2D/3D Geometry Morphing via L2 Semi-discrete Optimal Transport

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L2 semi-discrete optimal transport

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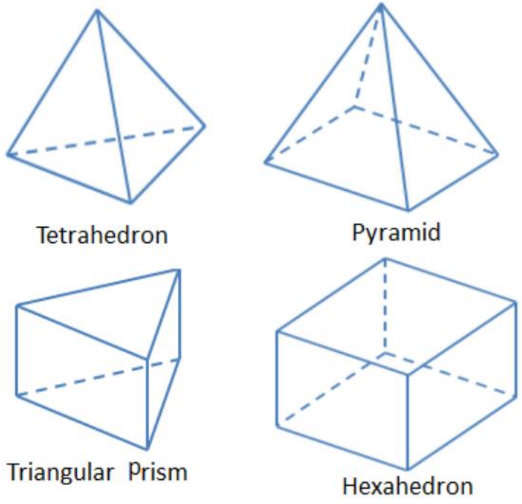


Figure 1: Different types of mesh elements

$$\max \int_\Omega \psi d\mu + \int_\Omega \phi d\nu$$
$$s.t.\ \phi(x) + \psi(y) \leq c(x,y), \forall (x,y) \in \Omega \times \Omega$$

Duality of Monge–Kantorovich form

$$c(x,y) = 1/2||x-y||^2$$

L2 distance cost

$$\nabla(\|x\|^2/2 - \psi(x)).$$

Optimal transport from Brenier Theorem

$$\nu = \sum_{i=1}^{k} \nu_i \delta_{p_i}$$

Dirac masses

$$\mu$$

Piece-wise linear density

$$t_1 = (1-t)x_1 + tT(x_1) \text{ and } t_2 = (1-t)x_2 + tT(x_2)$$

Collision free transport

Prescribed mass satisfied ⟹ **Theorem 2.1** *Given a measure μ with density, a set of discrete points y_i and prescribed masses ν_i such that $\sum_i \nu_i = \mu(\Omega)$, there exists a weight vector W such that $\mu(Pow_W(p_i)) = \nu_i$ and the T_W*

OTM as parameterized PD ⟹ *defined by the power diagram parameterized by W is an optimal transport map.*

$$\max \int_\Omega \psi d\mu + \int_\Omega \phi d\nu$$

$$s.t.\ \phi(x) + \psi(y) \leq c(x,y), \forall (x,y) \in \Omega \times \Omega$$

$$\phi^c(x) = \inf_y c(x,y) - \phi(y)$$
$$\phi(y_i) = 1/2\omega_i,$$

$$\max \sum_i \int_{\Omega_{y_i}} \|x - y_i\|^2 - w_i d\mu + \sum_i v_i w_i$$

$$T(x) \Longleftrightarrow Pow_W(y_i) := \{x \mid \|x - y_i\| - w_i < \|x - y_j\| - w_j, \forall i \neq j\}$$

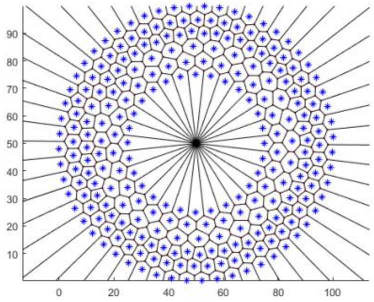
$$g(W) = \int_\Omega \|x - T(x)\|^2 d\mu - \sum_{i=1}^k w_i \mu(T^{-1}(y_i)) + \sum_{i=1}^k \omega_i \nu_i$$

$$\partial g / \partial w_i = -\mu(Pow_W(y_i)) + \nu_i$$ ⟹

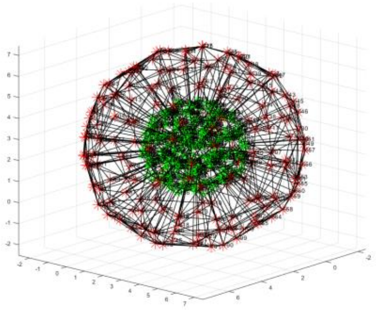
First-order gradient for optimization of weight vector

Power Diagram Construction

Unconstrained 2D/3D power diagrams



Power diagram of 2D points supporting "ring"



Power diagram of 3D points supporting "sphere"

Dual relation between RT(S) and PD(S)

A vertex y_i of $RT(S)$ is the the dual of power diagram cell $Pow_W(y_i)$,

An edge of $RT(S)$ is the dual of a face of $PD(S)$

A face of $RT(S)$ is the dual of an edge of $PD(S)$.

