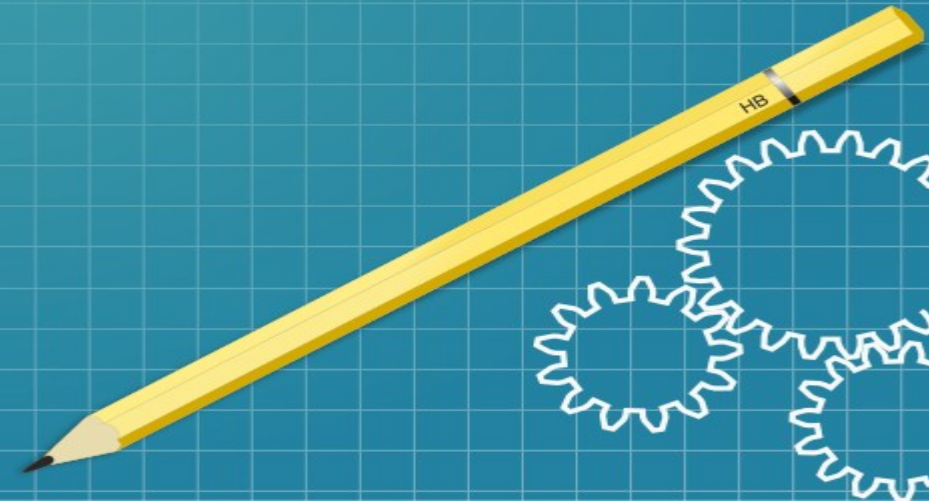
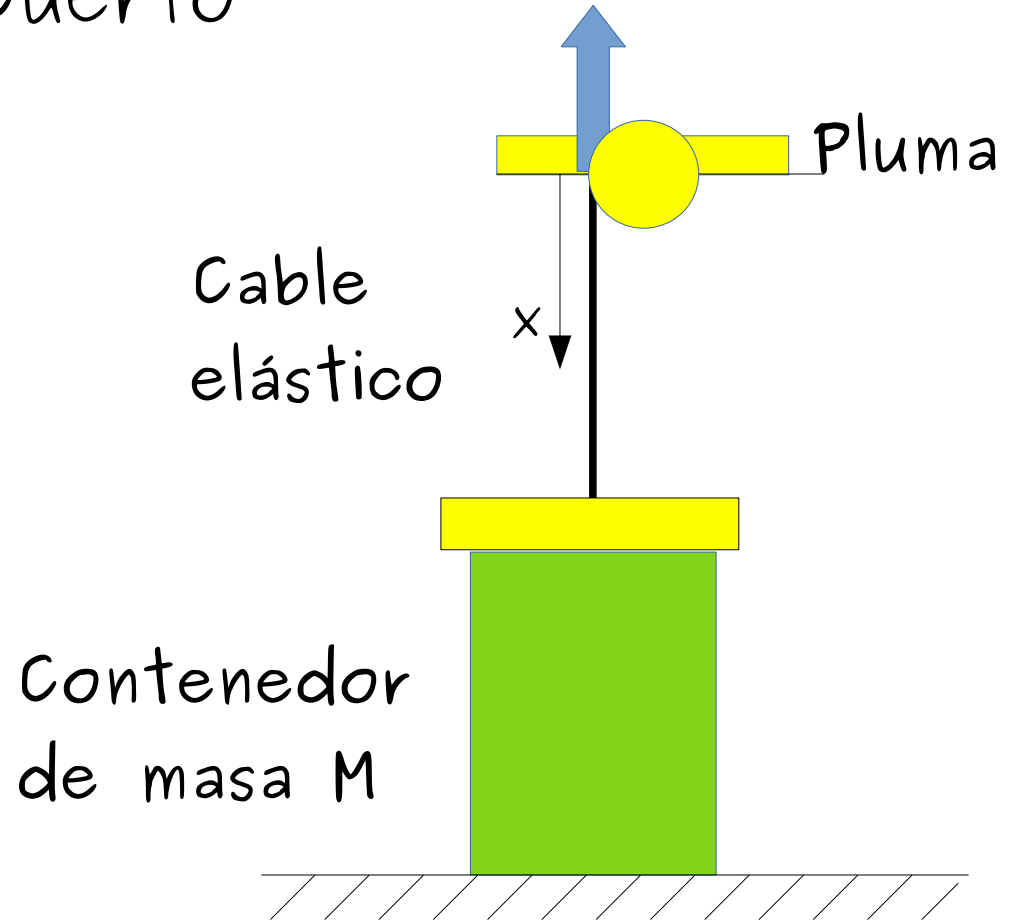


