Introduction to Astrophysics and Cosmology

Relativistic Cosmology

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Relativistic cosmology

Some topics in cosmology – especially those dealing with the analysis of high redshift observations – require general relativity for a proper understanding.

It is convenient to use the **co-moving coordinates in cosmology.** If we neglect the motions which galaxies may have with respect to the expanding space, then **a galaxy is at rest in this coordinate system and space is supposed to expand uniformly**, carrying the galaxies with it.

If matter is at rest in a coordinate system, then the **energy-momentum tensor** T_k^i in that system is **given by (12.88)**. We clearly expect this to be the energy-momentum tensor of the Universe in the co-moving coordinates. Since we had tacitly assumed the coordinate system to be Cartesian when putting the classical hydrodynamic equations in the form (12.78), one may wonder if (12.88) is the expression of the energy-momentum tensor only in Cartesian coordinates.

$$T_k^i = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \qquad (12.88)$$

Relativistic cosmology

We can use (12.81) **to introduce the generalized velocity in any coordinate system** and (12.84) **should give the general expression of the energy-momentum tensor in any coordinate system,** leading to (12.88) **in the special case when matter is at rest**. The energy-momentum tensor defined by (12.84) in different coordinate systems should transform according to the tensor transformation law (12.5).

$$\boxed{u^i = \frac{1}{c}\frac{dx^i}{d\tau}. \qquad (12.81)}$$

$$\boxed{\mathcal{T}^{ik} = \rho c^2 u^i u^k + P(g^{ik} + u^i u^k). \qquad (12.84)}$$

$$\boxed{\mathcal{T}^i_k = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \qquad (12.88)}$$

$$\boxed{\overline{T}^{ab..d}_{l..n} = T^{\alpha\beta..\delta}_{\lambda..\nu} \frac{\partial \overline{x}^a}{\partial x^\alpha} \frac{\partial \overline{x}^b}{\partial x^\beta} .. \frac{\partial \overline{x}^d}{\partial x^\delta} \frac{\partial x^\lambda}{\partial \overline{x}^l} .. \frac{\partial x^\nu}{\partial \overline{x}^n}. \qquad (12.5)}$$

Relativistic cosmology

We now want to apply Einstein's equation (12.96) to the Universe, which we write in the form

$$G^i_k = \frac{8\pi G}{c^4} T^i_k. \qquad (14.1)$$

Since we already know that T^i_k is given by (12.88), we only need to obtain the **Einstein tensor G^i_k of the Universe**.

For this we have to **start from the expression of the metric**. The metric of the Universe in the co-moving coordinates is the **Robertson–Walker metric** given by (10.19) or (10.20). We want to calculate the Einstein tensor for this metric. First we have to calculate the **various Christoffel symbols** by using (12.31) and then we have to calculate the **Ricci tensor** R_{ik} by using (12.41).

$$\boxed{ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right]. \qquad (10.19)}$$

$$\boxed{ds^2 = -c^2 dt^2 + a(t)^2 \left[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2) \right], \qquad (10.20)}$$

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The final result:

$$R_{tt} = -3\frac{\ddot{a}}{a}, \quad R_{\alpha\beta} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{kc^2}{a^2}\right)\frac{g_{\alpha\beta}}{c^2}, \tag{14.2}$$

where the Greek indices α or β can have values 1, 2, 3, which we take to be r,θ,ϕ in the present case. It follows from (14.2) that **the scalar curvature is given by**

$$R = R^i_i = \frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right). \tag{14.3}$$

By substituting (14.2) and (14.3) in (12.44), we finally get

$$G^t_t = -\frac{3}{c^2}\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right), \quad G^r_r = G^\theta_\theta = G^\phi_\phi = -\frac{1}{c^2}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right), \tag{14.4}$$

whereas **all the off-diagonal elements of the Einstein tensor are zero.**

Relativistic cosmology

Note that we have to raise an index in accordance with (12.18) and use (12.16) in these calculations. Finally we have to substitute (12.88) and (14.4) in (14.1). The *tt* component gives

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho, \tag{14.5}$$

whereas **the other three diagonal components give the identical equation**

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}P. \tag{14.6}$$

The above two equations are the **basic equations giving the dynamics of spacetime.**

Relativistic cosmology

$$\boxed{\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho, \qquad (10.27)}$$

From considerations of Newtonian cosmology in §10.4, we could write down **the Friedmann equation (10.27), which is identical with (14.5)**.

As we pointed out earlier, it is an astounding coincidence that we get exactly the same equation from a full general relativistic analysis and from very simple considerations of Newtonian cosmology (with some ad hoc assumptions). We now need to figure out the significance of the other equation (14.6). On differentiating (14.5) with respect to t, we get

$$\frac{2\dot{a}\ddot{a}}{a} - \frac{2\dot{a}^3}{a^2} - \frac{2kc^2\dot{a}}{a^2} = \frac{8\pi G}{3}\dot{\rho}a. \qquad (14.7)$$

Assuming the expansion of the Universe to be adiabatic, the first law of thermodynamics $dQ = dU + P\,dV$ suggests

$$\frac{d}{dt}(\rho c^2 a^3) + P\frac{d}{dt}(a^3) = 0, \qquad (14.8)$$

from which

$$c^2(\dot{\rho}a + 3\rho\dot{a}) = -3P\dot{a}.$$

Relativistic cosmology

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho, \qquad (14.5)$$

$$\frac{d}{dt}(\rho c^2 a^3) + P\frac{d}{dt}(a^3) = 0, \qquad (14.8)$$

Multiplying this by $8\pi G/3c^2$, we get

$$\frac{8\pi G}{3}\dot{\rho}a + 3\dot{a}\frac{8\pi G}{3}\rho = -\frac{8\pi G}{c^2}P\dot{a}.$$

We now substitute from (14.7) in the first term of this equation and substitute from (14.5) for $(8\pi G/3)\rho$ in the second term. A few steps of algebra then lead to (14.6). This means that (14.6) can be obtained from (14.5) and (14.8). We can, therefore, **regard (14.5) and (14.8) as our basic equations.**

(14.8) would lead to (10.37) if P is given by (10.36). For a Universe filled with matter and radiation, (10.37) becomes (10.50).

In other words, **the expression (10.50) for the density is equivalent to (14.8) for a Universe filled with matter and radiation.** We are thus finally led to the conclusion that (14.5) and (10.50) constitute our basic equations. We have already discussed the solutions of these equations in §10.6 and §10.7 for the cases of the **matter-dominated and the radiation-dominated Universe**, corresponding respectively to the later and earlier epochs in the thermal history of the Universe.

$$\rho = \rho_{\mathrm{M},0}\left(\frac{a_0}{a}\right)^3 + \rho_{\mathrm{R},0}\left(\frac{a_0}{a}\right)^4. \qquad (10.50)$$

Relativistic cosmology

There was **one important topic which could not be discussed satisfactorily within the framework of Newtonian cosmology. It is the propagation of light**.

We shall now study the propagation of light systematically.

The cosmological constant

It is possible to put **an extra term in Einstein's equation**, which would make it

$$\boxed{G_{ik} = \frac{8\pi G}{c^4} T_{ik} - \frac{\Lambda}{c^2} g_{ik}. \qquad (14.9)}$$

It follows from (12.27) that g_{ik} **is also a divergenceless tensor** like G_{ik} or T_{ik}. So, **on taking the divergence of (14.9), each term will give zero if Λ is a constant.** It is thus mathematically consistent to add the last term in (14.9). **The constant Λ is called the *cosmological constant***.

If Einstein's equation is extended to (14.9) by including the cosmological constant term, then (14.5) and (14.6) also get modified to

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \qquad (14.10)$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2} P + \Lambda. \qquad (14.11)$$

The cosmological constant

On subtracting (14.10) from (14.11), we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}. \tag{14.12}$$

If $\Lambda = 0$, it is clear from (14.12) that **a static solution in which *a* does not change with time is not possible.**

Einstein first applied general relativity to cosmology at a time when the expansion of the Universe had not yet been discovered. **Einstein wanted to construct a static solution of the Universe which is possible only with a non-zero Λ.** Assuming $P \ll \rho c^2$, if the density has the value

$$\rho_0 = \frac{\Lambda}{4\pi G}, \tag{14.13}$$

it is easily seen from (14.12) that **we get a static solution**. One, however, finds that **this static solution is unstable**. In other words, if this static Universe is disturbed from its initial static state, it will run away from this static state.

The cosmological constant

After the expansion of the Universe was reported, Einstein is said to have remarked that introducing the cosmological constant was the 'biggest blunder' of his life. Thereafter, **the cosmological constant was almost banished from the literature of astrophysically motivated cosmology** for several decades. Standard textbooks of cosmology would either ignore it or at most devote a small section to the cosmological constant as an odd curiosity.

Some recent observations, however, suggest that the cosmological constant may after all be non-zero. This is one of the most dramatic new developments in cosmology, leading to renewed interest in the cosmological constant. Here we present a brief discussion of how cosmological solutions get modified on including the cosmological constant.

Substituting for ρ from (10.50) in (14.10), we get

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\left[\rho_{\mathrm{M},0}\left(\frac{a_0}{a}\right)^3 + \rho_{\mathrm{R},0}\left(\frac{a_0}{a}\right)^4 + \rho_\Lambda\right], \tag{14.14}$$

where we have written

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}. \tag{14.15}$$

The cosmological constant

It follows from (10.28) that we can write

$$\frac{8\pi G}{3} = \frac{H_0^2}{\rho_{c,0}}. \tag{14.16}$$

Then (14.14) can be put in the form

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = H_0^2\left[\Omega_{\mathrm{M},0}\left(\frac{a_0}{a}\right)^3 + \Omega_{\mathrm{R},0}\left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda,0}\right], \tag{14.17}$$

where

$$\boxed{\Omega_{\mathrm{M},0} = \frac{\rho_{\mathrm{M},0}}{\rho_{c,0}}, \quad \Omega_{\mathrm{R},0} = \frac{\rho_{\mathrm{R},0}}{\rho_{c,0}}, \quad \Omega_{\Lambda,0} = \frac{\rho_{\Lambda}}{\rho_{c,0}} \qquad (14.18)}$$

give the fractional contributions at the present time of matter, radiation and the cosmological constant to the critical density.

$$\rho \propto \frac{1}{a^{3(1+w)}}. \tag{10.37}$$

The cosmological constant

It should be clear from (14.14) that the effect of **the cosmological constant is like a fluid whose density ρ_Λ does not change with the expansion of the Universe**.

It follows from (10.37) that we **should have w = −1 for such a fluid, suggesting a negative pressure** $P = -\rho c^2$ on the basis of (10.36). This strange fluid-like entity with a negative pressure is often referred to as the ***dark energy*.**

We know that the contribution of radiation density has been negligible ever since the Universe became matter-dominated. Observations suggest that the Universe is nearly flat. **If we neglect the radiation density and the curvature terms**, then (14.17) can be written as

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_{\mathrm{M},0} \left(\frac{a_0}{a} \right)^3 + \Omega_{\Lambda,0} \right]. \tag{14.19}$$

It is possible to write down an **analytical solution of this equation,** which is

$$\frac{a}{a_0} = \left(\frac{\Omega_{\mathrm{M},0}}{\Omega_{\Lambda,0}} \right)^{1/3} \sinh^{2/3} \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 t \right). \tag{14.20}$$

The cosmological constant

This can be verified by substituting this solution (14.20) into (14.19). It is instructive to consider **the early time and the late time limits of the solution** (14.20), which are

$$\frac{a}{a_0} \approx \left(\frac{3}{2}\Omega_{\mathrm{M},0}^{1/2} H_0 t\right)^{2/3} \quad \text{if} \quad \sqrt{\Omega_{\Lambda,0}} H_0 t \ll 1, \tag{14.21}$$

$$\frac{a}{a_0} \approx \left(\frac{\Omega_{\mathrm{M},0}}{4\Omega_{\Lambda,0}}\right)^{1/3} e^{\sqrt{\Omega_{\Lambda,0}} H_0 t} \quad \text{if} \quad \sqrt{\Omega_{\Lambda,0}} H_0 t \gg 1. \tag{14.22}$$

To understand the significance of these limiting solutions, we look at (14.19). **Since the matter density falls off as a^{-3}, it becomes less important with time as the Universe expands, whereas the term involving the cosmological constant Λ grows in relative importance.**

The solution (14.21) **at early times is the matter-dominated Universe solution**. The fact that $\Omega_{\Lambda,0}$ cancels out of the equation justifies our assertion that, even if Λ is non-zero, we do not make too much error in many calculations involving earlier times if we use the cosmological solution with $\Lambda = 0$.

