Numerical Methods GPS Project

Shenghan Yueh, Juju Ren, Yedidya Moise

November 22, 2022

1 Introduction

Global Positioning Systems provide relative locations to the user by making use of satellites. The location of a few satellites are known and using the time it took for a signal to travel from the user to the different satellites, the location of the user can be found. This is done by setting up a system of equations using the simple distance equations, time, and the speed of the signal traveling from user to satellite. While the equations are simple they are still non-linear and therefore, they require special methods of solving.
The overall goal of this project is to solve these equations and find the location of the user. Methods to solve the system will vary based on the number of satellites used. Additionally, we will study the how the inaccuracies in time measurement produce errors in the calculated position of the user.

2 Results and methodologies

2.1 Activity 1

We are given a system of equations

$$\sqrt{(x-A_1)^2+(y-B_1)^2+(z-C_1)^2}-c(t_1-d)=0$$
$$\sqrt{(x-A_2)^2+(y-B_2)^2+(z-C_2)^2}-c(t_2-d)=0$$
$$\sqrt{(x-A_3)^2+(y-B_3)^2+(z-C_3)^2}-c(t_3-d)=0$$
$$\sqrt{(x-A_4)^2+(y-B_4)^2+(z-C_4)^2}-c(t_4-d)=0$$

where x, y, z are the coordinates of the person on Earth which we are solving for. A_i, B_i, C_i are the coordinates of the satellites which we are given. t_i is the time it takes for a signal to reach the person on the ground to the ith satellite.

We first solve the system using Multivariate Newton's method. Using the initial conditions provided in the problem we create the F matrix and its Jacobian matrix DF and solve for $s = DF^{-1}F$. Instead of calculating the inverse of DF, we directly solve for s by solving the system $DFs = F$ with QR method. We use s in Multivariate Newton's method to correct x at each iteration. Using the initial conditions

$$(x_0, y_0, z_0, d_0) = (0, 0, 6370, 0)$$

With four satellites positioned at

$$(15600, 7540, 20140), (18760, 2750, 18610), (17610, 14630, 13480), (19170, 610, 18390)$$

We receive the same result as given in the book

$$(x, y, z, d) = (-41.772709570857543, -16.789194106537199, 6370.0595592233294, -0.0032015658295942305)$$

We use a while loop to let Multivariate Newton's method run until the error falls below our tolerance level. We calculate the error by using the norm of F. Ideally, F should be equal to zero so we try and get the norm of F as close to zero as possible. We set the tolerance level to be 10^{-16} because Multivariate Newton's method converges so quickly that this is not a huge cost and as we are dealing with positioning people we want to be as accurate as possible. The backward error is less than 10^{-16}. We include a table showing the infinity norm of F calculated at each iteration of Multivariate Newton's method.

Iterations	$\|\|F\|\|_\infty$
1	9.246606e+02
2	1.782459e-02
3	5.228503e-08
4	3.637979e-12
5	3.637979e-12
6	0.000000e+00

2.2 Activity 2

In this system of equations there are four unknowns: the three coordinates of the user (x, y, z) and the time at which the signal left the user d. When we use four satellites to find the position of the user, this system can be manipulated to put (x, y, z) in terms of d. We do this by taking the first equation (in 4.38 in the textbook) and subtracting from it the other three equations one at a time. This yields three linear equations of four variables. We then set d to be the free variable and put x, y and z in terms of d. Plugging the new forms of x, y and z values back into the first equation yields a quadratic equation of the variable d. This can be solved using the quadratic equation. Then, we plug the d back into the equations of x, y and z.
When solving the problem with a quadratic, there are, naturally, two solutions when we know that only one is the true solution. The way we can identify the correct solution is by realizing that the coordinate system is designed that the zero is the center of the earth. Therefore, by taking the Euclidean norm of the resulting coordinates we can identify which solution places the location of the user on the surface of the earth. This solution is the correct one when Euclidean norm of (x, y, z) is near the radius of the earth, 6370 km.
This method is much better than using Multivariate Newton's method for many reasons. Multivariate Newton's method requires a fairly accurate initial guess of the values to start the iterative process. Had we not been given such an initial value convergence would not have been guaranteed. Further, a simple quadratic calculation is much more computationally efficient than the iterative process of Multivariate Newton's method.

