Introduction to Astrophysics and Cosmology

Space-time dynamics of the Universe

Helga Dénes 2024 S2 Yachay Tech

hdenes@yachaytech.edu.ec

Space-time dynamics

How the Universe began is a fundamental question. However, only after Einstein's formulation of **general relativity**, did it become **possible to build theoretical models for the evolution of the Universe**.

Very simple considerations show that the Newtonian theory of gravity is inadequate for handling the Universe. Consider an infinite Universe uniformly filled with matter of density ρ. If we try to find the gravitational field **g** at a point on the basis of the Newtonian theory, we immediately run into inconsistencies. Since all directions are symmetric in an infinite uniform Universe, we expect the gravitation field to be zero at any point because there is no preferred direction in which the gravitational field vector can point. However, Newtonian theory of gravity leads to **the Poisson equation**

$$\boxed{\nabla . \mathbf{g} = -4\pi G\rho,}$$

which also must be satisfied. If **g** is zero at all points, then the left-hand side of this equation has to be zero, so that a non-zero ρ clearly leads to contradictions.

Space-time dynamics

In all theories of physics before the formulation of general relativity, **space-time** was supposed to provide an inert background against which one could study the dynamics of various systems. In other words, spacetime itself was not supposed to have any dynamics. However, **general relativity allowed spacetime also to have its own dynamics.**

We have discussed the recessive motions of galaxies in §9.3. The common sense interpretation of Hubble's law is that **galaxies are rushing away from each other through the empty space** of the Universe. However, general relativity provides a radically different viewpoint. **We assume that the galaxies stay put in space, but space itself is expanding** – thereby moving apart the galaxies which are embedded in space.

Space-time dynamics

The dynamics of spacetime, of course, can be handled properly only through general relativity and tensor analysis. However, we also need to analyse various physical phenomena taking place in the expanding space of the Universe, in order to understand why the Universe is what it is today. Much of this can be done without a detailed technical knowledge of general relativity. Even the dynamics of expanding space can be studied to some extent without introducing general relativity.

If we assume that we are at the centre of the Universe and the Universe is expanding around us in a spherically symmetric fashion, then the equation of motion derived from **Newtonian mechanics turns out to be essentially the same as the equation that follows from the detailed general relativistic analysis.** This is an amazing coincidence, and is often called ***Newtonian cosmology***. Many issues certainly remain unclear at a conceptual level in this approach and some assumptions have to be taken merely as ad hoc hypotheses without trying to justify them.

General relativity

General relativity provides a field theory of gravity. To explain what is meant by a field theory, let us consider the example of the other classical field theory, the theory of electromagnetic fields. We consider two charges q_1 and q_2 with $\mathbf{r}_{12}$ as the vectorial distance of q_2 from q_1. According to Coulomb's law, the **electric force** on q_2 due to q_1 is given by

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^3}\mathbf{r}_{12}. \tag{10.1}$$

In the action-at-a-distance approach, we only concern ourselves with the charges and the forces acting on them, without bothering about the surrounding space. On the other hand, the field approach suggests that the charge q_1 creates an **electric field** around it which is given at the distance $\mathbf{r}_{12}$ by the expression

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_{12}^3}\mathbf{r}_{12} \tag{10.2}$$

General relativity

and that the charge q_2 experiences the force

$$\mathbf{F}_{12} = q_2\mathbf{E} \tag{10.3}$$

by virtue of being present in the electric field. **As long as the charges are at rest, it is true that the field approach does not give us anything new compared to the action-at-a-distance approach**. However, the situation changes if the charges are in motion.

Suppose the **charge $\boldsymbol{q_1}$ suddenly starts moving** at an instant of time. What should we now substitute for $\mathbf{r}_{12}$ in the expression? You may think that we have to consider the simultaneous locations of the two charges at an instant of time and obtain $\mathbf{r}_{12}$ from that.

General relativity

However, **we know from special relativity that simultaneity in different regions of space is a subtle concept** and depends on the frame of reference. If we can somehow suitably define simultaneity in some frame of reference, then there is a more serious problem with (10.1).

Since the separation between q_1 and q_2 changes as soon as q_1 starts moving, the force $\mathbf{F}_{12}$ on q_2 also should change immediately according to (10.1). This means that the information that the charge q_1 has started moving has to be communicated to the other charge q_2 at infinite speed, which contradicts special relativity.

These difficulties disappear if the electromagnetic fields are treated with the help of **Maxwell's equations.**

It can be shown from Maxwell's equations that **the information about the motion of q_1 propagates at speed c and the charge q_2 starts getting affected only when the information reaches its location.** We thus see that the action-at-a-distance approach fails to provide a consistent theory of charges in motion and **we need a field approach.**

General relativity

Exactly the same considerations exist in the case of the gravitational field. **When two masses are at rest, Newton's law of gravitation gives the force between them.** But, **when the masses start moving, we have the same difficulties which we have with moving charges**.

After developing special relativity, Einstein realized that **Newton's theory of gravity is not consistent with special relativity.** We need to develop a field approach, which will ensure that **gravitational information does not propagate at a speed faster than c.**

For example: it takes time for gravitational waves to propagate trough space-time and reach Earth.

In contrast to two types of electric charges, we have only one type of mass.

A particle of charge q and mass m placed in an electric field $\mathbf{E}$ will have acceleration $(q/m)\mathbf{E}$. This acceleration will in general be different for different particles. In contrast, **all particles placed in a small region of gravitational field** (over which the variations of the field can be neglected) **have the same acceleration.** As a result, **a gravitational field is equivalent to an accelerated frame.**

Einstein realized that this ***equivalence principle*** would have to be an important part of a field theory of gravity.

General relativity

A second complication for a field theory of gravity is that it **has to be a nonlinear theory**, in contrast to the theory of the electromagnetic field which is linear.

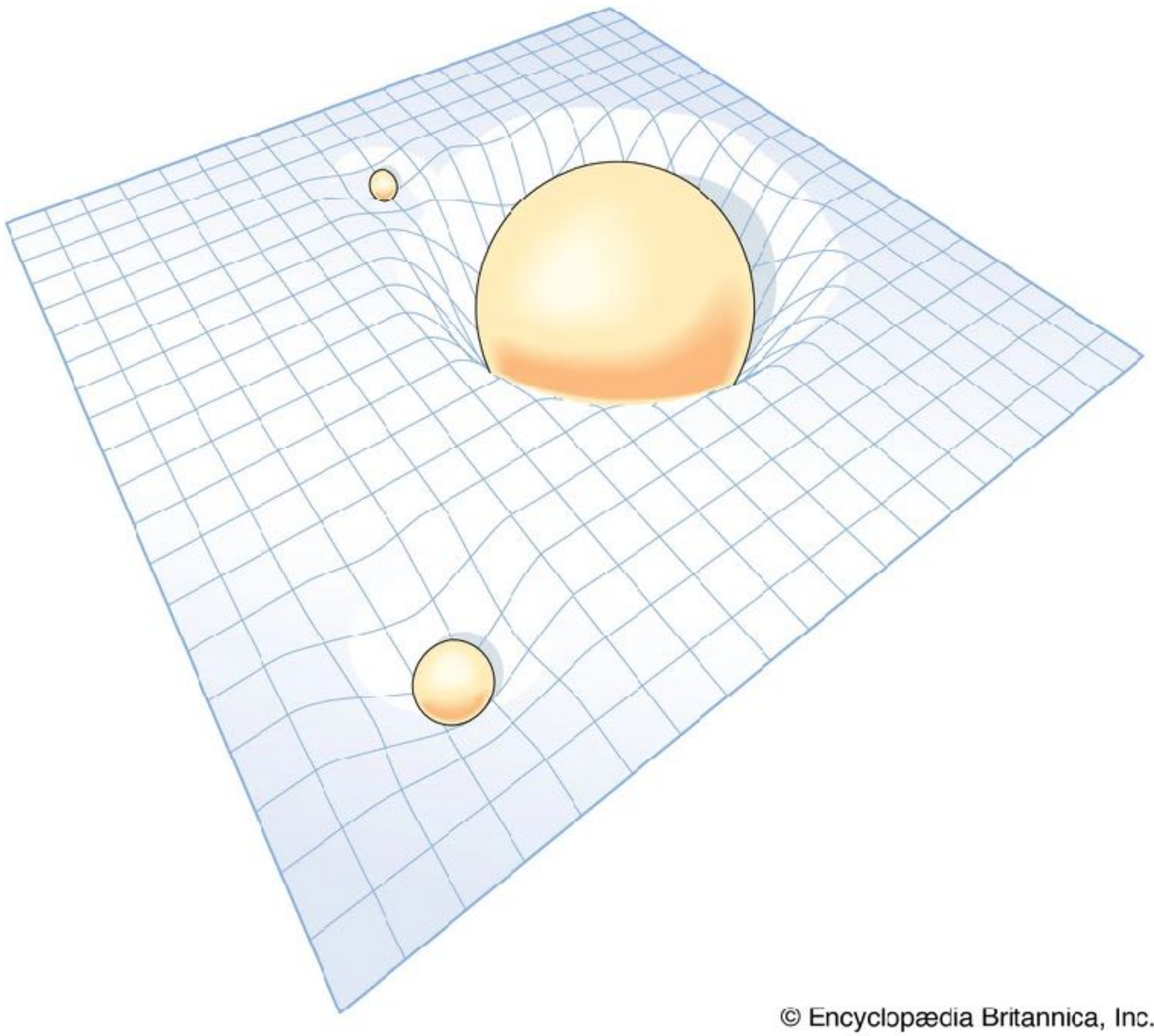
A simple argument for the nonlinearity: **Any field** such as a gravitational field **has some energy associated with it** and **energy is equivalent to mass** according to special relativity. Hence a gravitational field, having some equivalent mass associated with it, can itself act as a source of gravitational field. This is not the case in electromagnetic theory, where charges and currents are the sources of the electromagnetic field and the field itself cannot be its own source.