The cosmological constant

The values of $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$ suggest that **the present epoch of the Universe may actually be the epoch when the cosmological solution is making the transition from (14.21) to (14.22)**.

The solution (14.22) for late times is essentially what we would get on neglecting the matter density term in (14.19) and **keeping only the cosmological constant term**. This exponential part of (14.22) follows directly from (14.10) if we neglect the curvature and the density terms, noting that

$$\sqrt{\Omega_{\Lambda,0}}H_0 = \sqrt{\frac{\Lambda}{3}} \tag{14.23}$$

It is clear from (14.22) that **the cosmological constant is of the nature of a cosmic repulsion which makes the Universe expand exponentially when it is the dominant term over density and curvature.**

The cosmological constant

The different behaviours at early and late times can be understood by considering **how $\dot{a}$ changes with time.**

Figure 14.1 shows a plot of $\dot{a}$ obtained from the solution (14.20) plotted against the time t.

At early times, the matter density is dominant and pulls back on the expanding Universe, making the **expansion rate $\dot{a}$ decrease with time**.

On the other hand, **when the Λ term dominates at late times, the expansion of the Universe accelerates, making $\dot{a}$ increase with time.**

Observations suggest that at the present time the Universe may be making a transition from the matter-dominated era to the Λ-dominated era.

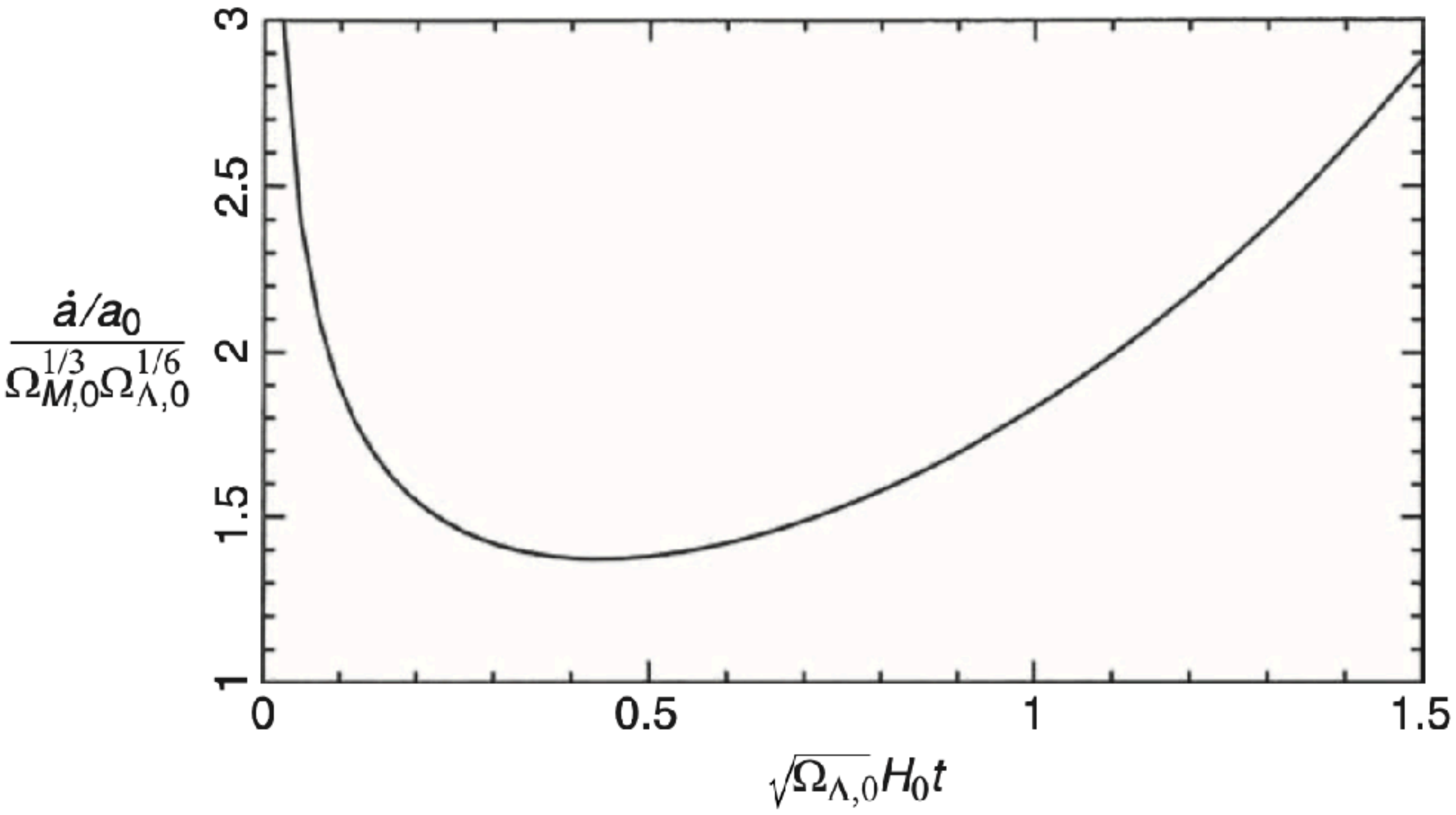


Fig. 14.1 A plot showing how the expansion rate $\dot{a}$ of a flat Universe ($k = 0$) with matter and a non-zero cosmological constant Λ changes with time.

Propagation of light in the expanding Universe

A light signal travels along a null geodesic, i.e. a special geodesic along which $ds^2 = 0$.

We now want to consider the **propagation of a light signal from a distant galaxy to us**. Let us take our position to be the **origin of our coordinate system** and let the **position of the distant galaxy be at the radial co-moving coordinate $r = S(\chi)$**, the metric of the Universe being the **Robertson–Walker metric** given by (10.19) or (10.20). Remember that $S(\chi)$ has to be χ, $\sin\chi$ or $\sinh\chi$ corresponding to the values 0, +1 or −1 of k. From considerations of symmetry, **we expect the light signal to propagate in the negative radial direction to reach us** from the distant galaxy so that **$d\theta = d\phi = 0$ along the path** of light propagation. Putting further $ds^2 = 0$, we conclude from (10.20) that the light signal propagation is given by

$$-c^2 dt^2 + a(t)^2 d\chi^2 = 0.$$

If we now replace t by the coordinate η defined through (10.33), then we have

$$a(t)^2[-d\eta^2 + d\chi^2] = 0, \qquad (14.24)$$

from which

$$d\chi = \pm d\eta.$$

Propagation of light in the expanding Universe

As the light signal propagates towards us, the **radial coordinate χ decreases with the increasing time.** So we choose the minus sign and **write the solution**

$$\chi = \eta_0 - \eta, \tag{14.25}$$

where η_0 is the constant of integration.

To understand the significance of this constant of integration, note that (14.25) gives the position χ of the light signal at time η. Since $\chi = 0$ at the time $\eta = \eta_0$, we easily see that η_0 **is the time when the light signal reaches us.**

We now consider **a monochromatic light wave starting from the galaxy at χ**. Let the two successive crests of the sinusoidal light wave leave the galaxy at times η and $\eta + \Delta\eta$. Suppose these successive crests reach us at η_0 and $\eta_0 + \Delta\eta_0$. For the second crest, it follows from (14.25) that

$$\chi = (\eta_0 + \Delta\eta_0) - (\eta - \Delta\eta).$$

Propagation of light in the expanding Universe

Let us subtract from this (14.25) which is satisfied by the first crest. Then we get

$$\Delta\eta_0 = \Delta\eta. \tag{14.26}$$

In other words, if we were to **use the time-like coordinate η to measure time, then the period of the wave would be the same when it was emitted and when it was received.**

But **η does not give the physical time.** For a stationary observer somewhere in the Universe, we have

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2$$

from (10.18). Thus the **proper time τ** at which the observer's clock runs **coincides with *t***. Suppose Δt and Δt_0 are the periods measured by physical clocks when the light was emitted and when the light was received, the **scale factors of the Universe being *a* and a_0** respectively at those times. It then follows from (10.33) that

$$c\Delta t = a\Delta\eta, \;\; c\Delta t_0 = a_0\Delta\eta_0,$$

Propagation of light in the expanding Universe

from which

$$\frac{\Delta t_0}{\Delta t} = \frac{a_0}{a} \tag{14.27}$$

on making use of (14.26).

Since an **observer in any location in the Universe would think that light propagates at speed c,** (14.27) suggests (10.24). The significance of (10.24) is that **light propagating in the expanding Universe gets stretched proportionately so that the wavelength of light expands the same way as the scale factor.** The frequency of a photon should fall as a^{-1}.

We can now prove a very important result which we have been using throughout our discussion of cosmology. **Blackbody radiation filling the expanding Universe continues to remain blackbody radiation even when there is no interaction with matter.**

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a_0}{a}. \tag{10.24}$$

Propagation of light in the expanding Universe

We consider radiation within the frequency range ν to $\nu + d\nu$ when the scale factor is a. If this radiation is initially blackbody radiation, then the **energy $U_\nu d\nu$ in unit volume is given by the Planck distribution** (2.1). **After the Universe has expanded to scale factor a',** the theory of light propagation suggests that the **frequencies will change to**

$$\nu \to \nu' = \nu \frac{a}{a'}, \quad \nu + d\nu \to \nu' + d\nu' = (\nu + d\nu)\frac{a}{a'}.$$

The radiation initially lying between frequencies ν, $\nu + d\nu$ would **now lie between frequencies ν' , $\nu' + d\nu'$** and the radiation initially occupying unit v**olume would now occupy a volume $(a'/a)^3$**, having **lost some energy in the work done for expansion**. We can write down the $dU + P\,dV = 0$ relation for this radiation, which will lead to something like (10.37) with w = 1/3. In other words, we must have

$$U'_{\nu'} d\nu' = U_\nu d\nu \left(\frac{a}{a'}\right)^4$$

where $U_\nu{}' d\nu'$ is the energy in unit volume lying in the frequency range from ν' to $\nu' + d\nu'$ when the Universe has expanded to scale factor a'.

Propagation of light in the expanding Universe

If we substitute $\nu = \nu'\,(a'/a)$ along with $T = T'\,(a'/a)$ in the expression of U_ν as given by (2.1), it is straightforward to see that U_ν' **will have the same functional dependence on** ν' **that** U_ν **had on** ν. This completes our proof that the **radiation continues to remain blackbody radiation** in the expanding Universe **with the temperature falling as** $T \propto a^{-1}$.

It is clear that **light arriving from a galaxy** at a certain location $r = S(\chi)$ **would show a certain redshift** z.

We now try to **find a functional relationship between** r **and** z. What we have discussed so far in this section holds whether the cosmological constant Λ is zero or non-zero.

Now we shall consider the case of a **matter-dominated Universe with** $\Lambda = 0$, since it is possible to derive an elegant analytical expression relating r with z only in this case and it is useful to discuss this case before a more general discussion with non-zero Λ.

