Introduction to Astrophysics and Cosmology

Radiation transfer

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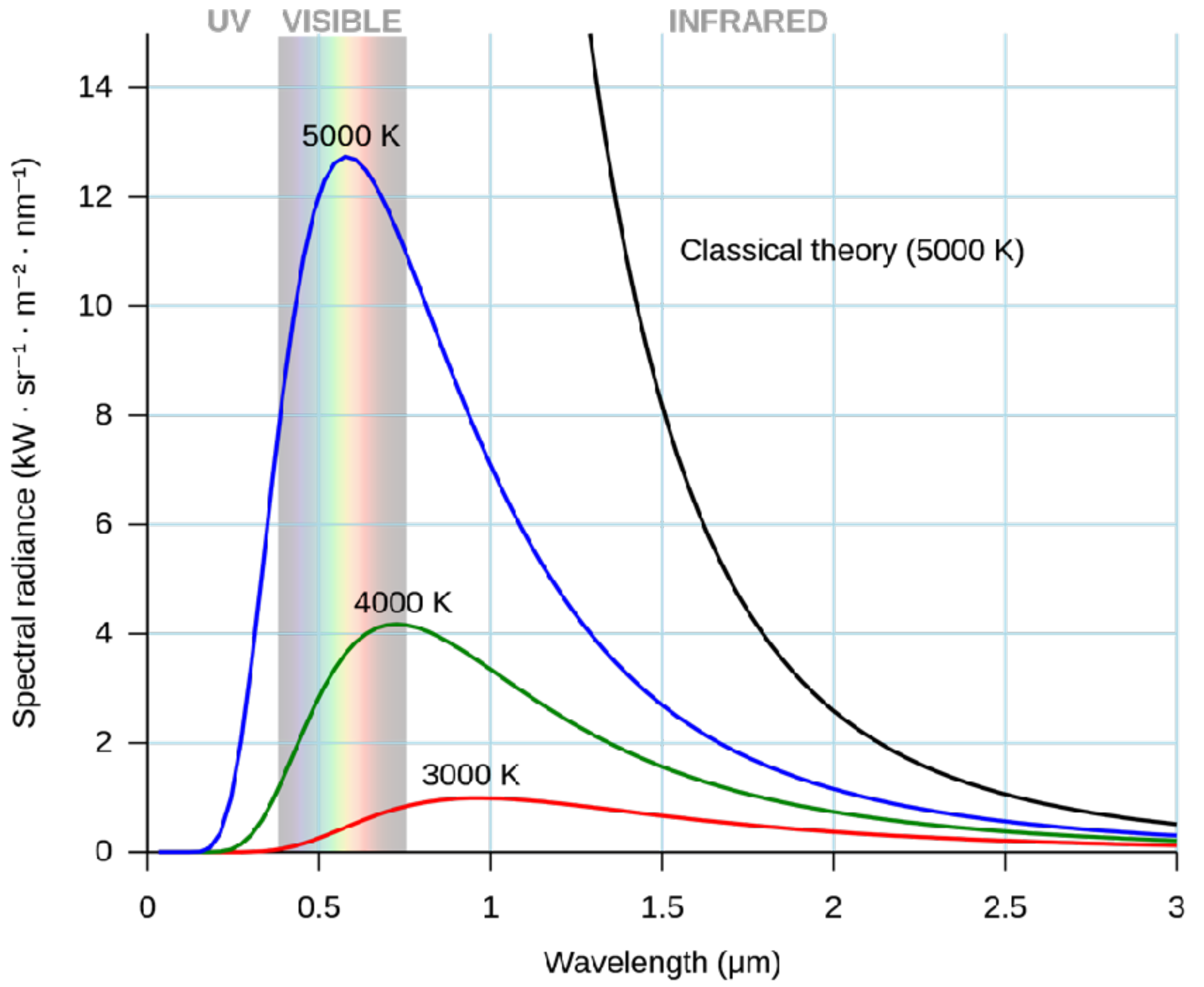
Radiation transfer

- **describes how radiation interacts with matter**
- Macroscopic: using emission and absorption coefficients
- Microscopic: calculating the emission and absorption coefficients

Radiation field
simples case: **blackbody radiation** (homogeneous and isotropic inside a container)

Planck's law - specifies energy density U_ν in given frequency range $\nu, \nu + d\nu$:

$$U_\nu d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{\kappa_B T}\right) - 1}.$$



https://en.wikipedia.org/wiki/Black-body_radiation

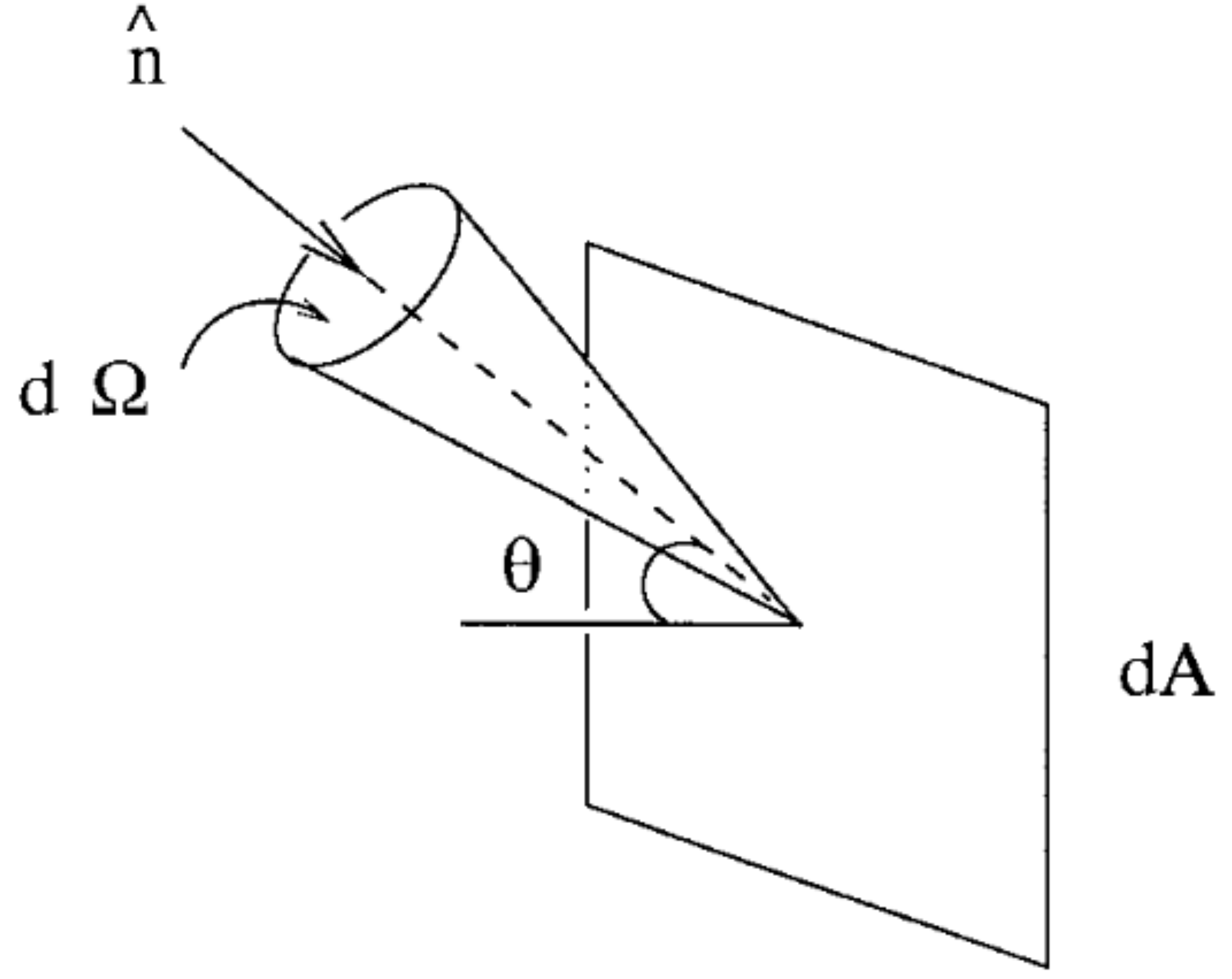
Radiation transfer

In general the radiation is not isotropic and we need to consider the direction.

We consider a small area dA, the amount of radiation is $dE_\nu d\nu$, passing trough this area in time dt, from the solid angle $d\Omega$ in the frequency range $\nu, \nu + d\nu$.

$dE_\nu d\nu$ is proportional to the projected area $dAcos\theta$ and to dt, $d\Omega$, $d\nu$

$$dE_\nu d\nu = I_\nu(\mathbf{r}, t, \hat{\mathbf{n}}) \cos\theta \, dA \, dt \, d\Omega \, d\nu,$$



- **n** is the unity vector that represents the direction of the radiation
- I_ν is the specific intensity.
- **If I_ν is specified in all directions in every point of a region at a specific time then we have a radiation field.**

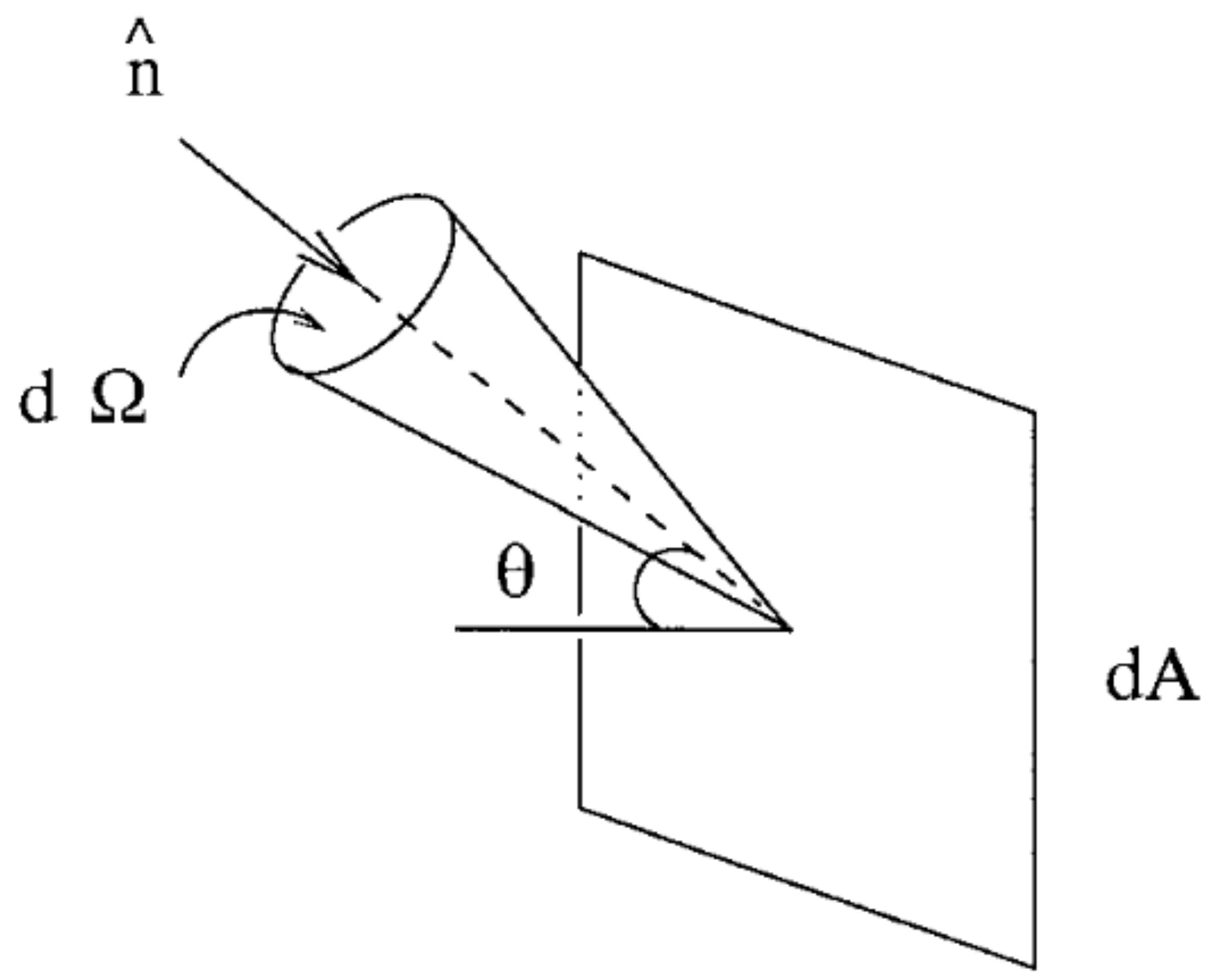
Radiation flux

We can calculate various quantities from the radiation field:

- Flux,
- energy density,
- radiation pressure.

