Introduction to Astrophysics and Cosmology

Relativistic Cosmology

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Time and length measurements

Before considering applications of general relativity, we need to understand how the **results of time and length measurements** can be expressed in terms of the quantities appearing in the mathematical theory.

We keep following the notation that **Roman indices *i*, *j*,... will run over the values 0, 1, 2, 3**, whereas the **Greek indices α, β,... will run over only 1, 2, 3**.

Writing the **time part of the metric separately**, we can write the **spacetime metric** in the form

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}\, dx^0\, dx^\alpha + g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad (13.1)$$

where $\boldsymbol{x^0}$ **is the time-like coordinate**.

The metric tensor component $g_{0\alpha}$ gives rise to **cross-terms between the time and space coordinates** in (13.1). We shall restrict ourselves to examples in which $g_{0\alpha} = 0$. The mathematical theory becomes much more complicated if $g_{0\alpha}$ **is not equal to zero,** which happens **when rotation is present** in a system. For **example, the metric around a rotating black hole** (known as the **Kerr metric**) has non-zero $g_{0\alpha}$.

Time and length measurements

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}\, dx^0\, dx^\alpha + g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{13.1}$$

If $g_{0\alpha} = 0$, then (13.1) reduces to

$$ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}\, dx^\alpha\, dx^\beta. \tag{13.2}$$

Suppose some **observer is at position x^α at time x^0 and at the position $x^\alpha + dx^\alpha$ at time $x^0 + dx^0$**. In the **special relativistic situation**, it is easy to show that the **physical time interval $d\tau$ measured by the observer's clock is related to ds** by

$$ds^2 = -c^2\, d\tau^2, \tag{13.3}$$

where ds^2 is given by the special relativistic metric (12.59) and (12.60).

Even in a **general relativistic situation**, we can **always introduce an inertial frame in a local region of spacetime where special relativity holds**. We thus expect that the physical time $d\tau$ measured by the observer's clock should satisfy **the same relation (13.3) in general relativity also**.

Time and length measurements

We now consider **an observer at rest such that the position x^α does not change**. It then follows from (13.2) and (13.3) that the **physical time measured by the observer's clock** is given by

$$\boxed{d\tau = \frac{1}{c}\sqrt{-g_{00}}\, dx^0. \qquad (13.4)}$$

This is our first important relation. If an **observer is at rest** in a coordinate frame and **an interval is dx^0** in the **time-like coordinate** x^0, then the **actual physical time** interval measured by a clock is given by **multiplying dx^0 by $\sqrt{-g_{00}}/c$**.

Time and length measurements

We now discuss how we can find the **physical distance between the neighbouring points** x^α **and** x^α **+** dx^α.

Suppose a **light signal is sent from the first point to the second point**, where it is **reflected back towards the first point immediately**. The **physical time interval** between the moment when the light signal leaves the first point and the moment when the light signal comes back there should be equal to $\mathbf{2}dl/c$**,** where dl **is the physical distance between the two neighbouring points,** if we assume that the light signal propagates at speed c. If two events are connected by a light signal, we know that in special relativity we have

$$ds^2 = 0. \tag{13.5}$$

The same consideration should hold in general relativity as well, since we can always introduce inertial frames in local regions. It follows from (13.2) and (13.5) that

$$dx^0 = \sqrt{\frac{-g_{\alpha\beta}\, dx^\alpha\, dx^\beta}{g_{00}}}. \tag{13.6}$$

Time and length measurements

A **light signal starting from the first point** x^α at $x^0 - dx^0$ should **reach the second point** $x^\alpha + dx^\alpha$ at x^0. If the light signal is **immediately reflected back** from the second point, it will again reach back to the first point at $x^0 + dx^0$.

Thus the moment when the light signal leaves the first point and the moment when the light signal comes back there **differ by** $\mathbf{2}dx^0$, with dx^0 given by (13.6).

To get the physical time interval, we have to **multiply this by** $\sqrt{-g_{00}}/c$, as suggested in (13.4). On equating this **physical time interval to** $\mathbf{2}dl/c$, we find

$$\boxed{dl = \sqrt{g_{\alpha\beta}\, dx^\alpha\, dx^\beta}. \qquad (13.7)}$$

This second important relation gives the **physical length between the neighbouring points** x^α and $x^\alpha + dx^\alpha$. The **length of a curve between two distant points can be obtained by integrating the length element given by (13.7)**, provided all the **components of** $g_{\alpha\beta}$ **are independent of time**.

Time and length measurements

If $g_{\alpha\beta}$ changes with time during the propagation of the light signal from one point to another distant point, then we can meaningfully **talk only about length elements *dl* along the path** and not the length of the whole path.

For the Robertson–Walker metric introduced in (10.19), the metric tensor evolves with time due to the time dependence of a(t).

Hence one has to take special care to treat the propagation of light in the expanding Universe or to talk about distances to faraway galaxies.

Time and length measurements

This also leads to the **concept of simultaneity**.

An observer at the first point x^α sees the light signal leaving at $x^0 - dx^0$ and returning back at $x^0 + dx^0$. Since the median value of the time-like coordinate between these two moments is x^0, this observer would expect the light signal to reach the second point at time x^0 in his clock. We have already pointed out that the signal reaches the second point $x^\alpha + dx^\alpha$ when the time-like coordinate at the second point has the value x^0. This means that x^0 at the first point is simultaneous with x^0 at the second point.

Extending this argument, **events taking place at different spatial points are simultaneous if the time-like coordinate has the same value x^0 for these events.**

The coordinate x^0 **is often called the *world time***.

We need to consider the world time to figure out whether different events are simultaneous, whereas **the physical time can be obtained from the world time by using (13.4)**. **Since g_{00} will in general have different values at different spatial points, it is clear that clocks will run at different rates at different spatial points.**

Gravitational redshift

We consider a **constant gravitational field** where the coordinates can be chosen in such a way that **the metric tensor components are independent of time**.

Suppose a **periodic signal** is sent from point ***A* to point *B***. Let a pulse be emitted at A at world time x_e^0 and reach B at world time x_r^0, the propagation time being $x_r^0 - x_e^0$. Suppose the next pulse is emitted at A at world time $x_e^0 + T^0$. For a **constant gravitational field, the propagation time of this pulse to *B* will be the same as the propagation time of the first pulse**. Hence the second pulse will reach B at the world time $(x_e^0 + T^0) + (x_r^0 - x_e^0) = x_r 0 + T^0$. This means that observers at A and B would both record the **same world time difference** T^0 **between the two pulses**.

Using (13.4) to relate the world time to the physical time, we conclude that the physical period T_A inferred by observer A and the physical period T_B inferred by observer B should be related by

$$\frac{T_A}{T_B} = \sqrt{\frac{-(g_{00})_A}{-(g_{00})_B}} = \frac{\omega_B}{\omega_A}, \tag{13.8}$$

Gravitational redshift

where ω_A and ω_B are the frequencies of some periodic signal measured at A and B respectively.

This is the **general expression showing how the frequency of a signal changes on propagating from one point in a gravitational field to another.** We shall now consider some simplifications for weak gravitational fields.

For a **weak gravitational field**, we can use (12.70) so that

$$\sqrt{-g_{00}} = \sqrt{1 + \frac{2\Phi}{c^2}} \approx 1 + \frac{\Phi}{c^2}.$$

On using this, (13.8) gives

$$\frac{\omega_B}{\omega_A} = \frac{1 + \frac{\Phi_A}{c^2}}{1 + \frac{\Phi_B}{c^2}}, \tag{13.9}$$

where Φ_A, Φ_B **are Newtonian gravitational potentials** at A, B.

Gravitational redshift

For a weak gravitational field, (13.9) can further be written as

$$\omega_B = \omega_A \left(1 + \frac{\Phi_A - \Phi_B}{c^2}\right). \tag{13.10}$$

Suppose the point ***B*** **is further away than *A* from the central region of the gravitational field**. It is easy to check that **the potential difference $\Phi_A - \Phi_B$ should be negative,** making $\omega_B < \omega_A$.

As a periodic signal **makes its way out of a gravitational field, the frequency will decrease.** For light coming out of a gravitational field, the spectrum should be **shifted towards the red.** This is the famous ***gravitational redshift*** predicted by general relativity. Pound and Rebka (1960) were able to verify the gravitational shift of wavelength by a brilliant terrestrial experiment, in which γ -rays from a source kept at the top of a tower were allowed to travel to the bottom where they were absorbed and analysed.

Gravitational redshift

Gravitational redshift is not the same as regular redshift due to the motion of the light source!

Let us consider a thought experiment with a laboratory suspended above the ground by a cable.

Monochromatic light of frequency ν_0 leaves a vertical flashlight on the floor at the same instant the cable holding the lab is severed.

The freely falling lab is again a **local inertial frame** where gravity has been abolished, and so the equivalence principle requires that a **frequency meter in the lab's ceiling record the *same* frequency,** ν_0, for the light that it receives.

But an **observer on the ground** sees a lab that is falling under the influence of gravity. As shown in Fig. 11, if the light has traveled upward a height h toward the meter in time t = h/c, then the **meter has gained a downward speed** toward the light of v = gt = gh/c since the cable was released.

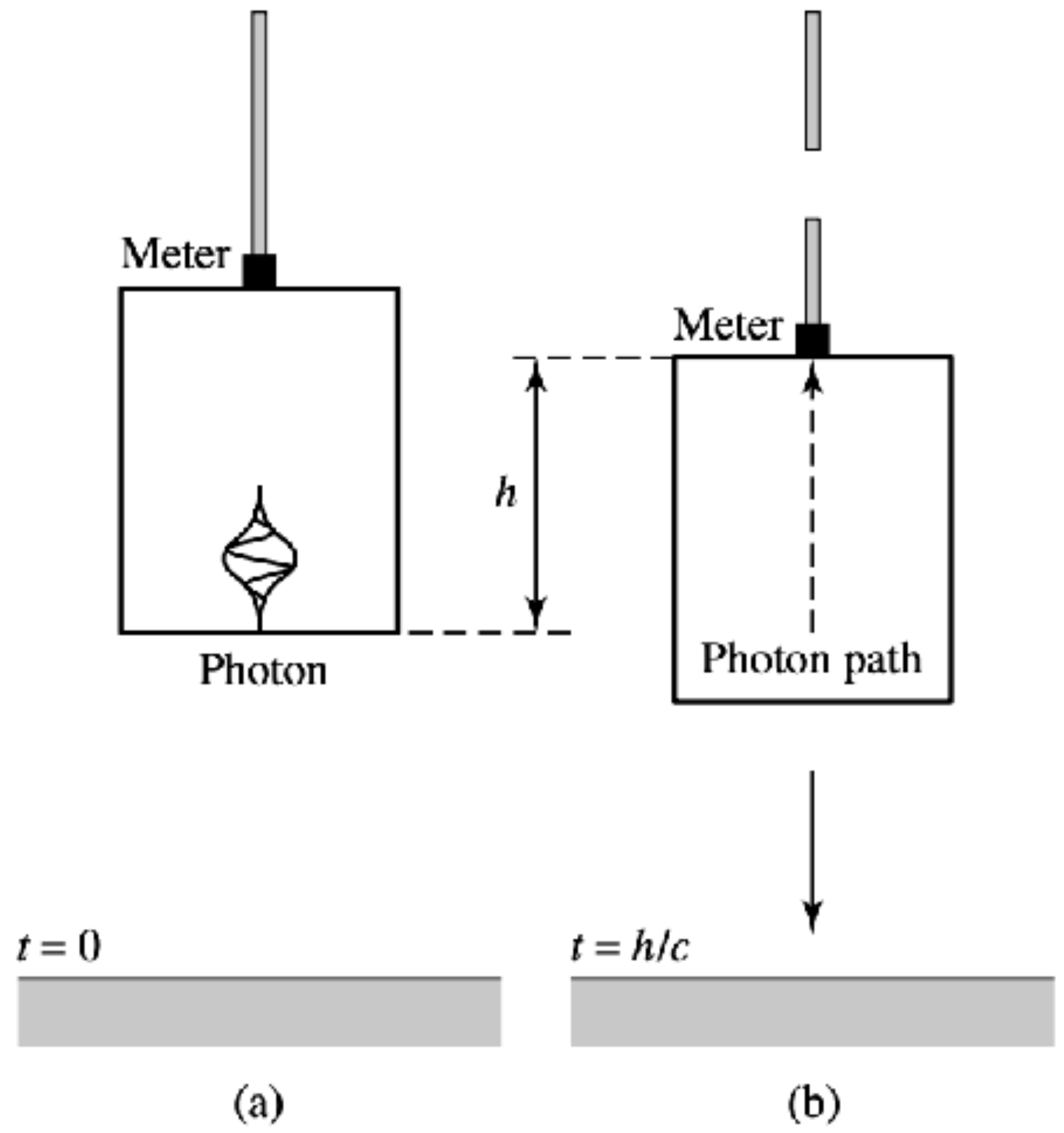


FIGURE 11 Equivalence principle for a vertically traveling light. The photon (a) leaves the floor at $t = 0$, and (b) arrives at the ceiling at $t = h/c$.