Let us first consider a **matter-dominated** $\Lambda = 0$ **Universe with positive curvature**, i.e. $k = +1$. This case has a solution given by (10.56) which, in conjunction with (10.24), gives

$$\frac{a}{a_0} = \frac{\Omega_{M,0}}{2(\Omega_{M,0}-1)}(1-\cos\eta), \qquad (10.56)$$

$$\frac{1}{1+z} = \frac{\Omega_{M,0}}{2(\Omega_{M,0}-1)}(1-\cos\eta).$$

Propagation of light in the expanding Universe

From this, we get

$$\cos\eta = \frac{\Omega_{M,0}(z-1)+2}{\Omega_{M,0}(1+z)}. \tag{14.28}$$

Applying the rule $\sin^2\eta + \cos^2\eta = 1$, we then find

$$\sin\eta = \frac{2\sqrt{\Omega_{M,0}-1}\,\sqrt{\Omega_{M,0}z+1}}{\Omega_{M,0}(1+z)}. \tag{14.29}$$

The light signal reaches us at time η_0, which means that $z = 0$ when $\eta = \eta_0$. Then (14.28) and (14.29) imply

$$\cos\eta_0 = \frac{2-\Omega_{M,0}}{\Omega_{M,0}}, \quad \sin\eta_0 = \frac{2\sqrt{\Omega_{M,0}-1}}{\Omega_{M,0}}. \tag{14.30}$$

In the $k = +1$ case we are considering, we have $r = \sin\chi$. Then (14.25) implies

$$r = \sin(\eta_0 - \eta) = \sin\eta_0\cos\eta - \sin\eta\cos\eta_0.$$

Propagation of light in the expanding Universe

Substituting from (14.28), (14.29) and (14.30), we get

$$r = \frac{2\sqrt{\Omega_{\mathrm{M},0} - 1}[\Omega_{\mathrm{M},0}(z-1) + 2 - \sqrt{\Omega_{\mathrm{M},0}z + 1}(2 - \Omega_{\mathrm{M},0})]}{\Omega_{\mathrm{M},0}^2(1+z)}. \qquad (14.31)$$

From (10.31), we can write

$$\sqrt{\Omega_{\mathrm{M},0} - 1} = \frac{c}{a_0 H_0}.$$

Substituting this in (14.31), we finally get

$$\boxed{r = \frac{2\Omega_{\mathrm{M},0}z + (2\Omega_{\mathrm{M},0} - 4)(\sqrt{\Omega_{\mathrm{M},0}z + 1} - 1)}{a_0 H_0 \Omega_{\mathrm{M},0}^2(1+z)/c}. \qquad (14.32)}$$

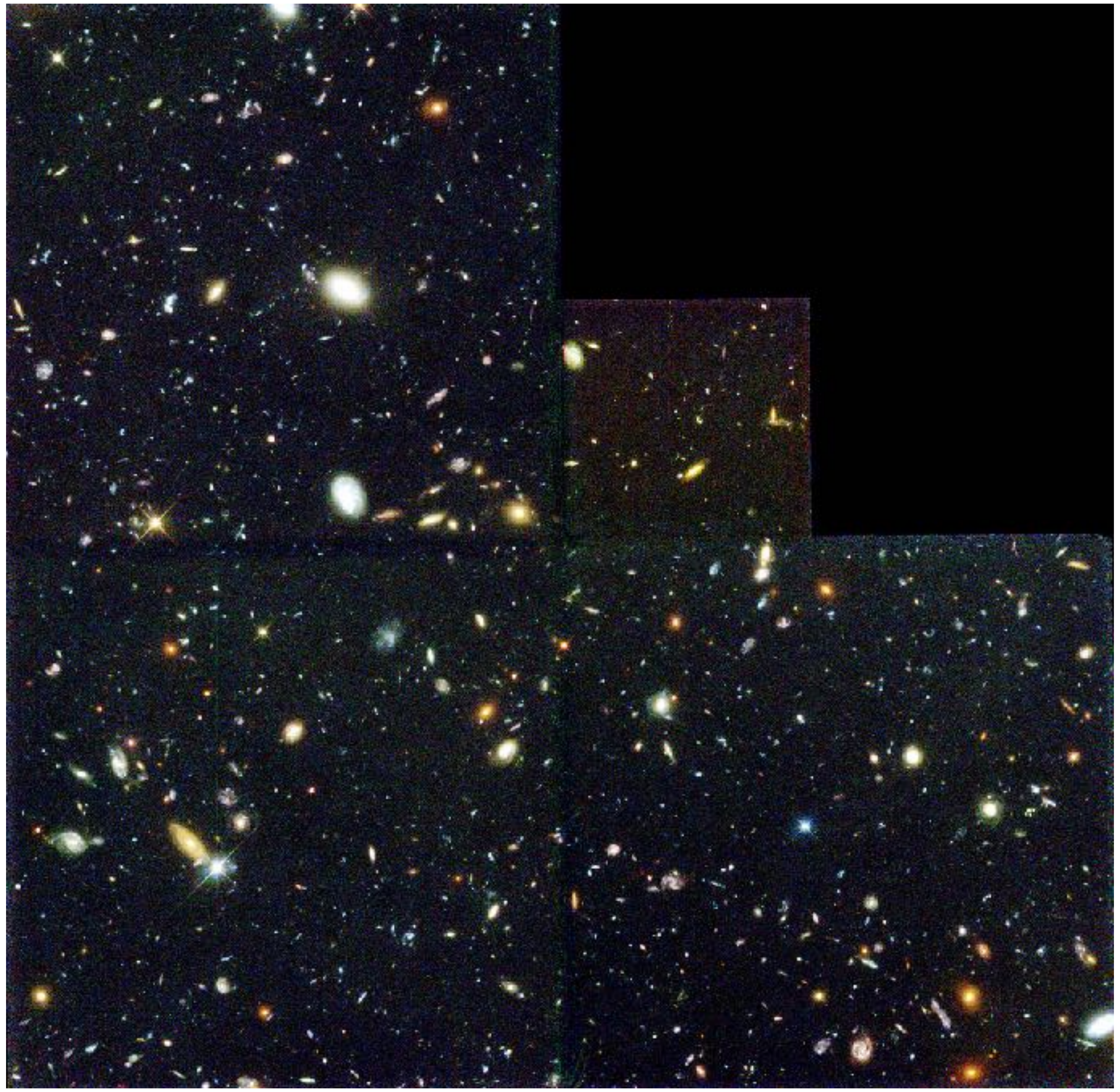
Propagation of light in the expanding Universe

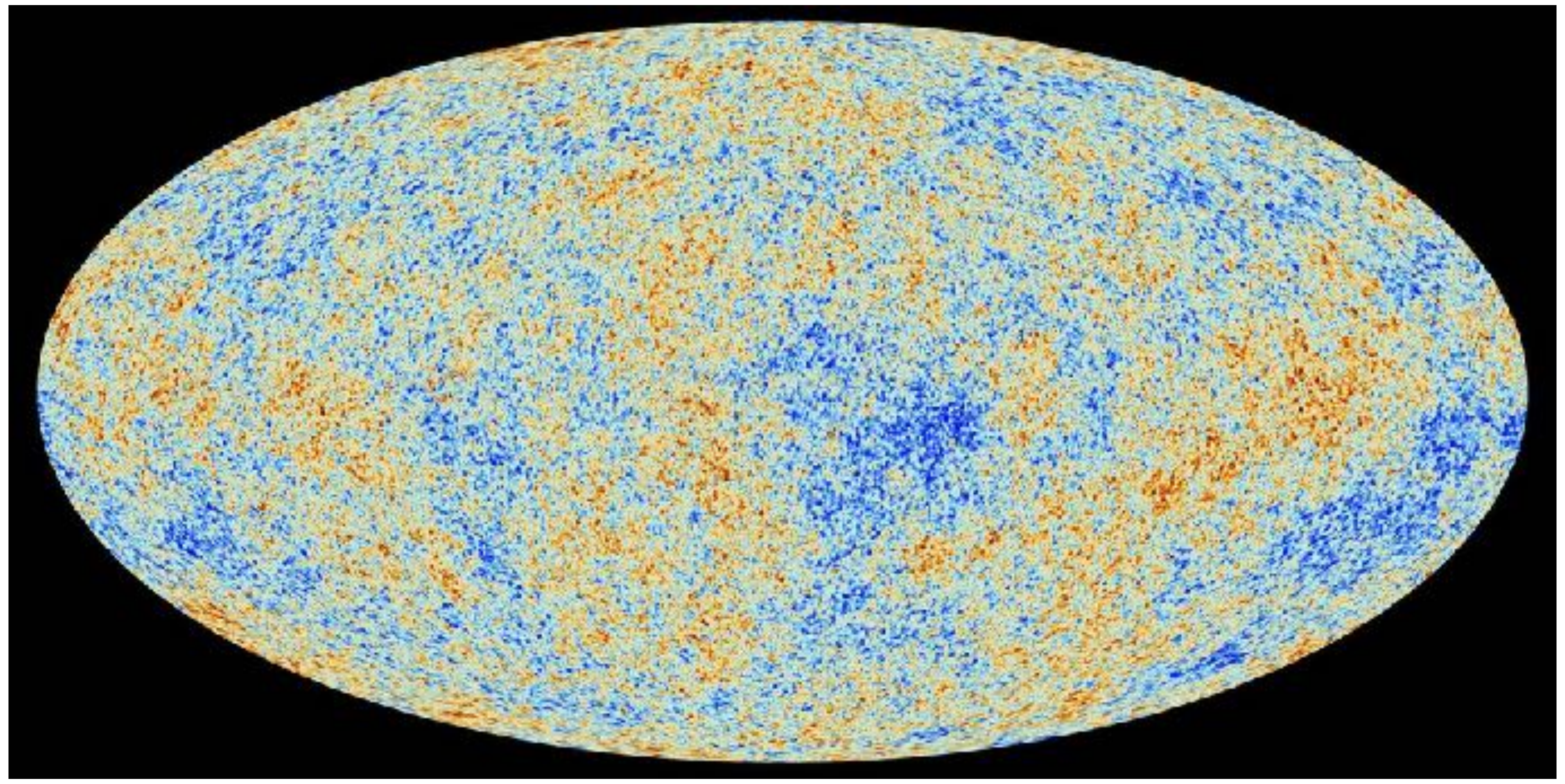
If one carries out a similar calculation for the $k = -1$ case starting from the solution (10.58) and taking $r = \sinh \chi$, then also one ends up with exactly the same relation (14.32) between r and z. Hence **we can take (14.32) as the general relation between r and z for a matter-dominated Universe with zero Λ**. This relation (14.32) is known as ***Mattig's formula***.

When Λ is non-zero, it is **not possible to derive such an analytical relation** between r and z. The relation between r and z has to be expressed in the form of an integral.

Tests of GR and cosmology

- Hubble test
- Angular size test
- Surface brightness test
- Supernova Ia
- Temperature anisotropies of the CMBR





Cosmological tests

One of the most important observational laws in cosmology is **Hubble's law** (9.13). This law was established from the study of galaxies having $z \ll 1$ and we find that **the linear relationship between distance and recession velocity holds at redshifts small compared to 1**.

An important question is **whether we would theoretically expect any departures from this linear relationship at redshifts $z \approx 1$**.

The first hurdle before us is to pose the question properly. Only in the case of low redshifts can we interpret the redshifts in terms of recession velocities given by (9.12). Even **the concept of distance involves complex issues when we consider faraway objects in the Universe. When the metric tensor is a function of time** as in the case of the Robertson–Walker metric, **only length elements** (given by (13.7)) **and not finite lengths have proper meaning**.

So we first have to **restate Hubble's law in terms of quantities which have precise meanings even when $z \approx 1$**, before we can talk about departures from linearity.

After restating Hubble's law in terms of suitable quantities, we shall see that the departures from linearity can give us clues about the values of important cosmological parameters like $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$.

$$v = H_0 l \tag{9.13}$$

$$v = zc. \tag{9.12}$$

Results for $\Lambda = 0$

To simplify things, **let us assume that all galaxies have the same intrinsic brightness and the same size.** Then a galaxy at a certain radial **co-moving coordinate *r***, which would correspond to a **certain redshift** z, will appear to have **a certain definite apparent luminosity and apparent size**.

We thus expect the **apparent luminosity and the apparent size to be functions of the redshift** z. We can **compare the theoretically derived functional relationships with observational data** to put constraints on the parameters in the theoretical model.

