Señales y Sistemas
Ingeniería de Sonido
Año 2018

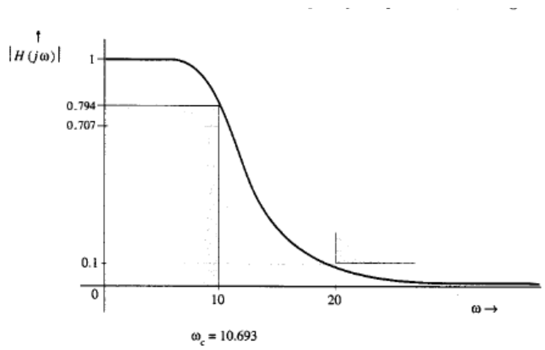
Filtros Analógicos

Ej1 Determinar el orden de un filtro pasa bajos de Butterwork las siguientes especificaciones:

i) Ganancia en la Banda de Paso entre $\ddot{G}_p = -0.5dB$ para $0 < \omega < 100$.

ii) Ganancia en la Banda Eliminada $\ddot{G}_s = -20dB$ para $\omega \geq 200$.

Ej2 Diseñe un filtro pasa bajos de Butterwork las especificaciones mostradas en la figura



i) Ganancia en la Banda de Paso entre 1 y $G_p = 0.794$ ($\ddot{G}_p = -2dB$) para $0 < \omega < 10$.

ii) Ganancia en la Banda Eliminada $G_s = 0.1$ $\left(\ddot{G}_s = -20dB\right)$para $\omega \geq 20$.

Paso 1: Determinar n

Paso 2: Determinar ω_c

Paso 3: Determinar $\mathcal{H}(s)$

Paso 4: Determinar $H(s)$ (Reemplazar en $\mathcal{H}(s)$ $s\ por\ {}^{s}/_{\omega_c}$

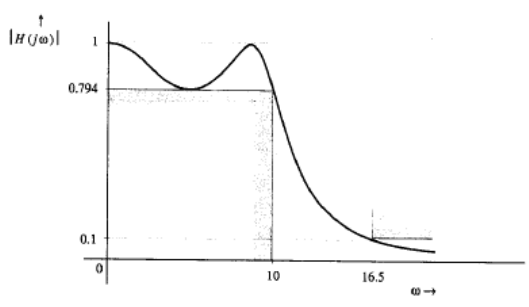
Ej3 Resolver el ejercicio Nº2 utilizando Matlab.

```
Wp=10;Ws=20;Gp=-2;Gs=-20;
[n,Wc]=buttord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=butter(n,Wc,'s');
```

E4 Determinar el orden de un filtro pasa bajos de Chebyshev y la función de transferencia que cumpla las siguientes especificaciones:

$r = 2dB;\ \ddot{G}_s = -20dB;\ \omega_p = 10\ rad/s, y\ w_s$=28 rad/s.

Ej5 Diseñe un filtro Pasa bajos de Chebyshev que cumpla los siguientes criterios (ver figura adjunta)



i) Ripple en la Banda de Paso $r \leq 2dB$; $0 < \omega < 10$.

ii) Ganancia en la Banda Eliminada $\ddot{G}_s = -20dB$ para $\omega \geq 16.5$ $(\omega_s = 16.5)$.

Observe que las especificaciones son las mismas que las del ejercicio Nº1, excepto la Banda de Transición.
Aquí la Banda de transición va de 10 a 16.5, mientras que en el ejemplo Nº1 va de 10 a 20. A pesar que en este ejemplo la banda de transición es más estrecha, encontraremos que el filtro de Chebyshev requiere un orden menos que el que requiere un filtro de Butterwork.

