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Cubic Normal Bi-Cayley Graphs and Bi-normal Cayley Graphs

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Abstract: A graph is called a Cayley graph (or bi-Cayley graph, respectively) over a group G if it has a group G of automorphisms acting semi-regularly on the vertices with exactly one orbit (or two orbits, respectively). We say that a Cayley graph or bi-Cayley graph Γ over a group G is normal if G is normal in the full automorphism group of Γ. A Cayley graph Γ over a group G is said to be bi-normal if the maximal normal subgroup of $\mathrm{Aut}(\Gamma)$ contained in G has index 2 in G. It is known that every bi-normal Cayley graph is a normal bi-Cayley graph. In this paper, a characterization is given of cubic normal bi-Cayley graphs which are also bi-normal Cayley. As an application, we give a classification of cubic non-normal Cayley graphs of order $2p^3$ for each prime $p > 3$.

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0 Introduction

In this paper we describe an investigation of the relationship between cubic normal bi-Cayley graphs and cubic bi-normal Cayley graphs. It is known that every bi-normal Cayley graph is a normal bi-Cayley graph. We give a characterization of cubic normal bi-Cayley graphs which are also bi-normal Cayley graphs. As an application, we give a classification of cubic non-normal Cayley graphs of order $2p^3$ for each prime $p > 3$.

We first set some notation and terminology. For a positive integer, let $\mathbb{Z}_n$ be the cyclic group of order n and $\mathbb{Z}_n^*$ be the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H of G. Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if G is transitive and semiregular.

For a finite, simple and undirected graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$, $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. A graph Γ is said to be *vertex-transitive* or *arc-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ and $A(\Gamma)$, respectively. An arc-transitive graph is also called a *symmetric* graph.

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Given a finite group G and a self-inverse subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G relative to S is a graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g) : x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well known that $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))$. We say that the Cayley graph $\mathrm{Cay}(G,S)$ is *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$ (see [10]). A connected Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ on a group G is said to be *bi-normal* if the maximal normal subgroup $\bigcap_{a\in A} R(G)^a$ of $\mathrm{Aut}(\Gamma)$ contained in $R(G)$ has index 2 in $R(G)$ (see [7]).

Let R, L and S be subsets of a group H such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* $\mathrm{BiCay}(H,R,L,S)$ over H relative to R, L, S is a graph having vertex set the union of the *right part* $H_0 = \{h_0 \mid h \in H\}$ and the *left part* $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the *right edges* $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the *spokes* $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. Let $\Gamma = \mathrm{BiCay}(H,R,L,S)$ be a bi-Cayley graph over a group H. For each $g \in H$, define a permutation of $V(\Gamma)$ as follows:

$$\mathcal{R}(g) : h_i \mapsto (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h \in H.$$

Set $\mathcal{R}(H) = \{\mathcal{R}(g) \mid g \in H\}$. Then $\mathcal{R}(H)$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with H_0 and H_1 as its two orbits. In general, a graph Γ is isomorphic to a bi-Cayley graph over a group H if and only if $\mathrm{Aut}(\Gamma)$ has a group of automorphisms which is isomorphic to H and acts semi-regularly on the vertices with two orbits. We say that the bi-Cayley graph $\Gamma = \mathrm{BiCay}(H,R,L,S)$ is a *normal bi-Cayley graph over* H if $\mathcal{R}(H)$ is normal in $\mathrm{Aut}(\Gamma)$ (see [13]).

Let $\Gamma = \mathrm{Cay}(G,S)$ be a bi-normal Cayley graph on a group G. Then the maximal normal subgroup $H = \bigcap_{a\in A} R(G)^a$ of $\mathrm{Aut}(\Gamma)$ contained in $R(G)$ has index 2 in $R(G)$. Clearly, H is semi-regular on $V(\Gamma)$ with exactly two orbits, and so Γ is a normal bi-Cayley graph over H. However, it is not necessarily true that a normal bi-Cayley graph is a bi-normal Cayley graph. In this paper, we give a characterization of cubic normal bi-Cayley graphs which are also bi-normal Cayley graphs. Before stating the results, we introduce some notation.

Let $\Gamma = \mathrm{BiCay}(H,R,L,S)$ be a bi-Cayley graph over a group H. For an automorphism α of H and $g \in H$, define a permutation of $V(\Gamma) = H_0 \cup H_1$ as follows:

$$\sigma_{\alpha,g} : h_0 \mapsto (h^\alpha)_0,\ h_1 \mapsto (gh^\alpha)_1, \quad \forall h \in H.$$

Theorem 0.1 Let $\Gamma = \mathrm{BiCay}(H,R,L,S)$ be a connected cubic vertex-transitive normal bi-Cayley graph over a group H with $1 \in S$. Let $A = \mathrm{Aut}(\Gamma)$. Suppose that A has a subgroup, say G such that $\mathcal{R}(H) \le G$ and G acts regularly on $V(\Gamma)$. Then Γ is a bi-normal Cayley graph on G if and only if one of the following holds.

(1) $R = L = \emptyset$, either $A/\mathcal{R}(H) \cong \mathrm{S}_3$ or $A/\mathcal{R}(H) \cong \mathrm{S}_3 \times \mathbb{Z}_2$ and $G/\mathcal{R}(H)$ is not in the center of $A/\mathcal{R}(H)$.