# Ejemplo de EDP

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# Problema: Vibraciones axiales en el cable de izaje de una grúa de puerto



Para modelar el problema se considera el equilibrio de las fuerzas que actúan sobre el cable.

$$\underbrace{\rho A \frac{\partial^2}{\partial t^2} u(x, t)}_{\text{Fuerza por efectos inerciales}} = \underbrace{EA \frac{\partial^2}{\partial x^2} u(x, t)}_{\text{Fuerza elástica del cable}}$$

Fuerza por  
efectos  
inerciales.

Fuerza elástica  
del cable.

$u(x, t)$  es el  
estiramiento en  
cada posición  
axial del cable  
e instante de  
tiempo

El modelo matemático queda definido por

$$\rho A \frac{\partial^2}{\partial t^2} u(x, t) = EA \frac{\partial^2}{\partial x^2} u(x, t)$$

$$\text{Dom } u(x, t) := \{(x, t) \in \mathbb{R}^2 / 0 \leq x \leq L \wedge 0 \leq t \leq tf\}$$

Gráficamente el dominio de la función solución  $u(x, t)$  es un rectángulo

# Las condiciones del problema

Diagram illustrating the conditions for a problem involving a beam of length  $L$  over time  $t$  from  $0$  to  $t_f$ .

The beam is represented by a blue rectangle. The left boundary is at  $x=0$ , the right boundary is at  $x=L$ , the start of time is at  $t=0$ , and the end of time is at  $t=t_f$ .

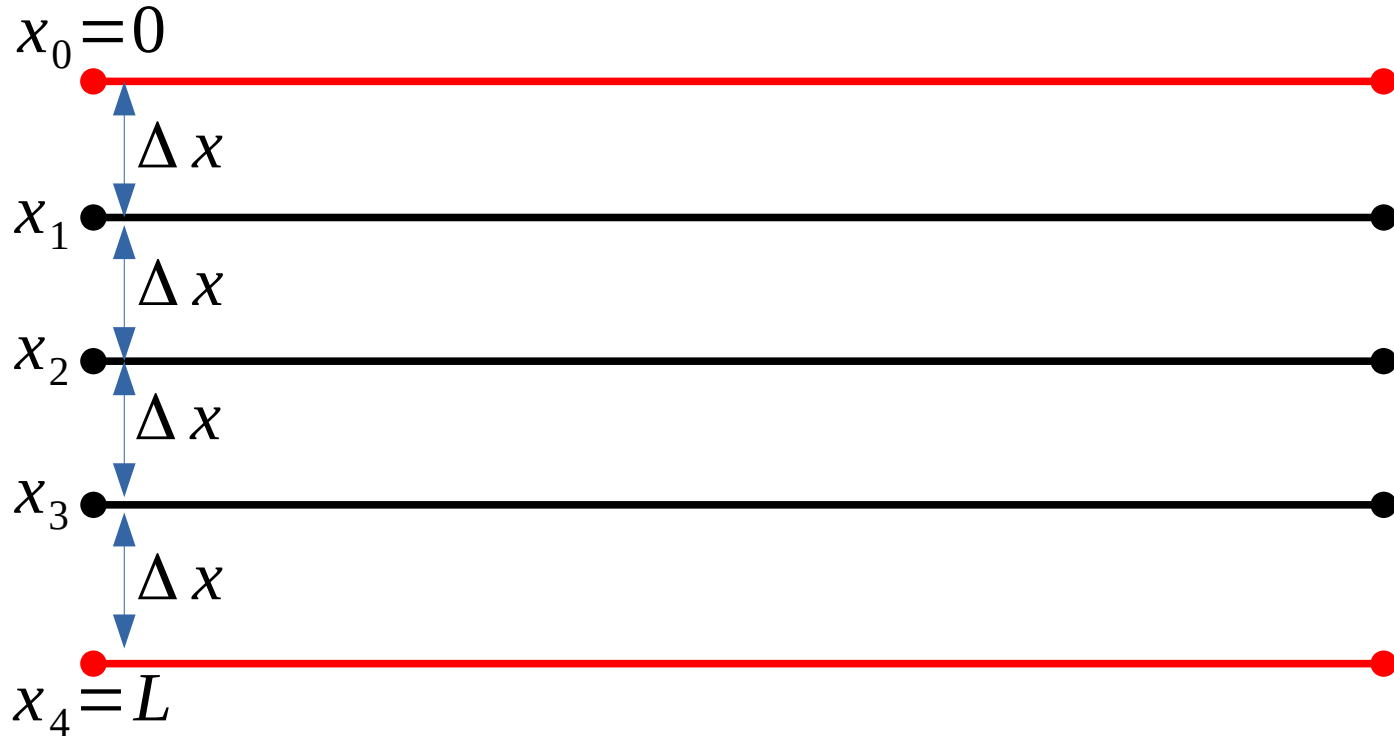
The conditions are:

- At  $x=0$ :
  - $u(0,t)=f(t)$  (Displacement boundary condition)
  - $u(x,0)=0$  (Initial displacement condition)
  - $\frac{\partial}{\partial t}u(x,0)=0$  (Initial velocity condition)
- At  $x=L$ :
  - $M \frac{\partial^2}{\partial t^2}u(L,t) = -EA \frac{\partial}{\partial x}u(L,t)$  (Dynamic boundary condition)
- Inside the beam:
  - $\rho A \frac{\partial^2}{\partial t^2}u(x,t) = EA \frac{\partial^2}{\partial x^2}u(x,t)$  (Wave equation)



# Aproximación numérica de la solución

# Discretización del espacio



El modelo matemático ahora describe un dominio discreto en cinco ubicaciones

$$u_0(t) = f(t) \quad \text{Condición de borde} \quad u_i(t) = u(x_i, t)$$

$$\rho A \frac{d^2}{dt^2} u_i(t) = EA \frac{\partial^2}{\partial x^2} u(x_i, t) \text{ para } i=1, 2, 3$$

$$M \frac{d^2}{dt^2} u_4(t) = -EA \frac{\partial}{\partial x} u(x_4, t) \quad \text{Condición de borde}$$



Aproximación de las derivadas en x

$$\frac{\partial^2}{\partial x^2} u(x_i, t) = \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{\Delta x^2} + O(\Delta x^2)$$

$$\rho A \frac{d^2}{dt^2} u_1(t) = EA \left( \frac{u_0(t) - 2u_1(t) + u_2(t)}{\Delta x^2} \right)$$

$$\rho A \frac{d^2}{dt^2} u_2(t) = EA \left( \frac{u_1(t) - 2u_2(t) + u_3(t)}{\Delta x^2} \right)$$

$$\rho A \frac{d^2}{dt^2} u_3(t) = EA \left( \frac{u_2(t) - 2u_3(t) + u_4(t)}{\Delta x^2} \right)$$

Aproximación de las derivadas en x

$$\frac{\partial}{\partial x} u(x_4, t) = \frac{3u_4(t) - 4u_3(t) + u_2(t)}{2\Delta x} + O(\Delta x^2)$$

$$M \frac{d^2}{dt^2} u_4(t) = -EA \left( \frac{3u_4(t) - 4u_3(t) + u_2(t)}{2\Delta x} \right)$$

Sistema EDO con condiciones iniciales

$$\rho A \frac{d^2}{dt^2} u_1(t) = EA \left( \frac{f(t) - 2u_1(t) + u_2(t)}{\Delta x^2} \right)$$

$$\rho A \frac{d^2}{dt^2} u_2(t) = EA \left( \frac{u_1(t) - 2u_2(t) + u_3(t)}{\Delta x^2} \right)$$

$$\rho A \frac{d^2}{dt^2} u_3(t) = EA \left( \frac{u_2(t) - 2u_3(t) + u_4(t)}{\Delta x^2} \right)$$

$$M \frac{d^2}{dt^2} u_4(t) = -EA \left( \frac{3u_4(t) - 4u_3(t) + u_2(t)}{2\Delta x} \right)$$

Sistema EDO con condiciones iniciales

$$\begin{bmatrix} \rho A & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 \\ 0 & 0 & \rho A & 0 \\ 0 & 0 & 0 & M \end{bmatrix} \begin{pmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \\ \ddot{u}_4(t) \end{pmatrix} = \frac{EA}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & \frac{-1}{2} \Delta x & 2 \Delta x & -\frac{3}{2} \Delta x \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{pmatrix} + \begin{pmatrix} \frac{EA}{\Delta x^2} f(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[M] \ddot{\mathbf{u}}(t) = [K] \mathbf{u}(t) + \mathbf{b}(t)$$

