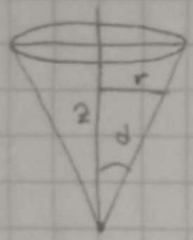
Primer Punto:



ds^2 en coordenadas cilíndricas:

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

Ligadura cono:

$$\tan(\alpha) = \frac{r}{z} \rightarrow z = r\cot(\alpha).$$

Reemplazando:

$$ds^2 = dr^2 + r^2 d\phi^2 + \cot^2(\alpha) dr^2$$

$$ds^2 = dr^2 \underbrace{(1 + \cot^2(\alpha))}_{\csc^2(\alpha)} + r^2 d\phi^2$$

trayectoria distancia más corta => función que optimisa S:

$$S = \int_{P_1}^{P_2} \left[\csc^2(\alpha) + r^2\left(\frac{d\phi}{dr}\right)^2\right]^{\frac{1}{2}} dr$$

El funcional f:

$$f = \left[\csc^2(\alpha) + r^2\left(\frac{d\phi}{dr}\right)^2\right]^{\frac{1}{2}}$$

Para optimizar el funcional, se cumplen las ecuaciones de Lagrange:

$$\frac{d}{dr}\left(\frac{\partial f}{\partial \phi'}\right) - \frac{\partial f}{\partial \phi} = 0 ; \quad \phi' = \frac{d\phi}{dr}$$

① $$\frac{\partial f}{\partial \phi'} = \frac{r^2 \phi'}{\left[\csc^2(\alpha) + r^2\phi'^2\right]^{\frac{1}{2}}}$$

Como $\frac{\partial f}{\partial \phi} = 0$, entonces:

$$\frac{d}{dr}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

Esto significa que:

$$\frac{\partial f}{\partial \phi'} = C$$

$$\frac{r^2 \phi'}{\sqrt{\csc^2(\alpha) + r^2(\phi')^2}} = C$$

$$r^4(\phi')^2 = C^2(\csc^2(\alpha) + r^2(\phi')^2)$$

$$r^2(\phi')^2[r^2 - C^2] = C^2 \csc^2(\alpha)$$

$$(\phi')^2 = C^2 \csc^2(\alpha)\left[\frac{1}{r^2[r^2 - C^2]}\right]$$

$$\frac{d\phi}{dr} = C \csc(\alpha)\left[\frac{1}{r\sqrt{r^2 - C^2}}\right]$$

$$d\phi = C \csc(\alpha)\left[\frac{1}{r\sqrt{r^2 - C^2}}\right] dr$$

Integrando:

$$\phi = C \csc(\alpha) \sec^{-1}\left(\frac{r}{C}\right)\left(\frac{1}{C}\right) + H$$

$$\phi = \csc(\alpha) \sec^{-1}\left(\frac{r}{C}\right) + H$$

$$(\phi - H)\operatorname{sen}(\alpha) = \sec^{-1}\left(\frac{r}{C}\right)$$

$$\boxed{C \sec[(\phi - H)\operatorname{sen}(\alpha)] = r(\phi)}$$

Trayectoria mínima distancia en un cono.

2. Calcule valor mínimo de la Integral

$$I = \int_0^1 [(y')^2 + 12xy]\, dx$$

donde la función $y(x)$ satisface $y(0) = 0$ y $y(1) = 1$.

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.}$$ Ecuación Euler.

$$F = (y')^2 + 12xy.$$

$$\frac{\partial f}{\partial y} = 12x$$

$$\frac{\partial f}{\partial y'} = 2\frac{dy}{\partial x}$$

$$12x - 2\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$$

$$12x = 2\ddot{y}$$

$$6x = \ddot{y}$$

$$\dot{y}(x) = 3x^2 + C_1$$

$$y(x) = x^3 + C_1x + C_2.$$

$$y(0) = C_2 = 0$$

$$y(1) = 1 + 1C_1 = 1;\ C_1 = 0$$

$$y(x) = x^3.$$

$$I = \int_0^1 (3x^2)^2 + 12x^4\, dx = \frac{9x^5}{5}\Big|_0^1 + \frac{12}{5}x^5\Big|_0^1 = \frac{9+12}{5} = \frac{21}{5}$$