(2) $|R| = |L| = 2$, $R \cap L = \emptyset$, $A_{1_0} = \langle\sigma_{\alpha,1}\rangle \times \langle\sigma_{\beta,1}\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\alpha \in \mathrm{Aut}(H)$ fixes R pointwise and swaps two elements in L and $\beta \in \mathrm{Aut}(H)$ fixes L pointwise and swaps two elements in R.

Let p be a prime. In [4], all non-normal Cayley graphs of order $2p$ are classified. In [11], all cubic non-normal Cayley graphs of order $2p^2$ are classified. In [5], all cubic symmetric graphs of order $2p^3$ and all such graphs are Cayley graphs. In [12], all cubic vertex-transitive non-Cayley graphs of order $2p^3$ are classified. In this paper, we prove that for each prime $p > 3$, every cubic Cayley graph of order $2p^3$ is a normal bi-Cayley graph on a group of order p^3, and then applying Theorem 0.1, we give a classification of cubic non-normal Cayley graphs of order $2p^3$ for each prime $p > 3$.

Theorem 0.2 Let $p > 3$ be a prime and let $\Gamma = \mathrm{Cay}(G, T)$ be a connected cubic Cayley graph on a group G of order $2p^3$. Then either $R(G) \trianglelefteq \mathrm{Aut}(\Gamma)$ or Γ is a bi-normal Cayley graph on G and one of the following holds:

(1) $G = P \rtimes \langle \delta\gamma \rangle$, $T = \{\delta\gamma, \delta\gamma x, \delta\gamma y\}$ and δ is an involution in $\langle \alpha, \beta \rangle \leq \mathrm{Aut}(P)$, where $P = \langle x, y, z \mid x^p = y^p = z^p = 1, z = [x, y], [x, z] = [y, z] = 1 \rangle$ and $\alpha, \beta, \gamma \in \mathrm{Aut}(P)$ such that

$$\alpha : x \mapsto x^{-1}z,\ y \mapsto x^{-1}; \quad \beta : x \mapsto y,\ y \mapsto x; \quad \gamma : x \mapsto x^{-1},\ y \mapsto y^{-1}.$$

(2) $G = P \rtimes \langle \gamma \rangle$, $T = \{\gamma, x, x^{-1}\}$, where $P = \langle x, y, z \mid x^p = y^p = z^p = 1, z = [x, y], [x, z] = [y, z] = 1 \rangle$ and $\gamma = \gamma_1$ or γ_2 and $\gamma_1, \gamma_2 \in \mathrm{Aut}(P)$ such that

$$\gamma_1 : x \mapsto y,\ y \mapsto x; \quad \gamma_2 : x \mapsto y^{-1},\ y \mapsto x^{-1}.$$

Remark 0.1 For $p = 2$ or 3, all cubic Cayley graphs of order 16 or 54 have been constructed in [9], and their normality can be easily determined by using Magma [2].

1 Preliminaries

1.1 Basic Properties of Cayley Graphs

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G. Then Γ is vertex-transitive due to $R(G) \leq \mathrm{Aut}(\Gamma)$. In general, we have the following proposition.

Proposition 1.1[1] A graph Γ is isomorphic to a Cayley graph on a group G if and only if $\mathrm{Aut}(\Gamma)$ has a subgroup isomorphic to G, acting regularly on the vertex set of Γ.

In 1981, Godsil[6] proved that the normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is $R(G) \rtimes \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise. Recall that a Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$.

Proposition 1.2[10] The Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is normal if and only if $A_1 = \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of the identity 1 of G in $\mathrm{Aut}(\Gamma)$.

1.2 Basic Properties of Bi-Cayley Graphs

In this subsection, we let Γ be a connected bi-Cayley graph $\mathrm{BiCay}(H, R, L, S)$ over a group H. It is easy to prove some basic properties of such a Γ, as in [13, Lemma 3.1].

Lemma 1.1 The following hold:

(1) H is generated by $R \cup L \cup S$.

(2) Up to graph isomorphism, S can be chosen to contain the identity of H.

Next, we collect several results about the automorphisms of bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$. For each $g \in H$, define a permutation as follows:

$$\mathcal{R}(g) : h_i \mapsto (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h \in H.$$

Set $\mathcal{R}(H)=\{\mathcal{R}(g)\mid g\in H\}$. Then $\mathcal{R}(H)$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with H_0 and H_1 as its two orbits.

For an automorphism α of H and $x,y,g\in H$, define two permutations of $V(\Gamma)=H_0\cup H_1$ as follows:

$$\begin{aligned}&\delta_{\alpha,x,y}: h_0\mapsto (xh^\alpha)_1,\quad h_1\mapsto (yh^\alpha)_0,\\ &\sigma_{\alpha,g}: h_0\mapsto (h^\alpha)_0,\quad h_1\mapsto (gh^\alpha)_1,\quad \forall h\in H.\end{aligned}\tag{1.1}$$

Set

$$\begin{aligned}&I=\{\delta_{\alpha,x,y}\mid \alpha\in\mathrm{Aut}(H)\text{ s.t. } R^\alpha=x^{-1}Lx,\ L^\alpha=y^{-1}Ry,\ S^\alpha=y^{-1}S^{-1}x\},\\ &F=\{\sigma_{\alpha,g}\mid \alpha\in\mathrm{Aut}(H)\text{ s.t. } R^\alpha=R,\ L^\alpha=g^{-1}Lg,\ S^\alpha=g^{-1}S\}.\end{aligned}$$

Theorem 1.1 [13, Theorem 1.1] Let $\Gamma=\mathrm{BiCay}(H,R,L,S)$ be a connected bi-Cayley graph over the group H. Then $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))=\mathcal{R}(H)\rtimes F$ if $I=\emptyset$; $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))=\mathcal{R}(H)\langle F,\delta_{\alpha,x,y}\rangle$ if $I\neq\emptyset$ and $\delta_{\alpha,x,y}\in I$. Furthermore, for any $\delta_{\alpha,x,y}\in I$, we have the following:

(1) $\langle\mathcal{R}(H),\delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$;

(2) if α has order 2 and $x=y=1$, then Γ is isomorphic to the Cayley graph $\mathrm{Cay}(\bar{H},R\cup\alpha S)$, where $\bar{H}=H\rtimes\langle\alpha\rangle$.

