

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2} \right)^n = 0$$

$$\lim_{n \rightarrow \infty} \frac{2n^2 + 3}{3n^3 - 7} = 0$$

$$\lim_{n \rightarrow \infty} \frac{2n^2 + 3}{3n^2 - 7} = \frac{2}{3}$$

$$a_n = \frac{n-1}{n} \Rightarrow \lim_{n \rightarrow \infty} a_n = 1$$

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Consider a sequence of random variables: X_1, X_2, X_3, \dots

$$\lim_{n \rightarrow \infty} X_n = ?$$

This is meaningless!

Convergence in Probability

The sequence of random variables:

$$X_1, X_2, X_3, \dots$$

converges in probability to a random variable X if, for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0$$

We write $X_n \xrightarrow{P} X$.

An Integral Notation:

Rewrite $\int_0^2 f(x) dx = \int_A f(x) dx$

where $A = \{x: 0 \leq x \leq 2\}$.

Suppose that X has the exponential distribution with rate λ .

How can we find $P(|\sin(X)| > 1/2)$?

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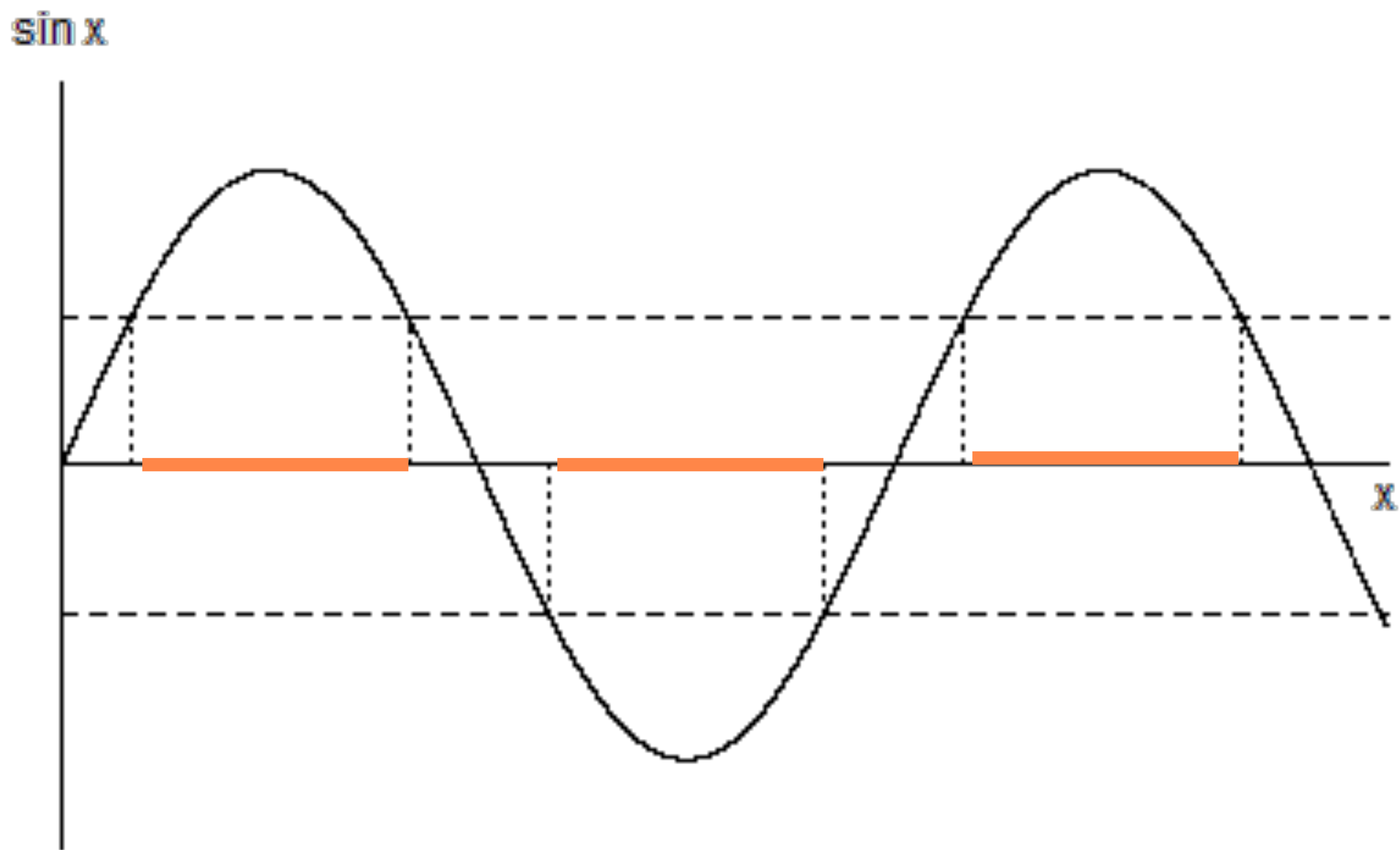
1. We can define a new random variable $Y = |\sin(X)|$, try to find its pdf and then

