Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the normal distribution with mean μ and variance σ^2.

Derive a test of size/level α for

$$H_0 : \sigma^2 \geq \sigma_0^2 \quad \text{vs.} \quad H_1 : \sigma^2 < \sigma_0^2$$

$$H_0 : \sigma^2 \geq \sigma_0^2$$
$$H_1 : \sigma^2 < \sigma_0^2$$

Step One:

Choose a statistic/estimator for σ^2.

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n - 1}$$

$$H_0 : \sigma^2 \geq \sigma_0^2$$
$$H_1 : \sigma^2 < \sigma_0^2$$

Step Two:

Give the form of the test.

Reject H_0, in favor of H_1, if

$$S^2 < c$$

for some c to be determined.

$$H_0 : \sigma^2 \geq \sigma_0^2$$
$$H_1 : \sigma^2 < \sigma_0^2$$

Step Three:

Find c using α.

$$\alpha = \max P(\text{ Type I Error })$$

$$= \max_{\sigma^2 \geq \sigma_0^2} P(\text{ Reject } H_0; \sigma^2)$$

$$= \max_{\sigma^2 \geq \sigma_0^2} P(S^2 < c; \sigma^2)$$

Distribution?

$$P(S^2 < c;\, \sigma^2)$$

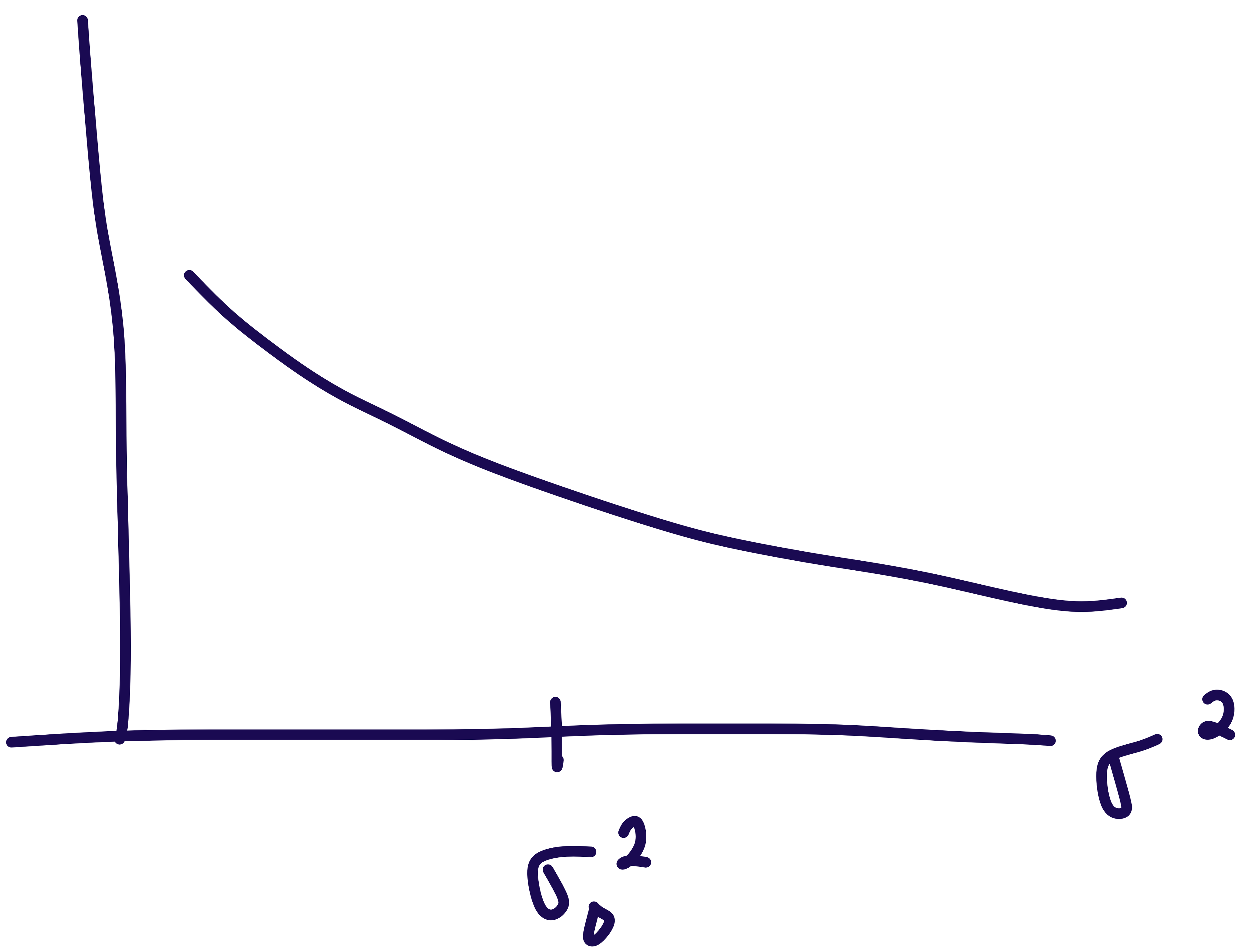
$$= P\left(\underbrace{\frac{(n-1)S^2}{\sigma^2}}_{\chi^2(n-1)} < \frac{(n-1)c}{\sigma^2};\, \sigma^2 \right)$$