Gravitational redshift

Accordingly, we would expect that from the point of view of the **ground observer**, the meter should have measured a **blueshifted frequency** greater than ν_0. For the slow free-fall speeds involved here, this expected *increase* in frequency is

$$\frac{\Delta\nu}{\nu_0} = \frac{v}{c} = \frac{gh}{c^2}.$$

But in fact, the meter recorded *no change* in frequency. Therefore there must be another effect of the light's upward journey through the curved spacetime around Earth that exactly compensates for this blueshift. This is a **gravitational redshift** that tends to *decrease* the frequency of the light as it travels upward a distance h, given by

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}.$$

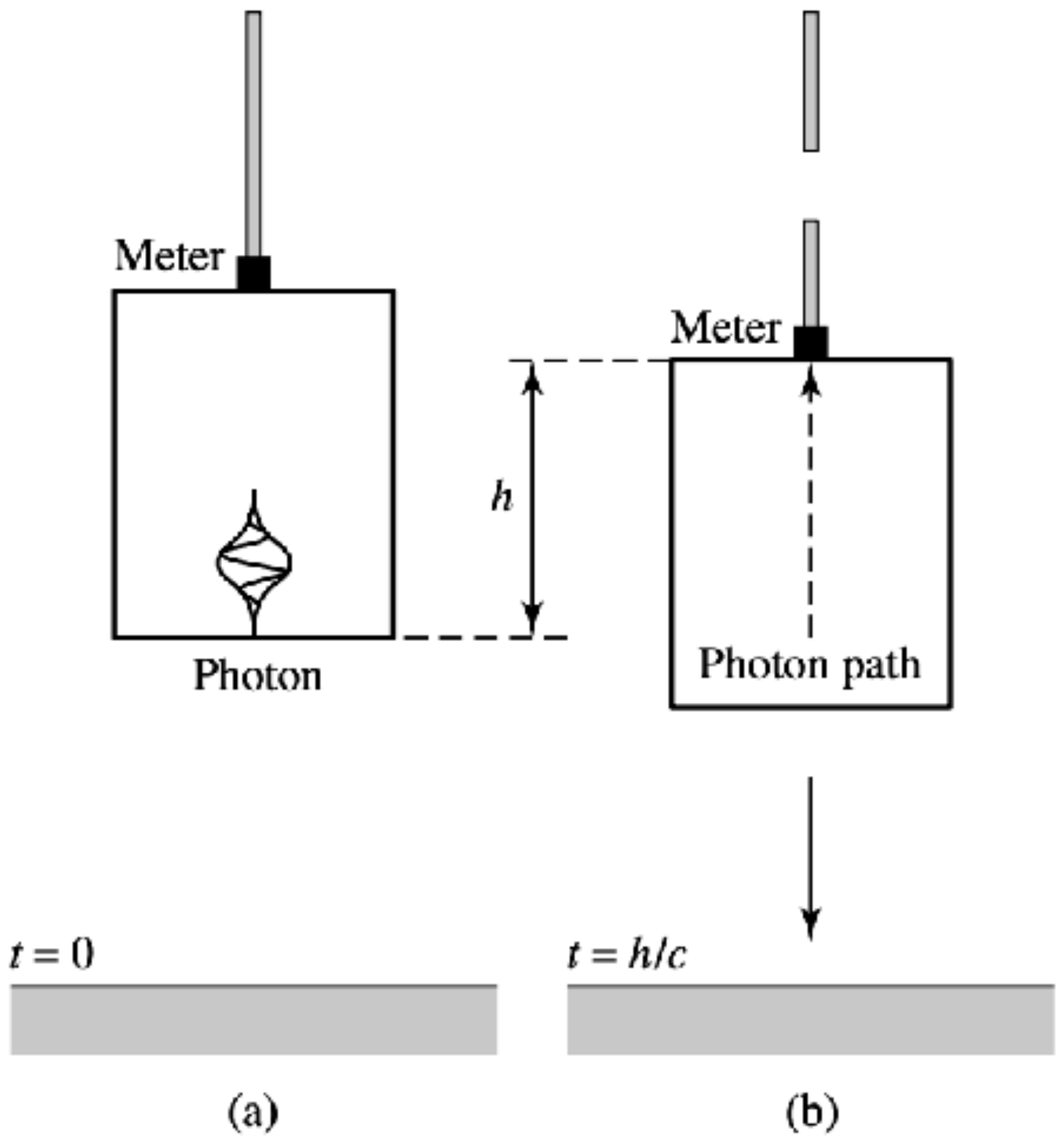


FIGURE 11 Equivalence principle for a vertically traveling light. The photon (a) leaves the floor at $t = 0$, and (b) arrives at the ceiling at $t = h/c$.

Gravitational redshift

An outside observer, not in free-fall inside the lab, would measure only this gravitational redshift. If the light were traveling downward, a corresponding blueshift would be measured. **This formula remains valid even if the light is traveling at an angle to the vertical, as long as h is taken to be the *vertical* distance covered by the light.**

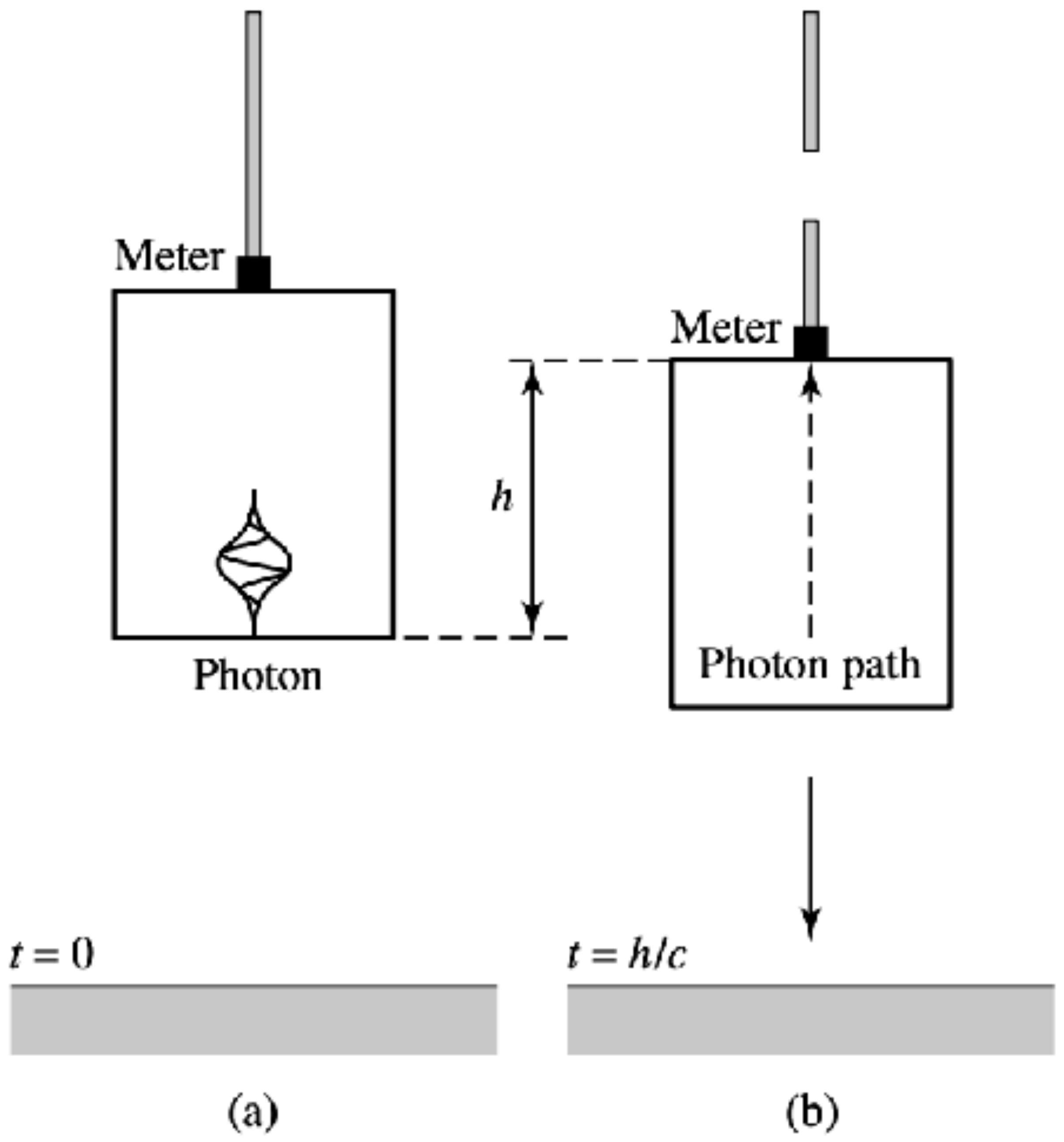


FIGURE 11 Equivalence principle for a vertically traveling light. The photon (a) leaves the floor at $t = 0$, and (b) arrives at the ceiling at $t = h/c$.

Gravitational redshift

Example experiment: In 1960, a test of the gravitational redshift formula was carried out.

A gamma ray was emitted by an unstable isotope of iron, ${}^{57}_{26}Fe$, at the bottom of a tower 22.6 m tall, and received at the top of the tower.

Using this value for h, the expected decrease in frequency of the gamma ray due to the gravitational redshift is

$$\frac{\Delta\nu}{\nu_0} = -\frac{gh}{c^2} = -2.46 \times 10^{-15},$$

in excellent agreement with the experimental result of $\frac{\Delta\nu}{\nu} = -(2.57 \pm 0.26) \times 10^{-15}$. More precise experiments carried out since that time have obtained agreement to within 0.007%.

The experiment was performed with both upward- and downward-traveling gamma rays, providing tests of both the gravitational redshift and blueshift.

Gravitational redshift

An approximate expression for the total gravitational redshift for a beam of light that escapes out to infinity can be calculated by integrating

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}.$$

from an initial position r_0 to infinity, using $g = GM/r^2$ (Newtonian gravity) and setting h equal to the differential radial element, dr for a spherical mass, M, located at the origin.

Some care must be taken when carrying out the integration, because

$$\frac{\Delta\nu}{\nu_0} = -\frac{gh}{c^2} = -2.46 \times 10^{-15},$$

was derived **using a *local* inertial reference frame. By integrating, we are really adding up the redshifts obtained for a chain of *different* frames.**

The radial coordinate r can be used to measure distances for these frames only if spacetime is nearly flat. In this case, the "stretching" of distances is not too severe, and we can integrate

$$\int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \simeq -\int_{r_0}^{\infty} \frac{GM}{r^2c^2}\, dr,$$

Gravitational redshift

The result is

$$\ln\left(\frac{\nu_\infty}{\nu_0}\right) \simeq -\frac{GM}{r_0 c^2},$$

which is valid **when gravity is weak** ($r_0/r_c = GM/r_0c^2 \ll 1$). This can be rewritten as

$$\frac{\nu_\infty}{\nu_0} \simeq e^{-GM/r_0c^2}.$$

Because the exponent is $\ll 1$, we use $e^{-x} \simeq 1 - x$ to get

$$\frac{\nu_\infty}{\nu_0} \simeq 1 - \frac{GM}{r_0 c^2}.$$

This approximation shows the **first-order correction** to the frequency of the photon.

Gravitational redshift

The exact result for the **gravitational redshift, valid even for a strong gravitational field**, is

$$\boxed{\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2} .}$$

When gravity is weak and the exponent is ≪1, we use $(1-x)^{1/2} \simeq 1 - x/2$ for the approximation.

The gravitational redshift can be incorporated into the redshift parameter, giving

$$\begin{aligned} z = \frac{\lambda_\infty - \lambda_0}{\lambda_0} &= \frac{\nu_0}{\nu_\infty} - 1 \\ &= \left(1 - \frac{2GM}{r_0 c^2}\right)^{-1/2} - 1 \\ &\simeq \frac{GM}{r_0 c^2}, \end{aligned}$$

valid only for a **weak gravitational field.**

Gravitational redshift

To understand the origin of the gravitational redshift, imagine a clock that is constructed to tick once with each vibration of a monochromatic light wave. The time between ticks is then equal to the period of the oscillation of the wave, $\Delta t = 1/\nu$. Then according to $\boxed{\dfrac{\nu_\infty}{\nu_0} = \left(1 - \dfrac{2GM}{r_0 c^2}\right)^{1/2}.}$, as seen from an infinite distance, the **gravitational redshift implies that the clock at r_0 will be observed to run more slowly than an identical clock at r = ∞**. If an amount of time Δt_0 passes at position r_0 outside a spherical mass, M, then the time Δt_∞ at r = ∞ is

$$\boxed{\frac{\Delta t_0}{\Delta t_\infty} = \frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2}}$$

For a weak field,

$$\frac{\Delta t_0}{\Delta t_\infty} \simeq 1 - \frac{GM}{r_0 c^2}.$$

Gravitational redshift

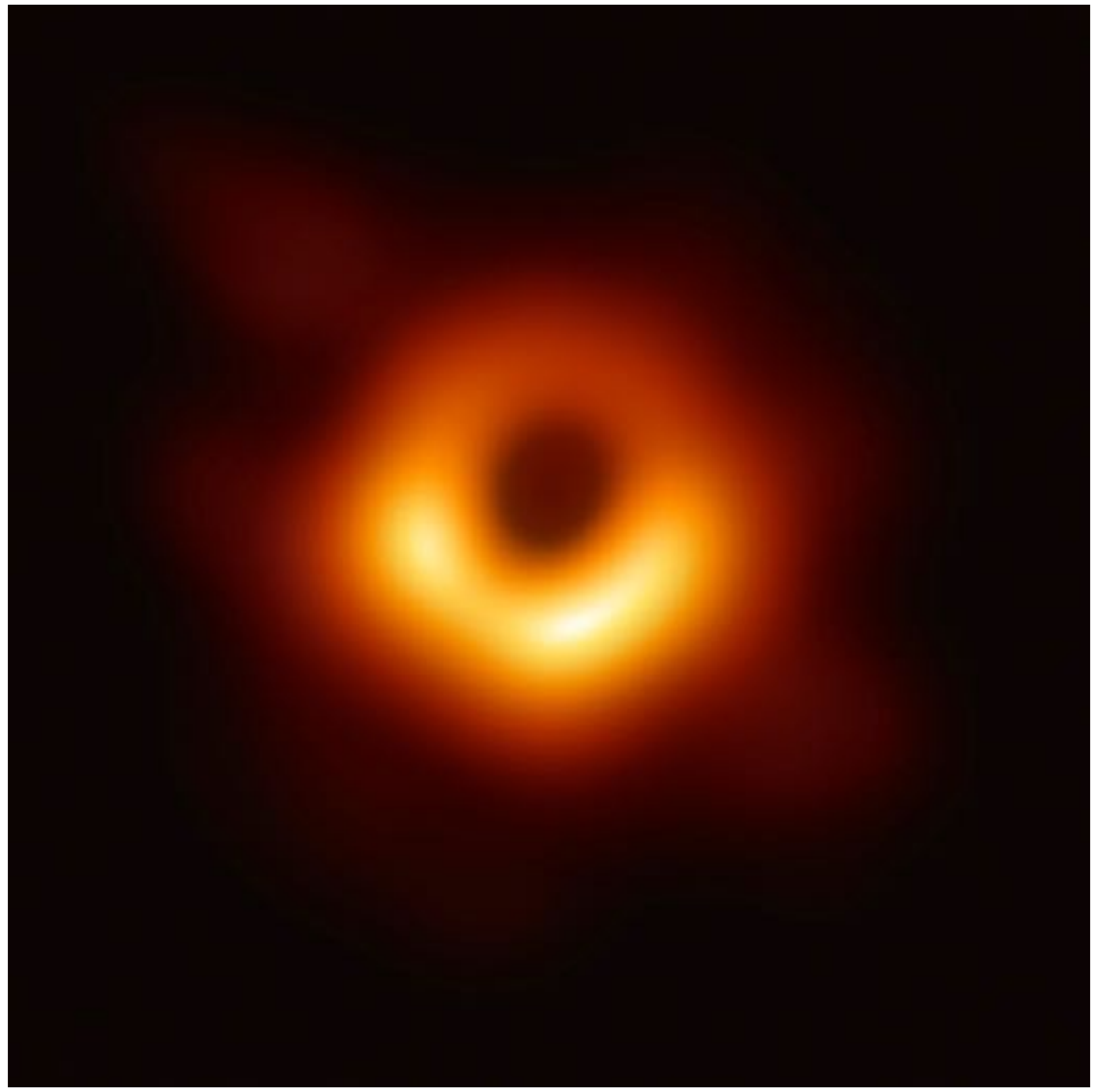
We must conclude that ***time passes more slowly as the surrounding spacetime becomes more curved*****,** an effect called **gravitational time dilation**. The gravitational redshift is therefore a consequence of time running at a slower rate near a massive object.

