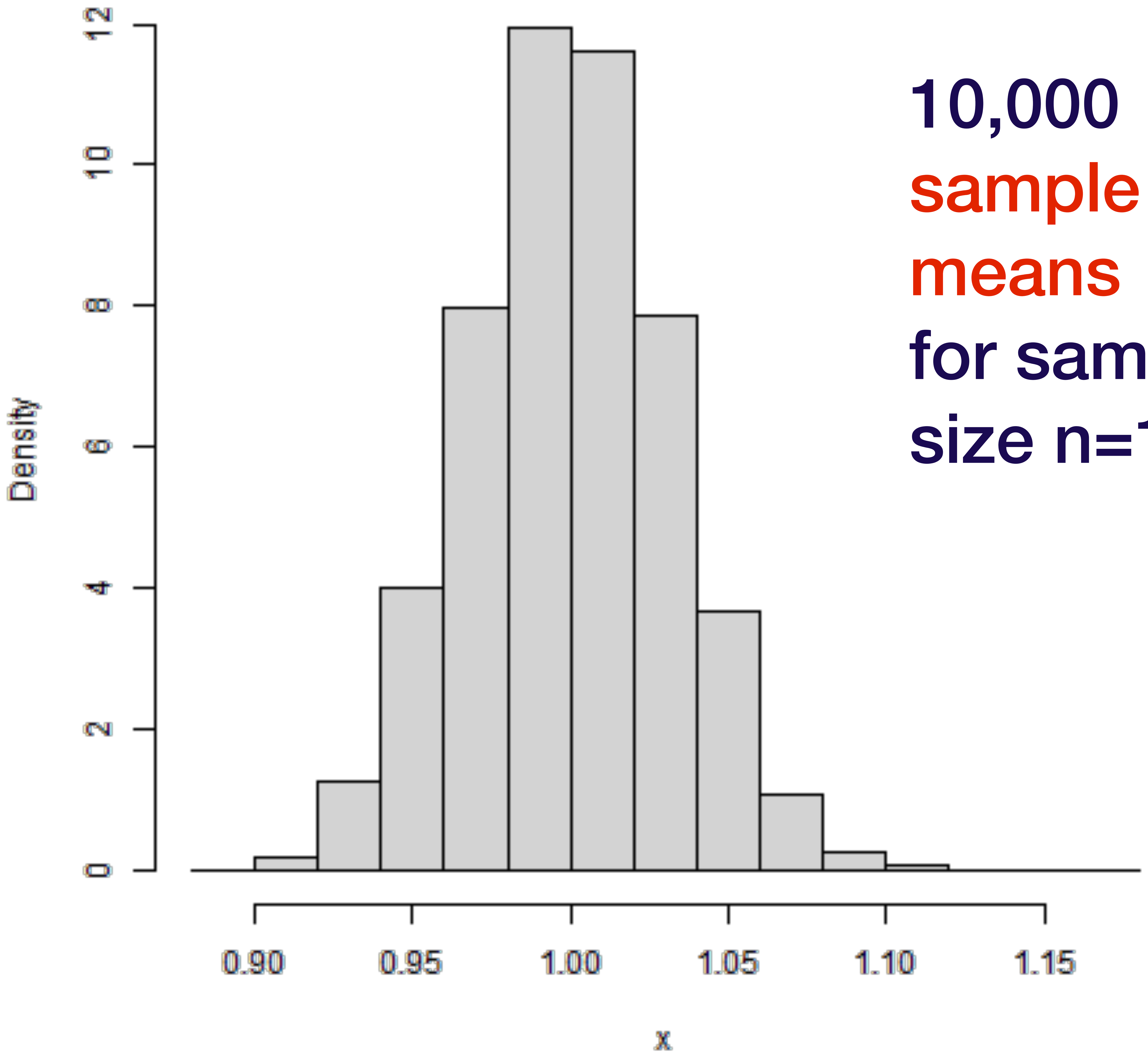
10,000 sample means for samples of size n=10

