

# Trace Pairing

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## Abstract

We show that the packing map of [NOZ26] that converts a vector of finite field elements to a single element of the ring of integers of a cyclotomic extension has an associated inner product formula. The key idea is that the trace pairing on number fields reduces to a perfect pairing on the rings of integers after reduction modulo an unramified prime. This specializes to the 2-power cyclotomic extension setting, which entails the existence of a self-dual basis that allows for efficient computation of the inner product.

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## 1 Trace form on finite étale algebras

**Definition 1** (Trace). *Let  $K$  be a field and  $A$  be a finite free  $K$ -algebra (unital) of rank  $n$ . For  $a \in A$ , the trace is*

$$\mathrm{Tr}_{A/K}(a) := \mathrm{tr}(\mu_a),$$

where  $\mu_a : A \rightarrow A$ ,  $x \mapsto ax$  is the  $K$ -linear multiplication map and  $\mathrm{tr}$  denotes the matrix trace. This is independent of the choice of  $K$ -basis [Lan02].<sup>1</sup>

**Definition 2** (Trace form). *The trace form is the  $K$ -bilinear pairing*

$$M_{A/K} : A \times A \rightarrow K, \quad M_{A/K}(a, b) := \mathrm{Tr}_{A/K}(ab).$$

**Theorem 1** (Nondegeneracy of the trace form). *Let  $A$  be a finite  $K$ -algebra of rank  $n$ . The following are equivalent:*

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<sup>1</sup>Readers familiar with arithmetic geometry will recognize this as Grothendieck duality applied to the morphism  $\mathrm{Spec} A \rightarrow \mathrm{Spec} K$ .

1.  $A$  is étale over  $K$  (i.e.  $A \cong \prod_{i=1}^r L_i$  with each  $L_i/K$  a finite separable extension),
2.  $B_{A/K}$  is a perfect pairing, i.e. the map  $A \rightarrow \text{Hom}_K(A, K)$ ,  $a \mapsto B_{A/K}(a, -)$  is an isomorphism,
3.  $\text{disc}(A/K) := \det(\text{Tr}_{A/K}(e_i e_j))_{i,j} \neq 0$  for some (equivalently, any)  $K$ -basis  $\{e_1, \dots, e_n\}$ .

*Proof.* (1)  $\Leftrightarrow$  (2) follows from [Aut, Lemma 49.3.1, Tag 0BJF]. (2)  $\Leftrightarrow$  (3): standard linear algebra — the bilinear form is nondegenerate iff its Gram matrix is invertible.  $\square$

**Corollary 1** (Dual basis). *If  $A/K$  is étale and  $\{e_1, \dots, e_n\}$  is a  $K$ -basis of  $A$ , there exists a unique dual basis  $\{e_1^*, \dots, e_n^*\}$  satisfying  $\text{Tr}_{A/K}(e_i \cdot e_j^*) = \delta_{ij}$ .*

*Proof.* Immediate from the isomorphism  $A \xrightarrow{\sim} \text{Hom}_K(A, K)$  of Theorem 1.  $\square$

**Proposition 1** (Inner product recovery — general case). *Let  $A/K$  be étale of rank  $n$  with basis  $\{e_i\}$  and dual basis  $\{e_i^*\}$ . Define the coordinate maps*

$$\psi : K^n \rightarrow A, \quad (a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i e_i, \quad \psi^* : K^n \rightarrow A, \quad (b_1, \dots, b_n) \mapsto \sum_{i=1}^n b_i e_i^*.$$

*Then for any  $\mathbf{a}, \mathbf{b} \in K^n$ ,*

$$\text{Tr}_{A/K}(\psi(\mathbf{a}) \cdot \psi^*(\mathbf{b})) = \langle \mathbf{a}, \mathbf{b} \rangle := \sum_{i=1}^n a_i b_i.$$

*Proof.* Bilinearity of the trace form and  $\text{Tr}_{A/K}(e_i \cdot e_j^*) = \delta_{ij}$ .  $\square$

**Remark 1.** *Proposition 1 holds for any finite étale  $K$ -algebra without hypotheses on the Galois group. In the cyclotomic setting of Section 1.1, the Gram matrix is scalar and  $\psi^* = (k/d) \cdot \sigma_{-1} \circ \psi$ ; for a non-abelian example where the Gram matrix is not scalar and no such involution  $\sigma_{-1}$  exists, see Appendix A.*