In other words, suppose two perfect, identical clocks are initially standing side by side, equally distant from a spherical mass. They are synchronized, and then one is slowly lowered below the other and then raised back to its original level. All observers will agree that when the clocks are again side by side, the clock that was lowered will be running behind the other because time in its vicinity passed more slowly while it was deeper in the mass's gravitational field.

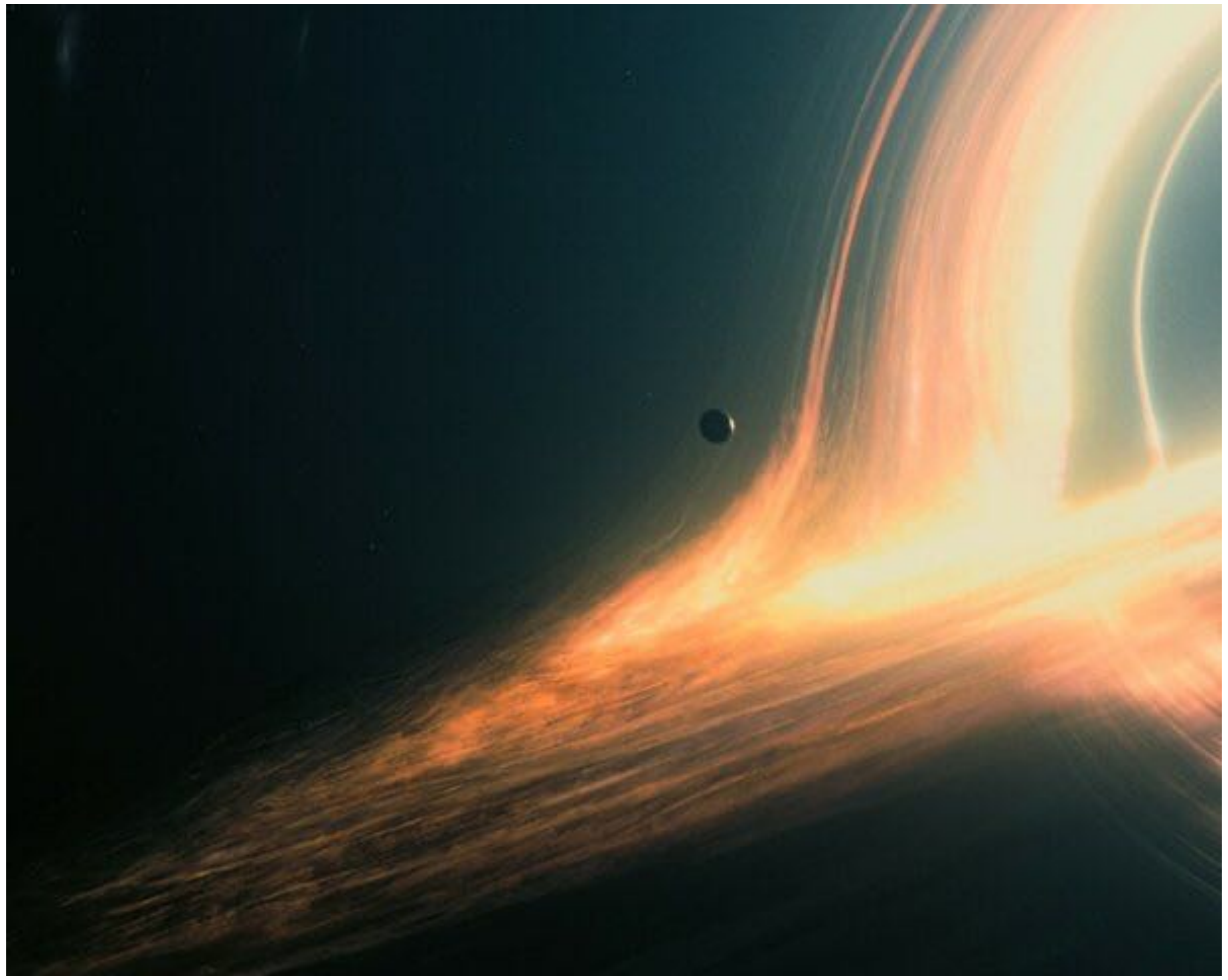
Any examples from movies?

Gravitational redshift

We must conclude that ***time passes more slowly as the surrounding spacetime becomes more curved***, an effect called **gravitational time dilation**. The gravitational redshift is therefore a consequence of time running at a slower rate near a massive object.



EHT image of the supermassive blackhole in M87



Planet "Miller" orbiting the black hole "Gargantua" in the Movie "Interstellar"

Gravitational redshift

Example: The **white dwarf** Sirius B has a radius of $R = 5.5 \times 10^6$ m and a mass of $M = 2.1 \times 10^{30}$ kg. The radius of curvature of the path of a horizontally traveling light beam near the surface of Sirius B is given by

$$r_c = \frac{c^2}{g} = \frac{R^2c^2}{GM} = 1.9 \times 10^{10} \text{ m}.$$

The fact that $GM/Rc^2 = R/rc \ll 1$ indicates that **the curvature of spacetime is not severe**. Even at the surface of a white dwarf, gravity is considered relatively weak in terms of its effect on the curvature of spacetime. The gravitational redshift suffered by a photon emitted at the star's surface is

$$z \simeq \frac{GM}{Rc^2} = 2.8 \times 10^{-4}.$$

This is in excellent **agreement with the measured gravitational redshift for Sirius B of $(3.0 \pm 0.5) \times 10^{-4}$**.

Gravitational redshift

To compare the **rate at which time passes at the surface of Sirius B** with the rate at a great distance, suppose that exactly one hour is measured by a distant clock. The time recorded by a clock at the surface of Sirius B would be *less* than one hour by an amount found using:

$$\Delta t_\infty - \Delta t_0 = \Delta t_\infty \left(1 - \frac{\Delta t_0}{\Delta t_\infty}\right) \simeq (3600\ \mathrm{s}) \left(\frac{GM}{Rc^2}\right) = 1.0\ \mathrm{s}.$$

The clock at the surface of Sirius B **runs more slowly by about one second per hour** compared to an identical clock far out in space.

Experimental results confirm the curvature of spacetime.

The Schwartzshield metric

Einstein's equation (12.96), is very difficult to solve and complete solutions are known only for a few cases of practical importance. The **simplest gravitational problem one can think of is to find the gravitational field due to an isolated point mass M**. Soon after Einstein's formulation of general relativity, Schwarzschild (1916) obtained the exact solution of this problem.

Let us choose the **position of the mass M as the origin of our coordinate system** and use spherical coordinates. According to the Newtonian theory of gravity, the **gravitational potential at a distance r** is given by

$$\Phi = -\frac{GM}{r}. \tag{13.11}$$

Far away from the mass point where the **gravitational field is weak,** the metric should be given by the expression (12.68) valid in the weak field limit. Substituting for Φ from (13.11) and using spherical coordinates, we write

$$ds^2(r \to \infty) = -\left(1 - \frac{2GM}{c^2 r}\right) c^2\, dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{13.12}$$

The Schwartzshield metric

Since there is no matter in space at points other than the point $r = 0$, the **energy-momentum tensor given by (12.88) should be zero at all points except $r = 0$**.

Then, according to Einstein's equation (12.96), the **Einstein tensor also must be zero at all points except $r = 0$.** Now we need to find a metric which tends to (13.12) as $r \to \infty$ and for which the Einstein tensor is zero at all points except $r = 0$. **The metric satisfying these requirements is the *Schwarzschild* metric:**

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2\, dt^2 + \frac{dr^2}{(1 - 2GM/c^2 r)} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (13.13)}$$

The first step in calculating the Einstein tensor is the **calculation of the Christoffel symbols**. The Christoffel symbols have 40 components in four-dimensional spacetime. Some of these components turn out to be zero. Still one has to take care of many non-zero components.

The Schwartzshield metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2\, dt^2 + \frac{dr^2}{(1 - 2GM/c^2 r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{13.13}$$

When r is very large, $2GM/c^2r$ becomes small compared to 1. If **we neglect this in the coefficient of dr^2 in (13.13), we are led to (13.12)**. One may wonder whether we ought to keep $2GM/c^2r$ in the coefficient of dt^2, while neglecting it in the coefficient of dr^2. It is not difficult to justify this. Suppose a particle is at the point r, θ, ϕ at time t and at the point $r + dr$, $\theta + d\theta$, $\phi + d\phi$ at time $t + dt$. If the particle is moving non-relativistically, then we must have $dr^2 \ll c^2dt^2$. Hence **the term involving dr^2 is itself small and a small term** in its coefficient is of second order of smallness. When we neglect this term, we still have to keep the similar term in the coefficient of dt^2.

It may be noted that the coefficient of dr^2 in (13.13) diverges when r has the value

$$r_S = \frac{2GM}{c^2}. \tag{13.14}$$

This is called the *Schwarzschild radius*.

The effect of general relativity can be neglected **if f** defined in (1.11), **which is equal to r_S/r, is small compared to 1**. This means that at **radial distances large compared to r_S general relativistic effects can be neglected**.

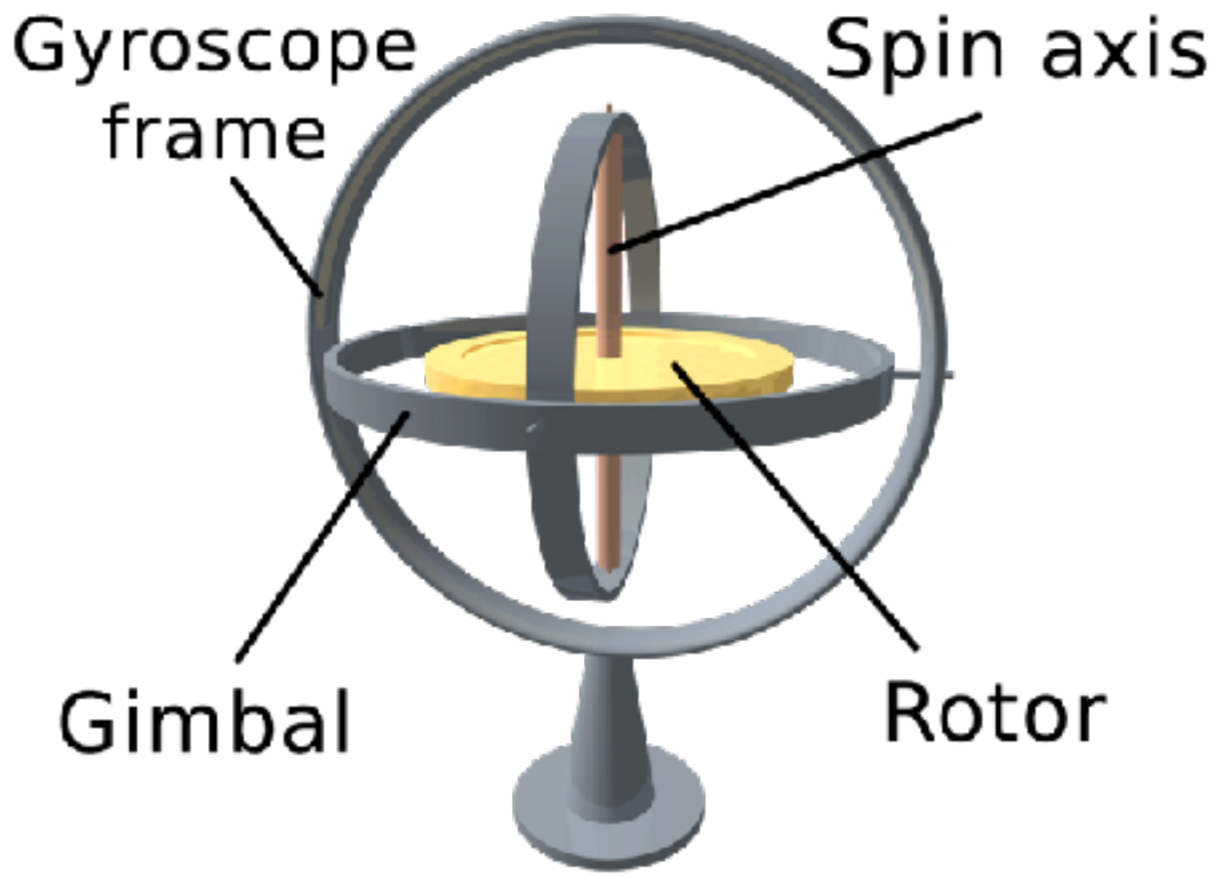
The Schwartzshield metric

This also follows from the fact that **the metric at such large distances can be approximated by (13.12)**, which leads to the same results as what we would get from the Newtonian theory of gravity.

