Ejemplo de EDP

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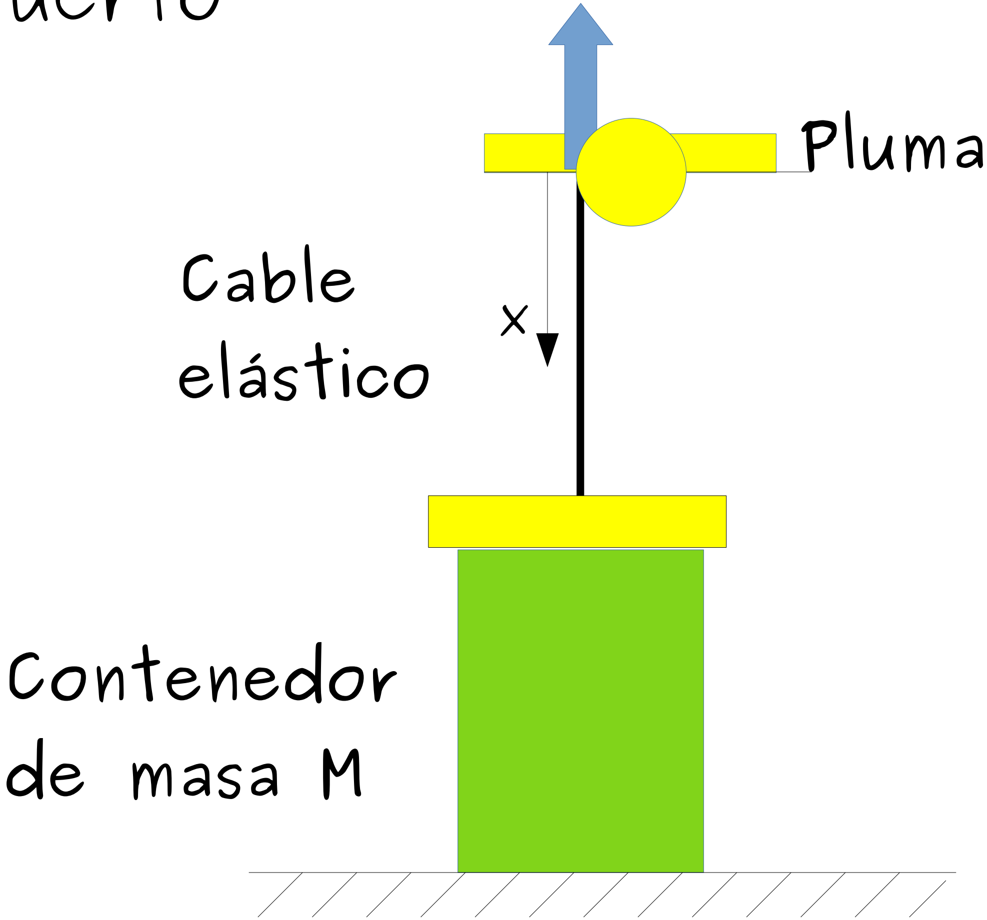
Problema: Vibraciones axiales en el cable de izaje de una grúa de puerto



Para modelar el problema se considera el equilibrio de las fuerzas que actúan sobre el cable.

$$\underbrace{\rho A \frac{\partial^2}{\partial t^2} u(x,t)}_{\text{Fuerza por efectos inerciales.}} = \underbrace{EA \frac{\partial^2}{\partial x^2} u(x,t)}_{\text{Fuerza elástica del cable.}}$$