$$X \sim \exp(\text{rate} = \lambda)$$



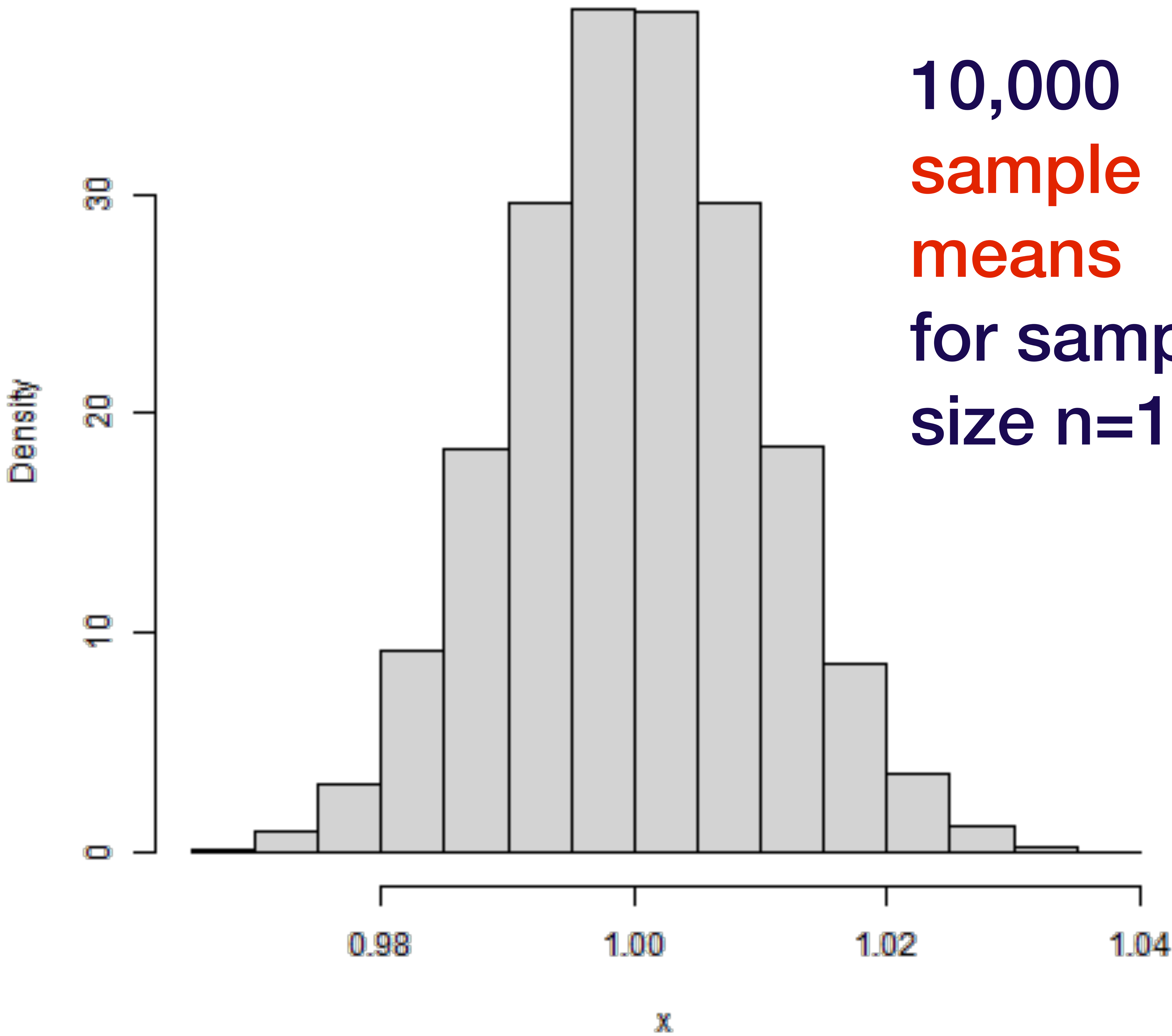
10,000
sample
means
for samples
of size n=100

$X \sim \exp(\text{rate} = \lambda)$



10,000
sample
means
for samples of
size n=1000

$$X \sim \exp(\text{rate} = \lambda)$$



10,000
sample
means
for samples of
size n=10000

The Central Limit Theorem

Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables from any distribution with mean μ and variance $\sigma^2 < \infty$.

Let

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1)$$

Definition/Notation:

A random variable X_n is **asymptotically normal** if there exists sequences $\{a_n\}$ and $\{b_n\}$ of real numbers such that

$$\frac{X_n - a_n}{\sqrt{b_n}} \xrightarrow{d} N(0, 1)$$

We write $X_n \overset{asymp}{\sim} N(a_n, b_n)$.

Note: This does not mean that $X_n \to N(a_n, b_n)$.

The CLT is saying that $\overline{X}_n$ is "asymptotically normal".

We write

$$\overline{X}_n \overset{\text{asymp}}{\sim} N(\mu, \sigma^2/n)$$

Example:

Let $\overline{X}$ be the sample mean for a random sample of size 100 from the $\Gamma(3, 2)$ distribution.

What is the approximate probability that $\overline{X}$ is greater than 1.49 ?

We already know that $\overline{X}$ has

- mean $E[\overline{X}] = E[X_1] = 3/2$
- variance $Var[\overline{X}] = \frac{Var[X_1]}{n} = \frac{3/4}{100} = \frac{3}{400}$

By the CLT, for this large sample" (n>30), the distribution of $\overline{X}$ is approximately normal.

So, we have that $\overline{X}$ has an approximately normal distribution with mean 3/2 and variance 3/400.

$$P(\overline{X} > 1.4) = P\left(\frac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} > \frac{1.4 - 1.5}{\sqrt{3/400}}\right)$$

$$\approx P(Z > -1.15)$$

$$\approx 0.87$$

$$
\begin{aligned}
P(Z > -1.15) &= 1 - P(Z \le -1.15) \\
&= 1 - \Phi(-1.15) \\
&\approx 0.8749
\end{aligned}
$$

Computed in R using:

```
1-pnorm(-1.15)
```

Note that:

$$\frac{\overline{X} - \mu_X}{\sigma_{\overline{X}}} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$