2.3 Activity 4

In this section, we will continue to use the quadratic method to solve for locations of the user. The overall goal of this activity is to ascertain the level of error in the location of the user that naturally will exist given the inaccuracy of clocks. To do this, we begin with a set location of the user and use four satellites with four different locations. These locations are set based on sine and cosine functions with the range limited to ensure the satellite is in the upper hemisphere of the Earth.
The positions for the satellites are calculated by setting the (x_i, y_i, z_i) coordinates equal to
$(\rho \cos\gamma_i \cos\theta_i, \rho\cos\gamma_i \sin\theta_i, \rho \sin\gamma_i)$ where $(\gamma_i, \theta_i) = (i\pi/8, (i-1)\pi/2)$ for $i = 1, 2, 3, 4$.

We then take these locations and use the distance formula and the speed of the signal to calculate the time it would take for the signal to reach each satellite from the user. We can now check our results by plugging the locations of the satellites and the time taken for the signal to reach each satellite into our quadratic function from Activity 2. That will give us the approximate position of the user. It should yield the same result as the location of the user we started this section with.
Now, we can get to the fun part. We have the locations of the user and satellites and time taken to for the signal to travel between the user and satellites. In order to identify the error that will naturally occur given the accuracy of clocks, we notice that clocks are accurate to 10^{-8} and add or subtract 10^{-8} to the various times calculated and then plug the new time into the quadratic function. We add error in the time for every satellite. For instance, we add 10^{-8} to the time of one satellite, subtract 10^{-8} from the time of the second satellite, and so on.
We take the new coordinates calculated and compare them with the true coordinates. We take the difference of the two vectors and take the infinity norm of the resulting error vector. This will be the numerator in calculating our error magnification factor. To calculate the denominator we take take the vector of added error in time for every satellite, take the infinity norm, and multiply by the speed

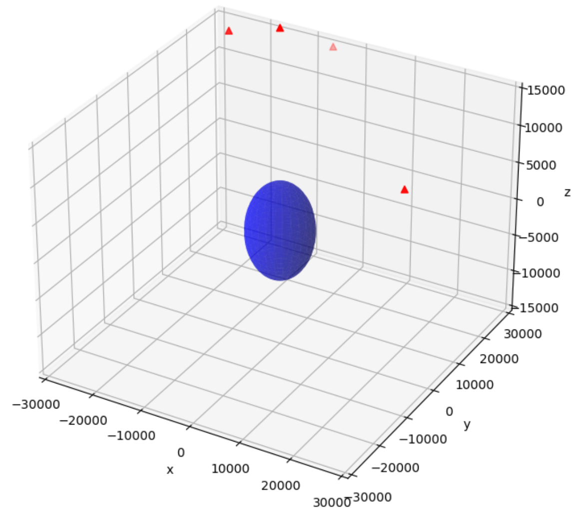

Figure 1: 3D image of satellites and Earth for Activity 4

of light (the approximate speed of the signal). This yields an error in terms of distance.

$$\frac{||\Delta x, \Delta y, \Delta z||_\infty}{c||\left(+\Delta t_1, .., \Delta t_m\right)||_\infty}$$

We tried four different combinations of time error introduced to the different satellites:

$$(+\Delta t_1, +\Delta t_2, +\Delta t_3, -\Delta t_4)$$

$$(+\Delta t_1, +\Delta t_2, -\Delta t_3, -\Delta t_4)$$

$$(+\Delta t_1, -\Delta t_2, +\Delta t_3, -\Delta t_4)$$

$$(+\Delta t_1, -\Delta t_2, -\Delta t_3, -\Delta t_4)$$

We calculate the EMF for each set of errors we placed in the system and find the maximum. The estimated condition number is the maximum EMF. We calculated that the maximum position error in meters was 20.282905136658467 m.

And the condition number 5.47529573895321

2.4 Activity 5

This activity is very similar to Activity four but the positions of the satellites are limited to ensure they are close together. We do this by setting $(\gamma_i, \theta_i) = (\pi/2 + (i-1)\frac{5}{100}\frac{\pi}{2}, (i-1)\frac{5}{100}2\pi)$ for $i = 1, 2, 3, 4$.