2 Proof of Theorem 0.1

In this section, we shall prove Theorem 0.1. We begin by proving the following result.

Proposition 2.1 Let $\Gamma=\mathrm{BiCay}(H,\emptyset,\emptyset,S)$ be a connected bi-Cayley graph over a group H. Let $N=N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$. Then N_v acts faithfully on $\Gamma(v)$ for each $v\in V(\Gamma)$.

Proof By Lemma 1.1, we may assume that S contains the identity 1 of H, and furthermore, the connectedness of Γ implies that $H=\langle S\rangle$. Since H_0 and H_1 are the two orbits of $\mathcal{R}(H)$ on $V(\Gamma)$, it suffices to prove that N_v acts faithfully both on $\Gamma(1_0)$ and $\Gamma(1_1)$. By the definition of bi-Cayley graph, we have $\Gamma(1_0)=\{s_1\mid s\in S\}$ and $\Gamma(1_1)=\{(s^{-1})_0\mid s\in S\}$.

From Theorem 1.1, we obtain that $N_{1_0}=\{\sigma_{\alpha,g}\mid\alpha\in\mathrm{Aut}(H),g\in H,S^\alpha=g^{-1}S\}$, where $\sigma_{\alpha,g}$ is defined in Eq. (1.1). Since $1\in S$, $(1_0,1_1)$ is an arc of Γ. Then $N_{1_01_1}=\langle\sigma_{\alpha,1}\mid\alpha\in\mathrm{Aut}(H),S^\alpha=S\rangle$. Clearly, for $v=1_0$ or 1_1, the kernel of N_v acting on $\Gamma(v)$ is just the kernel of $N_{1_01_1}$ acting on $\Gamma(v)$.

Let $f\in N_{1_01_1}$ be such that f fixes every element in $\Gamma(1_0)$. Then $f=\sigma_{\alpha,1}$, where $\alpha\in\mathrm{Aut}(H)$ fixes S pointwise. Since $H=\langle S\rangle$, f fixes every vertex of Γ, and so f is an identity permutation on $V(\Gamma)$. So N_{1_0} acts faithfully on $\Gamma(1_0)$.

Now let $f\in N_{1_01_1}$ be such that f fixes every element in $\Gamma(1_1)$. Then $f=\sigma_{\alpha,1}$, where $\alpha\in\mathrm{Aut}(H)$ fixes S^{-1} pointwise. Again, since $H=\langle S\rangle=\langle S^{-1}\rangle$, f fixes every vertex of Γ, and so f is an identity permutation on $V(\Gamma)$. So N_{1_1} acts faithfully on $\Gamma(1_1)$. □

Corollary 2.1 Let $\Gamma=\mathrm{BiCay}(H,\emptyset,\emptyset,S)$ be a connected cubic bi-Cayley graph over a group H. Let $N=N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$. Then $N_v\leq \mathrm{S}_3$ for each $v\in V(\Gamma)$.

The next lemma determines the vertex-stabilizers of the normalizer of $\mathcal{R}(H)$ in $\mathrm{Aut}(\Gamma)$, where $\Gamma=\mathrm{BiCay}(H,R,L,S)$ is a connected cubic bi-Cayley graph on a group H.

Proposition 2.2 Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected cubic bi-Cayley graph over a group H with $1 \in S$ and $|R| = |L|$. Let $N = N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$. Then either N_v acts faithfully on $\Gamma(v)$ for each $v \in V(\Gamma)$, or the following hold:

(1) $|R| = |L| = 2$, $R \cap L = \emptyset$;

(2) $N_{1_0} \le \langle \sigma_{\alpha,1} \rangle \times \langle \sigma_{\beta,1} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and either $\sigma_{\alpha,1} \in N_{1_0}$ or $\sigma_{\beta,1} \in N_{1_0}$, where $\alpha \in \mathrm{Aut}(H)$ fixes R pointwise and swaps two elements in L and $\beta \in \mathrm{Aut}(H)$ fixes L pointwise and swaps two elements in R.

Proof If $|R| = |L| = 0$, then by Proposition 2.1, N_v acts faithfully on $\Gamma(v)$ for each $v \in V(\Gamma)$.