General relativity

Einstein (1916) had the insight of **viewing gravity as a curvature of spacetime.**

In a region **without gravitational fields, the spacetime is flat** and a **body moves in a straight line**. However, in a **region where the spacetime is curved** (which implies the **presence of gravity**), **a body is deflected from a rectilinear path**. Thus, instead of saying that the gravitational force deflects the body, we say that **the curved spacetime makes it move in a curved path**.

This **geometrical interpretation of gravity** automatically provides an explanation for the equivalence principle. The curved path which a body would take in a region is determined merely by the curvature of spacetime and should be independent of the mass of the body. Hence bodies of different masses should follow the same curved path.



The Curvature of spacetime

Light also follows the curved path.

For example, light should follow a curved path near the Sun.

The passage of a ray of light near the Sun -> the bend of the photon's trajectory is small because the photon's speed carries it quickly through the curved space.

Can this effect be observed?

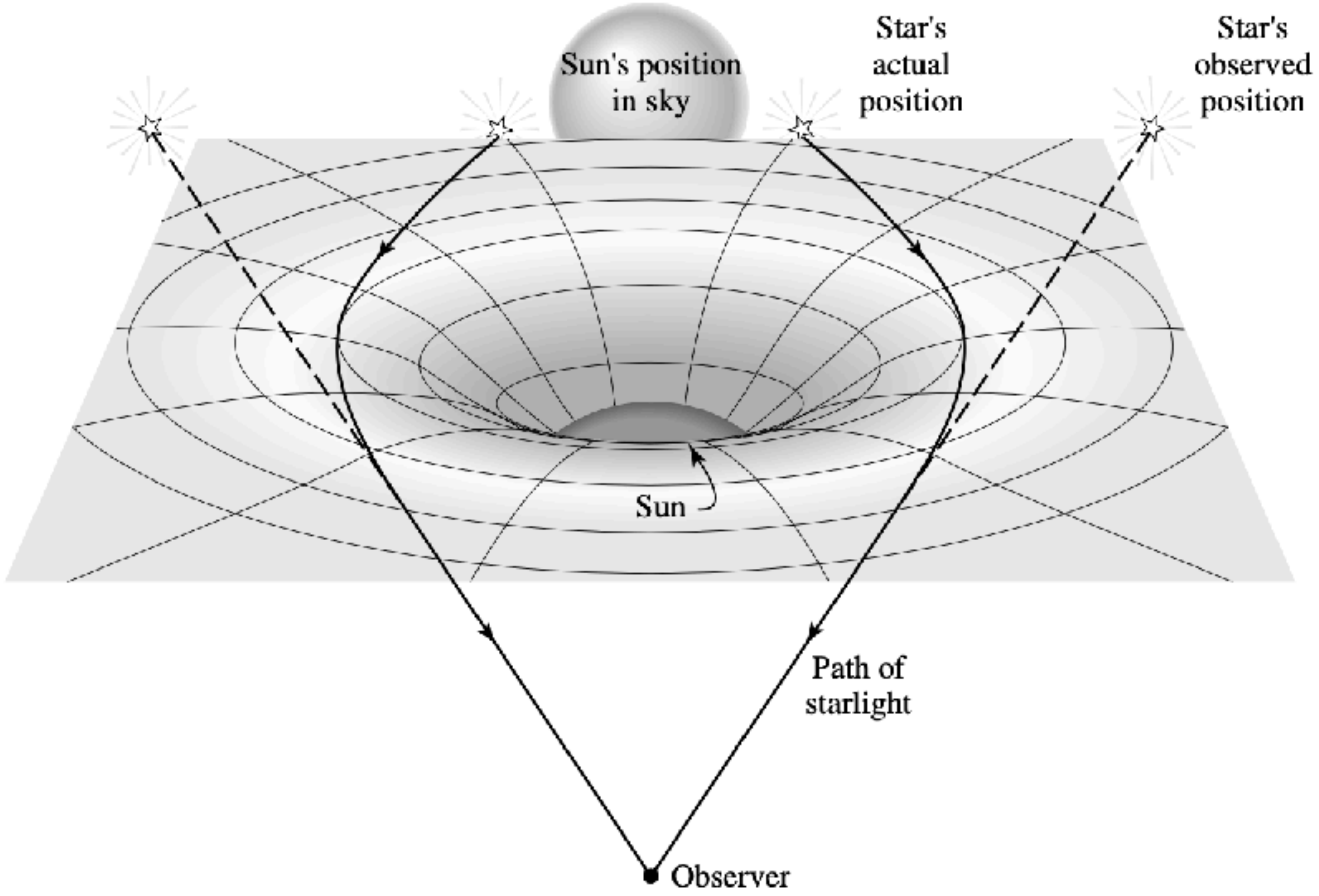


FIGURE 5 Bending of starlight measured during a solar eclipse.

The Curvature of spacetime

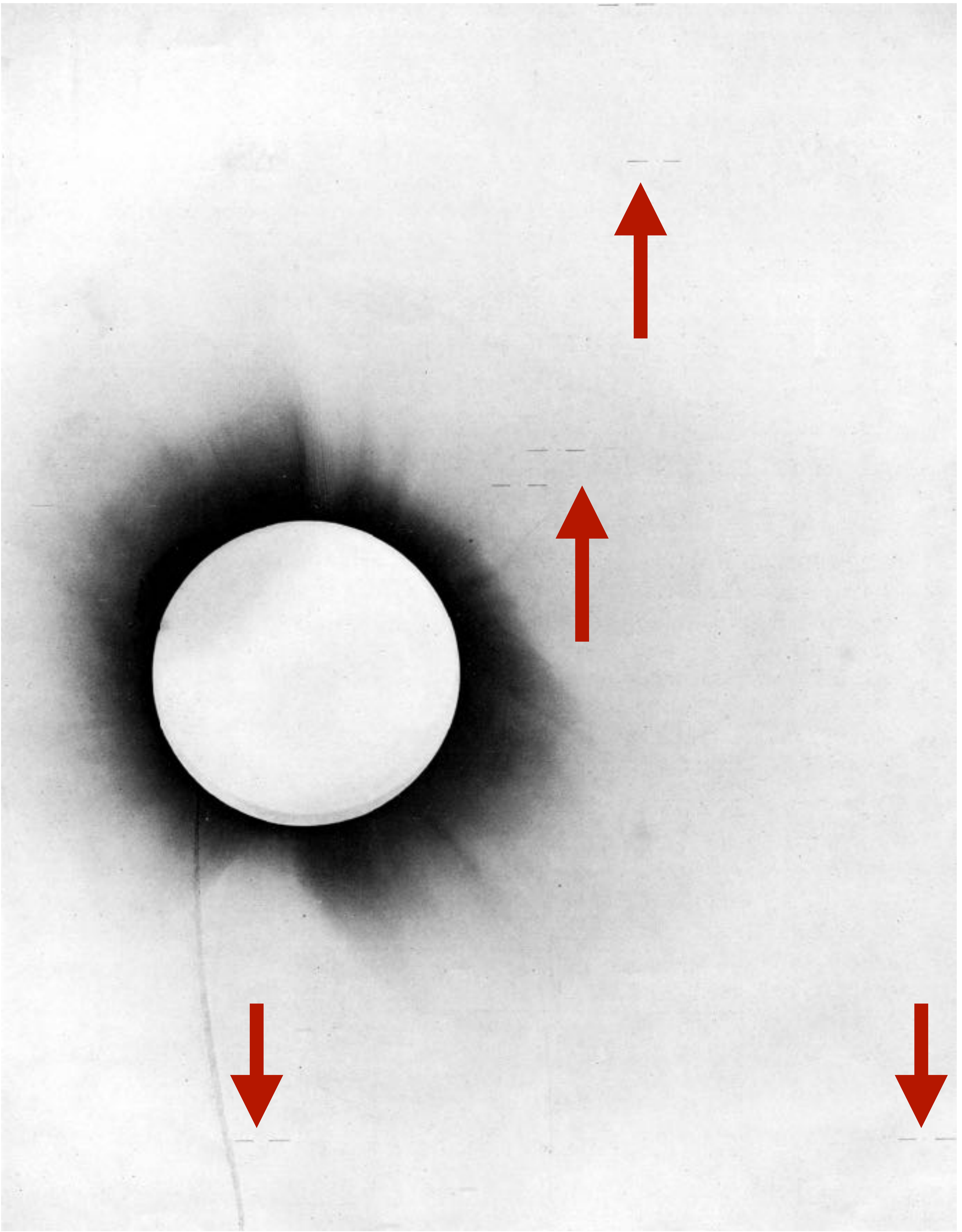
The passage of a ray of light near the Sun -> the bend of the photon's trajectory is small because the photon's speed carries it quickly through the curved space.