3. $P_1=(a,0,0)$ y $P_2(-a,0,\pi)$ superficie

$x^2+y^2-a^2=0$

$x^2+y^2=a^2 \rightarrow$ cilindro: radio a

$ds^2 = dr^2+r^2d\phi^2+dz^2$

$ds^2 = r^2d\phi^2+dz^2$

$r=a \Rightarrow ds^2 = a^2d\phi^2+dz^2$

$$L = \int_{P_1}^{P_2} ds = \int_{P_1}^{P_2} \sqrt{a^2 d\phi^2 + dz^2}$$

$$L = \int_{P_1}^{P_2} \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2} \, d\phi$$

$$f(z, z', \phi) = \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}$$

$$\frac{\partial f}{\partial z} - \frac{d}{d\phi}\left(\frac{\partial f}{\partial z'}\right) = 0$$

$$\frac{\partial f}{\partial z} = 0 \quad ; \quad \frac{\partial f}{\partial z'} = \frac{\frac{dz}{d\phi}}{\sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}}$$

$$0 - \frac{d}{d\phi}\left(\frac{\frac{dz}{d\phi}}{\sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}}\right) = 0$$

$$\hookrightarrow \frac{z'}{\sqrt{a^2 + (z')^2}} = C_1$$

$$(z')^2 = C_1^2 (a^2 + (z')^2)$$

$$(z')^2 [1 - C_1]^2 = C_1^2 a^2$$

$$(z')^2 = \frac{C_1^2 a^2}{1 - C_1^2}$$

$$\frac{dz}{d\phi} = \frac{Ga}{\sqrt{1-G^2}}$$

$$dz = \frac{Ga}{\sqrt{1-G^2}} = d\phi$$

$$z(\phi) = \frac{Ga}{(1-G^2)^{1/2}} \phi + K$$

$z = 0 \; ; \; \phi = 0$

⇒ Condiciones:

$x = -a = a\cos\phi \; ; \phi = \pi$

$y = 0 \quad = a \operatorname{sen}\phi$

$z = \pi$

$x = a \rightarrow \phi = 0$

$y = 0$

$z = 0$

$z(0) = K = 0$

$$z(\pi) = \frac{Ga}{(1-G^2)^{1/2}} \pi = \pi$$

$$Ga = (1-G^2)^{1/2}$$

$$G^2 a^2 = 1 - G^2$$

$$G^2(a^2+1) = 1$$

$$G^2 = \frac{1}{a^2+1}$$

$$G^2 = \frac{1}{(a^2+1)^{1/2}}$$

Norma

$$z(\phi) = \frac{\frac{a}{(a^2+1)^{1/2}}}{\left(1 - \frac{1}{a^2+1}\right)^{1/2}} \phi = \frac{\frac{a}{(a^2+1)^{1/2}}}{\sqrt{\frac{a^2}{a^2+1}}} = \phi$$

$$\Uparrow$$

función de la trayectoria mínima
o geodésica //

4. $y = h - g_1 t$; $y = h - \frac{1}{2} g_2 t^2$; $y = h - \frac{1}{4} g_3 t^3$

El lagrangiano para un cuerpo en caída libre es:

$$L = T - V = \frac{1}{2} m \dot{y}^2 - mgy$$

La acción S se define como:

$$S = \int_{t_1}^{t_2} L\,dt = \int_0^T \left(\frac{1}{2} m \dot{y}^2 - mgy\right) dt$$