$$= P\left(W < \frac{(n-1)c}{\sigma^2} \right)$$

where $W \sim \chi^2(n-1)$.

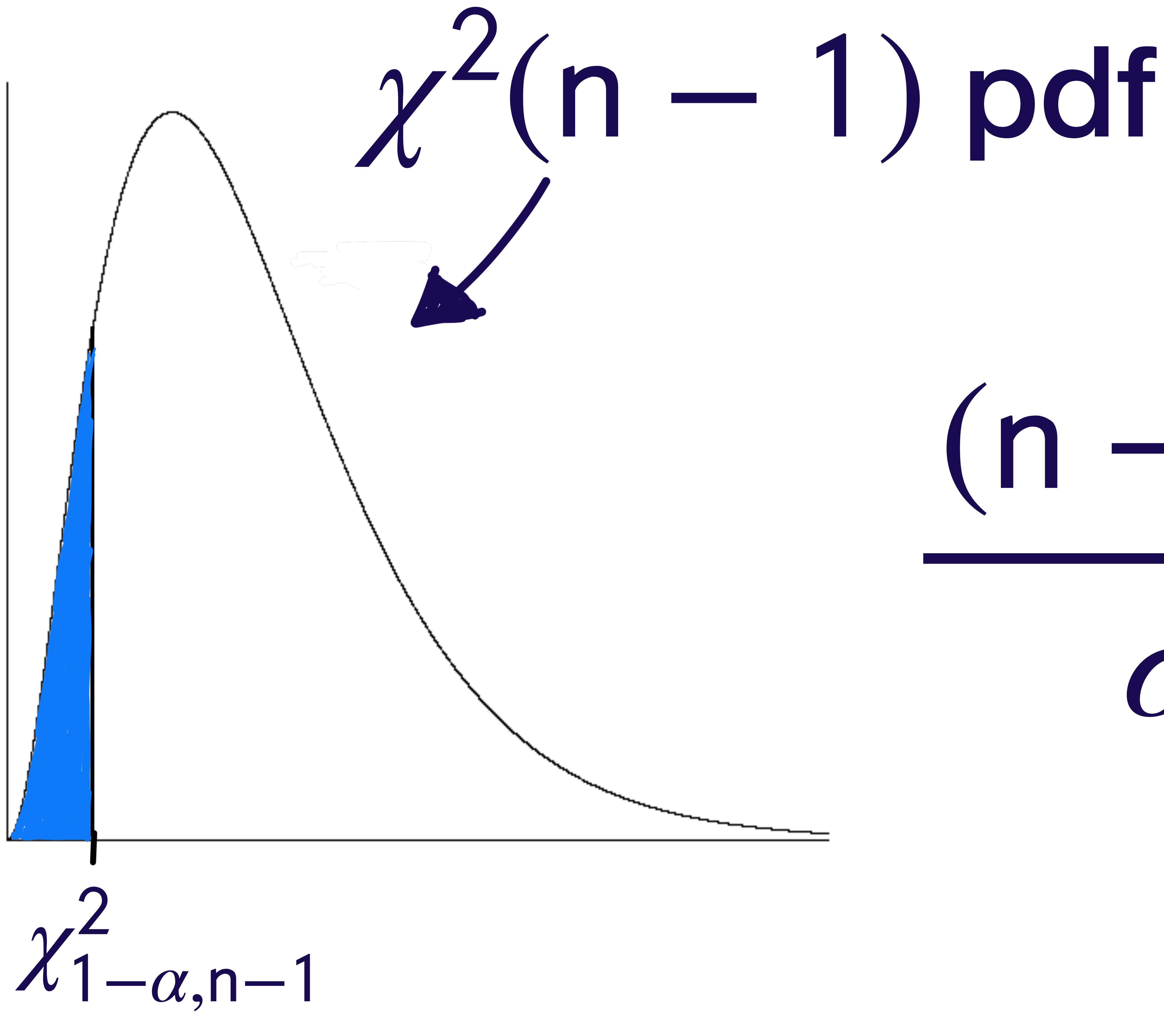
$$\alpha = \underbrace{\max_{\sigma^2 \geq \sigma_0^2} P\left(W < \frac{(n-1)c}{\sigma^2} \right)}_{\text{decreasing in } \sigma^2}$$

