Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf $f(x;\theta)$.

Let $\widehat{\theta}_n$ be an MLE for θ.

Under certain "regularity conditions" such as those needed for the CRLB.

- $\widehat{\theta}_n$ exists and is unique.
- $\widehat{\theta}_n \xrightarrow{P} \theta$. We say that $\widehat{\theta}_n$ is a consistent estimator of θ.

- $\widehat{\theta}_n$ is an asymptotically unbiased estimator of θ.

$$\text{i.e.} \qquad \lim_{n\to\infty} E[\widehat{\theta}_n] = \theta$$

- $\widehat{\theta}_n$ is asymptotically efficient.

$$\text{i.e.} \qquad \lim_{n\to\infty} \frac{CRLB_\theta}{Var[\widehat{\theta}_n]} = 1$$

- $\hat{\theta}_n \sim N(\theta, CRLB_\theta)$

i.e.

$$\frac{\hat{\theta}_n - \theta}{\sqrt{CRLB_\theta}} \xrightarrow{d} N(0, 1)$$

Example: (verifications)

$$X_1, X_2, \ldots, X_n \sim \exp(\text{rate} = \lambda)$$

We have seen that the MLE for λ is

$$\hat{\lambda} = \frac{1}{\overline{X}}$$

Existence and uniqueness ✓

Example: (continued)

We have seen that

$$E[\hat{\lambda}] = \frac{n}{n-1}\lambda$$

which goes to λ as $n \to \infty$.

Asymptotically unbiased ✔

Example: (continued)

We have seen that $\overline{X} \xrightarrow{P} E[X_1] = 1/\lambda$

Is it true that

$$\hat{\lambda} = \frac{1}{\overline{X}} \to \frac{1}{1/\lambda} = \lambda \qquad \textbf{?}$$

Suppose that $\{X_n\}$ and $\{Y_n\}$ be sequences of random variables such that $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ for random variables X and Y.

Then

- $X_n + Y_n \xrightarrow{P} X + Y$
- $X_n Y_n \xrightarrow{P} XY$
- $X_n / Y_n \xrightarrow{P} X / Y$ (if $P(Y \neq 0) = 1$)
- $g(X_n) \xrightarrow{P} g(X)$ (for g continuous)

Thus,

Using $g(x) = 1/x$, we do have that
$\overline{X} \xrightarrow{P} E[X_1] = 1/\lambda$

implies that

$$\hat{\lambda} = \frac{1}{\overline{X}} \xrightarrow{P} \frac{1}{1/\lambda} = \lambda$$

Consistent ✔

We saw that the CRLB for λ is

$$\mathrm{CRLB}_{\lambda} = \frac{\lambda^2}{n}$$

$$\frac{n}{n-1}\lambda$$

$$\mathrm{Var}[\hat{\lambda}] = \mathrm{Var}\left[\frac{1}{\overline{X}}\right]$$

$$= E\left[\left(\frac{1}{\overline{X}}\right)^2\right] - \left(E\left[\frac{1}{\overline{X}}\right]\right)^2$$

$$E\left[\left(\frac{1}{\overline{X}}\right)^2\right] = E\left[\frac{n^2}{Y^2}\right] \quad \text{where } Y \sim \Gamma(\alpha, \beta)$$

$$= n^2 \int_{-\infty}^{\infty} \frac{1}{y^2} f_Y(y)\, dy = n \int_0^{\infty} \frac{1}{y^2} \cdot \frac{1}{\Gamma(n)} \lambda^n y^{n-1} e^{-\lambda y}\, dy$$

$$= n \int_0^{\infty} \frac{1}{\Gamma(n)} \lambda^n \underbrace{y^{n-3} e^{-\lambda y}}\, dy$$