Can this effect be observed?

- Prediction: light bending 1.75 arcseconds for light that grazes the Sun.
- Observations: change in position of stars as they passed near the Sun, confirmed in 1919 during a solar eclipse

In general relativity, **gravity is the result of objects moving through curved spacetime, and everything that passes through, even massless particles such as photons, is affected.**

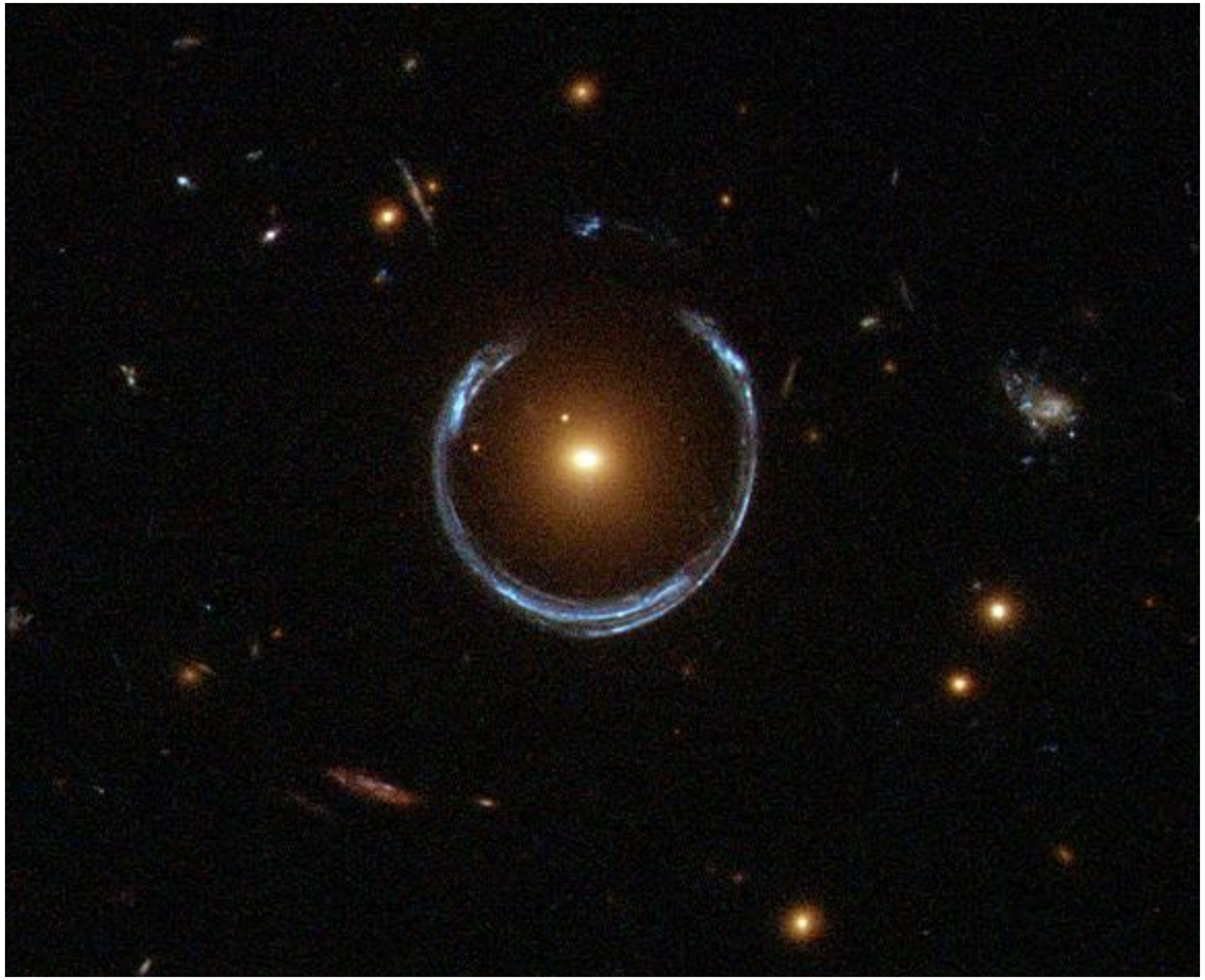
Image of the Solar eclipse in 1919 - shows the position of the stars



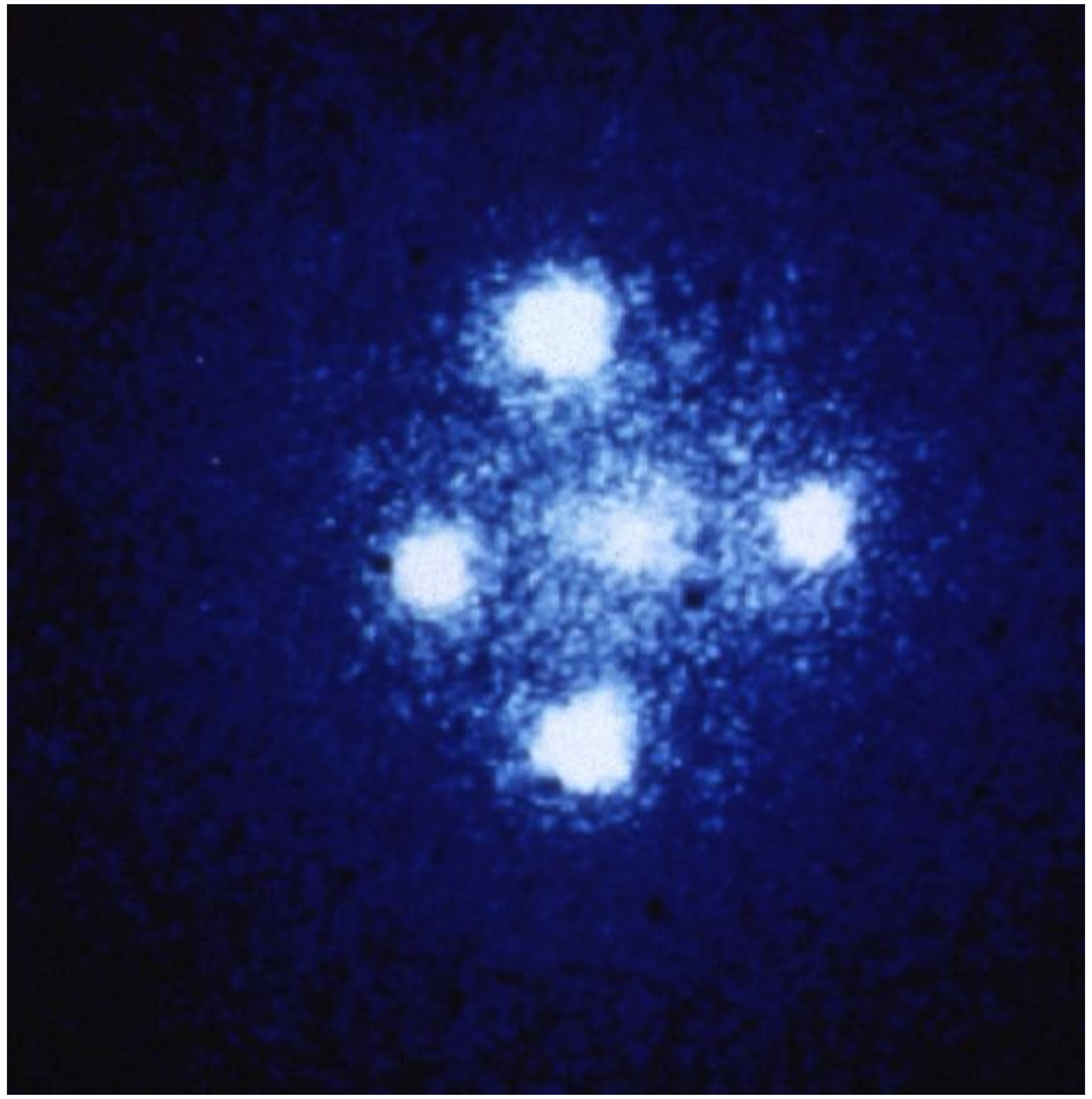
The Curvature of spacetime

More modern observations of the bending of light:
gravitational lensing

Rings like this are now known as Einstein Rings



Multiple images like this are called Einstein crosses.



General relativity

To develop a field theory of gravity, we therefore need mathematical methods to handle the curvature of spacetime.

Let us first consider the curvature of a two-dimensional surface. There are **two ways of looking at the curvature: extrinsic and intrinsic.**

When we perceive a surface as curved in a common sense way, we essentially perceive the surface embedded in three-dimensional space to be curved. This extrinsic way of looking at curvature is not so useful when we consider the curvature of four-dimensional spacetime. Do we have to think of it as being embedded in a five-dimensional something within which it is curved?