Since **not all galaxies have the same intrinsic luminosity and intrinsic size**, we expect the **observational data points to show some scatter** around the theoretically calculated functional relationships. However, an analysis involving a **large number of galaxies should have a good statistical significance** and should allow us to constrain the theoretical model.

Hubble test

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \qquad (10.19)$$

Consider a galaxy at position r . When light from this galaxy reaches us, the light passes through a spherical surface on which we lie and of which the galaxy is the centre. We first have to find the area of this spherical surface. By extending the discussion of length measurement, we can easily conclude that **an element of area on the spherical surface is** $a(t)^2r^2 \sin\theta\, d\theta\, d\phi$ if the metric of the Universe is **given by** (10.19). An **integration of this over the entire spherical surface** gives $4\pi a(t)^2r^2$. Since we are considering light falling on this surface at the present time, we have to take $a(t)$ to be the present value a_0 of the scale factor. If **L is the intrinsic luminosity** of the galaxy (i.e. the rate of energy emission per unit time), then we may think that the **flux received by us** should be given by

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi a_0^2 r^2}. \qquad (14.33)$$

But this is not yet the final correct answer. A photon which originally had energy hc/λ at the time of emission undergoes **redshift and has energy $hc/\lambda(1 + z)$ when it reaches us.** So we need to **divide** (14.33) **by the factor $1 + z$** to get the flux corrected for photon redshifts.

Hubble test

Another cause for concern is the **time dilation** given by (14.27). So the energy which was emitted by the galaxy in time Δt reaches us in time $\Delta t_0 = \Delta t\,(1 + z)$. This would further reduce the flux by another factor of 1 + z. The **correct expression of flux** is then given by

$$\boxed{\mathcal{F} = \frac{\mathcal{L}}{4\pi a_0^2 r^2 (1+z)^2} \qquad (14.34)}$$

We can write (14.34) in the form

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_{\mathrm{L}}^2}, \qquad (14.35)$$

where

$$\boxed{d_{\mathrm{L}} = a_0 r (1+z) \qquad (14.36)}$$

is called the ***luminosity distance*****. This is an observationally measurable quantity.**

Hubble test

Once you **measure the apparent luminosity, you can get d_L from** (14.35) **by assuming an average intrinsic luminosity L for all galaxies.**

When measuring the apparent luminosity, there is one other factor about which one has to be careful. Suppose you are **measuring the apparent luminosity with the help of the energy flux reaching you in the optical band** of the spectrum. For **a galaxy at a significant redshift z, the original photons which were emitted in the optical band may now be redshifted to the infrared**, whereas the photons **which were originally emitted in the ultraviolet may now appear in the optical** band and be detected by you. **If the galaxy is intrinsically less luminous in the ultraviolet compared to the optical band, then this shifting of photon wavelengths may make it appear dimmer** if you are measuring only the photons in the optical band. **Assuming a standard shape for the typical galactic spectrum, one can correct for this**. This is called the ***K correction***. Whenever we talk about luminosity distance, it should be assumed that we are talking about K corrected luminosity distance.

Substituting for r from (14.32) in (14.36), we get

$$H_0 d_L = \frac{c}{\Omega_{M,0}^2}[2\Omega_{M,0} z + (2\Omega_{M,0} - 4)(\sqrt{\Omega_{M,0} z + 1} - 1)]. \qquad (14.37)$$

Hubble test

This is the **functional relationship between d_L and z**, telling us what would be the redshift z of a galaxy which is located at luminosity distance d_L.

To see that Hubble's law follows from it at low redshifts, we consider $\Omega_{M,0} z < 1$. Then we can expand the square root by applying the binomial theorem. On keeping terms till z^2, it follows from (14.37) that

$$H_0 d_L \approx c \left[z + \frac{1}{4}(2 - \Omega_{M,0}) z^2 \right]. \qquad (14.38)$$

At low redshifts where we can neglect the z^2 term, we have a linear relation which is a restatement of Hubble's law in terms of the measurable quantities d_L and z. Theoretical considerations tell us that there should be a departure from the linear law at higher redshifts (unless $\Omega_{M,0} = 2$) and the amount of departure should depend on $\Omega_{M,0}$.

Hubble test

Figure 14.2 shows theoretical plots showing the **relation between d_L and z** for various values of $\Omega_{M,0}$, as obtained from (14.37).

It may now seem that it would be straightforward to estimate $\Omega_{M,0}$. **We have to determine the luminosity distances d_L of many galaxies having different redshifts z.**

Then we have to check **if the observational data points lie close to one of the curves** in Figure 14.2.

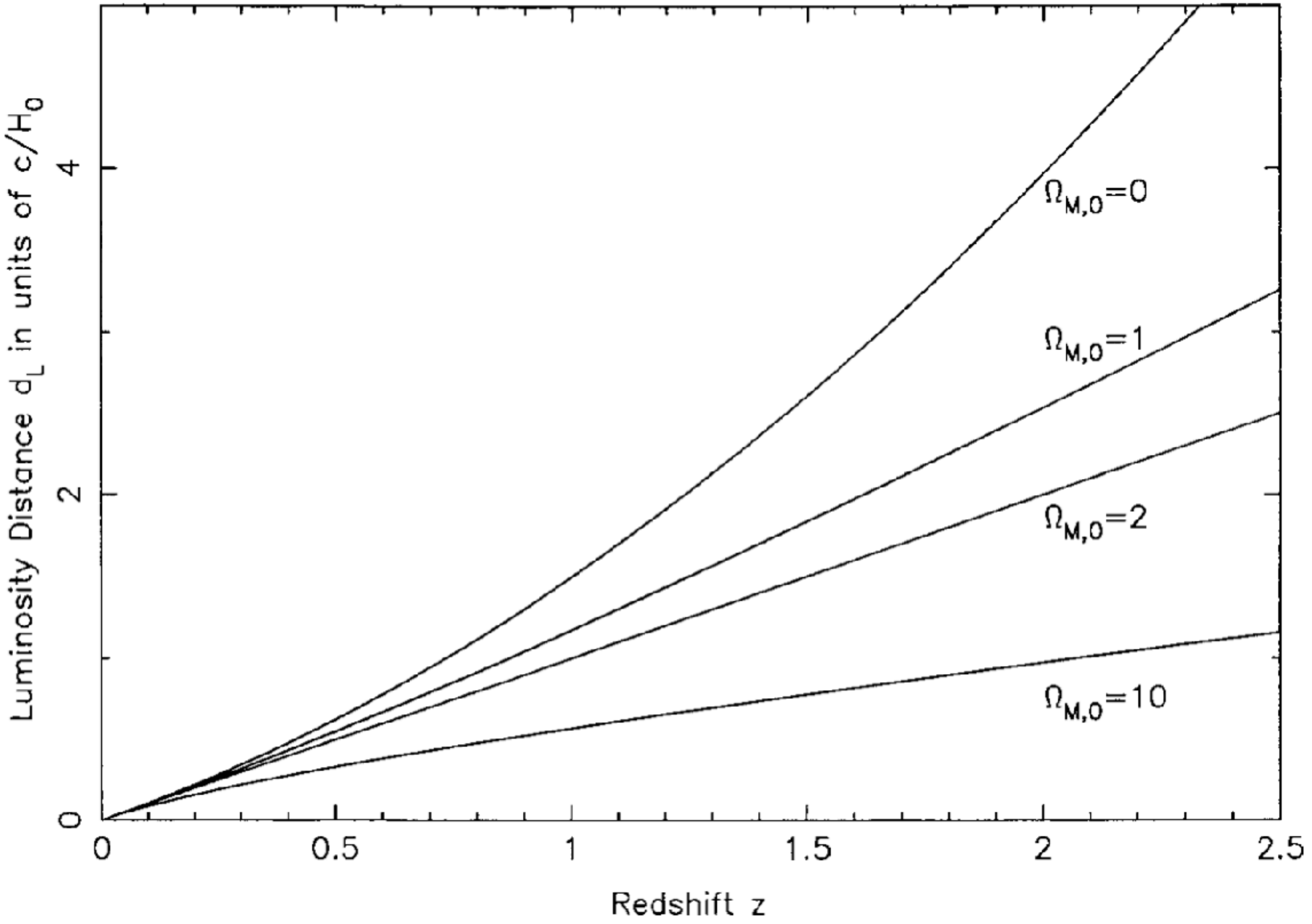


Fig. 14.2 The relation between the luminosity distance d_L and the redshift z of galaxies having the same intrinsic luminosity, for different values of $\Omega_{M,0}$, with $\Lambda = 0$.

Hubble test

The observational data seem not to fit any of the curves in Figure 14.2.

In other words, **a theoretical model with $\Lambda = 0$ does not provide a good fit to the observational data**.

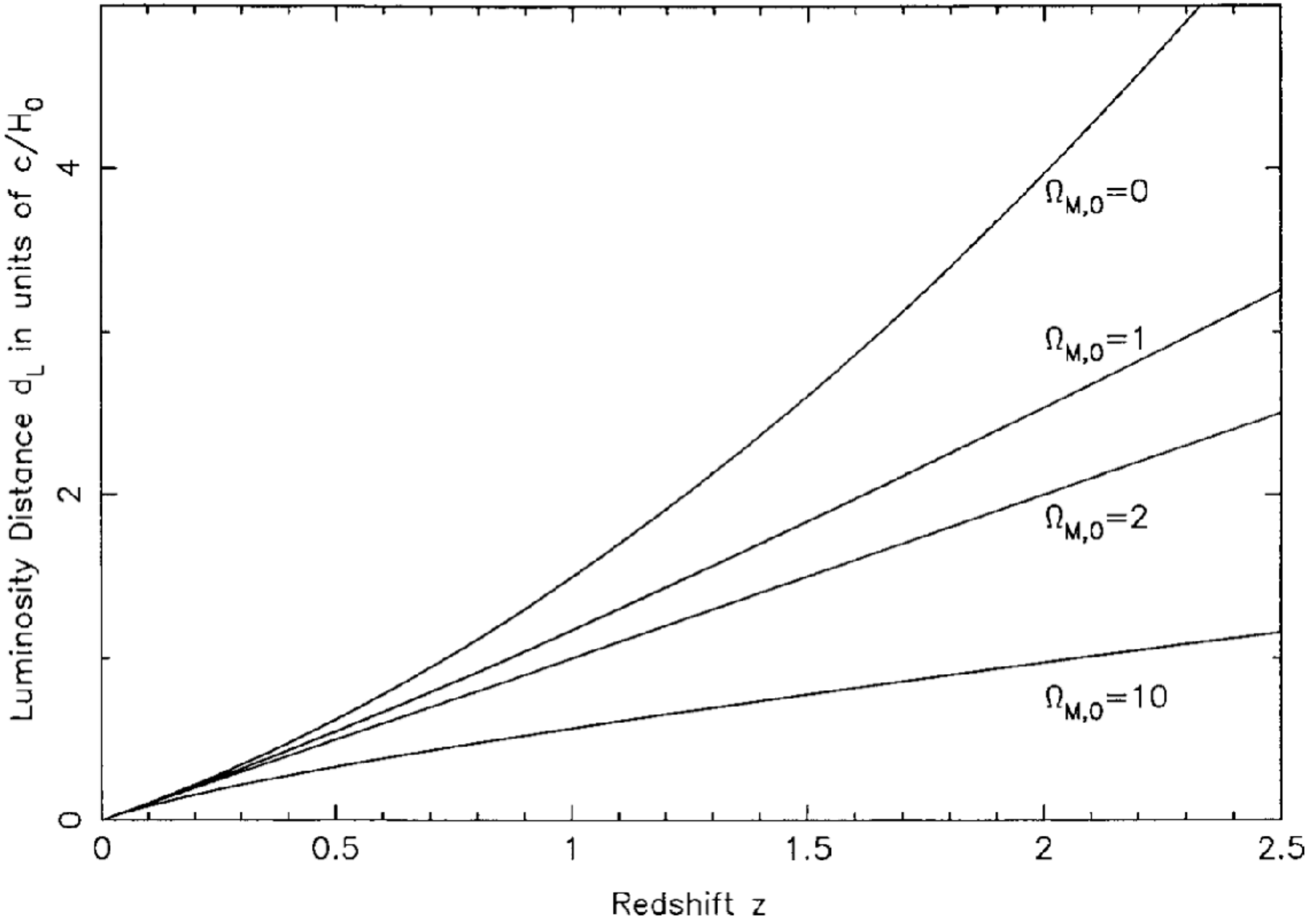


Fig. 14.2 The relation between the luminosity distance d_L and the redshift z of galaxies having the same intrinsic luminosity, for different values of $\Omega_{M,0}$, with $\Lambda = 0$.