Assume now that $|R| = |L| = 1$. Let $R = \{a\}$ and $L = \{b\}$ with $a, b \in H$. Since $R = R^{-1}$ and $L = L^{-1}$, a, b are involutions. By Lemma 1.1, we may assume that $S = \{1, c\}$. Let K be the kernel of N_v acting on $\Gamma(v)$ for $v = 1_0$ or 1_1. Since $1 \in S$, we have $\{1_0, 1_1\} \in E(\Gamma)$, and so $K \le N_{1_0 1_1}$. By Theorem 1.1, we obtain that

$$N_{1_0} = \{\sigma_{\alpha,g} \,|\, \alpha \in \mathrm{Aut}(H),\, g \in H,\, R^\alpha = R,\, L^\alpha = g^{-1}Lg,\, S^\alpha = g^{-1}S\},$$

where $\sigma_{\alpha,g}$ is defined in Eq. (1.1). For any $\sigma_{\alpha,g} \in N_{1_0 1_1}$, we have $1_1 = 1_1^{\sigma_{\alpha,g}} = g_1$, forcing that $g = 1$ and

$$N_{1_0 1_1} = \langle \sigma_{\alpha,1} \,|\, \alpha \in \mathrm{Aut}(H),\, c^\alpha = c,\, a^\alpha = a,\, b^\alpha = b \rangle.$$

Since Γ is connected, one has $H = \langle a, b, c \rangle$. It follows that $N_{1_0 1_1} = 1$, and so $K = 1$. Since H_0 and H_1 are the two orbits of $\mathcal{R}(H)$ on $V(\Gamma)$, N_v acts faithfully on $\Gamma(v)$ for each $v \in V(\Gamma)$.

Finally, assume that $|R| = |L| = 2$. In this case, we have $N_{1_0} = N_{1_1}$, and so

$$N_{1_0} = \{\sigma_{\alpha,1} \,|\, \alpha \in \mathrm{Aut}(H),\, R^\alpha = R,\, L^\alpha = L\}.$$

Let $R = \{a, b\}$ and $L = \{c, d\}$ with $a, b, c, d \in H \setminus \{1\}$. Since Γ is connected, one has $H = \langle a, b, c, d \rangle$. It follows that $N_{1_0} (= N_{1_1})$ acts faithfully on $\Gamma(1_0) \cup \Gamma(1_1) - \{1_0, 1_1\} = \{a_0, b_0, c_1, d_1\}$.

If $R = L$, then $H = \langle a, b \rangle = \langle c, d \rangle$ and then $N_{1_0} (= N_{1_1})$ acts faithfully on both $\{a_0, b_0\} (= \Gamma(1_0) - \{1_1\})$ and $\{c_1, d_1\} (= \Gamma(1_1) - \{1_0\})$. It follows that $N_{1_0} = N_{1_1} \le \mathrm{S}_2$. Clearly, N_v acts faithfully on $\Gamma(v)$ for $v = 1_0$ or 1_1. Thus, N_v acts faithfully on $\Gamma(v)$ for each $v \in V(\Gamma)$ as H_0 and H_1 are the two orbits of $\mathcal{R}(H)$ on $V(\Gamma)$.

If $|R \cap L| = 1$, then for any $\sigma_{\alpha,1} \in N_{1_0}$, we have $(R \cap L)^\alpha = R \cap L$, $R^\alpha = R$ and $L^\alpha = L$. Since $|R| = |L| = 2$, α fixes every element in $R \cup L$. Again, $H = \langle a, b, c, d \rangle$, and we have $\alpha = 1$. Consequently, we have $N_{1_0} = 1 = N_{1_1}$, and N_v acts faithfully on $\Gamma(v)$ for $v = 1_0$ or 1_1. So N_v acts faithfully on $\Gamma(v)$ for each $v \in V(\Gamma)$ as H_0 and H_1 are the two orbits of $\mathcal{R}(H)$ on $V(\Gamma)$.

Finally, let $|R \cap L| = 0$. Clearly, $\{\{a_0, b_0\}, \{c_1, d_1\}\}$ is a partition of $\{a_0, b_0, c_1, d_1\}$ which is invariant under the action of $N_{1_0} (= N_{1_1})$. Since $N_{1_0} (= N_{1_1})$ acts faithfully on $\Gamma(1_0) \cup \Gamma(1_1) - \{1_0, 1_1\} = \{a_0, b_0, c_1, d_1\}$, it follows that $N_{1_0} \le \mathbb{Z}_2 \times \mathbb{Z}_2$. If N_{1_0} is not faithful on $\Gamma(1_0)$, then there exists $\alpha \in \mathrm{Aut}(H)$ such that $\sigma_{\alpha,1}$ fixes a_0 and b_0 but interchanges c_1 and d_1. By the definition of $\sigma_{\alpha,1}$, we see that α fixes both a and b but interchanges c and d. Similarly, if N_{1_1} is not faithful on $\Gamma(1_1)$, then there exists $\beta \in \mathrm{Aut}(H)$ such that $\sigma_{\beta,1}$ fixes c_1 and d_1 but interchanges a_0 and

b_0. By the definition of $\sigma_{\beta,1}$, we see that β fixes both c and d but interchanges a and b. This completes the proof. □

Proof of Theorem 0.1 Assume first that Γ is arc-transitive. Since $\mathcal{R}(H) \trianglelefteq A$, H_0 and H_1 cannot contain edges of Γ, and so $R = L = \emptyset$. By Proposition 2.2, we have $\mathbb{Z}_3 \leq A_{1_0} \leq \mathrm{S}_3$, and hence $|A/\mathcal{R}(H)| = 6$ or 12. If $|A/\mathcal{R}(H)| = 6$, then $G/\mathcal{R}(H) \ntrianglelefteq A/\mathcal{R}(H)$ if and only if $A/\mathcal{R}(H) \cong \mathrm{S}_3$. If $|A/\mathcal{R}(H)| = 12$, then since $\mathrm{S}_3 \leq A/\mathcal{R}(H)$, one has $A/\mathcal{R}(H) \cong \mathrm{S}_3 \times \mathbb{Z}_2$, and so $G/\mathcal{R}(H) \ntrianglelefteq A/\mathcal{R}(H)$ if and only if $G/\mathcal{R}(H)$ is not in the center of $A/\mathcal{R}(H)$.