To explain the alternative **intrinsic way of viewing curvature**, let us again consider the example of the **two-dimensional surface**. Let there be some intelligent ants or other creatures living on this two-dimensional surface who have no conception of three-dimensional space. **Can they make some measurements within this two-dimensional surface to find out if it is curved or not?**

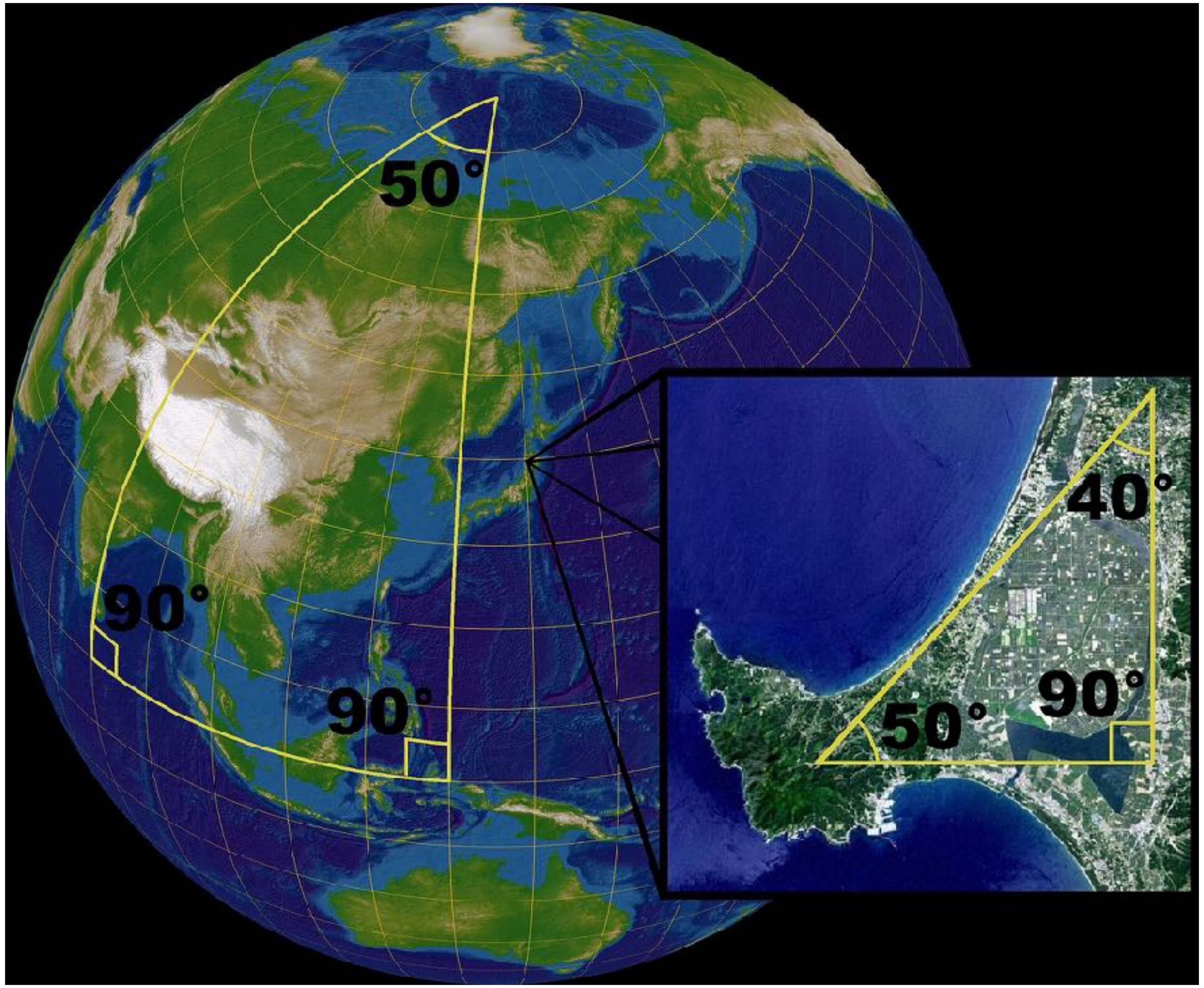
General relativity

Can they make some measurements within this two-dimensional surface to find out if it is curved or not?

If a **surface is plane,** we know that any arbitrary **triangle will have the sum of three angles equal to 180°**.

But this is not the case for a curved surface. In the case of a **curved surface**, a straight line has to be replaced by a **geodesic,** which is the **shortest path between two points** of the surface lying wholly on the surface. Great circles on a sphere are geodesics.

For example, a triangle on the Earth's surface made with two circles of longitude and the equator, as shown in the Figure. Clearly **this triangle made up of three geodesics has the sum of its angles larger than 180°**.



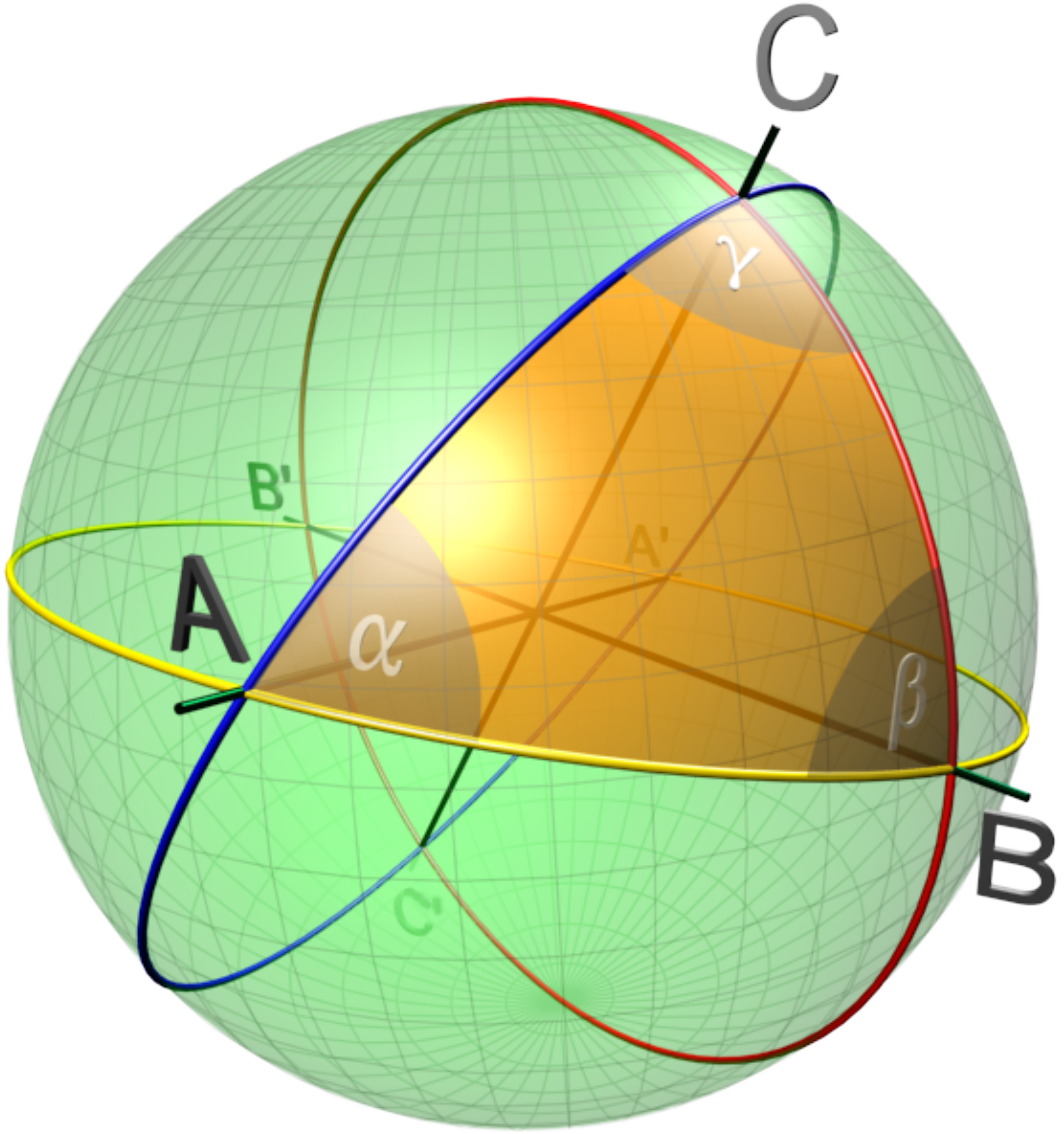
https://en.wikipedia.org/wiki/Spherical_geometry

General relativity

Can they make some measurements within this two-dimensional surface to find out if it is curved or not?

So the intelligent ants living on the two-dimensional surface **can determine whether the surface is curved by considering some arbitrary triangles** and measuring the sums of their angles.

When we consider the curvature of four-dimensional spacetime, we naturally have to look at the curvature from an intrinsic point of view.



https://en.wikipedia.org/wiki/Spherical_geometry

General relativity

The intrinsic view of curvature usually conforms to our everyday notion of curvature – but not always!

For example, we normally think of the **surface of a cylinder as a curved surface. But this surface can be unrolled to a plane** and has the same geometric properties as a plane surface. **All triangles drawn on a cylindrical surface have the sum of their angles equal to 180°**.