A triangle/tetrahedron of $RT(S)$ is the dual of a vertex of $PD(S)$.

$$Pow_W(y_i) := \{x | \|x - y_i\| - w_i < \|x - y_j\| - w_j, \forall i \neq j\}$$ Power diagram

$$Vor(y_i) := \{x | \|x - y_i\| < \|x - y_j\|, \forall i \neq j\}$$ Voronoi diagram

$$p^+ = (p_x, p_y, p_z, p_x^2 + p_y^2 + p_z^2 \text{-} \omega_p).$$ Lifted point

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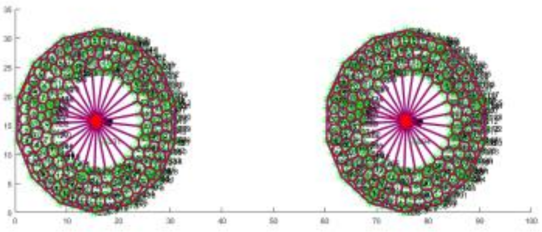
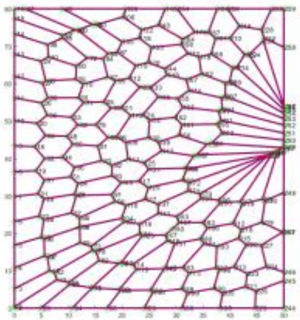
Constrained 2D/3D power diagrams



PD constrained by "2 circles"

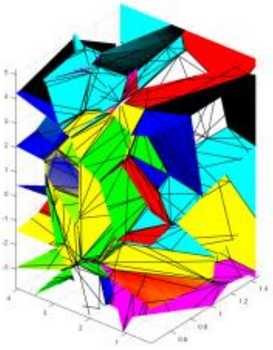


PD constrained by "rectangle"

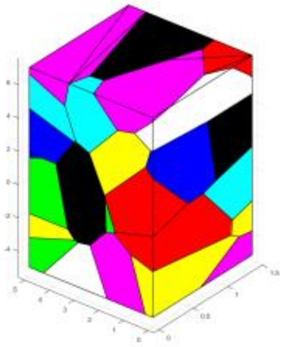
Intersection of mesh cells and power diagrams

More than 70% of total time

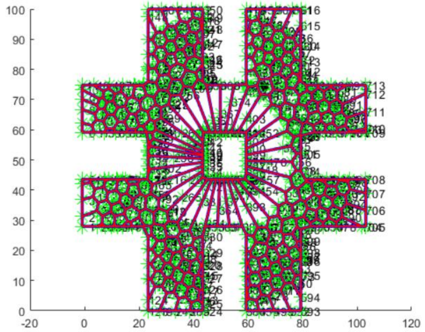
Convex hull V-H Rep computation by pre-compiled MEX file



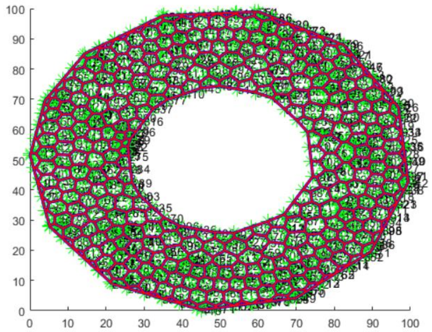
PD constrained by "cuboid" (interior)



PD constrained by "cuboid" (exterior)



PD constrained by "4 cross"



PD constrained by "ring"

Numerical Implementations

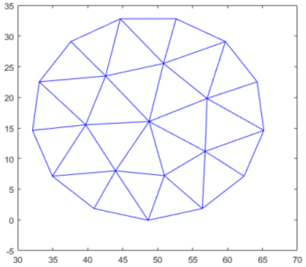
$$\frac{\partial^2 g}{\partial w_i \partial w_j} = \int_{Pow_W(y_i)|M \cap Pow_W(y_j)|M} \frac{1}{||y_i - y_j||} d\mu$$

