Introduction to Astrophysics and Cosmology

Radiation transfer

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Formation of spectral lines

- To understand how spectral lines form, the grey atmosphere model is not adequate, since the flux F_ν associated with frequency v is not in general a constant, even when the total flux F is a constant
- In the case of the general non-grey stellar atmosphere, we have frequency-dependent equations
- If there is no source or sink in the stellar atmosphere, then a constant energy passes through the layers of stellar atmosphere, but the energy continuously gets redistributed amongst different frequencies.
  - For example, in the interior of the Sun where the temperature is of order 10^7K, the radiation field is mainly made up of X-ray photons. By the time the energy flux reaches the solar surface, it mainly consists of visible light.

Formation of spectral lines

- In a rigorous treatment of stellar atmospheres, it is also necessary to split the absorption coefficient α_v into two parts:
  - scattering and
  - true absorption.
- In radiative equilibrium, true absorption is followed by complete re-emission of the absorbed radiation.
- One important difference between scattering and true absorption is that
  - in scattering the frequency of radiation does not change
  - in true absorption followed by re-emission the frequency changes. Also, Kirchhoff's law is applicable only in the case of true absorption.

Formation of spectral lines

$$I_\nu(0, 1) = B_\nu(\tau_\nu = 0) + \frac{dB_\nu}{d\tau_\nu}.$$

- If we expand $B_\nu(\tau_\nu = 1)$ in a Taylor series around $\tau_\nu = 0$ and keep only the linear term, then it becomes equal to the right-hand side of the above equation

$$I_\nu(0, 1) \approx B_\nu(\tau_\nu = 1).$$

- the specific intensity of radiation at a frequency ν coming out of a stellar atmosphere is approximately equal to the Planck function at a depth of the atmosphere where the optical depth for that frequency ν equals unity.

Formation of spectral lines

idealized situation: the absorption coefficient in the outer layers of a stellar atmosphere is equal to α_C at all frequencies except a narrow frequency range around ν_L where it has a larger value α_L

- For frequencies in the continuum outside the spectral line, the optical depth becomes unity at a depth α_C^{-1}. If the temperature there is T_C,

- the spectrum in the continuum region will be like the blackbody spectrum $B_\nu(T_C)$.

- For frequencies within the spectral line, the optical depth becomes unity at a shallower depth α_L^{-1}, where the temperature must have a lower value T_L
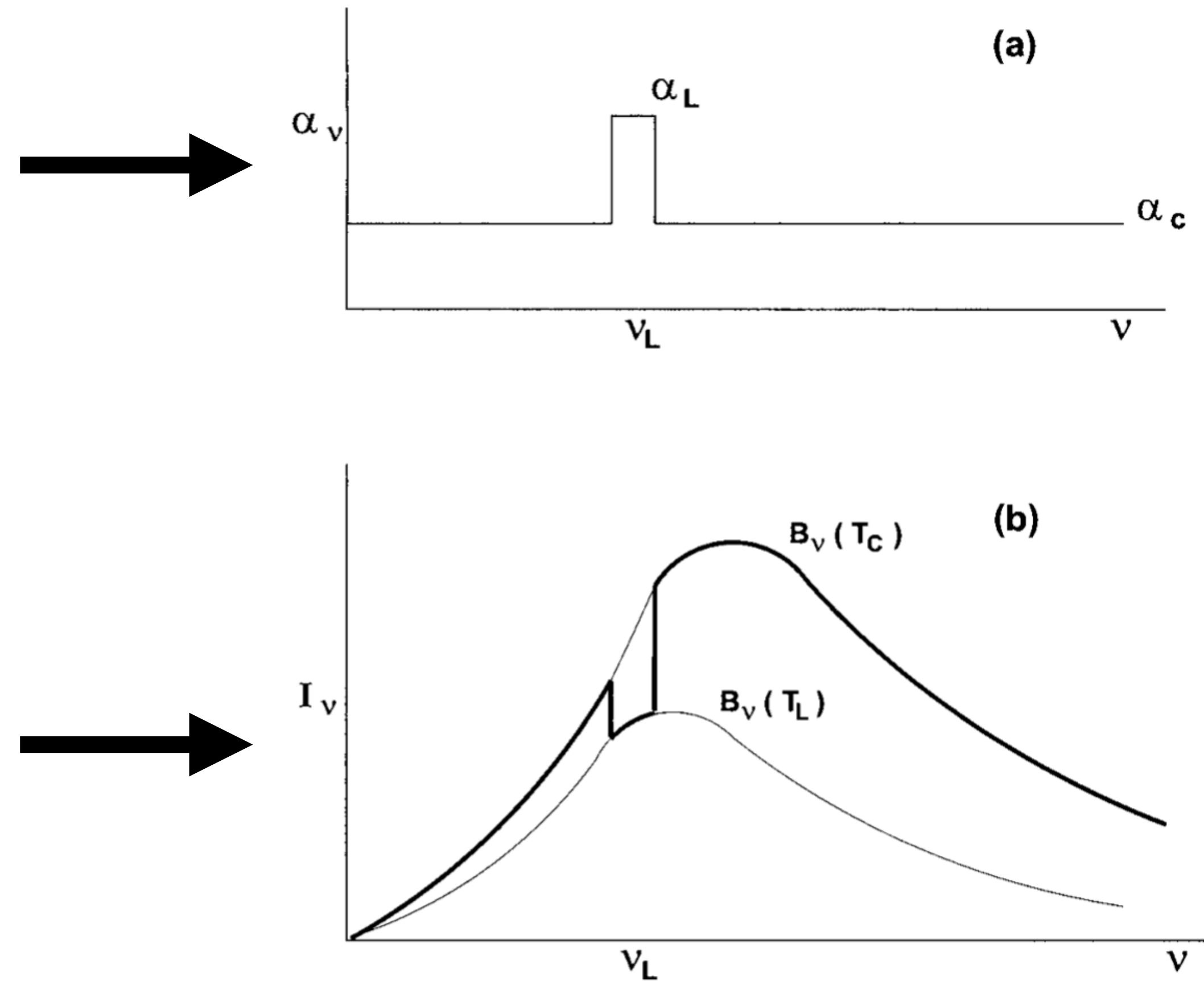
Fig. 2.7 For an absorption coefficient indicated in (a), the emergent spectrum is indicated in (b) by the dark line. The two blackbody spectra shown in (b) correspond to the temperatures T_C (upper curve) and T_L (lower curve) explained in the text.