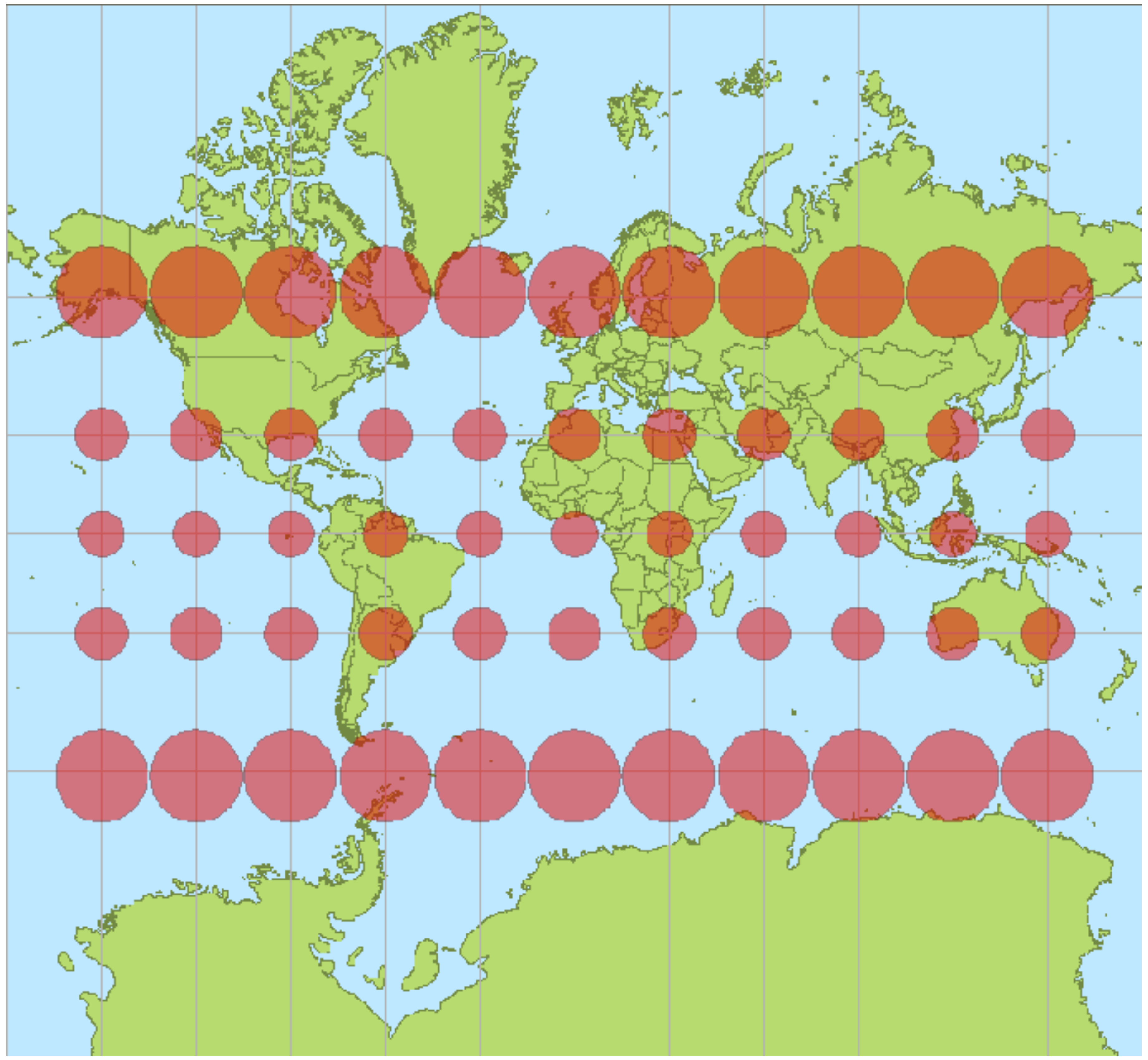
Therefore, when we are taking an intrinsic point of view, there is nothing to distinguish a cylindrical surface from a plane and we have to consider the surface of a cylinder as a plane or flat surface – a point of view which appears puzzling at first.

Now we come to the crucial question: **what determines if a two-dimensional surface is flat?**

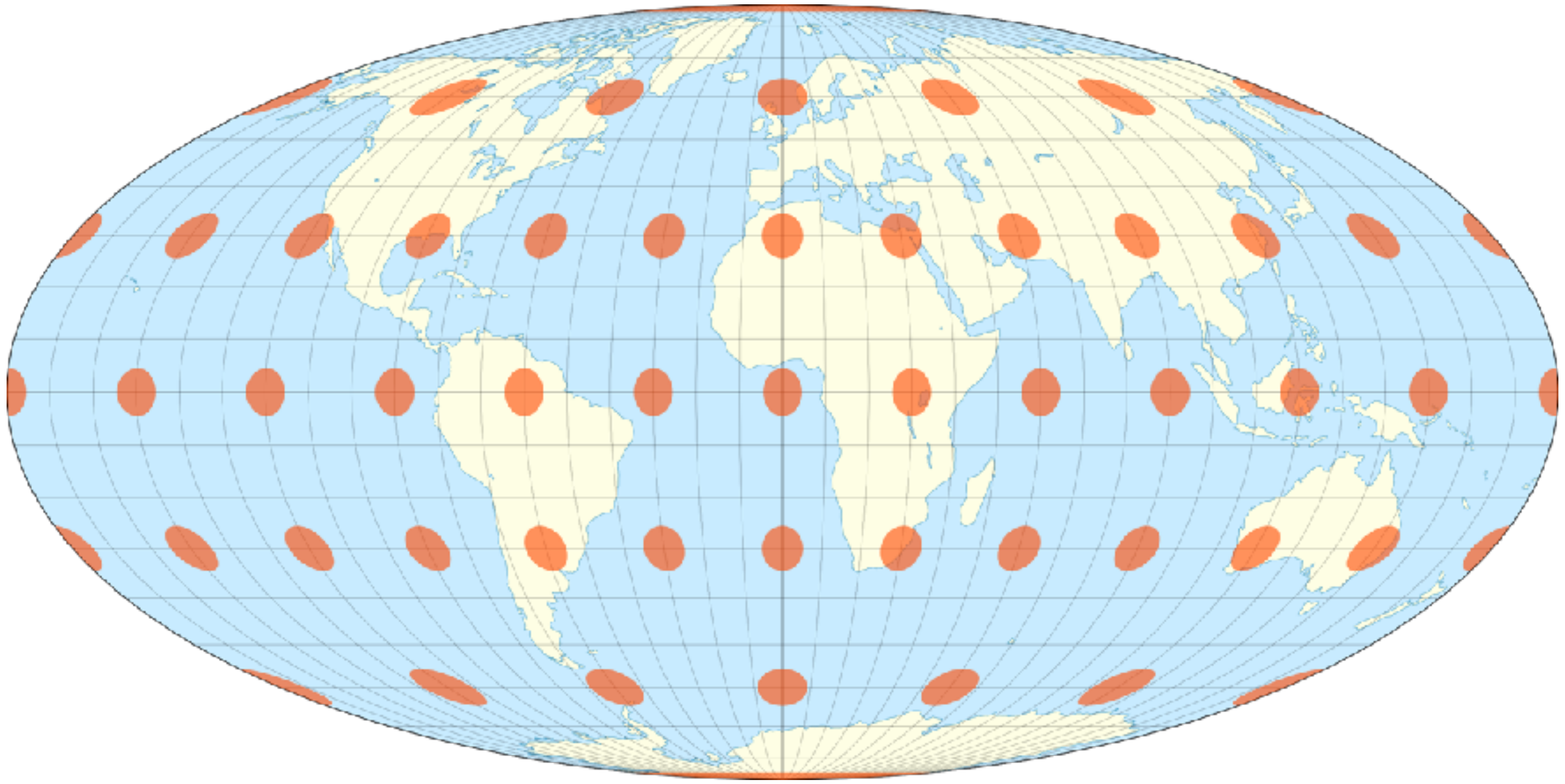
If the surface **can be spread on a plane *without stretching or shrinking*** anywhere, then it should be regarded as flat. This cannot be done with a spherical surface and that is why a spherical surface is not flat.

General relativity

When geographers have to **represent the map of the whole Earth on a flat sheet** of paper, they have to apply some projections which do drastic things like making Greenland appear as large as Africa or making some continents **highly distorted.**



Mercator projection: globe onto a cylinder. The circles illustrate the distortions.



Mollweide projection, often used in astrophysics since it is an equal area projection.

General relativity

When we stretch or shrink some portions of a surface, we change distances between various pairs of points on the surface. On the other hand, when we try to spread the surface on a plane without any stretching or shrinking, we **essentially apply transformations which keep distances between all possible pairs of points invariant.** Whether such transformations would enable us to spread the surface on a plane or not depends on **how the distances between various pairs of points are related to each other**.

If we introduce the standard (θ,ϕ) coordinates on a spherical surface of radius a, the **distance *ds* between two nearby points** (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ is given by

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{10.4}$$

In general, the **distance between two neighbouring points on a surface** is given by an expression of the form

$$\boxed{ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta}\, dx_\alpha\, dx_\beta, \qquad (10.5)}$$

General relativity

$$ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta}\, dx_\alpha\, dx_\beta,$$

where $\boldsymbol{g_{\alpha\beta}}$ **is called the metric tensor**. It is this tensor which determines how distances between various possible pairs of points are related. Hence, it is **this tensor $\boldsymbol{g_{\alpha\beta}}$** which decides whether it would be possible to spread the surface on the plane, i.e. **whether the surface is flat or not.**

These considerations carry over to spaces of higher dimensions as well. **The distances between nearby points in a higher dimensional space also can be written in the form** (10.5). It is again the metric tensor $g_{\alpha\beta}$ for the space which determines whether the space is flat or curved.