Looks like a $\Gamma(n-2, \lambda)$ pdf

$$= n^2 \lambda^2 \frac{\Gamma(n-2)}{\Gamma(n)} \underbrace{\int_0^{\infty} \frac{1}{\Gamma(n-2)} \lambda^{n-2} y^{n-3} e^{-\lambda y}\, dy}_{=1}$$

$$= \frac{n^2}{(n-1)(n-2)} \lambda^2$$

$$\mathrm{Var}\left[\frac{1}{\overline{X}}\right] = E\left[\left(\frac{1}{\overline{X}}\right)^2\right] - \left(E\left[\frac{1}{\overline{X}}\right]\right)^2$$

$$= \frac{n^2}{(n-1)(n-2)}\lambda^2 - \left(\frac{n}{n-1}\lambda^2\right)$$

$$= \frac{n^2}{(n-1)^2(n-2)}\lambda^2$$

$$\frac{\mathrm{CRLB}_\theta}{\mathrm{Var}[\widehat{\theta}_n]} = \frac{\frac{\lambda^2}{n}}{\frac{n^2\lambda^2}{(n-1)^2(n-2)}} = \frac{(n-1)^2(n-2)}{n^3} \to 1$$

Asymptotically Efficient as $n \to \infty$

Recall the Weak Law of Large Numbers where we showed that $\overline{X} \xrightarrow{P} \mu$.

We used:

- Chebyshev's inequality
- the fact that $\overline{X}$ is an unbiased estimator of the mean μ
- the fact that $\mathrm{Var}[\overline{X}] \to 0$

The exact same proof can be used to show the following.

If $\widehat{\theta}_n$ is an unbiased estimator of θ

and if $\lim_{n \to \infty} \mathrm{Var}[\widehat{\theta}_n] = 0$,

then $\widehat{\theta}_n \xrightarrow{P} \theta$.

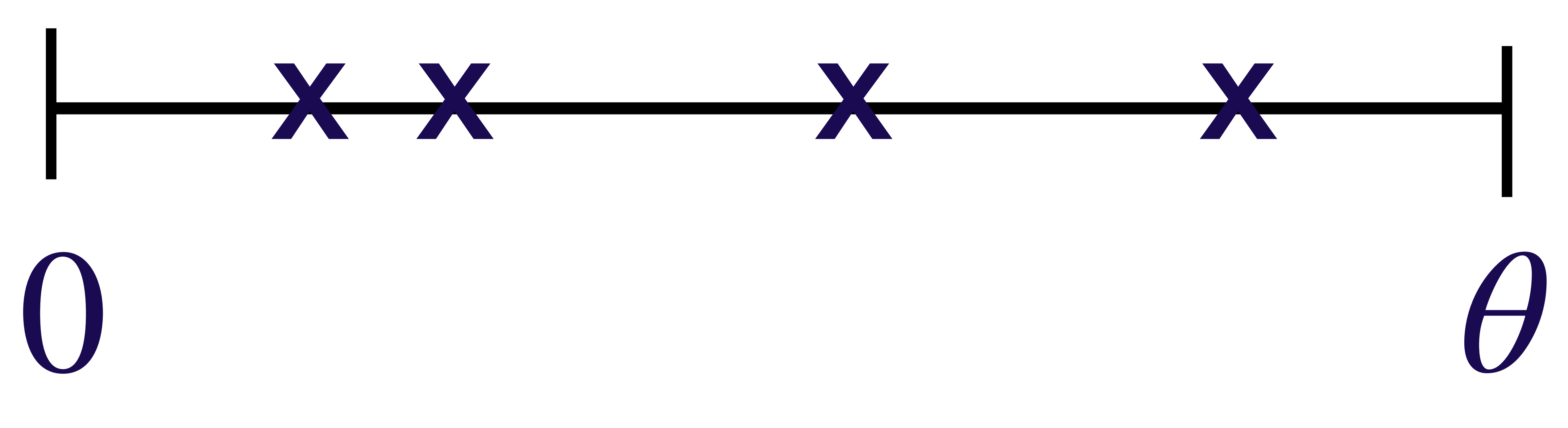
Using the generalized Markov inequality, we can show that this actually holds when “unbiased” is replaced by “asymptotically unbiased”.