Formation of spectral lines

the specific intensity in the continuum is given by $B_\nu(T_C)$ and the specific intensity in the spectral line by $B_\nu(T_L)$, the full spectrum looks as indicated by the dark line

Since any object radiating from the surface is expected to have a temperature gradient in the layers underneath the surface, the existence of spectral lines should be a very common occurrence.
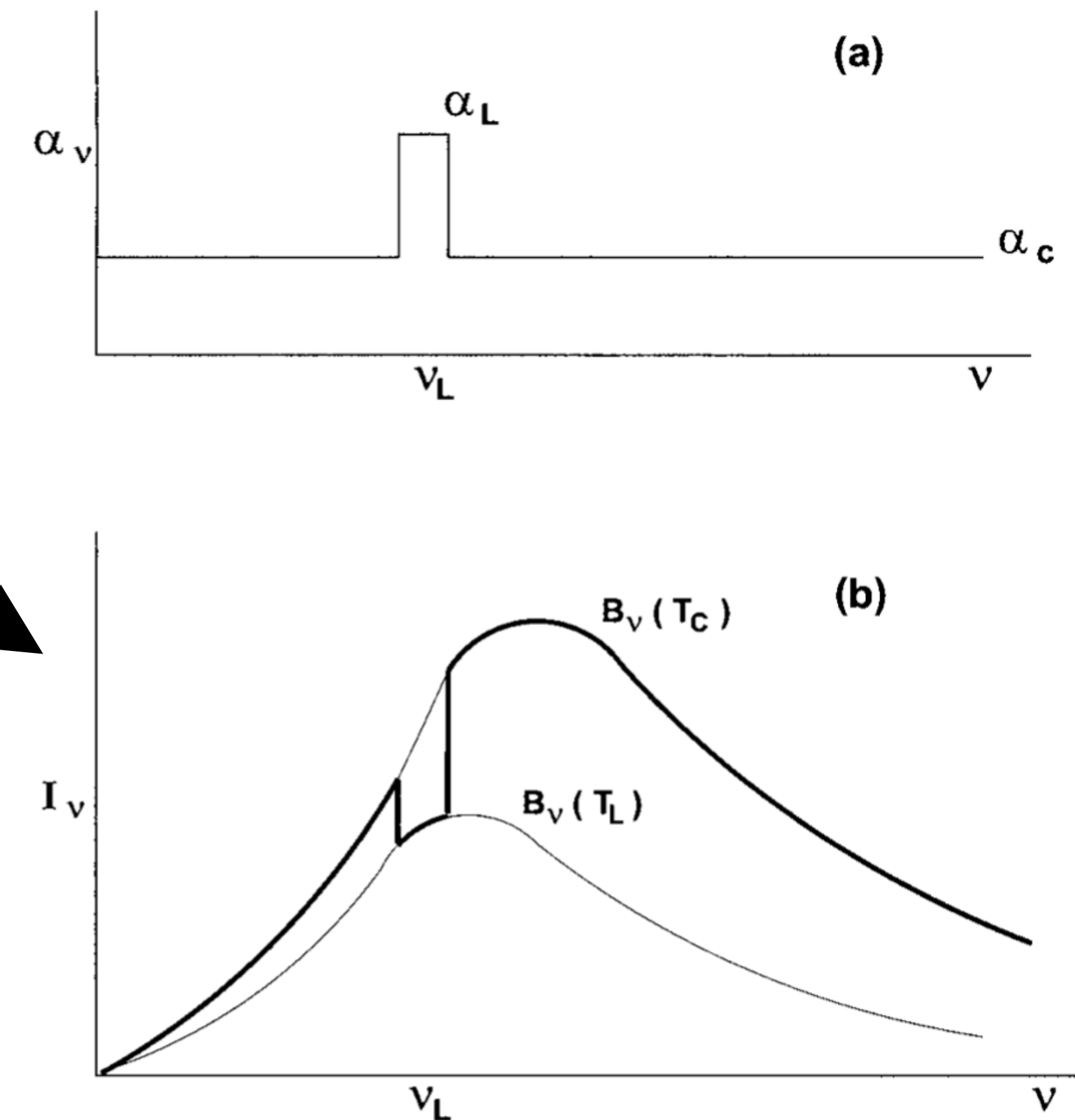