We will discuss the mathematical tools of tensor analysis later, these can be applied to a metric tensor $g_{\alpha\beta}$ to calculate the curvature of the space.

On using polar coordinates, the **metric tensor of a plane surface is given by**

$$ds^2 = dr^2 + r^2\, d\theta^2. \tag{10.6}$$

What type of metrics do you know about?

General relativity

Using the same coordinates (x_1, x_2), the two metrics (10.6) and (10.4) can be written as

$$ds^2 = a^2(dx_1^2 + x_1^2\, dx_2^2), \tag{10.7}$$

No curvature

$$ds^2 = a^2(dx_1^2 + \sin^2 x_1\, dx_2^2). \tag{10.8}$$

Curved

Another possible metric for a two-dimensional surface is

$$ds^2 = a^2(dx_1^2 + \sinh^2 x_1\, dx_2^2). \tag{10.9}$$

Curved

When we apply the technique of curvature calculation, we find that the metric (10.7) has zero curvature (i.e. it corresponds to a flat surface), whereas the metrics (10.8) and (10.9) correspond to uniform surfaces of constant curvature $2/a^2$ and $-2/a^2$ respectively.

Certainly a sphere, of which (10.8) is the metric, has a surface of uniform curvature. **But what kind of surface does (10.9) correspond to?**

General relativity

$$ds^2 = a^2(dx_1^2 + \sinh^2 x_1\, dx_2^2). \tag{10.9}$$

Figure 10.2 shows a **saddle-like surface**. It can be shown that the **saddle point P has negative curvature.** However, the saddle surface is not a uniform surface, but has its geometric properties changing from point to point.

The metric (10.9) corresponds to a uniform surface of which every point is a saddle point. Certainly there is no real two-dimensional surface embedded in three-dimensional space which has this property. However, one can mathematically postulate such a surface and study its properties by analysing (10.9).

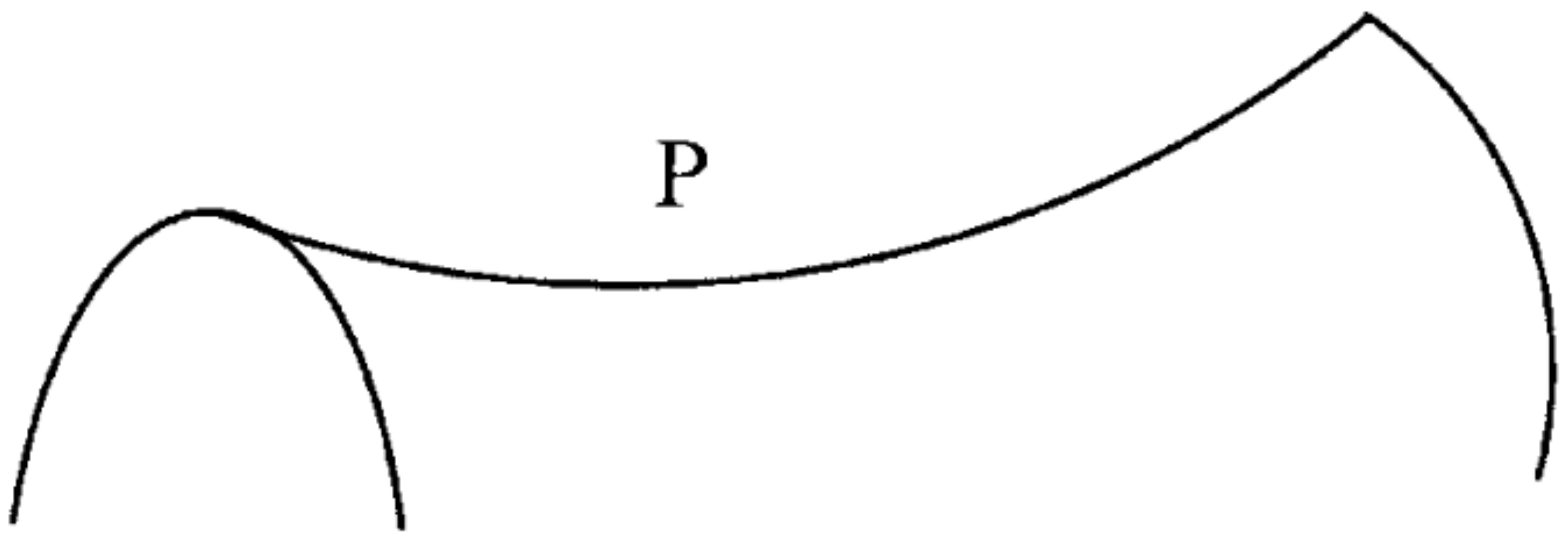


Fig. 10.2 Sketch of a saddle-like surface.

General relativity

Here we merely point out that the theoretical structure of general relativity has certain **analogies with the theory of electromagnetic fields.**

The basic idea of electromagnetic theory is that charges and currents produce electromagnetic fields. We have Maxwell's equations which tell us how the electromagnetic field can be obtained if we know the distribution of charges and currents.

In an analogous way, **general relativity suggests that mass and energy can give rise to the curvature of spacetime**, and the **central equation, known as Einstein's equation** (discussed later), **describes how the curvature of spacetime is related to mass-energy.**

Hence, **given the distribution of mass-energy, one can in principle find out the metric of spacetime from Einstein's equation and thereby determine the structure of spacetime**.

To complete electromagnetic theory, we need another equation describing how a charge moves in an electromagnetic field, which is the **Lorentz equation**

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

General relativity

We require an analogous equation in general relativity, which will **tell us how a mass will move in the curved spacetime**. The basic idea of general relativity is that a mass moves along a geodesic of spacetime if non-gravitational forces are absent. Hence, in the place of (10.10), we have the **equation of the geodesic** (to be discussed later) **in general relativity**.

Table 10.1 presents a comparison between electrodynamics and general relativity. In general relativity, we basically have to replace the Newtonian theory of gravity by the idea that **mass and energy create curvatures in spacetime and a particle moves along geodesics in this curved spacetime.**

Table 10.1 Analogy between electrodynamics and general relativity.

	ELECTRODYNAMICS	*GENERAL RELATIVITY*
Basic field equations	**Maxwell's equations:** {Charge, current} $\Rightarrow$ {Electromagnetic field}	**Einstein's equation:** {Mass, energy} $\Rightarrow$ {Curvature of spacetime}
Equation of motion in the field	The Lorentz equation (10.10)	Motion along geodesics

The metric of the Universe

We now discuss the possible structure of spacetime in the Universe. We have introduced **the *cosmological principle*, which states that space is homogeneous and isotropic.** Now, it is possible for space to be homogeneous and isotropic **only if it has uniform curvature everywhere.** We have seen that (10.7), (10.8) and (10.9) are the only possible metrics for a two-dimensional surface which is uniform (i.e. which has a constant curvature everywhere).

We now have to write down **similar metrics for a uniform three-dimensional space**. Using spherical coordinates (r, θ, ϕ), the distance between two nearby points in a flat space is given by

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{10.11}$$

One can explicitly calculate the curvature of this metric and show that it is zero.

Writing $r = a\chi$, this metric takes the form

$$ds^2 = a^2(d\chi^2 + \chi^2\, d\Omega^2), \tag{10.12}$$

The metric of the Universe

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{10.13}$$

It may be noted that the three-dimensional metric (10.12) looks very similar to the two-dimensional metric (10.7). In analogy with (10.8) and (10.9), we can **consider the following three-dimensional metrics**

$$ds^2 = a^2(d\chi^2 + \sin^2\chi\, d\Omega^2), \tag{10.14}$$

$$ds^2 = a^2(d\chi^2 + \sinh^2\chi\, d\Omega^2), \tag{10.15}$$

Uniform curvature

where $d\Omega^2$ is always given by (10.13).

If we apply the techniques of curvature calculations, we indeed find that the two metrics (10.14) and (10.15) have **uniform curvatures $6/a^2$ and $-6/a^2$** respectively. Just as all points of a sphere are equivalent, all points in the space described by metrics (10.14) and (10.15) must be equivalent, since the curvature has the same value at all points in either of these metrics.

The metric of the Universe

In fact, (10.12), (10.14) and (10.15) are the only possible forms of three-dimensional metrics for which all points are equivalent.

Hence, **if the cosmological principle has to be satisfied, then the spatial part of the metric of the Universe has to have one of these three forms.**

It is not difficult to show that the space described by the metric **(10.14) must have finite volume**.

Let us consider an element of volume with its sides along the three coordinate directions. Keeping in mind that $d\Omega^2$ is given by (10.13), it follows from (10.14) that **the sides of the volume element have lengths $a\, d\chi$, $a \sin\chi\, d\theta$ and $a \sin\chi \sin\theta\, d\phi$**. Hence the volume of this volume element is

$$dV = a^3 \sin^2\chi\, d\chi\, \sin\theta\, d\theta\, d\phi.$$

To get the total volume of the space, we have to integrate this over all possible values of χ, θ (0 to π) and ϕ (0 to 2π). **What is the range of values of χ?**

The factor $\sin^2\chi$ appearing in the metric takes the same values for $\chi = 0$ and $\chi = \pi$, beyond which there is a repetition of the same range.

