Introduction to Astrophysics and Cosmology

Relativistic Cosmology - Elements of GR

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The metric for weak gravitational field

After discussing the mathematics, now we are going to discuss the physics of general relativity.

Suppose we consider the **motion of a non-relativistic particle in a weak gravitational field**.

If the gravitational field **is sufficiently weak, we can certainly treat the motion of the particle by using the Newtonian theory of gravity**.

We can, however, **use general relativity also to solve this problem**. Both the Newtonian theory and general relativity **should give the same result in the regime where both are valid.**

We introduce some first concepts of general relativity by discussing how it can be applied to study the motion of a non-relativistic particle in a weak gravitational field for which the Newtonian theory is adequate.

The metric for weak gravitational field

The motion of **a particle in a gravitational field** is obtained in general relativity by using the fact that the **motion has to be along a geodesic**.

The geodesic between two spacetime points A and B is the path along which the path length $s = \int_A^B ds$ is an extremum, ds being given by (12.7). So the **path of a particle** A in the four-dimensional spacetime between the two points A and B can be obtained **in general relativity by making s an extremum**.

On the other hand, classical mechanics tells us that we **can solve the motion by applying Hamilton's principle that the action** $I = \int_A^B L dt$ has to be an extremum between A and A B, **where L is the Lagrangian**.

In this case, both classical mechanics and general relativity should give correct results, and the results should be identical.

The metric for weak gravitational field

In other words, the extremum of $s = \int_A^B ds$ and the extremum of $I = \int_A^B L dt$ should give the same path. This is possible **only if *ds* for a weak gravitational field in general relativity is essentially the same thing as *L dt* in classical mechanics** (except for any additive or multiplicative constants).

We know that the Lagrangian L for a particle moving in a gravitational potential Φ is given by

$$L = \frac{1}{2}mv^2 - m\Phi$$

Hence the **non-relativistic action** can be taken to be

$$I_{\mathrm{NR}} = \int_A^B \left[\frac{1}{2}mv^2 - m\Phi\right] dt - mc^2 \int_A^B dt. \qquad (12.57)$$

The metric for weak gravitational field

$$I_{\mathrm{NR}} = \int_A^B \left[\frac{1}{2}mv^2 - m\Phi\right] dt - mc^2 \int_A^B dt. \qquad (12.57)$$

The **last term has a constant value** $-mc^2(t_B - t_A)$ and **would not contribute** in the calculation of **the extremum**. As we proceed further, it will be clear why we are including this term.

We shall now try to figure out how to write ds in this situation such that the extremum of $\int_A^B ds$ gives the same result as the extremum of I_{NR} given by (12.57).

Let us first look at the case of **zero gravitational field.** In the absence of a gravitational field, **general relativity reduces to special relativity**.

We write the **spacetime coordinates** as $x^0 = ct, x^1, x^2, x^3$ in view of the fact that **their differentials make up a contravariant vector** and we should use superscripts. If ***ds* is the separation between two nearby spacetime events**, we know that $\boldsymbol{ds^2}$, which is often written as $-c^2 d\tau^2$, can be written in the special relativistic situation in the form

$$\boxed{ds^2 = -c^2 d\tau^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \qquad (12.58)}$$

The metric for weak gravitational field

$$\boxed{ds^2 = -c^2 d\tau^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \qquad (12.58)}$$

We can write this as

$$ds^2 = -c^2 d\tau^2 = \eta_{ik}\, dx^i\, dx^k, \tag{12.59}$$

where the **special relativistic metric η_{ik}** is given by

$$\eta_{ik} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12.60}$$

One can easily check that a **geodesic has to be a straight line** in this **spacetime (which is flat)**. A straight line in the special relativistic spacetime **corresponds to the world-line of a uniformly moving particle.**

Thus, if the geodesics give the paths which particles follow, we come to the conclusion that **particles move uniformly in the absence of a gravitational field.**

The metric for weak gravitational field

It is clear that the extremum of $\int_A^B d\tau$, with $d\tau$ given by (12.58), would give the geodesic in the special relativistic case (which is a straight line). Let us now multiply $\int_A^B d\tau$ by $-mc^2$ and argue that the resulting quantity

$$I_{\Phi=0} = -mc^2 \int_A^B d\tau \qquad (12.61)$$

should be **the action for a free particle** (i.e. a particle **in zero gravitational field) in classical mechanics**.