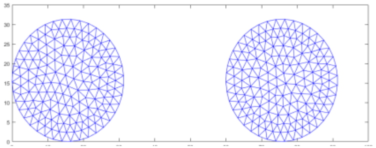
$$\frac{\partial^2 g}{\partial {w_i}^2} = - \sum_{k \neq i} \frac{\partial^2 g}{\partial w_i \partial w_k}$$

Hessian of loss function for Newton method

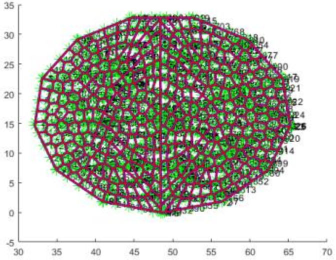
$$H_n = H + \delta I$$

Empirical item to avoid singularity

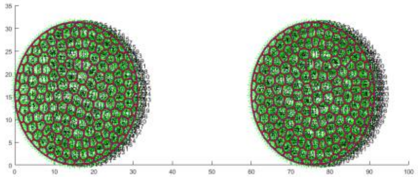




Start and end meshes



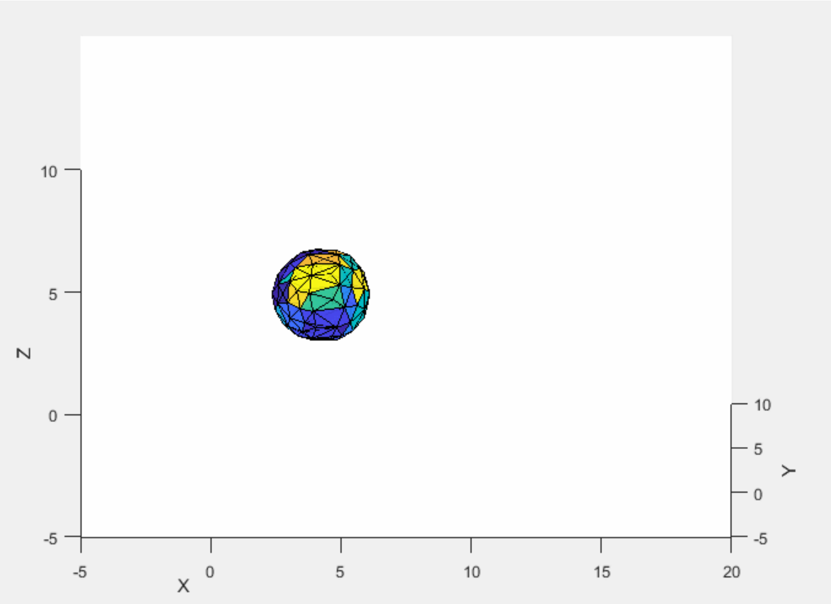
$Pow_{W^*}(Y)|M_{init}$



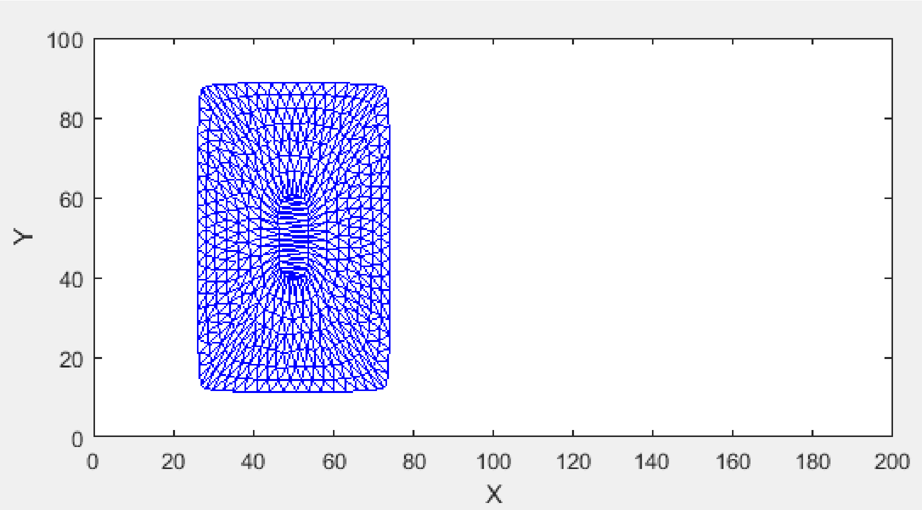
$Pow_{W=0}(Y)|M_{end}$

Initial and optimized constrained PD

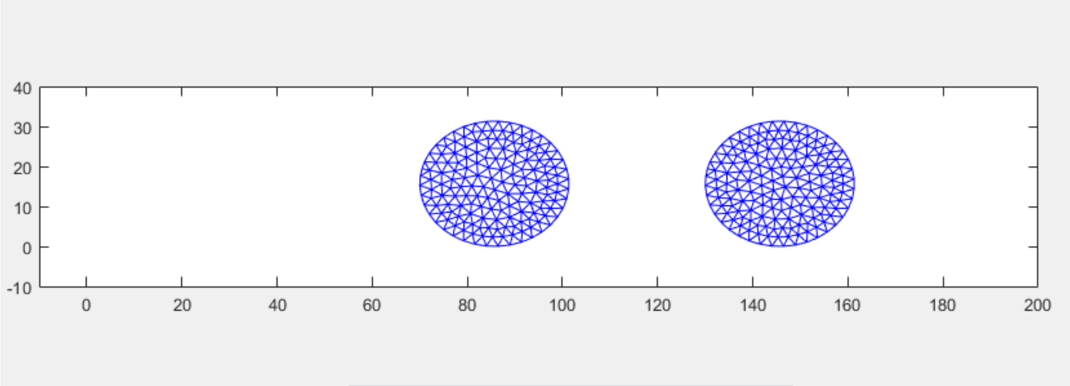
Animations



2 spheres to 1 sphere morphing



1 rectangle to 1 ring morphing



2 circles to 1 circle morphing

Acceleration

#1 Parallel computing acceleration

#2 Vertex to facet conversion optimization

Profile Summary (Total time: 4111.824 s)

▾ Flame Graph

Flame graph is not available because the number of function calls exceeds the current profiler history size of 5000000. Increase the value for the 'historysize' profiling option and rerun the Profiler. For more information, see profile.

Generated 30-Dec-2021 03:37:54 using performance time.

Function Name	Calls	Total Time (s)	Self Time* (s)	Total Time Plot (dark band = self time)