$$P(|\sin(X)| > 1/2) = P(Y > 1/2)$$

$$= \int_{1/2}^{\infty} f_Y(y) dy$$

How can we find $P(|\sin(X)| > 1/2)$?

2. We can integrate the pdf for X over the relevant region.

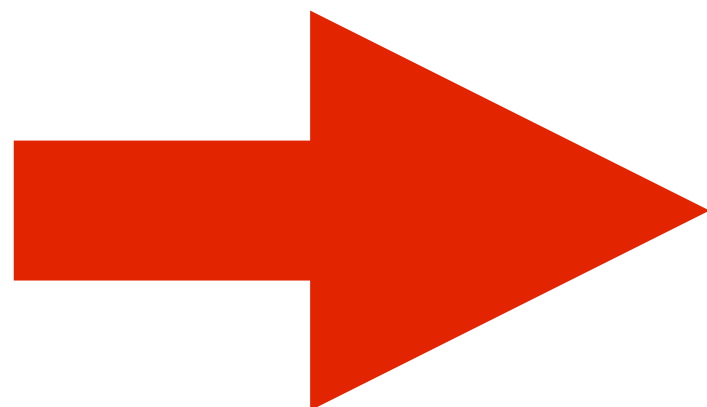


How can we find $P(|\sin(X)| > 1/2)$?

2. We can integrate the pdf for X over the relevant region.

$$P(|\sin(X)| > 1/2)$$

$$= \int_{\sin^{-1}(1/2)}^{\pi - \sin^{-1}(1/2)} f_X(x) dx + \int_{\pi + \sin^{-1}(1/2)}^{2\pi - \sin^{-1}(1/2)} f_X(x) dx + \dots$$

Notation 

$$= \int_{\{x : |\sin(x)| > 1/2\}} f_X(x) dx$$

An Inequality:

Let X be a random variable. Let g be a non-negative function and let $c > 0$.

Then

$$P(g(X) \geq c) \leq \frac{E[g(X)]}{c}$$

When $g(x) = |x|$,

this is known as **Markov's inequality**.

An Inequality:

$$P(g(X) \geq c) \leq \frac{E[g(X)]}{c}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$$

$$= \int_{\{x : g(x) \geq c\}} g(x)f_X(x) dx + \int_{\{x : g(x) < c\}} g(x)f_X(x) dx$$

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$$\geq \int_{\{x : g(x) \geq c\}} g(x) f_X(x) dx \geq \int_{\{x : g(x) \geq c\}} c f_X(x) dx$$

An Inequality:

$$P(g(X) \geq c) \leq \frac{E[g(X)]}{c}$$

$$E[g(X)] \geq \int_{\{x : g(x) \geq c\}} c f_X(x) dx$$

$$= c \int_{\{x : g(x) \geq c\}} f(x) dx = c P(g(X) \geq c)$$



Chebyshev's Inequality:

Let X be a random variable with mean μ and variance $\sigma^2 < \infty$. Let $k > 0$.