Radiation flux is the total energy of radiation coming from all directions per unit area, per unit time.

$$F_\nu = \int I_\nu \cos\theta \, d\Omega,$$



The **total radiation flux** is:

$$F = \int F_\nu \, d\nu.$$

Energy density

Energy passes to area dA in dt time in a certain direction **n**. The radiation travels a distance cdt in dt time, we expect the radiation to fill a cylinder with a bas of dA and length of cdt. The volume of such a cylinder is $cos\theta dAcdt$, from this the **energy density** is:

$$\frac{dE_\nu}{\cos\theta\, dA\, c\, dt} = \frac{I_\nu}{c} d\Omega$$

To get the **total energy density** at a point we need to integrate over all directions where radiation is coming from:

$$U_\nu = \int \frac{I_\nu}{c} d\Omega.$$

Energy density

Now we apply this to blackbody radiation. Where $B_\nu(T)$ is the specific intensity of blackbody radiation.

$$U_\nu = \frac{4\pi}{c} B_\nu(T),$$

From this, the specific intensity of blackbody ration is:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{\kappa_\mathrm{B} T}\right) - 1}.$$

Radiation pressure

The pressure of the radiation field over a surface is given by the flux of momentum perpendicular to that surface. The momentum associated with energy dE_ν is $\frac{dE_\nu}{c}$ and its component normal to the surface dA is $\frac{dE_\nu cos\theta}{c}$ by dividing this by $dA\ dt$ we get **the momentum flux associated with** dE_ν

$$\frac{dE_\nu \cos\theta}{c}\frac{1}{dA\,dt} = \frac{I_\nu}{c}\cos^2\theta\,d\Omega$$

The **pressure** is obtained by integrating over all directions:

$$P_\nu = \frac{1}{c}\int I_\nu \cos^2\theta\,d\Omega.$$

If the field is isotropic:

$$P_\nu = \frac{I_\nu}{c}\int \cos^2\theta\,d\Omega = \frac{4\pi}{3}\frac{I_\nu}{c}.$$

Radiation pressure

If the field is isotropic:

$$P_\nu = \frac{I_\nu}{c} \int \cos^2 \theta \, d\Omega = \frac{4\pi}{3} \frac{I_\nu}{c}.$$

This can also be written as:

$$U_\nu = 4\pi \frac{I_\nu}{c}$$

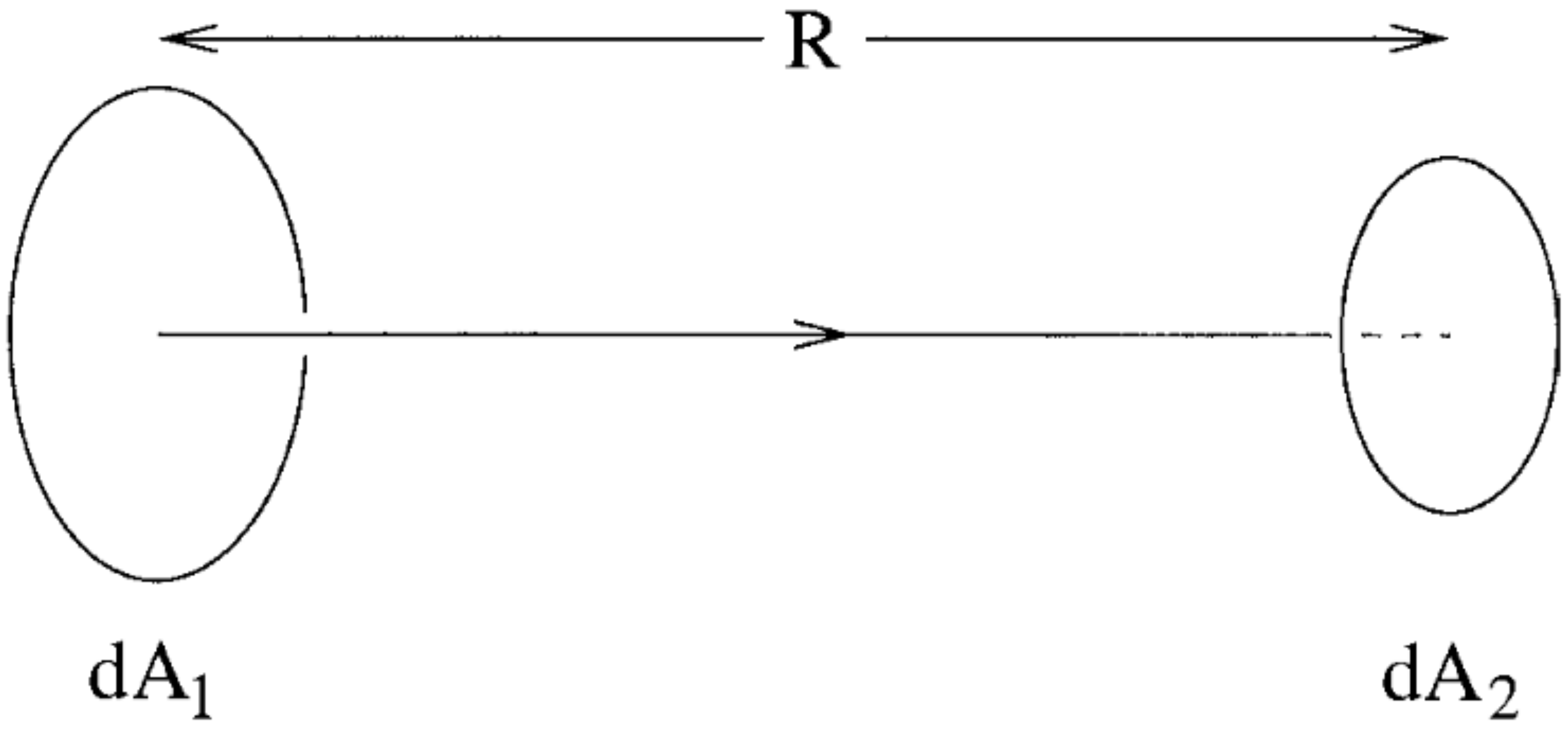
$$P_\nu = \frac{1}{3} U_\nu$$

Radiative transfer equation

If matter is present, then in general the specific intensity keeps changing as we move along a ray path.

Before we consider the effect of matter, first let us find out what happens to the ***specific intensity in empty space*** as we move along a ray path.

- Let dA_1 and dA_2 be two area elements separated by a distance *R* and placed perpendicularly to a ray path.
- Let $I_{\nu 1}$ and $I_{\nu 2}$ be the specific intensity of radiation in the direction of the ray path at dA_1 and dA_2.
- We want to find out the amount of radiation passing through both dA_1 and dA_2 in time *dt* in the frequency range ν, $\nu + d\nu$.



Radiative transfer equation

If $d\Omega_2$ is the solid angle subtended by dA_1 at dA_2, then the radiation falling on dA_2 in time *dt* after passing through dA_1 is

$$I_{\nu 2}\, dA_2\, dt\, d\Omega_2\, d\nu \quad = \quad I_{\nu 1}\, dA_1\, dt\, d\Omega_1\, d\nu,$$

where $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1. Equating these two expressions and noting that

$$d\Omega_1 = \frac{dA_2}{R^2}, \quad d\Omega_2 = \frac{dA_1}{R^2}, \quad \longrightarrow \quad \boxed{I_{\nu 1} = I_{\nu 2}}$$

In empty space the specific intensity along a ray path does not change as we move along the ray path.

Radiative transfer equation

If *s* is the distance measured along the ray path, then in empty space we can write: $$\frac{dI_\nu}{ds} = 0$$

- At first sight, this may appear like a surprising result.
- We know that the intensity falls off as we move further and further away from a source of radiation.

- **Can the specific intensity remain constant?**
- We need to keep in mind that the specific intensity due to a source is essentially its intensity divided by the solid angle it subtends, which means that **specific intensity is a measure of the surface brightness.**
- As we move further away from a source of radiation, **both its intensity and angular size fall as (distance)^2. Hence the surface brightness, which is the ratio of these two, does not change.**
- Suppose you are standing on a street in a dark night and are looking at the street lights. The lights further away would appear smaller in size, but their surfaces would appear as bright as the surfaces of nearby lights.

Radiative transfer equation

- This result has an important astronomical implication. If we neglect intergalactic extinction, then the surface brightness of a galaxy which is resolved by a telescope is independent of distance. **Whether the galaxy is nearby or far away, its surface would appear equally bright to us.**
- We may expect a similar consideration to hold for stars also. Then **why do distant stars look dimmer?**

Radiative transfer equation

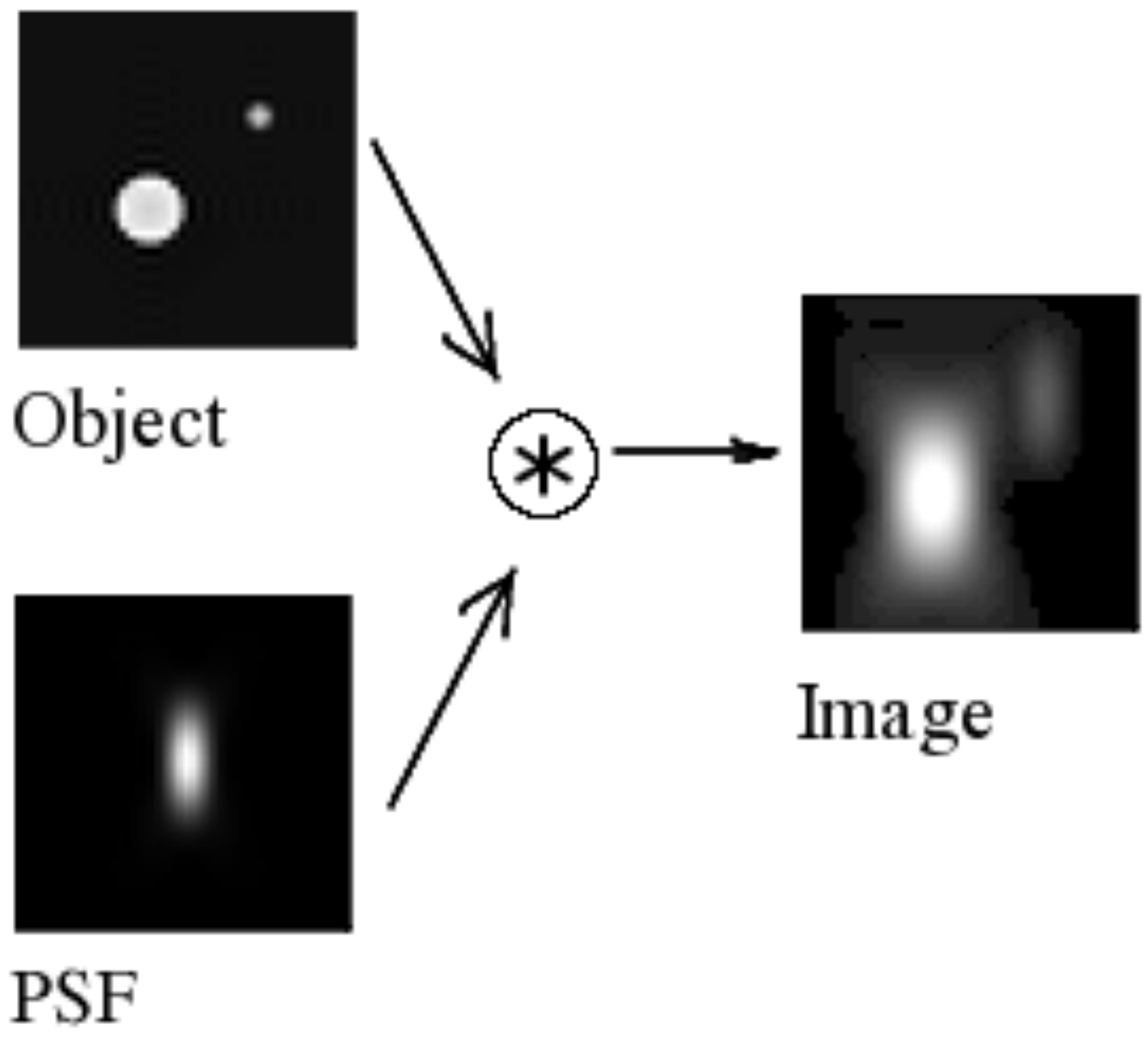
- This result has an important astronomical implication. If we neglect intergalactic extinction, then the surface brightness of a galaxy which is resolved by a telescope is independent of distance. **Whether the galaxy is nearby or far away, its surface would appear equally bright to us.**
- We may expect a similar consideration to hold for stars also. Then **why do distant stars look dimmer?**
- Since the theory of radiative transfer is based on the concept of ray path, we are assuming geometrical optics in all our derivations. So our results hold as long as geometrical optics is valid. **If the star is very far away, then its disk is not resolved and geometrical optics no longer holds.**
- The angular size of the star may be caused by the diffraction of light or the seeing. As the star is moved further away, its intensity diminishes, but the angular size due to diffraction does not change much. Hence **a decreasing amount of radiation gets spread over an image of the same angular size, making the star appear dimmer.**
- It may be noted that very faraway galaxies also look dimmer due to general relativistic effects.

