1 Simulation of the PDEs

1.1 Model Equations

Let $u = (u_1, u_2)$ be the velocity, where $u_1, u_2 : \Omega \times [0, \infty) \to \mathbb{R}$ and $p : \Omega \times [0, \infty) \to \mathbb{R}$ the pressure and $T : \Omega \times [0, \infty) \to \mathbb{R}$ the temperature variable. We then get the scalar equations:

$$\frac{\partial u_1}{\partial t} + \langle u, \nabla u_1 \rangle = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u_1 - g_1 \alpha (T - T_0) \tag{1}$$

$$\frac{\partial u_2}{\partial t} + \langle u, \nabla u_2 \rangle = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 u_2 - g_2 \alpha (T - T_0) \tag{2}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \varepsilon p \tag{3}$$

$$\frac{\partial T}{\partial t} + \langle u, \nabla T \rangle = \kappa \nabla^2 T \tag{4}$$

where

$$\begin{aligned}
g = (g_1, g_2)^T &: \text{gravitational acceleration} \\
\rho &: \text{density of the fluid} \\
\nu &: \text{kinematic viscosity} \\
\alpha &: \text{coefficient of expansion} \\
\kappa &: \text{thermal diffusivity} \\
\varepsilon &: \text{regularisation parameter} \\
T_0 &: \text{a reference temperature}
\end{aligned}$$

with boundary conditions

$$u_1 = u_2 = 0 \qquad \text{on } \Gamma \tag{5}$$

$$T = T_{\text{dir}} \qquad \text{on } \Gamma_1 \tag{6}$$

$$\partial_\eta T = \langle \nabla T, \eta \rangle = \gamma (T - T_{\text{out}}) \qquad \text{on } \Gamma \setminus \Gamma_1 \tag{7}$$

where

$$\begin{aligned}
\eta &: \text{outer normal of the boundary} \\
T_{\text{dir}} &: \text{heating} \\
T_{\text{out}} &: \text{outside temperature} \\
\gamma &: \text{thermal conductivity of the boundary}
\end{aligned}$$

1.2 Implicit Euler discretisation in time

If we have a differential equation, e.g for u_1

$$\frac{\partial u_1}{\partial t} = F(u_1, u_2, p, T)$$

we can apply implicit Euler with a timestep δt to get

$$\frac{u_1^{\text{new}} - u_1^{\text{old}}}{\delta t} = F(u_1^{\text{new}}, u_2, p, T) \tag{8}$$

where we now have find the root u_1 of the Function

$$u_1 - u_1^{\text{old}} - \delta t F(u_1, u_2, p, T)$$

1.3 Weak formulation

Testing each line $(1)-(4)$ with a test function $v_i, i \in \{1, \dots, 4\}$ and integrating over the domain Ω we can now define

$$F_1(u_1, u_2, p, T)v_1 := \int\limits_\Omega \left(-\langle u, \nabla u_1 \rangle - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u_1 - g_1 \alpha (T - T_0) \right) v_1 \mathrm{d}\omega \tag{9}$$

$$F_2(u_1, u_2, p, T)v_2 := \int\limits_\Omega \left(-\langle u, \nabla u_2 \rangle - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 u_2 - g_2 \alpha (T - T_0) \right) v_2 \mathrm{d}\omega \tag{10}$$

$$F_3(u_1, u_2, p, T)v_3 := \int\limits_\Omega \left(\varepsilon p - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) v_3 \mathrm{d}\omega \tag{11}$$

$$F_4(u_1, u_2, p, T)v_4 := \int\limits_\Omega \left(-\langle u, \nabla T \rangle + \kappa \nabla^2 T \right) v_4 \mathrm{d}\omega \tag{12}$$

partial integration and using the boundary conditions yields:
(remember, $\nabla^2 u_1 = \Delta u_1 = \sum\limits_i \frac{\partial^2 u_1}{\partial^2 x_i}$)