test_3d_pd1	1	4109.314	0.326	
compute_restricted_polytopes_3d	6	4094.581	88.962	
Polyhedron.intersect	161994	3350.475	20.955	
Polyhedron.minHRep	331047	3325.338	273.147	
mpt_solve	6340197	2601.685	350.470	
mpt_solve>sub_callsolver	6340197	2196.961	198.167	
mpt_call_lcp	6495945	1998.794	1272.759	
lcp (MEX-file)	6495945	606.017	606.017	
Polyhedron.computeHRep	323988	495.445	126.808	
Polyhedron.Polyhedron>Polyhedron.Polyhedron	811972	465.929	329.938	
Polyhedron.minVRep	331250	263.656	25.691	
Polyhedron.interiorPoint	1472470	249.915	80.400	
Polyhedron.isEmptySet	845874	247.905	40.147	
Polyhedron.Polyhedron>Polyhedron.get.V	1311881	212.523	5.663	
convhulln	667866	209.280	138.609	
Polyhedron.computeVRep	161994	206.859	6.042	
Polyhedron.chebyCenter	176111	148.254	23.107	
rank	6820136	128.718	128.718	
validate_polyhedron	486983	120.968	120.968	
unique	1694611	104.749	81.630	
Polyhedron.isFullDim	669559	104.140	20.202	
polyfun\private\mergeDuplicatePoints	667865	62.353	7.947	
mpt_solve>sub_fix_nan	6340197	54.255	54.255	

Figure 2: Profiler results of Algo. 2 of 6 iterations in serial mode

20.7 ms per intersection

×4 speed up !

Profile Summary (Total time: 3143.768 s)

▸ Flame Graph

Generated 08-Jan-2022 19:08:15 using performance time.