See also: point spread function: https://en.wikipedia.org/wiki/Point_spread_function

Radiative transfer equation

The point spread function (PSF) describes the response of a focused optical imaging system to a point source or point object. A more general term for the PSF is the system's impulse response; the PSF is the impulse response or impulse response function (IRF) of a focused optical imaging system. The PSF in many contexts can be thought of as the extended blob in an image that represents a single point object, that is considered as a spatial impulse. It is a useful concept in Fourier optics, astronomical imaging, medical imaging, electron microscopy and other imaging techniques such as 3D microscopy and fluorescence microscopy.

The degree of spreading (blurring) in the image of a point object for an imaging system is a measure of the quality of the imaging system.



See also: point spread function: https://en.wikipedia.org/wiki/Point_spread_function

Radiative transfer equation

- Let us now consider what happens **if matter is present along the ray path.**
- If the matter emits, we expect that it will add to the specific intensity.
    - This can be taken care of by adding an **emission coefficient** j_ν on the right-hand side.
    - On the other hand, absorption by matter would lead to a diminution of specific intensity and the diminution rate must be proportional to the specific intensity itself. In other words, the stronger the beam, the more energy there is for absorption.
- Hence the absorption term on the right-hand side should be negative and proportional to I_ν.
- Thus, in the presence of matter:

$$\boxed{\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu,}$$

where α_ν **is the absorption coefficient. This is the radiative transfer equation and provides the basis for our understanding of interaction between radiation and matter.**

Radiative transfer equation

- In the early years of spectral research, many astronomers held the view that the Sun was surrounded by a cool layer of gas which only absorbed radiation at certain frequencies to produce the dark lines. → This is not actually the case. **The same material can simultaneously emit and absorb radiation.**

- It is fairly trivial to solve the radiative transfer equation if either the emission coefficient or the absorption coefficient is zero.
    - Let us consider the case of $j_\nu = 0$, i.e. **matter is assumed to absorb only** but not to emit.

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu .$$

On integrating this equation over the ray path from s_0 to s, we get:

$$I_\nu(s) = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} \alpha_\nu(s')\, ds'\right]$$

Optical depth, solution of the radiative transfer equation

$$I_\nu(s) = I_\nu(s_0)\exp\left[-\int_{s_0}^{s} \underbrace{\alpha_\nu(s')\,ds'}\right]$$

We define ***optical depth*** τ_ν through the relation

$$\boxed{d\tau_\nu = \alpha_\nu\, ds}$$

such that the optical depth along the ray path between s_0 and *s* becomes

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')\,ds'.$$

the specific intensity along the ray path falls as, if the matter does not emit:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$$

Optical depth, solution of the radiative transfer equation

- If the optical depth $\tau_\nu \gg$ **1** along a ray path through an object, then the object is known as ***optically thick***.
- On the other hand, an object is known as ***optically thin*** **if** $\tau_\nu \ll$ **1** for a ray path through it.
- It follows that an optically thick object extinguishes the light of a source behind it, whereas an optically thin object does not decrease the light much.
- Hence **the terms optically thick and optically thin roughly mean opaque and transparent** at the frequency of electromagnetic radiation we are considering.

We now define the ***source function:***

$$\boxed{S_\nu = \frac{j_\nu}{\alpha_\nu}}$$

Optical depth, solution of the radiative transfer equation

Dividing the radiative transfer by α_ν, we get

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \longrightarrow \frac{d}{d\tau_\nu}(I_\nu e^{\tau_\nu}) = S_\nu e^{\tau_\nu}$$

Integrating this equation from optical path 0 to τ_ν (i.e. from s_0 to *s* along the ray path), we get

$$I_\nu(\tau_\nu) = \underbrace{I_\nu(0)\, e^{-\tau_\nu}}_{\text{Source}} + \underbrace{\int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu')\, d\tau_\nu'}_{\text{Path of the ray}}.$$

This is the general solution of the radiative transfer equation.

Optical depth, solution of the radiative transfer equation

If matter through which the radiation is passing has constant properties, then we can take S_ν constant and work out the integral. This gives

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + S_\nu\,(1 - e^{-\tau_\nu}).$$

We are now interested in studying the **emission and absorption properties of an object itself** without a source behind it. Then we take $I_\nu(0) = 0$ and write

$$I_\nu(\tau_\nu) = S_\nu\,(1 - e^{-\tau_\nu}).$$

Optical depth, solution of the radiative transfer equation

$$I_\nu(\tau_\nu) = S_\nu\,(1 - e^{-\tau_\nu}).$$

Let us consider the cases of optically thin and thick objects. If the object is **optically thin** (i.e. $\tau_\nu \ll 1$), then we write $1 - \tau_\nu$ for $e^{-\tau_\nu}$ such that

$$I_\nu(\tau_\nu) = S_\nu \tau_\nu.$$

For matter with constant properties, we take $\tau_\nu = \alpha_\nu L$, where L is the total length of the ray path. We get **optically thin:**

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \longrightarrow I_\nu = j_\nu L.$$

On the other hand, if the object is **optically thick**, then we neglect $e^{-\tau_\nu}$ compared to 1:

$$I_\nu = S_\nu.$$

Kirchof's law

Suppose we have a box kept in **thermodynamic equilibrium**. If we make a small hole on its side, we know that the radiation coming out of the hole will be **blackbody radiation**. Hence the specific intensity of radiation coming out of the hole is simply

$$I_\nu = B_\nu(T),$$

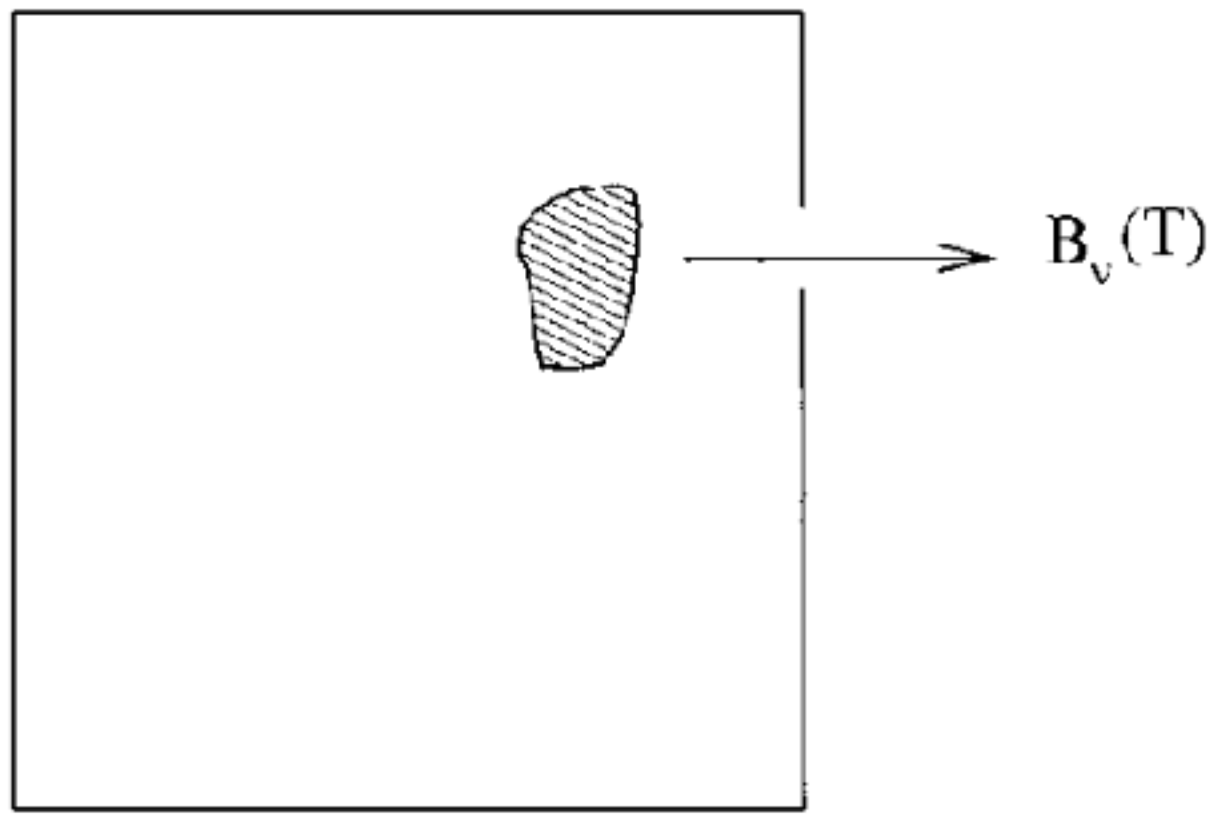


Fig. 2.3 Blackbody radiation coming out of a hole in a box with an optically thick obstacle placed behind the hole.

Kirchof’s law

- We now keep an optically thick object behind the hole as shown in the Figure. If this object is in thermodynamic equilibrium with the surroundings, then it will not disturb the environment and the **radiation coming out of the hole will still be blackbody radiation.**
- On the other hand, we have seen in that the **radiation coming out of an optically thick object has the specific intensity equal to the source function**.

$$S_\nu = B_\nu(T)$$

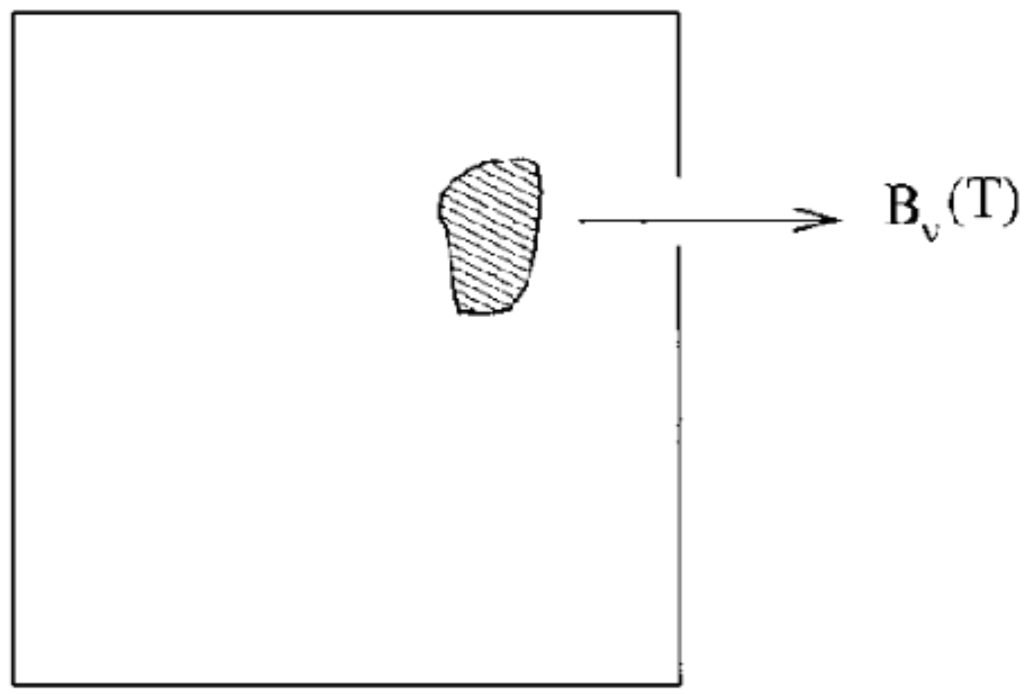


Fig. 2.3 Blackbody radiation coming out of a hole in a box with an optically thick obstacle placed behind the hole.