The metric of the Universe

Hence the total volume of the space is given by

$$V = a^3 \int_{\chi=0}^{\chi=\pi} d\chi \, \sin^2\chi \int_{\theta=0}^{\theta=\pi} d\theta \, \sin\theta \int_{\phi=0}^{\phi=2\pi} d\phi \; = \; 2\pi^2 a^3. \qquad (10.16)$$

Just as a sphere has a surface of finite area without any edges, **this space also similarly has a finite volume without any bounding surface**.

The metric of the Universe

$$ds^2 = a^2(d\chi^2 + \chi^2\, d\Omega^2), \tag{10.12}$$

$$ds^2 = a^2(d\chi^2 + \sinh^2\chi\, d\Omega^2), \tag{10.15}$$

The spaces given by the **metrics (10.12) or (10.15) have infinite volumes.**

Let us now introduce a slightly different notation. We substitute $\chi = r$ in (10.12), $\sin\chi = r$ in (10.14) and $\sinh\chi = r$ in (10.15) $\rightarrow$ (10.12), (10.14) and (10.15) now become

$$ds^2 = a^2(dr^2 + r^2\, d\Omega^2),$$

$$ds^2 = a^2\left(\frac{dr^2}{1-r^2} + r^2\, d\Omega^2\right),$$

$$ds^2 = a^2\left(\frac{dr^2}{1+r^2} + r^2\, d\Omega^2\right).$$

The metric of the Universe

These three equations can be written together in the combined compact form

$$\boxed{ds^2 = a^2\left(\frac{dr^2}{1-kr^2} + r^2\,d\Omega^2\right), \qquad (10.17)}$$

where k can have values 0, $+1$ and -1, respectively corresponding to uniform space with zero, positive and negative curvature.

So far we have considered the metric of three-dimensional space. **To describe the spacetime of the Universe, we need the metric of four-dimensional spacetime.**

We expect (10.17) to provide the spatial part of this metric. We now **have to add the time part.** Special relativity can guide us how to do this. We know that the **metric of special relativity** is given by

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.}$$

The metric of the Universe

Since $dx^2 + dy^2 + dz^2$ is the spatial part of the metric, it is clear that we get the full spacetime metric by adding $-c^2dt^2$ to it. In exactly the same fashion, we expect to get the four-dimensional spacetime metric of the Universe by adding $-c^2dt^2$ to (10.17). This gives

$$ds^2 = -c^2dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2\right). \tag{10.18}$$

Substituting for $d\Omega^2$ the full form of the metric is

$$\boxed{ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \qquad (10.19)}$$

Robertson-Walker metric

The metric of the Universe

This is known as the ***Robertson–Walker metric*** and is the only possible form of the metric for a Universe satisfying the cosmological principle (i.e. having the spatial part uniform).

The three **values 0, +1 and –1 of *k* give the three possible kinds of Universe** with **flat, positively curved and negatively curved space.**

Sometimes it is useful to write the Robertson– Walker metric in terms of the variable χ used in (10.12), (10.14) and (10.15) rather than *r:*

$$ds^2 = -c^2dt^2 + a(t)^2\left[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \qquad (10.20)$$

where the function $S(\chi)$ has to be χ, $\sin\chi$ or $\sinh\chi$ corresponding to the values 0, +1 or −1 of k.

The metric of the Universe

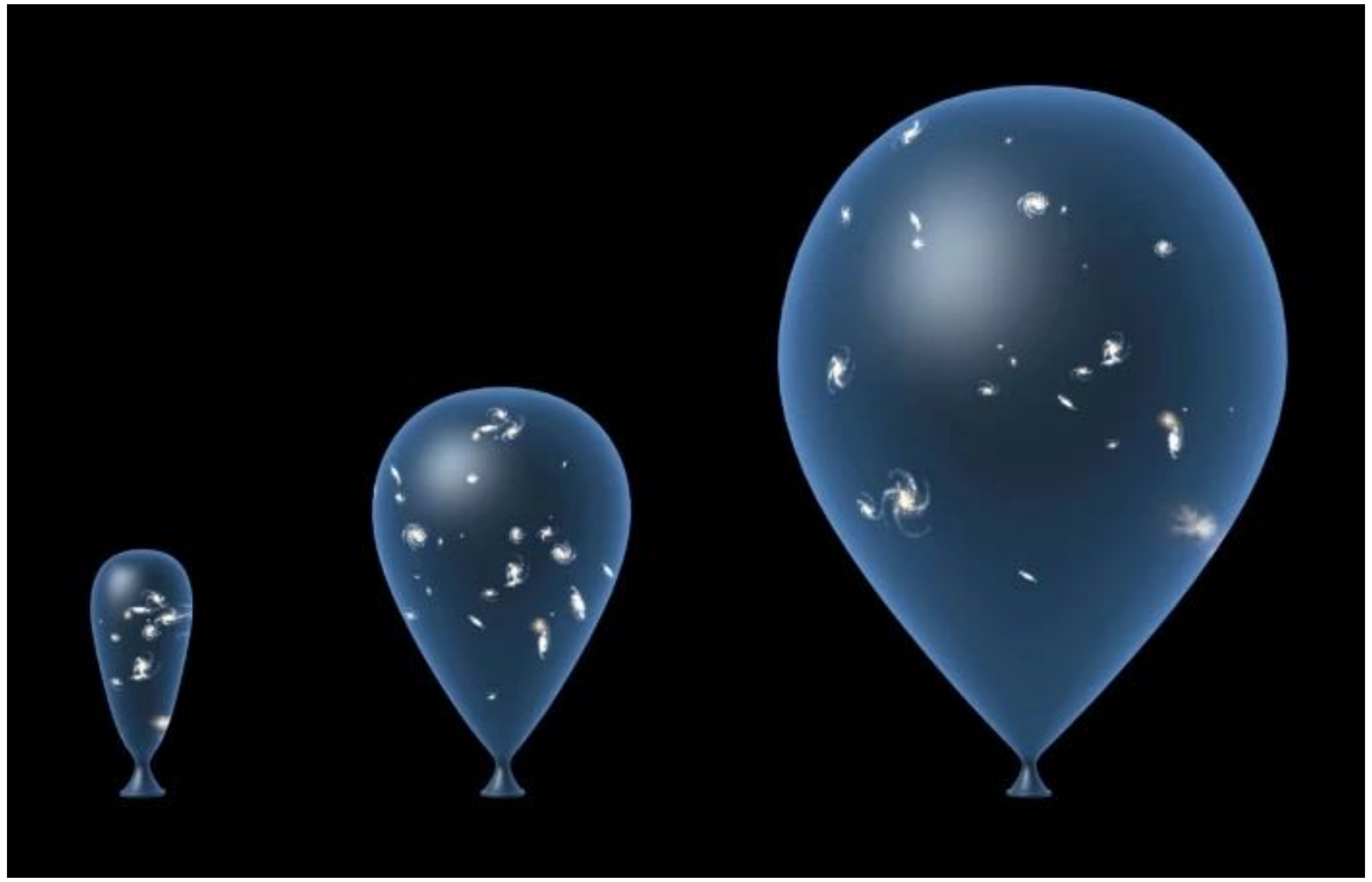
One very important point to note is that we have written a in (10.19) and (10.20) in the form $a(t)$ to make it explicit that **$a(t)$ can in general be a function of time**.

To understand the physical significance of this, let us look at the metric (10.4) for the surface of a **sphere.** There the **parameter a was the radius of the sphere** and **would increase with time if the sphere expanded.**

In exactly the same spirit, we can **regard $a(t)$ as a measure of the size of the Universe.** It is called the ***scale factor* of the Universe**.

A time evolution equation of $a(t)$ will tell us **how the Universe evolves with time**. Such an equation can be obtained by substituting (10.19) into Einstein's equation, the basic equation of general relativity. We shall carry out this later.

The metric of the Universe



https://www.nature.com/articles/nature.2013.13379

Another useful analogy with the metric (10.4) for the spherical surface: Suppose we **consider some marks on the spherical surface.**

The coordinates (θ,ϕ) for any particular mark will not change with time **if the sphere expands**. But the **marks will move away from each other** because the radius of the sphere is increasing and **the distance between any two marks** (along the great circle connecting them) **is proportional to the radius.**

In relativistic cosmology, we take a similar point of view that **galaxies are moving from each other because the scale factor of the Universe is increasing,** but the spatial coordinates (r,θ,ϕ) of a galaxy would not change (provided we neglect any motion of the galaxy with respect to the Hubble expansion).

In other words, a galaxy stays put at a point in space while space is expanding. **A coordinate system in which galaxies do not change their coordinates with the expansion of the Universe is called a *co-moving coordinate system*.**

The Robertson–Walker metric is usually assumed to be a metric corresponding to a co-moving coordinate system.

The metric of the Universe

Suppose we take our Galaxy to be the origin of our coordinate system and we **want to find the distance of a galaxy located at (r,θ,ϕ).**

One way of obtaining a measure of this distance is to **integrate the spatial part of *ds* between us and that galaxy**. If we are at the origin, this integration will clearly be in the radial direction and we easily conclude from (10.19) that this measure of distance to the galaxy is given by

$$l = a(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}}.$$

We should point out that the concept of distance in general relativity involves some subtleties. We shall present an analysis of length measurement later. However, the different distances as well as the distance measure given by (10.21) converge if the redshift z of the galaxy is small compared to 1 such that the curvature of the Universe is not important within the distance to the galaxy.

