Spacetime Dynamics - Summary

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This summary is based on the book Chapter 10 from Arnab Rai Choudhuri: Astrophysics for physicists.

1 General relativity (GR)

Newton's theory of gravity is not consistent with special relativity.

All particles placed in a small region of gravitational field (over which the variations of the field can be neglected) have the same acceleration. As a result, a gravitational field is equivalent to an accelerated frame. $\rightarrow$ **equivalence principle**

According to special relativity (SR) energy is equal to mass. $\rightarrow$ According to GR mass and energy are equivalent of curvature of spacetime. Einstein's equation, describes how the curvature of spacetime is related to mass-energy. Given the distribution of mass-energy, one can in principle find out the metric of spacetime from Einstein's equation and thereby determine the structure of spacetime. Mass and energy create curvatures in spacetime and a particle moves along geodesics in this curved spacetime.

In GR, gravity is the result of objects moving through curved spacetime, and everything that passes through, even massless particles such as photons, is affected. E.g. gravitational lenses

2 The metric of the Universe

The **cosmological principle** states that space is homogeneous and isotropic. This is only possible if the Universe has uniform curvature everywhere. The 4D metric that describes the Universe is also called the **Robertson-Walker metric.**

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2}r^2(d\theta^2 + sin^2\theta d\phi^2)\right]$$

where **k represents the curvature** of the Universe and can be -1, 0, or 1 $\rightarrow$ flat, positively curved and negatively curved space. **a(t)** is called the **scale factor** and represents the size of the Universe and how the Universe evolves with time.

Galaxies are moving from each other because the Universe is expanding and the scale factor of the Universe is increasing. A coordinate system in which galaxies do not change their coordinates with the expansion of the Universe is called a co-moving coordinate system. The Robertson-Walker metric is usually assumed to be a metric corresponding to a co-moving coordinate system.

Hubble constant: $H = \frac{v}{l} = \frac{\dot{a}}{a}$

Newtonian cosmology: A simplified version of relativistic cosmology using principles of Newtonian physics.

Redshift: the wavelength of light gets stretched with the expansion of the Universe. The scale factor a changes from when the light was emitted (a) to the present day (a_0).

$$1 - z = \frac{\lambda_{obs}}{\lambda_{emitted}} = \frac{a_0}{a}$$

3 The Friedmann equation

The Friedmann equation:

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

There is also a term due to the cosmological constant, this is the simplified version. The cosmological constant contributes to the acceleration of the Universe and is important at the present epoch. The cosmological constant becomes more dominant as the Universe becomes older.

If k = -1, then the Universe will expand forever. On the other hand, if k = +1, then the expansion of the Universe will eventually be halted, making the Universe fall back and collapse. Such a Universe will last for a finite time before it ends in a big crunch. → A Universe finite in space (with k = +1) should last for a finite time. On the other hand, a Universe infinite in space (with k = -1) will last for infinite time.

Whether the Universe will expand forever or not must depend on the density of the Universe, which determines the strength of the gravitational attraction. The value of density for which the Universe will lie exactly on the borderline between these two possibilities **(with k = 0)** is called the **critical density** and is denoted by ρ_c.

$$\rho_c = \frac{3H^2}{8\pi G}$$

The ratio of the density to the critical density is called the **density parameter** and is denoted by Ω

$$\Omega = \frac{\rho}{\rho_c}$$

k = 0, k = +1 and k = -1 respectively correspond to $\Omega = 1$, $\Omega > 1$, $\Omega < 1$.

Solving the Friedmann equation tells us how ρ and a change with time.

4 Contents of the Universe

To understand how the density ρ of the Universe varies with the scale factor a we have to measure the contents of the Universe.

There are 3 components contributing to the density:

- matter (Byronic matter and dark matter)
- relativistic particles (photons, neutrinos)
- the cosmological constant (dark energy)

Measurements indicate the following composition of the density:
$\Omega_M = 0.315 \pm 0.018$
$\Omega_R \approx 9.24 \times 10^{-5}$
$\Omega_\Lambda \approx 0.6817 \pm 0.0018$
$\Omega_M + \Omega_R + \Omega_\Lambda = 1.00 \pm 0.02$

Ω_M is estimated from the virial masses of galaxy clusters. Ω_R is estimated based on the CMB radiation.
The contribution of the cosmological constant becomes more important with time. At the present epoch, this term is comparable to the matter density term but at the early Universe its contribution can be neglected.

Based on current measurements the Universe seems to be spatially flat.

4.1 The CMB radiation

Hot matter emits radiation. the early Universe must have been dense and hot, it would have been filled with radiation existing in thermodynamic equilibrium with matter. Radiation in equilibrium with matter has to be blackbody radiation. As the Universe expanded and its density fell, at some stage the Universe became transparent to radiation and the radiation ceased to be in equilibrium with matter. As the Universe kept expanding after this decoupling, the radiation would undergo adiabatic expansion because it ceases to interact with matter. Blackbody radiation continues to remain blackbody radiation under adiabatic expansion, although its temperature keeps decreasing with expansion. The temperature of the CMB at the current time is 3K. The CMB is a remnant of the early Universe and its existence provided proof that there was a hot Big Bang from which the Universe was born.

The temperature of the CMB is changing inversely proportional to the scale factor $T \propto \frac{1}{a}$.

The radiation density (falling as a^{-4}) falls more rapidly than the matter density (falling as a^{-3}) with the expansion of the Universe. There was a past epoch when the matter and radiation densities were equal, an epoch of matter-radiation equality. This divides the history of the Universe into two distinct periods: **radiation-dominated** (first few thousand years) and **matter-dominated** (after the equity).

5 The evolution of the matter dominated Universe

The size of the Universe increases with time as $t^{2/3}$. This solution for the $\mathbf{k = 0}$ case is often called the **Einstein-de Sitter model.**

$$\frac{a}{a_0} = \left(\frac{3}{2}H_0 t\right)$$

$\mathbf{k = +1}$ a goes to zero when η(representing time) increases to 2π. This is a solution where the Universe eventually ends up in a big crunch.

k = -1 increases forever at a rate faster than the rate of increase of the critical solution k = 0.

All the solutions for all the three values of k behave very similarly at sufficiently early times. It is the curvature term kc^2/a^2 in the Friedmann equation which is responsible for making the three solutions different. At sufficiently early times, this term becomes negligible.

6 The evolution of the radiation dominated Universe

The very early Universe was radiation-dominated. The Universe would have become matter-dominated a few thousand years after the Big Bang. The current age of the Universe mostly depends on the value of $\Omega_{M,0}$ (this is an approximation, the full Ω would give the exact age). If we have a higher $\Omega_{M,0}$ the age of the Universe decreases.

The radiation-dominated Universe expanded with time as $t^{1/2}$ in contrast to the early matter-dominated Universe which expanded as $t^{2/3}$.

The temperature of the CMB falls as the inverse of a:

$$\frac{a}{a_0} = \frac{T_0}{T}$$

T can be written as

$$T(MeV) = \frac{1.31}{\sqrt{t}}$$

gives an indication of the typical energy a photon (or any other kind of particle) would have at time t after the Big Bang.