Let $X_1, X_2, \ldots, X_n$ be a random sample from any distribution with unknown parameter θ which takes values in a parameter space Θ.

We ultimately want to test

$$\boxed{\begin{array}{l} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta \backslash \Theta_0 \end{array}}$$

We also want the "best" possible test.

where Θ_0 is some subset of Θ.

The Power Function

$$\gamma(\theta) = \text{P(Reject H}_0 \text{ when the parameter is } \theta)$$

$$\gamma(\theta) = \text{P(Reject H}_0; \theta)$$

θ is an argument that can be anywhere in the parameter space Θ.

- it could be a θ from H_0
- it could be a θ from H_1

The Power Function

Note that

$$\alpha = \max \mathrm{P}(\text{Reject } \mathrm{H}_0 \text{ when true})$$

$$= \max_{\theta \in \Theta_0} \mathrm{P}(\text{Reject } \mathrm{H}_0; \theta)$$

$$= \max_{\theta \in \Theta_0} \gamma(\theta)$$

Other notation is $\max_{\theta \in \mathrm{H}_0}$

The Power Function

Note that

$$\beta = \max \mathrm{P}(\text{Fail to Reject } \mathrm{H}_0 \text{ when false})$$

$$= \max_{\theta \in \Theta \setminus \Theta_0} \mathrm{P}(\text{Fail to Reject } \mathrm{H}_0; \theta)$$

$$= \max_{\theta \in \Theta \setminus \Theta_0} \left[1 - \mathrm{P}(\text{Reject } \mathrm{H}_0; \theta)\right]$$

$$= \max_{\theta \in \Theta \setminus \Theta_0} \left[1 - \gamma(\theta)\right]$$

Other notation is $\max_{\theta \in \mathrm{H}_1}$

The Power Function

Power functions are useful for comparing two hypothesis tests.

Step One:

Choose an estimator for μ.

- What is an estimator?
- Really this should be "Choose a statistic that you can work with."

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and <u>known</u> variance σ^2.

Consider the hypotheses

$$H_0 : \mu \geq \mu_0 \qquad H_1 : \mu < \mu_0$$

where μ_0 is fixed and known.

Derive a test with level of significance (size) α.

Step One:

Choose an estimator for μ.

$$\hat{\mu} = \overline{X}$$

$$\Downarrow$$

Step Four: Conclusion

Reject H_0, in favor of H_1, if

$$\overline{X} < \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

Find the power function of this test.

$$\gamma(\mu) = \mathrm{P}(\text{Reject } \mathrm{H}_0; \mu)$$

$$= \mathrm{P}\left(\overline{\mathrm{X}} < \mu_0 + \mathrm{z}_{1-\alpha}\frac{\sigma}{\sqrt{\mathrm{n}}}; \mu \right)$$

$$= \mathrm{P}\left(\frac{\overline{\mathrm{X}} - \mu}{\sigma/\sqrt{\mathrm{n}}} < \frac{\mu_0 + \mathrm{z}_{1-\alpha}\sigma/\sqrt{\mathrm{n}} - \mu}{\sigma/\sqrt{\mathrm{n}}}; \mu \right)$$