Function Name	Calls	Total Time (s)	Self Time* (s)	Total Time Plot (dark band = self time)
evaluateCode	1	3143.622	0.033	
LiveEditorEvaluationHelperE1033564254	1	3143.487	0.114	
compute_restricted_polytopes_3d	23	3099.002	0.025	
parallel_function	23	3098.896	0.026	
parallel_function>distributed_execution	23	3097.592	0.130	
remoteparfor>remoteparfor.getCompleteIntervals	316	3096.930	0.743	
java.util.concurrent.LinkedBlockingQueue (Java method)	3397	3094.240	3094.240	
compute_hessian_3d	22	36.411	0.117	
compute_surface_3d	26450	16.329	1.930	
convhulln	51741	12.648	5.795	
PowerDiagramFunc	23	7.951	0.420	
ismember	331237	7.629	1.221	
unique	164875	7.508	5.514	
ismember>ismemberR2012a	331237	6.231	1.707	
randsample	25268	4.204	0.922	
intersect	26450	3.951	0.785	
polyfun\private\mergeDuplicatePoints	51741	3.934	0.498	
intersect>intersectR2012a	26450	3.166	0.810	
ismember>ismemberBuiltinTypes	331237	2.798	2.798	

Figure 4: Profiler results of Algo. 2 of 23 iterations in parallel mode

4.9 ms per intersection

×10 speed up !

Profile Summary (Total time: 1318.651 s)

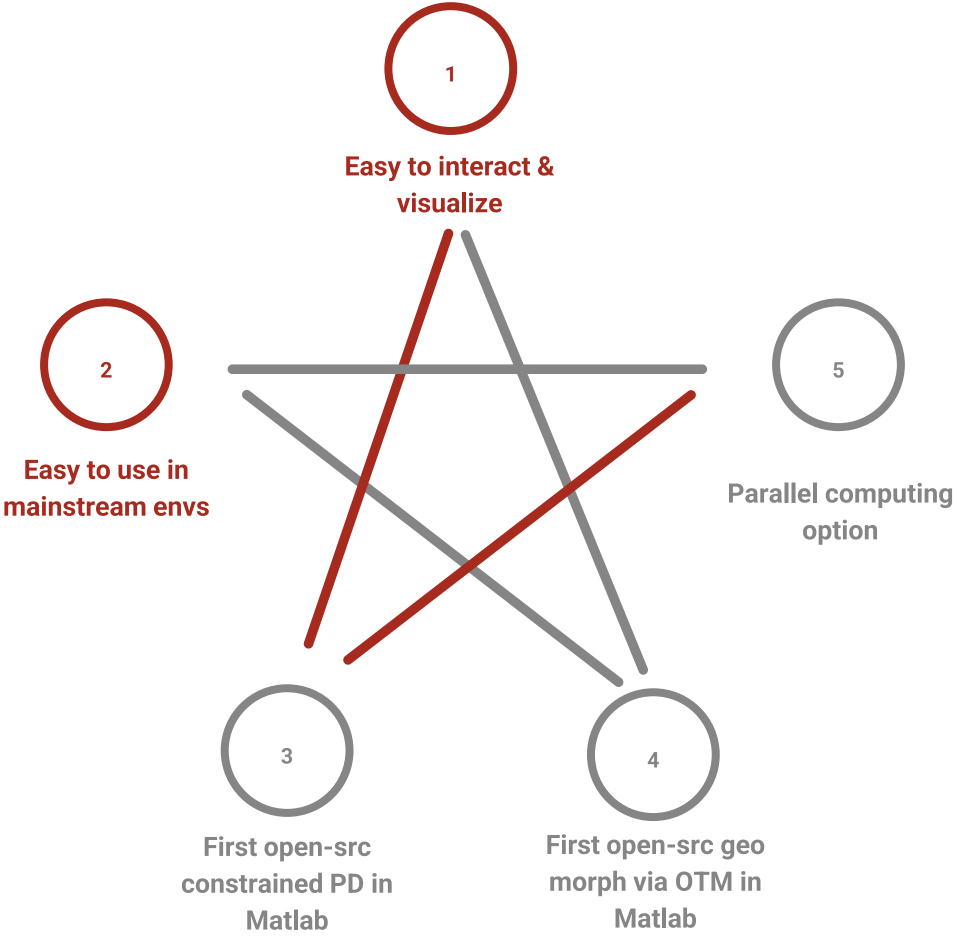
▸ Flame Graph

Generated 28-Jan-2022 10:26:19 using performance time.

