Scribe

TALLER 1

Calcule las siguientes transformadas de Laplace para las funciones $f(t)$.

1. $f(t) = t\cos(at)$ para $a \in \mathbb{R}$

$$\mathcal{L}\{t\cos(at)\} = \lim_{b\to\infty}\int_0^b t\cos(at)e^{-st}\,dt$$

$$\int \cos(at)e^{-st}dt = \frac{-e^{-st}}{s}\cos(at) - \int \frac{e^{-st}}{s}(a\,sen(at))dt$$

$u = \cos(at)$
$du = -a\,sen(at)dt$
$dv = e^{-st}dt$
$v = \frac{-e^{-st}}{s}$

$$\int \cos(at)e^{-st}dt = \frac{-e^{-st}}{s}\cos(at) - \frac{a}{s}\int e^{-st}sen(at)\,dt$$

$$\int \cos(at)e^{-st}dt = \frac{-e^{-st}}{s}\cos(at) - \frac{a}{s}\left[\frac{-e^{-st}}{s}sen(at) + \frac{a}{s}\int e^{-st}\cos(at)dt\right]$$

$u = sen(at)$
$du = a\cos(at)dt$
$dv = e^{-st}dt$
$v = \frac{-e^{-st}}{s}$

$$\int \cos(at)e^{-st}dt = \left[\frac{-e^{-st}}{s}\cos(at) + \frac{a e^{-st}}{s^2}sen(at) - \frac{a^2}{s^2}\int e^{-st}\cos(at)dt\right]$$

$$\frac{s^2+a^2}{s^2}\int \cos(at)e^{-st}dt = \frac{-e^{-st}}{s}\cos(at) + \frac{a e^{-st}}{s^2}sen(at)$$

$$\int \cos(at)e^{-st}dt = -\frac{s e^{-st}}{s^2+a^2}\cos(at) + \frac{a e^{-st}}{s^2+a^2}sen(at)$$

$$\iint \cos(at)e^{-st}dt = \frac{-s}{s^2+a^2}\left[-\frac{s e^{-st}}{s^2+a^2}\cos(at) + \frac{a e^{-st}}{s^2+a^2}sen(at)\right] + \frac{a}{s^2+a^2}\int e^{-st}sen(at)dt$$

$$\int e^{-st}sen(at)dt = \frac{-e^{-st}}{s}sen(at) + \frac{a}{s}\int e^{-st}\cos(at)dt$$

$$\int e^{-st}sen(at)dt = \frac{-e^{-st}}{s}sen(at) + \frac{a}{s}\left[\frac{-e^{-st}}{s}\cos(at) - \int \frac{e^{-st}}{s}[a\,sen(at)]dt\right]$$

$$\int e^{-st}sen(at)dt = \frac{-e^{-st}}{s}sen(at) - \frac{a e^{-st}}{s^2}\cos(at) - \frac{a^2}{s^2}\int e^{-st}sen(at)dt$$

$$\frac{s^2+a^2}{s^2}\int e^{-st}sen(at)dt = \frac{-e^{-st}}{s}sen(at) - \frac{a e^{-st}}{s^2}\cos(at)$$

$$\int e^{-st}\operatorname{sen}(at)\,dt = -\frac{s e^{-st}}{s^2+a^2}\operatorname{sen}(at) - \frac{a e^{-st}}{s^2+a^2}\cos(at)$$

$$\iint \cos(at)e^{-st}\,dt = \frac{s^2e^{-st}}{(s^2+a^2)^2}\cos(at) - \frac{a s e^{-st}}{(s^2+a^2)^2}\operatorname{sen}(at) - \frac{a s e^{-st}}{(s^2+a^2)^2}\operatorname{sen}(at) - \frac{a^2e^{-st}}{(s^2+a^2)^2}\cos(at)$$

$$\iint \cos(at)e^{-st}\,dt = \frac{s^2e^{-st}}{(s^2+a^2)^2}\cos(at) - \frac{2as e^{-st}}{(s^2+a^2)^2}\operatorname{sen}(at) - \frac{a^2e^{-st}}{(s^2+a^2)^2}\cos(at)$$