We would expect the **metric around a black hole to be given by (13.13). This is certainly true if the black hole is not rotating.** In the **Newtonian theory of gravity, the gravitational field due to a mass does not depend on whether the mass is rotating or not.**

One of the intriguing **results of general relativity is that a rotating mass tries to drag bodies around it to rotate with it** (Thirring and Lense, 1918). Kerr (1963) discovered the exact **metric for rotating black holes**. It has **cross-terms between time and space coordinates,** unlike the Schwarzschild metric which does not have such cross-terms. These cross-terms are **responsible for the rotational dragging**. This metric is called the ***Kerr metric*.**

Spacetime frame dragging



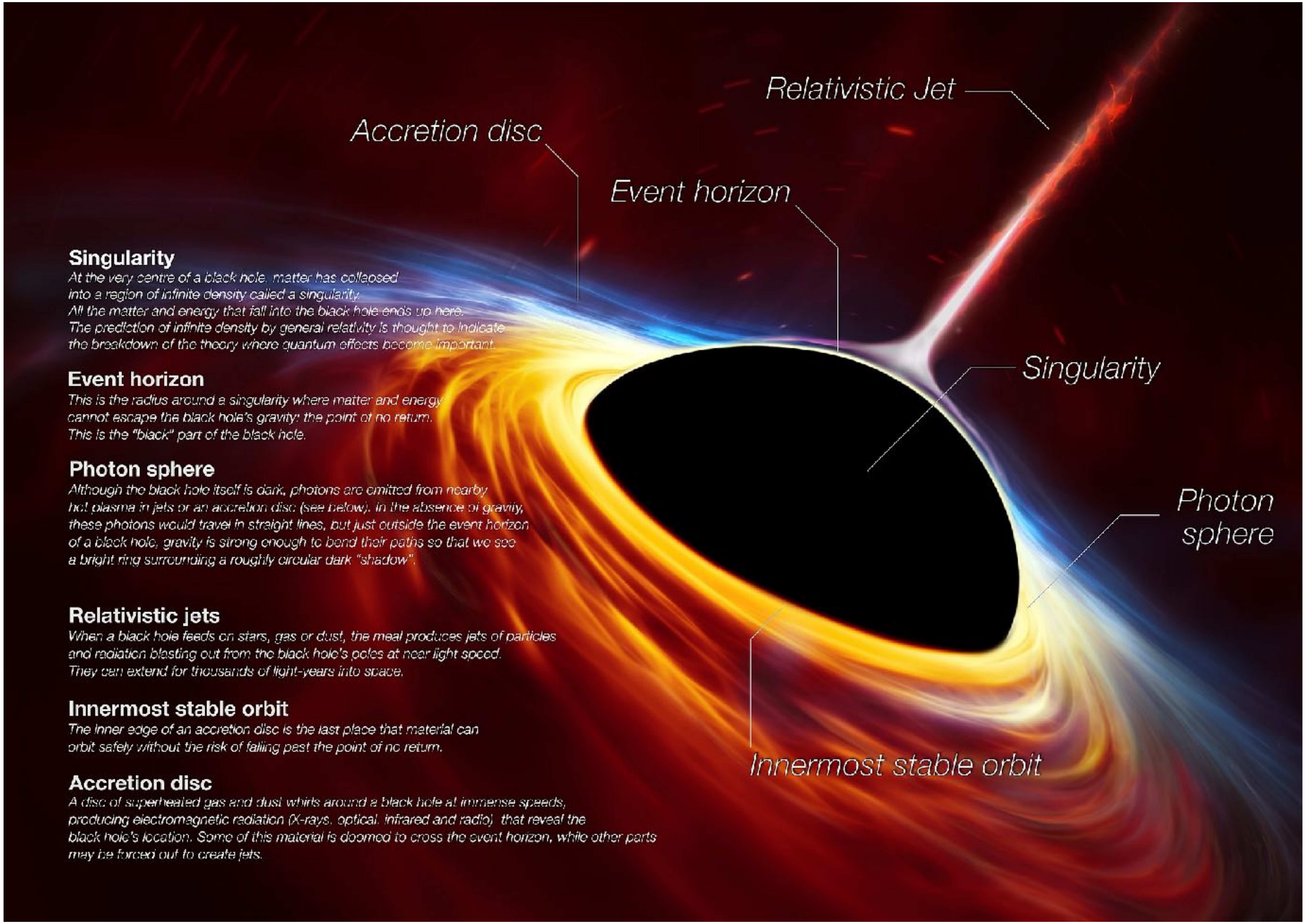
Experiment to measure frame dragging: Even Earth's rotation produces very weak frame dragging. Detecting the effect of frame dragging was the mission of the **Stanford Gravity Probe B** experiment.

Launched in April 2004 and ended data collection in October 2005.

The experiment employed four superconducting gyroscopes made of precisely shaped spheres of fused quartz 3.8 cm in diameter. The gyroscopes were so nearly freely rotating that they formed an **almost perfect spacetime reference frame**. Although the predicted precession rate of the gyroscopes was only $0.042'' \text{ yr}^{-1}$, the effect of frame dragging is cumulative. **Measured frame dragging within 15% of the predicted value.**

In addition to frame dragging, the experiment also measured the stronger geodetic effect.
The geodetic effect is an effect caused by **space-time being "curved" by the mass of the Earth**. A gyroscope's axis when parallel transported around the Earth in one complete revolution does not end up pointing in exactly the same direction as before.

Black hole



https://en.wikipedia.org/wiki/Black_hole

Black holes

First ideas:

In 1783 John Michell (1724–1793), an amateur astronomer, considered **if light were indeed a stream of particles, then it should be influenced by gravity**. In particular, the gravity of a star 500 times larger than the Sun, but with the Sun's average density, would be sufficiently strong that even light could not escape from it. Even if this Newtonian derivation were correct, the resulting radius of such a **star seemed unrealistically small**, and so it held little interest for astronomers until the middle of the twentieth century.

In 1939 American physicists J. Robert Oppenheimer and Hartland Snyder described the ultimate **gravitational collapse of a massive star that had exhausted its sources of nuclear fusion.** It was earlier that year that Oppenheimer and Volkoff had calculated the **first models of neutron stars.** We have seen that a **neutron star cannot be more massive than about 3** $M_\odot$. **What happens if a degenerate star exceed this limit?**

The upper mass limit of a neutron star is between 2.2 $M_\odot$ *and 2.9* $M_\odot$ *depending on the amount of rotation. We will adopt an approximate value of 3* $M_\odot$ *for the purposes of this discussion.*

The Schwarzschild radius

For the simplest case of a **nonrotating star,** the answer lies in the Schwarzschild metric

$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2.$$

When the radial coordinate of the star's surface has collapsed to

$$\boxed{R_S = 2GM/c^2,}$$

called the **Schwarzschild radius, the square roots in the metric go to zero**. The resulting behavior of space and time at $r = R_S$ is remarkable. For example, the proper time measured by a clock at the Schwarzschild radius is $d\tau = 0$. **Time has slowed to a complete stop**, as measured from a vantage point that is at rest a great distance away. From this viewpoint, *nothing ever happens at the Schwarzschild radius*!

The Schwarzschild radius

This behavior is quite curious; does it imply that even light is frozen in time? The speed of light measured by an observer suspended above the collapsed star must always be c. But **from far away, we can determine that light is delayed as it moves through curved spacetime.** (Example: The time delay of radio signals from the Viking lander on Mars.)

The **apparent speed of light**, the rate at which the spatial coordinates of a photon change, is called the ***coordinate speed of light*.** Starting with the Schwarzschild metric with ds = 0 for light,

$$0 = \left(c\,dt\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2,$$

we can calculate the coordinate speed of a vertically traveling photon. Inserting $d\theta = d\phi = 0$ shows that, in general, the coordinate speed of light in the radial direction is

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_S}{r}\right).$$

The Schwarzschild radius

When r ≫ R_S, dr/dt ≃ c, as expected in flat spacetime. However, at r = R_S, dr/dt = 0. **Light is frozen in time at the Schwarzschild radius.**

The spherical surface at r = R_S acts as a barrier and prevents our receiving any information from within. For this reason, **a star that has collapsed down within the Schwarzschild radius is called a black hole.**

It is enclosed by the **event horizon**, **the spherical surface at r = R_S**. Note that the event horizon is a mathematical surface and need not coincide with any physical surface.

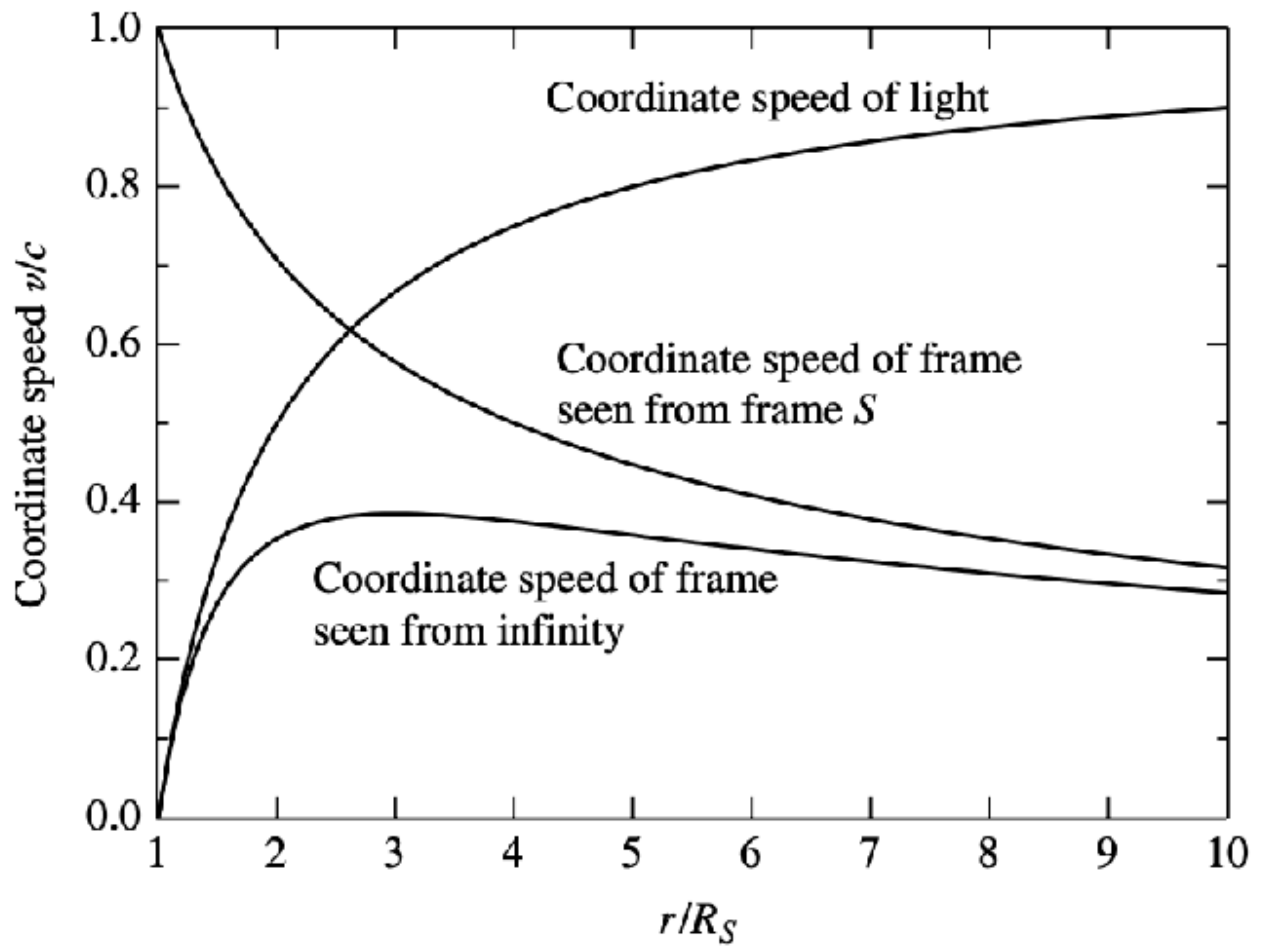


FIGURE 19 Coordinate speed of light, and coordinate speeds of a freely falling frame S seen by an observer at rest at infinity and by an observer in the frame S. The radial coordinates are in terms of R_S for a 10 $M_\odot$ black hole having a Schwarzschild radius of ≈ 30 km.

The Schwarzschild radius

Although the interior of a black hole, inside the event horizon, is a region that is forever hidden from us on the outside, its properties may still be calculated.

A nonrotating black hole has a particularly simple structure. At the center is the **singularity**, **a point of zero volume and infinite density where all of the black hole's mass is located.** Spacetime is infinitely curved at the singularity. Cloaking the central singularity is the event horizon, so **the singularity can never be observed**.

In fact, there is a hypothesis dubbed the ***"Law of Cosmic Censorship"*** that forbids a *naked singularity* from appearing unclothed (without an associated event horizon).

