📘 Module 1 (Part 2) – Axioms of Probability & Calculating Probabilities

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🎯 Learning Objectives

- Understand the **axioms of probability**
- Apply them to **calculate probabilities** of events
- Use **set operations**, **complements**, and **Venn diagrams**
- Learn key **formulas and consequences** from these axioms

🔢 What Is Probability?

- We define **P(A)** as the probability that event A occurs.
- This provides a **numerical measure** of the likelihood of an event.

📐 The Three Axioms of Probability

1. **Boundedness**
    - 0≤P(A)≤10 \leq P(A) \leq 10≤P(A)≤1
    - No negative or >1 probabilities allowed.
2. **Sample Space Completeness**
    - P(S)=1P(S) = 1P(S)=1
    - Some outcome from the sample space must occur.

3. **Additivity for Mutually Exclusive Events**
    - If $A_1, A_2, \ldots, A_n$ are mutually exclusive: $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$
    - This extends to **infinite mutually exclusive events**: $P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$

🎲 Example: Flipping a Coin Until First Tail Appears

- Sample space: {T, HT, HHT, HHHT, ...}
- Define A_n = Tail on the *n*-th flip
    - $P(A_1) = \frac{1}{2}$
    - $P(A_2) = \frac{1}{4}$
    - $P(A_5) = \frac{1}{32}$
    - $P(A_n) = \frac{1}{2^n}$
- Since all A_n are **mutually exclusive**, total probability: $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

❓Example: At Least 3 Flips to Get a Tail

- Define event **B** = get tail after at least 3 flips
- Then B^c = get tail on 1st or 2nd flip: $P(B^c) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
- So, $P(B) = 1 - P(B^c) = 1 - \frac{3}{4} = \frac{1}{4}$

📌 Consequences of the Axioms

1. **Complement Rule**:
   P(Ac)=1−P(A)P(A^c) = 1 - P(A)P(Ac)=1−P(A)
2. **Empty Set**:
   P(∅)=0P(\emptyset) = 0P(∅)=0
3. **Inclusion-Exclusion Principle (Two Events)**:
   P(A∪B)=P(A)+P(B)−P(A∩B)P(A \cup B) = P(A) + P(B) - P(A \cap B)P(A∪B)=P(A)+P(B)−P(A∩B)

🚗 Example: Car Defects

- 3 defects: engine (A), seatbelt (B), paint (C)
- Given:
  - P(A)=0.2P(A) = 0.2P(A)=0.2, P(B)=0.25P(B) = 0.25P(B)=0.25, P(C)=0.3P(C) = 0.3P(C)=0.3
  - P(A∩B)=0.05P(A \cap B) = 0.05P(A∩B)=0.05
  - P(B∩C)=0.075P(B \cap C) = 0.075P(B∩C)=0.075
  - P(A∩C)=0.06P(A \cap C) = 0.06P(A∩C)=0.06
  - P(A∩B∩C)=0.015P(A \cap B \cap C) = 0.015P(A∩B∩C)=0.015

🔍 Questions & Solutions

✅ 1. P(no Defect)

- No defect = complement of A∪B∪CA \cup B \cup CA∪B∪C
- Use inclusion-exclusion for 3 events:
  P(A∪B∪C)=P(A)+P(B)+P(C)−P(A∩B)−P(B∩C)−P(A∩C)+P(A∩B∩C)P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)P(A∪B∪C)=P(A)+P(B)+P(C)−P(A∩B)−P(B∩C)−P(A∩C)+P(A∩B∩C)
  =0.2+0.25+0.3−0.05−0.075−0.06+0.015=0.7= 0.2 + 0.25 + 0.3 - 0.05 - 0.075 - 0.06 + 0.015 = 0.7=0.2+0.25+0.3−0.05−0.075−0.06+0.015=0.7

- So:
  P(no defect)=1−0.7=0.3P(\text{no defect}) = 1 - 0.7 = 0.3P(no defect)=1−0.7=0.3

✅ **2. P(Defect 1 did not occur)**

- P(Ac)=1−P(A)=1−0.2=0.8P(A^c) = 1 - P(A) = 1 - 0.2 = 0.8P(Ac)=1−P(A)=1−0.2=0.8

✅ **3. P(Defect 1 AND 3, but NOT 2)**

- Target event: 1st and 3rd defect only → binary = (1, 0, 1)
- Subtract out full intersection from two-way:
  P(A∩C)−P(A∩B∩C)=0.06−0.015=0.045P(A \cap C) - P(A \cap B \cap C) = 0.06 - 0.015 = 0.045P(A∩C)−P(A∩B∩C)=0.06−0.015=0.045

📌 Summary: Key Formulas

Rule	Formula
Complement	P(Ac)=1−P(A)P(A^c) = 1 - P(A)P(Ac)=1−P(A)
Mutually Exclusive Additivity	P(A1∪A2∪…)=∑P(Ai)P(A_1 \cup A_2 \cup \dots) = \sum P(A_i)P(A1∪A2∪…)=∑P(Ai)
Inclusion-Exclusion (2 events)	P(A∪B)=P(A)+P(B)−P(A∩B)P(A \cup B) = P(A) + P(B) - P(A \cap B)P(A∪B)=P(A)+P(B)−P(A∩B)
Inclusion-Exclusion (3 events)	P(A∪B∪C)=P(A)+P(B)+P(C)P(A \cup B \cup C) = P(A) + P(B) + P(C)P(A∪B∪C)=P(A)+P(B)+P(C)
	−P(A∩B)−P(B∩C)−P(A∩C)- P(A \cap B) - P(B \cap C) - P(A \cap C)−P(A∩B)−P(B∩C)−P(A∩C)
	+P(A∩B∩C)+ P(A \cap B \cap C)+P(A∩B∩C)