u(x,t) es el estiramiento en cada posición axial del cable e instante de tiempo

El modelo matemático queda definido por

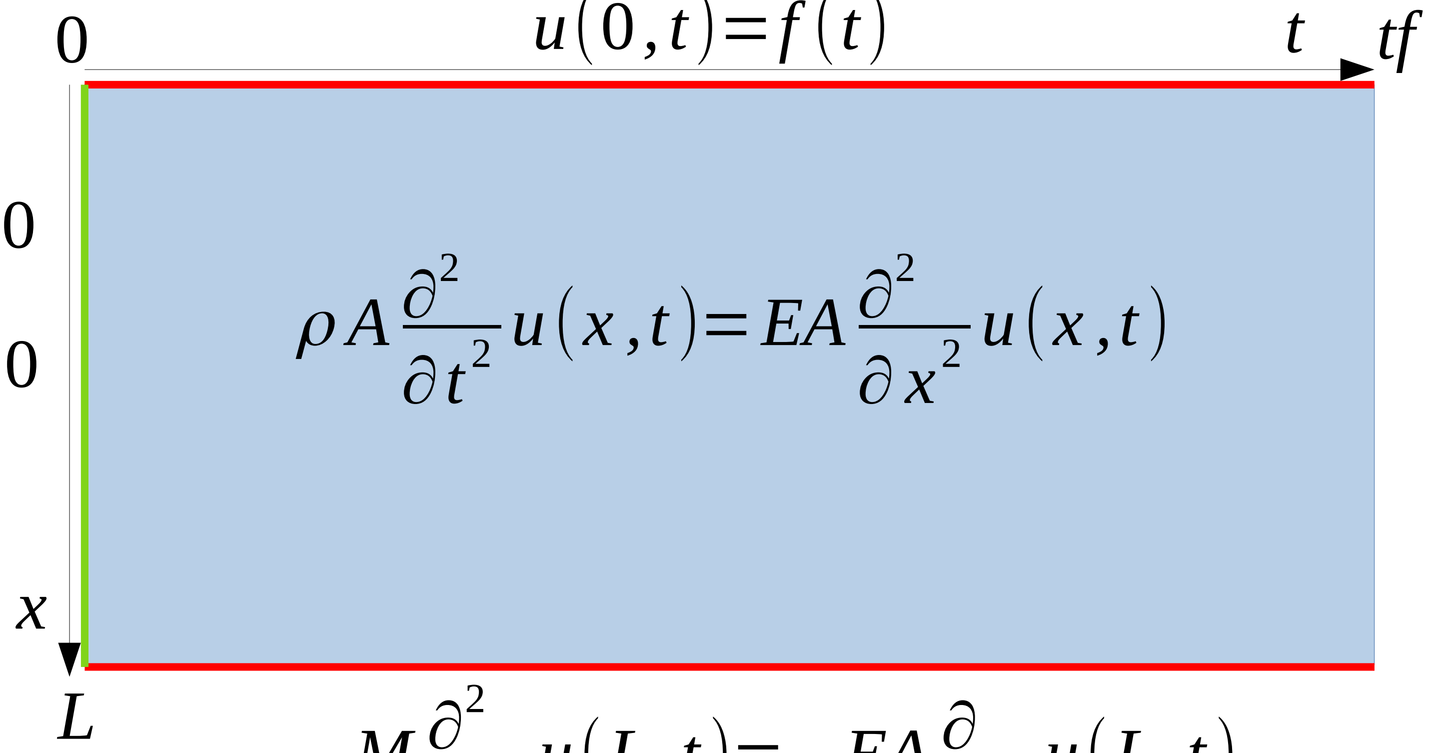
$$\rho A \frac{\partial^2}{\partial t^2} u(x,t) = EA \frac{\partial^2}{\partial x^2} u(x,t)$$

$$Dom\, u(x,t) := \left\{ (x,t) \in \Re^2 / 0 \leq x \leq L \wedge 0 \leq t \leq tf \right\}$$