The angular size test

Suppose a galaxy at redshift z has **linear size D.**

The galaxy will make up an arc of a circle passing through the galaxy with us at the centre. The angular size $\Delta\theta$ of the galaxy as seen by us can be obtained by equating $\Delta\theta/2\pi$ to the ratio of D to the circumference of this circle.

From the metric (10.19), it is easy to argue that the **circumference should be equal to $2\pi a(t)r$**. Since we are considering the circle to pass through the galaxy, the appropriate value of $a(t)$ **will be the scale factor a when light started from the galaxy**. This is equal to $a_0/(1 + z)$ so that the **circumference is $2\pi a_0 r/(1 + z)$**.

It is now easy to see that the **angular size is given by**

$$\boxed{\Delta\theta = \frac{\mathcal{D}(1+z)}{a_0 r}. \qquad (14.39)}$$

The angular size test

$$r = \frac{2\Omega_{M,0}z + (2\Omega_{M,0} - 4)(\sqrt{\Omega_{M,0}z + 1} - 1)}{a_0 H_0 \Omega_{M,0}^2 (1+z)/c}. \quad (14.32)$$

We can write this as

$$\Delta\theta = \frac{\mathcal{D}}{d_A}, \quad (14.40)$$

where

$$d_A = \frac{a_0 r}{1+z} \quad (14.41)$$

is called **the *angular size distance*.**

If we substitute for r from (14.32) in (14.39), it is easy to see that a_0 cancels out and we get an expression of $\Delta\theta$ as a function of z.

The angular size test

Figure 14.3 shows **$\Delta\theta$ as a function of z for different $\Omega_{M,0}$.**

This provides another possible test for the determination of $\Omega_{M,0}$. If we measure the angular sizes $\Delta\theta$ of many galaxies having different z, then we can try to **fit the observational data with the theoretical curves** in Figure 14.3, thereby allowing us to **estimate $\Omega_{M,0}$.**

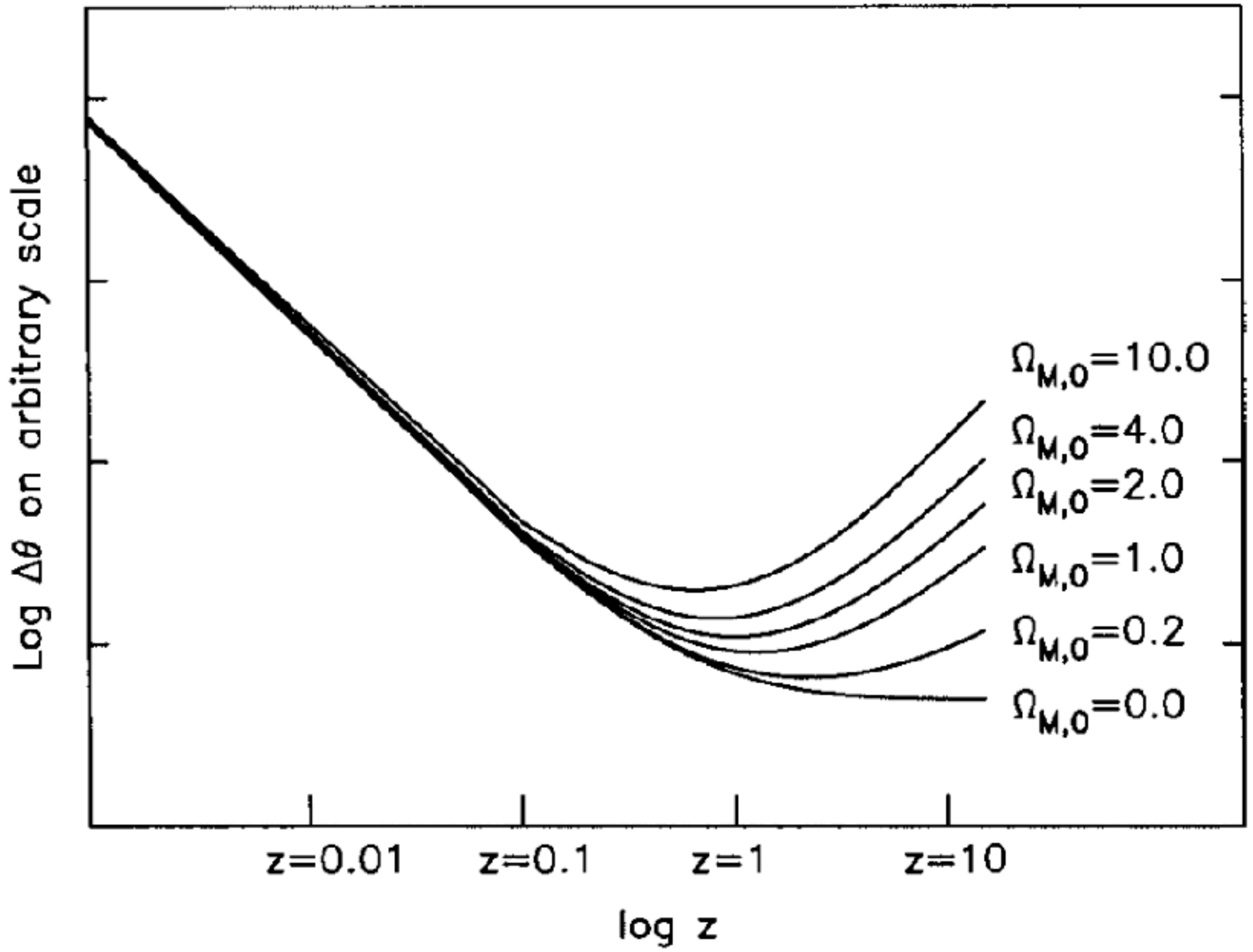


Fig. 14.3 The relation between the angular size $\Delta\theta$ and the redshift z of galaxies having the same intrinsic size, for different values of $\Omega_{M,0}$, with $\Lambda = 0$.

The surface brightness test

Assuming that **all galaxies have the same intrinsic surface brightness,** we now ask the question as to what would be the **apparent surface brightness of a galaxy at redshift z**.

The apparent surface brightness as seen by us is given by the quotient of the total flux received by us from the galaxy and the angular area of the galaxy as seen by us.

Since the **total flux received** by us goes as $\propto d_L^{-2}$ and the **angular area** goes as $\propto d_A^{-2}$, we expect that the **surface brightness should go as**

$$\mathcal{S} \propto \frac{d_{\mathrm{A}}^2}{d_{\mathrm{L}}^2}.$$

From (14.36) and (14.41), we then conclude

$$\mathcal{S} \propto \frac{1}{(1+z)^4}. \tag{14.42}$$

The surface brightness test

This is a model-independent relation**, which should hold if our basic ideas of the expanding Universe are correct and if galaxies are standard candles** (i.e. if intrinsic brightnesses of galaxies systematically do not change with time and hence with redshift).

The **surface brightness of an object should be independent of distance,** however this has to be **modified** when we **consider general relativistic effects** in an expanding Universe.

We have mainly discussed the application of the radiative transfer equation (2.12) to interiors of stars or to interstellar medium within a galaxy – situations where the general relativistic effect given by (14.42) is utterly negligible.

However, **when we venture into the extragalactic distances, (14.42) indicates that more distant galaxies should be dimmer**. This also allows us to get around the Olbers paradox discussed in §6.1.1. Because of the inverse fourth law dependence seen in (14.42), **the dimming of faraway galaxies is a rather drastic effect at high redshifts.** Even a **galaxy at a redshift of $z = 1$ would appear 16 times dimmer.** If we want to study galaxies at redshift $z \approx 6$, we need very long exposure times due to the extremely low value of the apparent surface brightness.

Results for $\Lambda \neq 0$

Some of the results for the $\Lambda = 0$ case get carried over to the $\Lambda \neq 0$ case.

For example, we define the **luminosity distance** d_L and the **angular size distance** d_A through (14.35) and (14.40) respectively even when the cosmological constant is non-zero (taking F as the observed flux and $\Delta\theta$ as the observed angular size).

The expressions for d_L and d_A will also still be given by (14.36) and (14.41) respectively. However, ***r* will no longer be related to the redshift *z*** by (14.32) and hence a relation like (14.37), based on (14.32), will no longer hold.

Before discussing how we relate r to z when $\Lambda \neq 0$, we point out that **the surface brightness would still fall as $(1 + z)^{-4}$ in accordance with (14.42).**

When the cosmological constant Λ **is non-zero**, it is not possible to write down an **analytical expression relating *r* with *z*.** The relation between them has to be expressed in the form of an integral, which we now derive. Remember that

$$r = S(\chi), \qquad (14.43)$$

where $S(\chi)$ has to be equal to $\sin\chi$, $\sinh\chi$ or χ, depending on whether k appearing in the Friedmann equation (10.27) is $+1$, -1 or 0.

Results for $\Lambda \neq 0$

$$a(t)^2[-d\eta^2 + d\chi^2] = 0, \tag{14.24}$$

$$\chi = \eta_0 - \eta, \tag{14.25}$$

$$c\,dt = a\,d\eta. \tag{10.33}$$

Since (14.24) and (14.25) describing **the propagation of light are valid even when $\Lambda \neq 0$**, we note from (14.25) that the position χ of a distant source of light is equal to the lapse in the time-like variable η between the emission of light by this source and its reception by us. From (10.33), it follows that

$$\eta_0 - \eta = c \int_{t_e}^{t_r} \frac{dt}{a(t)}, \tag{14.44}$$

where **t_e and t_r are** the values of **time t when the light was emitted and when it was received**.

Using (14.25), we can put (14.44) in the form

$$\frac{\chi}{c} = \int_z^0 \frac{dz'}{a} \frac{dt}{da} \frac{da}{dz'}, \tag{14.45}$$

where the limits of the integration over the redshift denoted by z' are z and 0 corresponding to the emission and the reception of the light signal.

Results for $\Lambda \neq 0$

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = H_0^2 \left[\Omega_{\mathrm{M},0} \left(\frac{a_0}{a} \right)^3 + \Omega_{\mathrm{R},0} \left(\frac{a_0}{a} \right)^4 + \Omega_{\Lambda,0} \right], \tag{14.17}$$

From the relation (10.24) between the redshift and the scale factor a, it follows that

$$\frac{da}{dz'} = -\frac{a^2}{a_0}.$$

Substituting this in (14.45), we get

$$\frac{\chi}{c} = \frac{1}{a_0} \int_0^z \frac{dz'}{(\dot{a}/a)}. \tag{14.46}$$

For $(\dot{a}/a)$ in (14.46) we now have to **substitute a general expression with non-zero Λ.** We use (14.17), in which we neglect the term involving $\Omega_{R,0}$ which is very small compared to the other terms when the Universe is matter-dominated. By making use of (10.24), we write (14.17) in the form

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_{\mathrm{M},0} (1+z)^3 + \Omega_{\Lambda,0} \right] + \kappa H_0^2 (1+z)^2, \tag{14.47}$$

Results for $\Lambda \neq 0$

where we have written $\kappa H_0{}^2$ for $-kc^2/a_0{}^2$ so that

$$\boxed{|\kappa| = \frac{c^2}{a_0^2 H_0^2}. \qquad (14.48)}$$

Since (14.47) is valid in our present epoch when $z = 0$ and $\dot{a}/a = H_0$, it easily follows from (14.47) that

$$\boxed{\kappa = 1 - \Omega_{\mathrm{M},0} - \Omega_{\Lambda,0}. \qquad (14.49)}$$

We can use (14.49) to determine κ when $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$ are given. From (14.46) and (14.47), we have

$$\chi = \frac{c}{a_0 H_0} \int_0^z [\Omega_{\mathrm{M},0}(1+z')^3 + \Omega_{\Lambda,0} + \kappa(1+z')^2]^{-1/2} dz'. \qquad (14.50)$$

Results for $\Lambda \neq 0$

From (14.36), (14.43) and (14.50), we get

$$d_\mathrm{L} = a_0(1+z)S\left(\frac{c}{a_0 H_0}\int_0^z [\Omega_{\mathrm{M},0}(1+z')^3 + \Omega_{\Lambda,0} + \kappa(1+z')^2]^{-1/2} dz'\right).$$

If we use (14.48) to eliminate a_0 which is not directly observable, then we finally get

$$\boxed{d_\mathrm{L} = \frac{(1+z)c}{H_0\sqrt{|\kappa|}} S\left(\sqrt{|\kappa|}\int_0^z [\Omega_{\mathrm{M},0}(1+z')^3 + \Omega_{\Lambda,0} + \kappa(1+z')^2]^{-1/2} dz'\right). \qquad (14.51)}$$

For given values of $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, we can **evaluate (14.51) numerically to determine $H_0 d_\mathrm{L}$ as a function of redshift z.**

Observationally we can **measure the redshifts z of a large number of galaxies and then determine their d_L from their observed apparent brightnesses** by using (14.35). By comparing the observational data with the theoretical results, we can determine the values of $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$.