Paso 1: Determinar n
Paso 2: Determinar $\mathcal{H}(s)$
Paso 3: Determinar $H(s)$ (Reemplazar en $\mathcal{H}(s)$, $s\ por\ {}^{s}/_{\omega_c}$

Ej6 Resolver el ejercicio Nº5 utilizando Matlab.

```
Wp=10;Ws=16.5;r=2;Gs=-20;
[n,Wp]=cheb1ord(Wp,Ws,r,-Gs,'s');
[num,den]=cheby1(n,r,Wp,'s');
```

Ej6 Resolver el ejercicio Nº5 para un filtro inverso de Chebyshev utilizando Matlab

```
Wp=10;Ws=16.5;Gp=-2;Gs=-20;
[n,Ws]=cheb2ord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=cheby2(n,-Gs,Ws,'s')
```

Formulas y Tablas

Filtros Butterwork

We can proceed in this way to find $\mathcal{H}(s)$ for any value of n. In general

$$\mathcal{H}(s) = \frac{1}{B_n(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1} \tag{7.36}$$

where $B_n(s)$ is the Butterworth polynomial of the nth order. Table 7.1 shows the coefficients $a_1, a_2, \ldots, a_{n-2}, a_{n-1}$ for various values of n; Table 7.2 shows $B_n(s)$ in factored form. In these Tables, we read for $n = 4$

$$\begin{aligned}\mathcal{H}(s) &= \frac{1}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1} \\ &= \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}\end{aligned}$$

Determination of n, the Filter Order

If $\hat{G}_x$ is the gain of a lowpass Butterworth filter in dB units at $\omega = \omega_x$, then according to Eq. (7.31)

$$\hat{G}_x = 20\log_{10}|H(j\omega_x)| = -10\log\left[1 + \left(\frac{\omega_x}{\omega_c}\right)^{2n}\right]$$

Substitution of the specifications in Fig. 7.19a (gains $\hat{G}_p$ at ω_p and $\hat{G}_s$ at ω_s) in this equation yields

$$\hat{G}_p = -10\log\left[1 + \left(\frac{\omega_p}{\omega_c}\right)^{2n}\right]$$

$$\hat{G}_s = -10\log\left[1 + \left(\frac{\omega_s}{\omega_c}\right)^{2n}\right]$$

or

$$\left(\frac{\omega_p}{\omega_c}\right)^{2n} = 10^{-\hat{G}_p/10} - 1 \tag{7.38a}$$

$$\left(\frac{\omega_s}{\omega_c}\right)^{2n} = 10^{-\hat{G}_s/10} - 1 \tag{7.38b}$$

Dividing (7.38b) by (7.38a), we obtain

$$\left(\frac{\omega_s}{\omega_p}\right)^{2n} = \left[\frac{10^{-\hat{G}_s/10} - 1}{10^{-\hat{G}_p/10} - 1}\right]$$

and

$$n = \frac{\log\left[\left(10^{-\hat{G}_s/10} - 1\right) / \left(10^{-\hat{G}_p/10} - 1\right)\right]}{2\log(\omega_s/\omega_p)} \tag{7.39}$$

Also from Eq. (7.38a)

$$\omega_c = \frac{\omega_p}{\left[10^{-\hat{G}_p/10} - 1\right]^{1/2n}} \tag{7.40}$$

Alternatively, from Eq. (7.38b)

$$\omega_c = \frac{\omega_s}{\left[10^{-\hat{G}_s/10} - 1\right]^{1/2n}} \tag{7.41}$$

Table 7.1: Coefficients of Butterworth Polynomial $B_n(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.41421356								
3	2.00000000	2.00000000							
4	2.61312593	3.41421356	2.61312593						
5	3.23606798	5.23606798	5.23606798	3.23606798					
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331				
7	4.49395921	10.09783468	14.59179389	14.59179389	10.09783468	4.49395921			
8	5.12583090	13.13707118	21.84615097	25.68835593	21.84615097	13.13707118	5.12583090		
9	5.75877048	16.58171874	31.16343748	41.98638573	41.98638573	31.16343748	16.58171874	5.75877048	
10	6.39245322	20.43172909	42.80206107	64.88239627	74.23342926	64.88239627	42.80206107	20.43172909	6.39245322