$$
\begin{aligned}
F_1(u_1,u_2,p,T)v_1 &= \int\limits_\Omega \left(-\langle u, \nabla u_1\rangle - \frac{1}{\rho}\frac{\partial p}{\partial x} - g_1\alpha(T-T_0)\right)v_1 + \nu\nabla^2 u_1 v_1 \mathrm{d}\omega \\
&= \int\limits_\Omega \left(-\langle u, \nabla u_1\rangle - \frac{1}{\rho}\frac{\partial p}{\partial x} - g_1\alpha(T-T_0)\right)v_1 - \nu\langle\nabla u_1, \nabla v_1\rangle \mathrm{d}\omega + \int\limits_\Gamma \nu\langle\nabla u_1, \eta\rangle v_1 \mathrm{d}s \\
&= \int\limits_\Omega \left(-\langle u, \nabla u_1\rangle - \frac{1}{\rho}\frac{\partial p}{\partial x} - g_1\alpha(T-T_0)\right)v_1 - \nu\langle\nabla u_1, \nabla v_1\rangle \mathrm{d}\omega, \quad \text{as } v_1 = 0 \text{ on } \Gamma \\
F_2(u_1,u_2,p,T)v_2 &= \int\limits_\Omega \left(-\langle u, \nabla u_2\rangle - \frac{1}{\rho}\frac{\partial p}{\partial y} - g_2\alpha(T-T_0)\right)v_2 - \nu\langle\nabla u_2, \nabla v_2\rangle \mathrm{d}\omega + \int\limits_\Gamma \nu\langle\nabla u_2, \eta\rangle v_2 \mathrm{d}s \\
&= \int\limits_\Omega \left(-\langle u, \nabla u_2\rangle - \frac{1}{\rho}\frac{\partial p}{\partial y} - g_2\alpha(T-T_0)\right)v_2 - \nu\langle\nabla u_2, \nabla v_2\rangle \mathrm{d}\omega \quad \text{as } v_2 = 0 \text{ on } \Gamma
\end{aligned}
$$

$$
\begin{aligned}
F_4(u_1,u_2,p,T)v_4 &= \int\limits_\Omega \left(-\langle u, \nabla T\rangle + \kappa\nabla^2 T\right)v_4 \mathrm{d}\omega \\
&= \int\limits_\Omega -\langle u, \nabla T\rangle v_4 - \kappa\langle\nabla T, \nabla v_4\rangle \mathrm{d}\omega + \int\limits_\Gamma \kappa\langle\nabla T, \eta\rangle v_4 \mathrm{d}s \\
&= \int\limits_\Omega -\langle u, \nabla T\rangle v_4 - \kappa\langle\nabla T, \nabla v_4\rangle \mathrm{d}\omega + \int\limits_{\Gamma_1} \kappa\langle\nabla T, \eta\rangle v_4 \mathrm{d}s + \int\limits_{\Gamma\setminus\Gamma_1} \kappa\langle\nabla T, \eta\rangle v_4 \mathrm{d}s \\
&= \int\limits_\Omega -\langle u, \nabla T\rangle v_4 - \kappa\langle\nabla T, \nabla v_4\rangle \mathrm{d}\omega + \int\limits_{\Gamma_1} \kappa\langle\nabla T, \eta\rangle v_4 \mathrm{d}s + \int\limits_{\Gamma\setminus\Gamma_1} \kappa\gamma(T - T_{\mathrm{out}}) v_4 \mathrm{d}s
\end{aligned}
$$

1.4 Newton's method on the implicit Euler equation

Using the considerations of the last two subsections, we define

$$
G(u_1,u_2,p,T) = \begin{pmatrix} \int\limits_\Omega (u_1 - u_1^{\mathrm{old}})v_1 \mathrm{d}\omega - \delta t F_1(u_1,u_2,p,T)v_1 \\ \int\limits_\Omega (u_2 - u_2^{\mathrm{old}})v_2 \mathrm{d}\omega - \delta t F_2(u_1,u_2,p,T)v_2 \\ F_3(u_1,u_2,p,T)v_3 \\ \int\limits_\Omega (T - T^{\mathrm{old}})v_4 \mathrm{d}\omega - \delta t F_4(u_1,u_2,p,T)v_4 \end{pmatrix}
$$