Observational data - Supernova

Type Ia supernovae are believed to be caused by **matter accreting onto a white dwarf** having mass close to the Chandrasekhar mass. So we expect the **maximum luminosity of a Type Ia supernova to have the same value everywhere and at all times**. Hence such a supernova can be used as a **standard candle**.

With modern telescopes, it has been possible to resolve and **study Type Ia supernovae in distant galaxies.** Once we measure the **maximum apparent luminosity** of the supernova when it is brightest, we can use (14.35) to **calculate the luminosity distance** d_L if we know the maximum absolute luminosity. From a **knowledge of the redshift z of the galaxy** in which the supernova took place, we get the z corresponding to this luminosity distance d_L.

Hence, in **a plot of d_L against z, each supernova will contribute one data point**. Sometimes, instead of d_L, one plots the **equivalent quantity $m - M$** which is the difference between the apparent and absolute magnitudes of the Type Ia supernova (when it was brightest).

By substituting d_L for d in (1.8), we can easily find how d_L is related to $m - M$. In a plot of $m - M$ against z in which the supernova data are represented by points, we can also put theoretical curves calculated from (14.51) for different combinations of $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, to find out which curve fits the observational data best.

Observational data - Supernova

High-z supernova data from HST were analysed by two independent groups who carried out this exercise. Figure 14.4 shows the result.

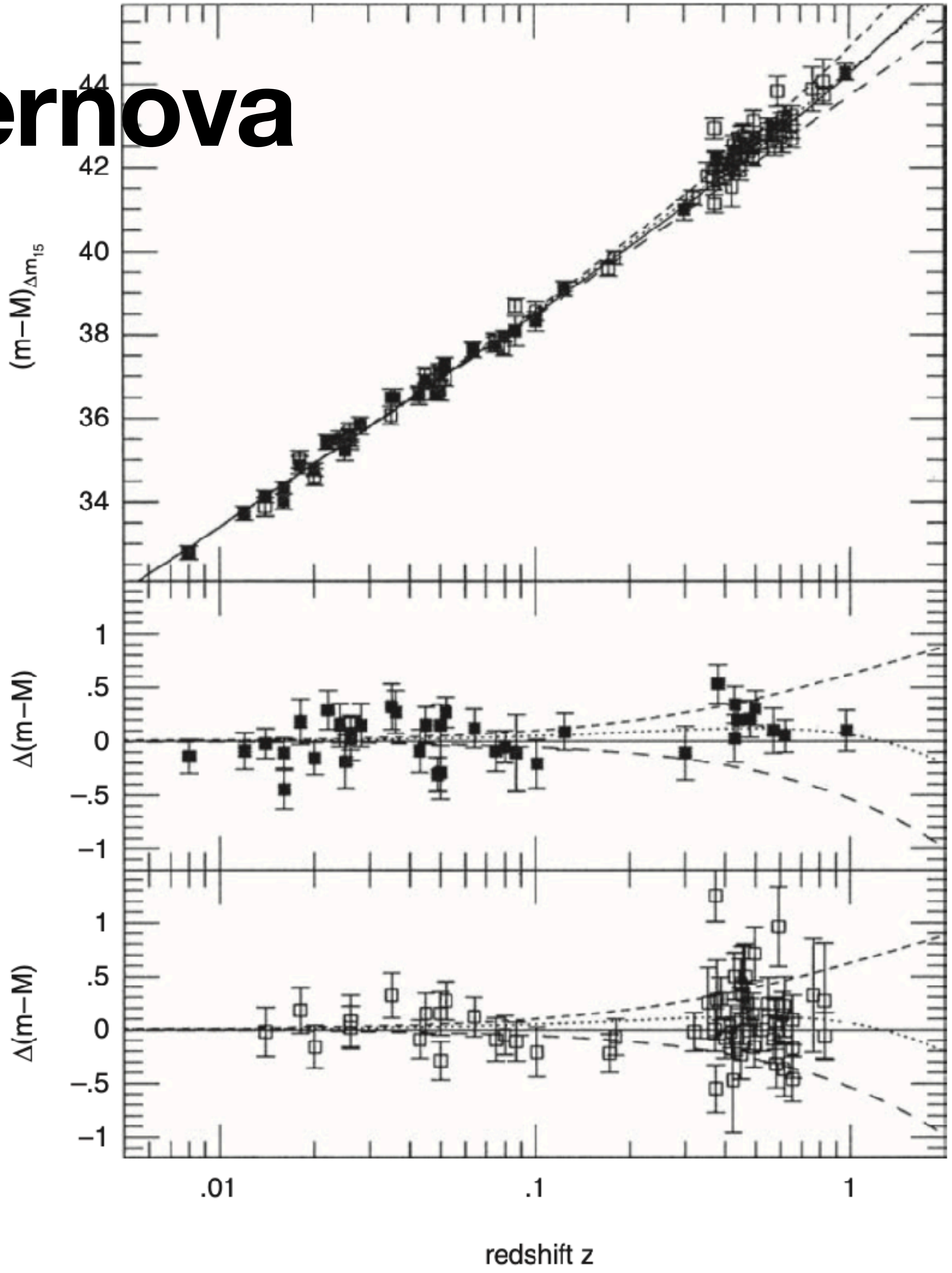


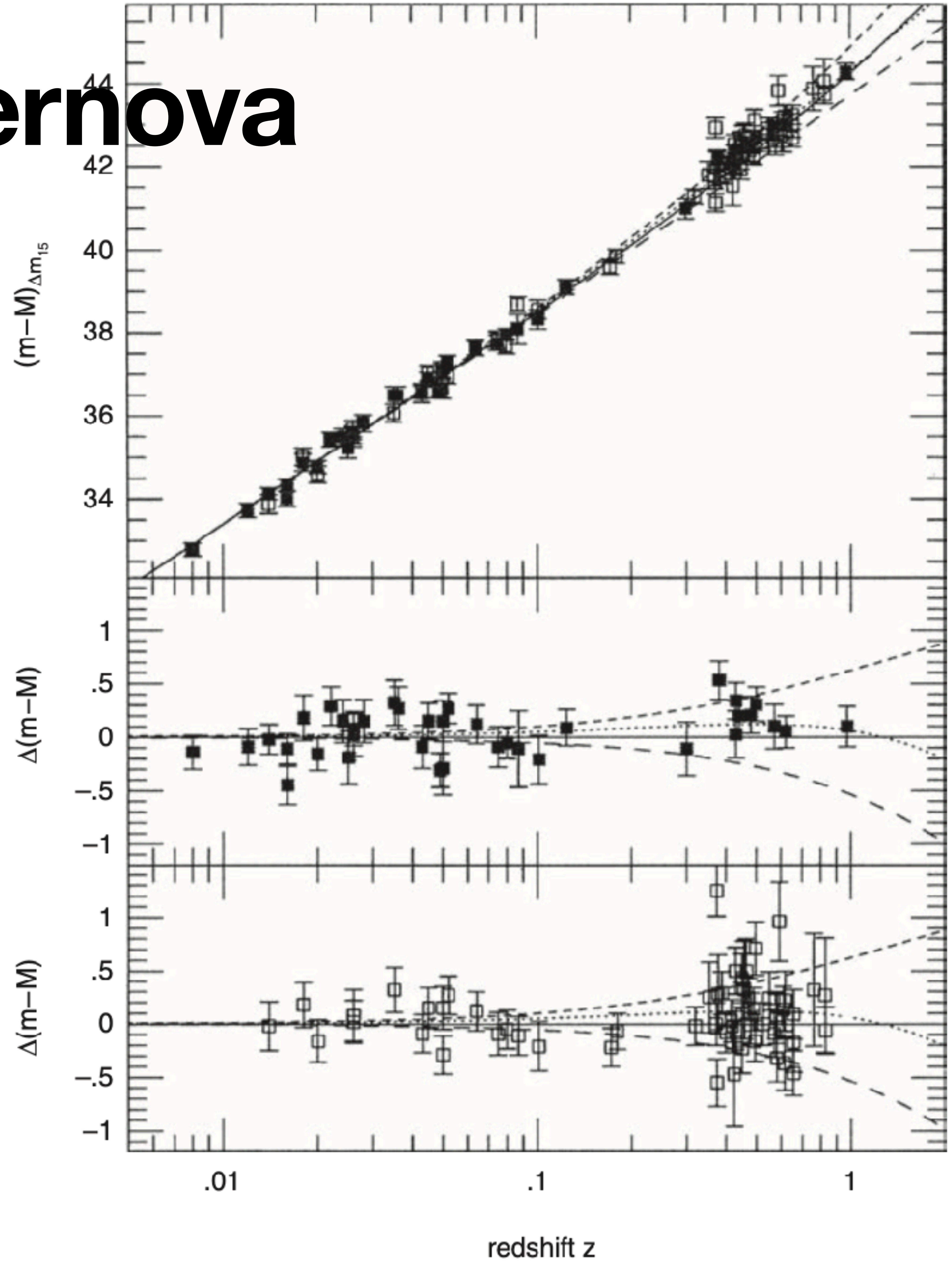
Fig. 14.4 The apparent luminosities of distant supernovae against redshifts z, along with theoretical curves for different combinations of the cosmological parameters. The cosmological parameters used for the different curves are: (i) solid line for $\Omega_{M,0} = 0$, $\Omega_{\Lambda,0} = 0$; (ii) long dashes for $\Omega_{M,0} = 1$, $\Omega_{\Lambda,0} = 0$; (iii) short dashes for $\Omega_{M,0} = 0$, $\Omega_{\Lambda,0} = 1$; (iv) dotted line for $\Omega_{M,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$. The lower two panels plot the deviation of $m - M$ from the solid line for $\Omega_{M,0} = 0$, $\Omega_{\Lambda,0} = 0$, showing the data of two groups separately: filled squares for the data of Riess *et al.* (1998) and open squares for the data of Perlmutter *et al.* (1999). From Leibundgut (2001). (©Annual Reviews Inc. Reproduced with permission from *Annual Reviews of Astronomy and Astrophysics*.)

Observational data - Supernova

The top panel with respect to the solid line for the empty Universe.

It appears that the dotted line corresponding to $\Omega_{M,0}$ **= 0.3 and $\Omega_{\Lambda,0}$ = 0.7 is the best theoretical fit** to the observational data.

This value of $\Omega_{M,0}$ agrees with what observers estimated from dynamical mass determinations of clusters of galaxies, as given by (10.42).



