Introduction to Astrophysics and Cosmology

Radiation transfer

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Kirchof’s law

- Suppose we have a box kept in thermodynamic equilibrium. If we make a small hole on its side, we know that the radiation coming out of the hole will be blackbody radiation. Hence the specific intensity of radiation coming out of the hole is simply

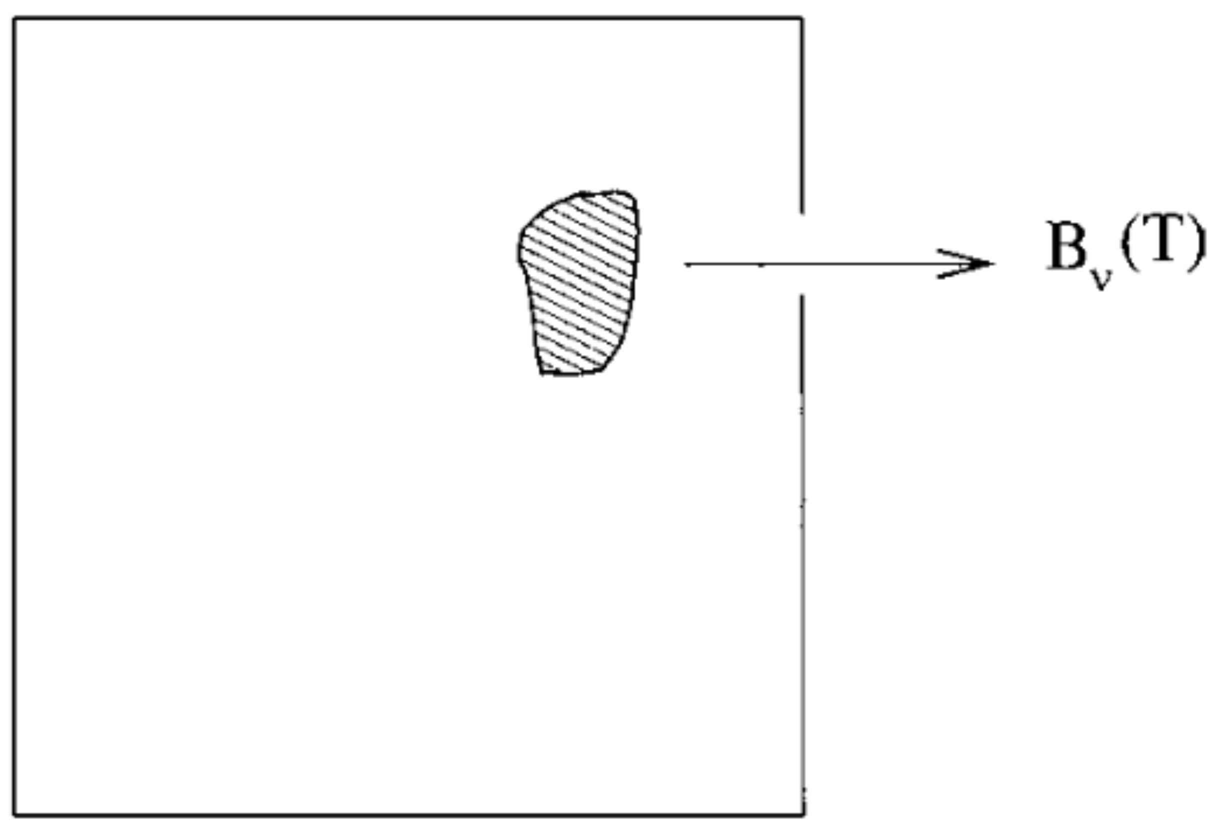


Fig. 2.3 Blackbody radiation coming out of a hole in a box with an optically thick obstacle placed behind the hole.

$$I_\nu = B_\nu(T),$$

Kirchof’s law

- We now keep an optically thick object behind the hole as shown in the Figure. If this object is in thermodynamic equilibrium with the surroundings, then it will not disturb the environment and the radiation coming out of the hole will still be blackbody radiation.
- On the other hand, we have seen in that the radiation coming out of an optically thick object has the specific intensity equal to the source function.

$$S_\nu = B_\nu(T)$$

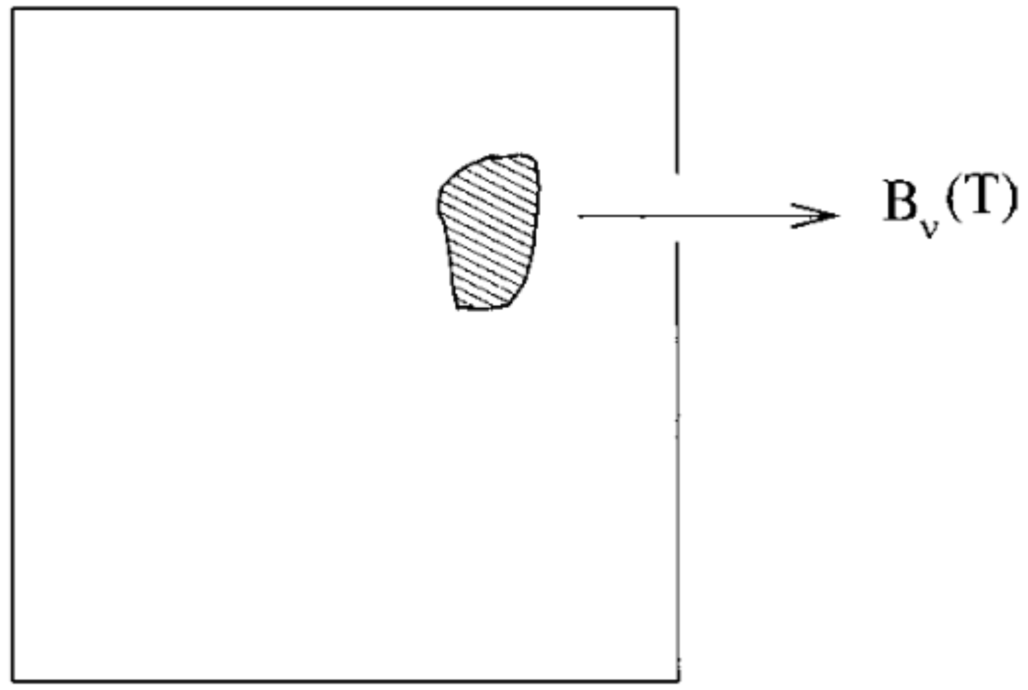


Fig. 2.3 Blackbody radiation coming out of a hole in a box with an optically thick obstacle placed behind the hole.

In thermodynamical equilibrium:

$$\boxed{j_\nu = \alpha_\nu B_\nu(T).}$$

Kirchof’s law

Kirchof's law

- Very often matter tends to emit and absorb more at specific frequencies corresponding to spectral lines. Hence both j_ν and α_ν are expected to have peaks at spectral lines.
- But, the ratio of these coefficients should be the smooth blackbody function $B_\nu(T)$.
- The radiation coming out of an optically thin source is essentially determined by its emission coefficient. Since the emission coefficient is expected to have peaks at spectral lines, we find the emission from an optically thin system like a hot transparent gas to be mainly in spectral lines.
- On the other hand, the specific intensity of radiation coming out of an optically thick source is its source function, which has been shown to be equal to the blackbody function $B_\nu(T)$. Hence we expect an optically thick object like a hot piece of iron to emit roughly like a blackbody.