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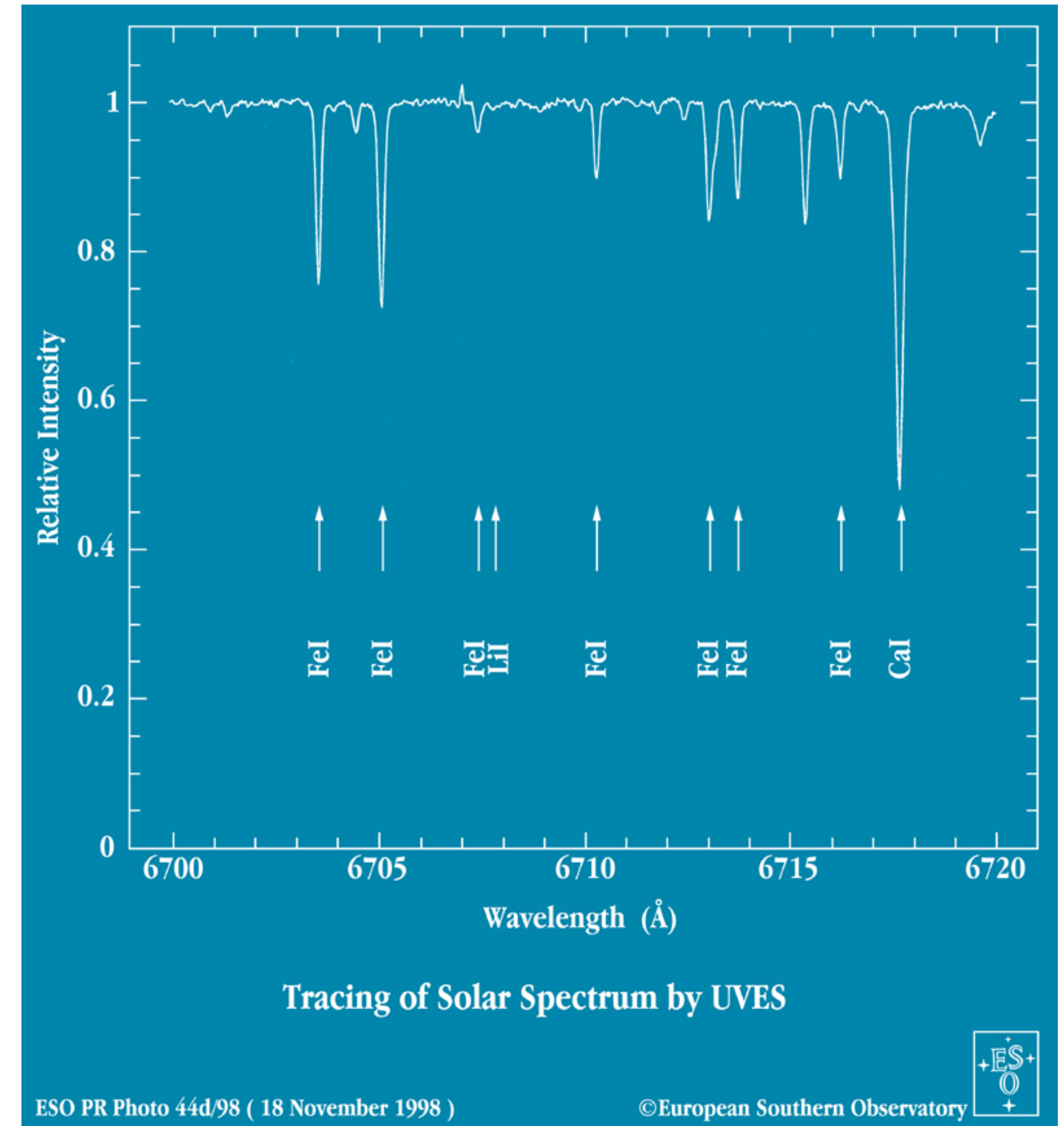
Formation of spectral lines

- One of the aims of stellar atmosphere studies is to estimate the abundances of various elements in an atmosphere from a spectroscopic analysis.

- Suppose an element has a spectral line at frequency ν_L. Then, the larger the number of atoms of that element per unit volume, the higher will be the value of the absorption coefficient α_L at that frequency and consequently the spectral line will be stronger.

- From the strengths of spectral lines, it is possible to calculate the abundances of elements.

Spectrum of an A0 star
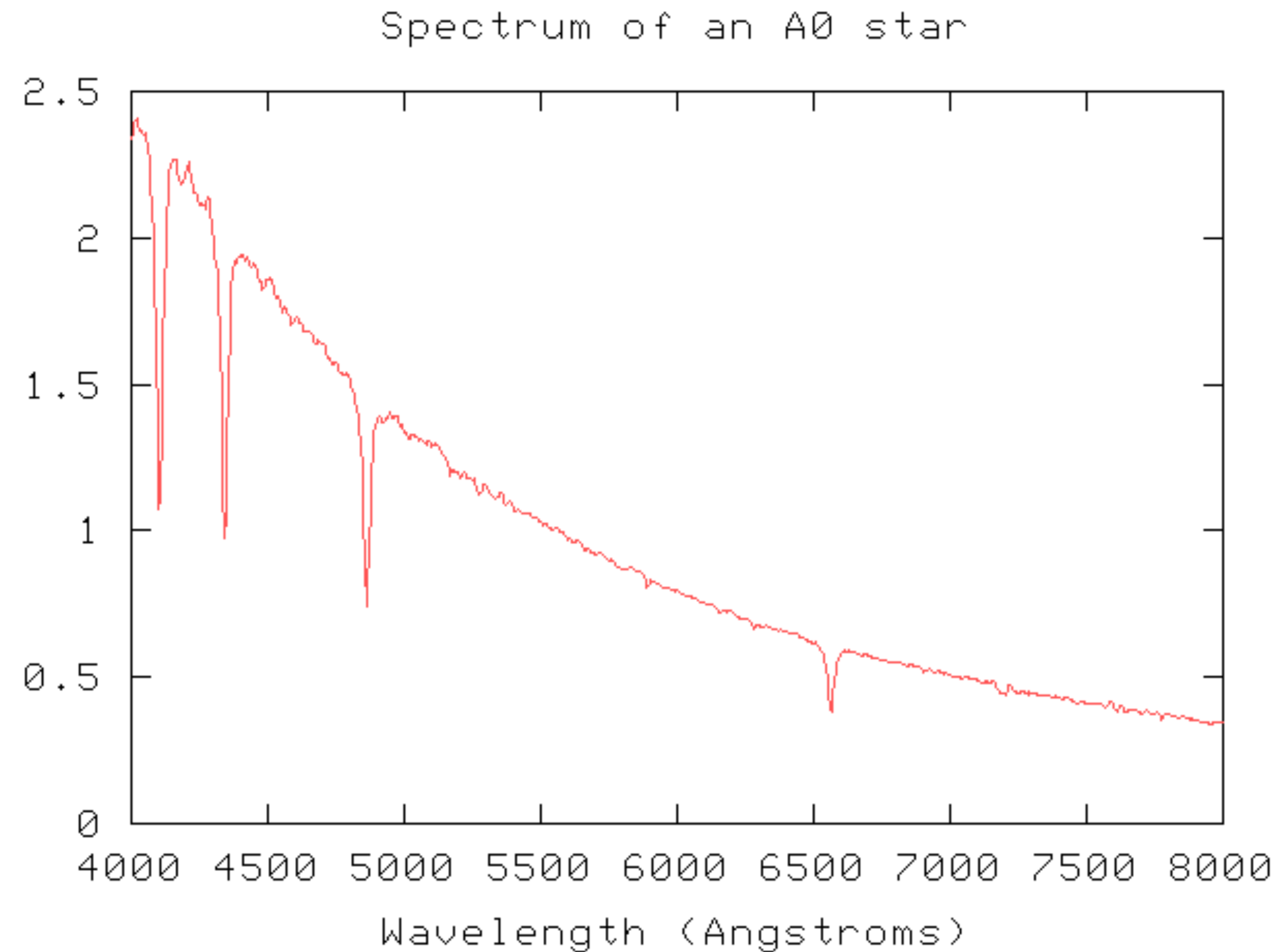
Formation of spectral lines