Table 7.2: Butterworth Polynomials in Factorized Form

n	$B_n(s)$
1	$s+1$
2	$s^2 + 1.41421356s + 1$
3	$(s+1)(s^2+s+1)$
4	$(s^2 + 0.76536686s + 1)(s^2 + 1.84775907s + 1)$
5	$(s+1)(s^2 + 0.61803399s + 1)(s^2 + 1.931803399s + 1)$
6	$(s^2 + 0.51763809s + 1)(s^2 + 1.41421356s + 1)(s^2 + 1.93185165s + 1)$
7	$(s+1)(s^2 + 0.44504187s + 1)(s^2 + 1.24697960s + 1)(s^2 + 1.80193774s + 1)$
8	$(s^2 + 0.39018064s + 1)(s^2 + 1.11114047s + 1)(s^2 + 1.66293922s + 1)(s^2 + 1.96157056s + 1)$
9	$(s+1)(s^2 + 0.34729636s + 1)(s^2 + s + 1)(s^2 + 1.53208889s + 1)(s^2 + 1.87938524s + 1)$
10	$(s^2 + 0.31286893s + 1)(s^2 + 0.90798100s + 1)(s^2 + 1.41421356s + 1)(s^2 + 1.78201305s + 1)(s^2 + 1.97537668s + 1)$

Filtros Chebyshev

The parameter ϵ controls the height of ripples. In the passband, r, the ratio of the maximum gain to the minimum gain is

$$r = \sqrt{1+\epsilon^2} \tag{7.46a}$$

This ratio r, specified in decibels, is

$$\hat{r} = 20\log\sqrt{1+\epsilon^2} = 10\log_{10}(1+\epsilon^2) \tag{7.46b}$$

so that

$$\epsilon^2 = 10^{\hat{r}/10} - 1 \tag{7.47}$$

Because all the ripples in the passband are of equal height, the Chebyshev polynomials $C_n(\omega)$ are known as **equal-ripple functions**.

Determination of n (Filter Order)

For a normalized Chebyshev filter, the gain $\hat{G}$ in dB [see Eq. (7.42)] is

$$\hat{G} = -10\log\left[1+\epsilon^2 {C_n}^2(\omega)\right]$$

The gain is $\hat{G}_s$ at ω_s. Therefore

$$\hat{G}_s = -10\log\left[1+\epsilon^2 {C_n}^2(\omega_s)\right] \tag{7.48}$$

or

$$\epsilon^2 {C_n}^2(\omega_s) = 10^{-\hat{G}_s/10} - 1$$

Use of Eq. (7.43b) and Eq.(7.47) in the above equation yields

$$\cosh\left[n\cosh^{-1}(\omega_s)\right] = \left[\frac{10^{-\hat{G}_s/10}-1}{10^{\hat{r}/10}-1}\right]^{1/2}$$

Hence

$$n = \frac{1}{\cosh^{-1}(\omega_s)}\cosh^{-1}\left[\frac{10^{-\hat{G}_s/10}-1}{10^{\hat{r}/10}-1}\right]^{1/2} \tag{7.49a}$$

Note that these equations are for normalized filters, where $\omega_p = 1$. For a general case, we replace ω_s with $\frac{\omega_s}{\omega_p}$ to obtain

$$n = \frac{1}{\cosh^{-1}(\omega_s/\omega_p)}\cosh^{-1}\left[\frac{10^{-\hat{G}_s/10}-1}{10^{\hat{r}/10}-1}\right]^{1/2} \tag{7.49b}$$

Pole Locations

We could follow the procedure of the Butterworth filter to obtain the pole locations of the Chebyshev filter. The procedure is straightforward but tedious and does not yield any special insight into our development. The Butterworth filter poles lie on a semicircle. We can show that the poles of an nth-order normalized Chebyshev filter lie on a semiellipse of the major and minor semiaxes $\cosh x$ and $\sinh x$, respectively, where[1]

$$x = \frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{7.50}$$

The Chebyshev filter poles are

$$s_k = -\sin\left[\frac{(2k-1)\pi}{2n}\right]\sinh x + j\cos\left[\frac{(2k-1)\pi}{2n}\right]\cosh x \quad k = 1, 2, \cdots, n \tag{7.51}$$

The transfer function $\mathcal{H}(s)$ of the normalized nth-order lowpass Chebyshev filter is

$$\mathcal{H}(s) = \frac{K_n}{C'_n(s)} = \frac{K_n}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{7.52}$$