We now want to find $(u_1, u_2, p, T)^T$ such that $G(u_1, u_2, p, T) = 0$. Applying Newton's method on the function G, we get the iteration

$$\begin{pmatrix} u_1^{i+1} \\ u_2^{i+1} \\ p^{i+1} \\ T^{i+1} \end{pmatrix} = \begin{pmatrix} u_1^{i} \\ u_2^{i} \\ p^{i} \\ T^{i} \end{pmatrix} + \begin{pmatrix} \delta u_1^{i} \\ \delta u_2^{i} \\ \delta p^{i} \\ \delta T^{i} \end{pmatrix}$$

where

$$G'(u_1^i, u_2^i, p^i, T^i) \begin{pmatrix} \delta u_1^{i} \\ \delta u_2^{i} \\ \delta p^{i} \\ \delta T^{i} \end{pmatrix} = -G(u_1^i, u_2^i, p^i, T^i)$$

and

$$G'(u_1^i, u_2^i, p^i, T^i) = \begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} & \frac{\partial G_1}{\partial p} & \frac{\partial G_1}{\partial T} \\ \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} & \frac{\partial G_2}{\partial p} & \frac{\partial G_2}{\partial T} \\ \frac{\partial G_3}{\partial u_1} & \frac{\partial G_3}{\partial u_2} & \frac{\partial G_3}{\partial p} & \frac{\partial G_3}{\partial T} \\ \frac{\partial G_4}{\partial u_1} & \frac{\partial G_4}{\partial u_2} & \frac{\partial G_4}{\partial p} & \frac{\partial G_4}{\partial T} \end{pmatrix}$$

1.5 Simplifications of the system

Consider a Function $G(z) = C + Az + N(z)$, where C is a constant, Az is the linear and $N(z)$ is the nonlinear part of the function. If we apply Newton's method to find a root of $G(z)$, using $\delta z^i = z^{i+1} - z^i$, we have to solve

$$G'(z^i)\delta z = -G(z^i) \tag{13}$$

$$\Leftrightarrow \quad (A + N'(z^i))(z^{i+1} - z^i) = -\big(C + Az^i + N(z^i)\big) \tag{14}$$

and solve this for z^{i+1} we obtain

$$(A + N'(z^i))(z^{i+1}) = -C - Az^i - N(z^i) + (A + N'(z^i))z_i = -C - N(z^i) + N'(z^i)z^i$$

In our context, we split up $G_i(u_1, u_2, p, T), i \in \{1, \dots, 4\}$ and get

$$C_{G_1} = -\int_\Omega u_1^{\mathrm{old}} v_1 \mathrm{d}\omega - \delta t C_{F_1} = -\int_\Omega u_1^{\mathrm{old}} v_1 \mathrm{d}\omega - \delta t \int_\Omega g_1 \alpha T_0 v_1 \mathrm{d}\omega$$

$$C_{G_2} = -\int_\Omega u_2^{\mathrm{old}} v_2 \mathrm{d}\omega - \delta t C_{F_2} = -\int_\Omega u_2^{\mathrm{old}} v_2 \mathrm{d}\omega - \delta t \int_\Omega g_2 \alpha T_0 v_2 \mathrm{d}\omega$$

$$C_{G_3} = C_{F_3} = 0$$

$$C_{G_4} = -\int_\Omega T^{\mathrm{old}} v_4 \mathrm{d}\omega - \delta t C_{F_4} = -\int_\Omega T^{\mathrm{old}} v_4 \mathrm{d}\omega + \delta t \int_{\Gamma \setminus \Gamma_1} \kappa \gamma T_{\mathrm{out}} v_3 \mathrm{d}\omega$$