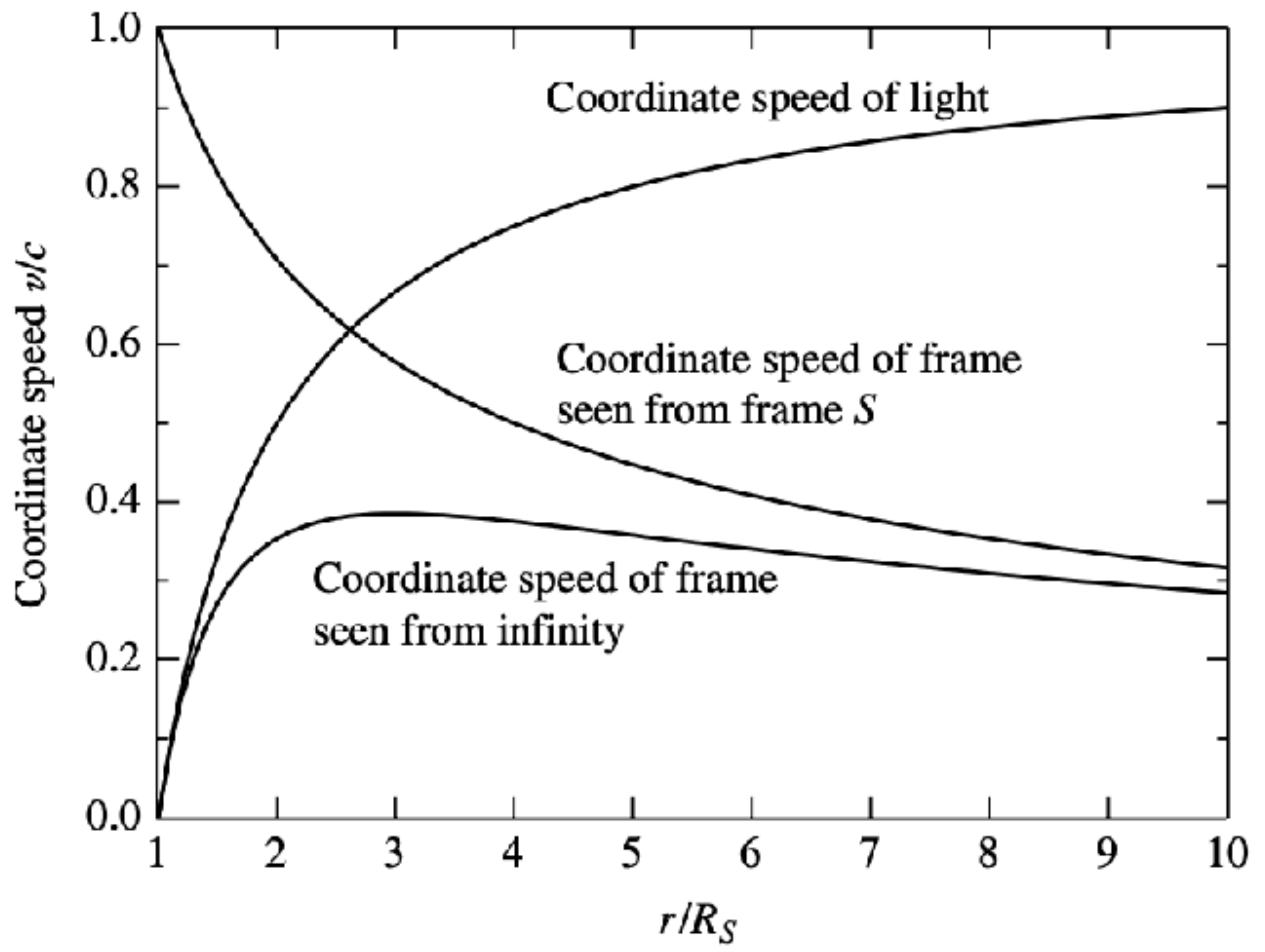


FIGURE 19 Coordinate speed of light, and coordinate speeds of a freely falling frame S seen by an observer at rest at infinity and by an observer in the frame S. The radial coordinates are in terms of R_S for a 10 $M_\odot$ black hole having a Schwarzschild radius of $\approx$ 30 km.

Black holes

Imagine an attempt to investigate the black hole by starting at a safe distance and reflecting a radio wave from an object at the event horizon. **How much time will it take for a radio photon (or any photon) to reach the event horizon from a radial coordinate r $\gg R_S$ and then return?**

Since the round trip is symmetric, it is necessary only to find the time for either the journey in or out and then double the answer. It is easiest to integrate the coordinate speed of light in the radial direction,

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_S}{r}\right).$$

between two arbitrary values of r_1 and r_2 to obtain the general answer,

$$\Delta t = \int_{r_1}^{r_2} \frac{dr}{dr/dt} = \int_{r_1}^{r_2} \frac{dr}{c(1 - R_S/r)} = \frac{r_2 - r_1}{c} + \frac{R_S}{c} \ln\left(\frac{r_2 - R_S}{r_1 - R_S}\right),$$

assuming that $r_1 < r_2$. Inserting $r_1 = R_S$ for the photon's original position, **we find that $\Delta t = \infty$**.

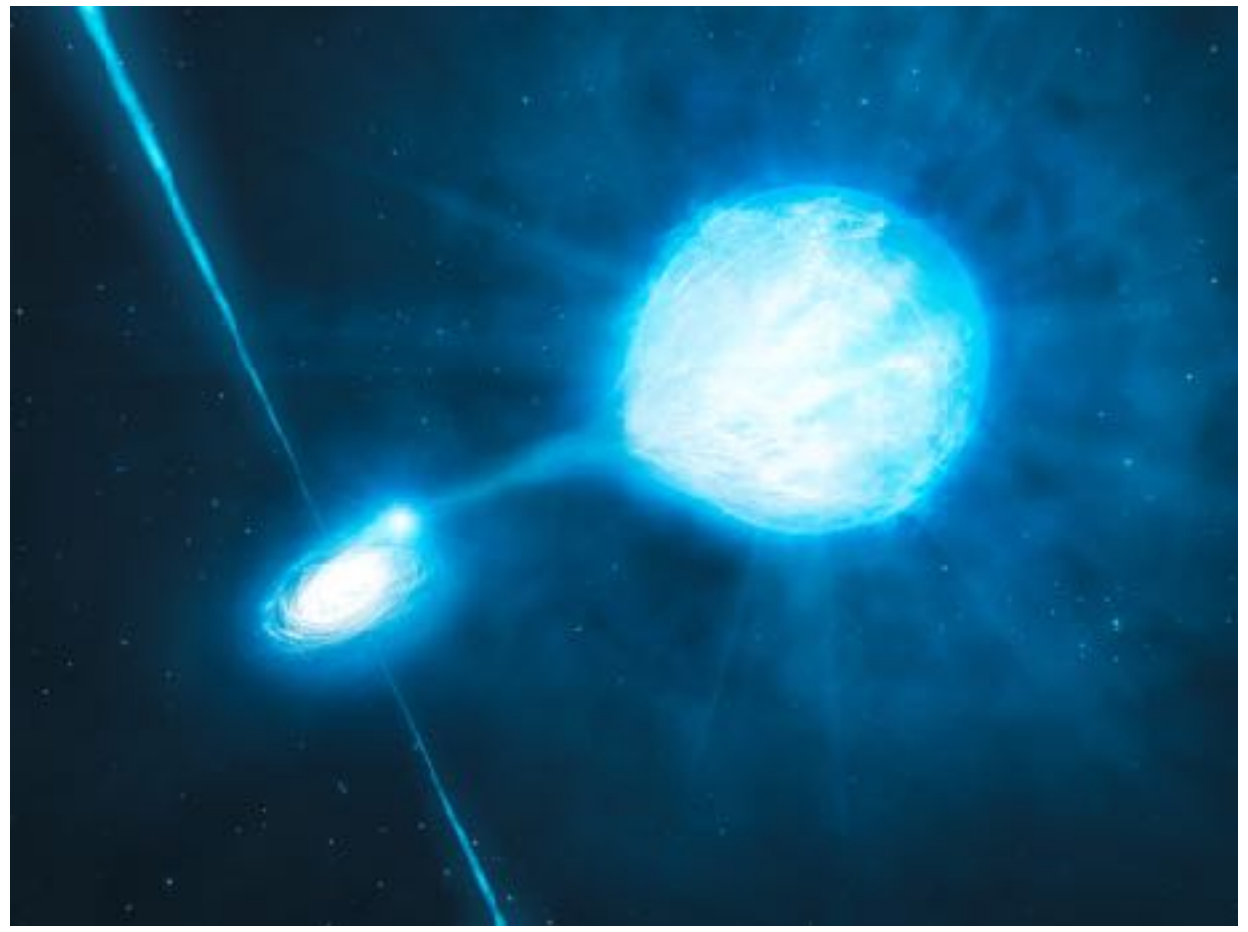
Black holes

Now, since the trip is symmetric, the same result applies if the photon started at R_S. **According to the distant observer, the radio photon will *never* reach the event horizon.**

Instead, **according to gravitational time dilation, the photon's coordinate velocity will slow down until it finally stops at the event horizon in the infinite future.** In fact, any object falling toward the event horizon will suffer the same fate. **Seen from the outside**, even the surface of the star that collapsed to form the event horizon would be frozen, and so a black hole is in this sense a *frozen star*.

However, in the **frame of the particle falling into the black hole** this is not the case. **Particles do fall trough the event horizon and after that into the singularity.**

Mass ranges of black holes



Artist's impression of a stellar-mass black hole (left) in the spiral galaxy NGC 300; it is associated with a Wolf–Rayet star

Black holes appear to exist with a range of masses.

Stellar-mass black holes, with masses in the range of **3 to 15 M⊙** or so, may form directly or indirectly as a consequence of the **corecollapse of a sufficiently massive supergiant star**.

It is also possible that a **neutron star in a close binary system** may gravitationally strip enough mass from its companion that the neutron star's self-gravity exceeds the ability of the degeneracy pressure to support it, again resulting in a black hole.

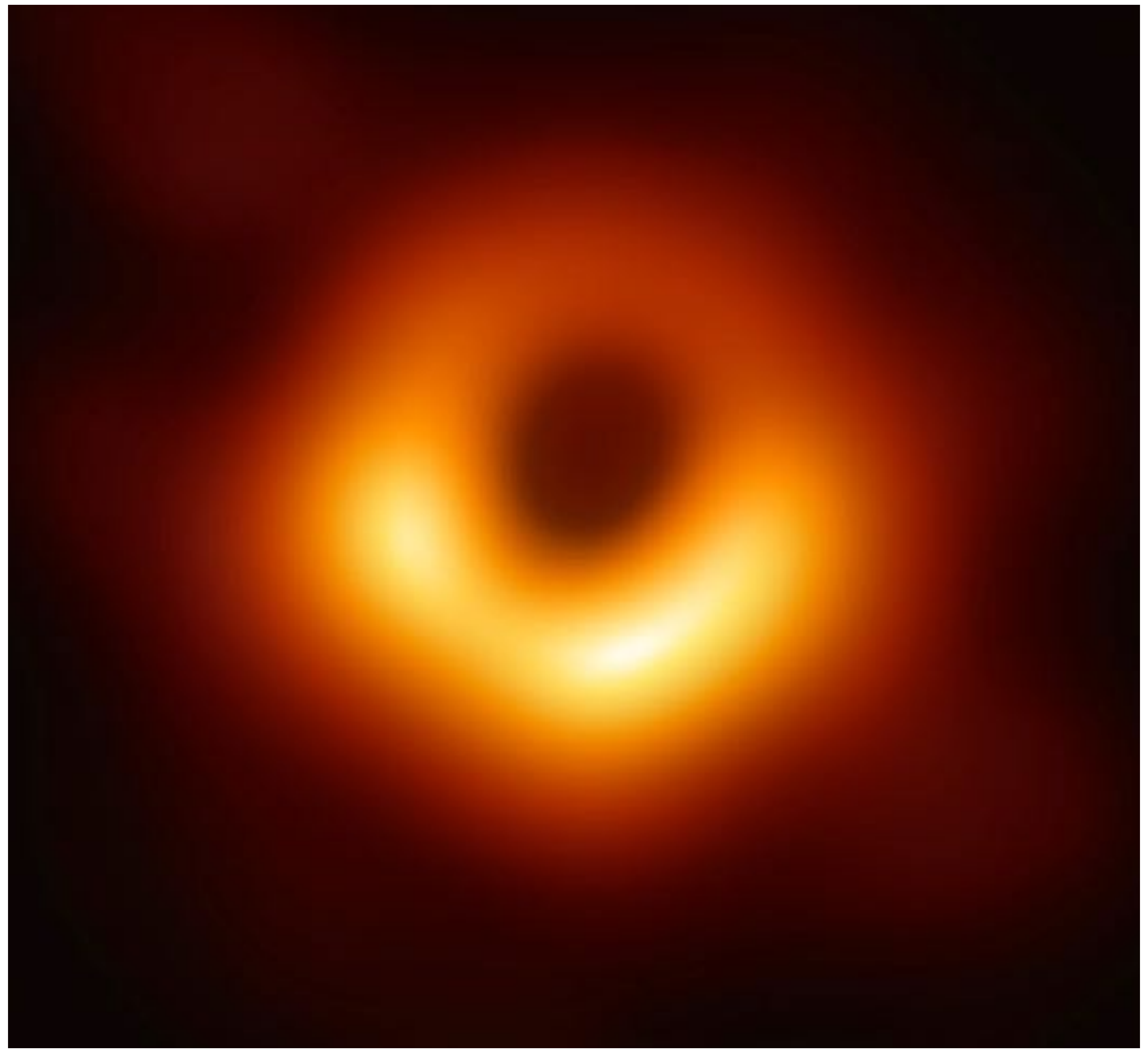
Intermediate-mass black holes (IMBHs) may exist that range in mass from **roughly 100 M⊙ to in excess of 1000 M⊙** (or perhaps even greater than 10^4 $M_\odot$). Evidence for them exists in the detection of sources known as **ultraluminous X-ray sources** (ULXs) that have been discovered by satellites such as Chandra and XMM-Newton. It is not entirely clear how these objects might form, although the **correlation of IMBHs with the cores of globular clusters** and low-mass galaxies **suggests** that they may develop in these dense stellar environments either by the **mergers of stars** to form a supermassive star that then core-collapses, or by the **merger of stellar-mass black holes.**