The constant K_n is selected to have proper dc gain, as shown in Eq. (7.45). As a result

$$K_n = \begin{cases} a_0 & n \text{ odd} \\ \dfrac{a_0}{\sqrt{1+\epsilon^2}} = \dfrac{a_0}{10^{\hat{r}/20}} & n \text{ even} \end{cases} \tag{7.53}$$

The design procedure is considerably simplified by ready-made tables of the polynomial $C'_n(s)$ in Eq. (7.52) or the pole locations of $\mathcal{H}(s)$. Table 7.4 lists the coefficients a_0, a_1, a_2, $\cdots$, a_{n-1} of the polynomial $C'_n(s)$ in Eq. (7.52) for $\hat{r} = 0.5$, 1, 2, and 3 dB ripples corresponding to the values of $\epsilon = 0.3493$, 0.5088, 0.7648, and 0.9976, respectively. Table 7.5 lists the poles of various Chebyshev filters for the same values of $\hat{r}$ (and ϵ). Tables listing more extensive values of $\hat{r}$ (or ϵ) can be found in the literature. We can also use MATLAB functions for this purpose.

Table 7.4: Chebyshev Filter Coefficients of the Denominator Polynomial $C'_n(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6
						0.5 db ripple ($\hat{r} = 0.5$)	
1	2.8627752						
2	1.5162026	1.4256245					
3	0.7156938	1.5348954	1.2529130				
4	0.3790506	1.0254553	1.7168662	1.1973856			
5	0.1789234	0.7525181	1.3095747	1.9373675	1.1724909		
6	0.0947626	0.4323669	1.1718613	1.5897635	2.1718446	1.1591761	
7	0.0447309	0.2820722	0.7556511	1.6479029	1.8694079	2.4126510	1.1512176
						1 db ripple ($\hat{r} = 1$)	
1	1.9652267						
2	1.1025103	1.0977343					
3	0.4913067	1.2384092	0.9883412				
4	0.2756276	0.7426194	1.4539248	0.9528114			
5	0.1228267	0.5805342	0.9743961	1.6888160	0.9368201		
6	0.0689069	0.3070808	0.9393461	1.2021409	1.9308256	0.9282510	
7	0.0307066	0.2136712	0.5486192	1.3575440	1.4287930	2.1760778	0.9231228
						2 db ripple ($\hat{r} = 2$)	
1	1.3075603						
2	0.8230604	0.8038164					
3	0.3268901	1.0221903	0.7378216				
4	0.2057651	0.5167981	1.2564819	0.7162150			
5	0.0817225	0.4593491	0.6934770	1.4995433	0.7064606		
6	0.0514413	0.2102706	0.7714618	0.8670149	1.7458587	0.7012257	
7	0.0204228	0.1660920	0.3825056	1.1444390	1.0392203	1.9935272	0.6978929
						3 db ripple ($\hat{r} = 3$)	
1	1.0023773						
2	0.7079478	0.6448996					
3	0.2505943	0.9283480	0.5972404				
4	0.1769869	0.4047679	1.1691176	0.5815799			
5	0.0626391	0.4079421	0.5488626	1.4149874	0.5744296		
6	0.0442467	0.1634299	6990977	6906098	1.6628481	0.5706979	
7	0.0156621	0.1461530	0.3000167	1.0518448	0.8314411	1.9115507	0.5684201