Signo	u	dv
+	t	$\cos(at)e^{-st}$
−	1	$\int \cos(at)e^{-st}$
+	0	$\iint \cos(at)e^{-st}$

$$\int t\cos(at)e^{-st}\,dt = t\int \cos(at)e^{-st}\,dt - \iint \cos(at)e^{-st}\,dt\,dt$$

$$\mathcal{L}\{t\cos(at)\} = \lim_{b\to\infty}\int_0^b t\cos(at)e^{-st}\,dt$$

$$= \lim_{b\to\infty}\left[t\int \cos(at)e^{-st}\,dt - \iint \cos(at)e^{-st}\,dt\,dt\right]_0^b$$

$$= \lim_{b\to\infty}\left[t\left(\frac{-se^{-st}}{s^2+a^2}\cos(at) + \frac{a e^{-st}}{s^2+a^2}\operatorname{sen}(at)\right) - \frac{s^2e^{-st}}{(s^2+a^2)^2}\cos(at) + \frac{2as e^{-st}}{(s^2+a^2)^2}\operatorname{sen}(at) + \frac{a^2e^{-st}}{(s^2+a^2)^2}\cos(at)\right]_0^b$$

$$= \lim_{b\to\infty}\left[b\left(\frac{-se^{-bt}}{s^2+a^2}\cos(ab) + \frac{a e^{-bt}}{s^2+a^2}\operatorname{sen}(ab)\right) - \frac{s^2e^{-bt}}{(s^2+a^2)^2}\cos(ab) + \frac{2as e^{-bt}}{(s^2+a^2)^2}\operatorname{sen}(ab) + \frac{a^2e^{-bt}}{(s^2+a^2)^2}\cos(ab)\right.$$

$$\left. + \frac{s^2}{(s^2+a^2)^2} - \frac{a^2}{(s^2+a^2)^2}\right]$$

$$\boxed{\mathcal{L}\{t\cos(at)\} = \frac{s^2-a^2}{(s^2+a^2)^2}} \qquad s>0$$

2. $f(t) = t\operatorname{sen}(at)$ para $a \in \mathbb{R}$

$$\mathcal{L}\{t\operatorname{sen}(at)\} = \lim_{b\to\infty}\int_0^b t\operatorname{sen}(at)e^{-st}\,dt$$

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signo	u	dv
+	t	$\text{sen}(at)\,e^{-st}$
−	1	$\int \text{sen}(at)\,e^{-st}$
+	0	$\iint \text{sen}(at)\,e^{-st}$

- $\int t\,\text{sen}(at)\,e^{-st}dt = t\int \text{sen}(at)e^{-st}dt - \iint \text{sen}(at)e^{-st}dt$

- $\int \text{sen}(at)e^{-st}dt = -\frac{e^{-st}}{s}\text{sen}(at) + \frac{a}{s}\int e^{-st}\cos(at)dt$

$u = \text{sen}(at)$
$du = a\cos(at)$
$dv = e^{-st}dt$
$v = -\frac{e^{-st}}{s}$

$\int \text{sen}(at)e^{-st}dt = -\frac{e^{-st}}{s}\text{sen}(at) + \frac{a}{s}\left[-\frac{e^{-st}}{s}\cos(at) - \frac{a}{s}\int e^{-st}\text{sen}(at)dt\right]$

$\frac{s^2+a^2}{s^2}\int \text{sen}(at)e^{-st}dt = -\frac{e^{-st}}{s}\text{sen}(at) - \frac{ae^{-st}}{s^2}\cos(at)$

$\int \text{sen}(at)e^{-st}dt = -\frac{se^{-st}}{s^2+a^2}\text{sen}(at) - \frac{ae^{-st}}{s^2+a^2}\cos(at)$