- This is a normalised spectrum -> the blackbody radiation component was removed to make it easier to measure the spectral lines

- The Solar atmosphere contains: iron (Fe), calcium (Ca) and lithium (Li)

- since these lines are only produced by neutral atoms (denoted by "I"), there must be a layer of the solar atmosphere which isn't hot enough to ionize these atoms

- detailed analysis of the intensities of these (and other) lines provides an estimate of the relative abundance of each element. In the Sun, about 25,000 iron atoms and 2,500 calcium atoms for every 1 lithium atom.

Tracing of Solar Spectrum by UVES

ESO PR Photo 44d/98 (18 November 1998)   ©European Southern Observatory

http://spiff.rit.edu/classes/phys230/lectures/spec_interp/spec_interp.html

Formation of spectral lines

- The upper blue portion of this graph shows the spectral lines ALMA detected in a star-forming region of the Cat's Paw Nebula. The lower black portion shows the lines detected by the European Space Agency's Herschel Space Observatory. The ALMA observations detected more than ten times as many spectral lines. Note that the Herschel data have been inverted for comparison. Two molecular lines are labeled for reference.
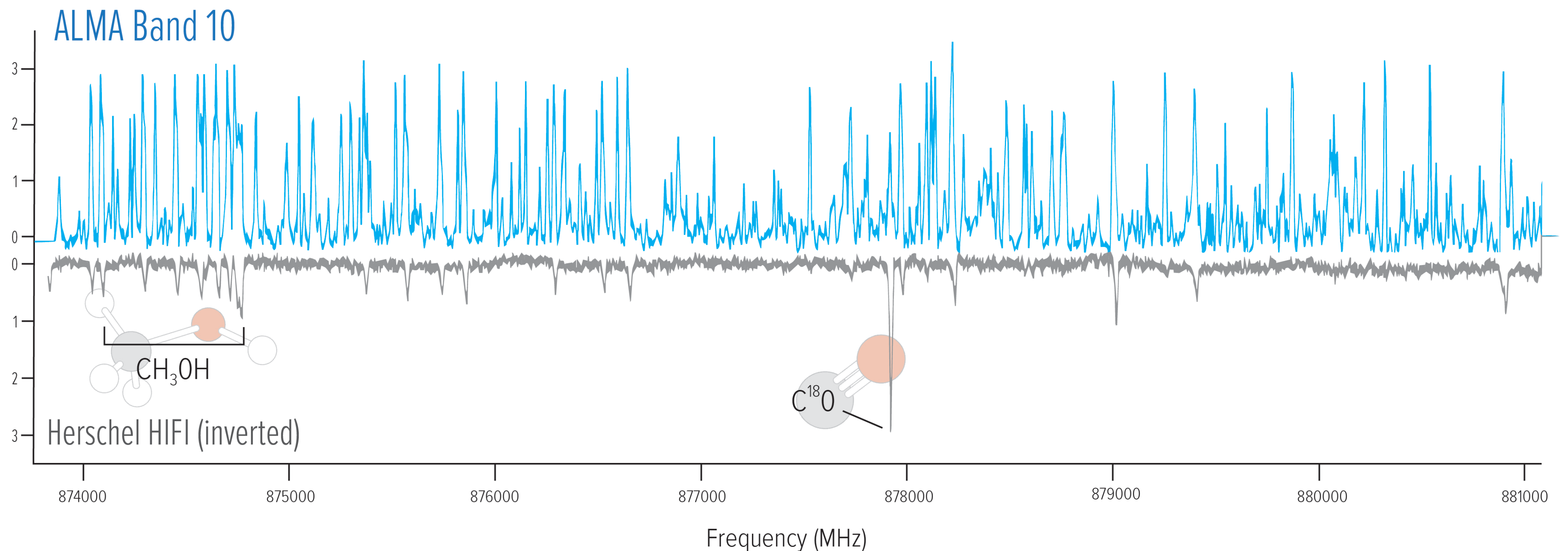
Radiative energy transport in the stellar interior

- In a typical star, energy is usually produced by nuclear reactions in the innermost core of the star. This energy in the form of radiation is then transported outward.
  - Energy can also be transported by convection (we will cover that in stellar physics)
- How an energy flux is driven outward by the gradient of the radiation field?
- for a non-grey atmosphere if an average of α_ν over ν is taken, then the relation between the radiation pressure and the flux is:

$$\frac{dP_\nu}{d\tau_\nu} = \frac{F_\nu}{c}, \quad \longrightarrow \quad F_\nu = -\frac{c}{\alpha_\nu}\frac{dP_\nu}{dz}$$

Integrating over all frequencies:
$$F = \int F_\nu d\nu = -c \int \frac{1}{\alpha_\nu}\frac{dP_\nu}{dz}d\nu.$$

Radiative energy transport in the stellar interior

- We can write the flux the following way, where α_R is an average of α_ν:

$$F = -c\frac{1}{\alpha_R}\frac{dP}{dz}, \longrightarrow \frac{1}{\alpha_R} = \frac{\int \frac{1}{\alpha_\nu}\frac{dP_\nu}{dz}d\nu}{\int \frac{dP_\nu}{dz}d\nu}.$$

$$\frac{dP_\nu}{dz} = \frac{4\pi}{3c}\frac{\partial B_\nu}{\partial T}\frac{dT}{dz}. \longrightarrow \frac{1}{\alpha_R} = \frac{\int \frac{1}{\alpha_\nu}\frac{\partial B_\nu}{\partial T}d\nu}{\int \frac{\partial B_\nu}{\partial T}d\nu}.$$

The mean absorption coefficient α_R defined in this way is known as the *Rosseland mean*

Radiative energy transport in the stellar interior

α_R is often expressed as: $\alpha_{\mathrm{R}} = \rho \chi,$

χ is the opacity of the stellar matter

$$\longrightarrow \qquad F = -\frac{c}{\chi \rho} \frac{d}{dz} \left(\frac{a_{\mathrm{B}}}{3} T^4 \right).$$

Opacity

- To build a model of the stellar interior, it is necessary to know the value of opacity χ.