In thermodynamical equilibrium:

$$\boxed{j_\nu = \alpha_\nu B_\nu(T).}$$

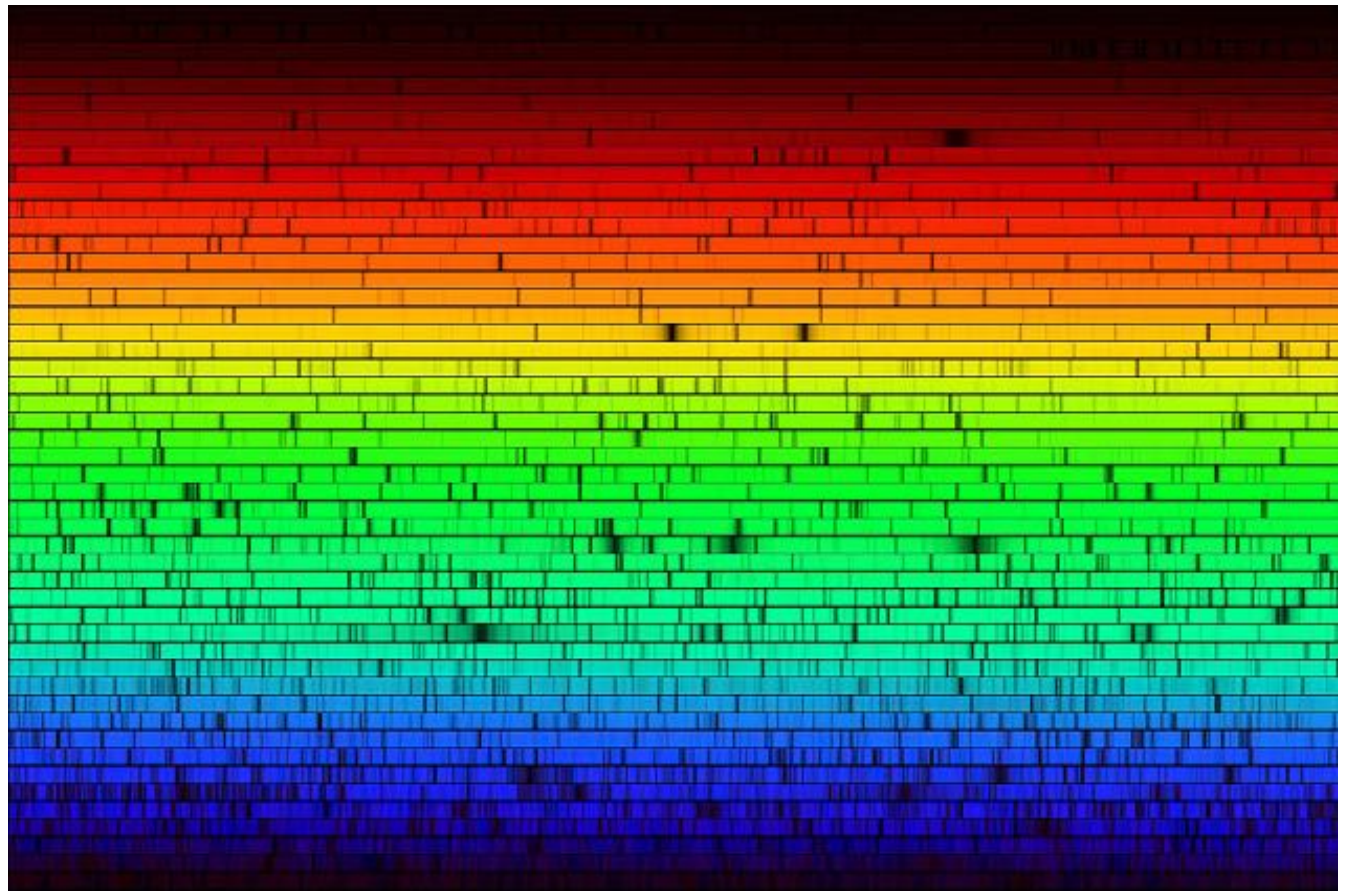
Kirchof’s law

Kirchof's law

- Very often matter tends to emit and absorb more at specific frequencies corresponding to spectral lines. Hence **both j_ν and α_ν are expected to have peaks at spectral lines.**
- But, the ratio of these coefficients should be the smooth blackbody function $B_\nu(T)$.
- The radiation coming out of an optically thin source is essentially determined by its emission coefficient. Since the emission coefficient is expected to have peaks at spectral lines, we find the **emission from an optically thin system** like a hot transparent gas to be **mainly in spectral lines**.
- On the other hand, the specific intensity of radiation coming out of an optically thick source is its source function, which has been shown to be equal to the blackbody function $B_\nu(T)$. Hence **we expect an optically thick object like a hot piece of iron to emit roughly like a blackbody**.

Kirchof’s law

- The theory of radiative transfer is important not only in astrophysics. If we want to understand rigorously and quantitatively many everyday phenomena such as why hot transparent gases emit in spectral lines whereas hot pieces of iron emit like blackbodies, then we need to invoke the theory of radiative transfer.
- **The nature of radiation from an astrophysical source crucially depends on whether the source is optically thin or optically thick.** Emission from a nebula is usually in spectral lines. On the other hand, a star emits very much like a blackbody.
- **Why is the radiation from a star not exactly blackbody radiation? Why do we see absorption lines?**



Spectrum of the Sun

Kirchof's law

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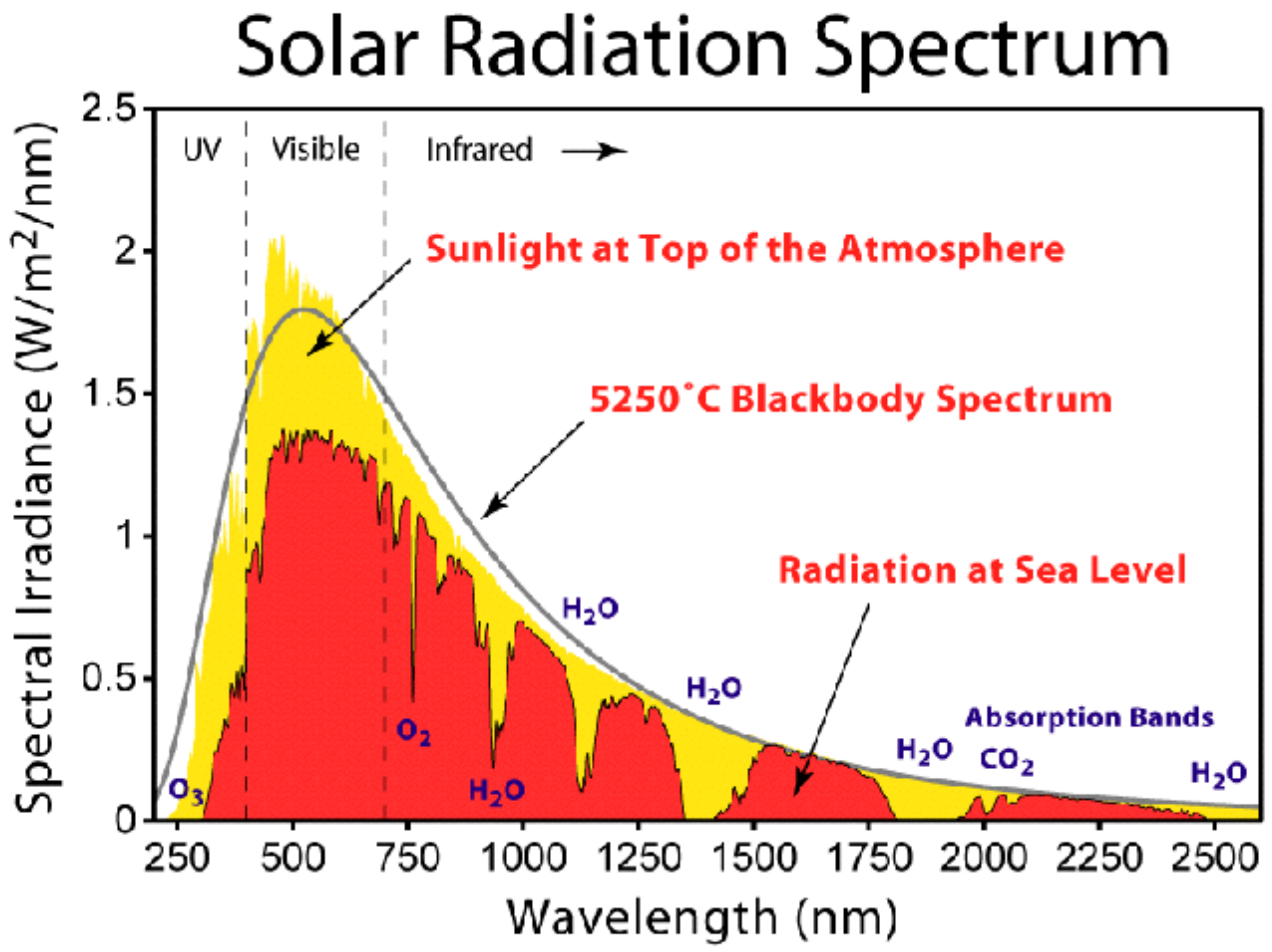
- **The nature of radiation from an astrophysical source crucially depends on whether the source is optically thin or optically thick.** Emission from a nebula is usually in spectral lines. On the other hand, a star emits very much like a blackbody.
- **Why is the radiation from a star not exactly blackbody radiation? Why do we see absorption lines?**
  - We derived the equations so far by assuming the source to have constant properties. This is certainly not true for a star.
  - As we go down from the star's surface, temperature keeps increasing. Hence $I_\nu = S_\nu$ should be only approximately true. It is the temperature gradient near the star's surface which gives rise to the absorption lines.