Kirchof’s law

- The theory of radiative transfer is important not only in astrophysics. If we want to understand rigorously and quantitatively many everyday phenomena such as why hot transparent gases emit in spectral lines whereas hot pieces of iron emit like blackbodies, then we need to invoke the theory of radiative transfer.
- The nature of radiation from an astrophysical source crucially depends on whether the source is optically thin or optically thick. Emission from a nebula is usually in spectral lines. On the other hand, a star emits very much like a blackbody.
- Why is the radiation from a star not exactly blackbody radiation? Why do we see absorption lines? We derived the equations so far by assuming the source to have constant properties. This is certainly not true for a star.
- As we go down from the star’s surface, temperature keeps increasing. Hence $I_\nu = S_\nu.$ should be only approximately true. It is the temperature gradient near the star’s surface which gives rise to the absorption lines.

Thermodynamical equilibrium

- By assuming thermodynamic equilibrium, we have derived that the source function should be equal to the blackbody function $B_\nu(T)$.
- In a realistic situation, we rarely have strict thermodynamic equilibrium. The temperature inside a star is not constant, but varies with its radius. In such a situation, will our assumptions hold?
- If a system is in thermodynamic equilibrium, then certain important principles of physics can be applied to that system.

Thermodynamical equilibrium

- **Maxwellian velocity distribution:** Different particles in a gas move around with different velocities. If the gas is in thermodynamic equilibrium with temperature T , then the number of particles having speeds between v and v + dv
- n is the number of particles per unit volume
- m is the mass of each particle.

$$dn_v = 4\pi n \left(\frac{m}{2\pi \kappa_\mathrm{B} T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2\kappa_\mathrm{B} T}\right) dv,$$

Thermodynamical equilibrium

- **Boltzmann and Saha equations**: We know that a hydrogen atom has several different energy levels. It can also be ionized.
- If a gas of hydrogen atoms is kept in thermodynamic equilibrium, then a certain fraction of the atoms will occupy a particular energy state and also a certain fraction will be ionized. The same considerations hold for other gases besides hydrogen.
- If n_0 is the number density of atoms in the ground state, then the number density n_e of atoms in an excited state with energy E above the ground state - this is called the Boltzmann distribution law.

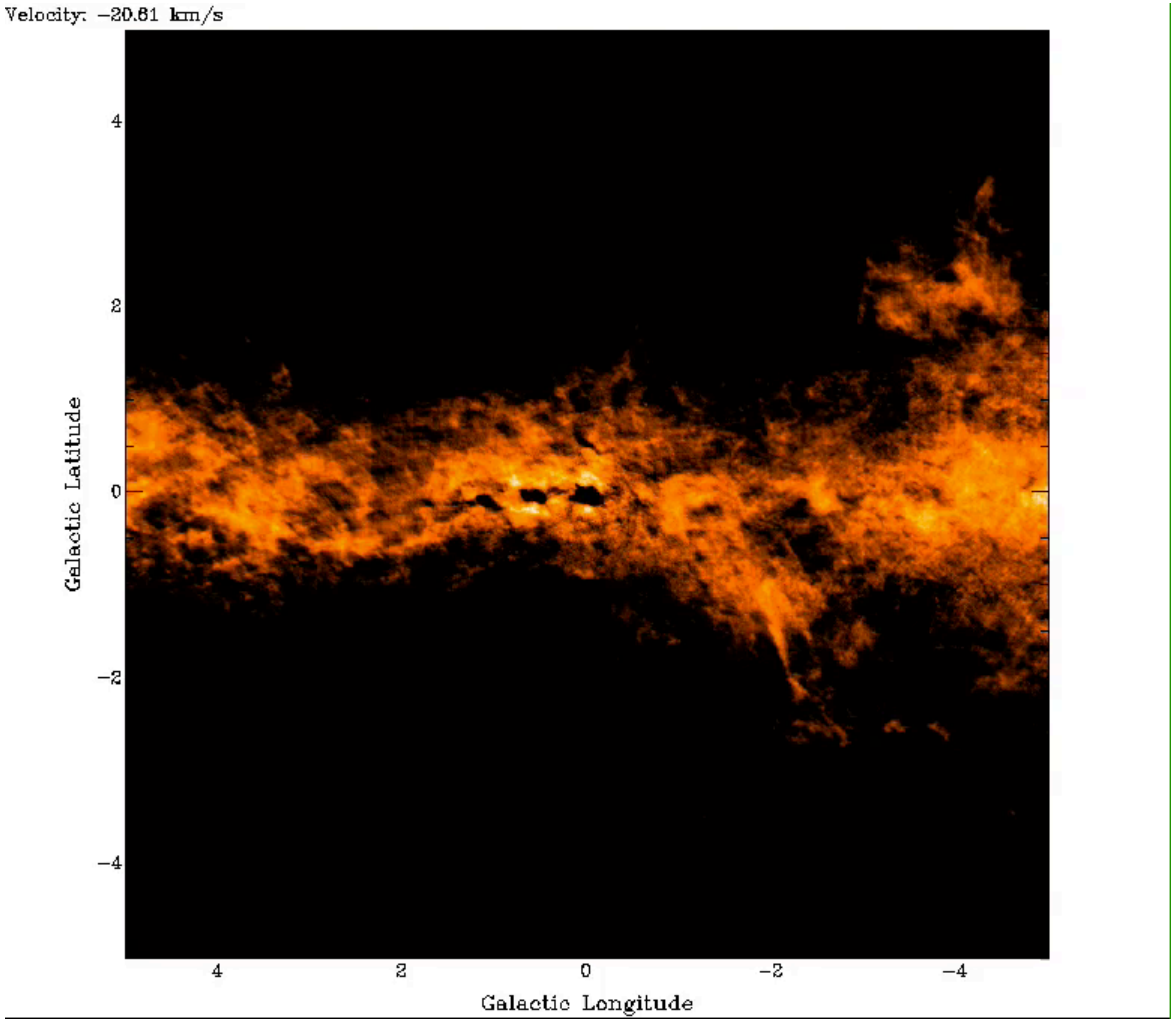
$$\frac{n_{\mathrm{e}}}{n_0} = \exp\left(-\frac{E}{\kappa_{\mathrm{B}} T}\right).$$

Thermodynamical equilibrium

- Saha (1920) derived the equation which tells us what **fraction of a gas will be ionized** at a certain temperature T and pressure P .
- If χ is the ionization potential (i.e. the amount of energy to be supplied to an atom to ionize it), h is Planck's constant and m_e is the mass of electron, then the fraction x of atoms which are ionized is given by:

$$\frac{x^2}{1-x} = \frac{(2\pi m_\mathrm{e})^{3/2}}{h^3} \frac{(\kappa_\mathrm{B} T)^{5/2}}{P} \exp\left(-\frac{\chi}{\kappa_\mathrm{B} T}\right),$$

Example for fractional ionisation:



- The Riegel-Crutcher cloud is a cold neutral hydrogen (HI) self-absorption cloud in the Galaxy
- Filamentary structure of the cold gas, which is aligned with the magnetic field
- The cold gas that is observed is neutral, however some fraction of the gas is ionised.