# Solución del sistema

Reducción de orden  
y métodos RK

$$[\mathbf{M}]\ddot{\mathbf{u}}(t)=[\mathbf{K}]\mathbf{u}(t)+\mathbf{b}(t)$$

Solución explícita por  
diferencia central


# Reducción de orden

Cambio de variables

$$\begin{aligned} \mathbf{z}_1(t) &= \mathbf{u}(t) \\ \mathbf{z}_2(t) &= \frac{d}{dt} \mathbf{u}(t) \end{aligned}$$

$$\begin{aligned} [\mathbf{M}] \ddot{\mathbf{u}}(t) &= [\mathbf{K}] \mathbf{u}(t) + \mathbf{b}(t) \\ \ddot{\mathbf{u}}(t) &= [\mathbf{M}]^{-1} [\mathbf{K}] \mathbf{u}(t) + [\mathbf{M}]^{-1} \mathbf{b}(t) \end{aligned}$$

Derivada del vector

$$\frac{d}{dt} \mathbf{Z}(t) = \begin{pmatrix} \frac{d}{dt} \mathbf{z}_1(t) \\ \frac{d}{dt} \mathbf{z}_2(t) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_2(t) \\ [\mathbf{M}]^{-1} [\mathbf{K}] \mathbf{z}_1(t) + [\mathbf{M}]^{-1} \mathbf{b}(t) \end{pmatrix}$$


Sistema de EDOs  
de primer orden

$$\frac{d}{dt} \mathbf{Z}(t) = \begin{bmatrix} [\mathbf{0}]_{4 \times 4} & [\mathbf{I}]_{4 \times 4} \\ [\mathbf{M}]^{-1} [\mathbf{K}] & [\mathbf{0}]_{4 \times 4} \end{bmatrix} \mathbf{Z}(t) + \begin{pmatrix} [\mathbf{0}]_{4 \times 1} \\ [\mathbf{M}]^{-1} \mathbf{b}(t) \end{pmatrix}$$

esto se carga en "evalfun"



## Diferencia central

Aproximación de la derivada  $\ddot{\mathbf{u}}(t) = \frac{\mathbf{u}(t-1) - 2\mathbf{u}(t) + \mathbf{u}(t+1)}{\Delta t^2}$

Reemplazo en EDO  $[\mathbf{M}] \left( \frac{\mathbf{u}(t-1) - 2\mathbf{u}(t) + \mathbf{u}(t+1)}{\Delta t^2} \right) = [\mathbf{K}] \mathbf{u}(t) + \mathbf{b}(t)$

Despejando el estado futuro

$$\underline{\mathbf{u}(t+1)} = (2[\mathbf{I}]_{4 \times 4} + \Delta t^2 [\mathbf{M}]^{-1} [\mathbf{K}]) \mathbf{u}(t) - \mathbf{u}(t-1) + \Delta t^2 [\mathbf{M}]^{-1} \mathbf{b}(t)$$

# Diferencia central - paso inicial

$$\mathbf{u}(1) = (2[\mathbf{I}]_{4 \times 4} + \Delta t^2 [\mathbf{M}]^{-1} [\mathbf{K}]) \mathbf{u}(0) - \underline{\mathbf{u}(-1)} + \Delta t^2 [\mathbf{M}]^{-1} \mathbf{b}(0)$$

Aproximación en  $t = -1$  con Taylor

$$\mathbf{u}(-1) = \underline{\mathbf{u}(0)} - \underline{\frac{d\mathbf{u}}{dt}(0)} \Delta t + \frac{d^2\mathbf{u}}{dt^2}(0) \Delta t^2 + O(\Delta t^3)$$

Condiciones iniciales

$$[\mathbf{M}] \ddot{\mathbf{u}} = [\mathbf{K}] \mathbf{u} + \mathbf{b}(0) \rightarrow \ddot{\mathbf{u}}(0) = [\mathbf{M}]^{-1} [\mathbf{K}] \mathbf{u}(0) + [\mathbf{M}]^{-1} \mathbf{b}(0)$$

$$\mathbf{u}(-1) = (\Delta t^2 [\mathbf{M}]^{-1} [\mathbf{K}] + [\mathbf{I}]_{4 \times 4}) \mathbf{u}(0) - \frac{d\mathbf{u}}{dt}(0) \Delta t + \Delta t^2 [\mathbf{M}]^{-1} \mathbf{b}(0)$$