Gráficamente el dominio de la función solución u(x,t) es un rectángulo

Las condiciones del problema

$$u(0,t)=f(t)$$

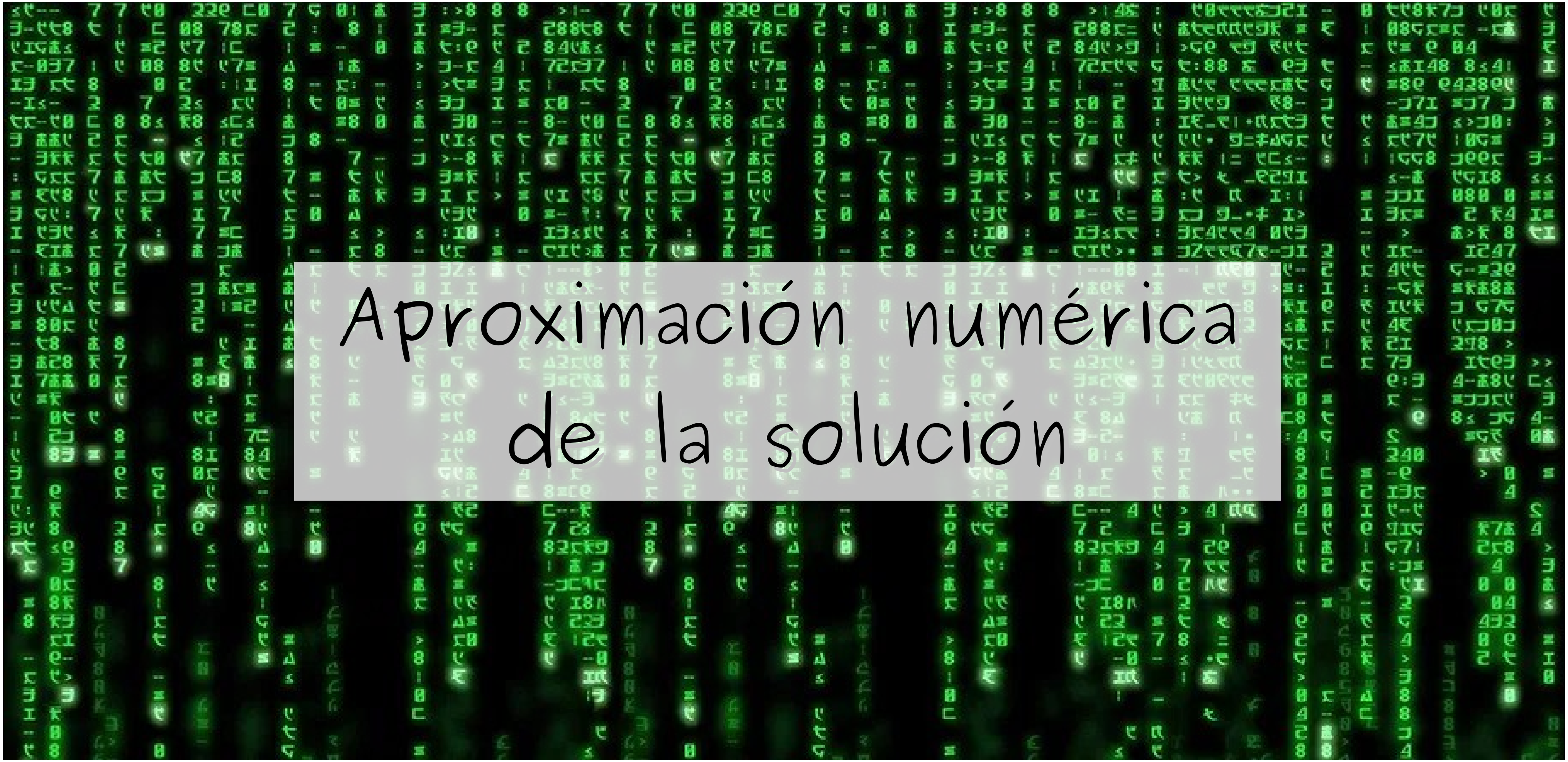


$$u(x,0)=0$$

$$\frac{\partial}{\partial t}u(x,0)=0$$

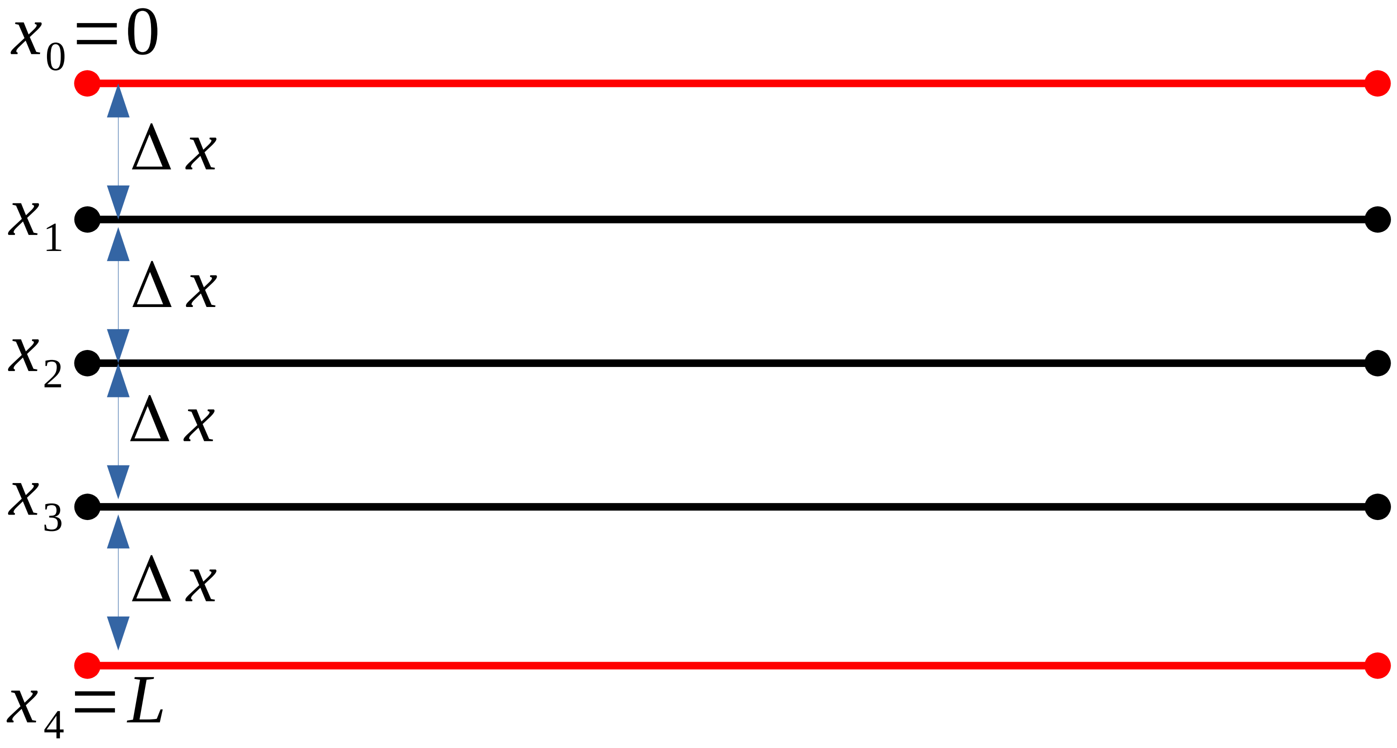
$$\rho A\frac{\partial^2}{\partial t^2}u(x,t)=EA\frac{\partial^2}{\partial x^2}u(x,t)$$