for the constant part and for the left hand side linear part

$$
\begin{aligned}
A_{G_1} z^{i+1} &= \int\limits_{\Omega} u_1^{i+1} v_1 \mathrm{d}\omega - \delta t A_{F_1} z^{i+1} \\
&= \int\limits_{\Omega} u_1^{i+1} v_1 \mathrm{d}\omega - \delta t \left[\int\limits_{\Omega} \left(-\frac{1}{\rho} \frac{\partial p^{i+1}}{\partial x} - g_1 \alpha T^{i+1} \right) v_1 - \nu \langle \nabla u_1^{i+1}, \nabla v_1 \rangle \mathrm{d}\omega \right] \\
&= \int\limits_{\Omega} u_1^{i+1} v_1 \mathrm{d}\omega + \delta t \int\limits_{\Omega} \left(\frac{1}{\rho} \frac{\partial p^{i+1}}{\partial x} + g_1 \alpha T^{i+1} \right) v_1 + \nu \langle \nabla u_1^{i+1}, \nabla v_1 \rangle \mathrm{d}\omega \\
A_{G_2} z^{i+1} &= \int\limits_{\Omega} u_2^{i+1} v_2 \mathrm{d}\omega - \delta t A_{F_2} z^{i+1} \\
&= \int\limits_{\Omega} u_2^{i+1} v_2 \mathrm{d}\omega - \delta t \left[\int\limits_{\Omega} \left(-\frac{1}{\rho} \frac{\partial p^{i+1}}{\partial y} - g_2 \alpha T^{i+1} \right) v_2 - \nu \langle \nabla u_2^{i+1}, \nabla v_2 \rangle \mathrm{d}\omega \right] \\
&= \int\limits_{\Omega} u_2^{i+1} v_2 \mathrm{d}\omega + \delta t \int\limits_{\Omega} \left(\frac{1}{\rho} \frac{\partial p^{i+1}}{\partial y} + g_2 \alpha T^{i+1} \right) v_2 + \nu \langle \nabla u_2^{i+1}, \nabla v_2 \rangle \mathrm{d}\omega \\
A_{G_3} z^{i+1} &= \delta t A_{F_3} z^{i+1} \\
&= \int\limits_{\Omega} (\varepsilon p^{i+1} - \frac{\partial u_1^{i+1}}{\partial x} - \frac{\partial u_2^{i+1}}{\partial y}) v_3 \\
A_{G_4} z^{i+1} &= \int\limits_{\Omega} T^{i+1} v_4 \mathrm{d}\omega - \delta t A_{F_4} z^{i+1} \\
&= \int\limits_{\Omega} T^{i+1} v_4 \mathrm{d}\omega - \delta t \left(\int\limits_{\Omega} -\kappa \langle \nabla T^{i+1}, \nabla v_4 \rangle \mathrm{d}\omega + \int\limits_{\Gamma_1} \kappa \langle \nabla T^{i+1}, \eta \rangle v_4 \mathrm{d}s + \int\limits_{\Gamma \setminus \Gamma_1} \kappa \gamma T^{i+1} v_4 \mathrm{d}s \right) \\
&= \int\limits_{\Omega} T^{i+1} v_4 \mathrm{d}\omega + \delta t \left(\int\limits_{\Omega} \kappa \langle \nabla T^{i+1}, \nabla v_4 \rangle \mathrm{d}\omega - \int\limits_{\Gamma_1} \kappa \langle \nabla T^{i+1}, \eta \rangle v_4 \mathrm{d}s - \int\limits_{\Gamma \setminus \Gamma_1} \kappa \gamma T^{i+1} v_4 \mathrm{d}s \right)
\end{aligned}
$$

To get the nonlinear part $N(z^i)$ and its linearization $N'(z^i)z^i$ and $N'(z^i)z^{i+1}$ respectively, we first compute with w being a test function

