Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and <u>known</u> variance σ^2.

Consider testing the simple versus simple hypotheses

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu < \mu_0$$

where μ_0 is fixed and known.

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

Step One:

Choose an estimator for μ.

$$\hat{\mu} = \overline{X}$$

Step Two:

Give the "form" of the test.

Reject H_0, in favor of H_1 if $\overline{X} < c$, where c is to be determined.

$$H_0 : \mu = \mu_0$$

Step Three:

$$H_1 : \mu < \mu_0$$

Find c.

$$\alpha = \max_{\mu=\mu_0} \mathrm{P}(\text{Type I Error})$$

$$= \max_{\mu=\mu_0} \mathrm{P}(\text{Reject } H_0; \mu)$$

$$= \mathrm{P}(\text{Reject } H_0; \mu_0)$$

$$= \mathrm{P}(\overline{X} < c; \mu_0)$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Three:

Find c.

$$\alpha = P(\overline{X} < c;\ \mu_0)$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Three:

Find c.

$$\alpha = P(\overline{X} < c; \mu_0)$$

$$\overline{X} \sim N(\mu_0, \sigma^2/n)$$

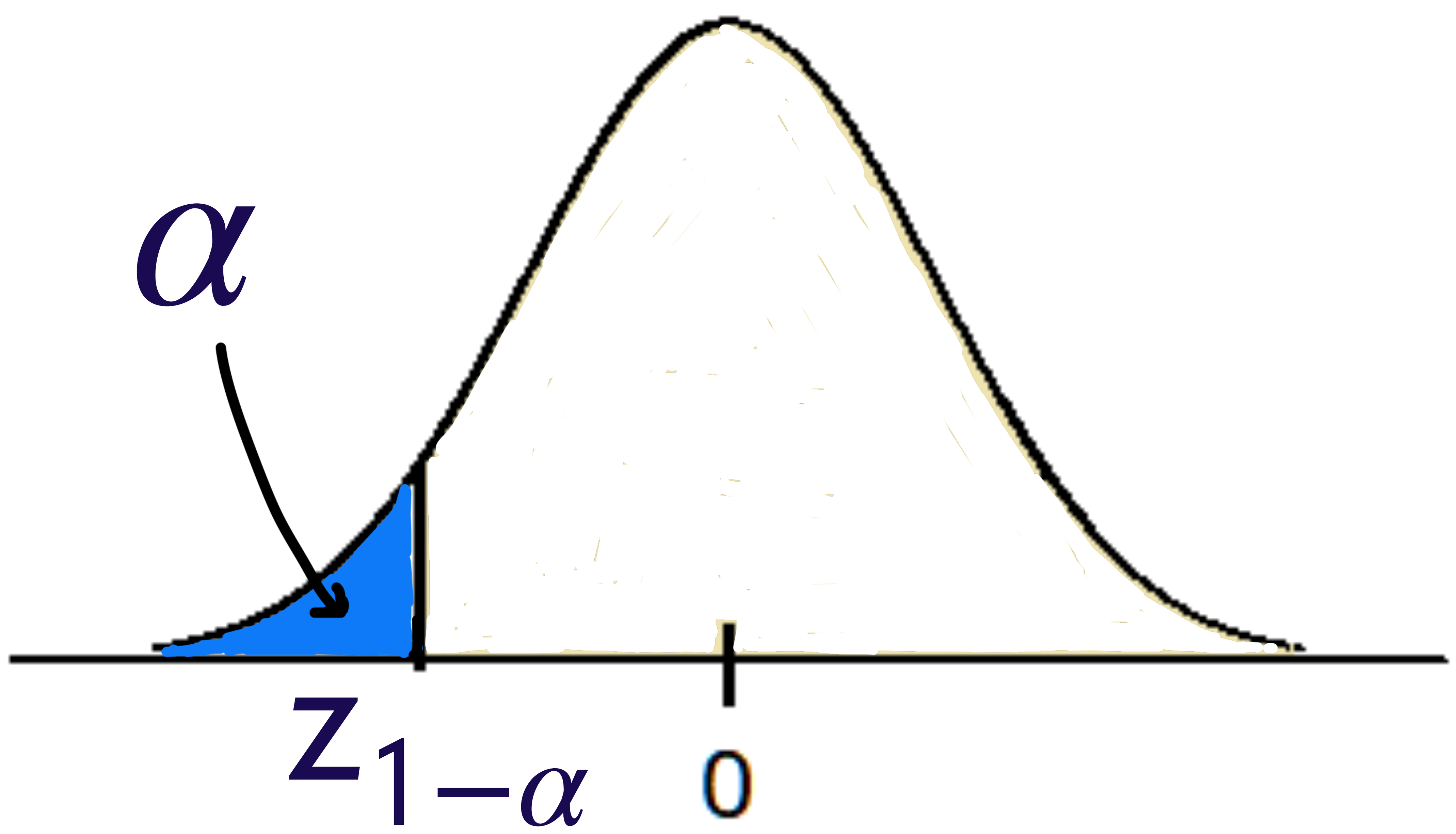
$$= P\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} < \frac{c - \mu_0}{\sigma/\sqrt{n}}; \mu_0 \right)$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Three:

Find c.

$$\alpha = P\left(Z < \frac{c - \mu_0}{\sigma/\sqrt{n}} \right)$$



$$\Rightarrow \quad \frac{c - \mu_0}{\sigma/\sqrt{n}} = z_{1-\alpha}$$

$$\Rightarrow \quad c = \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Four:

Conclusion:

Reject H_0, in favor of H_1, if

$$\overline{X} < \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

The Test:

Reject H_0**, in favor of** H_1**, if**

$$\overline{X} < \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

Question: What is β **?**

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = \max_{\mu<\mu_0} P(\text{Type II Error})$$

$$= \max_{\mu\in H_1} P(\text{Fail to Reject } H_0; \mu)$$

$$= \max_{\mu<\mu_0} P\left(\overline{X} \geq \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}; \mu\right)$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = \max_{\mu < \mu_0} P\left(\bar{X} \geq \mu_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}} ; \mu \right)$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = \max_{\mu < \mu_0} P\left(\bar{X} \geq \mu_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}} ; \mu \right)$$

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

$$\beta = \max_{\mu < \mu_0} P\left(\bar{X} \geq \mu_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}} ; \mu \right)$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = \max_{\mu<\mu_0} P\left(Z \geq \frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \right)$$

$$= \max_{\mu<\mu_0} \left[1 - \Phi\left(\underbrace{\frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}}}_{\textbf{decreasing in } \mu} \right) \right]$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = \max_{\mu<\mu_0} P\left(Z \geq \frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \right)$$

$$= \max_{\mu<\mu_0} \left[1 - \Phi\left(\frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \right) \right]$$

decreasing in μ

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = \max_{\mu<\mu_0} P\left(Z \geq \frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \right)$$

$$= \max_{\mu<\mu_0} \underbrace{\left[1 - \Phi\left(\frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \right) \right]}_{\text{increasing in } \mu}$$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = \max_{\mu<\mu_0} P\left(Z \geq \frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \right)$$

$$= \max_{\mu<\mu_0} \underbrace{\left[1 - \Phi\left(\frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \right) \right]}_{\text{increasing in } \mu}$$