- The gas in the interior of a star exists under such conditions of temperature and pressure which cannot be reproduced in the laboratory.

- The opacity χ , therefore, has to be calculated theoretically.

- This is a fairly complicated calculation. With improvements in stellar models, more and more accurate computations of opacity are demanded. This has become a highly specialized and technical subject. Astronomers generally use calculated values from tables published by experts on the field.

Opacity

- **Brief method to calculate the opacity:**

  - If we have a gas of a certain composition kept at a given density and temperature.

  - We can apply the Boltzmann law and the Saha equation to find out the numbers of atoms and electrons in various energy levels and in various stages of ionization.

  - When electromagnetic radiation of frequency ν enters the system, atoms can absorb this radiation if electrons associated with the atoms are pushed to levels which have energies higher by an amount $h\nu$ compared to previous levels.

  - We know from quantum mechanics that atomic energy levels can be either bound (discrete levels) or free (continuum). The absorption of radiation by an atom can be due to three kinds of upward electronic transitions: (i) bound-bound, (ii) bound-free and (iii) free-free.

  - One can apply Fermi's golden rule of quantum mechanics with a semiclassical treatment of radiation to calculate the absorption cross-sections for these processes

  - Finally one adds up the cross-sections for all atoms and electrons at different excitation and ionization levels present in a unit volume.

  - After including the effect of stimulated emission, one gets the absorption coefficient α_ν, from which the opacity is obtained

Opacity

If certain approximations are made, then it can be shown that both bound-free and free-free transitions (which are the dominant processes for the opacity) lead to an opacity which varies with density ρ and temperature T in the following way:

$$\chi \propto \frac{\rho}{T^{3.5}}.$$

This is called Kramers's law

This approximate law certainly could not be true for all temperatures.

For example, when the temperature is sufficiently low, most of the atoms will be in their lowest energy levels. In such a situation, it will be possible for radiation to be absorbed only if there are sufficiently energetic photons to knock off electrons from these lowest energy levels. Since the radiation falling on the system is very close to blackbody radiation and since blackbody radiation at a low temperature will not have many energetic photons to knock off the atomic electrons from the lowest levels, we conclude that not much radiation will be absorbed.

Hence opacity is expected to drop at low temperatures and to depart from Kramers's law.

Opacity

- Each curve is for a definite density and shows how the opacity varies with temperature for that density

- On the right side of the peak, we find that the opacity falls sharply with temperature and a $T^{-3.5}$ dependence in accordance with Kramers's law (indicated by the dashed line).

- However, Kramers's law would suggest that opacity should keep on going down with temperature and should be very small at high temperatures. But that does not seem to be happening. At high temperature, the opacity seems to become independent of density and reaches an asymptotic value.
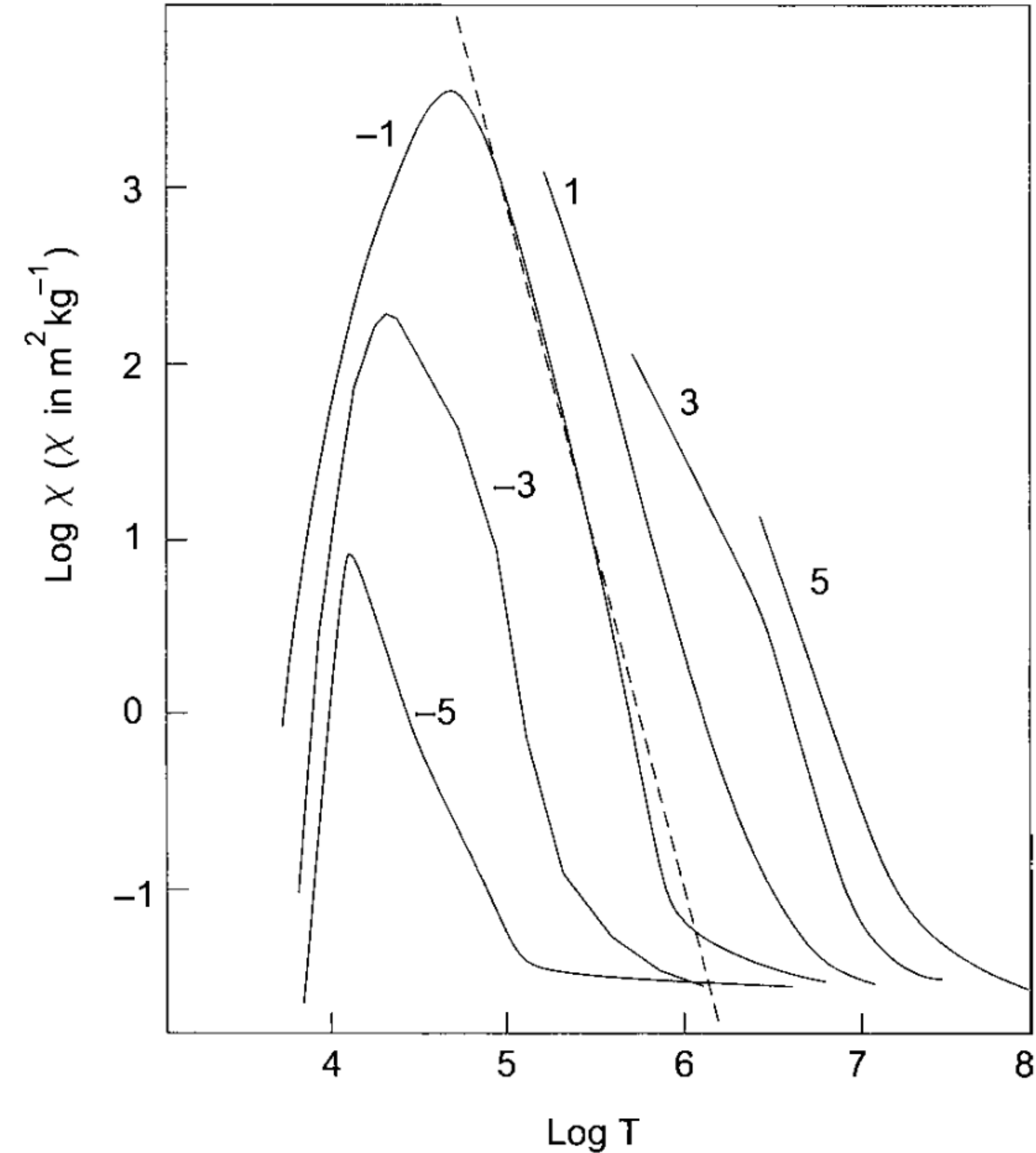