- $\iint \text{sen}(at)e^{-st}dt = -\frac{s}{s^2+a^2}\int e^{-st}\text{sen}(at)dt - \frac{a}{s^2+a^2}\int e^{-st}\cos(at)dt$

$= -\frac{s}{s^2+a^2}\left[-\frac{se^{-st}}{s^2+a^2}\text{sen}(at) - \frac{ae^{-st}}{s^2+a^2}\cos(at)\right]$

$-\frac{a}{s^2+a^2}\left[-\frac{se^{-st}}{s^2+a^2}\cos(at) + \frac{ae^{-st}}{s^2+a^2}\text{sen}(at)\right]$

$= \frac{s^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at) + \frac{ase^{-st}}{(s^2+a^2)^2}\cos(at) + \frac{ase^{-st}}{(s^2+a^2)^2}\cos(at) - \frac{a^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at)$

$= \frac{s^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at) + \frac{2ase^{-st}}{(s^2+a^2)^2}\cos(at) - \frac{a^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at)$

$\int t\,\text{sen}(at)e^{-st}dt = -\frac{tse^{-st}}{s^2+a^2}\text{sen}(at) - \frac{tae^{-st}}{s^2+a^2}\cos(at) - \frac{s^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at)$

$-\frac{2ase^{-st}}{(s^2+a^2)^2}\cos(at) + \frac{a^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at)$

$$\lim_{b\to\infty}\int_0^b t\,\text{sen}(at)e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{t\,s\,e^{-st}}{s^2+a^2}\text{sen}(at) - \frac{t\,a\,e^{-st}}{s^2+a^2}\cos(at) - \frac{s^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at)\right.$$

$$\left. - \frac{2as\,e^{-st}}{(s^2+a^2)^2}\cos(at) + \frac{a^2e^{-st}}{(s^2+a^2)^2}\text{sen}(at)\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{-bs\,e^{-sb}}{s^2+a^2}\text{sen}(ab) - \frac{ab\,e^{-sb}}{s^2+a^2}\cos(ab) - \frac{s^2e^{-sb}}{(s^2+a^2)^2}\text{sen}(ab)\right.$$

$$\left. - \frac{2as\,e^{-sb}}{(s^2+a^2)^2}\cos(ab) + \frac{a^2e^{-sb}}{(s^2+a^2)^2}\text{sen}(ab) + \frac{2as}{(s^2+a^2)^2}\right]$$

$$\mathcal{L}\{t\,\text{sen}(at)\} = \frac{2as}{(s^2+a^2)^2} \qquad s>0$$

continuación taller 1

3. $f(t) = e^{at}\cos(bt)$ para $a, b \in \mathbb{R}$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \lim_{B\to\infty}\int_0^B e^{-st}e^{at}\cos(bt)\,dt$$

$u = \cos(bt)$
$du = -b\,\text{sen}(bt)\,dt$
$dv = e^{-t(s-a)}\,dt$
$v = -\dfrac{e^{-t(s-a)}}{s-a}$

$$= \lim_{B\to\infty}\int_0^B e^{-t(s-a)}\cos(bt)\,dt$$

$$\int e^{-t(s-a)}\cos(bt) = -\frac{e^{-t(s-a)}}{s-a}\cos(bt) - \frac{b}{s-a}\int e^{-t(s-a)}\,\text{sen}(bt)\,dt$$

$$\int e^{-t(s-a)}\cos(bt) = \frac{-e^{-t(s-a)}}{s-a}\cos(bt) + \frac{b\,e^{-t(s-a)}}{(s-a)^2}\,\text{sen}(bt) - \frac{b^2}{(s-a)^2}\int e^{-t(s-a)}\cos(bt)\,dt$$

