Let $X_1, X_2, \ldots, X_n$ be a random sample from any distribution with mean μ and variance σ^2.

$$\sigma^2 = \mathrm{Var}[X] := \mathrm{E}[(X - \mu)^2]$$

To estimate this from the sample, we could use

$$\widetilde{S}^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n}$$

"sample variance"

- Currently, before numerical observations, this is a random variable.

- It has its own distribution, its own mean, and its own variance.

$$\sum_{i=1}^{n}(X_i-\overline{X})^2=\sum_{i=1}^{n}(X_i^2-2\overline{X}X_i+\overline{X}^2)$$

$$=\sum_{i=1}^{n}X_i^2-2\overline{X}\sum_{i=1}^{n}X_i+n\overline{X}^2$$

$n\overline{X}$

$$\sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n} X_i^2 - n\overline{X}^2$$

$$= \sum_{i=1}^{n} X_i^2 - n\left(\sum_{i=1}^{n} X_i \middle/ n\right)^2$$

$$= \sum_{i=1}^{n} X_i^2 - \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{n}$$

$$\mathrm{Var}[X] = \mathrm{E}[(X-\mu)^2] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$\Downarrow$$

$$\mathrm{E}[X^2] = \mathrm{Var}[X] + (\mathrm{E}[X])^2$$

$$= \sigma^2 + \mu^2$$

$$\Downarrow$$

$$\mathrm{E}[\widetilde{S}^2] = \cdots = \frac{n-1}{n}\sigma^2$$

"sample variance"

Variant of Sample Variance:

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n - 1}$$

$$S^2 = \frac{n}{n-1} \widetilde{S}^2$$

$$\Rightarrow E[S^2] = \frac{n}{n-1} E[\widetilde{S}^2]$$

$$= \frac{n}{n-1} \frac{n-1}{n} \sigma^2 = \sigma^2$$

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n - 1}$$

is an unbiased estimator of σ^2.

$$\left(\text{i.e. } E[S^2] = \sigma^2\right)$$

This is the sample variance that we will use.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$$

For the normal distribution, these are independent!

Aside:

$$X_1 \sim \chi^2(n_1) \text{ and } X_2 \sim \chi^2(n_2)$$

independent

$$\Downarrow$$

$$X_1 + X_2 \sim \chi^2(n_1 + n_2)$$

Aside:

$$X_1 \sim \chi^2(n_1) \text{ and } X_2 \sim \chi^2(n_2)$$

independent

$$\not\Downarrow$$

$$X_1 - X_2 \sim \chi^2(n_1 - n_2)$$

However,

$$X_1 \sim \chi^2(n_1) \text{ and } X_2 \sim \chi^2(n_2)$$

$$X_3 \sim ?$$

$$X_1 = X_2 + X_3$$

and X_2 and X_3 independent

$$\Downarrow$$

$$X_3 = X_1 - X_2 \sim \chi^2(n_1 - n_2)$$

$$\sum_{i=1}^{n}(X_i - \mu)^2$$

$$= \sum_{i=1}^{n}(X_i - \overline{X} + \overline{X} - \mu)^2$$

$$\sum X_i - n\overline{X}$$

$$\sum X_i - \sum X_i$$

$$= \sum_{i=1}^{n}(X_i - \overline{X})^2$$

$$+2(\overline{X} - \mu)\sum_{i=1}^{n}(X_i - \overline{X})$$

$$= 0$$

$$+n(\overline{X} - \mu)^2$$

$$\sum_{i=1}^{n} (X_i - \mu)^2$$

$$= \sum_{i=1}^{n} (X_i - \overline{X} + \overline{X} - \mu)^2$$

$$= \sum_{i=1}^{n} (X_i - \overline{X})^2 + n\,(\overline{X} - \mu)^2$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and variance σ^2.

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2 + n(\overline{X} - \mu)^2$$

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2}$$

$$=\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{\sigma^2}+\frac{n(\overline{X}-\mu)^2}{\sigma^2}$$

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2}=\boxed{\sum_{i=1}^{n}\boxed{\left(\underbrace{\frac{X_i-\mu}{\sigma}}_{N(0,1)}\right)^2}}$$

$\chi^2(n)$

$\chi^2(1)$

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2}$$

$$=\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{\sigma^2}+\frac{n(\overline{X}-\mu)^2}{\sigma^2}$$

$$\frac{n(\overline{X}-\mu)^2}{\sigma^2}=\boxed{\Bigg(\underbrace{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}_{N(0,1)}\Bigg)^2}$$

$\chi^2(1)$

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and variance σ^2.

$$= \frac{(n-1)S^2}{\sigma^2}$$

$$\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2} = \boxed{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2}}$$

$$+ \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and variance σ^2.

independent

$$W_1 = W_2 + W_3$$

$\chi^2(n)$ $\chi^2(1)$

Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and variance σ^2.

$$\Rightarrow \quad W_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$