decreasing in σ^2



maximize by plugging in $\sigma^2 = \sigma_0^2$

$$\alpha = P\left(W < \frac{(n-1)c}{\sigma_0^2} \right)$$



$$\Downarrow$$

$$\frac{(n-1)c}{\sigma_0^2} = \chi^2_{1-\alpha,n-1}$$

$$H_0 : \sigma^2 \geq \sigma_0^2$$
$$H_1 : \sigma^2 < \sigma_0^2$$

Step Four:

Conclusion.

Reject H_0, in favor of H_1, if

$$S^2 < \frac{\sigma_0^2 \chi^2_{1-\alpha, n-1}}{n-1}$$

Example:

A lawn care company has developed and wants to patent a new herbicide applicator spray nozzle.

For safety reasons, they need to ensure that the application is consistent and not highly variable.

The company selected a random sample of 10 nozzles and measured the application rate of the herbicide in gallons per acre

Example:

The measurements were recorded as

0.213, 0.185, 0.207, 0.163, 0.179

0.161, 0.208, 0.210, 0.188, 0.195

Assuming that the application rates are normally distributed, test the following hypotheses at level 0.04.

$$H_0 : \sigma^2 = 0.01 \qquad H_1 : \sigma^2 > 0.01$$

Example:

Get sample variance in R.

```
x<-c(0.213, 0.185, 0.207, 0.163, 0.179
     0.161, 0.208, 0.210, 0.188, 0.195)
```

or

```
x<-scan()
```

Hit <Enter> and then input numbers, one by one, hitting <Enter> in between and <Enter> at the end.

Example:

Compute variance by typing

```
var(x)
```

or

```
((sum(x^2)-(sum(x)^2)/10)/9
```

Result: 0.000364

Example:

Reject H_0, in favor of H_1, if $S^2 > c$.

$$\alpha = P(S^2 > c; \sigma^2 = 0.01)$$

$$= P\left(\frac{(n-1)S^2}{\sigma^2} > \frac{9c}{0.01}; \sigma^2 = 0.01\right)$$

$$= P\left(W > \frac{9c}{0.01}\right)$$

where $W \sim \chi^2(9)$.

Example:

Reject H_0, in favor of H_1, if $S^2 > c$.

$$0.04 = P\left(W > \frac{9c}{0.01}\right)$$

$$\Downarrow$$

$$\frac{9c}{0.01} = \chi^2_{0.04,9} = 17.61$$

qchisq(1-0.04,9)

Example:

Reject H_0, in favor of H_1, if $S^2 > c$.

$$c = (17.61)(0.01)/9 \approx 0.0196$$

$$s^2 = 0.000364$$

Fail to reject H_0, in favor of H_1, at level 0.04. There is not sufficient evidence in the data to suggest that $\sigma^2 > 0.01$.