If Γ is not arc-transitive, then by Proposition 2.2, we deduce that G is not normal in A if and only if Part (2) happens. □

3 Proof of Theorem 0.2

We begin by considering the cubic symmetric Cayley graphs of order $2p^3$ with $p > 3$ a prime.

Theorem 3.1 Let $p > 3$ be a prime and let $\Gamma = \mathrm{Cay}(G, T)$ be a connected cubic symmetric Cayley graph on a group G of order $2p^3$. Then either $R(G) \trianglelefteq \mathrm{Aut}(\Gamma)$ or Γ is a bi-normal Cayley graph on G such that $G = P \rtimes \langle \delta\gamma \rangle$, $T = \{\delta\gamma, \delta\gamma x, \delta\gamma y\}$ and δ is an involution in $\langle \alpha, \beta \rangle$, where $P = \langle x, y, z \mid x^p = y^p = z^p = 1, z = [x, y], [z, x] = [z, y] = 1 \rangle$ and $\alpha, \beta, \gamma \in \mathrm{Aut}(P)$ such that

$$\alpha : x \mapsto x^{-1}z,\ y \mapsto x^{-1}; \quad \beta : x \mapsto y,\ y \mapsto x; \quad \gamma : x \mapsto x^{-1},\ y \mapsto y^{-1}.$$

Proof By [5, Theorem 3.2], one of the following holds:

(i) $\Gamma \cong \mathcal{C}(\mathbb{Z}_{p^3}) = \mathrm{Cay}(D_{2p^3}, \{a, ab, ab^{-\lambda}\})$, $\mathrm{Aut}(D_{2p^3}, \{a, ab, ab^{-\lambda}\}) \cong \mathbb{Z}_3$ and $R(D_{2p^3}) \trianglelefteq \mathrm{Aut}(\mathcal{C}(\mathbb{Z}_{p^3}))$, where $D_{2p^3} = \langle a, b \mid a^{p^3} = b^2 = (ba)^2 = 1 \rangle$, $1 \leq \lambda \leq p^3 - 1$ and $\lambda^2 + \lambda + 1 \equiv 0 \pmod{n}$;

(ii) $\Gamma \cong \mathcal{C}(\mathbb{Z}_{p^2 \times p}) = \mathrm{Cay}(A_{2p^3}, \{a, ab, ab^k c\})$, $\mathrm{Aut}(A_{2p^3}, \{a, ab, ab^k c\}) \cong \mathbb{Z}_3$ and $R(A_{2p^3}) \trianglelefteq \mathrm{Aut}(\mathcal{C}(\mathbb{Z}_{p^2 \times p}))$, where $A_{2p^3} = \langle a, b, c \mid a^2 = b^{p^2} = c^p = [b, c] = 1, aba = b^{-1}, aca = c^{-2} \rangle$, $1 \leq k \leq p - 1$ and $k^2 + k + 1 \equiv 0 \pmod{p}$;

(iii) $\Gamma \cong \mathcal{C}(N(p \times p \times p)) = \mathrm{Cay}(B_{2p^3}, \{a, ax, ay\})$, $\mathrm{Aut}(B_{2p^3}, \{a, ax, ay\}) \cong \mathrm{S}_3$ and $R(B_{2p^3}) \trianglelefteq \mathrm{Aut}(\mathcal{C}(N(p \times p \times p)))$, where $B_{2p^3} = \langle a, x, y, z \mid a^2 = x^p = y^p = z^p = 1, z = [x, y], [a, z] = [x, z] = [y, z] = 1, axa = x^{-1}, aya = y^{-1} \rangle$.

In what follows, we assume $p > 3$. Let $A = \mathrm{Aut}(\Gamma)$ and let P be a Sylow p-subgroup of A. It can be easily checked that $P \trianglelefteq A$, and so Γ is a normal bi-Cayley graph on P.

For (i)–(ii), we have $A/P \cong \mathbb{Z}_6$. Clearly, $P \leq R(G)$, so $R(G)/P \trianglelefteq A/P$. Then $R(G) \trianglelefteq A$.

For (iii), we may let $\Gamma = \mathcal{C}(N(p \times p \times p)) = \mathrm{Cay}(B_{2p^3}, \{a, ax, ay\})$. In this case, we also have $P \trianglelefteq A$. Since $\mathrm{Aut}(B_{2p^3}, \{a, ax, ay\}) \cong \mathrm{S}_3$ and $R(B_{2p^3}) \trianglelefteq A$, we have $A/P \cong \mathrm{S}_3 \times \mathbb{Z}_2$. Furthermore, $\mathrm{Aut}(B_{2p^3}, \{a, ax, ay\}) = \langle \alpha' \rangle \rtimes \langle \beta' \rangle \cong \mathrm{S}_3$, where $\alpha', \beta' \in \mathrm{Aut}(B_{2p^3})$ such that

$$\alpha' : a \mapsto ax,\ x \mapsto x^{-1}y,\ y \mapsto x^{-1},\ z \mapsto z; \quad \beta' : a \mapsto a,\ x \mapsto y,\ y \mapsto x,\ z \mapsto z^{-1}.$$

