Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate $\lambda > 0$.

Derive a hypothesis test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

What statistic should we use?

Test 1: Using the Sample Mean

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step One:

Choose a statistic.

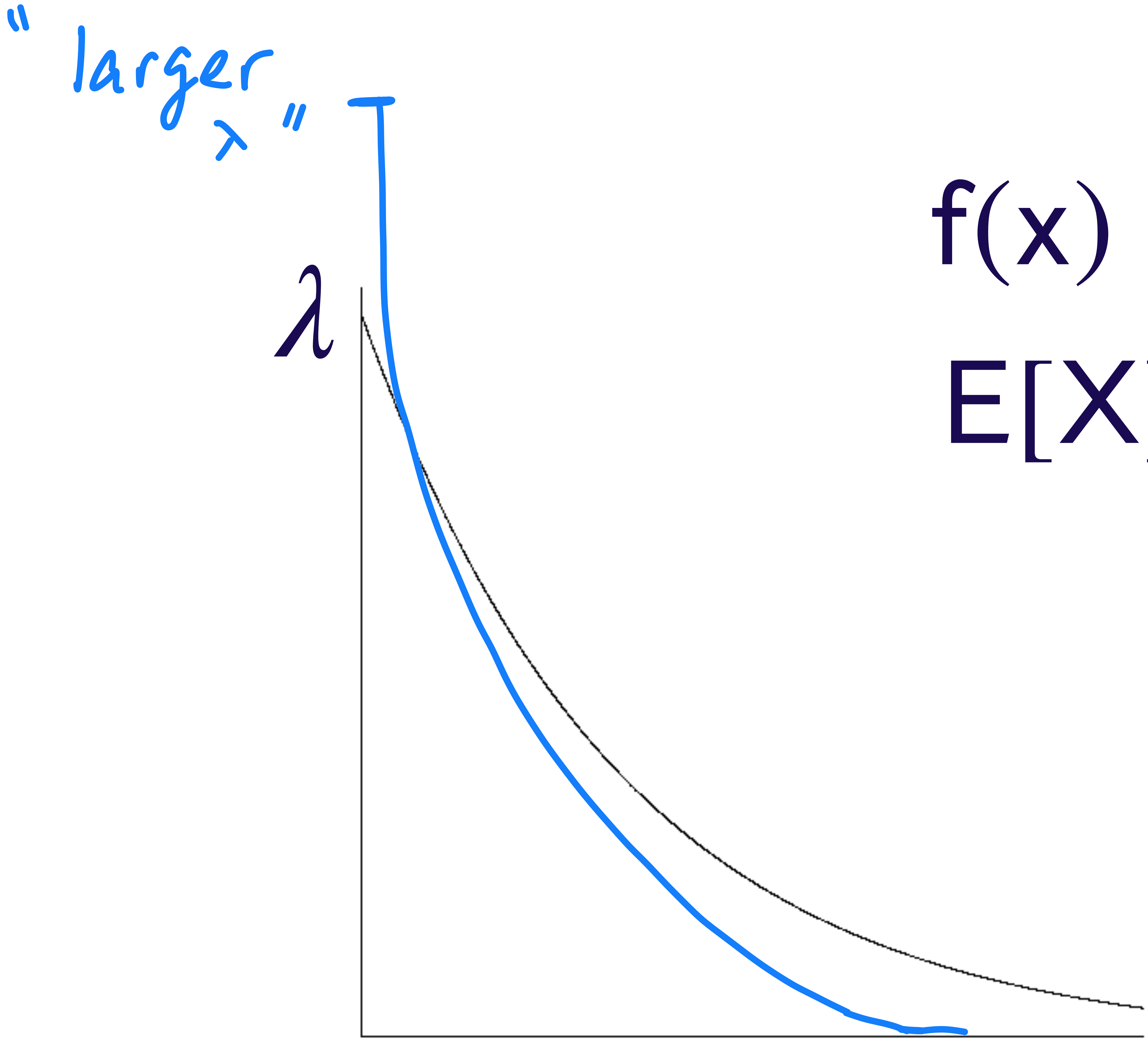
$$\overline{X}$$

Test 1: Using the Sample Mean

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Two:

Give the form of the test.



$$f(x) = \lambda e^{-\lambda x}$$

$$E[X] = 1/\lambda$$

Test 1: Using the Sample Mean

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Two:

Give the form of the test.