Fig. 2.8 Opacity of material of solar composition as a function of temperature. Different curves correspond to different densities, with the values of $\log \rho$ (ρ in kg m^{-3}) indicated next to the curves. The dashed line indicates the slope that would result if opacity varied as $T^{-3.5}$ for a fixed density. Adapted from Tayler (1994, p. 101).

Thomson scattering

- At sufficiently high temperatures, many atoms in a gas become ionized and there is a supply of free electrons. It is well known that a free electron can scatter radiation by a process called ***Thomson scattering***, which is extremely important in many astrophysical processes.

- If an electromagnetic wave of frequency ω falls on an electron bound to an atom by spring constant $M_e\omega_0^2$ where m_e is the mass of the electron. The equation of motion of the electron subject to an electric field $\mathbf{E}$ is

$$m_e\left(\frac{d^2\mathbf{x}}{dt^2} + \gamma\frac{d\mathbf{x}}{dt} + \omega_0^2\mathbf{x}\right) = -e\mathbf{E},$$

where γ is the damping constant. The electric field of the electromagnetic wave will force the electron to undergo an oscillatory motion.

We know that a charge in an oscillatory motion emits electromagnetic waves. The energy of this emitted wave must come from the energy of the incident electromagnetic wave.

In other words, some energy of the incident wave gets scattered in other directions.

Thomson scattering

A completely classical treatment shows that the scattering cross-section is given by

$$\sigma = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\,\epsilon_0 m_e c^2} \right)^2 \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}.$$

When the frequency of the incident electromagnetic wave is very high ($\omega \gg \omega 0, \gamma$), the electron is forced to move like a free electron and the cross-section reduces to the **Thomson cross-section for free electrons:**

$$\sigma_{\mathrm{T}} = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\,\epsilon_0 m_e c^2} \right)^2$$

Thomson scattering

If the electron being tightly bound to the atom ($\omega_0 \gg \omega, \gamma$). In that limit, reduces to

$$\sigma_R = \sigma_T \left(\frac{\omega}{\omega_0} \right)^4$$

- This is the **Rayleigh scattering**, in which the cross-section goes as ω^4 or as λ^{-4}, where λ is the wavelength of the incident electromagnetic wave.

- Rayleigh scattering provides explanations for many natural as well as astronomical phenomena.

- Examples:
  - In the visible spectrum, blue light is scattered more than red light because the wavelength of blue light is shorter. This explains why the setting Sun looks reddish. The rays of the setting Sun have to traverse through a larger distance of the atmosphere, where blue light is selectively scattered away, leaving more red light in the beam compared to the blue light. On the other hand, the daytime sky looks blue because the dust particles in the sky scatter more blue colour from the sunlight into our eyes.
  - When starlight passes through interstellar dust, it also becomes redder due to the selective scattering of blue light by the dust particles.

Thomson scattering

the numerical value of the Thomson cross-section is found to be : $\sigma_T = 6.65 \times 10^{-29} \ \mathrm{m}^2$

If there are n_e free electrons per unit volume, then the 'absorption coefficient' due to Thomson scattering is $n_e \sigma_T$ (scattering is not true absorption). Hence, the opacity χ_T due to Thomson scattering is given by

$$\chi_T = \frac{n_e}{\rho} \sigma_T.$$

At sufficiently high temperatures when a gas is fully ionized, n_e is proportional to the density and n_e/ρ depends on the composition alone. For example, for fully ionized hydrogen, n_e/ρ is equal to $1/m_H$ (where m_H is the mass of hydrogen atom) so that we conclude that the opacity for fully ionized hydrogen is

$$\chi_T = 3.98 \times 10^{-2} \ \mathrm{m}^2 \ \mathrm{kg}^{-1}$$

Thomson scattering

- While computating opacity, the contribution due to Thomson scattering is added to the contributions from bound-free and free-free transitions (keeping in mind that there is no stimulated emission associated with Thomson scattering).

- However, when Thomson scattering is present in an atmosphere, some special care has to be taken while solving the radiative transfer equation, since Kirchhoff's law will not hold for Thomson scattering.

- However at the typical temperatures of stellar surfaces Thomson scattering should contribute very little to the opacity. Hence, while studying radiative transfer through stellar atmospheres, one can usually neglect Thomson scattering.

- Other examples:
  - Thomson scattering plays a very important role in the early Universe. When all matter in the early Universe was ionized (due to the high temperature), matter was sufficiently opaque to keep the matter and radiation coupled together.
  - The solar K-corona is the result of the Thomson scattering of solar radiation from solar coronal electrons.

Thomson scattering

Thomson Scattering around the Sun visualizes the trajectory of charged particles in visible light.

A solar mystery

- Thomson The temperature of the solar surface is about 6000 K. It appears that the solar surface is sufficiently opaque and we cannot see anything underneath it.