$$
\begin{aligned}
&\frac{\partial}{\partial u_1} (\langle u, \nabla u_1 \rangle) w \\
=&\frac{\partial}{\partial u_1} (u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y}) w \\
=&w \frac{\partial u_1}{\partial x} + u_1 \frac{\partial w}{\partial x} + u_2 \frac{\partial w}{\partial y} \qquad \text{product rule}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial u_2}(\langle u, \nabla u_2\rangle)w\\
=&\frac{\partial}{\partial u_2}(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y})w\\
=&u_1\frac{\partial w}{\partial x}+w\frac{\partial u_2}{\partial y}+u_2\frac{\partial w}{\partial y} && \text{product rule}
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
N_{G_1}(z^i)=&-\delta t N_{F_1}(z^i)\\
=&\int_\Omega \langle u^i, \nabla u_1^i\rangle v_1 \mathrm{d}\omega\\
N'_{G_1}(z^i)z^{i+1}=&-\delta t N'_{F_1}(z^i)z^{i+1}\\
=&\delta t\int_\Omega (\underbrace{u_1^{i+1}\frac{\partial u_1^i}{\partial x}+u_1^i\frac{\partial u_1^{i+1}}{\partial x}+u_2^i\frac{\partial u_1^{i+1}}{\partial y}}_{-(\frac{\partial N_{F_1}(z_i)}{\partial u_1})u_1^{i+1}}+\underbrace{u_2^{i+1}\frac{\partial u_1^i}{\partial y}}_{-(\frac{\partial N_{F_1}(z_i)}{\partial u_2})u_2^{i+1}})v_1\mathrm{d}\omega\\
N_{G_2}(z^i)=&-\delta t N_{F_2}(z^i)\\
=&\int_\Omega \langle u^i, \nabla u_2^i\rangle v_2 \mathrm{d}\omega\\
N'_{G_2}(z^i)z^{i+1}=&-\delta t N'_{F_2}(z^i)z^{i+1}\\
=&\delta t\int_\Omega (\underbrace{u_1^{i}\frac{\partial u_2^{i+1}}{\partial x}+u_2^{i+1}\frac{\partial u_2^{i}}{\partial y}+u_2^i\frac{\partial u_2^{i+1}}{\partial y}}_{-(\frac{\partial N_{F_2}(z_i)}{\partial u_2})u_2^{i+1}}+\underbrace{u_1^{i+1}\frac{\partial u_2^i}{\partial x}}_{-(\frac{\partial N_{F_2}(z_i)}{\partial u_1})u_1^{i+1}})v_2\mathrm{d}\omega\\
N_{G_3}(z^i)=&0\\
N_{G_4}(z^i)=&-\delta t N_{F_4}(z^i)\\
=&\delta t\int_\Omega \langle u, \nabla T\rangle v_4 \mathrm{d}\omega\\
N'_{G_4}(z^i)z^{i+1}=&-\delta t N'_{F_4}(z^i)z^{i+1}\\
=&\delta t\int_\Omega (\underbrace{u_1^{i+1}\frac{\partial T^i}{\partial x}}_{-(\frac{\partial N_{F_4}(z^i)}{\partial u_1})u_1^{i+1}}+\underbrace{u_2^{i+1}\frac{\partial T^i}{\partial y}}_{-(\frac{\partial N_{F_4}(z^i)}{\partial u_2})u_2^{i+1}}+\underbrace{\langle u^i, \nabla T^{i+1}\rangle}_{-(\frac{\partial N_{F_4}(z^i)}{\partial T})T^{i+1}})v_4\mathrm{d}\omega
\end{aligned}
$$

1.6 Formulation as linear equation system

Using a Finite Element discretisation, we obtain finite dimensional spaces $V_i, i \in \{1, \ldots, 4\}$, where we can just use the Basis functions $\Phi_j, j \in \{1, \ldots, N\}$ for each space as test functions $v_i, \in \{1, \dots, 4\}$. This gives the lines of the equation system. Furthermore, the variables u_1, u_2, p and T are linear combinations of these basis functions. Hence we have, e.g for the temperature T^{i+1} in the next Newton step

$$T^{i+1} = \sum_{j=1}^{N} \bar{T}_j^{i+1} \Phi_j^{i+1}$$

with coefficients $\bar{T}_j^{i+1} \in \mathbb{R}$.

1.7 Solutions of the linear equation system

1.7.1 No Temperature

If we consider the case without the temperature, we obtain a linear system with a saddle point matrix

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} u^{i+1} \\ p^{i+1} \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix},$$

where $u = (u_1, u_2)$.
Note, that $M_{22} = 0$, if $\varepsilon = 0$ in the continuity equation. In this case, we obtain a unsymmetric, indefinite saddle point matrix. where M_{12} is the gradient and M_{21} the diffusion operator (with boundary conditions), where M_{11} stems from the (nonlinear) advection diffusion part.
An indeph overview of numerical methods can be found in [1], choices for preconditioners in [3, 2]. In [2] the SIMPLE and SIMPLER method for preconditioning are presented.
A possible solution of these systems is by the GMRES method, augmented with an inexact Schur preconditioner $P : X^\star \to X$, namely performs the operation