Reject H_0, in favor of H_1, if

$$\overline{X} < c$$

for some c to be determined.

Test 1: Using the Sample Mean

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Three:

Find c.

$$\alpha = P(\text{ Type I Error })$$

$$= P(\text{ Reject } H_0; \lambda_0)$$

$$= P(\overline{X} < c; \lambda_0)$$

Test 1: Using the Sample Mean

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Three:

Find c.

$$\alpha = P(\,\overline{X} < c;\ \lambda_0\,)$$
$$= P(\,2n\lambda_0\overline{X} < 2n\lambda_0 c;\ \lambda_0\,)$$
$$= P(\,W < 2n\lambda_0 c;\ \lambda_0\,)$$

where $W \sim \chi^2(2n)$.

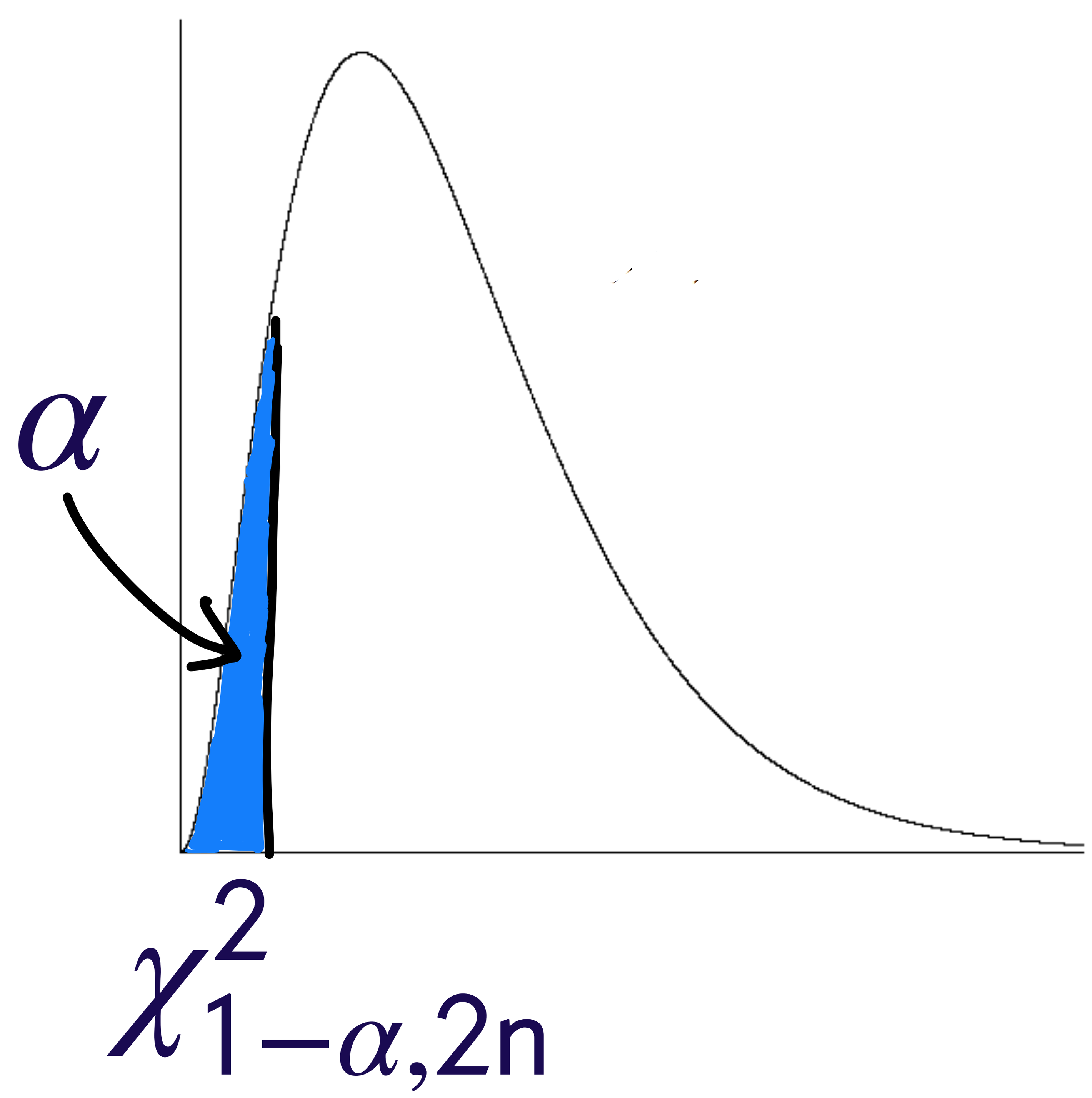
Test 1: Using the Sample Mean

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Three:

Find c.

$$\alpha = P(\, W < 2n\lambda_0 c; \,)$$



We want

$$2n\lambda_0 c = \chi^2_{1-\alpha,2n}$$

Test 1: Using the Sample Mean

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Four:

Conclusion.