If (12.61) indeed gives the classical action, then it would follow that **a geodesic is a path along which the action is also an extremum** and the path followed by the particle according to classical mechanics would then be a **geodesic in spacetime**.

Suppose we consider a **moving particle** which is at spacetime points (x^0, x^1, x^2, x^3) and $(x^0+dx^0, x^1+dx^1, x^2+dx^2, x^3 + dx^3)$ at the beginning and the end of a short interval. The **velocity of the particle** is given by

$$v^2 = \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{dt^2} = c^2 \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(dx^0)^2}, \qquad (12.62)$$

The metric for weak gravitational field

since $dx^0 = cdt$. It then follows from (12.58) that

$$d\tau = \frac{dx^0}{c}\sqrt{1-\frac{v^2}{c^2}} = dt\sqrt{1-\frac{v^2}{c^2}}. \tag{12.63}$$

On substituting this in (12.61), we have

$$I_{\Phi=0} = -mc^2\int_A^B dt\sqrt{1-\frac{v^2}{c^2}}. \tag{12.64}$$

If $\mathbf{v}^2 \ll c^2$, then we can make a Taylor series expansion of the square root, which would give

$$I_{\Phi=0} \approx -mc^2\int_A^B dt + \int_A^B \frac{1}{2}mv^2\, dt.$$

The metric for weak gravitational field

This is the same as I_{NR} given by (12.57) after setting $\Phi = 0$. Thus **the non-relativistic limit of (12.61) gives the usual non-relativistic action for a free particle**.

So, if we take (12.61) as our action, we shall get the correct path in the non-relativistic situation, which would be a geodesic in spacetime. Although we are here discussing the motion of a non-relativistic particle, it may be mentioned that **(12.61) is the correct expression for the action of a free particle even when the particle moves relativistically**.

When a **weak gravitational field is present,** we need to **add a part due to gravitational interaction in (12.61).** Let us take

$$I = -mc^2 \int_A^B d\tau - \int_A^B m\Phi\, dt$$

which, by virtue of (12.63), becomes

$$I = -\int_A^B dt \left[mc^2 \sqrt{1 - \frac{v^2}{c^2}} + m\Phi \right]. \qquad (12.65)$$

The metric for weak gravitational field

The form of (12.65) clarifies the **quantitative meaning of a weak gravitational field.** The **extra term** in the action **due to the gravitational field should be small** compared to the rest of the action.

It follows from (12.65) that the condition for a weak gravitational field is

$$\boxed{\Phi \ll c^2. \qquad (12.66)}$$

The Newtonian theory of gravity to be adequate is that f defined by (1.11) is small compared to 1.

It is easy to see that this is effectively the same condition as (12.66). When (12.66) is satisfied, (12.65) must be approximately equivalent to

$$I \approx -mc^2 \int_A^B dt \sqrt{1 - \frac{v^2}{c^2} + \frac{2\Phi}{c^2}}.$$

$$\boxed{f = \frac{2GM}{c^2 r} \qquad (1.11)}$$

The metric for weak gravitational field

$$I \approx -mc^2 \int_A^B dt \sqrt{1 - \frac{v^2}{c^2} + \frac{2\Phi}{c^2}}.$$

This can be written as

$$I \approx -mc \int_A^B \sqrt{\left(1 + \frac{2\Phi}{c^2}\right) c^2\, dt^2 - v^2\, dt^2}.$$

On making use of (12.62), this becomes

$$I \approx -mc \int_A^B \sqrt{\left(1 + \frac{2\Phi}{c^2}\right) (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}. \quad (12.67)$$

If we take the **metric of the four-dimensional spacetime** to be

$$\boxed{ds^2 = -c^2 d\tau^2 = -\left(1 + \frac{2\Phi}{c^2}\right) (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad (12.68)}$$

The metric for weak gravitational field

then the **geodesics in this spacetime will coincide with the paths we would get by making the action given by (12.67) an extremum.**

Since (12.67) in turn reduces to (12.57) in the non-relativistic situation with a weak gravitational field, we conclude that **a particle moving non-relativistically in a weak gravitational field**, of which the motion can be found by making (12.57) an extremum, **should follow a geodesic in the spacetime described by (12.68)**.

