Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate $\lambda > 0$.

Derive a hypothesis test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

where $\lambda_1 > \lambda_0$.

What statistic should we use?

One test has rejection rule:

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

“Denote” this by $\blacksquare$

$$\alpha = P\left(\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0} ; \lambda_0 \right)$$

$$= P\left(\blacksquare ; \lambda_0 \right)$$

One another has rejection rule:

$$\min(X_1, X_2, \ldots, X_n) < \frac{-\ln(1-\alpha)}{n\lambda_0}$$

"Denote" this by $\blacksquare$

$$\alpha = P\left(\min(X_1, X_2, \ldots, X_n) < \frac{-\ln(1-\alpha)}{n\lambda_0};\ \lambda_0\right)$$

$$= P\left(\ \blacksquare\ ;\ \lambda_0\right)$$

Consider "all" tests:

$$P(\blacksquare\,;\lambda_0) = \alpha$$

$$P(\blacksquare\,;\lambda_0) = \alpha$$

$$P(\blacksquare\,;\lambda_0) = \alpha$$

$$P(\blacksquare\,;\lambda_0) = \alpha$$

$$P(\blacksquare\,;\lambda_0) = \alpha$$

$$\vdots$$

When H_1 is true:

$P(\ \blacksquare\ ;\lambda_1) = ?$

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$P(\ \blacksquare\ ;\lambda_1) = ?$

$P(\ \blacksquare\ ;\lambda_1) = ?$

$P(\ \blacksquare\ ;\lambda_1) = ?$

⋮

Want these numbers to be large!!!

When H_1 is true:

$P(\blacksquare\,; \lambda_1) = ?$

$P(\blacksquare\,; \lambda_1) = ?$

$P(\blacksquare\,; \lambda_1) = ?$

$P(\blacksquare\,; \lambda_1) = ?$

$P(\blacksquare\,; \lambda_1) = ?$

$\vdots$

The best test will be largest!!!

Tests are defined by rejection regions.

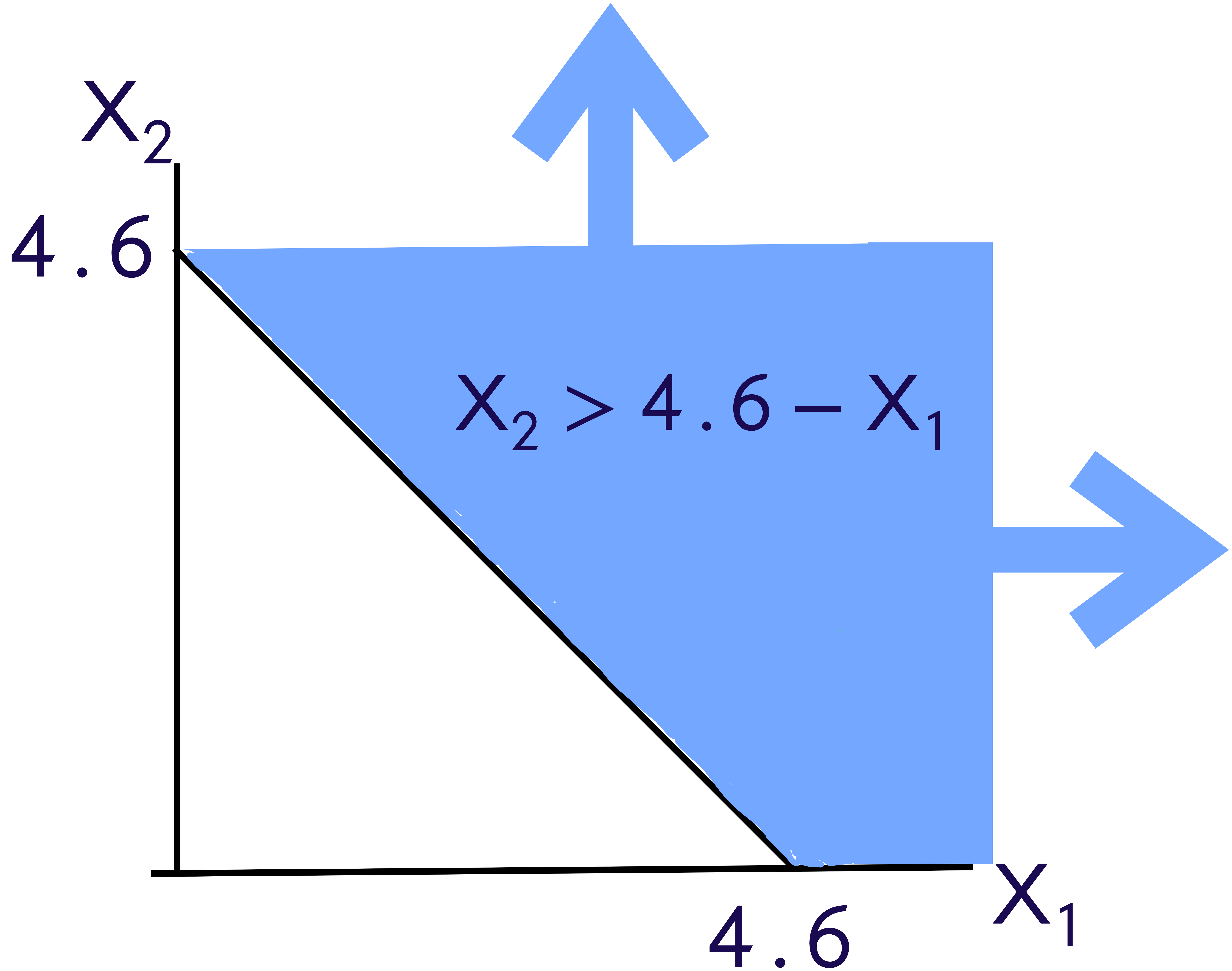
For example, when n=2:

Reject H_0 if $\overline{X} > 2.3$

$$\Leftrightarrow \quad \frac{X_1 + X_2}{2} > 2.3$$

$$\Leftrightarrow \quad X_2 > 4.6 - X_1$$

Reject H_0 if (X_1, X_2) is in this region



$$H_0 : \theta = \theta_0 \quad \textbf{vs.} \quad H_1 : \theta = \theta_1$$

$$P({\color{deepskyblue}\blacksquare}\ ; \lambda_0) = \alpha \qquad P(\overrightarrow{X} \in R_1; \lambda_0) = \alpha$$

$$P({\color{orange}\blacksquare}\ ; \lambda_0) = \alpha \qquad P(\overrightarrow{X} \in R_2; \lambda_0) = \alpha$$

$$P({\color{limegreen}\blacksquare}\ ; \lambda_0) = \alpha \qquad P(\overrightarrow{X} \in R_3; \lambda_0) = \alpha$$

$$P({\color{indigo}\blacksquare}\ ; \lambda_0) = \alpha \qquad P(\overrightarrow{X} \in R_4; \lambda_0) = \alpha$$

$$P({\color{pink}\blacksquare}\ ; \lambda_0) = \alpha \qquad P(\overrightarrow{X} \in R_5; \lambda_0) = \alpha$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1$$

$\boxed{\text{Definition:}}$

A test R^* is a best test of size/level α for the above hypotheses if

1. $P(\overrightarrow{X} \in R^*; \theta_0) = \alpha$ and

2. If R represents <u>any</u> other test with

$$P(\overrightarrow{X} \in R; \theta_0) = \alpha,$$

then

$$P(\overrightarrow{X} \in R^*; \theta_1) \geq P(\overrightarrow{X} \in R; \theta_1).$$

The Neyman-Pearson Lemma (setup)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf f which depends on an unknown parameter θ.

Write the joint pdf as

$$f(\overrightarrow{x};\theta) \stackrel{\text{iid}}{=} \prod_{i=1}^{n} f(x_i;\theta)$$

The Neyman-Pearson Lemma

$$H_0 : \theta = \theta_0 \quad \textbf{vs.} \quad H_1 : \theta = \theta_1$$

The best test of size/level α is to reject H_0, in favor of H_1 if $\overrightarrow{X} \in R^*$ where

$$R^* = \left\{ \overrightarrow{x} : \frac{f(\overrightarrow{x};\theta_0)}{f(\overrightarrow{x};\theta_1)} \leq c \right\}$$

For discrete $X_1, X_2, \ldots, X_n$

$$f(\overrightarrow{x}; \theta) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n; \theta)$$

$$R^* = \left\{ \overrightarrow{x} : \frac{f(\overrightarrow{x}; \theta_0)}{f(\overrightarrow{x}; \theta_1)} \leq c \right\}$$

- If H_0 is true and H_1 is false, the ratio is large.
- If H_0 is false and H_1 is true, the ratio is small.