1. solve $M_{21} \tilde{M_{11}}^{-1} M_{12} p = D \tilde{M_{11}}^{-1} r_1$
2. solve $\tilde{M_{11}} u = r_1 - Gp$

where $\tilde{M_{11}}$ could be the diagonal of M_{11} (SIMPLE method). Note that, if we used $\tilde{M_{11}} = M_{11}$, we converge in one step, as the precondition operation yields the solution. Using the library libmesh, with PETsc as backend, this can be achieved by passing

- -pc_type fieldsplit $\to$ use fieldsplit preconditioner
- -pc_fieldsplit_type schur $\to$ use Schur-comp as preconditioner
- -pc_fieldsplit_detect_saddle_point $\to$ automatic detection of blocks
- -pc_fieldsplit_schur_fact_type <full,diag,upper...> $\to$ how does $\tilde{M_{11}}$ evolve from M_{11}.

as command line arguments for setting the preconditioner, and

- -ksp_type <richardson,chebyshev,cg,bicg,gmres..>

as Krylov solver. GMRES is the default here.

2 Affine covariant Newton algorithms

Defined by 13, we will compute the Newton step δz at iteration point z. Afterwards, we will compute another (simplified) Newton step, via the solution of

$$G'(z)\bar{\delta z} = -\left(G(z+\delta z) - G(z) - G'(z)\delta z\right)$$

In the absence of damping, i.e $-G(z)-G'(z)\delta z = 0$, this yields a second Newton step after δz, with derivative information at z. The idea is now, to estimate the Newton contraction

$$\Theta_{z_\star}(x_k) := \frac{\|z_{k+1} - z_\star\|}{\|z_k - z_\star\|} = \frac{\|G'(z_k)^{-1}\left(G'(z_k)(z_\star - z_k) - (G(z_\star) - G(z_k)\right)\|}{\|z_k - z_\star\|}$$

by $z_+ = z - G'(z)^{-1}G(z) = z + \delta z$:

$$\Theta_{z_+}(x) := \frac{\|G'(z)^{-1}\left(G'(z)(z_+ - z) - (G(z_+) - G(z_k)\right)\|}{\|z_k - z_\star\|}$$

and to extend this to a model around z with

$$[\Theta]_{z_+}(\xi) = \frac{[\omega]}{2}\|\xi - z_+\|$$

and to choose $[\omega]$ such that

$$[\Theta]_{z_+}(z) = \Theta_{x_+}(x) = \frac{\|\bar{\delta z}\|}{\|\delta z\|} \qquad \Rightarrow \qquad [\omega] = \frac{2\|\bar{\delta z}\|}{\|\delta z\|^2}$$

2.1 Globalization by damping

The problem $G(z) = 0$ might be too difficult, and our starting guess (even the one computed with the explicit Euler) might be too far away from the solution.

Algorithm 1 Damped Newton's method

Require: iterate $z, [\omega]$

repeat// step computation loop

solve $G'(z)\delta z + G(z) = 0.$

do

compute $\lambda = \min\{1, \frac{1}{\|\delta z\|[\omega]}\}$

solve $G'(z)\bar{\delta z} + G(z + \lambda \delta z) = 0.$

compute $[\omega] = \frac{\|\bar{\delta z} - (1-\lambda)\delta z\|}{\|\lambda \delta z\|^2}$

$\Theta_{z+\lambda\delta z}(z) = \frac{[\omega]}{2}\|\lambda \delta z\|$

while $\Theta_{z+\lambda\delta z}(z) \geq 1$ (divergence)

$z_+ \leftarrow z + \lambda \delta z$

if $\lambda = 1$ and $\Theta_{z_+}(z) \leq 0.5$ and $\|\delta z\| \leq \text{TOL}$ **then**

$z_+ + \bar{\delta z} \leftarrow$ terminate "desired accuracy needed"

end if

$z \leftarrow z_+$

until

References

[1] M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta Numerica*, (14):1–137, 2005.

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