Thermodynamical equilibrium

- By assuming thermodynamic equilibrium, we have derived that the source function should be equal to the blackbody function $B_\nu(T)$.
- In a realistic situation, we rarely have strict thermodynamic equilibrium. The temperature inside a star is not constant, but varies with its radius. In such a situation, will our assumptions hold?
  - If a system is in thermodynamic equilibrium, then certain important principles of physics can be applied to that system.
  - **Maxwellian velocity distribution:** Different particles in a gas move around with different velocities. If the gas is in thermodynamic equilibrium with temperature T, then the number of particles having speeds between v and v + dv
  - n is the number of particles per unit volume
  - m is the mass of each particle.

$$dn_v = 4\pi n \left(\frac{m}{2\pi \kappa_B T} \right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2\kappa_B T} \right) dv,$$

Thermodynamical equilibrium

- **Boltzmann and Saha equations**: We know that a hydrogen atom has several different energy levels. It can also be ionized.
- If a gas of hydrogen atoms is kept in **thermodynamic equilibrium, then a certain fraction of the atoms will occupy a particular energy state and also a certain fraction will be ionized.** The same considerations hold for other gases besides hydrogen.
- If n_0 **is the number density of atoms in the ground state, then the number density** n_e **of atoms in an excited state with energy** E **above the ground state** - this is called the **Boltzmann distribution law.**

$$\boxed{\frac{n_\mathrm{e}}{n_0} = \exp\left(-\frac{E}{\kappa_\mathrm{B} T}\right).}$$

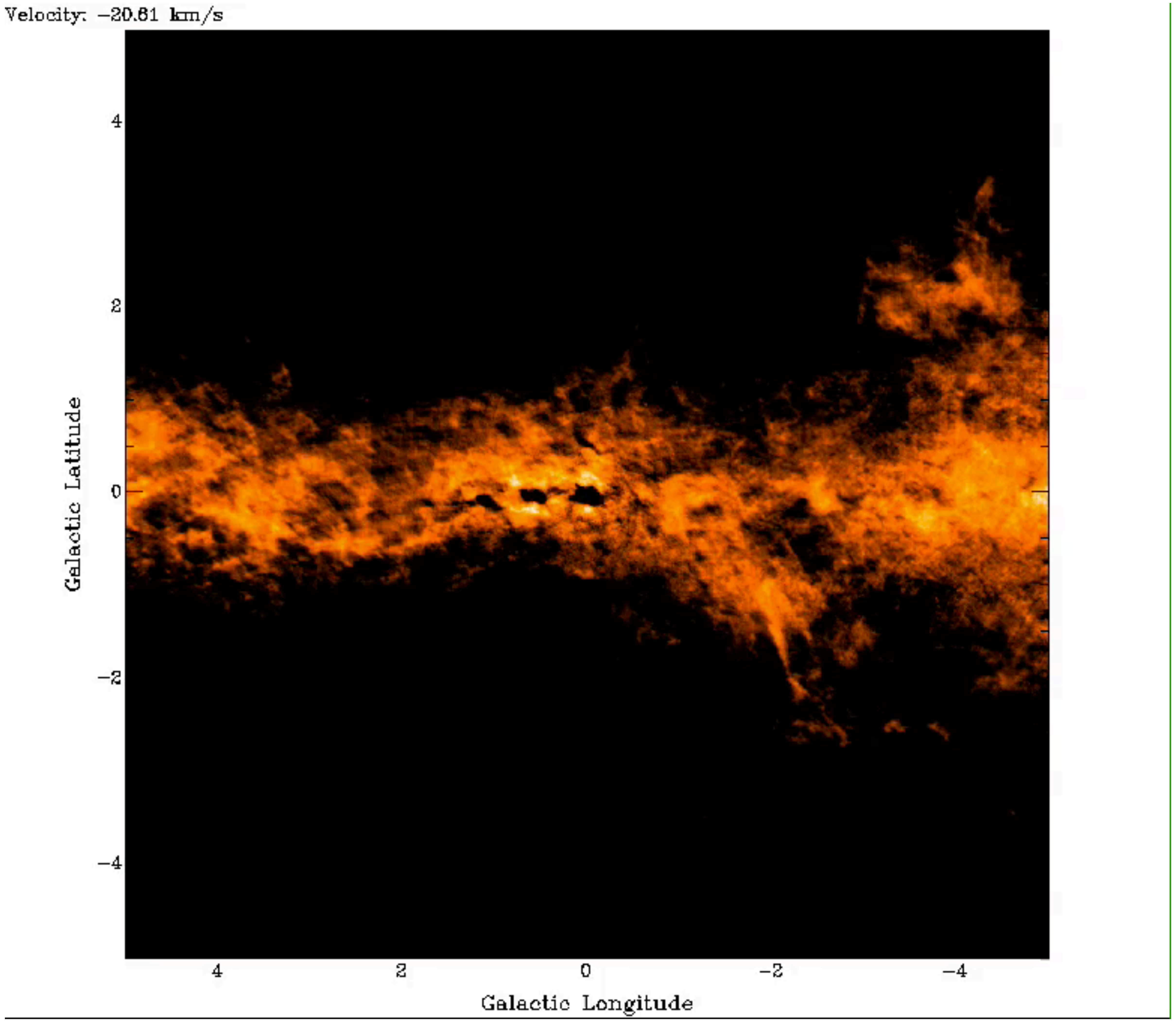
Thermodynamical equilibrium

- Saha (1920) derived the equation which tells us what **fraction of a gas will be ionized** at a certain temperature T and pressure P .
- If **χ is the ionization potential** (i.e. the amount of energy to be supplied to an atom to ionize it), ***h* is Planck's constant** and m_e is the mass of electron, then the fraction x of atoms which are ionized is given by:

The Saha equation

$$\boxed{\frac{x^2}{1-x} = \frac{(2\pi m_{\mathrm{e}})^{3/2}}{h^3} \frac{(\kappa_{\mathrm{B}} T)^{5/2}}{P} \exp\left(-\frac{\chi}{\kappa_{\mathrm{B}} T}\right),}$$

Example for fractional ionisation:



SGPS data - McClure-Griffith et al. 2006

- The Riegel-Crutcher cloud is a cold neutral hydrogen (HI) self-absorption cloud in the Galaxy. This is a cold hydrogen cloud ($\tau > 1$) that absorbes the radiation from warm hydrogen ($\tau < 1$) behind it.

- The cold hydrogen gas has filamentary structure, which is aligned with the magnetic field.

- The cold gas that is shown in this image is neutral gas (not ionised), however there must be some ionised gas present as well.

- The Saha equation describes the fraction of incised to non ionised gas.

Example for fractional ionisation:

- Clark et al. 2014 - filaments are identified with a method called the Rolling Hugh Transform. The magnetic field is calculated based on starlight polarisation.
- The magnetic field seems aligned with the gas filaments.

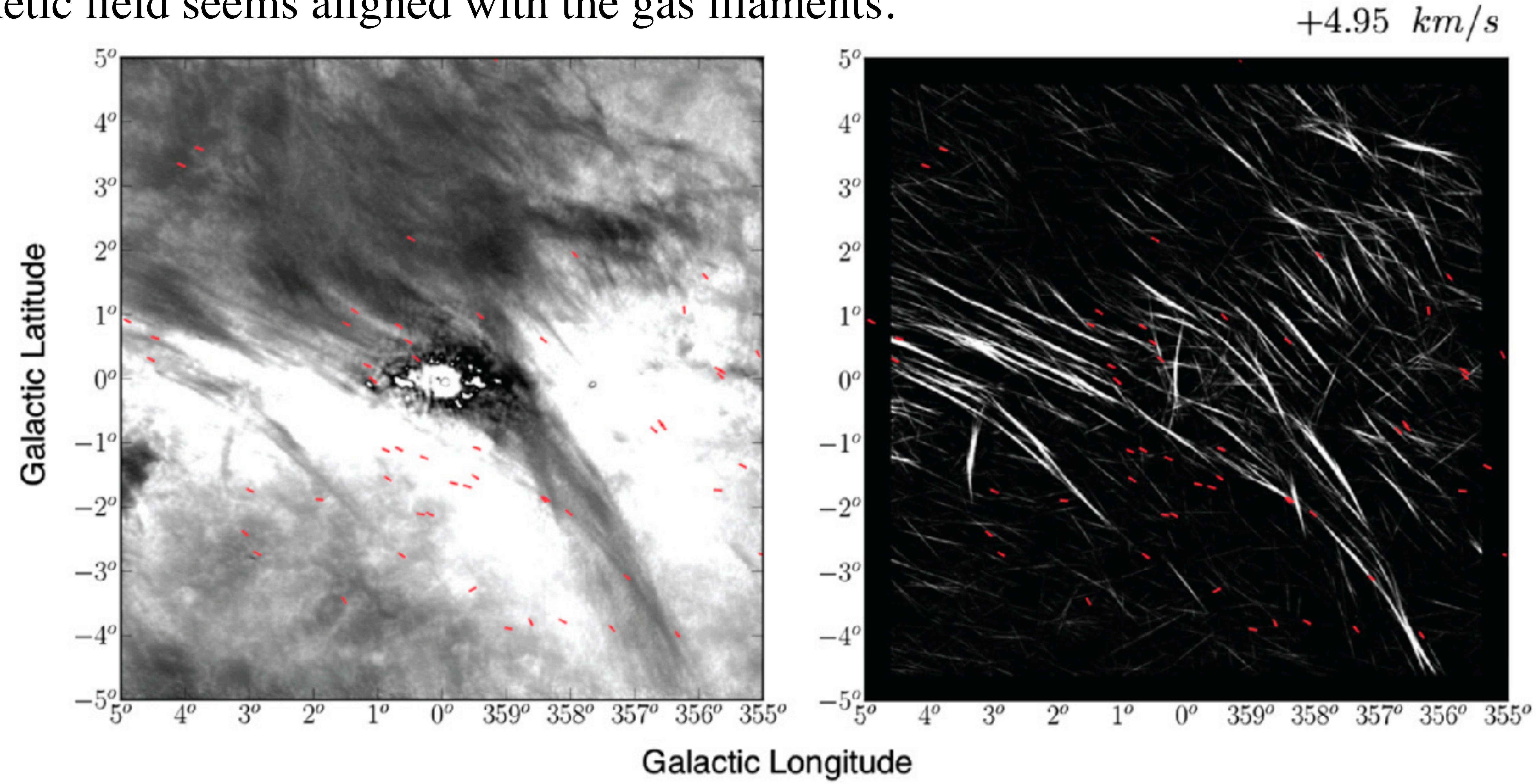


Figure 10. Riegel–Crutcher cloud (Section 6) in H I absorption (left) and RHT backprojection (right). Overlaid pseudovectors represent polarization angle measurements from the Heiles (2000) compilation. In the left panel, the intensity scale is linear from −20 K (white) to −120 K (black).

Thermodynamical equilibrium

- Question: when **can we expect a system to be in thermodynamic equilibrium** and when can we expect the above principles (Maxwellian velocity distribution, Boltzmann equation, Saha equation, Planck's law) to hold?
    - If a box filled with gas and radiation is kept isolated from the surroundings, then we know that thermodynamic equilibrium will get established inside, and all the above principles will hold.
    - However, a realistic system is always more complicated. Inside a star, the temperature keeps decreasing as we go from the central region to the outside. **Can the above principles be applied in such a situation?**

Thermodynamical equilibrium

- We again consider a box filled with gas and radiations. Even if the gas particles initially do not obey the Maxwellian distribution, they will relax to it after undergoing a few collisions.
- In other words, collisions or **interactions** amongst the constituents of the system are vital in **establishing thermodynamic equilibrium.**
- When collisions are frequent, the **mean free path** turns out to be small, then particles in a gas will interact with each other more effectively and we expect that principles like the Maxwellian velocity distribution, the Boltzmann equation or the Saha equation will hold.
- If the mean free path is small and the temperature does not vary much over that distance, then we shall have the Maxwellian velocity distribution.
- **The condition of validity of the Maxwellian velocity distribution (as well as the Boltzmann equation and the Saha equation) is that the mean free path has to be small enough such that the temperature does not vary much over the mean free path.**

Thermodynamical equilibrium

- For Planck's law to be established for radiation, the radiation has to be in equilibrium with matter. This is possible only when radiation interacts efficiently with matter.
- The **absorption coefficient** α_ν in the radiative transfer equation **is a measure of the interaction between radiation and matter**
- α_ν has the dimension of inverse length.
- **Its inverse α_ν^{-1} gives the distance over which a significant part of a beam of radiation would get absorbed by matter. This is referred to as the mean free path of photons**
- The typical distance a photon is expected to traverse freely before interacting with an atom. The smaller the value of α_ν^{-1}, the more efficient is the interaction between matter and radiation.
- **If α_ν^{-1} is sufficiently small such that the temperature can be taken as constant over such distances, then we expect Planck's law of blackbody radiation to hold.**