$$M\frac{\partial^2}{\partial t^2}u(L,t)=-EA\frac{\partial}{\partial x}u(L,t)$$



Aproximación numérica de la solución

Discretización del espacio



El modelo matemático ahora describe un dominio discreto en cinco ubicaciones

$u_0(t)=f(t)$ Condición de borde

$u_i(t)=u(x_i,t)$

$\rho A \frac{d^2}{dt^2} u_i(t) = EA \frac{\partial^2}{\partial x^2} u(x_i,t)$ para $i=1,2,3$

$M \frac{d^2}{dt^2} u_4(t) = -EA \frac{\partial}{\partial x} u(x_4,t)$ Condición de borde

Aproximación de las derivadas en x

$$\frac{\partial^2}{\partial x^2} u(x_i, t) = \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{\Delta x^2} + O(\Delta x^2)$$

$$\rho A \frac{d^2}{dt^2} u_1(t) = EA \left(\frac{u_0(t) - 2u_1(t) + u_2(t)}{\Delta x^2} \right)$$

$$\rho A \frac{d^2}{dt^2} u_2(t) = EA \left(\frac{u_1(t) - 2u_2(t) + u_3(t)}{\Delta x^2} \right)$$

$$\rho A \frac{d^2}{dt^2} u_3(t) = EA \left(\frac{u_2(t) - 2u_3(t) + u_4(t)}{\Delta x^2} \right)$$

Aproximación de las derivadas en x

$$\frac{\partial}{\partial x} u(x_4, t) = \frac{3u_4(t) - 4u_3(t) + u_2(t)}{2\Delta x} + O(\Delta x^2)$$

$$M \frac{d^2}{dt^2} u_4(t) = -EA\left(\frac{3u_4(t) - 4u_3(t) + u_2(t)}{2\Delta x}\right)$$

Sistema EDO con condiciones iniciales

$$\rho A \frac{d^2}{dt^2} u_1(t) = EA\left(\frac{f(t) - 2u_1(t) + u_2(t)}{\Delta x^2}\right)$$

$$\rho A \frac{d^2}{dt^2} u_2(t) = EA\left(\frac{u_1(t) - 2u_2(t) + u_3(t)}{\Delta x^2}\right)$$

$$\rho A \frac{d^2}{dt^2} u_3(t) = EA\left(\frac{u_2(t) - 2u_3(t) + u_4(t)}{\Delta x^2}\right)$$

$$M \frac{d^2}{dt^2} u_4(t) = -EA\left(\frac{3u_4(t) - 4u_3(t) + u_2(t)}{2\Delta x}\right)$$

Sistema EDO con condiciones iniciales

$$\begin{bmatrix} \rho A & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 \\ 0 & 0 & \rho A & 0 \\ 0 & 0 & 0 & M \end{bmatrix} \begin{pmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \\ \ddot{u}_4(t) \end{pmatrix} = \frac{EA}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & \frac{-1}{2}\Delta x & 2\Delta x & -\frac{3}{2}\Delta x \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{pmatrix} + \begin{pmatrix} \frac{EA}{\Delta x^2} f(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[\boldsymbol{M}]\ddot{\boldsymbol{u}}(t) = [\boldsymbol{K}]\boldsymbol{u}(t) + \boldsymbol{b}(t)$$