Mass ranges of black holes

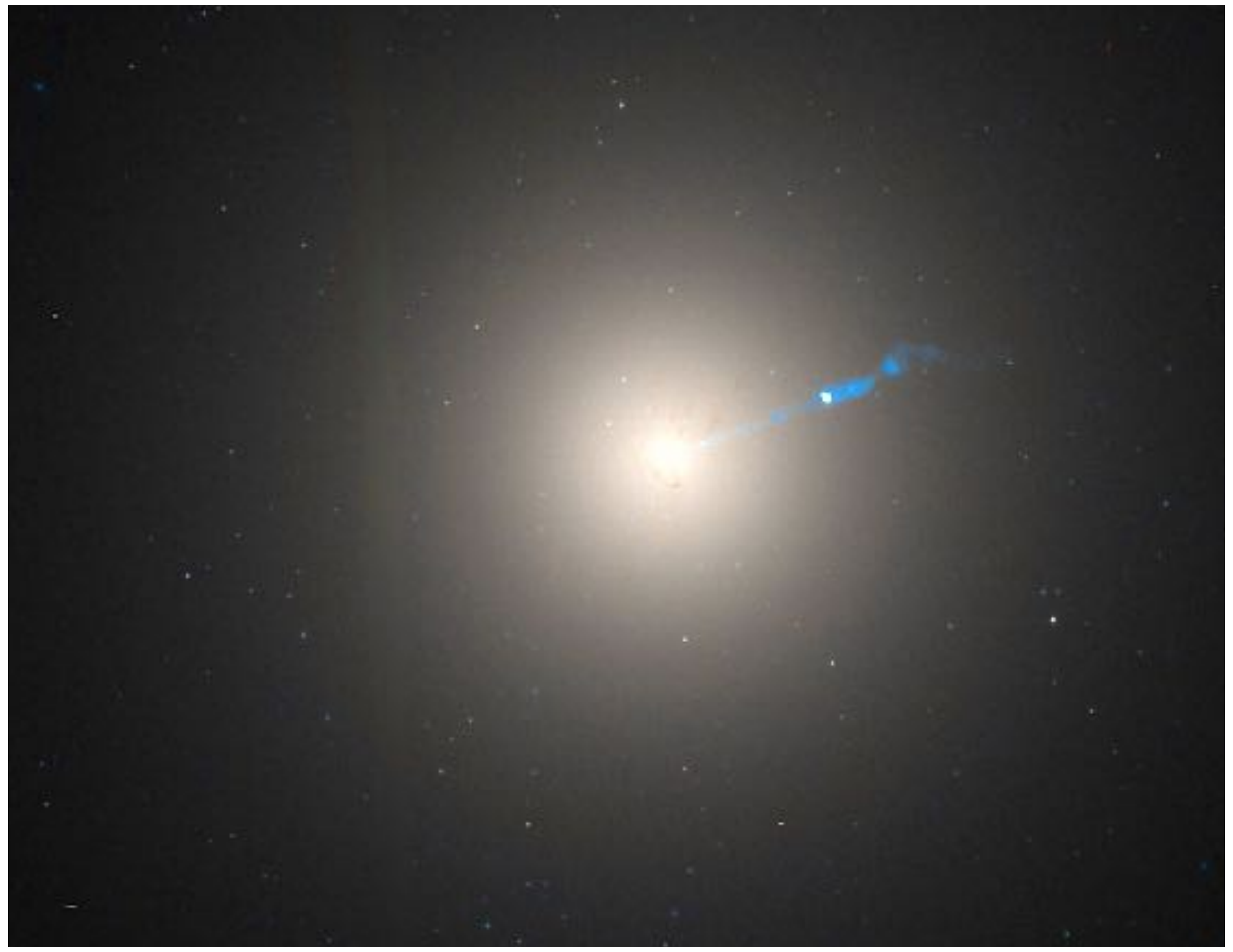
Supermassive black holes (SMBH) are known to **exist at the centers of most galaxies.** These enormous black holes range in mass from **10^5 $M_\odot$ to 10^9 $M_\odot$** (our own Milky Way Galaxy has a central black hole of mass $M = 3.7 \pm 0.2 \times 10^6\ M_\odot$).

How these behemoths formed remains an open question. One popular suggestion is that they formed from **collisions between galaxies**; another is that they formed as an **extension of the formation process of IMBHs**. Whatever the process, SMBHs appear to be **closely linked with some bulk properties of galaxies, implying an important connection between galaxy formation and the formation of SMBHs.**

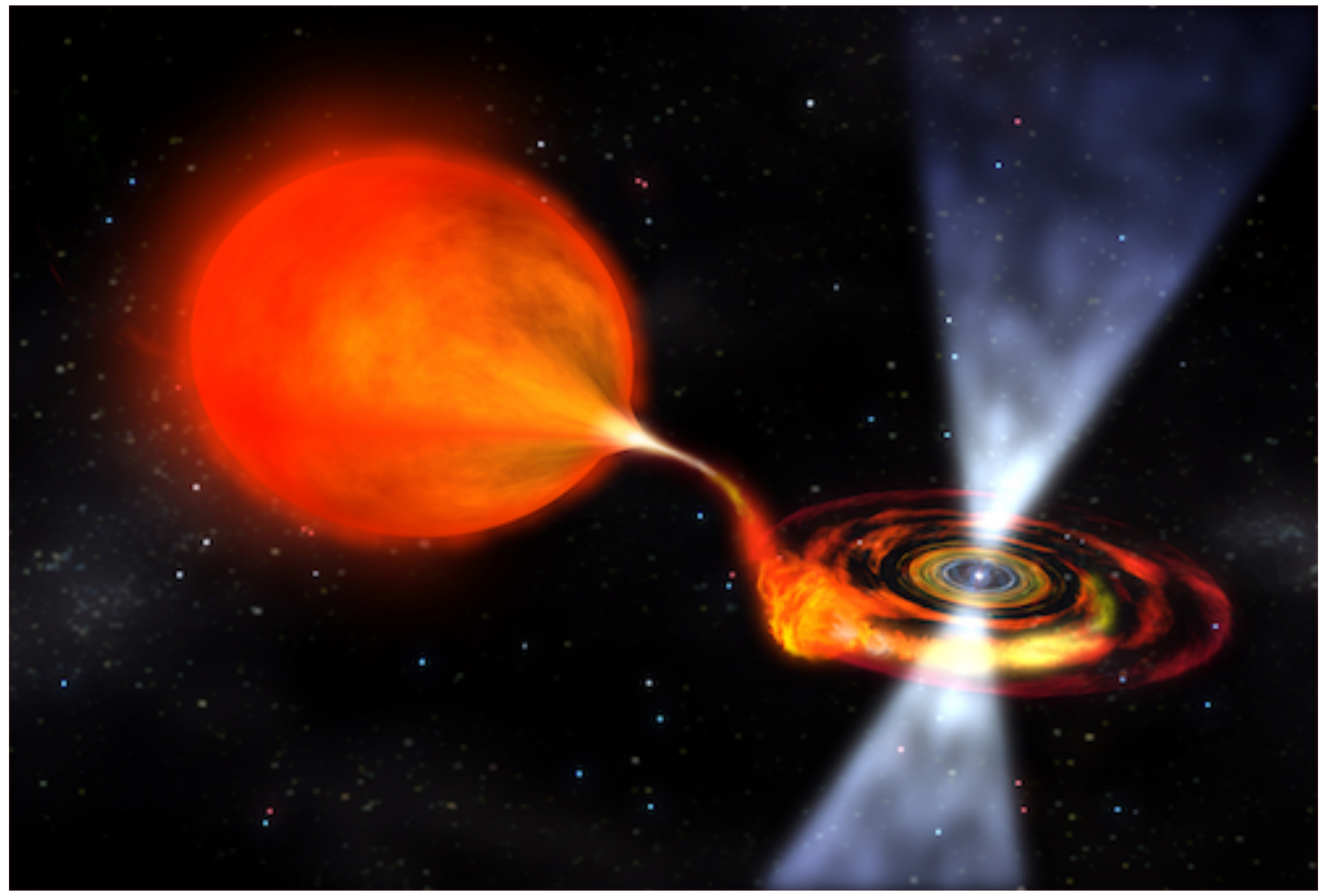
Black holes may have also been manufactured in the earliest instants of the universe. Presumably, these **primordial black holes** would have been formed with a wide range of masses, from 10^{-8} kg to 10^5 $M_\odot$. The only criterion for a black hole is that its entire mass must lie within the Schwarzschild radius.



M87



Binary X-ray sources



- What are binary X-ray sources?
- **Binary X-ray sources** can be **neutron stars or stellar mass black holes accreting matter** from inflated binary companions.
- In a **binary system** at some stage, the binary companion may become a red giant and fill up the Roche lobe. This would lead to a **transfer of mass** from the inflated companion star to the neutron star or the black hole.
- The matter accreting onto the neutron star from its companion will carry a considerable amount of angular momentum. This is **expected to increase the angular velocity of the accreting neutron star or black hole.**
- Eventually, when the red giant phase of the companion star is over (it may become a white dwarf or another neutron star), **the neutron star which has been spun up** by accreting matter with angular momentum **becomes visible as a millisecond pulsar**.
- Many X-ray sources were **found in the galactic plane**.
- They are also called **microquasars.**

Binary X-ray sources

- Many X-ray sources were **found in the galactic plane**. -> needs to be objects related to stars