$Z \sim N(0,1)$

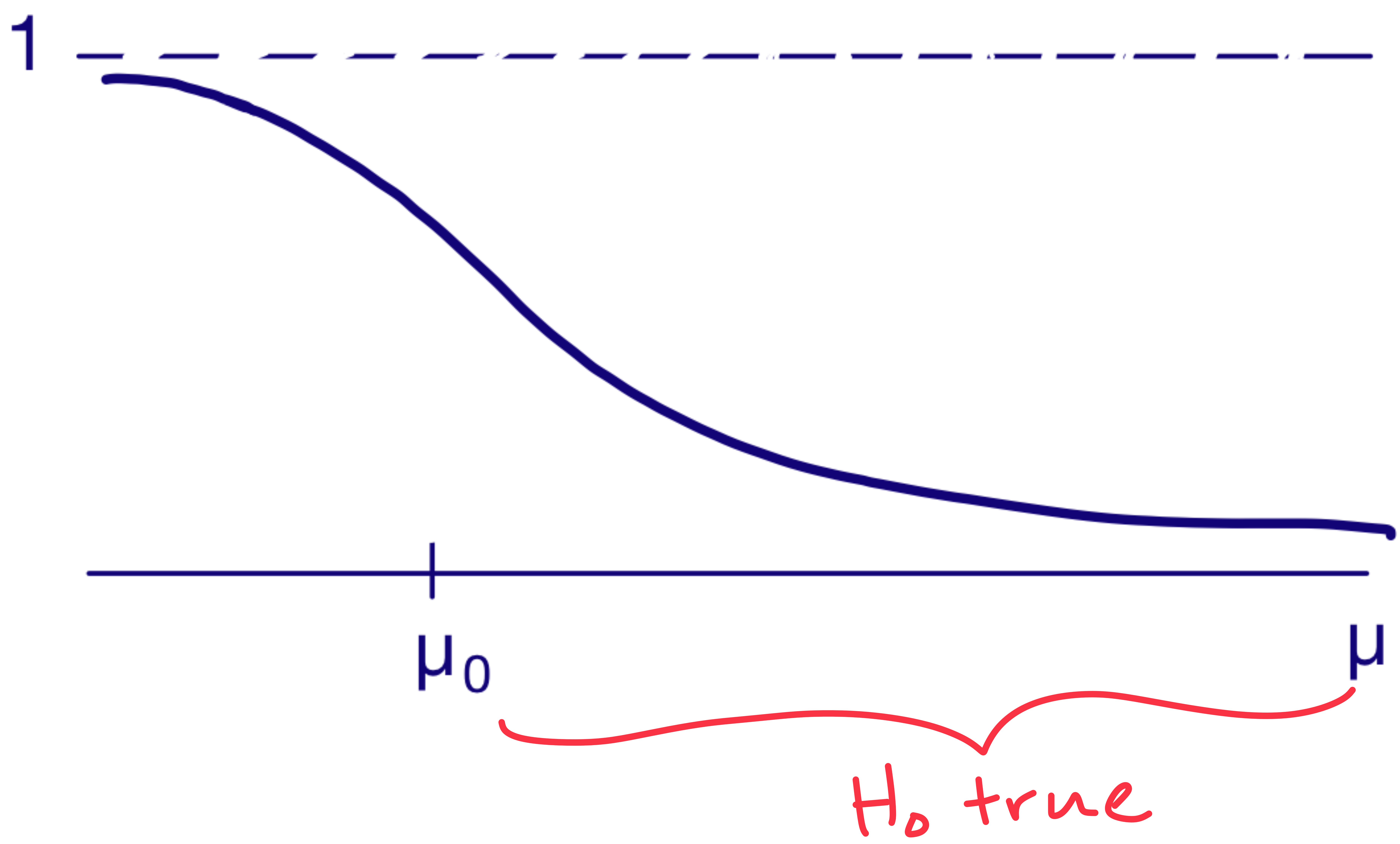
Find the power function of this test.

$$\gamma(\mu) = \mathrm{P}\left(Z < \frac{\mu_0 + z_{1-\alpha}\sigma/\sqrt{n} - \mu}{\sigma/\sqrt{n}} ; \mu \right)$$

$$= \Phi\left(\frac{\mu_0 + z_{1-\alpha}\sigma/\sqrt{n} - \mu}{\sigma/\sqrt{n}} \right)$$

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

What does the power function for a "good" test of these hypotheses look like?