Caso 1: $y = h - g_1 t$

$$\dot{y} = \frac{dy}{dt} = -g_1$$

$$T = \frac{1}{2} m \dot{y}^2 = \frac{1}{2} m g_1^2$$

$$V = mgy = mg(h - g_1 t)$$

$$L = T - V = \frac{1}{2} m g_1^2 - mg(h - g_1 t)$$

$$S_1 = \int_0^T \left(\frac{1}{2} m g_1^2 - mg(h - g_1 t)\right) dt$$

$$S_1 = \int_0^T \frac{1}{2} m g_1^2 \, dt - \int_0^T mg(h - g_1 t)\, dt$$

$\bullet \int_0^T \frac{1}{2} m g_1^2 \, dt = \frac{1}{2} m g_1^2 T$

$\bullet \int_0^T mg(h - g_1 t) \, dt = mghT - \frac{1}{2} mg g_1 T^2$

$\Rightarrow S_1 = \frac{1}{2} m g_1^2 T - mghT + \frac{1}{2} mg g_1 T^2$

Caso 2: $y = h - \frac{1}{2} g_2 t^2$

$\dot{y} = -g_2 t$

$T = \frac{1}{2} m g_2^2 t^2$

$V = mg(h - \frac{1}{2} g_2 t^2)$

$L = \frac{1}{2} m g_2^2 t^2 - mg(h - \frac{1}{2} g_2 t^2)$

$S_2 = \int_0^T (\frac{1}{2} m g_2^2 t^2 - mg(h - \frac{1}{2} g_2 t^2)) \, dt$

$S_2 = \int_0^T (\frac{1}{2} m g_2^2 t^2 - mgh + \frac{1}{2} mg g_2 t^2) \, dt$

$S_2 = \int_0^T (\frac{1}{2} m (g_2^2 + g g_2) t^2 - mgh) \, dt$

$$S_2 = \frac{1}{2} m (g_2^2 + g g_2) \frac{T^3}{3} - mghT$$

$$S_2 = \frac{1}{6} m (g_2^2 + g g_2) T^3 - mghT$$

Caso 3: $y = h - \frac{1}{4} g_3 t^3$

$$\dot{y} = -\frac{3}{4} g_3 t^2$$

$$T = \frac{9}{32} m g_3^2 t^4$$

$$V = mg\left(h - \frac{1}{4} g_3 t^3\right)$$

$$L = \frac{9}{32} m g_3^2 t^4 - mg\left(h - \frac{1}{4} g_3 t^3\right)$$

$$S_3 = \int_0^T \left(\frac{9}{32} m g_3^2 t^4 - mgh + \frac{1}{4} m g g_3 t^3\right) dt$$

$$S_3 = \frac{9}{160} m g_3^2 T^5 - mghT + \frac{1}{16} m g g_3 T^4$$

⇒ Utilizando las ecuaciones de Lagrange para un cuerpo en caída libre

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad ; \quad L = T - V$$

$$T = \frac{1}{2} m \dot{y}^2 \quad ; \quad V = mgy$$

$$L = T - V = \frac{1}{2} m \dot{y}^2 - mgy$$

i) $\frac{\partial L}{\partial \dot{y}}$

$$\frac{\partial L}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left(\frac{1}{2} m \dot{y}^2 - mgy\right) = m\dot{y}$$

ii) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$

$$\frac{d}{dt}(m\dot{y}) = m\ddot{y}$$

iii) $\frac{\partial L}{\partial y}$

$$\frac{\partial}{\partial y}\left(\frac{1}{2} m \dot{y}^2 - mgy\right) = -mg$$

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$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$

$$m\ddot{y} - (-mg) = 0$$

$$m\ddot{y} + mg = 0$$

$$\ddot{y} + g = 0$$

$\Rightarrow$ E.P. $\quad \ddot{y} = -g$

$$\int \ddot{y}\, dt = -\int g\, dt$$

$$\dot{y} = -gt + c_1$$

Si: $\dot{y}(0) = 0$

$$0 = -g(0) + c_1$$

$$c_1 = 0$$

$\Rightarrow \quad \dot{y} = -gt$

$$\int \dot{y}\, dt = -\int gt\, dt$$

$$y = -\frac{1}{2}gt^2 + c_2$$

Si: $y(0) = h$

$$h = -\frac{1}{2}g(0)^2 + C_2$$

$$C_2 = h$$

$\Rightarrow$ Por lo tanto, la ecuación de posición es:

$$y = -\frac{1}{2}gt^2 + h$$ //

— // —— // —— // —— // —— // ——

Ahora se calcula g_1, g_2 y g_3 cuando $y(T) = 0$

i) $y = h - g_1 t$

$$0 = h - g_1 T$$

$$g_1 = \frac{h}{T}$$ //

ii) $y = h - \frac{1}{2}g_2 t^2$

$$0 = h - \frac{1}{2}g_2 T^2$$

$$g_2 = \frac{2h}{T^2}$$ //

iii) $y = h - \frac{1}{4}g_3 t^3$

$$0 = h - \frac{1}{4}g_3 T^3$$

$$y_3 = \frac{4h}{T^3}$$

Se reemplazan y_1, y_2 y y_3 en S_1, S_2 y S_3 respectivamente, dando como resultado:

$$S_1 = \frac{1}{2} m \frac{h^2}{T} - \frac{1}{2} mghT$$

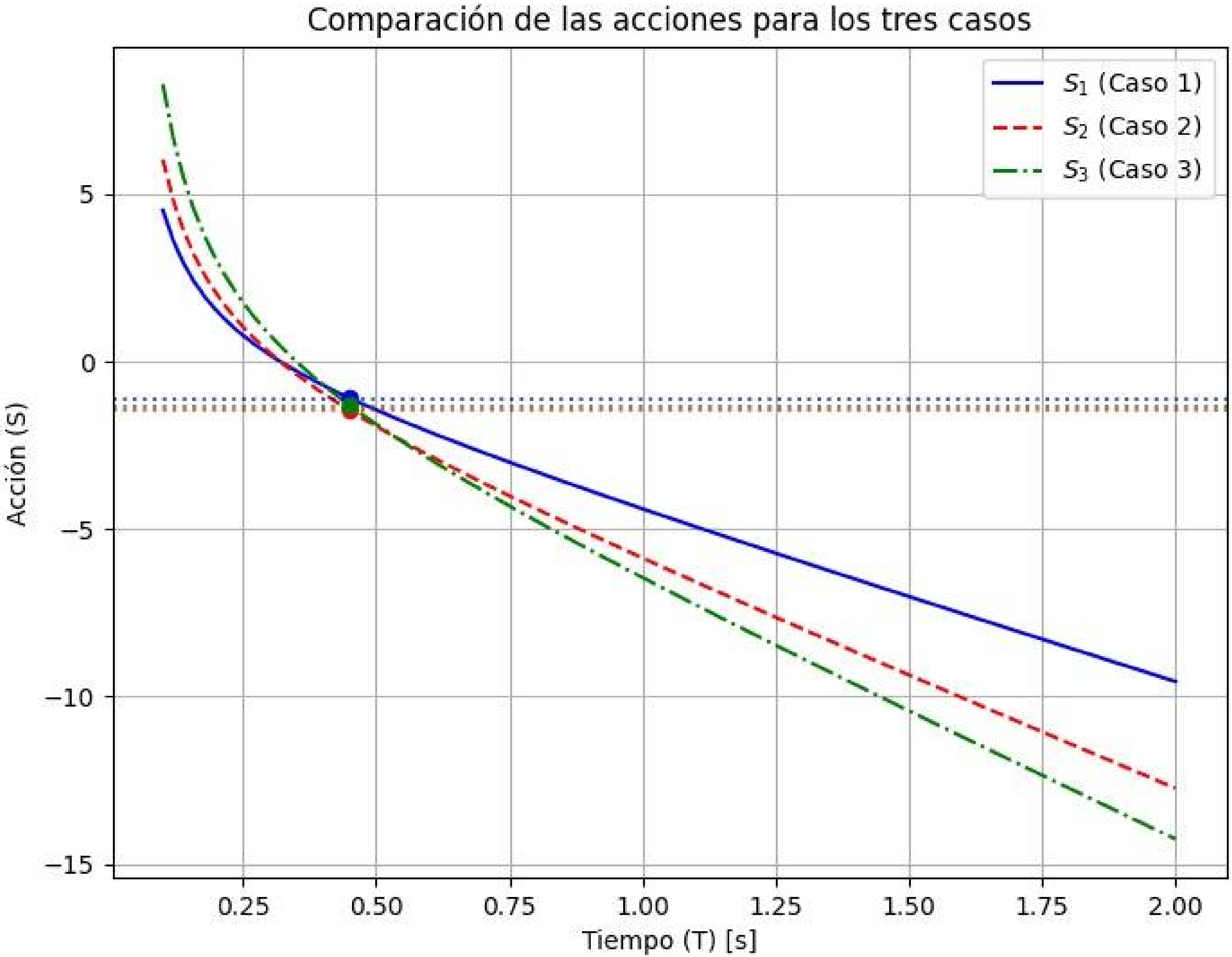
$$S_2 = \frac{2}{3} m \frac{h^2}{T} - \frac{2}{3} mghT$$