Temperature anisotropies in the CMBR

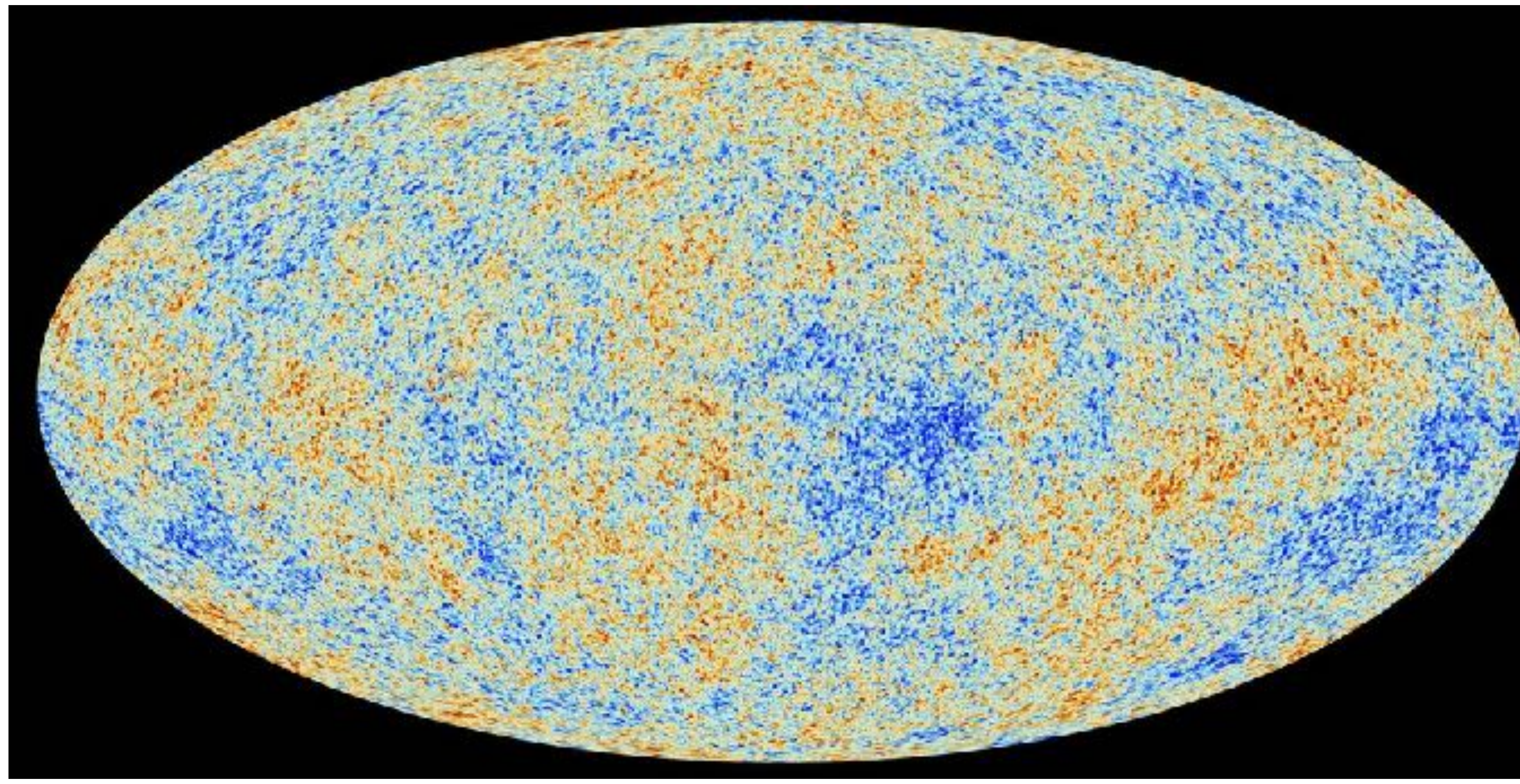
The measured **temperature variation of the CMBR** as a function of the galactic coordinates is shown in the Figure. Let us consider the temperature variation $\Delta T/T$ in a direction ψ . The temperature variation $\Delta T / T$ in a nearby direction $\psi + \theta$ is expected to be very similar if θ is sufficiently small, but will not be correlated with the temperature variation in direction ψ if θ is large. **The angular correlation of the CMBR anisotropy is given by**

$$C(\theta) = \left\langle \frac{\Delta T}{T}(\psi) \frac{\Delta T}{T}(\psi + \theta) \right\rangle, \tag{14.52}$$

where the averaging is supposed to have been done for all possible values of ψ and possible values of θ around them. Since this correlation function $C(\theta)$ is a function of θ, we can expand it in Legendre polynomials, i.e.

$$C(\theta) = \sum_l \frac{(2l+1)}{4\pi} C_l P_l(\cos\theta), \tag{14.53}$$

where C_l is the coefficient of the l-th Legendre polynomial.



Temperature anisotropies in the CMBR

Figure 14.5 plots C_l as a function of l. It is clear that **there is a maximum around $l \approx$ 250.**

To understand the **significance of the maximum,** note that $P_l(\cos\theta)$ has l nodes between 0 and π. When l is large, the first node is approximately located at $\Delta\theta \approx \pi/l$. A value of **$l \approx$ 250 would give a value of $\Delta\theta$ somewhat less than 1°**. This is the **typical angular scale of the temperature anisotropies in CMBR.**

We now come to the question of what determines this angular scale.

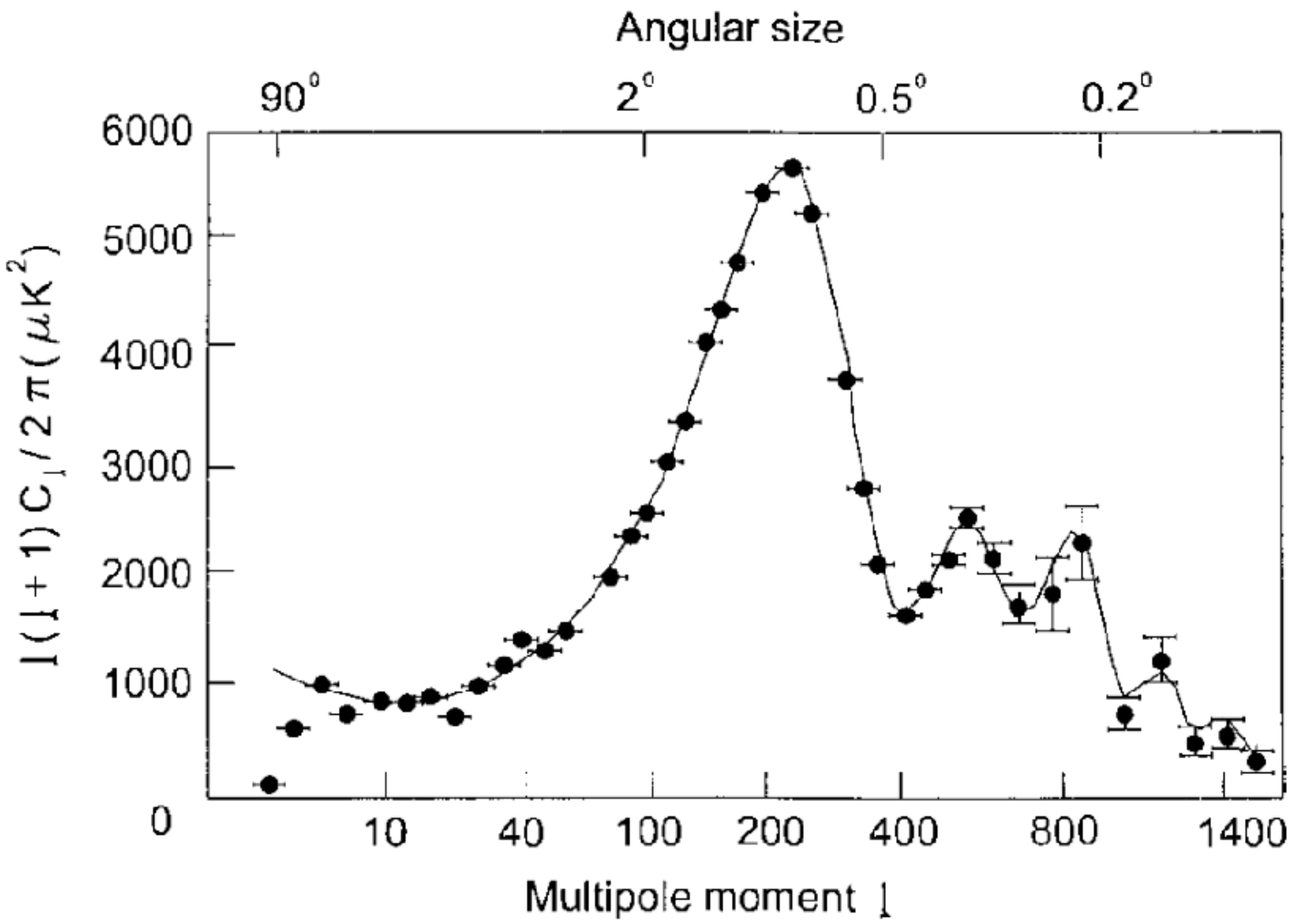


Fig. 14.5 The values of the coefficients C_l in the Legendre polynomial expansion of the angular correlation $C(\theta)$ of temperature anisotropies in the CMBR. Data for $l < 800$ come from WMAP, whereas data for higher l come from other experiments. The top of the figure indicates angular sizes corresponding to different values of l. Adapted from Bennett *et al.* (2003).

Temperature anisotropies in the CMBR

Let us consider an **object of linear size D** on the last scattering surface from where the CMBR photons come and which is at redshift $z_{\text{dec}} \approx 1100$.

We now **figure out the angular size Δθ which this object of size D will produce in the sky.** The relation between D and Δθ is given by (14.39). Since we are considering the possibility of Λ not being zero, we should substitute for r from (14.43) and (14.50). To have a rough idea of how things go, let us substitute for r from (14.32) appropriate for the $\Lambda = 0$ case, since this will allow us to make some estimates analytically and a non-zero Λ does not introduce too much error when calculating quantities relevant for early epochs.

When $z \gg 1$, it follows from (14.32) that

$$r \to \frac{2c}{a_0 H_0 \Omega_{\mathrm{M},0}}.$$

Substituting this for r , we obtain from (14.39) that

$$\Delta\theta \approx \frac{\Omega_{\mathrm{M},0}}{2} \cdot \frac{\mathcal{D} z}{c H_0^{-1}}.$$

Temperature anisotropies in the CMBR

On substituting for H_0 from (9.17), this becomes

$$\Delta\theta \approx 34.4''(\Omega_{\mathrm{M},0}h)\left(\frac{\mathcal{D}z}{1\ \mathrm{Mpc}}\right). \tag{14.54}$$

By putting $z_{\mathrm{dec}} \approx 1100$ in (14.54), we can determine the linear size D of an object on the last scattering surface which would subtend an angle **slightly less than 1° in the sky.**

Only perturbations larger than the Jeans length grow. So we may **expect the Jeans length to give sizes of the typical perturbations on the last scattering surface**. According to (11.40), the Jeans length was **of the order of the horizon size *ct* till the decoupling of matter and radiation** at $z_{\mathrm{dec}} \approx 1100$.