By following the same process we find that the maximum position error in meters is 7683.097776366875. The condition number is 2510.92145830439. We see that the error in solving for the location of the user is greatly increased when the satellites are close to one another. We can understand this intuitively by realizing that to estimate the location of the user we "compare" the time it takes for a signal to reach each satellite. This comparison is done mathematically but the essential idea is that we are looking for a position on the surface of the Earth that accounts for the time it took for a signal to reach each satellite at its given location. Thus, when the satellites are close together, there is less variance in the information given. This makes it more difficult to "compare" the data and identify a unique location on the Earth that would explain the time it took for a signal to reach each satellite from the user.

2.5 Activity 6

In this Activity, we add four more satellites to see how the added satellites affect the accuracy of our results. Once we solve for the location of the user using the extra satellites we will calculate the EMF

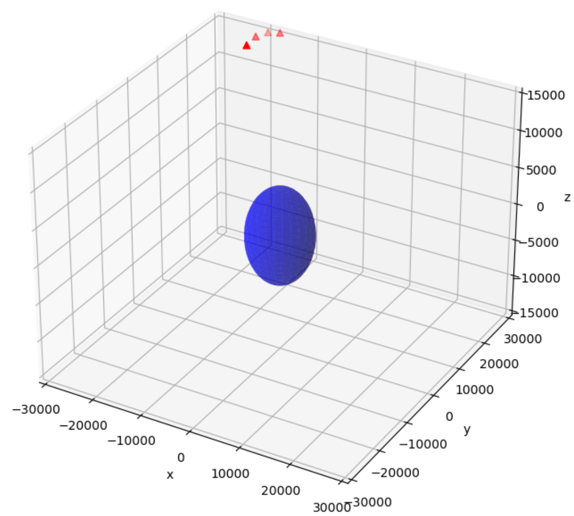

Figure 2: 3D image of satellites and Earth for Activity 5

and condition number in the same way as in Activity 4. To solve for the location itself we employ the use of the Gauss-Newton method. We set up three criteria, TOL, maximum iteration cycle, and the s, which was solved by $A^TAs = A^Tr$ with QR method, to stop the iteration.
The Gauss-Newton method finds the (x, y, z, d) that yields a zero of the derivative of F. This can yield either maxima, minima, or saddle points of a function. We are obviously looking for the minimum of F as we want $F = 0$. To check if the vector given produces a minimum, we can plug it into the F and take the norm. If the result is close to zero then we know that we have found a minimum.
When calculating the coordinates of the eight satellites we use the same $(x_i, y_i, z_i) =$ $(\rho\cos\gamma_i\cos\theta_i, \rho\cos\gamma_i\sin\theta_i, \rho\sin\gamma_i)$ (x_i, y_i, z_i) but where $(\gamma_i, \theta_i) = (i\pi/16, (i-1)\pi/4)$ for $i = 1, 2, .., 8$
We now want to see how well this method deals with error in the time. To do this we do the same thing as in Activity 4: we add in different errors in the time for each satellite and recalculate the position of the location. We then take the infinity norm of the difference of these two locations to see how much error was produced by the small time perturbations. We do this a total of four times, each time changing how exactly we perturb the time of each satellite.
We add the following vectors of perturbations to the time vector containing all the times recording the signal traveling from the user to each satellite.

$$(+\Delta t_1, +\Delta t_2, +\Delta t_3, -\Delta t_4, +\Delta t_5, +\Delta t_6, +\Delta t_7, -\Delta t_8)$$
$$(+\Delta t_1, +\Delta t_2, -\Delta t_3, -\Delta t_4, +\Delta t_5, +\Delta t_6, -\Delta t_7, -\Delta t_8)$$
$$(+\Delta t_1, -\Delta t_2, +\Delta t_3, -\Delta t_4, +\Delta t_5, -\Delta t_6, +\Delta t_7, -\Delta t_8)$$
$$(+\Delta t_1, -\Delta t_2, -\Delta t_3, -\Delta t_4, +\Delta t_5, -\Delta t_6, -\Delta t_7, -\Delta t_8)$$