$$\frac{(s-a)^2+b^2}{(s-a)^2}\int e^{-t(s-a)}\cos(bt) = \frac{-e^{-t(s-a)}}{s-a}\cos(bt) + \frac{b\,e^{-t(s-a)}}{(s-a)^2}\,\text{sen}(bt)$$

$$\int e^{-t(s-a)}\cos(bt) = \frac{-(s-a)e^{-t(s-a)}}{(s-a)^2+b^2}\cos(bt) + \frac{b\,e^{-t(s-a)}}{(s-a)^2+b^2}\,\text{sen}(bt)$$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \lim_{B\to\infty}\left[\frac{(a-s)e^{-t(s-a)}}{(s-a)^2+b^2}\cos(bt) + \frac{b\,e^{-t(s-a)}}{(s-a)^2+b^2}\,\text{sen}(bt)\right]_0^B$$

$$= \lim_{B\to\infty}\left[\frac{(a-s)e^{-B(s-a)}}{(s-a)^2+b^2}\cos(Bb) + \frac{b\,e^{-B(s-a)}}{(s-a)^2+b^2}\,\text{sen}(Bt) - \frac{(a-s)}{(s-a)^2+b^2}\right]$$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \left.\frac{s}{s^2+b^2}\right|_{s\to s-a} = \frac{s-a}{(s-a)^2+b^2}$$

4. $f(t) = t^2\cosh(at)$ para $a \in \mathbb{R}$

$$\mathcal{L}\{t^2\cosh(at)\} = \lim_{b\to\infty}\int_0^b t^2\cosh(at)\,e^{-st}\,dt = (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2-a^2}\right]$$

$$\mathcal{L}\{t^2\cosh(at)\} = \frac{2s(s^2+3a^2)}{(s^2-a^2)^3}$$

Signo	u	dv
+	t^2	$\cosh(at)\,e^{-st}$
-	$2t$	$\int \cosh(at)\,e^{-st}\,dt$
+	2	$\iint \cosh(at)\,e^{-st}\,dt$
-	0	$\iiint \cosh(at)\,e^{-st}\,dt$

- $\int \cosh(at)\,e^{-st}\,dt = \int \left(\frac{e^{at}}{2} + \frac{e^{-at}}{2}\right) e^{-st}\,dt$

$$= \frac{1}{2}\int e^{-t(s-a)}\,dt + \frac{1}{2}\int e^{-t(s+a)}\,dt$$

$$= -\frac{e^{-t(s-a)}}{2(s-a)} - \frac{e^{-t(s+a)}}{2(s+a)}$$

- $\iint \cosh(at)\,e^{-st}\,dt = -\frac{1}{2(s-a)}\int e^{-t(s-a)}\,dt - \frac{1}{2(s+a)}\int e^{-t(s+a)}\,dt$

$$= \frac{e^{-t(s-a)}}{2(s-a)^2} + \frac{e^{-t(s+a)}}{2(s+a)^2}$$

- $\iiint \cosh(at)\,e^{-st}\,dt = \frac{1}{2(s-a)^2}\int e^{-t(s-a)}\,dt - \frac{1}{2(s+a)^2}\int e^{-t(s+a)}\,dt$

$$= \frac{-e^{-t(s-a)}}{2(s-a)^3} - \frac{e^{-t(s+a)}}{2(s+a)^3}$$

$$\mathcal{L}\{t^2\cosh(at)\} = \lim_{b\to\infty}\left[t^2\left(-\frac{e^{-t(s-a)}}{2(s-a)} - \frac{e^{-t(s+a)}}{2(s+a)}\right) - 2t\left(\frac{e^{-t(s-a)}}{2(s-a)^2} + \frac{e^{-t(s+a)}}{2(s+a)^2}\right) - \frac{e^{-t(s-a)}}{(s-a)^3} - \frac{e^{-t(s+a)}}{(s+a)^3}\right]_0^b$$

$$= \lim_{b\to\infty}\left[b^2\left(-\frac{e^{-b(s-a)}}{2(s-a)} - \frac{e^{-b(s+a)}}{2(s+a)}\right) - b\left(\frac{e^{-b(s-a)}}{(s-a)^2} + \frac{e^{-b(s+a)}}{(s+a)^2}\right) - \frac{e^{-b(s-a)}}{(s-a)^3} - \frac{e^{-b(s+a)}}{(s+a)^3} + \frac{1}{(s-a)^3} + \frac{1}{(s+a)^3}\right]$$