SGPS data - McClure-Griffith et al. 2006

Example for fractional ionisation:

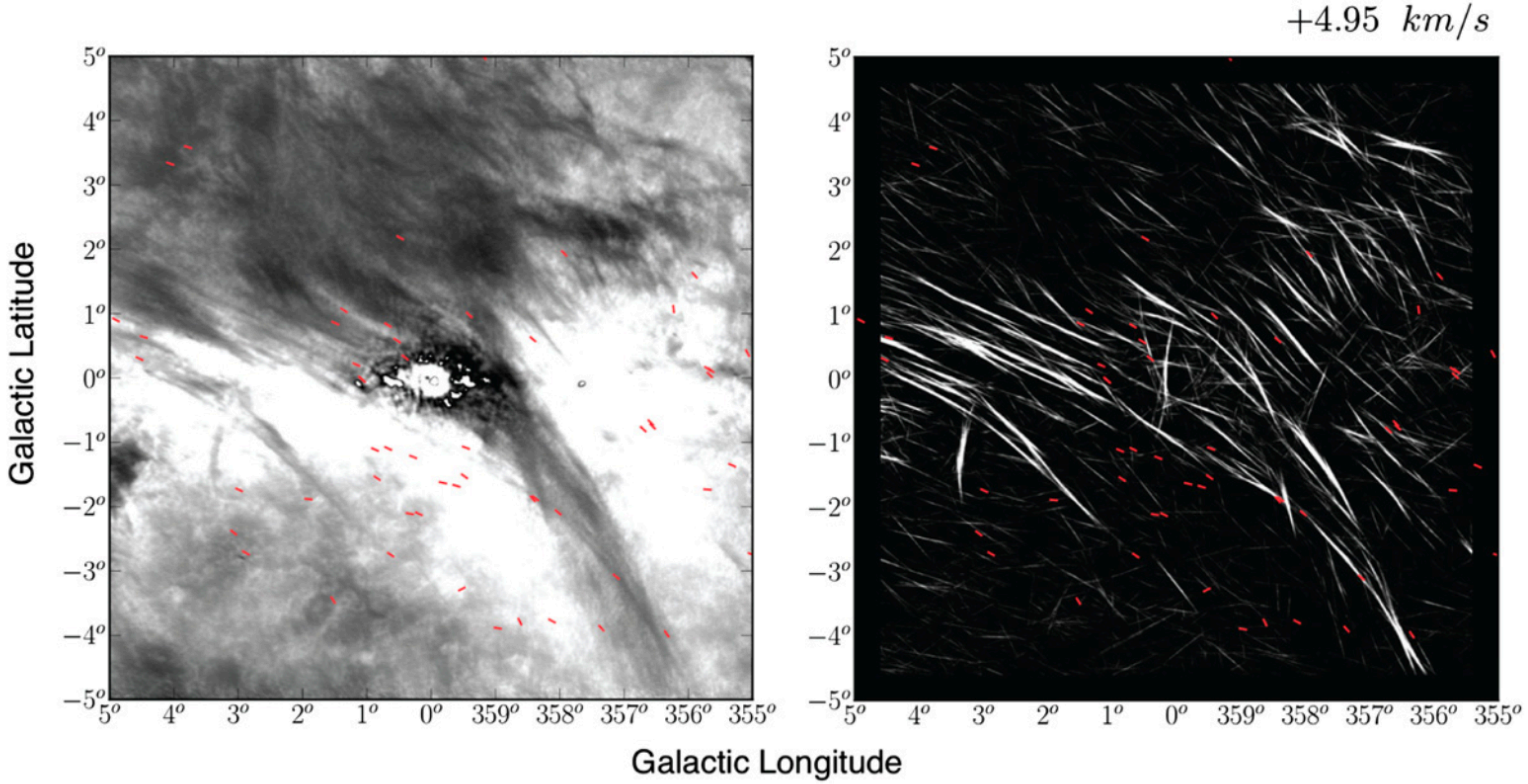


Figure 10. Riegel–Crutcher cloud (Section 6) in H I absorption (left) and RHT backprojection (right). Overlaid pseudovectors represent polarization angle measurements from the Heiles (2000) compilation. In the left panel, the intensity scale is linear from −20 K (white) to −120 K (black).

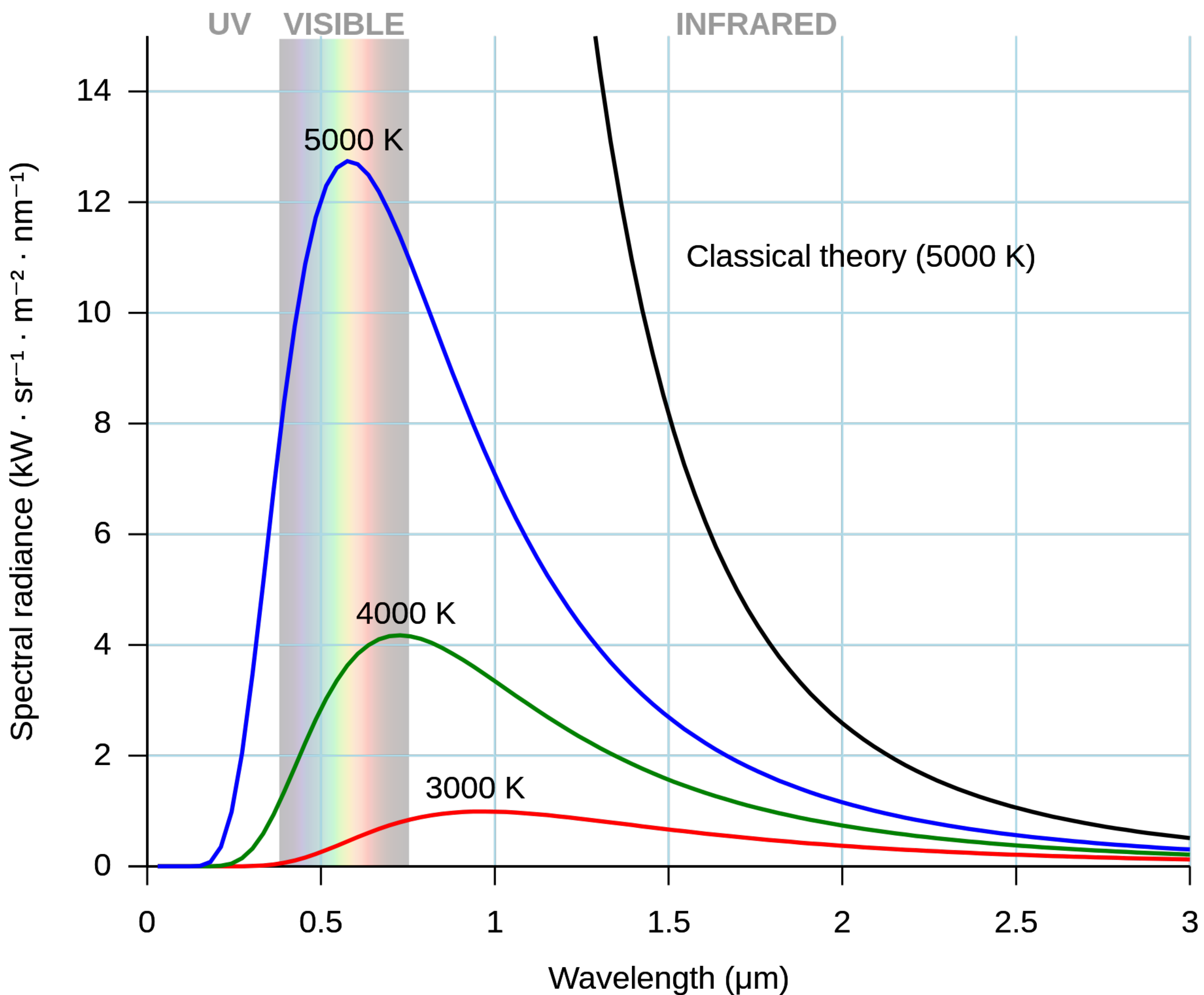
- Clark et al. 2014 - filaments are identified with a method called the Rolling Hugh Transform. The magnetic field is calculated based on starlight polarisation. There a good statistical match with alignment.