$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

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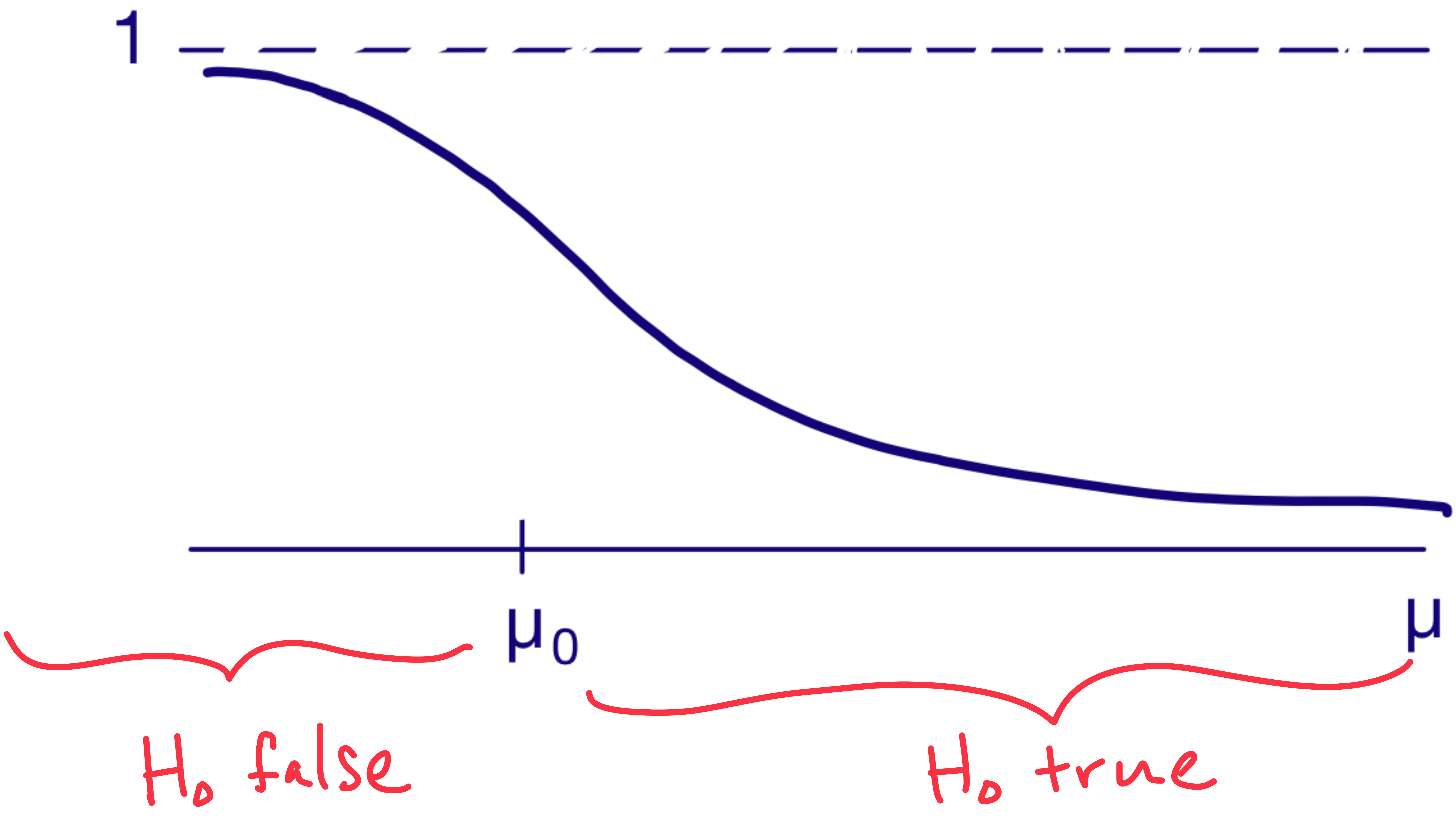
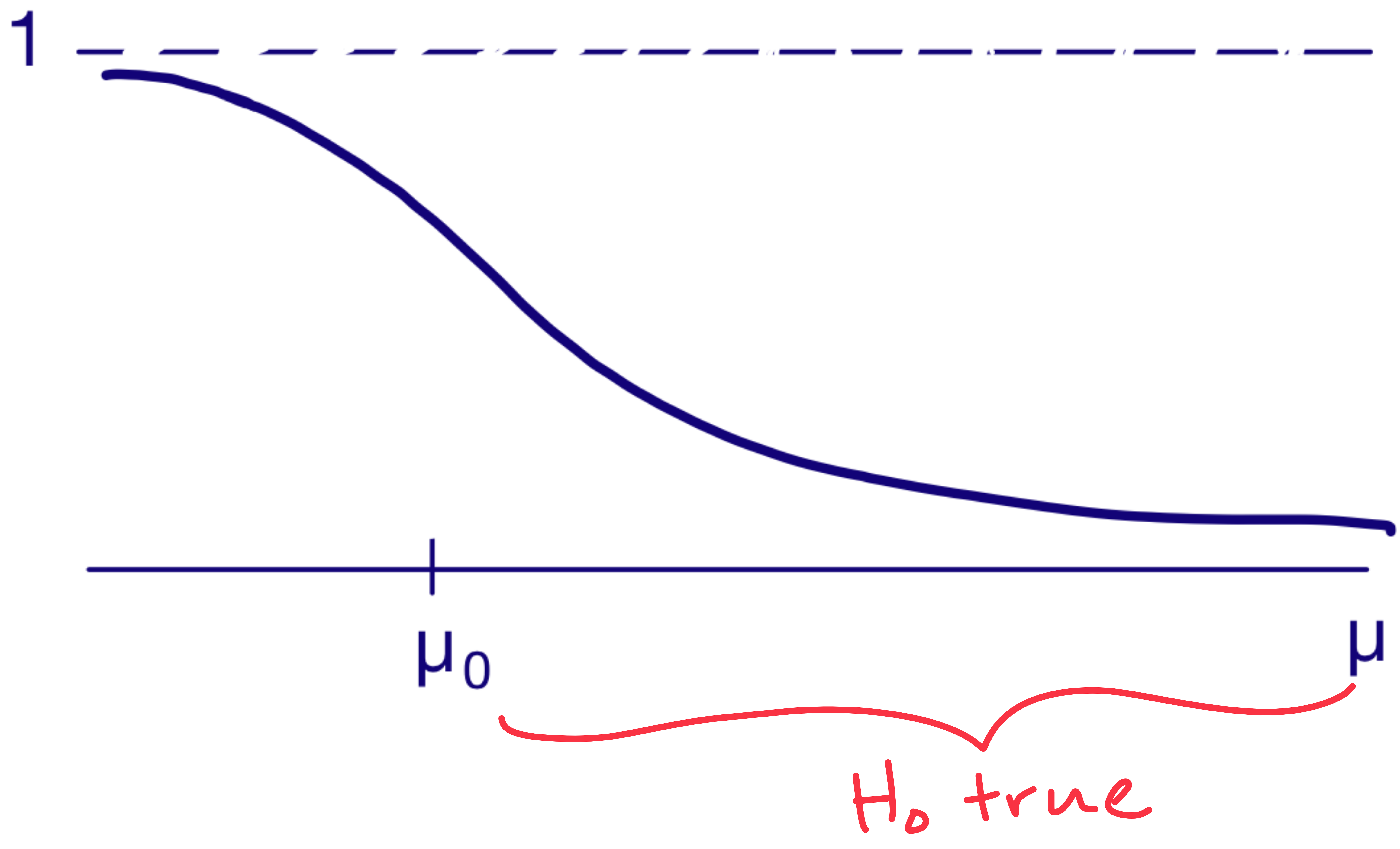


