Logistic Regression Algorithm(single variable)

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December 13, 2017

The **Logistic Regression Algorithm** is a Machine learning technique used for classification. It is a binary decision curve, which means that for some input x (in the *one-variable* case), it returns a value between 0-1 representing the probability of belonging to the set.

The base function for the logistic regression is the sigmoid function, which has the form

$$f(x) = \frac{1}{1 + e^{-(ax+b)}} \tag{1}$$

Different from the Linear Regression function, this is not linear. That means that the error function can't be linear either. As the distribution of the dataset points is binary, it follows the general form for binomial distribution. Thus, we get:

$$Error = E(a, b) = \sum_{i=0}^{N} f^y (1 - f)^{1-y} \tag{2}$$

or what is equivalent:

$$E(a, b) = \sum_{i=0}^{N} \frac{e^{(y-1)(ax+b)}}{1 + e^{-(ax+b)}} \tag{3}$$

Both forms of the error value represent two expressions of the same object. Equation (2), however, is computationally more desirable, as it is more compact and clear. On the other hand, (3) is used better when calculating the gradients for the coefficients, as it is more explicit.

Having the error function, we can have an expression for the gradient to have a value of change for each learning step. We proceed to find those gradients on which the value variation is depending on:

$$\frac{\partial E(a,b)}{\partial a} = \sum_{i=0}^{N} x_i f(x_i)[y_i - f(x_i)] \left[e^{-(ax_i+b)}\right]^{1-y_i} \tag{4}$$

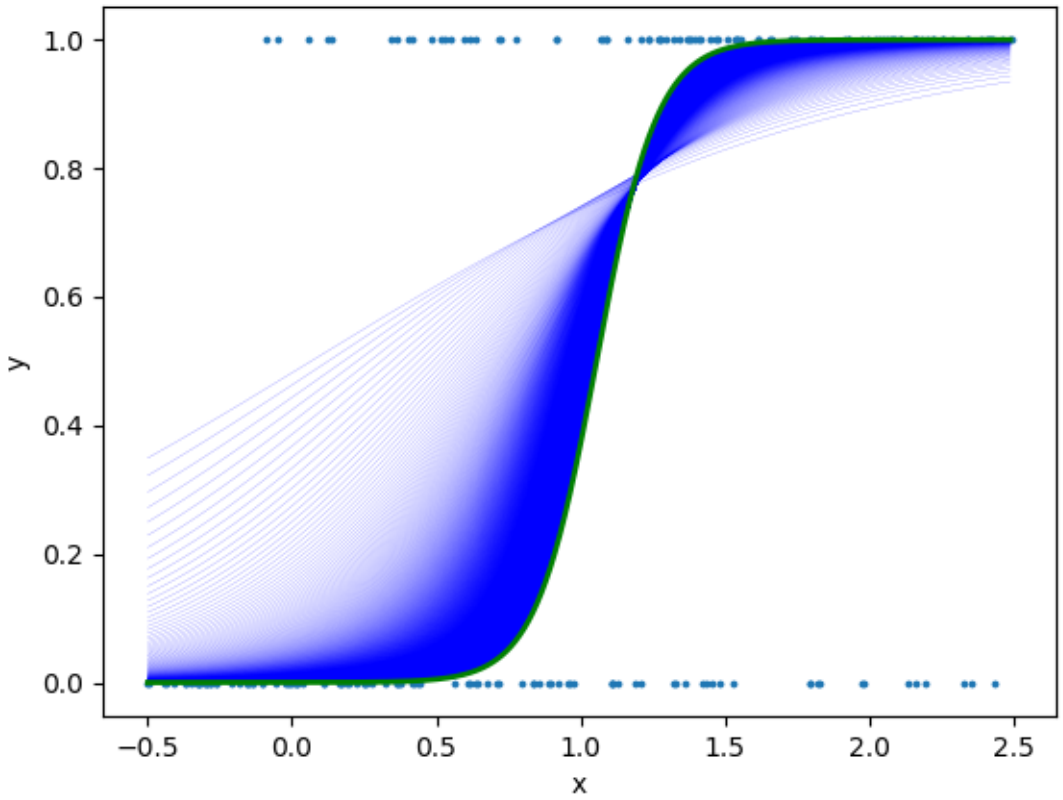
$$\frac{\partial E(a,b)}{\partial b} = \sum_{i=0}^{N} f(x_i)[y_i - f(x_i)] \left[e^{-(ax_i+b)}\right]^{1-y_i} \tag{5}$$

With (4) and (5) we solve the problem of the direction and magnitude of the movement. To avoid the divergence of the *learning process*, in the code we define a *learning_rate* which will control the movement of the values. now, for the setting of the new values, let p be the step number in which the system is at certain point. That way, we have, for each step:

$$a_{p+1} = a_p + \frac{\partial E}{\partial a_p} \times learning\_rate \tag{6}$$