Clearly, $P \leq R(G)$, so $R(G)/P \leq A/P = R(B_{2p^3})/P \times \mathrm{Aut}(B_{2p^3}, \{a, ax, ay\})P/P \cong \mathbb{Z}_2 \times \mathrm{S}_3$. Then $R(G)/P = \langle R(a)P \rangle$ or $\langle \delta R(a)P \rangle$ with δ an involution in $\mathrm{Aut}(B_{2p^3}, \{a, ax, ay\})$. Clearly, if $R(G)/P = \langle R(a)P \rangle$, then $R(G)/P = R(B_{2p^3})/P \trianglelefteq A/P$, and then $G = B_{2p^3}$ and $T = \{a, ax, ay\}$. If $R(G)$ is not normal in A, then $R(G)/P = \langle \delta R(a)P \rangle$ with δ an involution in

$\mathrm{Aut}(B_{2p^3}, \{a, ax, ay\})$. Since δ fixes 1, one has $1^{\delta R(a)} = a$, $1^{\delta R(a)R(x)} = ax$ and $1^{\delta R(a)R(y)} = ay$. So we may let $G = \langle x, y, z\rangle \rtimes \langle \delta a\rangle$, and $T = \{\delta a, \delta ax, \delta ay\}$. Clearly, $\alpha', \beta', \delta a$ induce the automorphisms α, β, γ, respectively, of P as given in the statement of our theorem. This completes the proof. $\square$

In the following, we shall deal cubic non-symmetric Cayley graphs of order $2p^3$ with $p > 3$ a prime. We first introduce the concept of quotient graph. Let Γ be a graph and let $G \leq \mathrm{Aut}(\Gamma)$ be transitive on $V(\Gamma)$. For a normal subgroup N of G, the *quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two different orbits adjacent if there exists an edge in Γ between the vertices lying in those two orbits.

Lemma 3.1 Let Γ be a connected cubic Cayley graph of order $2p^3$ with $p > 3$ a prime. Let P be a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$. If Γ is not symmetric, then $P \trianglelefteq \mathrm{Aut}(\Gamma)$.

Proof Let $A = \mathrm{Aut}(\Gamma)$. For any $v \in V(\Gamma)$, let A_v be the stabilizer of v in A. Since Γ has valency 3 and is not symmetric, A_v is a 2-group. It follows that $|A| = 2^{\ell+1} \cdot p^3$ for some integer $\ell \geq 1$. By Burnside $p^a q^b$-theorem, A is solvable, and so every minimal normal subgroup of A is an elementary abelian r-subgroup, where $r = 2$ or p.

Let N be a maximal normal 2-subgroup of A. Suppose that $N > 1$. Then every orbit of N on $V(\Gamma)$ has size 2, and since Γ has $2p^3$ vertices with $p > 3$ a prime, the quotient graph Γ_N of Γ relative to N has order p^3. As $N \trianglelefteq A$, Γ_N has valency at most 3, and since Γ_N has p^3 vertices, Γ_N is a cycle of length p^3 and every orbit of N is also an edge of Γ. Let K be the kernel of A acting on $V(\Gamma_N)$. Clearly, $N \leq K$. Let $B_0 = \{u_0, v_0\}$ be an orbit of N on $V(\Gamma)$, and let $B_1 = \{u_1, v_1\}$ and $B_{-1} = \{u_{-1}, v_{-1}\}$ be two orbits of N which are adjacent to B_0 in Γ_N. Since $B_0 \in E(\Gamma)$, each of the three orbits B_0, B_1 and B_{-1} contains exactly one neighbor of u_0. So K_{u_0} fixes every neighbor of u_0. By the connectedness of Γ, K_{u_0} fixes every vertex of Γ and so $K_{u_0} = 1$. Then $K = N \cong \mathbb{Z}_2$. Note that $A/K \leq \mathrm{Aut}(\Gamma_N) \cong D_{2p^3}$. Then $PK/K \trianglelefteq A/K$ and then $PK \trianglelefteq A$. Since $|K| = 2$, one has $P \trianglelefteq PK$ and so P is the unique Sylow p-subgroup of PK. Then $P \trianglelefteq A$ due to $PK \trianglelefteq A$.

In what follows, assume that A has no normal 2-subgroups. Let M be the maximal normal p-subgroup of A. If $|M| = p^3$, then M is just a Sylow p-subgroup of A, and so $P = M \trianglelefteq A$. Suppose that $|M| \leq p^2$. Then M is abelian, and so $M \leq C_A(M)$. If $C_A(M) > M$, then let N/M be a minimal normal subgroup A/M contained in $C_A(M)/M$. Then N/M is a 2- or p-group. If N/M is a 2-group, then $N = M \times Q$, where Q is a Sylow 2-subgroup of N. So Q is characteristic in N, and then $Q \trianglelefteq A$ since $N \trianglelefteq A$. This is contrary to our assumption that A has no normal 2-subgroups. If N/M is a p-group, then N is also a normal p-subgroup of A, contrary to the maximality of M. Thus, $C_A(M) = M$ and $A/M = A/C_A(M) \leq \mathrm{Aut}(M)$. If $|M| = p$, then M is contained in the center of P, and so $P \leq C_A(M)$, forcing $C_A(M) > M$, a contradiction. Thus, $|M| = p^2$ and then $M \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. For the former, we have $A/M \leq \mathrm{Aut}(\mathbb{Z}_{p^2}) \cong \mathbb{Z}_{p(p-1)}$, and then $P/M \trianglelefteq A/M$, forcing $P \trianglelefteq A$, a contradiction. For the latter, we have $A/M \leq \mathrm{Aut}(\mathbb{Z}_p^2) \cong \mathrm{GL}(2, p)$. Note that $|A/M| = 2^{\ell+1}p$ for some integer ℓ. Let $H = A/M$ and view H as a subgroup of $\mathrm{GL}(2, p)$. Then $H \cap \mathrm{SL}(2, p) \trianglelefteq H$ and $H/(H \cap \mathrm{SL}(2, p)) \cong H\mathrm{SL}(2, p)/\mathrm{SL}(2, p) \leq \mathrm{GL}(2, p)/\mathrm{SL}(2, p) \cong \mathbb{Z}_{p-1}$. So $p \mid |H \cap \mathrm{SL}(2, p)|$ and $|H \cap \mathrm{SL}(2, p)| \mid 2^{\ell+1}p$.