This suggests that (12.68) is the metric of spacetime with a weak gravitational field, where **Φ is the classical gravitational potential**. We can write (12.68) as

$$ds^2 = g_{ik}\, dx^i\, dx^k, \tag{12.69}$$

where the **metric tensor g_{ik}** in the presence of a **weak gravitational field** is given by

$$g_{ik} = \begin{pmatrix} -\left[1 + \frac{2\Phi}{c^2}\right] & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12.70}$$

The metric for weak gravitational field

It is obvious that the general relativistic metric tensor (12.70) reduces to the special relativistic metric tensor (12.60) in the absence of the gravitational field.

It is instructive to work out the **geodesic equation (12.51) for the metric tensor** given by (12.70) and to show that it is the same as the classical equation for non-relativistic motion in a weak gravitational field.

In the following discussion, we shall use the superscript **α to indicate indices 1, 2 or 3,** but not 0. For non-relativistic motion, the change dx^α in the particle's position during an interval dt **has to be much smaller than $dx^0 = cdt$.** Hence the dominant terms in (12.51) are

$$\frac{d^2x^m}{ds^2} = -\Gamma^m_{00}\frac{dx^0}{ds}\frac{dx^0}{ds}, \qquad (12.71)$$

where there is no summation now.

To proceed further, we **have to calculate the Christoffel symbol** Γ^m_{00} , which involves derivatives of the metric tensor as seen in (12.31).

The metric for weak gravitational field

If the gravitational potential Φ is independent of time, then the only non-zero derivative of the metric tensor (12.70) is

$$\frac{\partial g_{00}}{\partial x^\alpha} = -\frac{2}{c^2}\frac{\partial \Phi}{\partial x^\alpha}.$$

On using this, we find from (12.31) that

$$\Gamma^\alpha_{00} = \frac{1}{c^2}\frac{\partial \Phi}{\partial x_\alpha}, \tag{12.72}$$

if we neglect terms quadratic in the small quantity Φ. Substituting this in (12.71) and making use of the fact $ds = i\, cd\tau$, we get

$$\frac{d^2 x^\alpha}{d\tau^2} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial x_\alpha}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}. \tag{12.73}$$

The metric for weak gravitational field

It follows from (12.62) and (12.68) that

$$c\,d\tau = dx^0\sqrt{1+\frac{2\Phi}{c^2}-\frac{v^2}{c^2}}. \tag{12.74}$$

For non-relativistic motions in a weak gravitational field for which we neglect quadratics of Φ, we need to substitute

$$\frac{dx^0}{d\tau} = c$$

in (12.73) and take $d\tau = dt$. Then (12.73) gives

$$\frac{d^2x^\alpha}{dt^2} = -\frac{\partial\Phi}{\partial x_\alpha},$$

which is the classical equation of motion.

This completes our proof that general relativity and ordinary classical mechanics would give the same result if (12.68) is the metric for a weak gravitational field.

General relativity

$$\boxed{G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R. \qquad (12.44)}$$

We are now ready to present the complete formulation of general relativity.

The central equation of general relativity is Einstein's equation telling us **how the curvature of spacetime is related to the density of mass-energy present in spacetime.**

We have introduced several tensors associated with the curvature of space. It is the **Einstein tensor G_{ik}** defined in (12.44) which turns out to be a particularly convenient tensor in the basic formulation of the theory.

Suppose we are able to find a suitable **second-rank tensor describing the mass-energy density.** Then **making G_{ik} proportional to this tensor would give us an equation that would imply that the curvature of spacetime is caused by mass-energy**.

Since **the divergence of G_{ik} is zero** according to (12.45), **the divergence of the tensor giving the mass-energy density also has to be zero** for the sake of consistency.

This tensor is called the *energy-momentum tensor*.