- **What makes the solar gases so opaque at a temperature of 6000 K?**

- At that temperature, hydrogen and helium atoms are not ionized and mostly occupy the lowest energy levels. To force transitions to higher energy levels, one needs photons having energy of the order of a few eV.

- Blackbody radiation at 6000 K does not have enough photons with such energy. So, it seems that matter at 6000 K should not be able to absorb radiation and should be transparent.

- It mystified astrophysicists for some time.

Negative hydrogen ion

- Solution: The electron inside a hydrogen atom is not able to screen the electrostatic force of the nucleus completely. So it is possible for the hydrogen atom to attract an additional electron and form a loosely bound negative ion H^-.

- The binding energy of the negative hydrogen ion is only about 0.75 eV – much smaller than the ionization energy of 13.6 eV.

- So blackbody radiation at 6000 K has enough photons to knock off this loosely bound electron and can get absorbed in this process.

Spectral lines

- A quantitative theory of spectral lines in a stellar atmosphere involves certain difficulties because we need to consider both absorption and emission at spectral lines in the outer layers of the star.
- A simpler problem is to consider the passage of radiation through a medium which absorbs only at a spectral line and does not emit.
- For example, we may consider the passage of visible light from a star through the interstellar medium.
- Since parts of the interstellar medium are made up of gas having fairly low temperatures like 100 K, there is negligible emission of visible light from this gas which may absorb starlight at particular frequencies.
- Since this gas is cold, it produces spectral lines which are much narrower than typical spectral lines produced in the stellar atmosphere. The extreme narrowness of a line in a stellar spectrum is indicative that it is produced during the passage of light through the interstellar medium rather than in the stellar atmosphere.

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Spectral lines

Let n be the number density of atoms of a certain kind in the absorbing medium having energy levels differing by $h\nu_0$. We expect these atoms to absorb at the frequency ν_0 and produce a spectral line. It is customary to write the absorption cross-section of the atom as

$$\sigma = \frac{e^2}{4\epsilon_0 m_e c} f$$

where f is called the ***oscillator strength***. Each spectral line will be characterized by an oscillator strength f. The larger the value of f, the stronger the spectral line is expected to be. We also expect the absorption coefficient to have a normalized profile $\phi(\Delta\nu)$ where $\Delta\nu$ is the departure of the frequency from the line centre at ν_0 and $\int \phi(\Delta\nu)d\nu = 1$. Then the absorption coefficient is given by

$$\alpha_\nu = n\sigma\phi(\Delta v) = \frac{e^2}{4\epsilon_0 m_e c} nf\phi(\Delta v)$$

Spectral lines

The optical depth through the absorbing medium, as given by

$$\tau_\nu = \frac{e^2}{4\epsilon_0 m_e c} N f \phi(\Delta \nu).$$

where $N = \int n ds$ is the column density of the atoms along the line of sight through the absorbing medium.

One has to subtract the effect of induced emission in a full calculation of the absorption coefficient.

For visible light passing through a gas at temperature of order 100 K, the induced emission is negligible (because of the very low population of the upper level) and we do not consider it here.

If we assume that there is no emission in the medium, then the intensity will be:

$$I_\nu = I_c e^{-\tau_\nu}$$

Where I_c is the continuum emission

Spectral lines

It is clear that $(I_c - I_\nu)/I_c$ is the fractional dip in intensity at some frequency ν inside the spectral line.

We can get an estimate of the strength of the spectral line by integrating this fractional dip over the spectral line. This is called the **equivalent width** of the spectral line, defined as

$$W_\lambda = \int \frac{I_c - I_\nu}{I_c} d\lambda.$$

$$W_\lambda = \frac{\lambda^2}{c} \int [1 - e^{-\tau_\nu}] \, d\nu,$$

if the spectral line is weak, when we can take $e^{-\tau_\nu} \approx 1 - \tau_\nu$ so: $\qquad W_\lambda = \frac{\lambda^2}{c} \int \tau_\nu \, d\nu.$

Spectral lines

Substituting:
$$\frac{W_\lambda}{\lambda} = \frac{e^2}{4\epsilon_0 m_e c^2} N f \lambda.$$

For a weak spectral line of which we know the oscillator strength f, we can determine the column density N of the atoms producing the spectral line when we have a measurement of the equivalent width W_λ.

Suppose the absorbing medium has certain atoms producing several spectral lines with different oscillator strengths f. -> W_λ / λ will be proportional to $N f \lambda$ for weak spectral lines.

Spectral lines

Even if the spectral lines are not weak, we can plot W_λ / λ as a function of $N f \lambda$ for all the spectral lines. The curve passing through the data points is called the **curve of growth**. The left side of the curve of growth shows a linearly increasing regime corresponding to weak spectral lines for which we have the proportionality to $N f \lambda$

