Example:

Fifth grade students from two neighboring counties took a placement exam.

- Group 1, from County 1, consisted of 57 students. The sample mean score for these students was 77.2 and the true variance is known to be 15.3.

- Group 2, from County 2, consisted of 63 students and had a sample mean score of 75.3 and the true variance is known to be 19.7.

Example:

From previous years of data, it is believed that the scores for both counties are normally distributed.

Derive a test to determine whether or not the two population means are the same.

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

- Suppose that $X_{1,1}, X_{1,2}, \ldots, X_{1,n_1}$ is a random sample of size n_1 from the normal distribution with mean μ_1 and variance σ_1^2.

- Suppose that $X_{2,1}, X_{2,2}, \ldots, X_{2,n_2}$ is a random sample of size n_2 from the normal distribution with mean μ_2 and variance σ_2^2.

- Suppose that σ_1^2 and σ_2^2 are known and that the samples are independent.

$$H_0 : \mu_1 = \mu_2 \qquad H_1 : \mu_1 \neq \mu_2$$

$$\Downarrow$$

$$H_0 : \mu_1 - \mu_2 = 0$$

$$H_1 : \mu_1 - \mu_2 \neq 0$$

Think of this as

$\theta = 0$ **versus** $\theta \neq 0$

for

$$\theta = \mu_1 - \mu_2$$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Step One:

Choose an estimator for $\theta = \mu_1 - \mu_2$.

$$\widehat{\theta} = \overline{X}_1 - \overline{X}_2$$

Step Two:

Give the "form" of the test.

Reject H_0, in favor of H_1 if either

$$\widehat{\theta} > c \quad \text{or} \quad \widehat{\theta} < -c$$

for some c to be determined.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Step Three:

Find c using α.

Will be working with the random variable

$$\overline{X}_1 - \overline{X}_2$$

We need to know its distribution…

Distribution of the statistic:

- $\overline{X}_1 - \overline{X}_2$ is normally distributed

Mean:

$$E\left[\overline{X}_1 - \overline{X}_2\right] = E[\overline{X}_1] - E[\overline{X}_2]$$

$$= \mu_1 - \mu_2$$

Distribution of the statistic:

- $\overline{X}_1 - \overline{X}_2$ is normally distributed

Variance:

$$\mathrm{Var}\left[\overline{X}_1 - \overline{X}_2\right] = \mathrm{Var}[\overline{X}_1 + (-1)\overline{X}_2]$$

$$\overset{\text{indep}}{=} \mathrm{Var}[\overline{X}_1] + \mathrm{Var}[(-1)\overline{X}_2]$$

$$= \mathrm{Var}[\overline{X}_1] + (-1)^2\mathrm{Var}[\overline{X}_2]$$

$$= \mathrm{Var}[\overline{X}_1] + \mathrm{Var}[\overline{X}_2]$$

Distribution of the statistic:

- $\overline{X}_1 - \overline{X}_2$ is normally distributed

Variance:

$$\text{Var}\left[\overline{X}_1 - \overline{X}_2\right] \overset{\text{indep}}{=} \text{Var}[\overline{X}_1] + \text{Var}[\overline{X}_2]$$

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Step Three:

Find c using α.

$\overline{X}_1 - \overline{X}_2$ **is normally distributed**

$$\overline{X}_1 - \overline{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_1}\right)$$

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Step Three:

$$\alpha = P(\text{ Type I Error })$$

$$= P(\text{ Reject } H_0 \; ; \theta = 0)$$

$$= P(\overline{X}_1 - \overline{X}_2 > c \text{ or } \overline{X}_1 - \overline{X}_2 < -c ; \theta = 0)$$

$$= 1 - P(-c \leq \overline{X}_1 - \overline{X}_2 \leq c \; ; \underbrace{\theta = 0}_{\mu_1 - \mu_2 = 0})$$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Step Three:

$$= 1 - P(-c \leq \overline{X}_1 - \overline{X}_2 \leq c \; ; \theta = 0)$$

- **Subtract** $\mu_1 - \mu_2$ **(which is 0)**
- **Divide by**

$$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Step Three:

$$\alpha = 1 - P\left(\frac{-c}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq Z \leq \frac{c}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right)$$

$$1 - \alpha = P\left(\frac{-c}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq Z \leq \frac{c}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right)$$

$$H_0 : \mu_1 = \mu_2$$
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Step Three:

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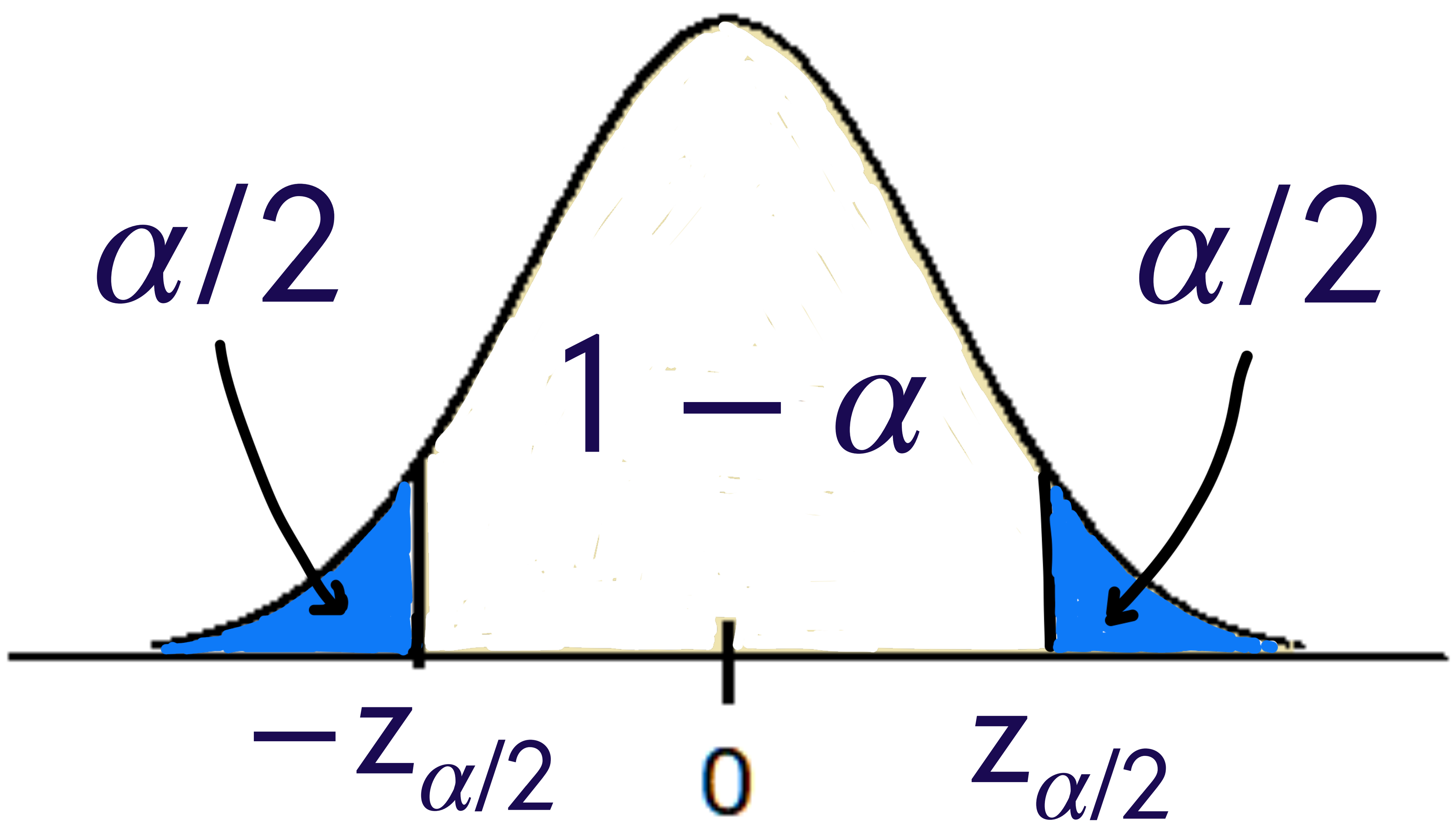
$$1 - \alpha = P\left(\frac{-c}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq Z \leq \frac{c}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right)$$

Step Three:

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$



$$\Rightarrow \quad \frac{c}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = z_{\alpha/2}$$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Step Four:

Conclusion:

Reject H_0**, in favor of** H_1**, if**

$$\overline{X}_1 - \overline{X}_2 > z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

or

$$\overline{X}_1 - \overline{X}_2 < - z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Example:

Fifth grade students from two neighboring counties took a placement exam.

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From previous years of data, it is believed that the scores for both counties are normally distributed, and that the variances of scores from Counties A and B, respectively, are 15.3 and 19.7.

Example:

Fifth grade students from two neighboring counties took a placement exam.

Derive a test to figure out whether or not the two population means are the same.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Example:

$$n_1 = 57 \qquad n_2 = 63$$

$$\bar{x}_1 = 77.2 \qquad \bar{x}_2 = 75.3$$

$$\sigma_1^2 = 15.3 \qquad \sigma_2^2 = 19.7$$

Suppose that $\alpha = 0.05$**.**

$$z_{\alpha/2} = z_{0.025} = 1.96$$

$$z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = 1.49$$

Example:

$$z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} = 1.49$$

$$\bar{x}_1 - \bar{x}_2 = 77.2 - 75.3 = 1.9$$

So,

$$\bar{x}_1 - \bar{x}_2 > z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

and we reject H_0. The data suggests that the true mean scores for the counties are different!