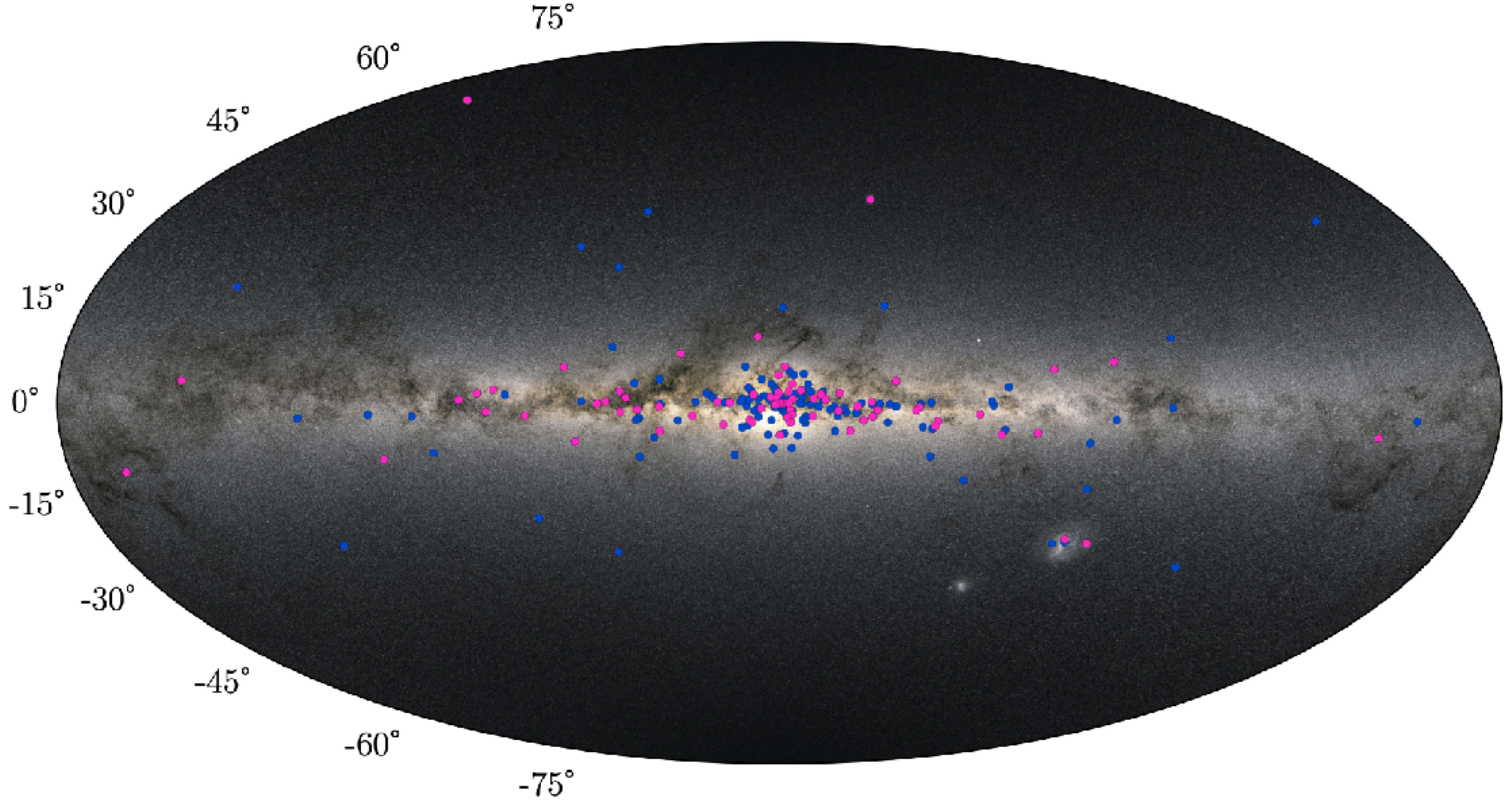


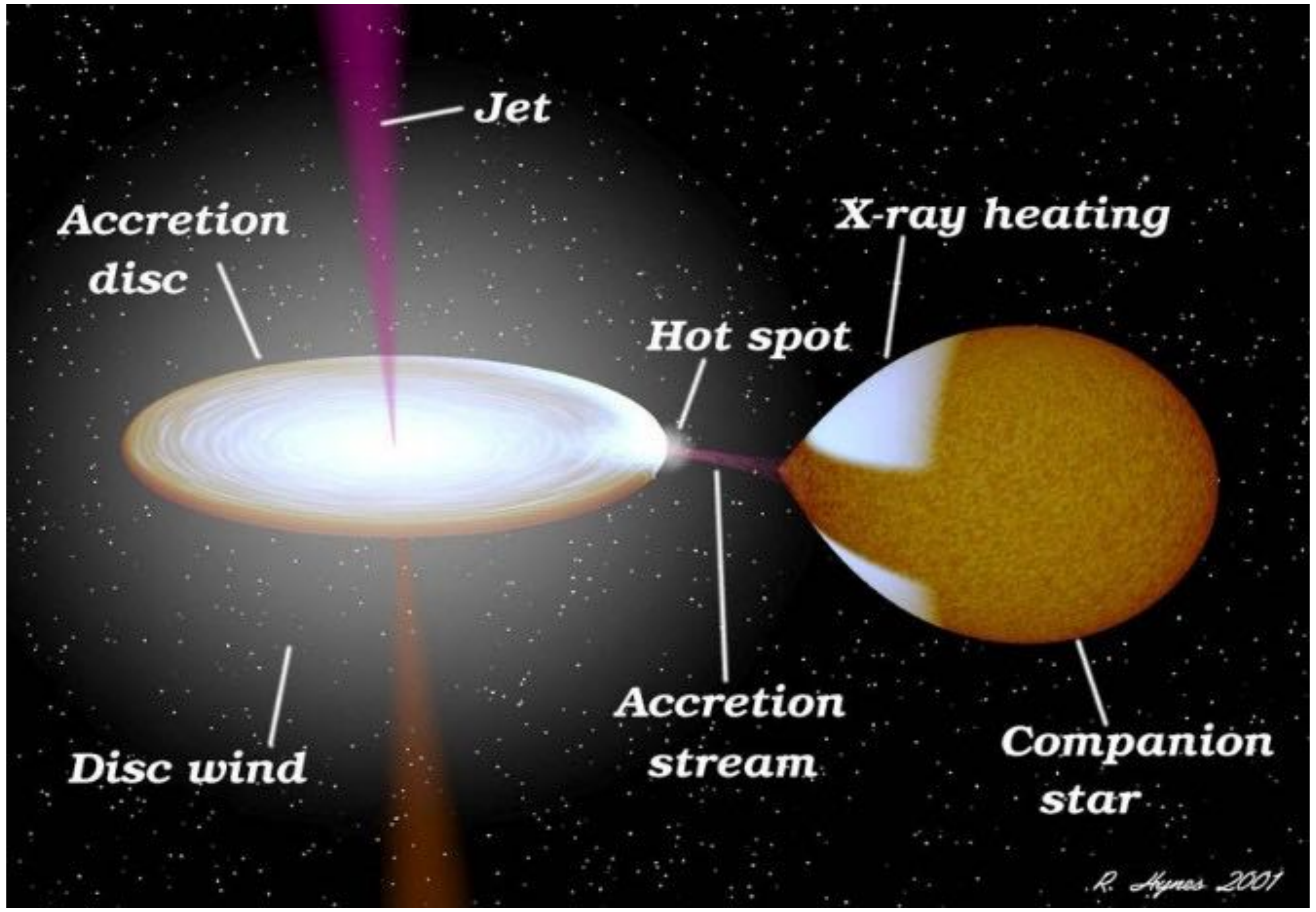
Fig. 9 Distribution of a subset of known LMXBs in the sky overlaid on the Gaia all-sky image.

Bahramian and Degenaar 2022

Binary X-ray sources

- Mass transfer between the two stars in a binary system:
- If, the mass m is dropped from infinity to a star of mass M and radius R, then the gravitational energy lost is:

$$\frac{GM}{R}m = \frac{GM}{c^2 R}mc^2.$$

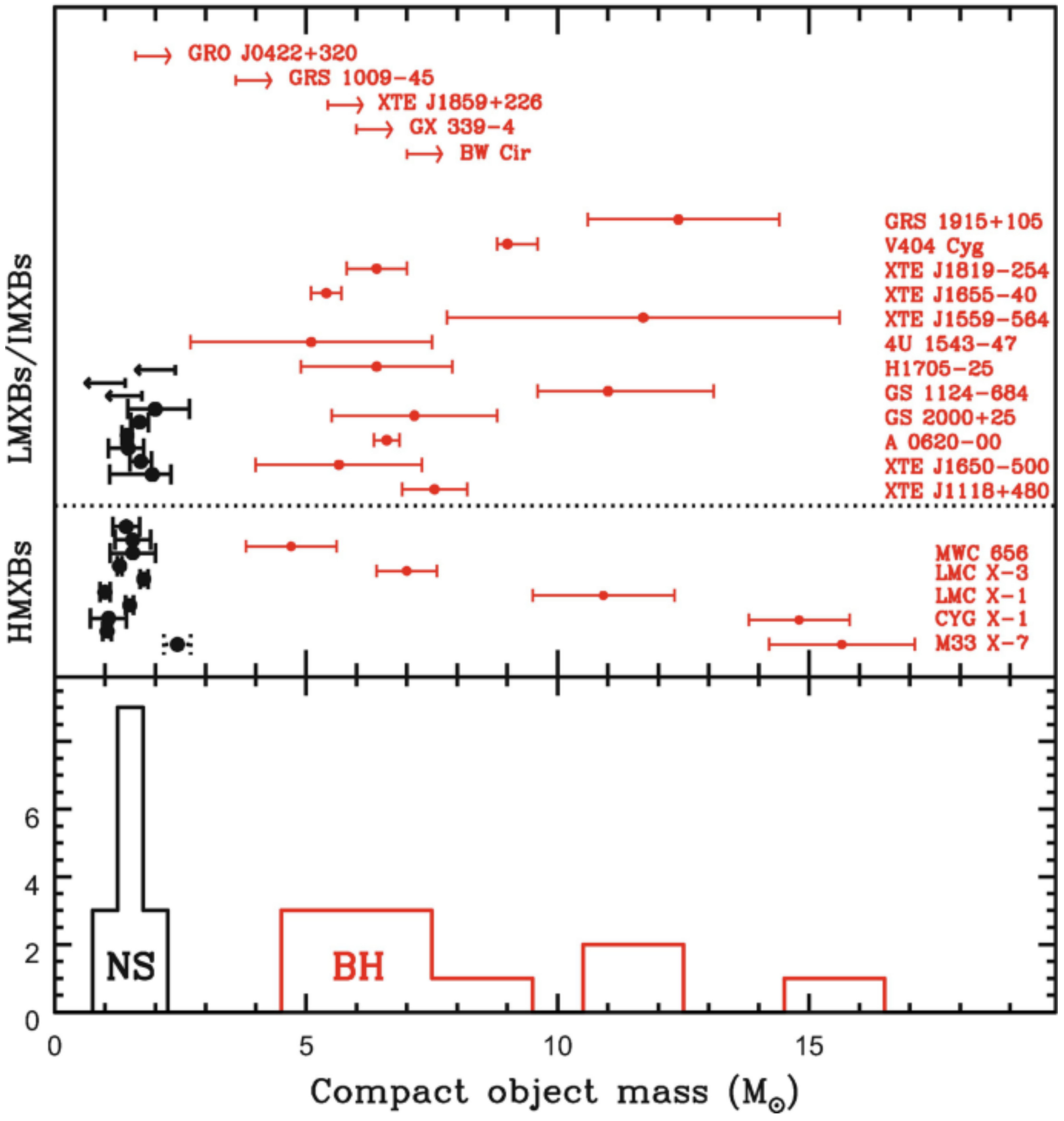


- For a typical neutron star of mass $1M_\odot$ and radius 10 km, the factor $GM/c^2 R$ turns out to be about 0.15.
- The **loss of gravitational energy** may be a very significant fraction of the rest mass energy, making such an infall of matter into the deep gravitational well of a compact object like a neutron star or a black hole a very **efficient process for energy release. -> radiation**
- Intense X-ray emission is released from the **inner region of the accretion disk** where it falls onto the collapsed star. The infalling material is heated to over a million degrees.

https://www.cosmos.esa.int/web/cesar/x-ray-binaries-monitoring

Binary X-ray sources

- **How do we know if the compact object is a neutron star or a black hole?**
- The Figure shows the masses of several neutron stars in binary systems.
- **There are binary X- ray sources with accreting objects which have masses higher than** $3M_{\odot}$. E.g. Vela X-1.
- The **central accreting object is a black hole,** since its estimated mass is well above the neutron star mass limit.



https://link.springer.com/referenceworkentry/10.1007/978-3-319-21846-5_111

Mass ranges of black holes

Example: If Earth could somehow (miraculously) be compressed sufficiently to become a black hole, its radius would only be $R_S = 2GM_{\oplus}/c^2 =$ **0.009 m.**

A primordial black hole could be this size.

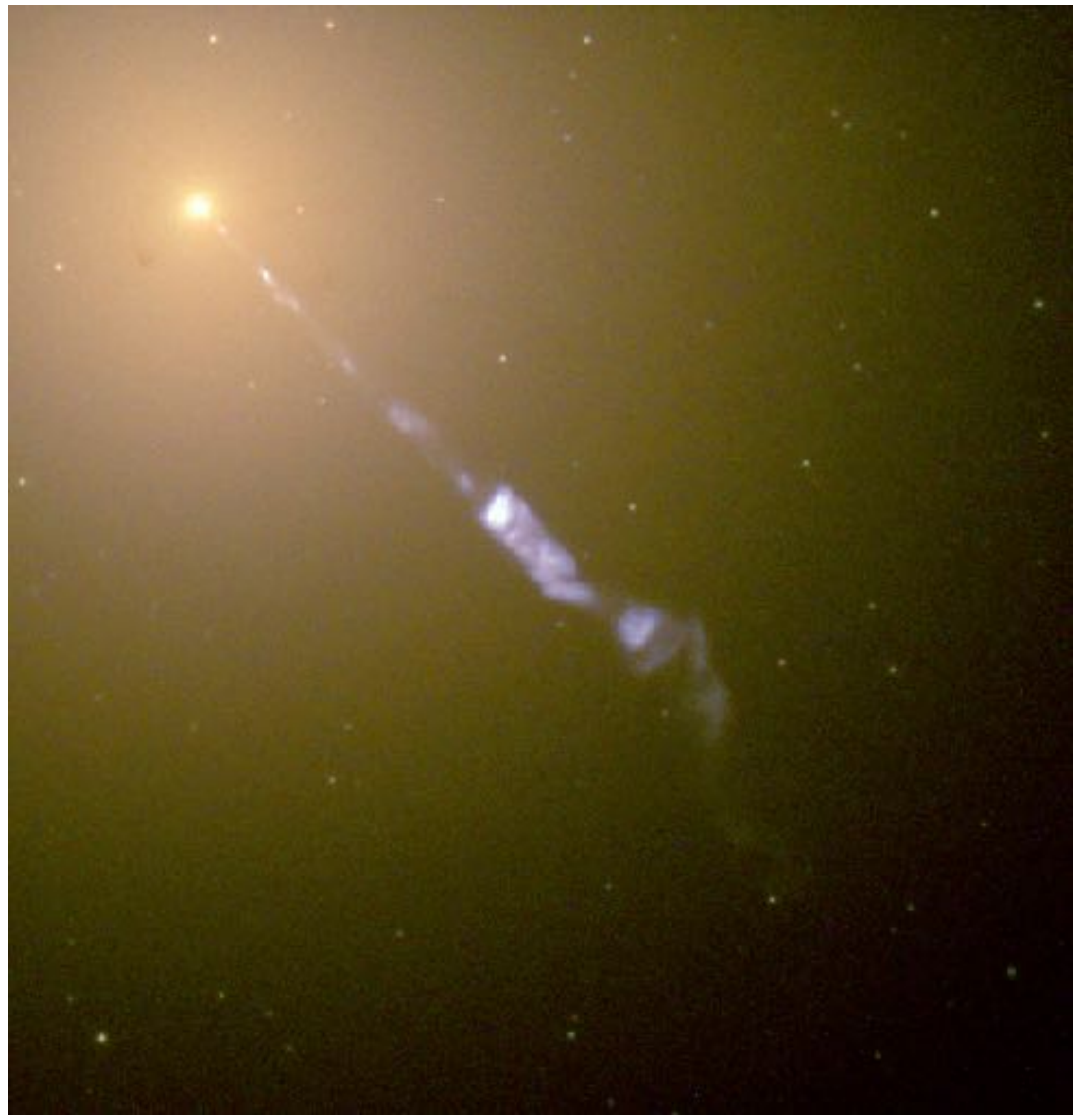
Supermassive black holes

Found in the centres of galaxies.

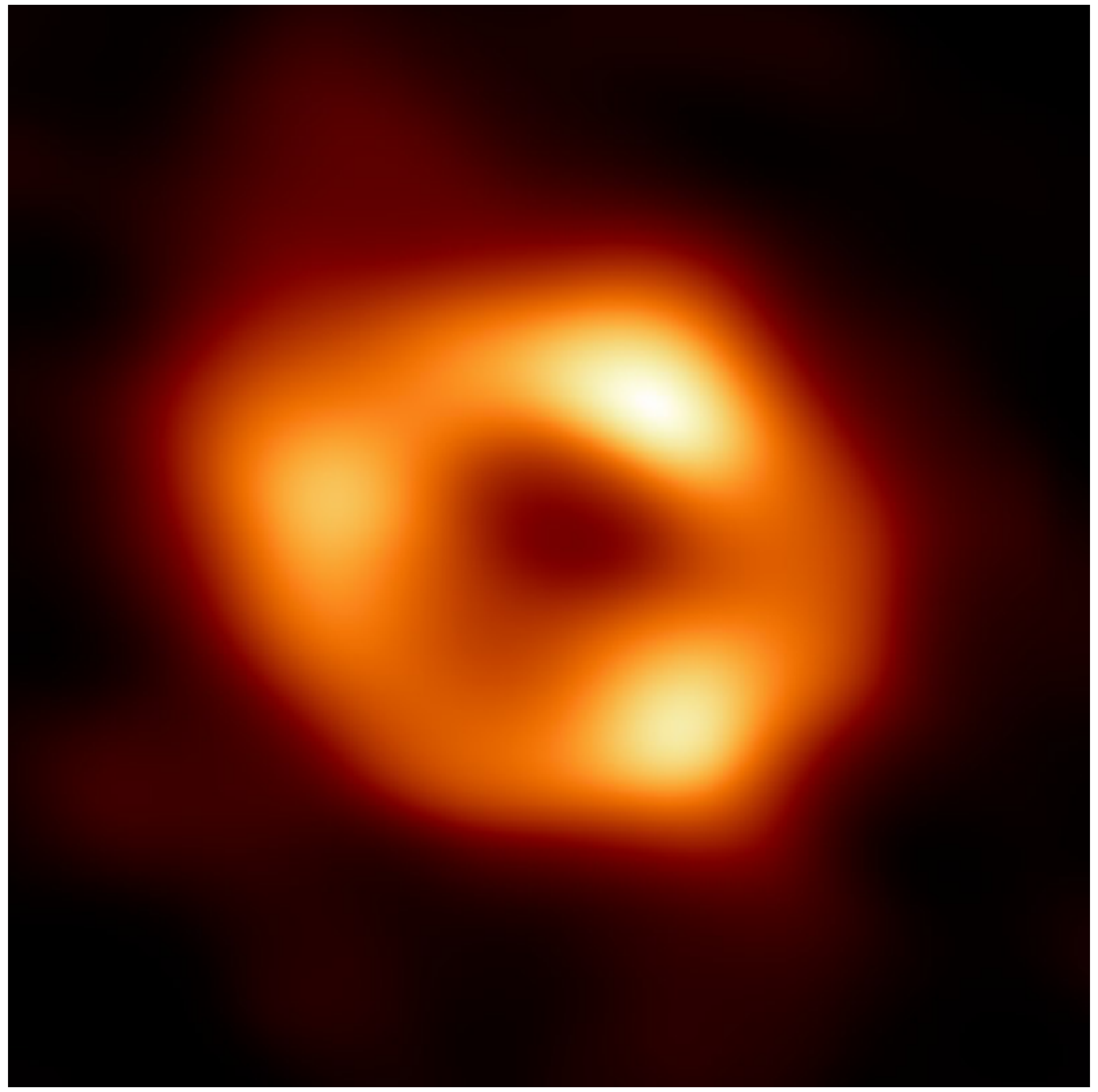
Observational evidence:

- AGN activity
- The motion of stars in the centre of the Milky Way
- Images of the Event Horizon telescope

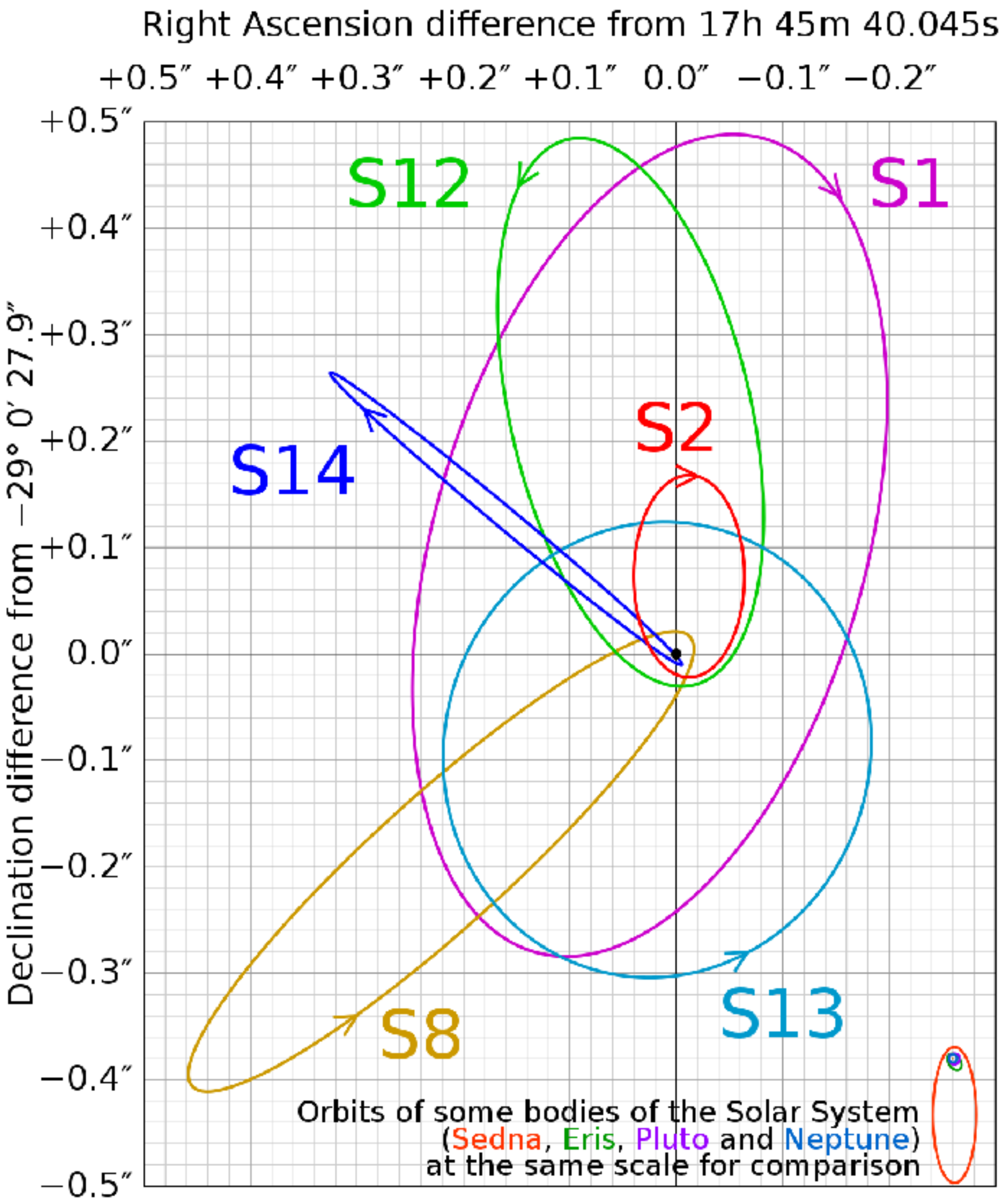
M87 optical jet



SgrA* in radio
with the Event Horizon Telescope



Motion of stars around SgrA*

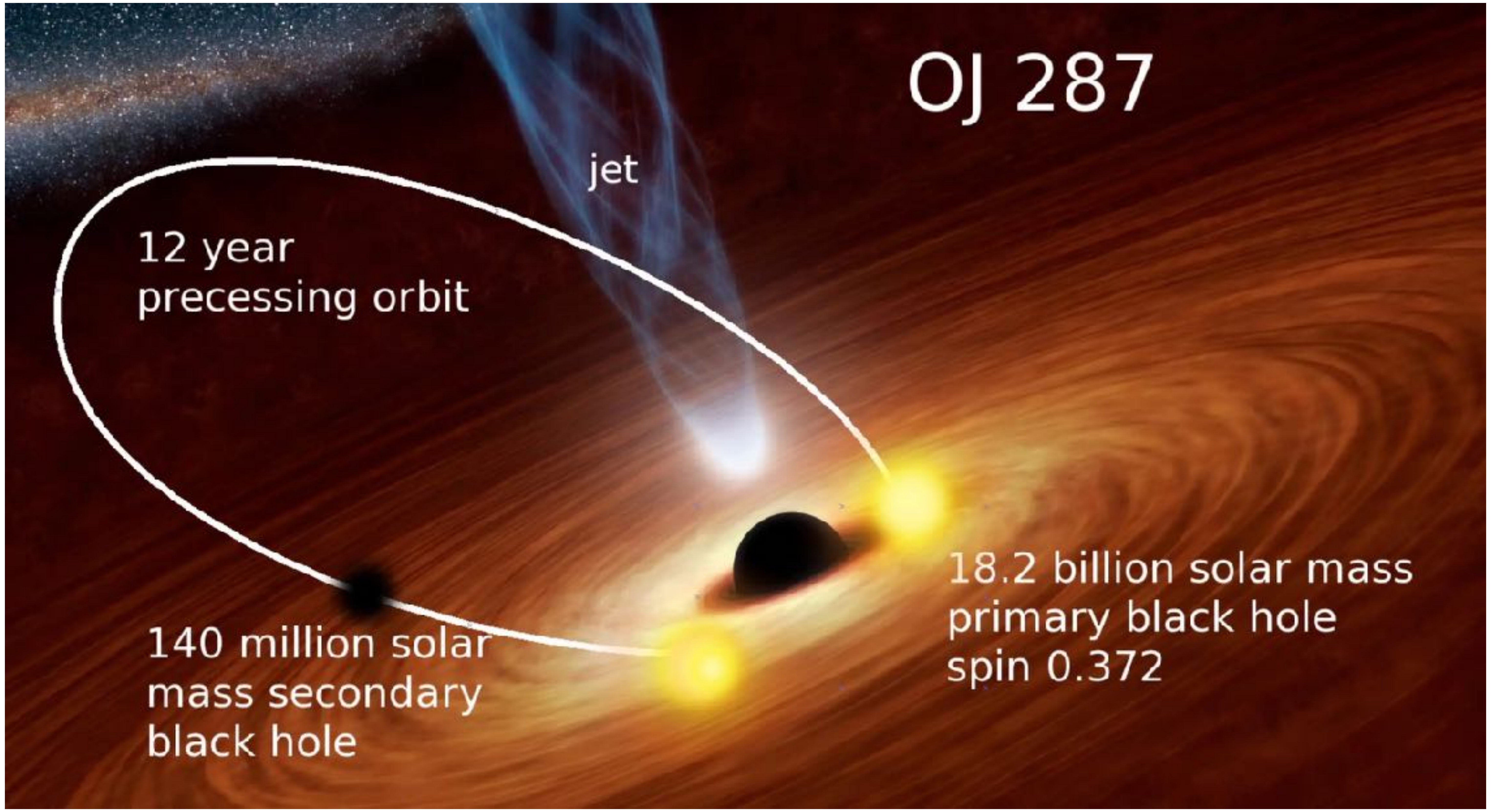


Supermassive black holes

Two supermassive black holes orbiting each other at the core of the distant galaxy OJ 287.

The orbital motion is revealed by a **series of flares** that arise when the **secondary black hole plunges regularly through the accretion disk of the primary black hole** at speeds that are a fraction slower than the speed of light.

This **heats the disk material** and the hot gas is **released as expanding bubbles.** These hot bubbles take months to cool while they radiate and cause a flash of light – a flare – that lasts roughly a fortnight and is brighter than a trillion stars.

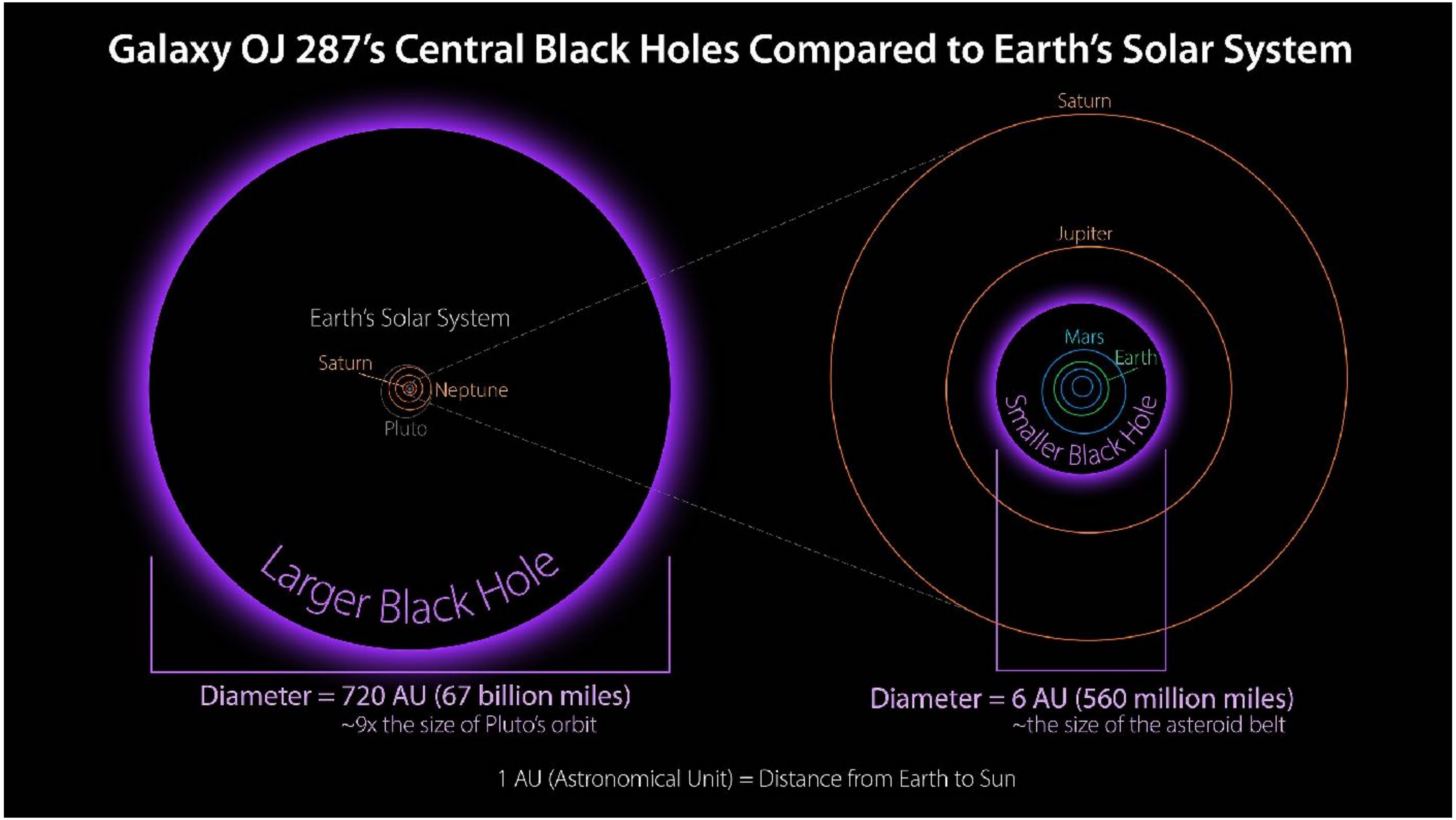


https://scitechdaily.com/flash-of-light-brighter-than-a-trillion-stars-leads-to-supermassive-black-hole-breakthrough/

Supermassive black holes

Two supermassive black holes orbiting each other at the core of the distant galaxy OJ 287.

The larger one, with about 18 billion times the mass of our sun (left), the smaller one is about 150 million times the mass of our sun (right).



https://en.wikipedia.org/wiki/Supermassive_black_hole