Thermodynamical equilibrium

- If the temperature is varying within a system, then it is not in full thermodynamic equilibrium.
- However, we can have a situation **where both α_ν^{-1} and the mean free path of particles are small compared to the length over which the temperature varies significantly**. In a such situation, all the important laws of thermodynamic equilibrium are expected to hold within a local region, provided we use the local temperature T in the expressions.
- Such a situation is known as ***local thermodynamic equilibrium***, abbreviated as **LTE**.
- For example: ***Inside a star, we expect LTE*** to be a very good approximation and we can solve the radiative transfer equation inside the star.
    - In the ***outermost atmosphere of a star, LTE may fail*** and it often becomes necessary to consider departures from LTE.
- We also often assume ***LTE in the interstellar medium.***

Radiative transfer trough stellar atmospheres

- **Plane parallel atmosphere:** When we **focus on a local region of a stellar atmosphere, we can neglect the curvature and assume the various thermodynamic quantities like the temperature *T* to be constant over horizontal planes.**
- Let us take the z axis in the vertical direction, with z increasing above. Any thermodynamic variable of the atmosphere can be a function of z alone. Let us consider an element of a ray path ds as shown in the Figure
- If dz is the change in z corresponding to ds, then we have

$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu},$$

$$\theta = \cos^{-1}\mu$$

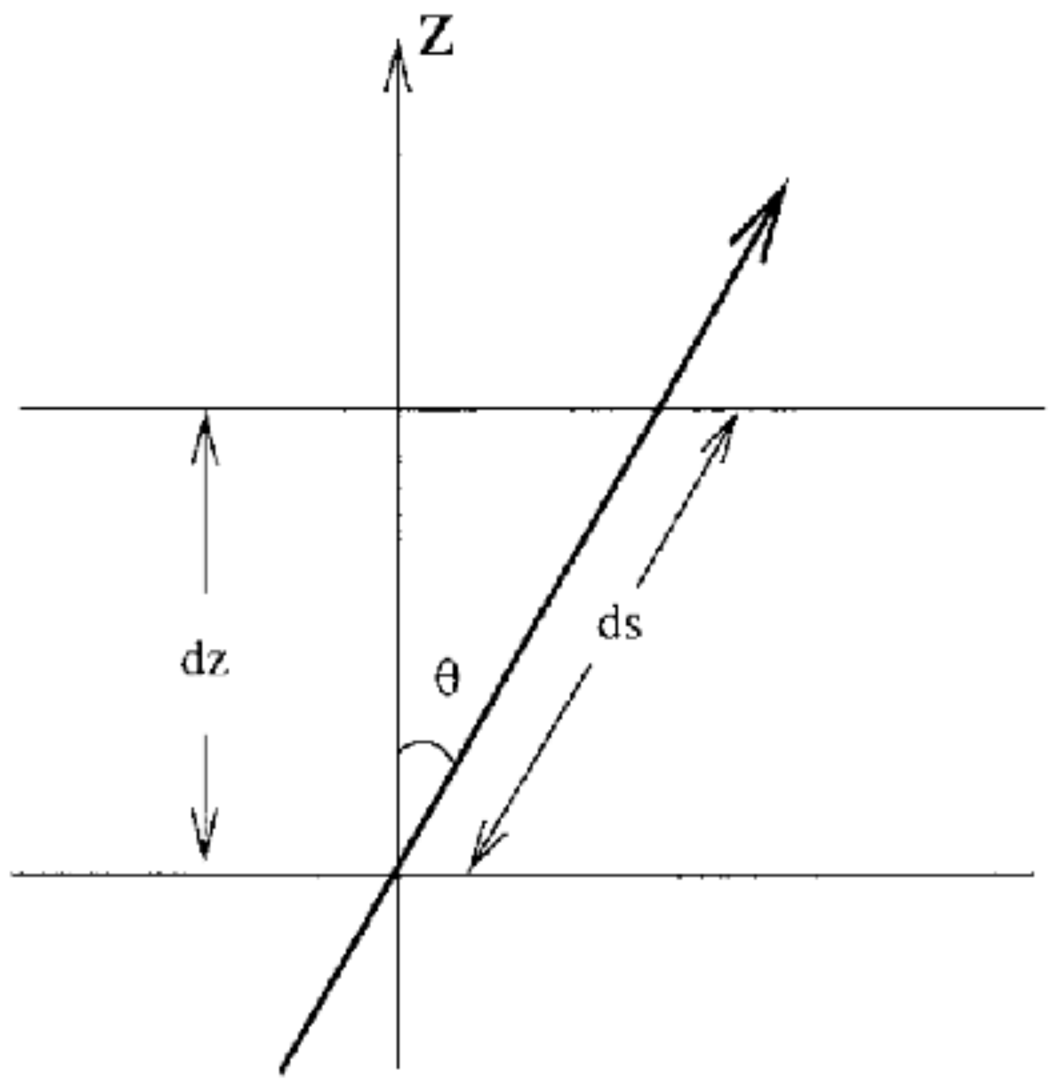


Fig. 2.4 A ray path through a plane parallel atmosphere.

Radiative transfer trough stellar atmospheres

- We have seen in that the specific intensity $I_\nu(\mathbf{r}, t, \mathbf{n})$ can in general be a function of position, time and direction.
- We are **considering a static situation**. In the plane parallel stellar atmosphere, **nothing varies in the horizontal directions** and all direction vectors lying on a cone around the vertical axis are symmetrical.
- -> we expect the specific intensity $I_\nu(z, \mu)$ to be a function of z and $\mu = \cos\theta$ only.
- The radiative transfer gives us the equation for the plane parallel atmosphere problem.

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu, \quad \longrightarrow \quad \mu \frac{\partial I_\nu(z, \mu)}{\partial z} = j_\nu - \alpha_\nu I_\nu$$

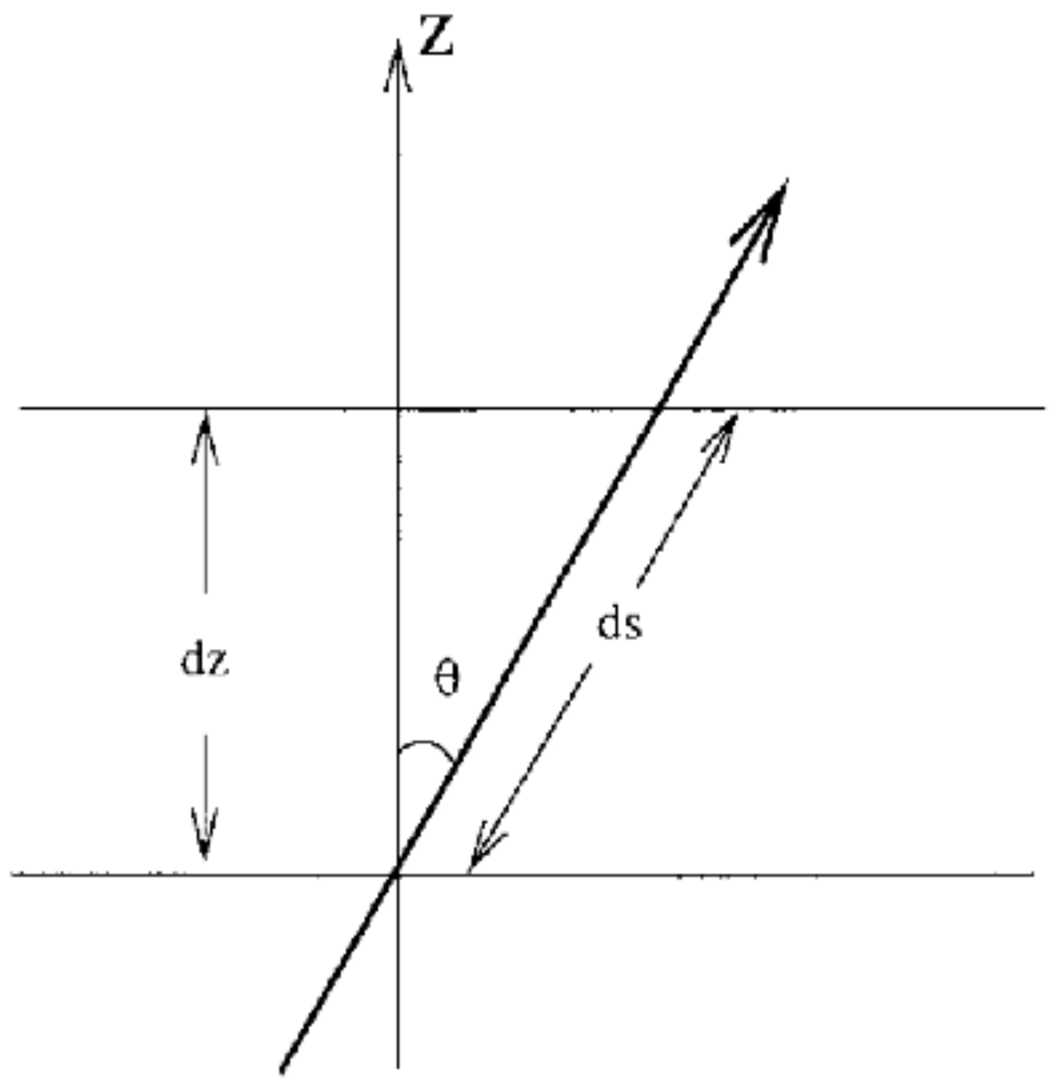


Fig. 2.4 A ray path through a plane parallel atmosphere.

Radiative transfer trough stellar atmospheres

- Optical depth for a plane parallel atmosphere problem. $d\tau_\nu = -\alpha_\nu \, dz.$
- the optical depth is now defined as a function of the vertical distance z rather than the distance along ray path s.
- The negative sign in implies that the **optical depth increases as we go deeper down in the star.**
- The normal convention is to take τ_ν **= 0 at the top of the stellar atmosphere**.

$$\mu \frac{\partial I_\nu(\tau_\nu, \mu)}{\partial \tau_\nu} = I_\nu - S_\nu.$$

$$\mu \frac{d}{d\tau_\nu} \left(I_\nu e^{-\frac{\tau_\nu}{\mu}} \right) = -S_\nu e^{-\frac{\tau_\nu}{\mu}}.$$

$$I_\nu e^{-\frac{t_\nu}{\mu}} \Big|_{\tau_{\nu,0}}^{\tau_\nu} = -\int_{\tau_{\nu,0}}^{\tau_\nu} \frac{S_\nu}{\mu} e^{-\frac{t_\nu}{\mu}} dt_\nu.$$