Then

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

or, equivalently,

$$P(|X - \mu| < k\sigma) > 1 - \frac{1}{k^2}$$

“the probability that X is within k standard deviations of its mean”

Chebyshev's Inequality:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Proof:

$$P(|X - \mu| \geq k\sigma) = P(\underbrace{(X - \mu)^2}_{g(x)} \geq \underbrace{k^2\sigma^2}_c)$$

$$\leq \frac{\underbrace{E[g(X)]}_c}{c} = \frac{E[(X - \mu)^2]}{k^2\sigma^2} = \frac{\sigma^2}{k^2\sigma^2}$$

$$= 1/k^2 \quad \checkmark$$

Convergence in Probability ($X_n \xrightarrow{P} X$)

The sequence of random variables:

$$X_1, X_2, X_3, \dots$$

converges in probability to a random variable X if, for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0$$

could be a
constant!

Ex: $X=3$ w.p.1

$$P(X = x) = \begin{cases} 1, & \text{if } x=3 \\ 0, & \text{otherwise} \end{cases}$$

The Weak Law of Large Numbers

Suppose that X_1, X_2, X_3, \dots is a sequence of iid random variables from any distribution with mean μ and variance $\sigma^2 < \infty$.

Then

$$\bar{X} \xrightarrow{P} \mu$$

-
- $E[\bar{X}] = \mu$
 - $\text{Var}[\bar{X}] = \sigma^2/n$

The Weak Law of Large Numbers

Proof: Let $\varepsilon > 0$.

$$\bar{X} \xrightarrow{P} \mu$$

Chebyshev:
$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Here:

$$P(|\bar{X} - \mu_{\bar{X}}| \geq k\sigma_{\bar{X}}) \leq \frac{1}{k^2}$$

Which is:

$$P(|\bar{X} - \mu| \geq k\sigma/\sqrt{n}) \leq \frac{1}{k^2}$$

Choose k so that this is ε .

The Weak Law of Large Numbers

$$P(|\bar{X} - \mu| \geq \varepsilon) \leq \frac{1}{(\varepsilon\sqrt{n}/\sigma)^2} = \frac{\sigma^2}{\varepsilon^2 n}$$

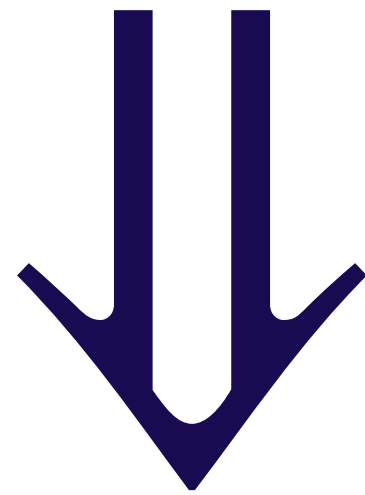
$$\lim_{n \rightarrow \infty} P(|\bar{X} - \mu| \geq \varepsilon) \leq \lim_{n \rightarrow \infty} \frac{\sigma^2}{\varepsilon^2 n} = 0$$

$$\lim_{n \rightarrow \infty} P(|\bar{X} - \mu| \geq \varepsilon) = 0$$

$$\bar{X} \xrightarrow{P} \mu$$

The Weak Law of Large Numbers

$$\lim_{n \rightarrow \infty} P(|\bar{X} - \mu| \geq \varepsilon) = 0$$



$$\bar{X} \xrightarrow{P} \mu$$

Example:

$$X_1, X_2, X_3, \dots \stackrel{\text{iid}}{\sim} \exp(\text{rate} = \lambda)$$

$$\overline{X} \xrightarrow{\text{P}} 1/\lambda$$

$$X_1, X_2, X_3, \dots \stackrel{\text{iid}}{\sim} \Gamma(\alpha, \beta)$$

$$\overline{X} \xrightarrow{\text{P}} \alpha/\beta$$