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0. \quad (8.3)$$

The energy-momentum tensor

For the time being, let us forget about relativity and show that **the classical hydrodynamic equations can be put in a form such that the divergence of a second-rank tensor is zero**. Then we shall consider how to generalize this tensor to general relativity and thereby obtain the energy-momentum tensor.

As in the previous section, let us write **x^0 for ct and x^1, x^2, x^3 for the three spatial coordinates**. The **Roman** indices **i, j,... will run over the values 0, 1, 2, 3,** whereas the **Greek indices α, β, . . . will run over only 1, 2, 3.**

We have pointed out in §12.2.1 that **the generalized velocity transforms as a contravariant vector.** So we shall write the **velocity components with indices at the top indicative of contravariant tensors**. It is easy to see that the **continuity equation (8.3)** can be written in the form

$$\frac{\partial S^i}{\partial x^i} = 0, \quad (12.75)$$

where S^i is a 4-vector with components $(\rho c, \rho \mathrm{v}^1, \rho \mathrm{v}^2, \rho \mathrm{v}^3)$ and the index i repeated twice implies that we are summing over $0, 1, 2, 3$. We have

$$\frac{\partial}{\partial t}(\rho v^\alpha) = v^\alpha \frac{\partial \rho}{\partial t} + \rho \frac{\partial v^\alpha}{\partial t}.$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \mathbf{F}. \qquad (8.9)$$

The energy-momentum tensor

Let us now **substitute for $\partial\rho/\partial t$ from the continuity** equation (8.3) and for **$\partial v^\alpha/\partial t$ from the Euler** equation (8.9) after setting the external force **F** to zero. This gives

$$\frac{\partial}{\partial t}(\rho v^\alpha) = -v^\alpha \frac{\partial}{\partial x^\beta}(\rho v^\beta) - \rho v^\beta \frac{\partial v^\alpha}{\partial x^\beta} - \frac{\partial P}{\partial x_\alpha},$$

where a Greek index α or β repeated twice signifies summation over only the spatial components 1, 2, 3. It is easy to see that the above equation **can be written in the form**

$$\frac{\partial}{\partial t}(\rho v^\alpha) + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0, \qquad (12.76)$$

where

$$T^{\alpha\beta} = P\,\delta^{\alpha\beta} + \rho v^\alpha v^\beta. \qquad (12.77)$$

The energy-momentum tensor

We can now combine (12.75) and (12.76) in the compact form

$$\frac{\partial(\mathcal{T}_{\mathrm{NR}})^{ik}}{\partial x^k} = 0, \tag{12.78}$$

where $(\mathrm{T}_{\mathrm{NR}})^{ik}$ is the **non-relativistic four-dimensional energy-momentum tensor** of which the various **components** are given by

$$(\mathcal{T}_{\mathrm{NR}})^{00} = \rho c^2, \quad (\mathcal{T}_{\mathrm{NR}})^{0\alpha} = (\mathcal{T}_{\mathrm{NR}})^{\alpha 0} = \rho c v^\alpha, \quad (\mathcal{T}_{\mathrm{NR}})^{\alpha\beta} = T^{\alpha\beta}. \tag{12.79}$$

It should be noted that we have not invoked relativity in obtaining (12.78). Writing x^0 for ct has been merely a matter of notation. **The equation (12.78) combines the equations of continuity and motion of classical hydrodynamics**, showing that we can have a divergenceless second-rank tensor $(\mathrm{T}_{\mathrm{NR}})^{ik}$ in the non-relativistic situation. We now **have to generalize $(\mathrm{T}_{\mathrm{NR}})^{ik}$ to obtain the fully relativistic energy-momentum tensor.**

The energy-momentum tensor

Let us first consider how we **generalize the concept of velocity in general relativity**. Suppose a particle has positions x^i and $x^i + dx^i$ before and after an infinitesimal interval. **The difference dx^i is a 4-vector** and the quotient obtained by dividing it by a scalar will be a 4-vector as well. As in (12.58) and (12.68), we introduce the **time-like interval $d\tau$** defined through

$$ds^2 = -c^2 d\tau^2 = g_{ik}\, dx^i\, dx^k. \tag{12.80}$$

$d\tau \to dt$ for non-relativistic motion in a weak gravitational field. Since $d\tau$ as introduced in (12.80) **must be a scalar**, dx^i divided by $d\tau$ should give us a 4-vector.