This is when we should reject H_0!

Example:

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate λ>0.

FInd the best test of size/level α for testing

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

where $\lambda_1 > \lambda_0$.

pdf: $f(x;\lambda) = \lambda e^{-\lambda x}$

joint pdf: $f(\vec{x};\lambda) \overset{\text{iid}}{=} \prod_{i=1}^{n} f(x_i;\lambda)$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

"likelihood ratio": $\dfrac{f(\vec{x};\lambda_0)}{f(\vec{x};\lambda_1)}$

"likelihood ratio":

$$\frac{f(\overrightarrow{x};\lambda_0)}{f(\overrightarrow{x};\lambda_1)} = \frac{\lambda_0^n \, e^{-\lambda_0 \sum_{i=1}^n x_i}}{\lambda_1^2 \, e^{-\lambda_1 \sum_{i=1}^n x_i}}$$

$$= \left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)\sum_{i=1}^n x_i}$$

The Neyman-Pearson Lemma says:

"Reject H_0, in favor of H_1, if

$$\left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)\sum_{i=1}^n x_i} \leq c$$

where c is to be determined."

The rejection rule

$$\left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)\sum_{i=1}^n x_i} \leq c$$

is equivalent to the rule

"Reject H_0, in favor of H_1, if

$$e^{-(\lambda_0-\lambda_1)\sum_{i=1}^n x_i} \leq \left(\frac{\lambda_1}{\lambda_0}\right)^n c$$

This is a new constant.

The Neyman-Pearson Lemma says:

"Reject H_0, in favor of H_1, if

$$e^{-(\lambda_0-\lambda_1)\sum_{i=1}^{n} x_i} \leq c_1$$

where c_1 is to be determined."

$$e^{-(\lambda_0-\lambda_1)\sum_{i=1}^{n} x_i} \leq c_1$$

Taking the log of both sides, this is equivalent to

$$-(\lambda_0 - \lambda_1)\sum_{i=1}^{n} x_i \leq c_2$$

(Neyman-Pearson says to reject H_0 if this happens.)

$$-(\lambda_0 - \lambda_1) \sum_{i=1}^{n} x_i \leq c_2$$

Divide both sides by $-(\lambda_0 - \lambda_1)$.

Note that $\lambda_1 > \lambda_0$, so $-(\lambda_0 - \lambda_1) > 0$.

This means that the inequality won't flip.

$$\sum_{i=1}^{n} x_i \leq c_3$$

The Neyman-Pearson Lemma says:

"Reject H_0, in favor of H_1, if

$$\sum_{i=1}^{n} X_i \leq c_3$$

where c_3 is to be determined."

Let's find c_3.

$$\alpha = P(\text{ Type I Error })$$

$$= P(\text{ Reject } H_0 \, \lambda_0 \,)$$

$$= P(\sum_{i=1}^{n} X_i < c_3;\, \lambda_0 \,)$$

Wait a minute… this is equivalent to

$$= P(\,\overline{X} < c_4;\, \lambda_0 \,)$$

where $c_4 = c_3/n$.

$$\alpha = P(\overline{X} < c_4; \lambda_0)$$

We already did this in the previous video!

$$c_4 = \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

Conclusion:

The best test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

where $\lambda_1 > \lambda_0$,

is to reject H_0, in favor of H_1 if

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

Remember, R^* is the best test of size α if

$$P(\overrightarrow{X} \in R^*; \theta_0) = \alpha$$

and $$P(\overrightarrow{X} \in R^*; \theta_1) \geq P(\overrightarrow{X} \in R; \theta_1)$$

for any other test of size α.

$$P(\overrightarrow{X} \in R^*; \theta_1) \geq P(\overrightarrow{X} \in R; \theta_1)$$

Each of these tests has it's own power function.

$$\gamma_R(\theta) = P(\text{ Reject } H_0; \theta)$$

$$= P(\overrightarrow{X} \in R; \theta)$$

$$P(\overrightarrow{X} \in R^*; \theta_1) \geq P(\overrightarrow{X} \in R; \theta_1)$$

becomes

$$\gamma_{R^*}(\theta_1) \geq \gamma_R(\theta_1)$$

for any test described by R with

$$P(\overrightarrow{X} \in R; \theta_0) = \alpha$$

The best test of has <u>highest power</u> when H_1 is true.

"Best Test"

||

"Most Powerful Test"