Planck law of blackbody spectrum

- When radiation is in thermodynamic equilibrium with matter, it is called blackbody radiation. The spectral distribution of energy in blackbody radiation is given by the Planck law.

Planck's law - specifies energy density U_ν in given frequency range $\nu, \nu + d\nu$:

$$U_\nu d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{\kappa_B T}\right) - 1}.$$



Thermodynamical equilibrium

- Question: when **can we expect a system to be in thermodynamic equilibrium** and when can we expect the above principles (Maxwellian velocity distribution, Boltzmann equation, Saha equation, Planck's law) to hold?
- If a box filled with gas and radiation is kept isolated from the surroundings, then we know that thermodynamic equilibrium will get established inside, and all the above principles will hold.
- However, a realistic system is always more complicated. Inside a star, the temperature keeps decreasing as we go from the central region to the outside. Can the above principles be applied in such a situation?

Thermodynamical equilibrium

- We again consider a box filled with gas and radiations. Even if the gas particles initially do not obey the Maxwellian distribution, they will relax to it after undergoing a few collisions.
- In other words, collisions or **interactions** amongst the constituents of the system are vital in **establishing thermodynamic equilibrium.**
- When collisions are frequent, the **mean free path** turns out to be small, then particles in a gas will interact with each other more effectively and we expect that principles like the Maxwellian velocity distribution, the Boltzmann equation or the Saha equation will hold.
- If the mean free path is small and the temperature does not vary much over that distance, then we shall have the Maxwellian velocity distribution.
- **The condition of validity of the Maxwellian velocity distribution (as well as the Boltzmann equation and the Saha equation) is that the mean free path has to be small enough such that the temperature does not vary much over the mean free path.**

Thermodynamical equilibrium

- For Planck's law to be established for radiation, the radiation has to be in equilibrium with matter. This is possible only when radiation interacts efficiently with matter.
- The absorption coefficient α_ν in the radiative transfer equation is a measure of the interaction between radiation and matter
- α_ν has the dimension of inverse length.
- Its inverse α_ν^{-1} gives the distance over which a significant part of a beam of radiation would get absorbed by matter. This is referred to as the **mean free path of photons**
- The typical distance a photon is expected to traverse freely before interacting with an atom. The smaller the value of α_ν^{-1}, the more efficient is the interaction between matter and radiation.
- If α_ν^{-1} is sufficiently small such that the temperature can be taken as constant over such distances, then we expect Planck's law of blackbody radiation to hold.

Thermodynamical equilibrium

- If the temperature is varying within a system, then it is not in full thermodynamic equilibrium.
- However, we can have a situation where both α_ν^{-1} and the mean free path of particles are small compared to the length over which the temperature varies appreciably. In a such situation, all the important laws of thermodynamic equilibrium are expected to hold within a local region, provided we use the local temperature T in the expressions.
- Such a situation is known as ***local thermodynamic equilibrium*,** abbreviated as LTE.
- For example: Inside a star, we expect LTE to be a very good approximation and we can solve the radiative transfer equation inside the star.
    - In the outermost atmosphere of a star, LTE may fail and it often becomes necessary to consider departures from LTE.
- We also often assume LTE in the interstellar medium.

Radiative transfer trough stellar atmospheres

- Plane parallel atmosphere :When we focus our attention on the local region of a stellar atmosphere, we can neglect the curvature and assume the various thermodynamic quantities like the temperature T to be constant over horizontal planes.
- Let us take the z axis in the vertical direction, with z increasing above. Any thermodynamic variable of the atmosphere can be a function of z alone. Let us consider an element of a ray path ds as shown in the Figure
- If dz is the change in z corresponding to ds, then we have