We now estimate the angle Δθ which the horizon on the last scattering surface would subtend to us today. We can use (10.60) to get the time when the decoupling took place, which turns out to be of order

$$t_{\mathrm{dec}} \approx H_0^{-1}\Omega_{\mathrm{M},0}^{-1/2} z_{\mathrm{dec}}^{-3/2}. \tag{14.55}$$

Temperature anisotropies in the CMBR

We would get the horizon by multiplying this by c. On substituting ct_{dec} for D in (14.54), we get

$$\Delta\theta \approx 0.87^\circ\ \Omega_{\mathrm{M},0}^{1/2} \left(\frac{z_{\mathrm{dec}}}{1100}\right)^{-1/2} . \tag{14.56}$$

If we take $\Omega_{M,0}$ of order 1, then (14.56) gives an angular size comparable to the angular scale of anisotropies in the CMBR data. **This suggests that the irregularities that we see on the last scattering surface correspond to the Jeans length,** which was **of the same order as the horizon** till that time. It follows from (14.56) that a **larger value of $\Omega_{M,0}$ would make $\Delta\theta$ larger**, causing the peak in Figure 14.5 to shift leftward. **The position of the peak would thus give the value of $\Omega_{M,0}$.**

When we assume $\Lambda \neq 0$, the analysis becomes much more complicated and has to be done numerically. We shall not present that analysis here. The more complicated analysis with $\Lambda \neq 0$ suggests that the **position of the peak in Figure 14.5 depends on $\Omega_{\Lambda,0} + \Omega_{M,0}$**. The observed position of the peak turns out to be consistent with

$$\Omega_{\Lambda,0} + \Omega_{\mathrm{M},0} = 1. \tag{14.57}$$

Combined constrains

Figure 14.6 **indicates the likely values of $\Omega_{\Lambda,0}$ and $\Omega_{M,0}$.** The straight line corresponds to $\Omega_{\Lambda,0} + \Omega_{M,0} = 1$ concluded from the temperature anisotropies of CMBR. On the other hand, the ellipses indicate the best possible combinations of $\Omega_{\Lambda,0}$ and $\Omega_{M,0}$which fit the data of distant supernovae. Constraints arising out of these two different sets of observational data are simultaneously satisfied if the values of our basic cosmological parameters are around

$$\Omega_{\Lambda,0} \approx 0.7, \quad \Omega_{M,0} \approx 0.3. \tag{14.58}$$

At the present time, these seem to be the values of these parameters accepted.

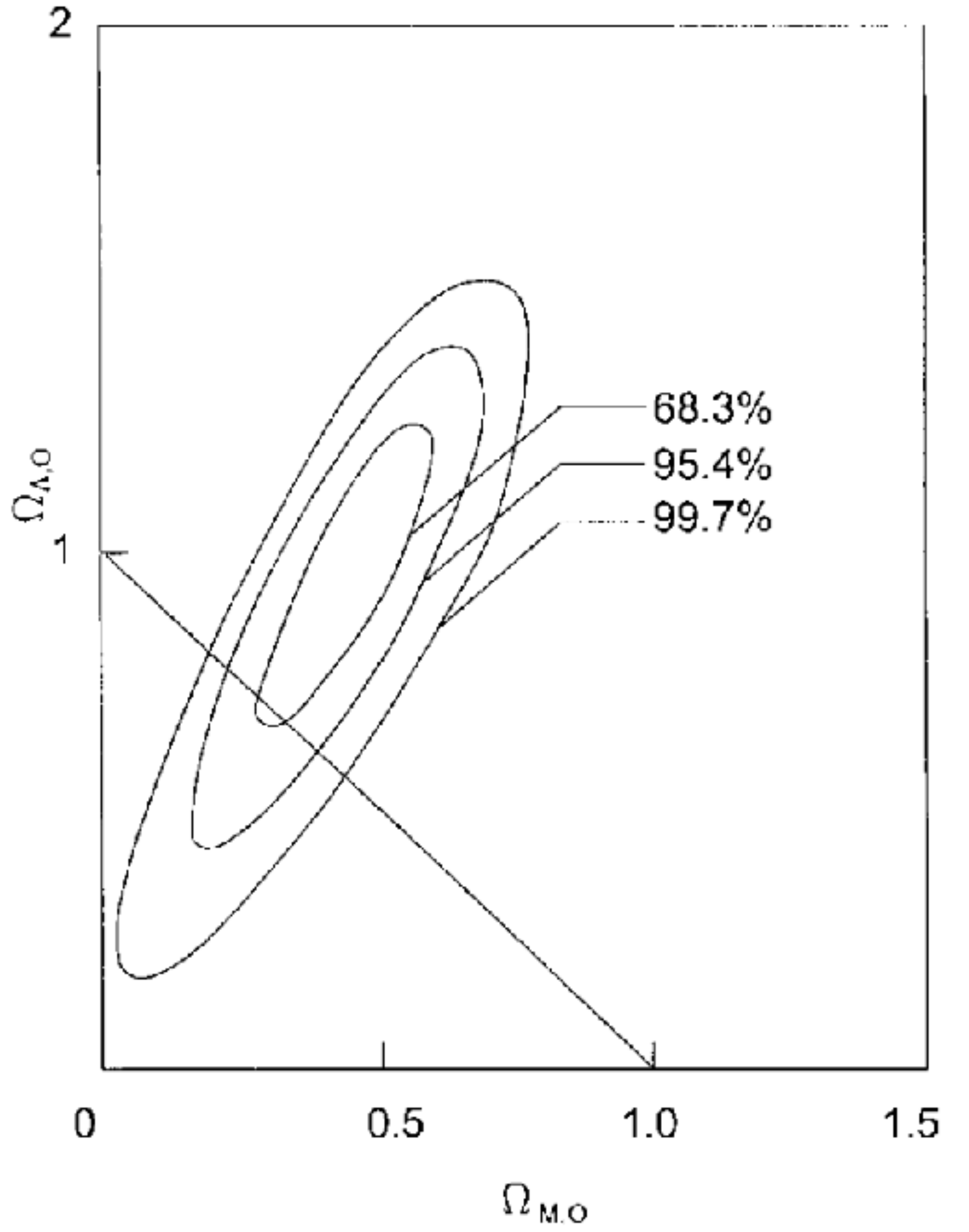
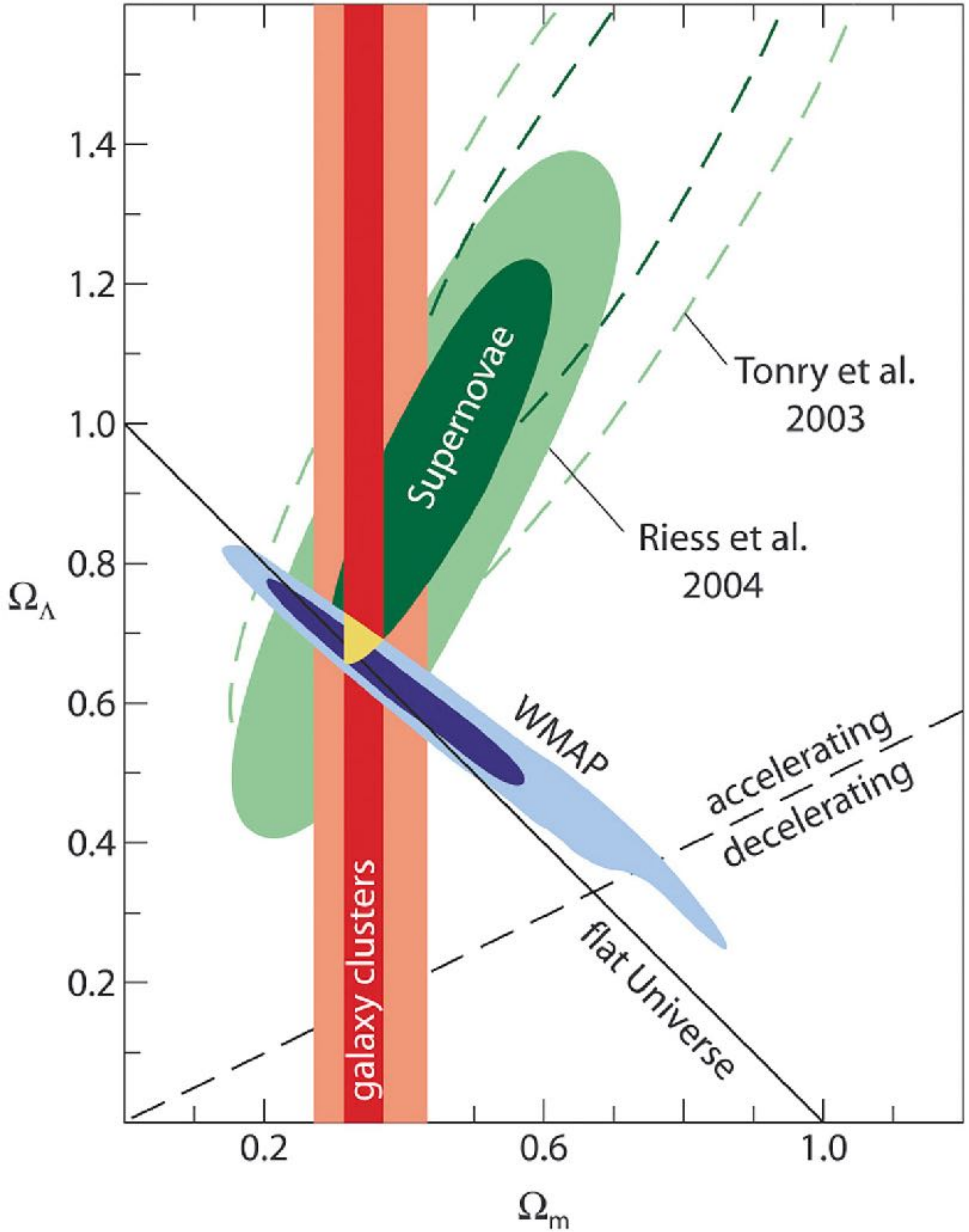


Fig. 14.6 Constraints on $\Omega_{\Lambda,0}$ and $\Omega_{M,0}$ jointly coming from the temperature anisotropies of the CMBR (the straight line corresponding to $\Omega_{\Lambda,0} + \Omega_{M,0} = 1$) and the data of distant Type Ia supernovae (the ellipses within which the values would lie at particular confidence levels). Adapted from Riess *et al.* (2004).

Combined constrains

Current observational constraints on the cosmic density of all matter including dark matter (Ω_M) and the dark energy (Ω_Λ) relative to the density of a critical-density Universe.

All three observational tests by means of supernovae (green), the cosmic microwave background (blue) and galaxy clusters converge at a Universe around $\Omega_M \sim 0.3$ and $\Omega_\Lambda \sim 0.7$. The dark red region for the galaxy cluster determination corresponds to 95% certainty (2-sigma statistical deviation) when assuming good knowledge of all other cosmological parameters, and the light red region assumes a minimum knowledge. For the supernovae and WMAP results, the inner and outer regions corespond to 68% (1-sigma) and 95% certainty, respectively. References: Schuecker et al. 2003; Tonry et al. 2003; Riess et al. 2004



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Combined constrains

Another **independent confirmation of the above value of $\Omega_{M,0}$ comes from the virial mass estimates of clusters of galaxies.**

Two important conclusions follow from the values quoted in (14.58). Firstly, it follows from (14.49) that $\kappa \approx 0$, which means that our Universe must be nearly flat with very little curvature. Secondly, since **$\Omega_{\Lambda,0}$ and $\Omega_{M,0}$ are of comparable values at the present time,** it follows from (14.17) that the matter density was more dominant in the past when a was smaller than a_0, whereas **the cosmological constant term will be more dominant in the future**. It thus seems that we live in a flat Universe which is at present in the process of making a **transition from a matter-dominated epoch to a Λ-dominated epoch.**

Alcubierre metric

The **Alcubierre drive is a speculative warp drive idea** according to which a spacecraft could achieve apparent faster-than-light travel by contracting space in front of it and expanding space behind it, under the assumption that a configurable energy-density field lower than that of vacuum (that is, negative mass) could be created. The Alcubierre drive is based on a solution of Einstein's field equations. Since those solutions are metric tensors, the Alcubierre drive is also referred to as Alcubierre metric.

Objects cannot accelerate to the speed of light within normal spacetime; instead, t**he Alcubierre drive shifts space around an object** so that the object would arrive at its destination more quickly than light would in normal space without breaking any physical laws.

Although the metric proposed by Alcubierre is **consistent with the Einstein field equations**, construction of such a drive is not necessarily possible. The proposed mechanism of the Alcubierre drive implies a negative energy density and therefore **requires exotic matter or manipulation of dark energy. Currently we do not know what is dark energy, so this is a purely theoretical idea.**

https://en.wikipedia.org/wiki/Alcubierre_drive