We now define the **relativistic velocity 4-vector** as

$$u^i = \frac{1}{c}\frac{dx^i}{d\tau}. \tag{12.81}$$

In the **non-relativistic limit,** this clearly reduces to

$$u^i \to \left(1, \frac{v^1}{c}, \frac{v^2}{c}, \frac{v^3}{c}\right). \tag{12.82}$$

$$T^{\alpha\beta} = P\,\delta^{\alpha\beta} + \rho v^\alpha v^\beta. \tag{12.77}$$

The energy-momentum tensor

One interesting property of the velocity 4-vector is that

$$u^i u_i = -1. \tag{12.83}$$

This can be easily proved if we use (12.14) to obtain u_i from u^i and then use (12.80)–(12.81).

We now define the **energy-momentum tensor**

$$\boxed{\mathcal{T}^{ik} = \rho c^2 u^i u^k + P(g^{ik} + u^i u^k). \qquad (12.84)}$$

The quantities like ρ **and** P **are defined with respect to the rest frame**. Since T^{ik} **is a proper relativistic second-rank tensor** and reduces to the non-relativistic expression in the appropriate limit, we take it to be the **relativistic generalization of the energy-momentum tensor.**

The energy-momentum tensor

$$\frac{\partial(\mathcal{T}_{\mathrm{NR}})^{ik}}{\partial x^k} = 0, \tag{12.78}$$

Now we can **generalize the** equation (12.78) also. We need to replace the ordinary derivative by the covariant derivative apart from replacing $(\mathbf{T}_{\mathbf{NR}})^{ik}$ by T^{ik}. This gives

$$\boxed{\frac{D\mathcal{T}^{ik}}{Dx^k} = 0. \qquad (12.85)}$$

Now we have a properly relativistic divergenceless second-rank tensor, which can be presumed to **act as the source of curvature of spacetime.**

Before leaving the discussion of the energy-momentum tensor, let us consider **one important special case of this tensor**. We consider **a fluid at rest**. Then the spatial **components of the 4-velocity must be zero**, i.e.

$$u^i = (u^0, 0, 0, 0). \tag{12.86}$$

$$\boxed{ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \qquad (10.19)}$$

The energy-momentum tensor

It then follows from (12.83) that

$$u^0 u_0 = -1. \qquad (12.87)$$

On lowering the index k in (12.84) by the usual procedure (12.14), we have

$$\mathcal{T}^i_k = \rho c^2 u^i u_k + P\,(\delta^i_k + u^i u_k).$$

On making use of (12.86) and (12.87), this gives

$$\boxed{\mathcal{T}^i_k = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \qquad (12.88)}$$

The Robertson–Walker metric (10.19) corresponds to a co-moving coordinate system in which the material of the Universe is assumed to be at rest. So **the energy-momentum tensor of the Universe is given by (12.88) when we use the co-moving coordinate system.**

Einstein's equation

Since G_{ik} defined in (12.44) and T^{ik} defined in (12.84) are **both divergenceless tensors,** one being a **measure of the curvature of spacetime** and the other being a **measure of energy-momentum density,** it is tempting to write

$$G_{ik} = \kappa T_{ik}, \qquad (12.89)$$

where κ is a constant.

This equation would imply that the curvature of spacetime is produced by the energy-momentum density, which is a basic **requirement of general relativity**.

It should be emphasized that we have not 'derived' (12.89), but have given a string of arguments that such an equation may be expected. Since the divergences of both sides are zero, there would not be any mathematical inconsistency when we take a divergence of (12.89). **Whether (12.89) is a really correct equation can be determined only by checking if results are confirmed by experiments.**

To complete our discussion, we need to determine the constant κ. If we can determine the value of κ in a special case, then that would be the true value of κ in all situations if it is a universal constant. We have discussed in §12.3 how general relativity can be formulated in the case of a weak gravitational field. We shall now **determine κ by applying (12.89) to a weak gravitational field** and comparing it with the Newtonian theory of gravity.