where $\Delta t_i = 10^{-8}$.
As in Activity 4, we then calculate the EMF and condition number. We calculated that the maximum position error in meters was 11.298820637670085, and the condition number was 3.1559204167377692.
One difficulty we ran into applying Gauss-Newton's method is finding a good initial vector. In a real-life scenario we would have the data of time it takes for the signal to reach the satellites from the user. Then we could take a four satellites and calculate a fairly accurate location of the user using the quadratic method in Activity 2. We could then use that as an initial vector to plug into Gauss-Newton method for a more accurate location with a much smaller error. However, in this case we are missing both an initial vector and a time vector so we must make an initial vector and use that to create a time vector as spelled out in the instructions in Activity 4. As we use satellites in the same region as in Activity 4 we also use the same initial vector as given in Activity 4 $(0, 0, 6370, 0.0001)$.
We also calculated EMF for all the combinations of Δt_i.The value changes per two Δt_i combinations.

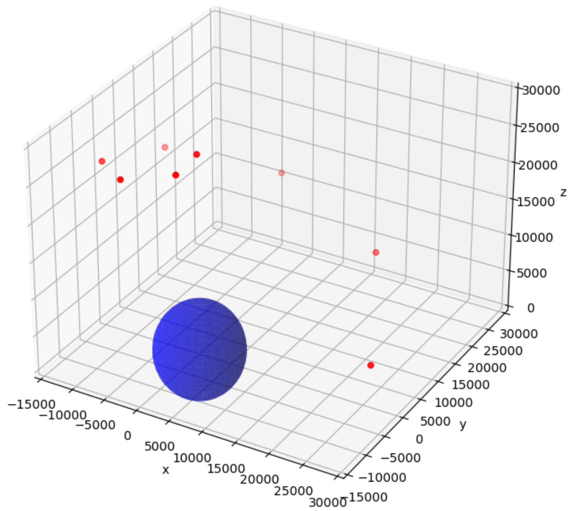

Figure 3: 3D image of satellites and Earth for Activity 6

The two combinations with all the opposite signs have a similar EMF. When all the Δt_i are the same, the EMF are close to 3.34×10^{-6} (as the following condition).

$$(-\Delta t_1, -\Delta t_2, -\Delta t_3, -\Delta t_4, -\Delta t_5, -\Delta t_6, -\Delta t_7, -\Delta t_8)$$

$$(+\Delta t_1, +\Delta t_2, +\Delta t_3, +\Delta t_4, +\Delta t_5, +\Delta t_6, +\Delta t_7, +\Delta t_8)$$

The maximum EMF, 4.32, is when Δt_i are:

$$(-\Delta t_1, +\Delta t_2, +\Delta t_2, -\Delta t_4, -\Delta t_5, -\Delta t_6, +\Delta t_7, +\Delta t_8)$$

$$(+\Delta t_1, -\Delta t_2, -\Delta t_3, +\Delta t_4, +\Delta t_5, +\Delta t_6, -\Delta t_7, -\Delta t_8).$$

3 Conclusion

In this project we simulated how satellites can be used to solve for the coordinates of someone on Earth simply by using the coordinates of the satellites and the time it takes for a signal to reach each satellite from the person. We started with using four satellites for the four unknowns: the three coordinates of the person and the d the difference between the synchronized time on the satellite clocks and the receiver-clock. We did this with Multivariate Newton and then simplified the process of finding the location by transforming our system of equations into a simple quadratic and then solving for the coordinates.

We then examined how much error in distance is produced when we introduce error into the time the signal takes to travel from the person to the satellites. We did this for four unbunched satellites, four bunched satellites, and eight unbunched satellites.

We found that the error from four bunched satellites is much higher than four unbunched satellites and that eight unbunched satellites had less error than four unbunched satellites. We summarize our results in the table below. It is clear that unbunched satellites yield much more accurate results than bunched, and the more satellites we use the more accurate the results are.

Satellites	Condition Number	Maximum Position Error in Meters
4 (Unbunched)	5.47529573895321	20.282905136658467
4 (Bunched)	2510.92145830439	7683.097776366875
8 (unbunched)	3.1559204167377692	11.298820637670085

References

1.https://github.com/snazrul1/PyRevolution/blob/master/matplotlib/3D
Used for the 3D plots.

2.https://www.geeksforgeeks.org/permutation-and-combination-in python/
Used to try different combinations of Δt_i.

3.https://pandas.pydata.org/docs/reference/api/pandas.DataFrame.sort_values.html
Used to sort lists of EMF values to find maximum and minimum.