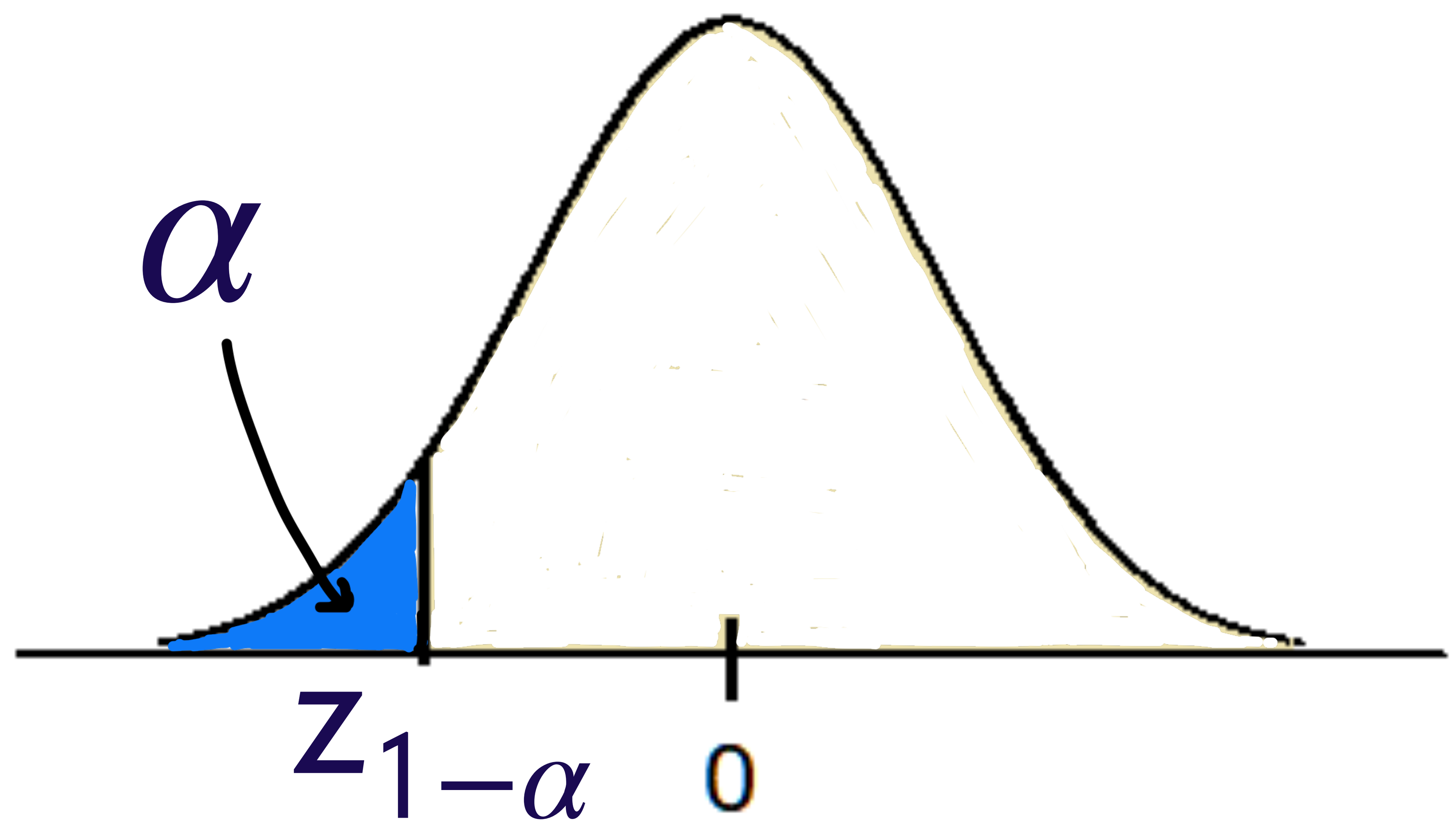
maxed at $\mu = \mu_0$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

$$\beta = 1 - \Phi\left(\frac{\mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \mu_0}{\sigma/\sqrt{n}}\right)$$

$$= 1 - \Phi(z_{1-\alpha})$$

$$= 1 - \alpha$$



$$\max\{x : 0 \leq x \leq 1\} = 1$$

$$\max\{x : 0 \leq x < 1\} = ?$$

$$\sup\{x : 0 \leq x < 1\} = 1$$

maximum versus “supremum”

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and known variance σ^2.

Consider testing the hypotheses

$$\boxed{H_0 : \mu \geq \mu_0} \qquad H_1 : \mu < \mu_0$$

where μ_0 is fixed and known.

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

Step One:

Choose an estimator for μ.

$$\hat{\mu} = \overline{X}$$

Step Two:

Give the "form" of the test.

Reject H_0, in favor of H_1 if $\overline{X} < c$, where c is to be determined.