By [3, Tables 8.2–8.3], we see that $H \cap \mathrm{SL}(2,p) \le \mathbb{Z}_p : \mathbb{Z}_{p-1}$. So the Sylow p-subgroup of $H \cap \mathrm{SL}(2,p)$ is normal, and since $H \cap \mathrm{SL}(2,p) \trianglelefteq H$, the Sylow p-subgroup of $H \cap \mathrm{SL}(2,p)$ is also a normal Sylow p-subgroup of H. Recall that $H = A/M$ and P/M is a Sylow p-subgroup of A/M. So $P/M \trianglelefteq A/M$ and hence $P \trianglelefteq A$, a contradiction. □

Theorem 3.2 Let $p > 3$ be a prime and let $\Gamma = \mathrm{Cay}(G,T)$ be a connected cubic non-symmetric Cayley graph on a group G of order $2p^3$. Then either $R(G) \trianglelefteq \mathrm{Aut}(\Gamma)$ or Γ is a bi-normal Cayley graph on G such that $G = P \rtimes \langle\gamma\rangle$, $T = \{\gamma, x, x^{-1}\}$, where $P = \langle x, y, z \mid x^p = y^p = z^p = 1, z = [x,y], [x,z] = [y,z] = 1\rangle$ and $\gamma = \gamma_1$ or γ_2 and $\gamma_1, \gamma_2 \in \mathrm{Aut}(P)$ such that

$$\gamma_1 : a \mapsto b,\ b \mapsto a; \quad \gamma_2 : a \mapsto b^{-1},\ b \mapsto a^{-1}.$$

Proof Let $A = \mathrm{Aut}(\Gamma)$ and let P be a Sylow p-subgroup of A. By Lemma 3.1, we obtain that $P \trianglelefteq A$. Since $p > 3$, Γ is a normal bi-Cayley graph over P. By Lemma 1.1, we may let $\Gamma = \mathrm{BiCay}(P, R, L, S)$ such that S contains the identity 1 of P. Clearly, $P \le R(G)$. Since Γ is not symmetric, by Theorem 0.1, $R(G)$ is not normal in A if and only if $|R| = |L| = 2$, $R \cap L = \emptyset$, $A_{1_0} = \langle\sigma_{\alpha,1}\rangle \times \langle\sigma_{\beta,1}\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\alpha \in \mathrm{Aut}(P)$ fixes R pointwise and swaps two elements in L and $\beta \in \mathrm{Aut}(P)$ fixes L pointwise and swaps two elements in R. Since $|P| = p^3$ is odd, $R = R^{-1}$ and $L = L^{-1}$ imply that $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$ for some $a, b \in P$. Since Γ is connected, we have $P = \langle R, L\rangle = \langle a, b\rangle$.

Suppose that $R(G)$ is not normal in A. Then there exist $\alpha, \beta \in \mathrm{Aut}(P)$ such that

$$\alpha : a \mapsto a,\ b \mapsto b^{-1}; \quad \beta : a \mapsto a^{-1},\ b \mapsto b.$$

So $\langle\alpha,\beta\rangle \lesssim \mathrm{Aut}(P)$. It is also easy to check that $\langle\alpha,\beta\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Since $P = \langle a, b\rangle$, if P is abelian, then $P \cong \mathbb{Z}_{p^3}$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, and if P is not abelian, then we have $P = \langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1}\rangle$ or $P = \langle x, y, z \mid x^p = y^p = z^p = 1, z = [x,y], [x,z] = [y,z] = 1\rangle$.

If $P \cong \mathbb{Z}_{p^3}$, then we have $\mathrm{Aut}(P) \cong \mathbb{Z}_{p^2(p-1)}$. This is impossible because $\mathbb{Z}_2 \times \mathbb{Z}_2 \lesssim \mathrm{Aut}(P)$. If $P \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then since $P = \langle a, b\rangle$ and a, b have the same order, we may let $P = \langle a\rangle \times \langle c\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ with $b = a^i c$ for some $i \in \mathbb{Z}_p^*$. Since $a^\alpha = a$ and $b^\alpha = b^{-1}$, we have $a^{-i}c^{-1} = b^{-1} = b^\alpha = (a^i c)^\alpha = a^i c^\alpha$, and hence $c^\alpha = a^{-2i}c^{-1}$. Consequently, we have $(a^{-2i})^p = 1$. However, since $p > 3$ and $i \in \mathbb{Z}_p^*$, we have $2i \in \mathbb{Z}_p^*$ and hence a^{-2i} has order p^2, a contradiction. If $P = \langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1}\rangle$, then by [8, Theorem 2.8], the Sylow 2-subgroup of $\mathrm{Aut}(P)$ is cyclic, contrary to $\mathbb{Z}_2 \times \mathbb{Z}_2 \lesssim \mathrm{Aut}(P)$.