$s-a>0$ $s>a$
$s+a>0$ $s>-a$

$$= \frac{(s+a)^3 + (s-a)^3}{(s-a)^3(s+a)^3} = \frac{s^3 + 3s^2a + 3sa^2 + a^3 + s^3 - 3s^2a + 3sa^2 - a^3}{(s^3 - 3s^2a + 3sa^2 - a^3)(s^3 + 3s^2a + 3sa^2 + a^3)}$$

$$= \frac{2s^3 + 6sa^2}{s^6 + 3s^5a + 3s^4a^2 + s^3a^3 - 3s^5a - 9s^4a^2 - 9s^3a^3 - 3s^2a^4 + 3s^4a^2 + 9s^3a^3 + 9s^2a^4 + 3sa^5 - s^3a^3 - 3s^2a^4 - 3sa^5 - a^6}$$

$$= \frac{2s(s^2+3a^2)}{s^6 - 3s^4a^2 + 3s^2a^4 - a^6} = \frac{2s(s^2+3a^2)}{(s^2-a^2)^3}$$

5. $f(t) = \operatorname{senh}(at)$ para $a \in \mathbb{R}$

$$\mathcal{L}\{\operatorname{senh}(at)\} = \lim_{b\to\infty} \int_0^b \operatorname{senh}(at)\, e^{-st}\, dt$$

$$= \lim_{b\to\infty} \int_0^b \left(\frac{e^{at} - e^{-at}}{2}\right) e^{-st}\, dt$$

$$= \lim_{b\to\infty} \left[\int_0^b \frac{e^{-t(s-a)}}{2}\, dt - \int_0^b \frac{e^{-t(s+a)}}{2}\, dt\right]$$

$$= \lim_{b\to\infty} \left[-\frac{e^{-t(s-a)}}{2(s-a)} + \frac{e^{-t(s+a)}}{2(s+a)}\right]_0^b$$

$$= \lim_{b\to\infty} \left[\frac{-e^{-b(s-a)}}{2(s-a)} + \frac{e^{-b(s+a)}}{2(s+a)} + \frac{1}{2(s-a)} - \frac{1}{2(s+a)}\right]$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)}$$

$s - a > 0$
$s > a$

$s + a > 0$
$s > -a$

$$\mathcal{L}\{\operatorname{senh}(at)\} = \frac{2[(s+a)-(s-a)]}{4(s^2-a^2)} = \frac{a}{s^2-a^2}$$

6. $f(t) = t^n e^{at}$ para $a \in \mathbb{R}$

$$\mathcal{L}\{t^n e^{at}\} = \left.\frac{n!}{s^{n+1}}\right|_{s \to s-a}$$

$$= \frac{n!}{(s-a)^{n+1}}$$

$s - a > 0$
$s > a$

$$\mathcal{L}\{t^n e^{at}\} = \lim_{b\to\infty} \int_0^b t^n e^{at} e^{-st}\, dt$$

$$= \lim_{b\to\infty} \int_0^b t^n e^{-t(s-a)}\, dt$$

$w = s - a$

$$= \lim_{b\to\infty} \int_0^b t^n e^{-wt}\, dt$$

$$= \frac{n!}{w^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

Utilice la trasformada de laplace para calcular la solución de cada una de los siguientes ecuaciones diferenciales en el espacio $y(s)$.