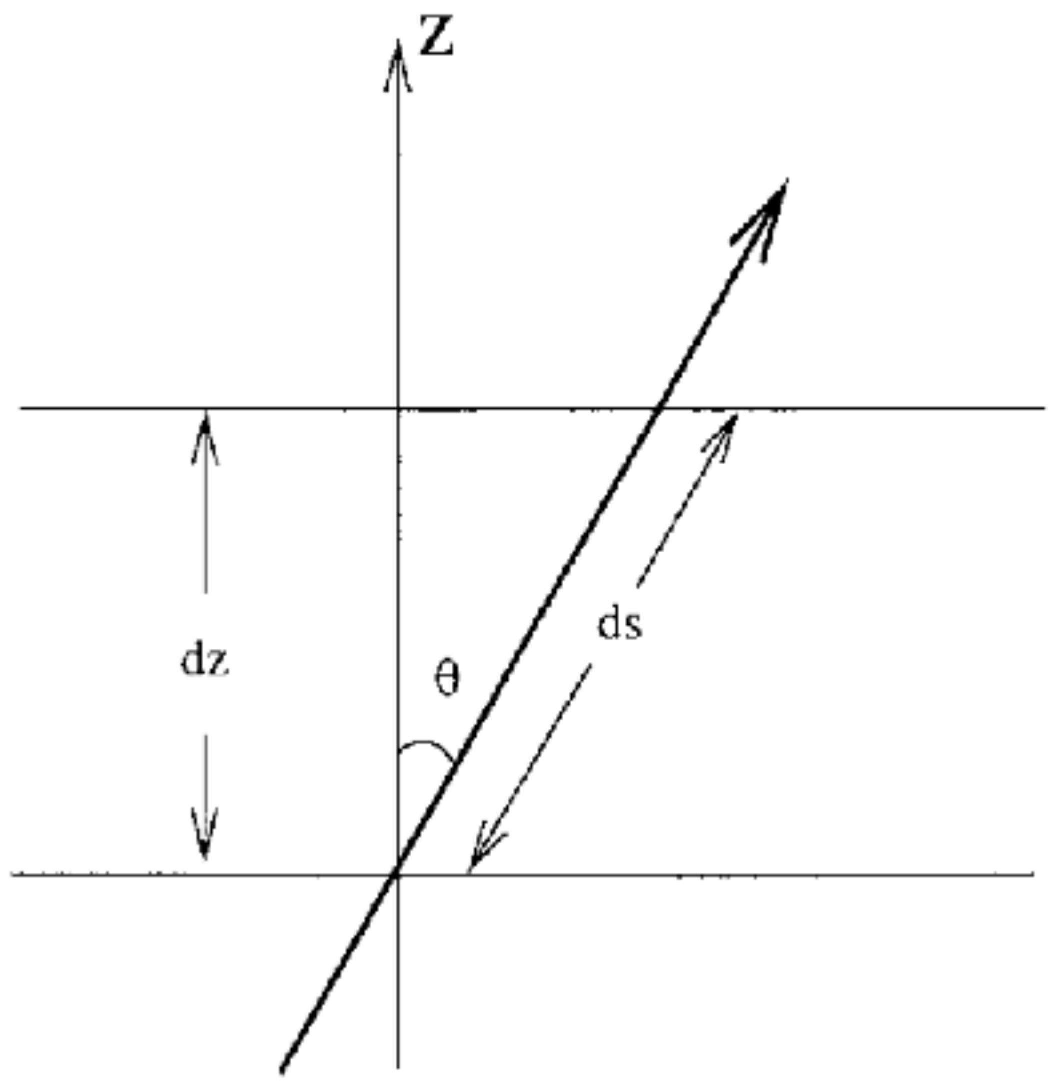


Fig. 2.4 A ray path through a plane parallel atmosphere.

Radiative transfer trough stellar atmospheres

- We now consider two cases separately:
  - (I) $0 \leq \mu \leq 1$, which corresponds to a **ray path proceeding in the outward** direction in the stellar atmosphere; and the ray path can be assumed to begin from a great depth inside the star and we can take $\tau_{\nu,0} = \infty$.
  - (II) $-1 \leq \mu \leq 0$, which corresponds to a **ray path going inward** in the stellar atmosphere and we can take $\tau_{\nu,0} = 0$.

$$0 \leq \mu \leq 1: \quad I_\nu(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)/\mu} \frac{dt_\nu}{\mu},$$

$$-1 \leq \mu \leq 0: \quad I_\nu(\tau_\nu, \mu) = \int_{0}^{\tau_\nu} S_\nu e^{-(\tau_\nu - t_\nu)/(-\mu)} \frac{dt_\nu}{(-\mu)}.$$

Radiative transfer trough stellar atmospheres

- We Let us assume **LTE** throughout the stellar atmosphere so that the **source function everywhere is equal to the Planck function** there, in accordance with Kirchhoff's law.
- Suppose we want to **find out the radiation field at some optical depth** τ_ν. The source function there is given by $B_\nu(T(\tau_\nu))$ which we write as $B_\nu(\tau_\nu)$ for simplification.
- The source function at a nearby optical depth t_ν can be written in the form of a Taylor expansion around the optical depth τ_ν, i.e.

$$S_\nu(t_\nu) = B_\nu(\tau_\nu) + (t_\nu - \tau_\nu)\frac{dB_\nu}{d\tau_\nu} + \dots$$

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu\frac{dB_\nu}{d\tau_\nu},$$

Radiative transfer trough stellar atmospheres

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu},$$

- Provided the point considered is sufficiently inside the atmosphere such that $\tau_\nu \gg 1$ and we can take $e^{-\tau_\nu}$ to be zero.
- The **second term** on the right-hand side depends on μ and makes the **radiation field anisotropic**.
- **If there was no variation of temperature within the stellar atmosphere**, then $dB_\nu/d\tau_\nu$ would vanish and the **radiation field would become an isotropic blackbody radiation.**
- The presence of a **temperature gradient** in the atmosphere **makes the radiation field anisotropic.**

Radiative transfer trough stellar atmospheres

- the **radiation flux, the energy density and the pressure** of a radiation field can be calculated from the specific intensity
- In the case of a plane parallel atmosphere, the integration over the solid angle becomes simplified due to symmetry. If $A(\cos\theta)$ is any function of angle in a plane parallel atmosphere, then

$$U_\nu = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \, d\mu, \qquad U_\nu = \frac{4\pi}{c} B_\nu(\tau_\nu),$$

$$F_\nu = 2\pi \int_{-1}^{1} I_\nu \, \mu \, d\mu, \quad \longrightarrow \quad F_\nu = \frac{4\pi}{3}\frac{dB_\nu}{d\tau_\nu},$$

$$P_\nu = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \, \mu^2 \, d\mu. \qquad P_\nu = \frac{4\pi}{3c} B_\nu(\tau_\nu).$$

Radiative transfer trough stellar atmospheres

- the **ratio of the anisotropic part** in the radiation field **to the isotropic part** is of order

$$\frac{dB_\nu/d\tau_\nu}{B_\nu} \approx \frac{3F_\nu}{cU_\nu}$$

Approximating F_ν/U_ν by F/U , where F and U are respectively the total flux and the total energy density integrated over all wavelengths

$$\frac{\text{Anisotropic term}}{\text{Isotropic term}} \approx \frac{3F}{c\,U}.$$

Radiative transfer trough stellar atmospheres

- thermal physics that the total energy density of blackbody radiation is given by

$$U = a_B T^4,$$

The total flux is the flux which eventually emerges out of the surface and is given by the Stefan–Boltzmann law:

$$\boxed{F = \sigma T_{\mathrm{eff}}^4,} \qquad \sigma = \frac{c a_B}{4}$$

T_{eff} is the temperature on the surface of the star

$$\frac{\text{Anisotropic term}}{\text{Isotropic term}} \approx \frac{3}{4}\left(\frac{T_{\mathrm{eff}}}{T}\right)^4$$

Radiative transfer trough stellar atmospheres

- **As we go deeper in a stellar atmosphere, T becomes much larger than T_{eff}, making the anisotropic term negligible compared to the isotropic term.**
- The **radiation field is nearly isotropic in sufficiently deep layers of a stellar atmosphere** where the temperature is considerably higher than the surface temperature.
- If we knew how temperature varied with depth we could calculate the Planck function $B_\nu(\tau_\nu)$ at different depths. This is not known in general and the real **challenge of studying stellar atmospheres is to solve the radiation field and the temperature structure** of the stellar atmosphere simultaneously.

Grey atmosphere

- **If the absorption coefficient α_ν is constant for all frequencies, then the atmosphere is called a grey atmosphere.**
- There is no real stellar atmosphere which has this property. The grey atmosphere is an **idealised mathematical model** which is much simpler to treat than a more realistic stellar atmosphere
- If α_ν is independent of frequency, then it follows that the value of optical depth at some physical depth will be the same for all frequencies.

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I - S,$$

$$I = \int I_\nu \, d\nu$$

$$S = \int S_\nu \, d\nu$$

Grey atmosphere

- We also define the **average specific intensity *J*** averaged over all angles

$$J = \frac{1}{2}\int_{-1}^{1} I \, d\mu . \qquad J = \frac{c}{4\pi} U .$$

- Using these expressions of J, we obtain two important moment equations:

$$\boxed{\frac{1}{4\pi}\frac{dF}{d\tau} = J - S} \qquad \boxed{\frac{dP}{d\tau} = \frac{F}{c} .}$$

- *I* is a function of both τ and μ, it may be noted that *F* and *P* are functions of τ alone

Grey atmosphere

- **The energy generated in the stellar interior passes out in the form of a constant energy flux** through the outer layers of the stellar atmosphere. In other words, *F* has to be independent of depth. It follows that this is possible only if J = S
- **the average specific intensity has to be equal to the source function. This is called the condition of *radiative equilibrium*,** and can also be expressed as:

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I - \frac{1}{2} \int_{-1}^{1} I \, d\mu$$

There are techniques for solving this equation exactly and obtaining I for all τ and μ.

Grey atmosphere

- Since F is constant under the condition of radiative equilibrium, we can obtain

$$P = \frac{F}{c}(\tau + q),$$

where q **is the constant of integration**. The total pressure and total energy density of an isotropic radiation field are related by

$$P = \frac{1}{3}U.$$

- The radiation field becomes nearly isotropic as we go somewhat below the surface. Just underneath the surface, however, we do not expect isotropy.
- **If we assume isotropy to hold everywhere, then finding a full solution to our problem becomes straightforward. This is known as the Eddington approximation.** Under this approximation, we can combine the equations to obtain

$$S = \frac{3F}{4\pi}(\tau + q).$$

Grey atmosphere

- We have seen from equations that the specific intensity can easily be written down if the source function is given.
- The main challenge is to obtain the solution for the source function (which depends on the temperature) at different depths along with the specific intensity.
- **If we can evaluate the constant of integration q, then we can get the solution for the source function for the grey atmosphere problem (under the Eddington approximation).**
- q can be evaluated by calculating the flux from this equation, and setting it equal to F

$$S = \frac{3F}{4\pi}(\tau + q).$$

Grey atmosphere

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} = I - S, \longrightarrow \quad I(\tau, \mu \geq 0) = \int_\tau^\infty S\, e^{-(t-\tau)/\mu} \frac{dt}{\mu}.$$

Setting $\tau = 0$ on the surface of the star $\longrightarrow$ $$I(0,\mu) = \int_0^\infty S\, e^{-t/\mu} \frac{dt}{\mu}.$$

Substituting:

$$S = \frac{3F}{4\pi}(\tau + q). \longrightarrow \quad I(0,\mu) = \frac{3F}{4\pi}\int_0^\infty (t+q) e^{-t/\mu} \frac{dt}{\mu} = \frac{3F}{4\pi}(\mu + q).$$

The flux coming out of the upper surface of the stellar atmosphere is:

$$F = 2\pi \int_0^1 I\, \mu\, d\mu. \longrightarrow \quad F = \frac{3F}{2}\left(\frac{1}{3} + \frac{q}{2}\right), \longrightarrow \quad q = \frac{2}{3}.$$

Grey atmosphere

On putting this value of q into the equation, the source function as a function of depth inside the stellar atmosphere is finally given by

$$S = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right) \qquad cU = 3F\left(\tau + \frac{2}{3}\right).$$

Tells us how temperature varies inside a grey atmosphere:

$$T^4 = \frac{3}{4}T_{\mathrm{eff}}^4\left(\tau + \frac{2}{3}\right).$$

Finally we derive an important result for **radiation coming out from the stellar surface.**

$$I(0,\mu) = \frac{3F}{4\pi}\left(\mu + \frac{2}{3}\right), \quad \longrightarrow \quad \boxed{\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right).}$$

Limb darkening

Limb darkening

- Suppose we consider the **intensity of radiation coming from different points on the disk of the Sun** as seen by us.
- The ray coming from the central point of the solar disk emerges out of the solar surface in the vertical direction and the specific intensity for this ray will be $I\,(0, 1)$.
- the ray coming from an off-centre point must emerge from the solar surface at an angle $\theta = \cos^{-1}\mu$ with the vertical, as seen in the Figure, and the corresponding specific intensity will be $I\,(0, \mu)$.