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Three:

Find c.

$$\alpha = \max_{\mu \geq \mu_0} P(\text{Type I Error})$$

$$= \max_{\mu \geq \mu_0} P(\text{Reject } H_0; \mu)$$

$$= \max_{\mu \geq \mu_0} P(\overline{X} < c; \mu)$$

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Three:

Find c.

$$\alpha = \max_{\mu \geq \mu_0} P(\overline{X} < c;\ \mu)$$

$$= \max_{\mu \geq \mu_0} P\left(Z < \frac{c - \mu}{\sigma/\sqrt{n}} \right)$$

$$= \max_{\mu \geq \mu_0} \Phi\left(\frac{c - \mu}{\sigma/\sqrt{n}} \right)$$

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Three: **Find c.**

$$\alpha = \max_{\mu \geq \mu_0} P(\overline{X} < c;\ \mu)$$

$$= \max_{\mu \geq \mu_0} P\left(Z < \frac{c - \mu}{\sigma/\sqrt{n}} \right)$$

$$= \max_{\mu \geq \mu_0} \Phi\left(\frac{c - \mu}{\sigma/\sqrt{n}} \right)$$

decreasing in μ

$H_0 : \mu \geq \mu_0$
$H_1 : \mu < \mu_0$

Step Three: Find c.

$$\alpha = \max_{\mu \geq \mu_0} P(\overline{X} < c; \mu)$$

$$= \max_{\mu \geq \mu_0} P\left(Z < \frac{c - \mu}{\sigma/\sqrt{n}} \right)$$

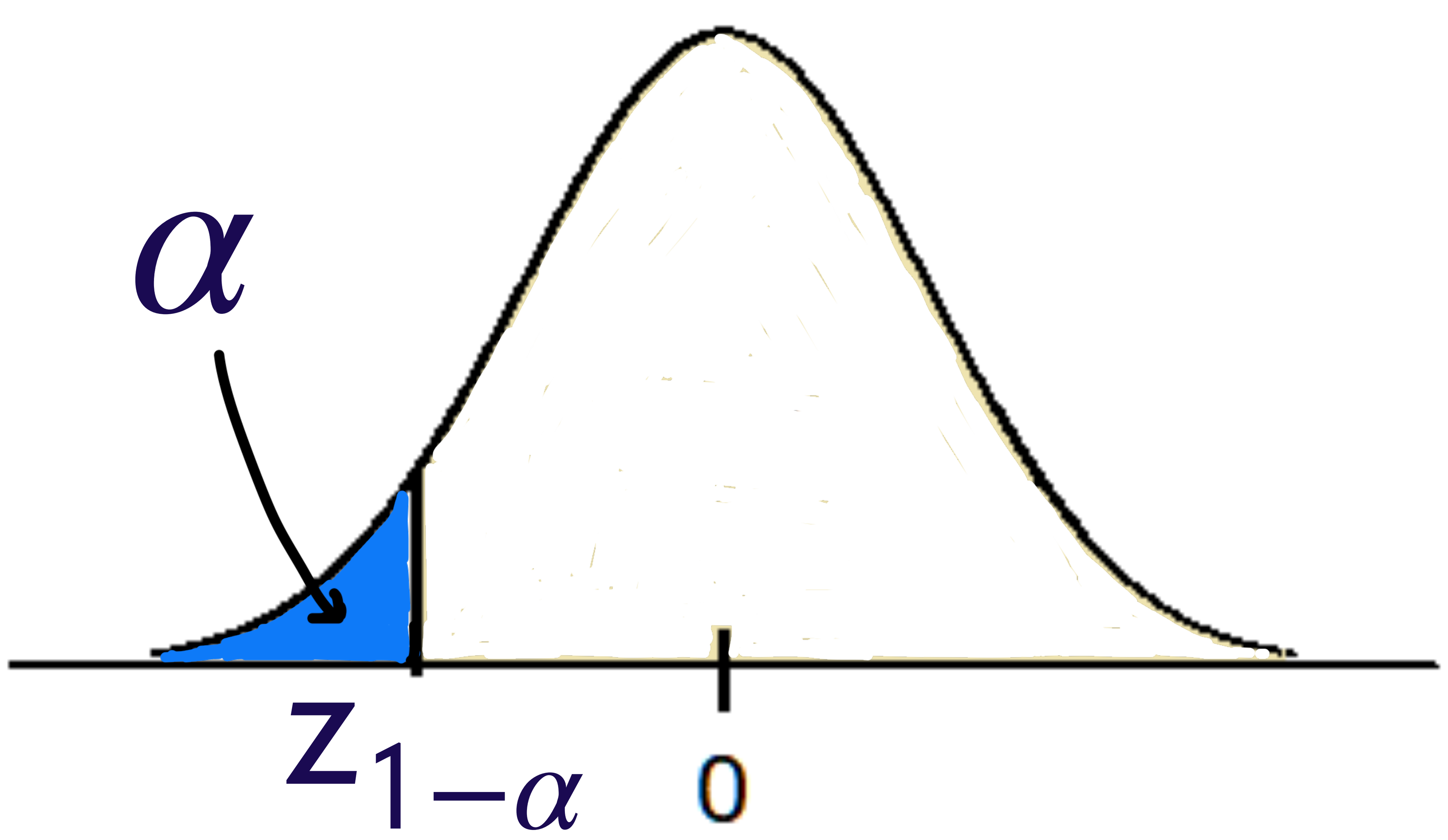
$$= \max_{\mu \geq \mu_0} \Phi\left(\frac{c - \mu}{\sigma/\sqrt{n}} \right)$$