$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu},$$

$$\theta = \cos^{-1}\mu$$

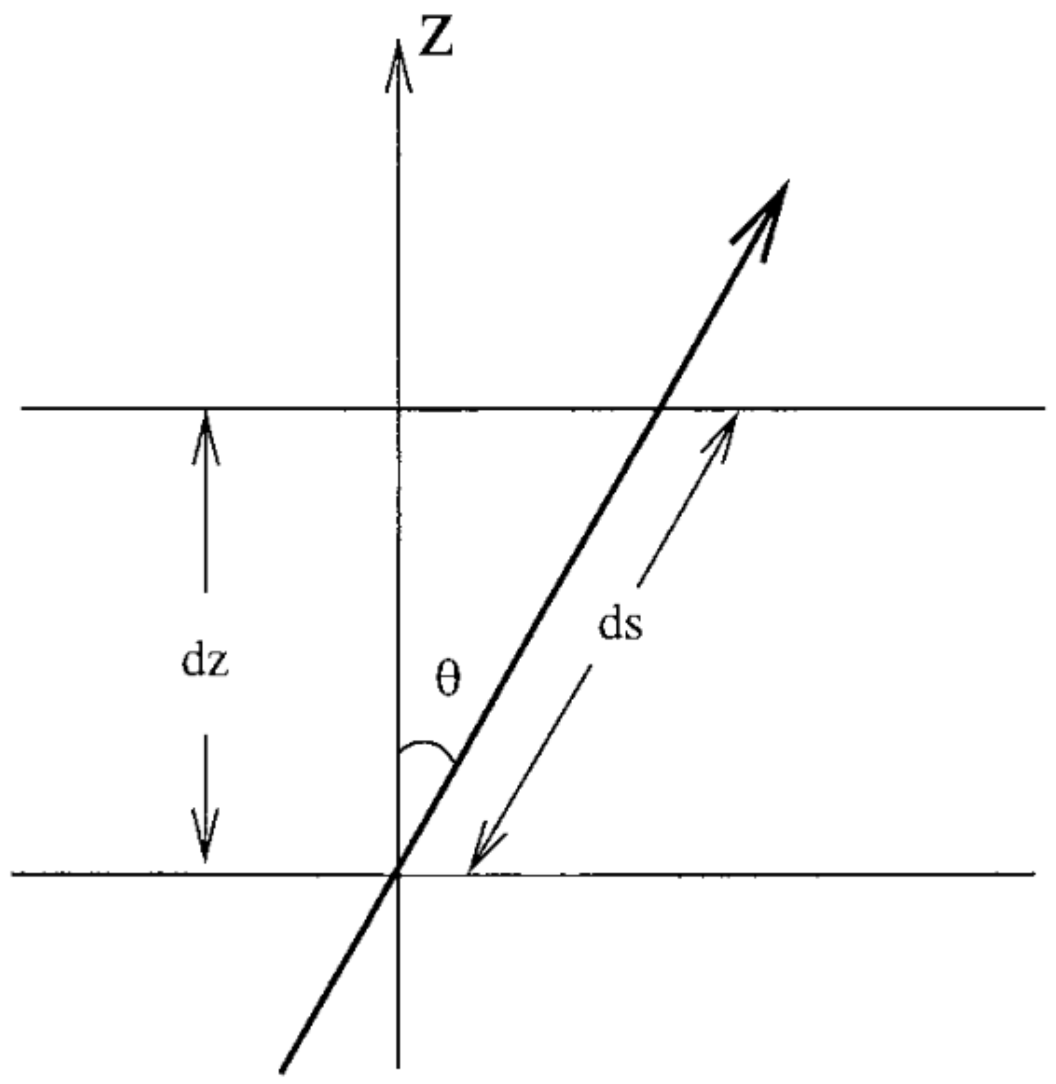


Fig. 2.4 A ray path through a plane parallel atmosphere.

Radiative transfer trough stellar atmospheres

- We have seen in that the specific intensity $I_\nu(\mathbf{r},t,\mathbf{n})$ can in general be a function of position, time and direction.
- We are considering a static situation. In the plane parallel stellar atmosphere, nothing varies in the horizontal directions and all direction vectors lying on a cone around the vertical axis are symmetrical.
- -> we expect the specific intensity $I_\nu(z,\mu)$ to be a function of z and $\mu = \cos\theta$ only.
- The radiative transfer gives us the equation for the plane parallel atmosphere problem.

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu, \quad \longrightarrow \quad \mu \frac{\partial I_\nu(z,\mu)}{\partial z} = j_\nu - \alpha_\nu I_\nu$$

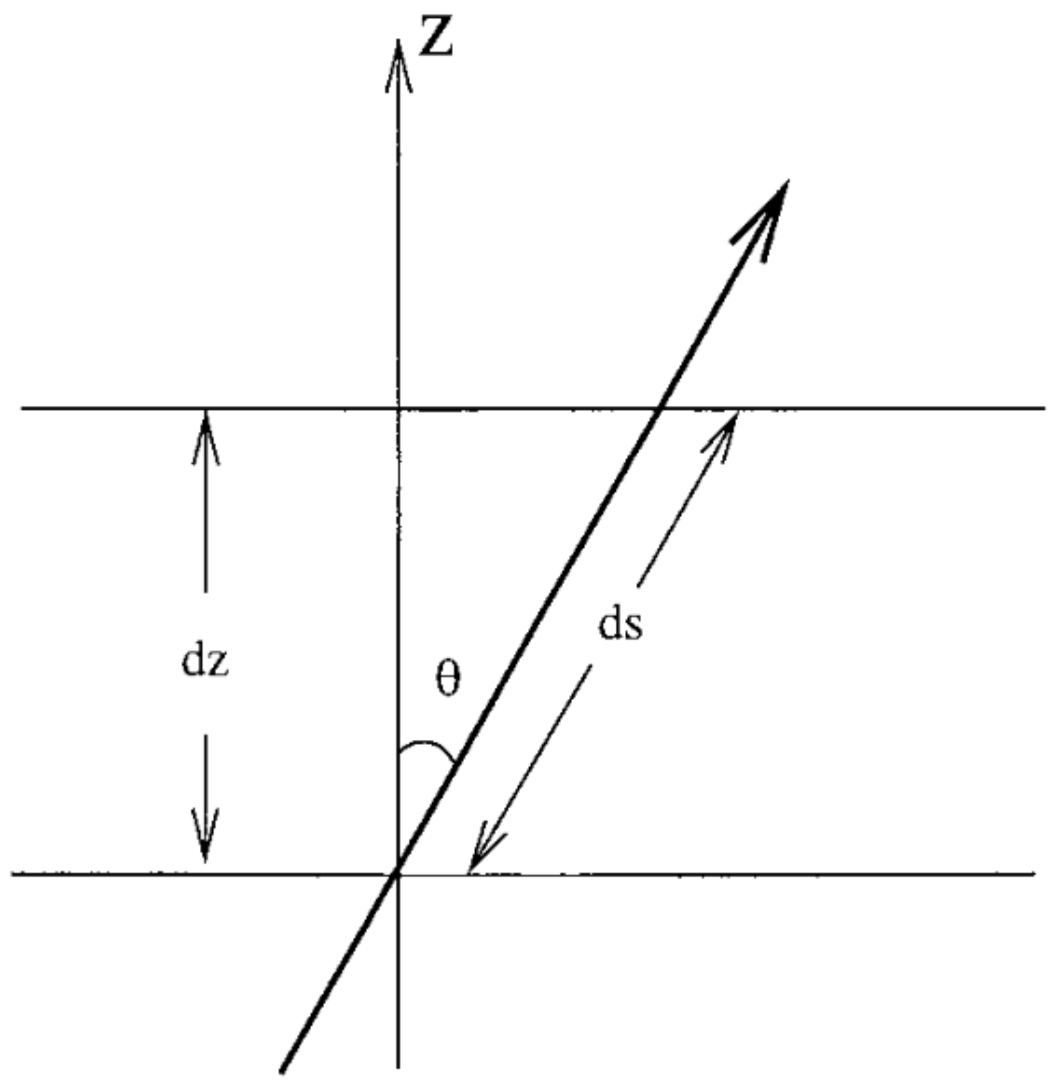


Fig. 2.4 A ray path through a plane parallel atmosphere.

Radiative transfer trough stellar atmospheres

- Optical depth for a plane parallel atmosphere problem. $$d\tau_\nu = -\alpha_\nu \, dz.$$
- the optical depth is now defined as a function of the vertical distance z rather than the distance along ray path s.
- The negative sign in implies that the optical depth increases as we go deeper down in the star.
- The normal convention is to take **τ_ν = 0 at the top of the stellar atmosphere**.

$$\mu \frac{\partial I_\nu(\tau_\nu, \mu)}{\partial \tau_\nu} = I_\nu - S_\nu.$$

$$\mu \frac{d}{d\tau_\nu} \left(I_\nu e^{-\frac{\tau_\nu}{\mu}} \right) = -S_\nu e^{-\frac{\tau_\nu}{\mu}}.$$

$$I_\nu e^{-\frac{t_\nu}{\mu}} \Big|_{\tau_{\nu,0}}^{\tau_\nu} = -\int_{\tau_{\nu,0}}^{\tau_\nu} \frac{S_\nu}{\mu} e^{-\frac{t_\nu}{\mu}} dt_\nu.$$