7. $y'' - 2y' - 2y = 0$; $y(0) = 2$; $y'(0) = 0$

$$\mathcal{L}\{y'' - 2y' - 2y\} = \mathcal{L}\{0\}$$
$$\mathcal{L}\{y''\} - \mathcal{L}\{2y'\} - \mathcal{L}\{2y\} = 0$$
$$s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0) - 2[s\,\mathcal{L}\{y(t)\} - y(0)] - 2\mathcal{L}\{y(t)\} = 0$$
$$s^2 y(s) - 2s - 2[s\,y(s) - 2] - 2y(s) = 0$$
$$s^2 y(s) - 2s - 2s\,y(s) + 4 - 2y(s) = 0$$
$$y(s)[s^2 - 2s - 2] = 2s - 4$$
$$y(s) = \frac{2s - 4}{s^2 - 2s - 2}$$

8. $y'' + r^2 y = \cos(2t)$; $r^2 \neq 4$; $y(0) = 1$; $y'(0) = 0$

$$\mathcal{L}\{y'' + r^2 y\} = \mathcal{L}\{\cos(2t)\}$$
$$\mathcal{L}\{y''\} + \mathcal{L}\{r^2 y\} = \frac{s}{s^2+4}$$
$$s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0) + r^2 y(s) = \frac{s}{s^2+4}$$
$$s^2 y(s) - s + r^2 y(s) = \frac{s}{s^2+4}$$
$$y(s)[s^2 + r^2] = \frac{s}{s^2+4} + s$$
$$y(s) = \frac{s + s(s^2+4)}{(s^2+4)(s^2+r^2)}$$
$$y(s) = \frac{s(1 + s^2 + 4)}{(s^2+4)(s^2+r^2)}$$
$$y(s) = \frac{s(s^2+5)}{(s^2+4)(s^2+r^2)}$$

9. $y'' - 2y' + 2y = \cos(t)$; $y(0) = 1$; $y'(0) = 0$

$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{\cos(t)\}$

$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{s}{s^2+1}$

$s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0) - 2[s\mathcal{L}\{y(t)\} - y(0)] + 2y(s) = \frac{s}{s^2+1}$

$s^2 y(s) - s - 2s\,y(s) + 2 + 2y(s) = \frac{s}{s^2+1}$

$y(s)[s^2 - 2s + 2] = \frac{s}{s^2+1} + s - 2$

$y(s) = \frac{s + s(s^2+1) - 2(s^2+1)}{(s^2+1)(s^2-2s+2)} = \frac{s + (s^2+1)(s-2)}{(s^2+1)(s^2-2s+2)}$

10. $y'' - 2y' + 2y = e^{-t}$; $y(0) = 0$; $y'(0) = 1$

$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{e^{-t}\}$

$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s+1}$

$s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0) - 2[s\mathcal{L}\{y(t)\} - y(0)\} + 2y(s) = \frac{1}{s+1}$

$s^2 y(s) - 1 - 2s\,y(s) + 2y(s) = \frac{1}{s+1}$

$y(s)[s^2 - 2s + 2] = \frac{1}{s+1} + 1$

$y(s) = \frac{1 + s + 1}{(s+1)(s^2-2s+2)}$

$y(s) = \frac{s+2}{(s+1)(s^2-2s+2)}$

11. $y'' - 3y' + 2y = \operatorname{senh}(t)$; $y(0) = 0$; $y'(0) = 0$

$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{\operatorname{senh}(t)\}$

$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \dfrac{1}{s^2-1}$

$s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0) - 3[s\mathcal{L}\{y(t)\} - y(0)\} + 2\,y(s) = \dfrac{1}{s^2-1}$

$s^2 y(s) - 3s\,y(s) + 2\,y(s) = \dfrac{1}{s^2-1}$

$y(s)[s^2 - 3s + 2] = \dfrac{1}{s^2-1}$

$y(s) = \dfrac{1}{(s^2-1)(s-2)(s-1)}$

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