## 1.1 Trace pairing on number fields

Consider an extension of number fields  $L/K/\mathbb{Q}$ . The trace form  $M(a, b) = \text{Tr}_{L/K}(ab)$  is a perfect pairing, which restricts to a bilinear pairing on rings of integers:  $\text{Tr}_{\mathcal{O}_L/\mathcal{O}_K} : \mathcal{O}_L \rightarrow \mathcal{O}_K$ . Let  $H$  be the Galois group of  $L/K$  and suppose a prime  $p$  is unramified in  $L/K$ , then the pairing  $\text{Tr}_{\mathcal{O}_L/\mathcal{O}_K}$  reduces to a perfect pairing after reduction modulo  $p$ , which we denote by  $\text{Tr}_H : \mathcal{O}_L/p\mathcal{O}_L \rightarrow \mathcal{O}_K/p\mathcal{O}_K$ .

**Remark 2.** *Notice that the notation implicitly assumes that we are working with the field extension  $L/K$ .*

**Remark 3.** *In the case that  $L = \mathbb{Q}(\zeta_{2^a})$ , then every odd prime  $p$  is unramified.*

## 1.2 2-power cyclotomic specialization

We now specialize to the setting of [NOZ26]. Let  $d = 2^\alpha$ ,  $\mathcal{O} = \mathbb{Z}[X]/(X^d + 1)$ , and let  $G \subseteq \text{Aut}(\mathcal{O}/p\mathcal{O})$  be a subgroup of automorphisms. We first state and prove the following lemma:

**Lemma 1.**  $\langle \sigma_{4k+1} \rangle = \{ \sigma_{4k\alpha+1} : 0 \leq \alpha \leq d/(2k) - 1 \}$  as a set.

*Proof.* The containment  $\subseteq$  follows from the binomial theorem:  $(4k+1)^n = \sum_{j=0}^n \binom{n}{j} (4k)^j \equiv 1 \pmod{4k}$  for all  $n \geq 0$ , so every power of  $4k+1$  has the form  $4k\alpha + 1$ . To see equality, by [LS118, Lemma 2.4], the left hand side has cardinality  $d/(2k)$ , which matches the right hand side.  $\square$

Let  $\mathcal{B} = \{e_0, \dots, e_{2n-1}\} = \{X^i\}_{0 \leq i < \frac{d}{2k}} \cup \{X^{\frac{d}{2}+i}\}_{0 \leq i < \frac{d}{2k}}$ , with the obvious indices and  $n = \frac{d}{2k}$ , be a basis of  $\mathcal{O}/p\mathcal{O}$  over  $\mathcal{O}^H/p\mathcal{O}^H$ . The key result in the proof is the following:

**Proposition 2.** For all  $a, b \in \{0, \dots, 2n-1\}$ ,

$$\text{Tr}_H(e_a \cdot \sigma_{-1}(e_b)) = \frac{d}{k} \cdot \delta_{ab}.$$

*Proof.* Suppose  $e_a = X^i$ ,  $e_b = X^j$ , and  $i \neq j$ , then  $e_a \cdot \sigma_{-1}(e_b) = X^i \cdot \sigma_{-1}(X^j) = X^{i-j}$ . By inspection, this is equal to  $X^m$  for some  $m \in \{0, \dots, \frac{d}{2k} - 1\} \cup \{\pm \frac{d}{2} + 0, \dots, \pm \frac{d}{2} + \frac{d}{2k} - 1\}$ . By the structure theorem  $(\mathbb{Z}/2d\mathbb{Z})^\times \cong \langle -1 \rangle \times \langle g \rangle$  where  $g$  has order  $d/2$  and  $\langle g \rangle$  consists of elements  $\equiv 1 \pmod{4}$ . Since  $4k+1 \equiv 1 \pmod{4}$ , we have  $\sigma_{4k+1} \in \langle g \rangle$ , generating the unique subgroup of order  $d/(2k)$ . Thus  $H = \langle \sigma_{-1} \rangle \times \langle \sigma_{4k+1} \rangle$  and  $\text{Tr}_H = \text{Tr}_{\langle \sigma_{-1} \rangle} \circ \text{Tr}_{\langle \sigma_{4k+1} \rangle}$ .