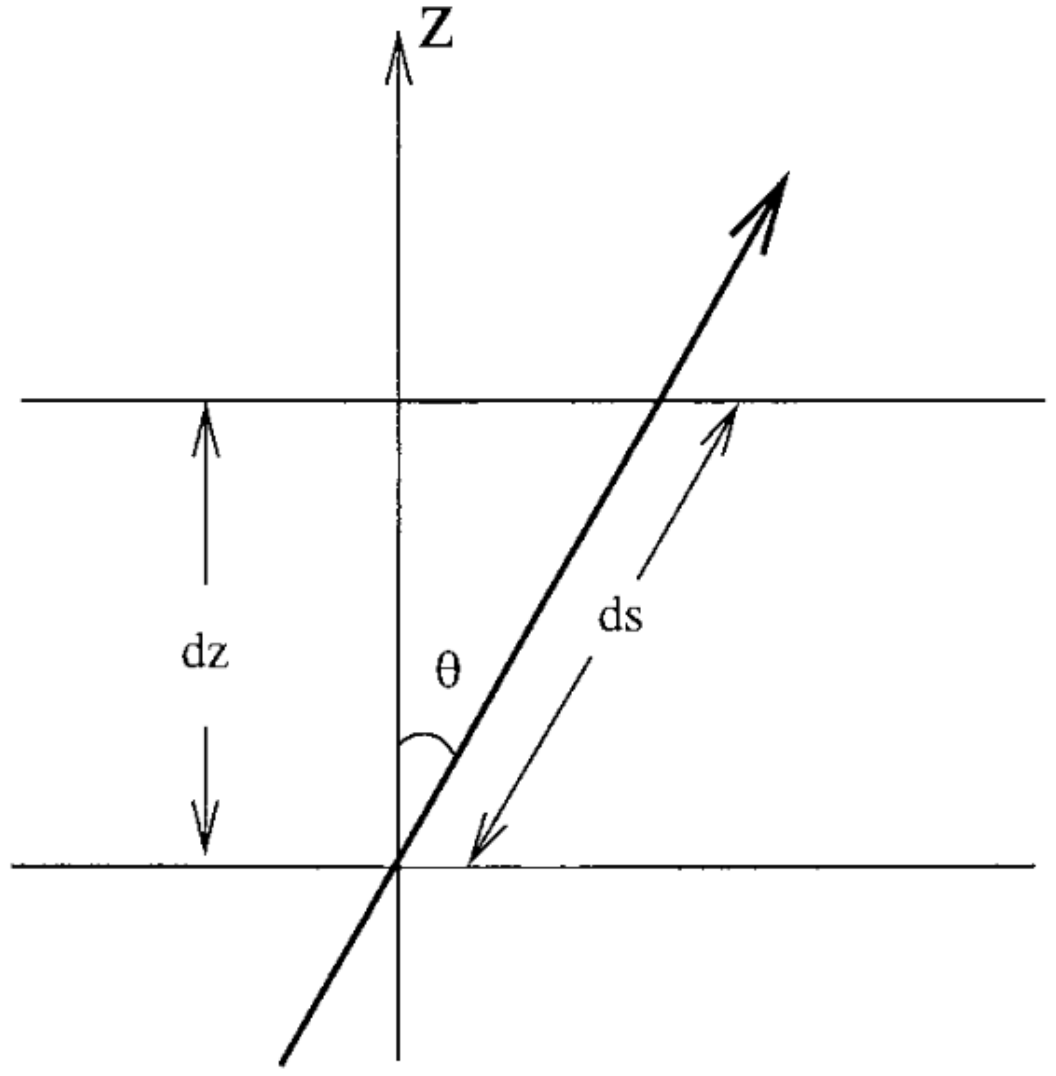


Fig. 2.4 A ray path through a plane parallel atmosphere.

Radiative transfer trough stellar atmospheres

- We now consider two cases separately:
  - (I) $0 \leq \mu \leq 1$, which corresponds to a ray path proceeding in the outward direction in the stellar atmosphere; and the ray path can be assumed to begin from a great depth inside the star and we can take $\tau_{\nu,0} = \infty$.
  - (II) $-1 \leq \mu \leq 0$, which corresponds to a ray path going inward in the stellar atmosphere and we can take $\tau_{\nu,0} = 0$.

$$0 \leq \mu \leq 1: \quad I_\nu(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)/\mu} \frac{dt_\nu}{\mu},$$

$$-1 \leq \mu \leq 0: \quad I_\nu(\tau_\nu, \mu) = \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu - t_\nu)/(-\mu)} \frac{dt_\nu}{(-\mu)}.$$

Radiative transfer trough stellar atmospheres

- We Let us assume **LTE** throughout the stellar atmosphere so that the **source function everywhere is equal to the Planck function** there, in accordance with Kirchhoff's law.
- Suppose we want to find out the radiation field at some optical depth τ_ν. The source function there is given by $B_\nu(T(\tau_\nu))$ which we write as $B_\nu(\tau_\nu)$ for simplification.
- The source function at a nearby optical depth $t_\nu$can be written in the form of a Taylor expansion around the optical depth τ_ν, i.e.

$$S_\nu(t_\nu) = B_\nu(\tau_\nu) + (t_\nu - \tau_\nu)\frac{dB_\nu}{d\tau_\nu} + \ldots$$

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu\frac{dB_\nu}{d\tau_\nu},$$

Radiative transfer trough stellar atmospheres

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu\frac{dB_\nu}{d\tau_\nu},$$

- Provided the point considered is sufficiently inside the atmosphere such that $\tau_\nu \gg 1$ and we can take $e^{-\tau_\nu}$ to be zero.
- The second term on the right-hand side depends on μ and makes the **radiation field anisotropic**.
- If there was no variation of temperature within the stellar atmosphere, then $dB_\nu/d\tau_\nu$ would vanish and the radiation field would become an isotropic blackbody radiation.
- The presence of a temperature gradient in the atmosphere which makes the radiation field depart from the Planckian distribution, making it anisotropic.

Radiative transfer trough stellar atmospheres

- the radiation flux, the energy density and the pressure of a radiation field can be calculated from the specific intensity
- In the case of a plane parallel atmosphere, the integration over the solid angle becomes simplified due to symmetry. If $A(\cos\theta)$ is any function of angle in a plane parallel atmosphere, then

$$\begin{aligned}\int A(\cos\theta)\,d\Omega &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} A(\cos\theta)\ \sin\theta\,d\theta\,d\phi \\ &= 2\pi\int_{+1}^{-1} A(\mu)\,d(-\mu) = 2\pi\int_{-1}^{+1} A(\mu)\,d\mu.\end{aligned}$$

Radiative transfer trough stellar atmospheres

- the **radiation flux, the energy density and the pressure** of a radiation field can be calculated from the specific intensity
- In the case of a plane parallel atmosphere, the integration over the solid angle becomes simplified due to symmetry. If $A(\cos\theta)$ is any function of angle in a plane parallel atmosphere, then

$$U_\nu = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \, d\mu, \qquad U_\nu = \frac{4\pi}{c} B_\nu(\tau_\nu),$$

$$F_\nu = 2\pi \int_{-1}^{1} I_\nu \, \mu \, d\mu, \quad \longrightarrow \quad F_\nu = \frac{4\pi}{3}\frac{dB_\nu}{d\tau_\nu},$$

$$P_\nu = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \, \mu^2 \, d\mu. \qquad P_\nu = \frac{4\pi}{3c} B_\nu(\tau_\nu).$$