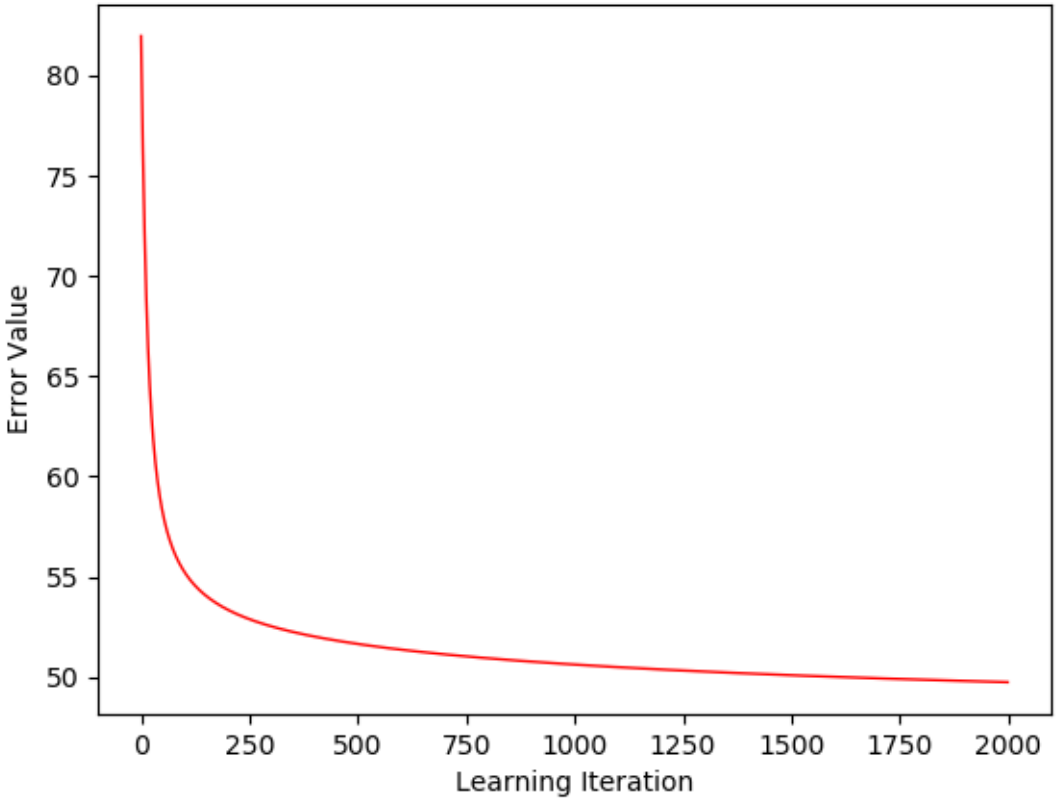
$$b_{p+1} = b_p + \frac{\partial E}{\partial b_p} \times learning\_rate \tag{7}$$

1 Result graph



This graph summarizes visually the process of learning for the classifier. The blue points are the input data; the blue lines represent the state of the model in each learning iteration, thus showing how the model changes through the learning process; and the green curve represents the last state of the model, after the learning process.

2 Error graph



Another key point to understand the model through its process is the variability of the error value. This graph shows after each learning iteration the value for the last model. As shown, the error value decreases, as expected. This information can be used to determine the value of iterations optimal for the system, as a very low number of iterations might lead to a wrong model, and a high number of them to a very long-time processing. It might give a clue among the over-adjustment problem.

3 Condensed Formula set

$$\mathbf{L} = -(a\mathbf{X} + b)$$

$$\mathbf{F} = \frac{1}{1 + e^{\mathbf{L}}}$$

$$\mathbf{E} = 1 - \mathbf{F}^{\mathbf{Y}} \left[1 - \mathbf{F}\right]^{1-\mathbf{Y}}$$

$$\boldsymbol{\Delta} = \mathbf{F} \left[\mathbf{Y} - \mathbf{F}\right] \left[e^{\mathbf{L}}\right]^{[1-\mathbf{Y}]}$$

$$\frac{\partial \mathbf{E}}{\partial a} = \mathbf{X} \cdot \boldsymbol{\Delta}$$

$$\frac{\partial \mathbf{E}}{\partial b} = \sum_{i=0}^{N-1} \boldsymbol{\Delta}_i$$

For this formula set, N represents the length of the input data. The styled-letters (i.e. $\mathbf{L}, \mathbf{X}, \boldsymbol{\Delta}$, and so on) represent vector objects. It is important to note that all the operations between vector objects are done as element-by-element (excluding the dot product, which operates as usual). This applies also with the conventional scalar to vector operations.

For example, when saying $\mathbf{F}^{a\mathbf{Y}}$, what should be interpreted is:
$\mathbf{F}^{a\mathbf{Y}} = F_i^{(aY_i)}, \quad \forall i \in [0, N-1], a \in \mathbb{R}$. If the dot product symbol $(\cdot)$ does not appear, it is assumed element-by-element multiplication.