By Lemma 1,  $\text{Tr}_{\langle \sigma_{4k+1} \rangle}(X^m) = \sum_{\alpha=0}^{d/(2k)-1} X^{(4k\alpha+1)m} = X^m \sum_{\alpha=0}^{d/(2k)-1} \omega^\alpha$  where  $\omega = X^{4km}$ . The rest of the proof follows from a straightforward calculation:

- *Case  $m = 0$ :* Then  $\text{Tr}(X^m) = \text{Tr}(1) = |H| = d/k$ .
- *Case  $\frac{d}{2k} \nmid m$ :* Then  $\omega \neq 1$  and  $\omega^{d/(2k)} = X^{2dm} = 1$ , so  $\omega$  is a primitive  $\ell$ -th root of unity for some  $\ell \mid \frac{d}{2k}$ ,  $\ell > 1$ . The sum of all  $\frac{d}{2k}$ -th powers of such a root vanishes.
- *Case  $\frac{d}{2k} \mid m$ :* Then we must have  $m = \pm \frac{d}{2}$  by inspection, and hence  $\omega = 1$ , so  $\text{Tr}_{\langle \sigma_{4k+1} \rangle}(\pm X^{d/2}) = \pm \frac{d}{2k} X^{d/2}$ . Then  $\text{Tr}_{\langle \sigma_{-1} \rangle}(\pm X^{d/2}) = \pm X^{d/2} + \sigma_{-1}(\pm X^{d/2}) = 0$ , since  $X^d = -1$ .

$\square$

**Corollary 2.** The set  $\mathcal{B}$  is an  $\mathcal{O}^H/p\mathcal{O}^H$ -module basis of  $\mathcal{O}/p\mathcal{O}$

*Proof.* For a 2-power cyclotomic extension, every odd prime is unramified, so  $\mathcal{O}^H/p\mathcal{O}^H = K_1 \times \dots \times K_r$  is a finite product of fields, and hence  $\mathcal{O}/p\mathcal{O}$  is a finite product of  $K_i$ -vector spaces and therefore a free  $\mathcal{O}^H/p\mathcal{O}^H$ -module. From standard algebraic number theory, the rank of  $\mathcal{O}/p\mathcal{O}$  as an  $\mathcal{O}^H/p\mathcal{O}^H$ -module is  $|\mathcal{Q}(\zeta_{2^d}) : \mathbb{Q}(\zeta_{2^d})^H| = \frac{d}{k} = |\mathcal{B}|$ . From Proposition 2, the determinant of the Gram matrix associated to the trace pairing and  $\mathcal{B}$  is a unit, so  $\mathcal{B}$  spans  $\mathcal{O}/p\mathcal{O}$  as an  $\mathcal{O}^H/p\mathcal{O}^H$ -module.  $\square$

**Corollary 3** (Self-dual inner product identity). *With  $\psi$  as in Theorem 2 of [NOZ26],*

$$\psi(\mathbf{a}) = \sum_{i=0}^{d/2k-1} a_i X^i + X^{d/2} \sum_{i=0}^{d/2k-1} a_{d/2k+i} X^i, \quad (1)$$

*the dual basis is  $(k/d) \cdot \{\sigma_{-1}(e_i)\}$ , so  $\psi^* = (k/d) \cdot \sigma_{-1} \circ \psi$ . Substituting into Proposition 1:*

$$\mathrm{Tr}_H(\psi(\mathbf{a}) \cdot \sigma_{-1}(\psi(\mathbf{b}))) = \frac{d}{k} \cdot \langle \mathbf{a}, \mathbf{b} \rangle.$$

## Appendix A Non-abelian example: $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$

The minimal polynomial  $x^3 - 2$  has roots  $\sqrt[3]{2}, \omega \sqrt[3]{2}, \omega^2 \sqrt[3]{2}$ , where  $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$ . The three embeddings  $\sigma_j : \mathbb{Q}(\sqrt[3]{2}) \hookrightarrow \mathbb{C}$  send  $\sqrt[3]{2} \mapsto \omega^{j-1} \sqrt[3]{2}$ , and since  $1 + \omega + \omega^2 = 0$ , the trace  $\mathrm{Tr}(\sqrt[3]{2}^k) = \sqrt[3]{2}^k (1 + \omega^k + \omega^{2k})$  vanishes unless  $3 \mid k$ . For the power basis  $e_1 = 1, e_2 = \sqrt[3]{2}, e_3 = \sqrt[3]{4}$ :

$$G = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 1/6 \\ 0 & 1/6 & 0 \end{pmatrix}, \quad \det(G) = -108 \neq 0.$$

The dual basis is  $e_1^* = \frac{1}{3}, e_2^* = \frac{1}{6} \sqrt[3]{4}, e_3^* = \frac{1}{6} \sqrt[3]{2}$ .

## References

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