Radiative transfer trough stellar atmospheres

- the ratio of the anisotropic part in the radiation field to the isotropic part is of order

$$\frac{dB_\nu/d\tau_\nu}{B_\nu} \approx \frac{3F_\nu}{cU_\nu}$$

Approximating $F\nu/U\nu$ by F/U , where F and U are respectively the total flux and the total energy density integrated over all wavelengths

$$\frac{\text{Anisotropic term}}{\text{Isotropic term}} \approx \frac{3F}{c\,U}.$$

Radiative transfer trough stellar atmospheres

- thermal physics that the total energy density of blackbody radiation is given by

$$U = a_{\mathrm{B}} T^4,$$

the total flux is the flux which eventually emerges out of the surface and is given by the Stefan–Boltzmann law:

$$F = \sigma T_{\mathrm{eff}}^4, \qquad \sigma = \frac{c a_{\mathrm{B}}}{4}$$

T_{eff} is the temperature on the surface of the star

$$\frac{\text{Anisotropic term}}{\text{Isotropic term}} \approx \frac{3}{4} \left(\frac{T_{\mathrm{eff}}}{T} \right)^4$$

Radiative transfer trough stellar atmospheres

- As we go deeper in a stellar atmosphere, T becomes much larger than T_{eff}, making the anisotropic term negligible compared to the isotropic term.
- The radiation field is nearly isotropic in sufficiently deep layers of a stellar atmosphere where the temperature is considerably higher than the surface temperature.
- If we knew how temperature varied with depth and could calculate the Planck function $B_\nu(\tau_\nu)$ at different depths. This is not known in general and the real challenge of studying stellar atmospheres is to solve the radiation field and the temperature structure of the stellar atmosphere simultaneously.

Grey atmosphere

- If the absorption coefficient α_ν is constant for all frequencies, then the atmosphere is called a **grey atmosphere.**
- There is no real stellar atmosphere which has this property. The grey atmosphere is an idealized mathematical model which is much simpler to treat than a more realistic stellar atmosphere and gives us some insight into the nature of the problem
- If α_ν is independent of frequency, then it follows that the value of optical depth at some physical depth will be the same for all frequencies.

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I - S,$$

$$I = \int I_\nu \, d\nu$$

$$S = \int S_\nu \, d\nu$$

Grey atmosphere

- We also define the average specific intensity *J* averaged over all angles

$$J = \frac{1}{2}\int_{-1}^{1} I \, d\mu. \qquad\qquad J = \frac{c}{4\pi}U.$$

- Using these expressions of J, we obtain two important moment equations:

$$\frac{1}{4\pi}\frac{dF}{d\tau} = J - S \qquad\qquad \frac{dP}{d\tau} = \frac{F}{c}.$$

- *I* is a function of both τ and μ, it may be noted that *F* and *P* are functions of τ alone

Grey atmosphere

- The energy generated in the stellar interior passes out in the form of a constant energy flux through the outer layers of the stellar atmosphere. In other words, F has to be independent of depth. It follows that this is possible only if J = S
- the average specific intensity has to be equal to the source function. This is called the condition of *radiative equilibrium,* and can also be expressed as:

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I - \frac{1}{2} \int_{-1}^{1} I \, d\mu$$

There are techniques for solving this equation exactly and obtaining I for all τ and μ.

Grey atmosphere

- Since F is constant under the condition of radiative equilibrium, we can obtain

$$P = \frac{F}{c}(\tau + q),$$

where q is the constant of integration. The total pressure and total energy density of an isotropic radiation field are related by

$$P = \frac{1}{3}U.$$

Grey atmosphere

- We note that the radiation field becomes nearly isotropic as we go somewhat below the surface. Just underneath the surface, however, we do not expect isotropy.
- If we assume isotropy to hold everywhere, then finding a full solution to our problem becomes straightforward. This is known as the Eddington approximation. Under this approximation, we can combine the equations to obtain

$$S = \frac{3F}{4\pi}(\tau + q).$$

Grey atmosphere

- We have seen from equations that the specific intensity can easily be written down if the source function is given.
- The main challenge is to obtain the solution for the source function (which depends on the temperature) at different depths along with the specific intensity.
- If we can evaluate the constant of integration q, then we can get the solution for the source function for the grey atmosphere problem (under the Eddington approximation).
- q can be evaluated by calculating the flux from this equation and setting it equal to F

$$S = \frac{3F}{4\pi}(\tau + q).$$

Grey atmosphere

$$I(\tau, \mu \geq 0) = \int_{\tau}^{\infty} S\, e^{-(t-\tau)/\mu} \frac{dt}{\mu}.$$

Setting $\tau = 0$ on the surface of the star

$$I(0, \mu) = \int_{0}^{\infty} S\, e^{-t/\mu} \frac{dt}{\mu}.$$

$$I(0, \mu) = \frac{3F}{4\pi} \int_{0}^{\infty} (t+q) e^{-t/\mu} \frac{dt}{\mu} = \frac{3F}{4\pi} (\mu + q).$$

Grey atmosphere

The flux coming out of the upper surface of the stellar atmosphere is

$$F = 2\pi \int_0^1 I \, \mu \, d\mu .$$

$$F = \frac{3F}{2}\left(\frac{1}{3} + \frac{q}{2}\right),$$

$$q = \frac{2}{3}.$$

Grey atmosphere

On putting this value of q in (2.64), the source function as a function of depth inside the stellar atmosphere is finally given by

$$S = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right)$$

$$cU = 3F\left(\tau + \frac{2}{3}\right).$$

Tells us how temperature varies inside a grey atmosphere:

$$T^4 = \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right).$$

Grey atmosphere

Finally we derive an important result for radiation coming out from the stellar surface.

$$I(0, \mu) = \frac{3F}{4\pi}\left(\mu + \frac{2}{3}\right),$$

$$\frac{I(0, \mu)}{I(0, 1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right).$$

Grey atmosphere

- Suppose we consider the intensity of radiation coming from different points on the disk of the Sun as seen by us.
- The ray coming from the central point of the solar disk emerges out of the solar surface in the vertical direction and the specific intensity for this ray will be $I\,(0, 1)$.
- the ray coming from an off-centre point must emerge from the solar surface at an angle $\theta = \cos^{-1}\mu$ with the vertical, as seen in the Figure, and the corresponding specific intensity will be $I\,(0, \mu)$.