We can use this to show, for example, that if $X_1, X_2, \ldots, X_n \sim \text{unif}(0, \theta)$,

The maximum

$$Y_n = \max(X_1, X_2, \ldots, X_n)$$

is a consistent estimator of θ.

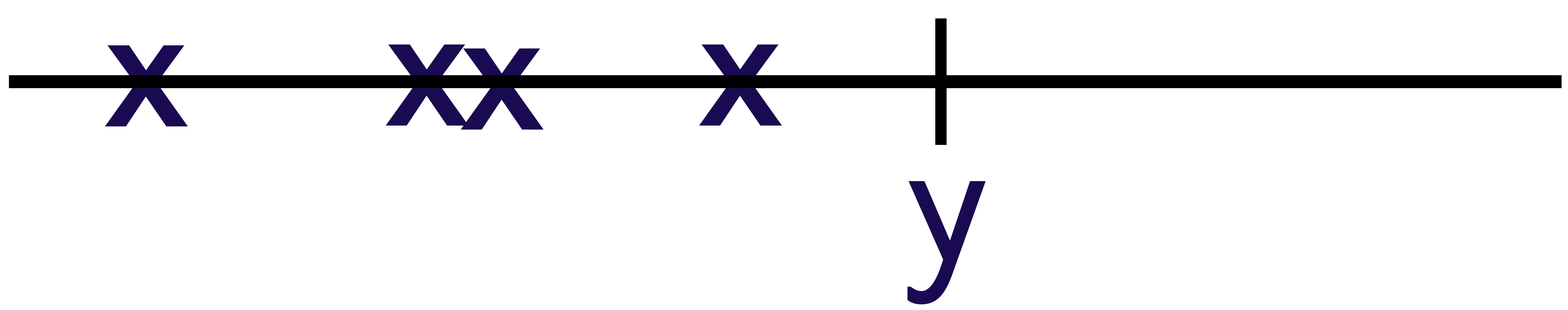


$$X_1, X_2, \ldots, X_n \sim \text{unif}(0, \theta)$$

$$Y_n = \max(X_1, X_2, \ldots, X_n)$$

What is the distribution of Y?

$$P(Y_n \leq y) = P(\max(X_1, X_2, \ldots, X_n \leq y)$$



$$= P(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y)$$

$$X_1, X_2, \ldots, X_n \sim \text{unif}(0, \theta)$$

$$Y_n = \max(X_1, X_2, \ldots, X_n)$$

$$\begin{aligned}
P(Y_n \le y) \\
&= P(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \\
&= P(X_1 \le y)\, P(X_2 \le y) \cdots P(X_n \le y) \\
&= [P(X_1 \le y)]^n = \left[\frac{y}{\theta}\right]^n
\end{aligned}$$

$$\text{for } 0 \le y \le \theta.$$

The pdf for $Y_n = \max(X_1, X_2, \ldots, X_n)$ is

$$f_{Y_n}(y) = \frac{d}{dy}F_{Y_n}(y) = \frac{d}{dy}\left[\frac{y}{\theta}\right]^n = \frac{n}{\theta^n}y^{n-1}$$

for $0 \le y \le \theta$.

The expected value of the maximum is then

$$E[Y_n] = \int_{-\infty}^{\infty} y\, f_{Y_n}(y)\, dy = \int_0^{\theta} \frac{n}{\theta^n} y^n\, dy = \frac{n}{n+1}\theta$$

$$X_1, X_2, \ldots, X_n \sim \mathrm{unif}(0, \theta)$$

$$Y_n = \max(X_1, X_2, \ldots, X_n)$$

$$E[Y_n] = \frac{n}{n+1}\theta$$

$$\mathrm{Var}[Y_n] = \frac{n}{(n+1)^2(n+2)}\theta^2$$

Consistent ✔