$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

Where is α on this graph?



$$H_0 : \mu \geq \mu_0$$
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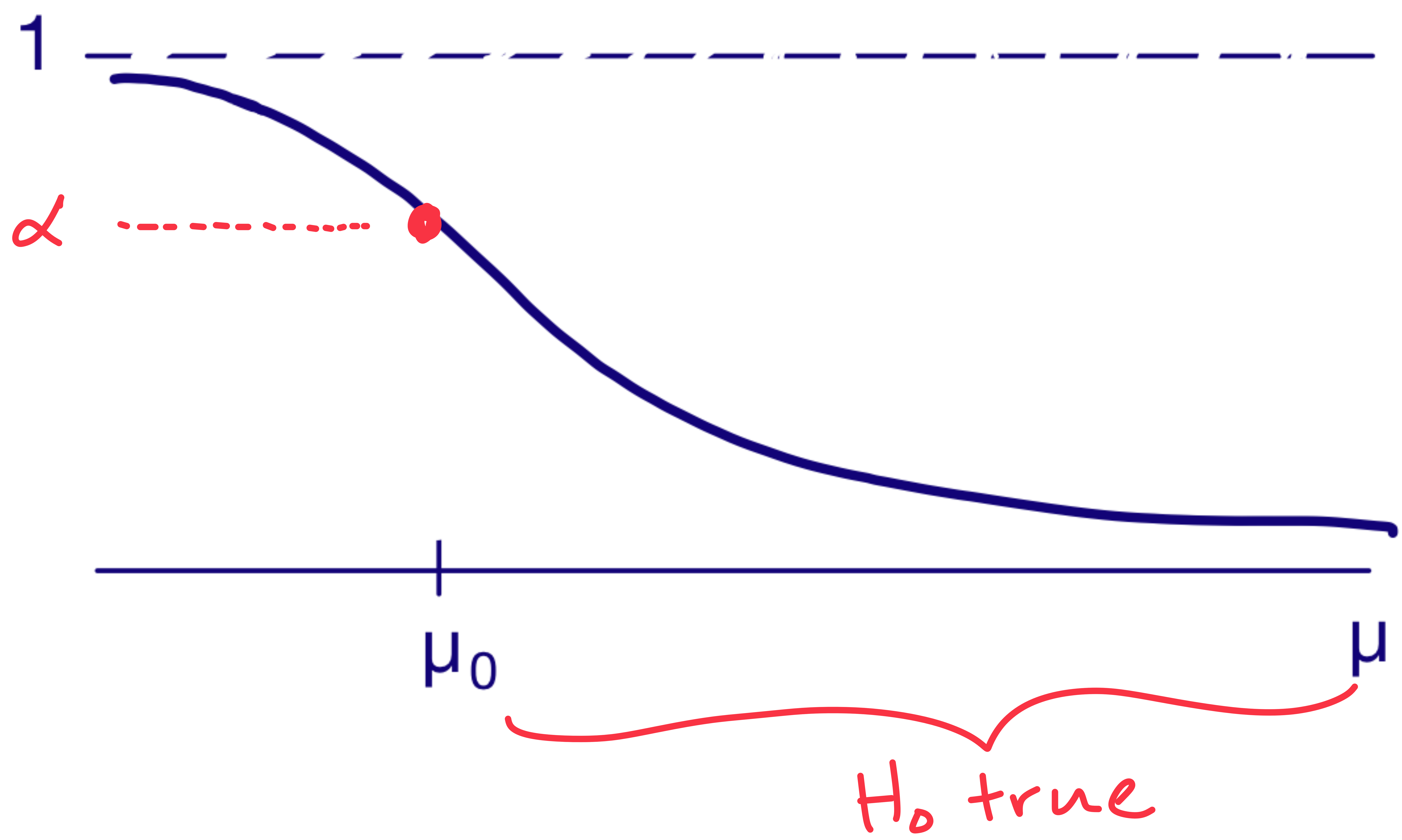

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$

A Second Test of Size α

Step One:

Choose an "estimator" for μ.

$$\widehat{\mu} = \max(X_1, X_2, \ldots, X_n)$$

Step Two: Form of the Test

Reject H_0, in favor of H_1, if

$$\max(X_1, X_2, \ldots, X_n) < c$$

for some c to be determined.

$$H_0 : \mu \geq \mu_0$$
$$H_1 : \mu < \mu_0$$