Now let $P = \langle x, y, z \mid x^p = y^p = z^p = 1, z = [x,y], [x,z] = [y,z] = 1\rangle$. Then every element in P has order at most p. Since $P = \langle a, b\rangle$, we have $a^p = b^p = 1$. Let $c = [a,b]$. Since P is non-abelian, c has order p and $\langle c\rangle$ is just the derived subgroup of P. Then c is in the center of P. So we may let $a = x$, $b = y$ and $c = z$. Since $R(G)$ acts transitively on $V(\Gamma)$ and $|R(G) : P| = 2$, there exists an involution in $R(G)$ interchanging 1_0 and 1_1, and by Theorem 1.1, we may assume this involution as $\delta_{\gamma,1,1}$, where $\gamma \in \mathrm{Aut}(P)$ and it swaps R with L. Recalling that $R = \{x, x^{-1}\}$ and $L = \{y, y^{-1}\}$, we have $\gamma = \gamma_1$ or γ_2, where

$$\gamma_1 : x \mapsto y,\ y \mapsto x; \quad \gamma_2 : x \mapsto y^{-1},\ y \mapsto x^{-1}.$$

Again, by Theorem 1.1, we have either $R(G) = P \rtimes \langle\gamma_1\rangle$, $T = \{x, x^{-1}, \gamma_1\}$ or $R(G) = P \rtimes \langle\gamma_2\rangle$, $T = \{x, x^{-1}, \gamma_1\}$.

It is easy to check that the following two maps

$$\alpha : a \mapsto a,\ b \mapsto b^{-1};\quad \beta : a \mapsto a^{-1},\ b \mapsto b$$

indeed induce automorphisms of P. By Theorem 0.1, we have $A_{1_0} = \langle\sigma_{\alpha,1}\rangle \times \langle\sigma_{\beta,1}\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So both $P \rtimes \langle\gamma_1\rangle$ and $P \rtimes \langle\gamma_2\rangle$ are not normal in A. This completes the proof. □

Combining Theorems 3.1–3.2, we obtain Theorem 0.2.

References

[1] Biggs, N., Algebraic Graph Theory, 2nd Ed., Cambridge Mathematical Library, Cambridge: Cambridge University Press, 1993.

[2] Bosma, W., Cannon, J. and Playoust, C., The Magma algebra system, I: The user language, *J. Symbolic Comput.*, 1997, 24(3/4): 235–265.

[3] Bray, J.N., Holt, D.F. and Roney-Dougal, C.M., The Maximal Subgroups of the Low-dimensional Finite Classical Groups, London Mathematical Society Lecture Note Series, Vol. 407, Cambridge: Cambridge University Press, 2013.

[4] Du, S.F., Wang, R.J. and Xu, M.Y., On the normality of Cayley digraphs of groups of order twice a prime, *Australas. J. Combin.*, 1998, 18: 227–234.

[5] Feng, Y.Q., Kwak, J.H. and Xu, M.Y., Cubic s-regular graphs of order $2p^3$, *J. Graph Theory*, 2006, 52(4): 341–352.

[6] Godsil, C.D., On the full automorphism group of a graph, *Combinatorica*, 1981, 1(3): 243–256.

[7] Li, C.H., Finite s-arc transitive Cayley graphs and flag-transitive projective planes, *Proc. Amer. Math. Soc.*, 2005, 133(1): 31–41.

[8] Li, C.H. and Sim, H.-S., On half-transitive metacirculant graphs of prime-power order, *J. Combin. Theory Ser. B*, 2001, 81(1): 45–57.

[9] Potočnik, P., Spiga, P. and Verret, G., Cubic vertex-transitive graphs on up to 1280 vertices, *J. Symbolic Comput.*, 2013, 50: 465–477.

[10] Xu, M.Y., Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.*, 1998, 182(1/2/3): 309–319.

[11] Yu, H.X., Zhou, J.X. and Wang, L.L., Cubic Cayley graph of order $2p^2$, *Adv. Math.* (*China*), 2006, 35(5): 581–589.

[12] Zhou, J.X. and Feng, Y.Q., Cubic non-Cayley vertex-transitive bi-Cayley graphs over regular a p-group, *Electron. J. Combin.*, 2016, 23(3): Paper 3.34, 10 pp.

[13] Zhou, J.X. and Feng, Y.Q., The automorphisms of bi-Cayley graphs, *J. Combin. Theory Ser. B*, 2016, 116: 504–532.

三度正规双凯莱图与双正规凯莱图

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摘要:　一个图称为群 G 上的凯莱图 (或双凯莱图), 如果它的自同构群有一个同构于 G 的半正则子群在图的顶点集合上作用有一个 (或两个) 轨道. 称群 G 上的凯莱图或双凯莱图 Γ 是正规的, 如果群 G 在图 Γ 的全自同构群中是正规的. 称群 G 上的凯莱图 Γ 为双正规的, 如果 $\mathrm{Aut}(\Gamma)$ 的包含在 G 中的极大正规子群在 G 中的指数为 2. 由定义可知, 每个双正规凯莱图都是正规双凯莱图. 本文给出了三度正规双凯莱图同时也是双正规凯莱图的一个刻画. 作为应用, 给出了 $2p^3$ 阶的三度非正规凯莱图的分类, 这里 $p > 3$ 为素数.

关键词:　双凯莱图; 双正规凯莱图; 凯莱图