Reject H_0, in favor of H_1, if

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

$$\chi^2_{\alpha,\mathrm{n}}$$

In R, get

$$\chi^2_{0.10,6}$$

by typing

```
qchisq(0.90,6)
```

Test 2: Using the Sample Minimum

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step One:

Choose a statistic.

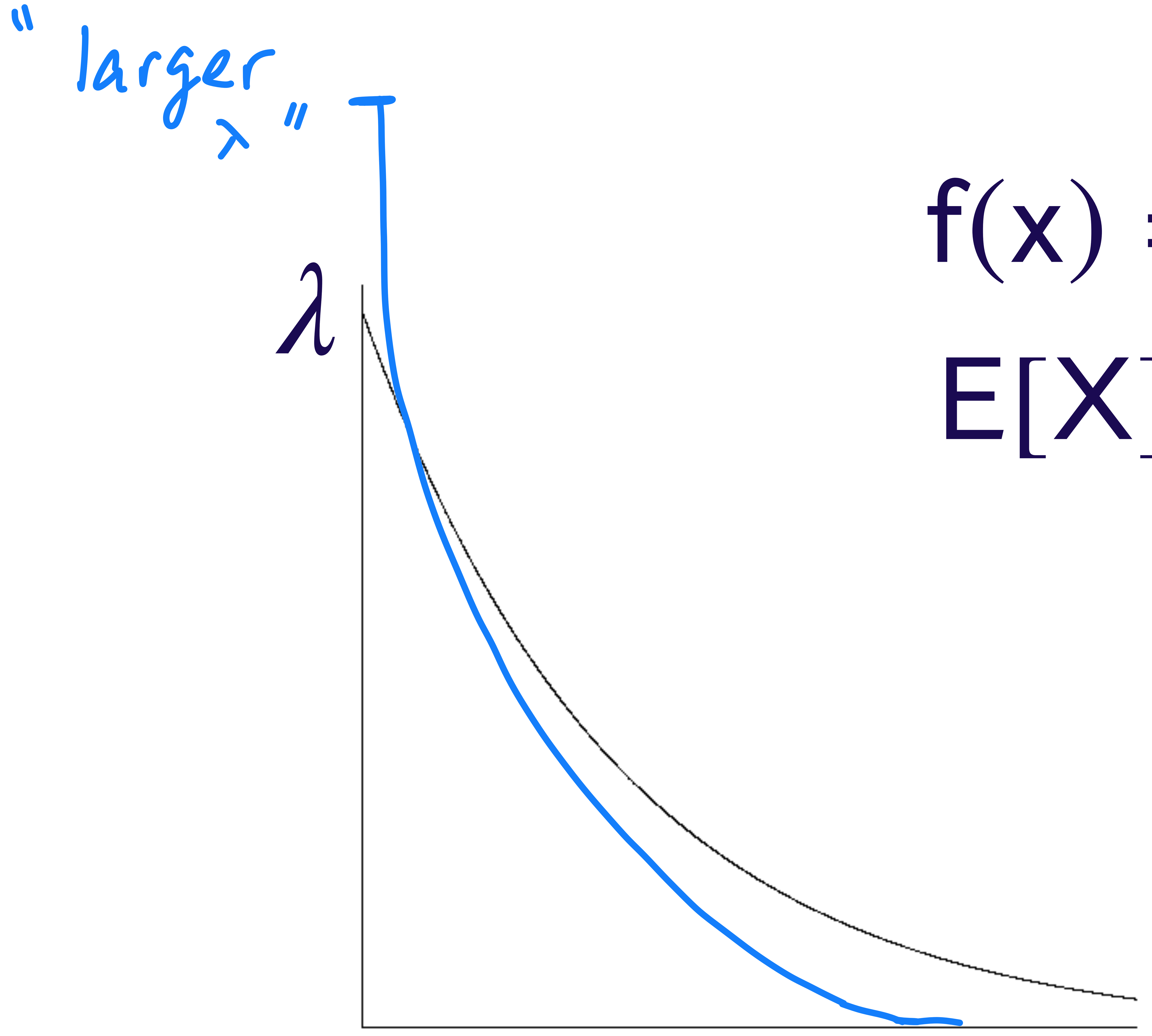
$$Y_n = \min(X_1, X_2, \ldots, X_n)$$

$\boxed{\text{Test 2: Using the Sample Minimum}}$

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Two:

Give the form of the test.



$$f(x) = \lambda e^{-\lambda x}$$

$$E[X] = 1/\lambda$$

Test 2: Using the Sample Minimum

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Two:

Give the form of the test.

Reject H_0, in favor of H_1, if

$$Y_n < c$$

for some c to be determined.

Test 2: Using the Sample Minimum

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Three:

Find c.

$$\alpha = P(\text{ Type I Error })$$
$$= P(\text{ Reject } H_0;\ \lambda_0\ )$$
$$= P(\ Y_n < c;\ \lambda_0\ )$$

Test 2: Using the Sample Minimum

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Step Three:

Find c.

$$\alpha = P(Y_n < c; \lambda_0)$$

$$Y_n \sim \exp(\text{rate} = n\lambda_0)$$

$$n\lambda_0 Y_n \sim \exp(\text{rate} = 1)$$

Test 2: Using the Sample Minimum

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

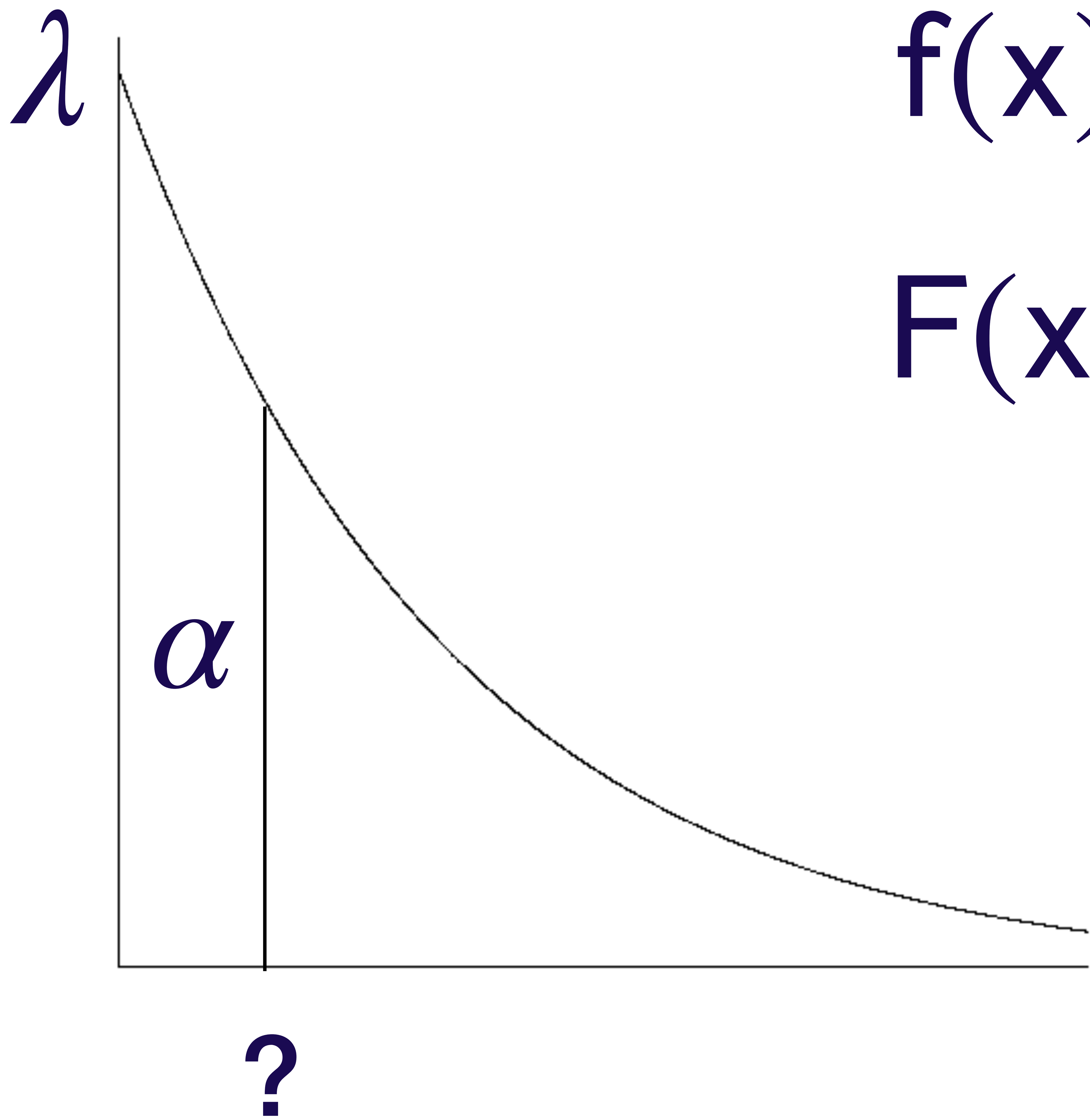
Step Three:

Find c.

$$\alpha = P(\, n\lambda_0 Y_n < cn\lambda_0;\, \lambda_0\,)$$

$$= P(\, X < cn\lambda_0\,)$$

where $X \sim \exp(rate = 1)$.



$$f(x) = e^{-x}$$

$$F(x) = P(X \leq x)$$
$$= 1 - e^{-x}$$

$$1 - e^{-?} = \alpha$$

$$\Rightarrow \quad ? = -\ln(1 - \alpha)$$

$$\alpha = P(X < cn\lambda_0)$$

$$\Downarrow$$

$$cn\lambda_0 = -\ln(1-\alpha)$$

$$\Downarrow$$

$$c = \frac{-\ln(1-\alpha)}{n\lambda_0}$$

Test 2: Using the Sample Minimum

Step Four:

$$H_0 : \lambda = \lambda_0$$
$$H_1 : \lambda > \lambda_0$$

Conclusion.

Reject H_0, in favor of H_1, if

$$Y_n = \min(X_1, X_2, \ldots, X_n) < \frac{-\ln(1-\alpha)}{n\lambda_0}$$

Compare the Tests:

Test 1: based on $\overline{X}$

$$\gamma_1(\lambda) = P(\text{ Reject } H_0 \;; \lambda\,)$$

$$= P\left(\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0} ; \lambda \right)$$

$$= P\left(2n\lambda\overline{X} < 2n\lambda\frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0} ; \lambda \right)$$

$$= P\left(W < \frac{\lambda}{\lambda_0}\chi^2_{1-\alpha,2n} \right) \qquad W \sim \chi^2(2n)$$