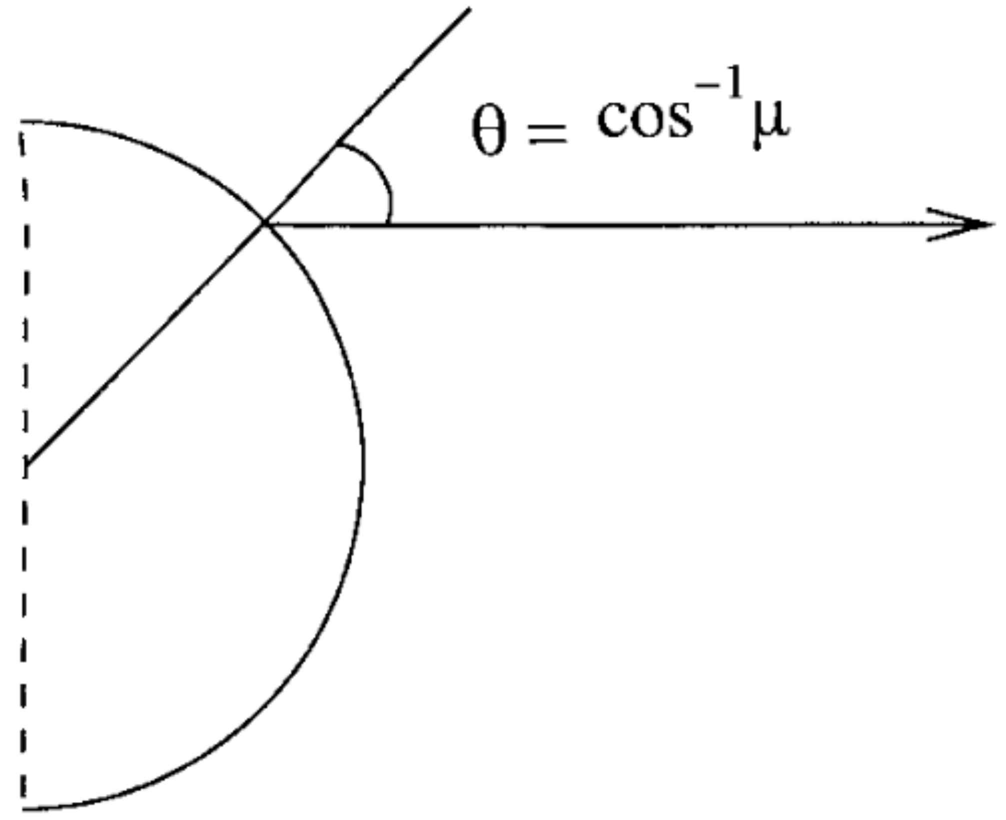


Fig. 2.5 A ray coming to an observer from the solar disk.

Grey atmosphere

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right).$$

- Gives the variation of intensity on the solar disk as we move from the centre to the edge.
- The region near the edge of the solar disk is referred to as the limb of the Sun.
- Giving the variation of intensity over the solar disk is called the *limb-darkening law*.
- The theoretical limb-darkening law predicts that the intensity at the edge of the solar disk will be about 40% of the intensity at the centre.
- In Figure 2.6, we show the observationally determined limb-darkening along with the plot of (2.70) obtained by the Eddington approximation as well as the theoretical limb- darkening law derived by an exact solution of the grey atmosphere problem (i.e. derived by solving the integro-differential equation (2.61) exactly). Although theory matches the observational data reasonably well, the discrepancy between the two is due to the fact that the solar atmosphere is not grey.

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Grey atmosphere

- The Figure shows the observationally determined limb-darkening along with the plot obtained by the Eddington approximation as well as the theoretical limb-darkening law derived by an exact solution of the grey atmosphere problem
- The theory matches the observational data reasonably well, the discrepancy between the two is due to the fact that the solar atmosphere is not grey.

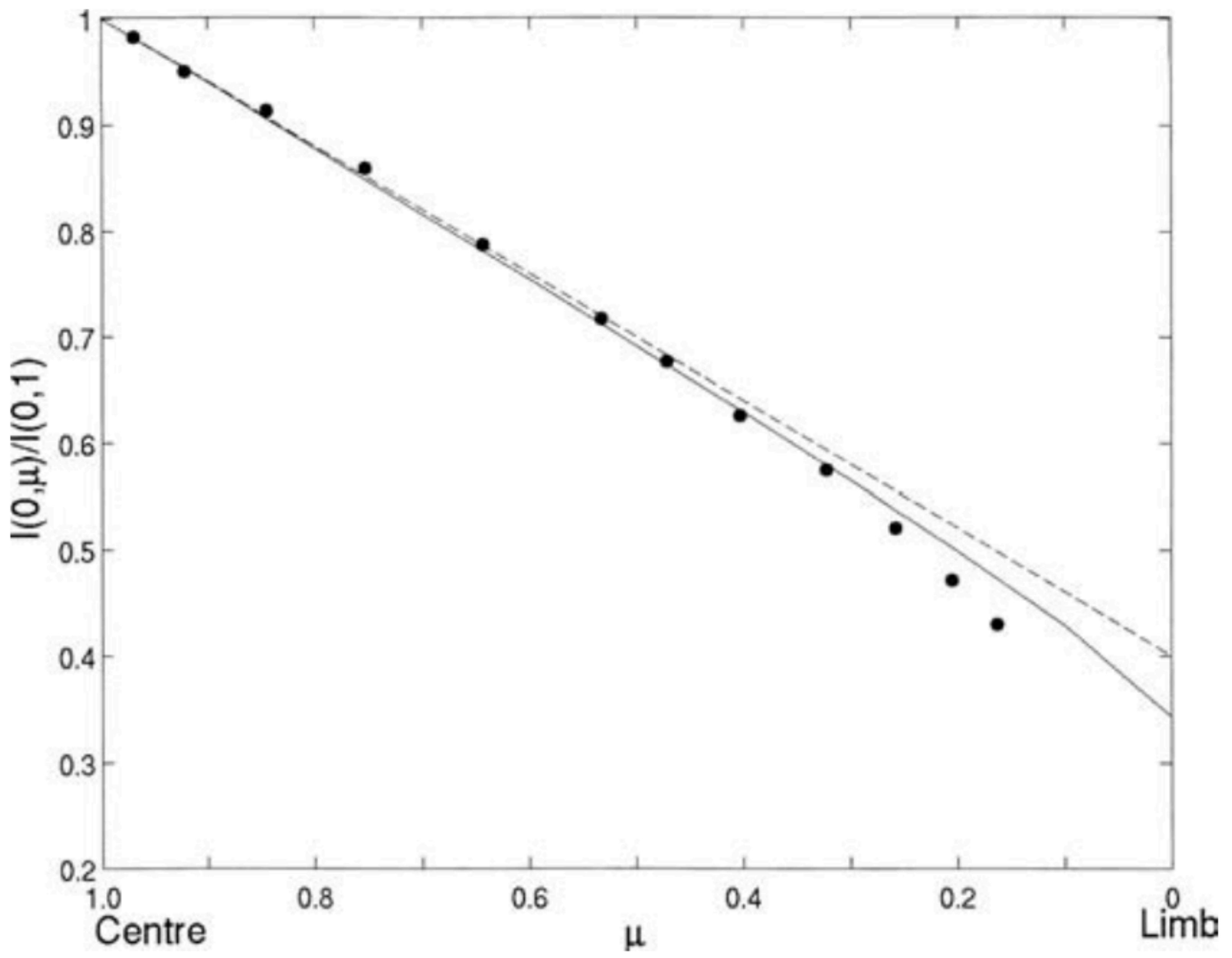

Fig. 2.6 The observed limb-darkening of the solar disk (indicated by dots) along with theoretical limb-darkening laws obtained by the Eddington approximation (dashed line) and by exact solution of the grey atmosphere problem (solid line). The observational data (indicated by dots) are for wavelength $\lambda = 5485$ Å as given by Pierce *et al.* (1950).