Probability - Set Theory

Notation :

Sets will be denoted by capital letters:

↳ $A, B, C, \ldots$ will be sets

↳ Special sets

$\mathcal{U}$: Universal set

$\emptyset$: Empty set

Operations between sets :

Let A and B be sets. Then

Intersection : $A \cap B = \{ x : x \in A \wedge x \in B \}$

Union : $A \cup B = \{ x : x \in A \vee x \in B \}$

Complement : $A^c = \{ x : x \notin A \}$

Set Difference: $A - B = \{ x : x \in A \wedge x \notin B \}$

$$A - B = A \cap B^c$$

Sometimes, you will also see it written as $A \setminus B$.

Symmetric Difference of two sets: $A \Delta B = (A - B) \vee (B - A)$.

It is the set of elements that belong to either A or B but not to both.

Collection of Sets

To define a collection of sets, a set of indexes will be used:

- T : Set of indexes
- A collection of subsets of $\mathcal{U}$ is defined as

$$\{A_t\}_{t \in T}$$

Operations associated with $\{A_t\}_{t\in T}$

- $\displaystyle\bigcap_{t\in T} A_t = \{ x : x \in A_t \text{ for every } t \in T \}$

↳ $\bigcup_{t \in T} A_t = \{ x : x \in A_t \text{ for any } t \in T \}$

↳ If $A_1, A_2, A_3, \ldots$ are subsets of $\mathcal{U}$ that satisfy $A_i \cap A_j = \emptyset$ for all $i \neq j$, then it is customary to write

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i$$

Definition Indicator Function:

Let A be a set. The indicator function of A is defined as:

$$\chi_A(x) = \mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition liminf and limsup:

Let $\Omega \neq \emptyset$ be a set and $\{A_n\}_n$ be a sucession of subsets of Ω. We define:

↳ $A_* = \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$

↳ $A^* = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$

↳ If A_* and A^* are equal then it is told that the limit of $\{A_n\}_n$ exists and it is written

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A$$

Definition Monotonically Increasing and Decreasing

1 Monotonically Increasing

Let $(A_n)_{n\geq 1}$ a succession of subsets of Ω. It is said that the sucession $(A_n)_{n\geq 1}$ is monotonically increasing, if and only if,

$$A_n \subseteq A_{n+1}$$

and it is denoted $(A_n)\uparrow = A_n\uparrow$

2. Monotonically Decreasing

Let $(A_n)_{n\geq 1}$ be a succession of subsets of Ω. It is said that $(A_n)_{n\geq 1}$ is monotonically decreasing, if and only if,

$$A_n \supseteq A_{n+1} \quad \text{for all } n$$

It's written $(A_n)\searrow = A_n \searrow$

Preposition:

Let $(A_n)_{n\geq 1}$ be a succession of subsets of Ω.

a) If $A_n \uparrow$ then $\lim\limits_{n\to\infty} A_n = \bigcup\limits_{n=1}^{\infty} A_n$

b) If $A_n \searrow$ then $\lim\limits_{n\to\infty} A_n = \bigcap\limits_{n=1}^{\infty} A_n$

Proof:

a) Since, by hypothesis, $A_n \subset A_{n+1}$ then

↳ $A_n \cap A_{n+1} = A_n$

↳ $A_n \cup A_{n+1} = A_{n+1}$

We need to show that

$$A_* = A^* = \bigcup_{n=1}^{\infty} A_n$$

↳ Let's begin with $A_* = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$

Notice that $\bigcap_{k=n}^{\infty} A_k = A_n \cap A_{n+1} \cap \cdots = A_n$

as $A_n \uparrow$.

Then, $A_* = \bigcup_{n=1}^{\infty} A_n$

↳ Let's continue with $A^* = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$

This one is more tricky. Let's define a new succession named $(C_n)_{n=1}$ where

$$C_n = \bigcup_{k=n}^{\infty} A_k$$

Notice that

$$C_1 = \bigcup_{k=1}^{\infty} A_k$$

$$C_2 = \bigcup_{k=2}^{\infty} A_k = C_1 \quad \text{as} \quad A_1 \cup A_2 = A_2$$

$$C_3 = \bigcup_{k=3}^{\infty} A_k = C_2 = C_1 \quad \text{as} \quad A_1 \cup A_2 \cup A_3 = A_3$$

$\vdots$

$$C_n = \bigcup_{k=n}^{\infty} A_k = C_1 \quad \text{as} \quad \bigcup_{n=1}^{k} A_n = A_k$$