For stronger spectral lines, the curve of growth saturates to a horizontal plateau. The reason behind this is that the fractional dip $(I_C - I_\nu)/I_C$ can never be more than 1, no matter how strong the spectral line is.
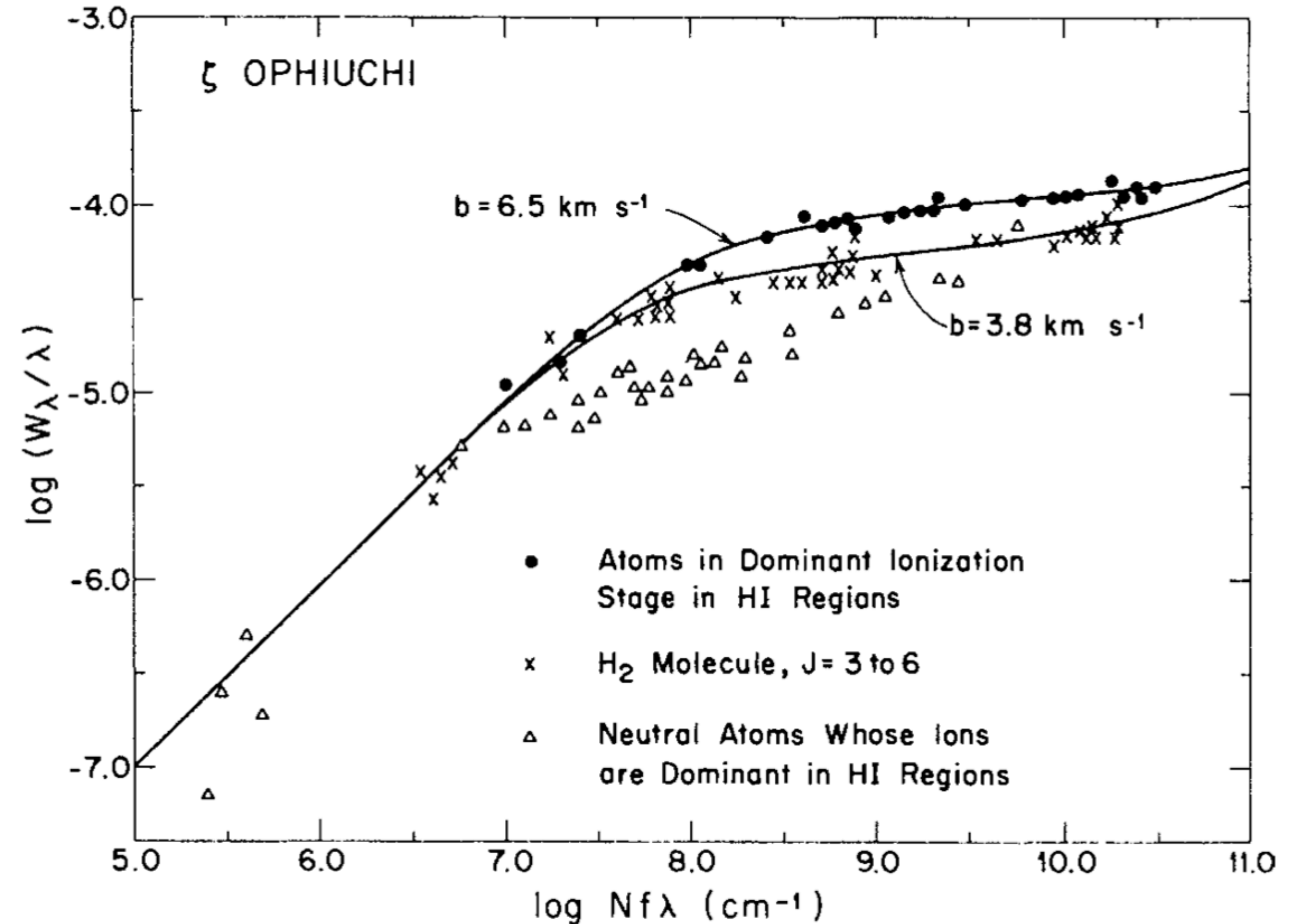

Fig. 2.9 Equivalent widths of various spectral lines produced in the spectrum of the star ζ Ophiuchi by absorption in the interstellar medium, plotted against $N f \lambda$. The curves of growth for hydrogen atoms and hydrogen molecules are shown. From Spitzer and Jenkins (1975). (©Annual Reviews Inc. Reproduced with permission from *Annual Reviews of Astronomy and Astrophysics*.)

Photon diffusion inside the Sun

Suppose the energy generation rate at the centre of the Sun were to increase or decrease suddenly due to some reason. We expect that eventually the surface of the Sun will become brighter or dimmer as a consequence of this. **How much time will it take before the effect of this sudden change at the centre becomes visible at the surface?**

Photon diffusion inside the Sun

The photons created at the centre of the Sun interact with the neighbouring atoms. Atoms which have absorbed photons will de-excite by giving out photons.

In this process, photons diffuse from the centre of the Sun towards the surface.

The absorption and re-emission of photons by atoms can be far from simple. The atom may spend some time in the excited state and when it de-excites, it may not come back to exactly the same state in which it was originally in. As a result, the emitted photon may have a frequency different from what was the frequency of the absorbed photon.

This is necessary because the initial photons at the centre at temperature of order 10^7 would typically be X-ray photons, whereas the photons which reach the outer surface are more likely to be photons of visible light.

We now make a rough estimate of the diffusion time by making a simplifying assumption that photons merely do a random walk through stellar matter where an encounter with an atom simply changes the direction of flight of the photon.

Photon diffusion inside the Sun

Let us first make an estimate of the mean free path of photons between encounters with atoms.

The inverse of the absorption coefficient gives this mean free path. So we can take $\alpha^{-1} = (\varrho\chi)^{-1}$ to be the R mean free path.

While this mean free path would be a function of radius, we simplify our life further by using an approximate average value. 10^{-1} m^2 kg^{-1} would be an appropriate value for χ to use in the solar interior. Taking an average density of order 10^3 kg m^{-3}, we get a mean free path of about 1 cm.

Suppose an average photon has to take N steps to diffuse from the centre to the surface . If $l_1, l_2, \ldots l_n$ are the displacements in these steps, then the total displacement is

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \cdots + \mathbf{l}_N.$$

$$\langle L^2 \rangle = \langle l_1^2 \rangle + \langle l_2^2 \rangle + \cdots + \langle l_N^2 \rangle.$$

$$\langle L^2 \rangle = N \langle l^2 \rangle$$

Photon diffusion inside the Sun

Taking $l = 1$cm and L to be equal to the solar radius, N turns out to be of order 10^{22}. With a step size of 1cm, an average photon would travel over a distance 10^{20} m in order to escape from the centre to the surface. Dividing this by the speed of light, we get a diffusion time of order 10^4 years. A more careful calculation shows that the photon diffusion time inside the Sun is actually a few times larger than this.

If the energy generation rate at the centre of the Sun were to change suddenly, that information will take tens of thousands years to reach the surface.

Exercises

Consider hydrogen gas having the same density as the density of air under normal temperature and pressure ($\rho = 1.29$ kg m^{-3}). Given the fact that the ionization potential χ of hydrogen is 13.6 eV, use the Saha equation to calculate the fraction of ionization x at different temperatures T and make a plot of x versus T.