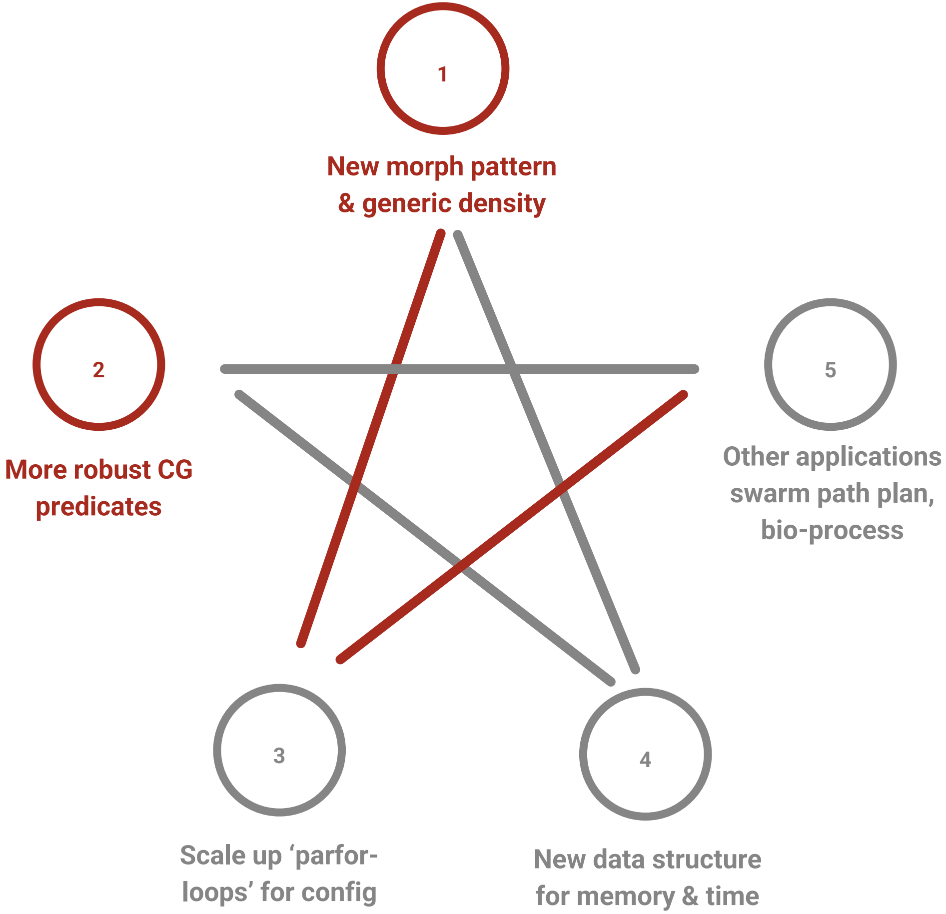
Function Name	Calls	Total Time (s)	Self Time* (s)	Total Time Plot (dark band = self time)
evaluateCode	1	1318.512	0.021	
LiveEditorEvaluationHelperE1585049703	1	1318.436	0.207	
compute_restricted_polytopes_3d	75	1105.655	0.355	
parallel_function	225	1104.848	0.129	
parallel_function>distributed_execution	225	1098.787	1.255	
remoteparfor>remoteparfor.getCompleteIntervals	3215	1089.965	2.882	
java.util.concurrent.LinkedBlockingQueue (Java method)	6670	1061.723	1061.723	
compute_hessian_3d	74	169.563	37.928	
compute_surface_3d	98090	71.942	7.367	
convhulln	191985	63.567	38.594	
PowerDiagramFunc	75	42.996	2.258	
ismember	1116537	41.371	6.597	
unique	596427	37.641	27.726	
ismember>ismemberR2012a	1116537	34.328	8.740	
remoteparfor>remoteparfor.deserialize	5903	24.184	0.488	
deserialize	5903	23.696	1.041	
distcompdeserialize (MEX-file)	5903	22.372	6.800	
intersect	98090	18.637	4.107	

0.52 ms per intersection

Contributions



Outlooks


1
New morph pattern & generic density
2
More robust CG predicates
3
Scale up ‘parfor-loops’ for config
4
New data structure for memory & time
5
Other applications swarm path plan, bio-process

Thank you for listening !

Q & A Sessions