Solución del sistema

$$[\boldsymbol{M}]\ddot{\boldsymbol{u}}(t)=[\boldsymbol{K}]\boldsymbol{u}(t)+\boldsymbol{b}(t)$$

Reducción de orden y métodos RK

Solución explícita por diferencia central

Reducción de orden

Cambio de variables

$$\mathbf{z}_1(t)=\mathbf{u}(t)$$
$$\mathbf{z}_2(t)=\frac{d}{dt}\mathbf{u}(t)$$

$$[\mathbf{M}]\ddot{\mathbf{u}}(t)=[\mathbf{K}]\mathbf{u}(t)+\mathbf{b}(t)$$
$$\ddot{\mathbf{u}}(t)=[\mathbf{M}]^{-1}[\mathbf{K}]\mathbf{u}(t)+[\mathbf{M}]^{-1}\mathbf{b}(t)$$

Derivada del vector

$$\frac{d}{dt}\mathbf{Z}(t)=\begin{pmatrix}\frac{d}{dt}\mathbf{z}_1(t)\\ \frac{d}{dt}\mathbf{z}_2(t)\end{pmatrix}=\begin{pmatrix}\mathbf{z}_2(t)\\ [\mathbf{M}]^{-1}[\mathbf{K}]\mathbf{z}_1(t)+[\mathbf{M}]^{-1}\mathbf{b}(t)\end{pmatrix}$$

Sistema de EDOs de primer orden

$$\boxed{\frac{d}{dt}\mathbf{Z}(t)=\begin{bmatrix}[\mathbf{0}]_{4x4} & [\mathbf{I}]_{4x4}\\ [\mathbf{M}]^{-1}[\mathbf{K}] & [\mathbf{0}]_{4x4}\end{bmatrix}\mathbf{Z}(t)+\begin{pmatrix}[\mathbf{0}]_{4x1}\\ [\mathbf{M}]^{-1}\mathbf{b}(t)\end{pmatrix}}$$

ésto se carga en "evalfun"

Diferencia central

Aproximación de la derivada $\ddot{\mathbf{u}}(t)=\dfrac{\mathbf{u}(t-1)-2\mathbf{u}(t)+\mathbf{u}(t+1)}{\Delta t^2}$

Reemplazo en EDO $[\boldsymbol{M}]\left(\dfrac{\mathbf{u}(t-1)-2\mathbf{u}(t)+\mathbf{u}(t+1)}{\Delta t^2}\right)=[\boldsymbol{K}]\mathbf{u}(t)+\mathbf{b}(t)$

Despejando el estado futuro

$$\boxed{\underline{\mathbf{u}(t+1)}=(2[\boldsymbol{I}]_{4x4}+\Delta t^2[\boldsymbol{M}]^{-1}[\boldsymbol{K}])\mathbf{u}(t)-\mathbf{u}(t-1)+\Delta t^2[\boldsymbol{M}]^{-1}\mathbf{b}(t)}$$

Diferencia central - paso inicial

$$\boxed{\boldsymbol{u}(1)=(2[\boldsymbol{I}]_{4x4}+\Delta t^2[\boldsymbol{M}]^{-1}[\boldsymbol{K}])\boldsymbol{u}(0)-\underline{\boldsymbol{u}(-1)}+\Delta t^2[\boldsymbol{M}]^{-1}\boldsymbol{b}(0)}$$

Aproximación en t=-1 con Taylor

$$\boldsymbol{u}(-1)=\underline{\boldsymbol{u}(0)}-\underline{\frac{d\boldsymbol{u}}{dt}(0)}\Delta t+\frac{d^2\boldsymbol{u}}{dt^2}(0)\Delta t^2+O(\Delta t^3)$$

Condiciones iniciales

$$[\boldsymbol{M}]\ddot{\boldsymbol{u}}=[\boldsymbol{K}]\boldsymbol{u}+\boldsymbol{b}(0)\rightarrow\ddot{\boldsymbol{u}}(0)=[\boldsymbol{M}]^{-1}[\boldsymbol{K}]\boldsymbol{u}(0)+[\boldsymbol{M}]^{-1}\boldsymbol{b}(0)$$

$$\boxed{\boldsymbol{u}(-1)=(\Delta t^2[\boldsymbol{M}]^{-1}[\boldsymbol{K}]+[\boldsymbol{I}]_{4x4})\boldsymbol{u}(0)-\frac{d\boldsymbol{u}}{dt}(0)\Delta t+\Delta t^2[\boldsymbol{M}]^{-1}\boldsymbol{b}(0)}$$