decreasing in μ

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

Step Three: Find c.

$$\alpha = \Phi\left(\frac{c - \mu_0}{\sigma/\sqrt{n}}\right)$$



$$\Rightarrow \frac{c - \mu_0}{\sigma/\sqrt{n}} = z_{1-\alpha}$$

$$\Rightarrow c = \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

Step Four: Conclusion

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

Reject H_0, in favor of H_1, if

$$\overline{X} < \mu_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}}$$

Example:

In 2019, the average health care annual premium for a family of 4 in the United States, was reported to be $6,015.

[The Kaiser Family Foundation, "Employer Health Benefits 2019 Annual Survey"]

In a more recent survey, 100 randomly sampled families of 4 reported an average annual health care premium of $6,537.

Can we say that the true average is currently greater than $6,015 for all families of 4?

Example:

Assume that annual health care premiums are normally distributed with a standard deviation of $814.

Let μ be the true average for all families of 4.

Step Zero:

Set up the hypotheses.

$$H_0 : \mu = 6015 \qquad H_1 : \mu > 6015$$

Step Zero:

Set up the hypotheses.

$$H_0 : \mu = 6015 \qquad H_1 : \mu > 6015$$

Decide on a level of significance.

$$\alpha = 0.10$$

Step One:

Choose an estimator for μ.

$$\hat{\mu} = \overline{X}$$

Step Two:

Give the form of the test.

Reject H_0, in favor of H_1, if

$$\overline{X} > c$$

for some c to be determined.

Step Three: Find c.