$$S_3 = \frac{9}{10} m \frac{h^2}{T} - \frac{3}{4} mghT$$

Ahora se grafica y se calcula computacionalmente el valor de S cuando T es crítico, es decir, cuando cae al suelo la partícula:

$$T_{critico} = \sqrt{\frac{2h}{g}}$$

Al realizar el cálculo el S que minimiza la acción en el T crítico es S_2, que concuerda con lo obtenido por medio de la ecuación de Lagrange.

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El valor de S1 en T crítico es: -1.1067971810589328
El valor de S2 en T crítico es: -1.4757295747452432
El valor de S3 en T crítico es: -1.3281566172707195
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5. $L = \frac{m^2 \dot{x}^4}{12} + m\dot{x}^2 f(x) - f^2(x)$

$f(x) \Rightarrow$ función difereciable de x

$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

i) $\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{m^2 \dot{x}^4}{12} + m\dot{x}^2 f(x) - f^2(x)\right)$

$= \frac{m^2 \dot{x}^3}{3} + 2m\dot{x} f(x)$

ii) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$

$\frac{d}{dt}\left(\frac{m^2 \dot{x}^3}{3} + 2m\dot{x} f(x)\right)$

$= m^2 \dot{x}^2 \ddot{x} + 2m\ddot{x} f(x) + 2m\dot{x} \frac{df}{dt}$

$= m^2 \dot{x}^2 \ddot{x} + 2m\ddot{x} f(x) + 2m\dot{x} \frac{df}{dx} \frac{dx}{dt}$

$= m^2 \dot{x}^2 \ddot{x} + 2m\ddot{x} f(x) + 2m\dot{x}^2 \frac{df(x)}{dx}$

iii) $\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(\frac{m^2\dot{x}^4}{12} + m\dot{x}^2 f(x) - f^2(x)\right)$

$$= m\dot{x}^2\frac{df}{dx} - 2f(x)\frac{df}{dx}$$

Sustituyendo se obtiene:

$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + 2m\dot{x}^2\frac{df}{dx}$$

$$-m\dot{x}^2\frac{df}{dx} + 2f(x)\frac{df}{dx} = 0$$

$$\Rightarrow m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + m\dot{x}^2\frac{df}{dx} + 2f(x)\frac{df}{dx} = 0 \quad //$$

6^{to} Punto:

Lagrangiano de un sistema con n grados de libertad puramente cinético

$$\mathcal{L} = \frac{1}{2} g_{ab}(q_c)\dot{q}^a\dot{q}^b$$

1. Convenio suma Einstein: $\frac{1}{2} g_{ab}(q_c)\dot{q}^a\dot{q}^b = \frac{1}{2} g_{cb}(q_c)\dot{q}^c\dot{q}^b$

2. Funciones simétricas: $g_{ab} = g_{ba}$

3. $\det(g_{ab}) \neq 0 \Rightarrow g^{ab}$ existe tal que $g^{ab} g_{bc} = \delta^a_c \begin{cases} 1 ; & a = c \\ 0 ; & a \neq c \end{cases}$

Ecuaciones Lagrange:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) - \frac{\partial \mathcal{L}}{\partial q^a} = 0$$

①. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \overbrace{\frac{1}{2} g_{cb}(q_c)}^{\text{Independiente de } \dot{q}^a}\left[\frac{\partial \dot{q}^c}{\partial \dot{q}^a}\dot{q}^b + \dot{q}^c \frac{\partial \dot{q}^b}{\partial \dot{q}^a}\right]$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} g_{cb}(q_c)\left[\delta^c_a \dot{q}^b + \dot{q}^c \delta^b_a\right]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} g_{cb}(q_c)\delta^c_a \dot{q}^b + \frac{1}{2} g_{cb}(q_c)\dot{q}^c\delta^b_a$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} g_{ab}(q_c)\dot{q}^b + \frac{1}{2} g_{ca}(q_c)\dot{q}^c$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} g_{ab}(q_c)\dot{q}^b + \frac{1}{2} g_{ac}(q_c)\dot{q}^c$$

→ c índices libre

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = g_{ab}(q_c)\dot{q}^b = g_{ab}\dot{q}^b$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \frac{d}{dt}(g_{ab}\dot{q}^b) = \frac{d}{dt}(g_{ab})\dot{q}^b + g_{ab}\frac{d}{dt}(\dot{q}^b)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \frac{\partial g_{ab}}{\partial q^c}\dot{q}^c\dot{q}^b + g_{ab}\ddot{q}^b$$ (Regla de la cadena)

② $$\frac{\partial L}{\partial q^a} = \frac{1}{2}\dot{q}^c\dot{q}^b\frac{\partial g_{cb}}{\partial q^a}$$

Ecuación Lagrange:

$$g_{ab}\ddot{q}^b + \frac{\partial g_{ab}}{\partial q^c}\dot{q}^c\dot{q}^b - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^a}\dot{q}^c\dot{q}^b = 0$$

$$g_{ab}\ddot{q}^b + \left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^a}\right)\dot{q}^c\dot{q}^b = 0$$

Multiplicamos g^{da} por izquierda a ambos lados:

$$\underbrace{g^{da}g_{ab}}_{\delta^d_b}\ddot{q}^b + g^{da}\left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^a}\right)\dot{q}^c\dot{q}^b = 0$$

$$\delta^d_b\ddot{q}^b + g^{da}\left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^a}\right)\dot{q}^c\dot{q}^b = 0$$

$$\ddot{q}^d + \frac{1}{2}g^{da}\left(2\frac{\partial g_{ab}}{\partial q^c} - \frac{\partial g_{cb}}{\partial q^a}\right)\dot{q}^c\dot{q}^b = 0$$

$$\ddot{q}^d + \frac{1}{2}g^{da}\left(\frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{cb}}{\partial q^a}\right)\dot{q}^c\dot{q}^b = 0$$

Esto debido que son índices doomies.

Intercambiando $d \leftrightarrow a$:

$$\ddot{q}^a + \frac{1}{2}g^{ad}\left(\frac{\partial g_{db}}{\partial q^c} + \frac{\partial g_{dc}}{\partial q^b} - \frac{\partial g_{cb}}{\partial q^d}\right)\dot{q}^c\dot{q}^b = 0$$

$$\ddot{q}^a + \frac{1}{2}g^{ad}\left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d}\right)\dot{q}^b\dot{q}^c = 0$$

$$\ddot{q}^a + \Gamma^a_{bc}\dot{q}^b\dot{q}^c = 0$$; Γ^a_{bc} := Símbolos de Christoffel.