$$n = 10 \qquad \lambda_0 = 1 \qquad \alpha = 0.05$$

$$\chi^2_{1-\alpha,2n} = \chi^2_{0.95,20} = 10.851$$

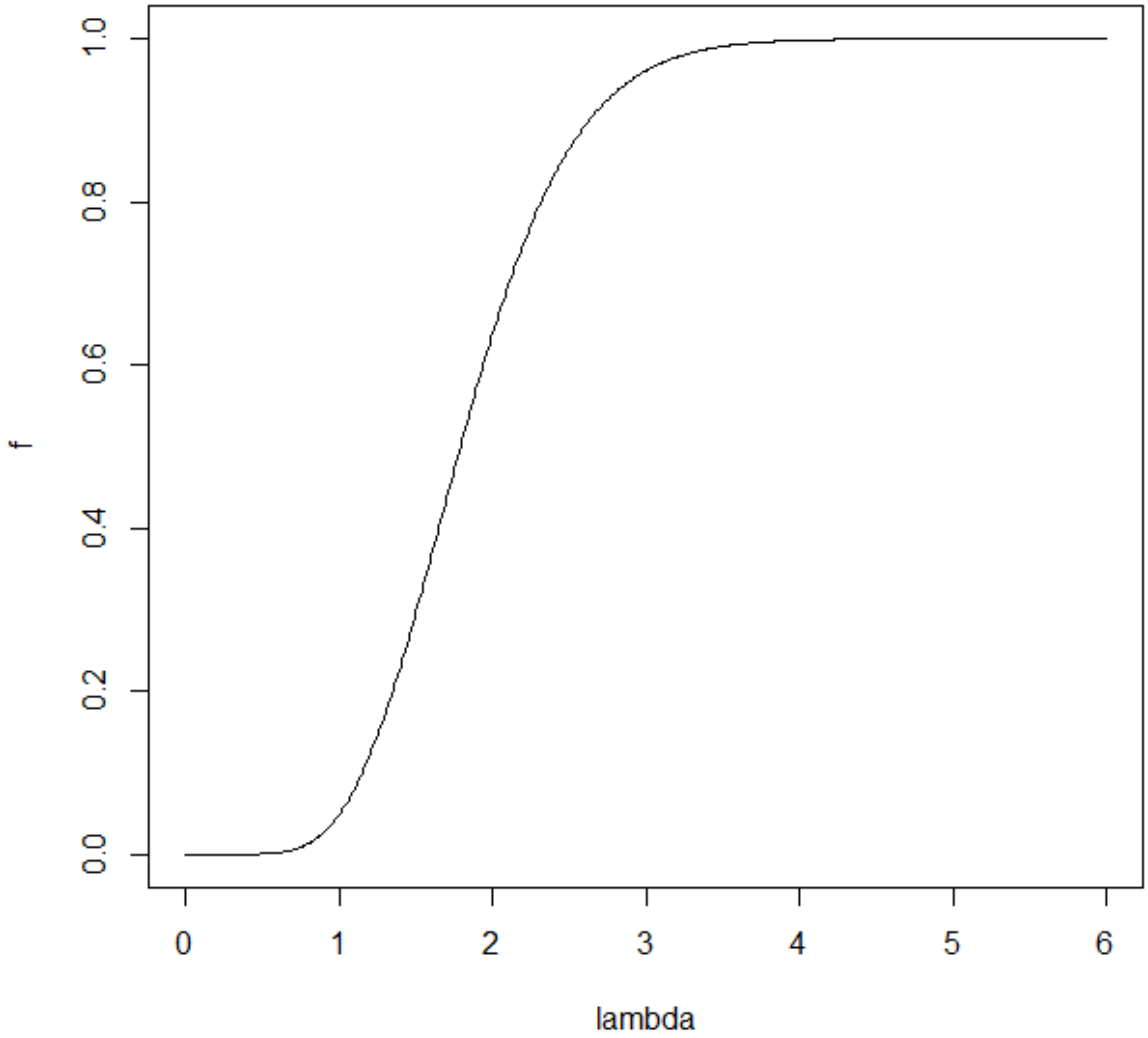
qchisq(0.05, 20)

$$\gamma_1(\lambda) = P\left(W < \frac{\lambda}{\lambda_0}\chi^2_{1-\alpha,2n} \right)$$

$$= P(W < 10.851\lambda)$$

Plot in R:

```
lambda<-seq(0,6,0.01)
f<-pchisq(10.81*lambda,20)
```



Compare the Tests:

Test 2: based on

$exp(rate = n\lambda)$

$$Y_n = \min(X_1, X_2, \ldots, X_n)$$

$$\gamma_2(\lambda) = P(\text{ Reject } H_0 \; ; \lambda \;)$$

$$= P\left(Y_n < \frac{-\ln(1-\alpha)}{n\lambda_0} \; ; \lambda \right)$$

$$= 1 - e^{-n\lambda(-\ln(1-\alpha)/n\lambda_0)}$$

$$= 1 - e^{-\lambda(-\ln(1-\alpha)/\lambda_0)}$$

Compare the Tests:

Test 2: based on

$$Y_n = \min(X_1, X_2, \ldots, X_n)$$

$$\gamma_2(\lambda) = P(\text{ Reject } H_0 \; ; \lambda \,)$$

$$= P\left(Y_n < \frac{-\ln(1-\alpha)}{n\lambda_0} \; ; \lambda \right)$$

$$= 1 - e^{-n\lambda(-\ln(1-\alpha)/n\lambda_0)}$$

$$= 1 - e^{-\lambda(-\ln(1-\alpha)/\lambda_0)}$$

Compare the Tests:

Test 2: based on

$$\gamma_2(\lambda) = 1 - e^{-\lambda(-\ln(1-\alpha)/\lambda_0)}$$

$$= 1 - (1 - \alpha)^{\lambda/\lambda_0}$$

$$H_0 : \lambda = 1 \qquad H_1 : \lambda > 1$$