$$\alpha = \max_{\mu = \mu_0} P(\text{Type I Error}; \mu)$$

$$= P(\text{Type I Error}; \mu_0)$$

$$\alpha = P(\text{Reject } H_0; \mu_0)$$

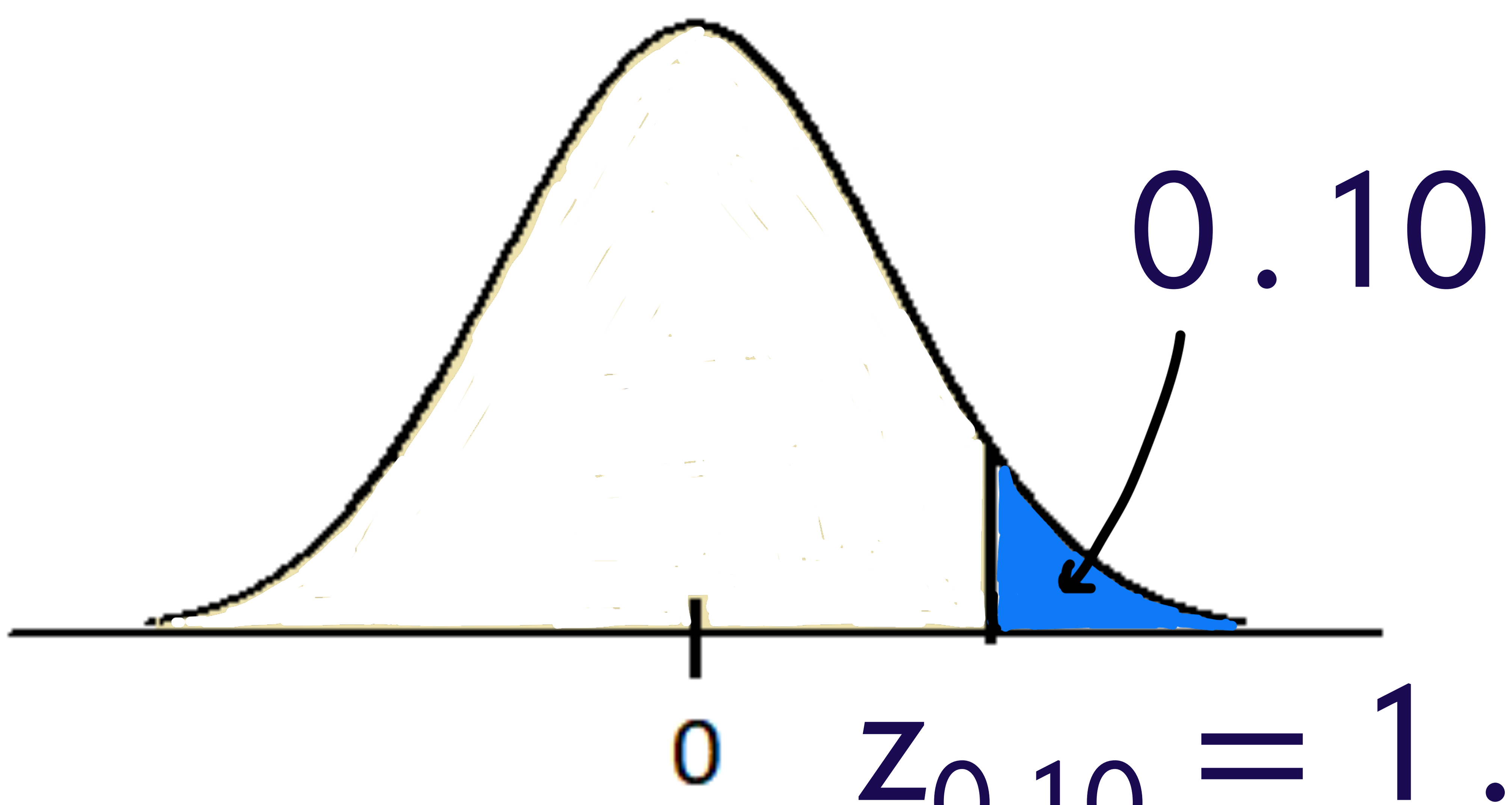
When it's true!

$$= P(\bar{X} > c; \mu_0)$$

$$= P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > \frac{c - 6015}{814/\sqrt{100}}; \mu_0 \right)$$

$$= P\left(Z > \frac{c - 6015}{814/\sqrt{100}} \right)$$

$$0.10 = P\left(Z > \frac{c - 6015}{814/\sqrt{100}}\right)$$



$z_{0.10} = 1.28$

qnorm(0.90)

$$\Rightarrow \quad \frac{c - 6015}{814/\sqrt{100}} = 1.28$$

$$\Rightarrow \quad c = 6119.19$$

Step Four:

Conclusion.

Reject H_0, in favor of H_1, if

$$\overline{X} > 6119.19$$

From the data, where $\overline{x} = 6537$, we reject H_0 in favor of H_1.

The data suggests that the true mean annual health care premium is greater than $6015.