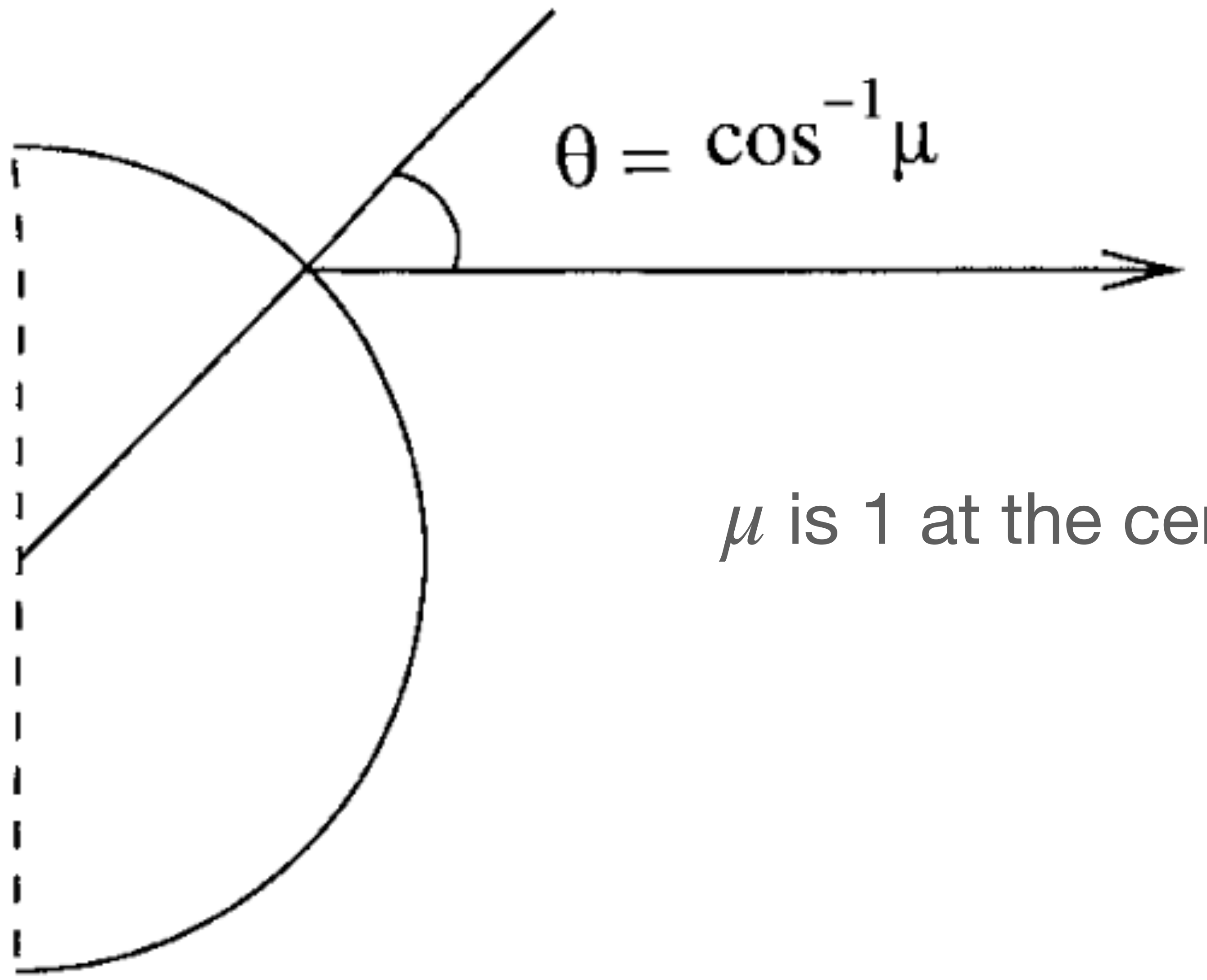


Fig. 2.5 A ray coming to an observer from the solar disk.

μ is 1 at the centre and 0 at the limb

Limb darkening

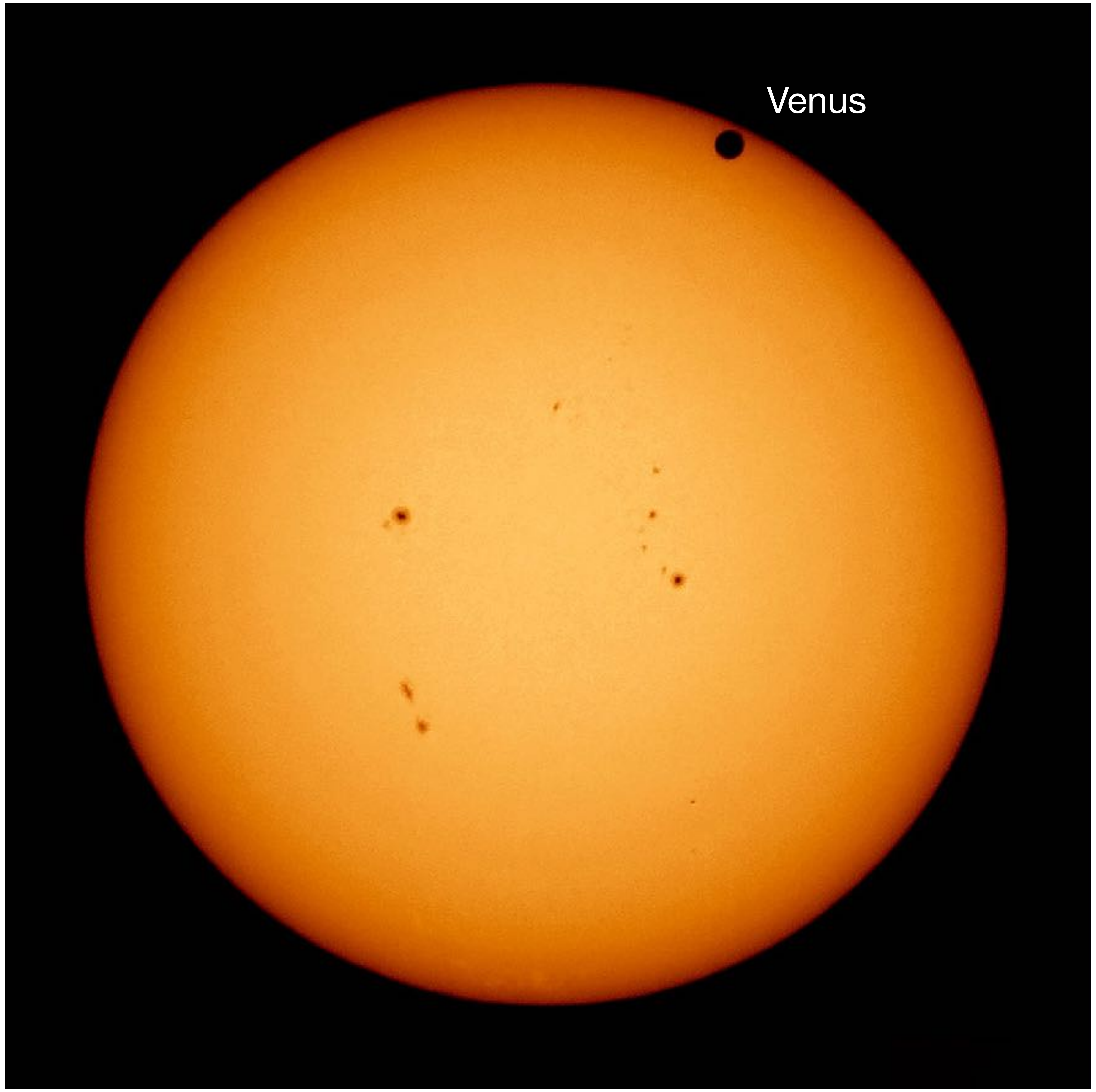
Intensity ratio between the centre and the limb (edge)

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right).$$

- Gives the variation of intensity on the solar disk as we move from the centre to the edge.
- The ***region near the edge of the solar disk is referred to as the limb of the Sun***.
- Giving the **variation of intensity over the solar disk is called the *limb-darkening law*.**
- The theoretical limb-darkening law predicts that the intensity at the edge of the solar disk will be about 40% of the intensity at the centre.

Image of the Sun, during the Venus transit in 2012

Notice the darker edges!



Limb darkening

- The Figure shows the observationally determined limb-darkening along with the plot obtained by the Eddington approximation (dashed) as well as the theoretical limb-darkening law (solid line) derived by an exact solution of the grey atmosphere problem
- The theory matches the observational data (dots) reasonably well, the discrepancy between the two is due to the fact that the solar atmosphere is not grey.

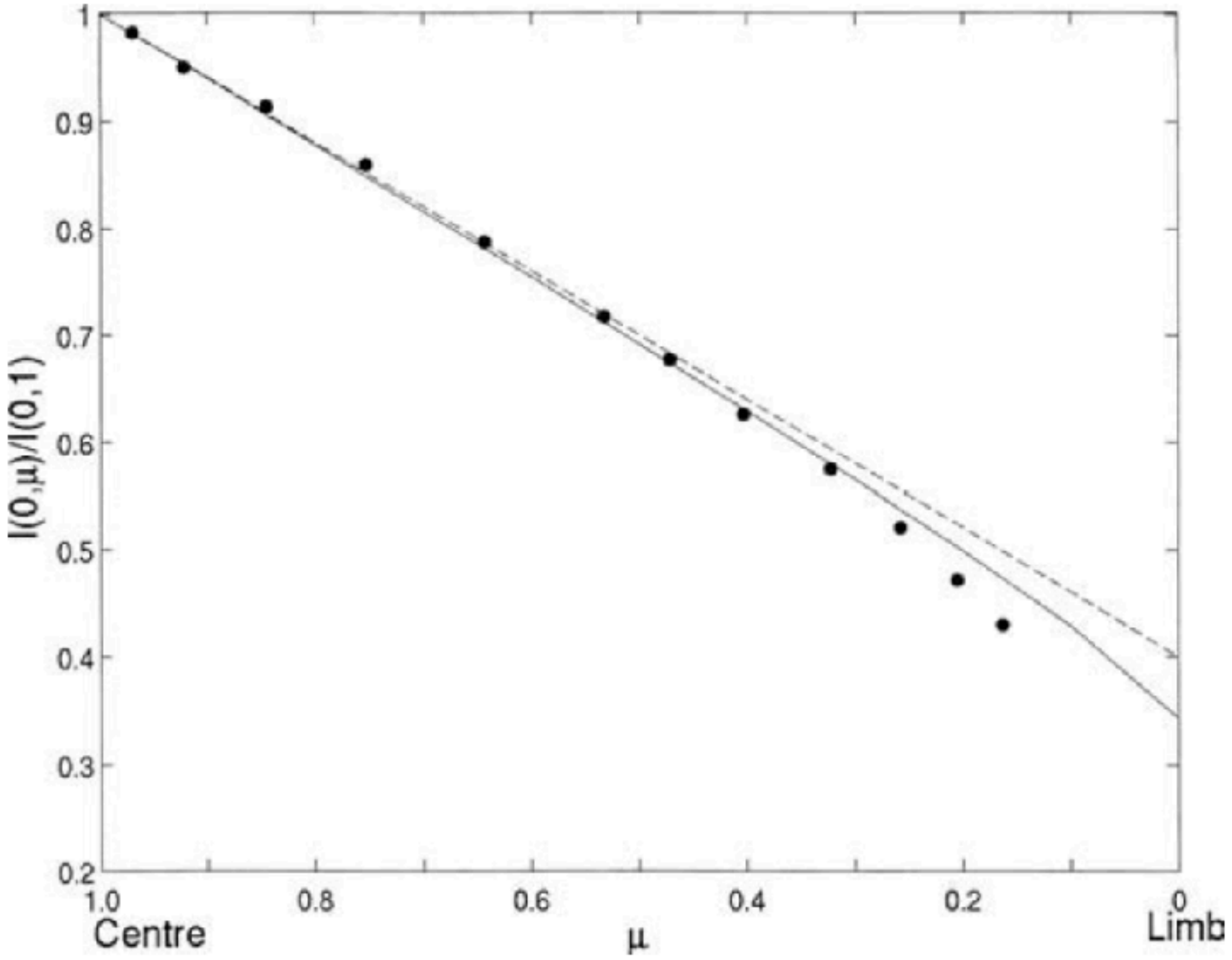


Image of the Sun, during the Venus transit in 2012

Notice the darker edges!