Table 7.5: Chebyshev Filter Pole Locations

n	$\hat{r} = 0.5$	$\hat{r} = 1$	$\hat{r} = 2$	$\hat{r} = 3$
1	-2.8628	-1.9652	-1.3076	-1.0024
2	$-0.7128 \pm j1.0040$	$-0.5489 \pm j0.8951$	$-0.4019 \pm j0.8133$	$-0.3224 \pm j0.7772$
3	-0.6265	-0.4942	-0.3689	-0.2986
	$-0.3132 \pm j1.0219$	$-0.2471 \pm j0.9660$	$-0.1845 \pm j0.9231$	$-0.1493 \pm j0.9038$
4	$-0.1754 \pm j1.0163$	$-0.1395 \pm j0.9834$	$-0.1049 \pm j0.9580$	$-0.0852 \pm j0.9465$
	$-0.4233 \pm j0.4209$	$-0.3369 \pm j0.4073$	$-0.2532 \pm j0.3968$	$-0.2056 \pm j0.3920$
5	-0.3623	-0.2895	-0.2183	-0.1775
	$-0.1120 \pm j1.0116$	$-0.0895 \pm j0.9901$	$-0.0675 \pm j0.9735$	$-0.0549 \pm j0.9659$
	$-0.2931 \pm j0.6252$	$-0.2342 \pm j0.6119$	$-0.1766 \pm j0.6016$	$-0.1436 \pm j0.5970$
6	$-0.0777 \pm j1.0085$	$-0.0622 \pm j0.9934$	$-0.0470 \pm j0.9817$	$-0.0382 \pm j0.9764$
	$-0.2121 \pm j0.7382$	$-0.1699 \pm j0.7272$	$-0.1283 \pm j0.7187$	$-0.1044 \pm j0.7148$
	$-0.2898 \pm j0.2702$	$-0.2321 \pm j0.2662$	$-0.1753 \pm j0.2630$	$-0.1427 \pm j0.2616$
7	-0.2562	-0.2054	-0.1553	-0.1265
	$-0.0570 \pm j1.0064$	$-0.0457 \pm j0.9953$	$-0.0346 \pm j0.9866$	$-0.0281 \pm j0.9827$
	$-0.1597 \pm j0.8071$	$-0.1281 \pm j0.7982$	$-0.0969 \pm j0.7912$	$-0.0789 \pm j0.7881$
	$-0.2308 \pm j0.4479$	$-0.1851 \pm j0.4429$	$-0.1400 \pm j0.4391$	$-0.1140 \pm j0.4373$
8	$-0.0436 \pm j1.0050$	$-0.0350 \pm j0.9965$	$-0.0265 \pm j0.9898$	$-0.0216 \pm j0.9868$
	$-0.1242 \pm j0.8520$	$-0.0997 \pm j0.8447$	$-0.0754 \pm j0.8391$	$-0.0614 \pm j0.8365$
	$-0.1859 \pm j0.5693$	$-0.1492 \pm j0.5644$	$-0.1129 \pm j0.5607$	$-0.0920 \pm j0.5590$
	$-0.2193 \pm j0.1999$	$-0.1760 \pm j0.1982$	$-0.1332 \pm j0.1969$	$-0.1085 \pm j0.1962$
9	-0.1984	-0.1593	-0.1206	-0.0983
	$-0.0345 \pm j1.0040$	$-0.0277 \pm j0.9972$	$-0.0209 \pm j0.9919$	$-0.0171 \pm j0.9896$
	$-0.0992 \pm j0.8829$	$-0.0797 \pm j0.8769$	$-0.0603 \pm j0.8723$	$-0.0491 \pm j0.8702$
	$-0.1520 \pm j0.6553$	$-0.1221 \pm j0.6509$	$-0.0924 \pm j0.6474$	$-0.0753 \pm j0.6459$
	$-0.1864 \pm j0.3487$	$-0.1497 \pm j0.3463$	$-0.1134 \pm j0.3445$	$-0.0923 \pm j0.3437$
10	$-0.0279 \pm j1.0033$	$-0.0224 \pm j0.9978$	$-0.0170 \pm j0.9935$	$-0.0138 \pm j0.9915$
	$-0.0810 \pm j0.9051$	$-0.1013 \pm j0.7143$	$-0.0767 \pm j0.7113$	$-0.0401 \pm j0.8945$
	$-0.1261 \pm j0.7183$	$-0.0650 \pm j0.9001$	$-0.0493 \pm j0.8962$	$-0.0625 \pm j0.7099$
	$-0.1589 \pm j0.4612$	$-0.1277 \pm j0.4586$	$-0.0967 \pm j0.4567$	$-0.0788 \pm j0.4558$
	$-0.1761 \pm j0.1589$	$-0.1415 \pm j0.1580$	$-0.1072 \pm j0.1574$	$-0.0873 \pm j0.1570$