Einstein’s equation

Using (12.44) we can write (12.89) in the form

$$R^i_k - \frac{1}{2}\delta^i_k R = \kappa \mathcal{T}^i_k. \tag{12.90}$$

We now carry on a **contraction between the indices *i* and *k***, keeping in mind that $\delta_i = 4$ because of the four dimensions of spacetime, allowing i to have values 0, 1, 2, 3. This gives

$$-R = \kappa \mathcal{T},$$

where $\mathscr{T} = \mathscr{T}^i$. On substituting $-\kappa\mathscr{T}$ for R in (12.90), we have

$$R^i_k = \kappa \left(\mathcal{T}^i_k - \frac{1}{2}\delta^i_k \mathcal{T} \right).$$

Einstein's equation

We shall now consider the following particular component of this equation

$$R_0^0 = \kappa \left(\mathcal{T}_0^0 - \frac{1}{2}\delta_0^0 \mathcal{T} \right). \tag{12.91}$$

Let us **apply** (12.91) **to the case of a weak gravitational field** produced by a distribution of matter at rest in our coordinate system. The expression for the energy-momentum tensor for matter at rest is given by (12.88). If $P \ll \rho c^2$, then

$$\mathcal{T} \approx \mathcal{T}_0^0 = -\rho c^2.$$

On substituting this in (12.91), we get

$$R_0^0 = -\frac{1}{2}\kappa\rho c^2. \tag{12.92}$$

Einstein's equation

Now we need to find the expression for R_0 from the metric (12.68) for the weak gravitational field. It follows from (12.41) that

$$R_{00} = R^i_{0i0} = \frac{\partial \Gamma^i_{00}}{\partial x^i} - \frac{\partial \Gamma^i_{0i}}{\partial x^0}, \tag{12.93}$$

if we **neglect the quadratic terms in Christoffel symbols** for a weak gravitational field.

Since we are considering **matter to be at rest**, the **field should be independent of time and any derivative with respect to $x^0 = ct$ should give zero**. Then (12.93) reduces to

$$R_{00} = \frac{\partial \Gamma^\alpha_{00}}{\partial x^\alpha},$$

where α repeated twice implies summation over 1, 2, 3 as usual.

The Christoffel symbol Γ^α_{00} for a weak gravitational field was already determined in (12.72). On substituting this,

$$R_{00} = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{c^2}\frac{\partial \Phi}{\partial x_\alpha}\right) = \frac{1}{c^2}\nabla^2\Phi.$$

Einstein's equation

To get R_0, we merely have to multiply R_{00} by $g^{00} \approx -1$, since terms quadratic in Φ are neglected. It then follows from (12.92) that

$$\nabla^2 \Phi = \frac{\kappa c^4}{2} \rho. \tag{12.94}$$

The **Newtonian theory of gravity** leads to the gravitational **Poisson equation**

$$\nabla^2 \Phi = 4\pi G \rho.$$

Comparing this with (12.94), we finally conclude that

$$\boxed{\kappa = \frac{8\pi G}{c^4}. \qquad (12.95)}$$

On substituting the value of κ in (12.89), we have the famous ***Einstein equation*** and tells **how matter-energy acts as a source of the curvature of spacetime.**

$$\boxed{G_{ik} = \frac{8\pi G}{c^4} T_{ik}. \qquad (12.96)}$$

Einstein's equation

The compact tensorial notation makes Einstein's equation (12.96) appear deceptively simple. It happens to be one of the most difficult equations to handle. Since **particles move along geodesics** and we **need a knowledge of the metric tensor to determine the geodesics,** most of the practical **problems in general relativity require the determination of the metric tensor for a given matter-energy distribution.**

The connection between the metric tensor g_{ik} and the Einstein tensor G_{ik} follows from (12.31), (12.41) and (12.44). If we know the energy-momentum tensor T_{ik} in a particular situation, Einstein's equation (12.96) at once gives us the Einstein tensor G_{ik}.

But **determining the metric tensor g_{ik} after that is not an easy** job. There are very few cases of practical importance where one can determine the metric tensor that would satisfy Einstein's equation.