Then, we've got that

$$C_n = \bigcup_{k=n}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k = C_1$$

Plugging this result into A^*, we obtain that

$$A^* = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} C_1 = C_1$$

$$A^* = C_1 = \bigcup_{n=1}^{\infty} A_n$$

So, we can now conclude that if $A_n \nearrow$ then

$$\lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

a) More formal proof:

Recalling that $A_* = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$, let's define

$$C_n = \bigcap_{k=n}^{\infty} A_k .$$

Notice that, because $A_n \nearrow$,

$$C_1 = A_1 \cap A_2 \cap \cdots = A_1$$

$$C_2 = A_2 \cap A_3 \cap \cdots = A_2$$

$$C_3 = A_3 \cap A_4 \cap \cdots = A_3$$

$$\vdots$$

$$C_n = A_n \cap A_{n+1} \cap \cdots = A_n$$

Then,

$$A_* = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n$$

On the other hand, let's define D_n as

$$D_n = \bigcup_{k=n}^{\infty} A_n$$

Note that $A^* = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ and

$$A^* = \bigcap_{n=1}^{\infty} D_n \subseteq D_n \quad \text{for any } D_n$$

In particular, $D_1 = \bigcup_{n=1}^{\infty} A_n = A_x$

So,

$$A^* = \bigcap_{n=1}^{\infty} D_n \subset A_x$$

And given that $A_x \subset A^*$, we have that

$A^* \subset A_x$ and $A_x \subset A^*$, which means that

$$A^* = A_x = \bigcup_{n=1}^{\infty} A_n$$

b) Notice that:

✓ If $A_n \searrow$, then $A_n^c \nearrow$

✓ Based on the part a of this theorem.

$$\liminf A_n^c = \limsup A_n^c = \bigcup_{n=1}^{\infty} A_n^c$$

$$(A_n^c)_* = (A_n^c)^* = \bigcup_{n=1}^{\infty} A_n^c$$

✓ Notice that in general

↳ $\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$

$$\Rightarrow (\liminf A_n)^c = \bigcap_{n=1}^{\infty} \Big\{ \bigcap_{k=n}^{\infty} A_k \Big\}^c = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c = \limsup A_n^c$$

↳ $\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$

$$\Rightarrow (\liminf A_n)^c = \bigcup_{n=1}^{\infty} \Big\{ \bigcup_{k=n}^{\infty} A_k \Big\}^c = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c = \liminf A_n^c$$

Therefore, we can conclude that

$$(\liminf A_n)^c = \limsup A_n^c = \bigcup_{n=1}^{\infty} A_n^c$$

$$\Rightarrow \quad \liminf A_n = \left\{ \bigcup_{n=1}^{\infty} A_n^c \right\} = \bigcap_{n=1}^{\infty} A_n$$

and

$$(\limsup A_n)^c = \liminf A_n^c = \bigcup_{n=1}^{\infty} A_n^c$$

$$\Rightarrow \quad \limsup A_n = \left\{ \bigcup_{n=1}^{\infty} A_n^c \right\}^c = \bigcap_{n=1}^{\infty} A_n$$

Finally

$$\lim A_n = \bigcap_{n=1}^{\infty} A_n.$$

Observation

Let $C_n := \bigcup_{k=n}^{\infty} A_n$. It is clear that

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots \quad \text{meaning} \quad C_n \downarrow$$

Then

$$\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n$$

i.e

$$\lim_{n\to\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup A_n$$

We define

$$\sup_{k\geq n} A_k := \bigcup_{k=n}^{\infty} A_k. \quad \text{Which leads to}$$

$$\limsup_{n\to\infty} A_n = \lim_{n\to\infty} \left(\sup_{k\geq n} A_k \right)$$

Similarly, let $D_n := \bigcap_{k=n}^{\infty} A_n$. It is clear that

$$D_1 \subseteq D_2 \subseteq D_3 \cdots \quad \text{meaning} \quad D_n \nearrow$$

Then

$$\lim_{n\to\infty} D_n = \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} D_n = \liminf_{n\to\infty} A_n$$

We define

$$\inf_{k\geq n} A_k := \bigcap_{k=n}^{\infty} A_k \quad \text{which leads to}$$

$$\liminf_{n\to\infty} A_n = \lim_{n\to\infty} \left\{ \inf_{k\geq n} A_k \right\}$$