The metric of the Universe

Taking l given by (10.21) as the distance to the galaxy, the **recession velocity of the galaxy with the expansion of the Universe** is

$$v = \dot{a}(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}},$$

where a dot represents differentiation with t throughout this chapter.

It follows from (9.13) that the **Hubble constant** is given by

$$\boxed{H = \frac{v}{l} = \frac{\dot{a}}{a}.}$$

The metric of the Universe

We have introduced some elementary concepts of relativistic cosmology.

Now we shall use only Newtonian mechanics in the rest of this chapter and see how far we can go with it. We shall find that it is possible to study the mathematical equations for the evolution of the Universe without getting into the details of general relativity. However, our treatment will lack self-consistency and will not be satisfactory at a deep conceptual level if we shy away from general relativity.

Table 10.2 lists the **main differences between relativistic cosmology and the so-called Newtonian cosmology.**

Table 10.2 Conceptual differences between relativistic cosmology and Newtonian cosmology.

	RELATIVISTIC COSMOLOGY	*NEWTONIAN COSMOLOGY*
1	All points in space are equivalent	We are at the centre of the Universe
2	Space is expanding with galaxies	Galaxies are moving away in space
3	Redshift caused by stretching of light wavelength due to expansion	Redshift caused by Doppler effect due to recession of galaxies

The metric of the Universe

Just as all points on a spherical surface are equivalent, all points in a uniform three-dimensional space described by the metric (10.19) are equivalent. In **Newtonian cosmology, however, we shall take the point of view that we are at the centre and the Universe is expanding radially outwards with us at the centre.**

In the co-moving coordinate system introduced in relativistic cosmology, **space is expanding** and carrying the galaxies with it. On the other hand, we shall have to assume in Newtonian cosmology that **galaxies are moving away in space,** which is regarded as the inert background without any dynamics.

Another problematic aspect of Newtonian cosmology is the **interpretation of the redshift of spectral lines**, which is regarded as a simple Doppler shift due to the recession of the galaxies. **When the redshift** z defined in (9.11) **is of the order of 1 or larger** (which is the case for many objects found through the most powerful telescopes), **this interpretation does not make sense**. We shall see later that **the wavelength of light gets stretched with the expansion of the Universe**.

The metric of the Universe

Thus, if ***a* were the scale factor of the Universe when light started from a distant galaxy and if a_0 is the present scale factor** (which must be larger than *a* for an expanding Universe), it follows from relativistic considerations that

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a_0}{a}.$$

We shall see that the expressions of most of the observable quantities in cosmology will involve the ratio of scale factors rather than the scale factor itself.

Friedman equation for the scale factor

We now want to derive an equation describing how the scale factor $a(t)$ appearing in (10.19) evolves with time.

We shall use some simple considerations of Newtonian mechanics. As we already pointed out, we have to assume the Universe to be spherically symmetric around us located at the centre of a uniform expansion, as sketched in Figure 10.3, and the equation we shall get miraculously turns out to be the same as what one gets by putting the Robertson–Walker metric in Einstein's equation.

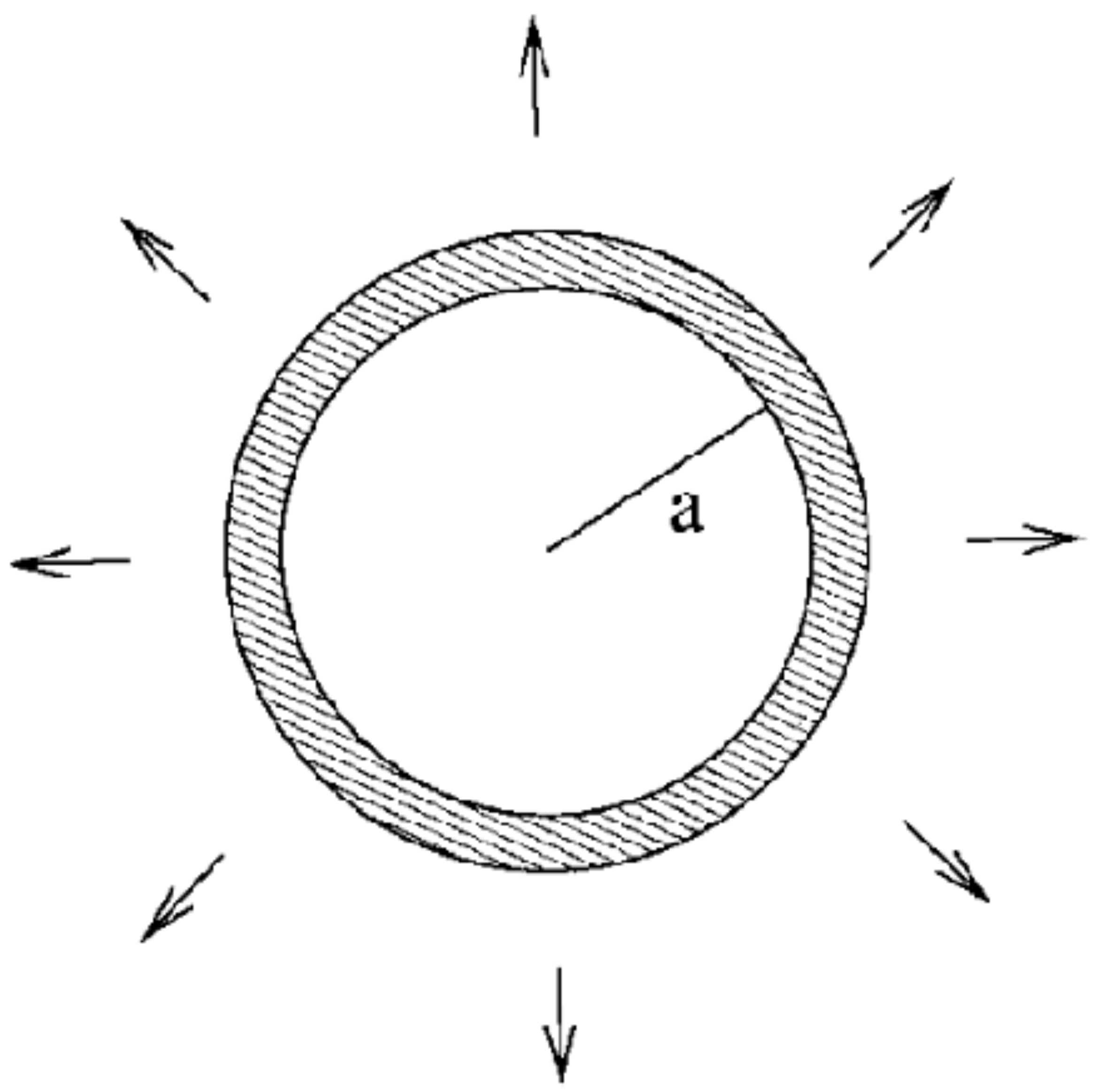


Fig. 10.3 Sketch of a spherical shell in a region of spherical expansion.

Friedman equation for the scale factor

Let us consider a spherical shell of radius a indicated in Figure 10.3. The **kinetic energy** per unit mass of the shell is $\frac{1}{1}\dot{a}^2$ and the **potential energy per unit mass** is $-GM/a$, where M is the mass enclosed within this shell.

Assuming a uniform density ρ, the **potential energy** turns out to be

$$-\frac{\frac{4}{3}\pi G\rho a^3}{a} = -\frac{4}{3}\pi G\rho a^2$$

so that the total mechanical energy E, which is a constant of motion, is given by

$$E = \frac{1}{2}\dot{a}^2 - \frac{4}{3}\pi G\rho a^2. \tag{10.25}$$

Friedman equation for the scale factor

If we know how ρ depends on a, then this equation can be solved to find how a will evolve with time. As we shall see later, on substituting the Robertson– Walker metric into Einstein's equation, we get essentially the same equation as (10.25) with E given by

$$E = -\frac{kc^2}{2}, \tag{10.26}$$

where k is the same k that appears in the Robertson–Walker metric and can have values $+1$, -1 or 0.

On substituting (10.26) in (10.25), we get

$$\boxed{\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho, \qquad (10.27)}$$

which is known as the ***Friedmann equation*****.**

Friedman equation for the scale factor

It may be pointed out that Einstein's equation allows the possibility of an **extra term (called the *cosmological constant*)**, which causes an **extra acceleration of the Universe.**

Recent redshift data of distant supernovae indicate that the **Universe may be accelerating**, which suggests that the Friedmann equation (10.27) may not be the complete equation and an **additional cosmological constant term may be present**.

It is useful to study the consequences of (10.27) before getting into a discussion of the effects of the cosmological constant. All the discussions in this chapter will assume a zero cosmological constant and will be based on (10.27).

In a later unit, we will discuss the modifications of the theory necessitated by the cosmological constant. Since the cosmological constant was believed to be zero until a few years ago, most of the standard textbooks of cosmology written till about 2000 assumed (10.27) to be the complete equation and discussed its solutions. We shall see in §14.2 that **the cosmological constant becomes more dominant as the Universe becomes older.** It appears that the Universe is right now passing through the phase when the cosmological constant term has become as large as the other terms in the equation. For a study of the Universe when it was young, (10.27) is adequate.

Friedman equation for the scale factor

For a projectile moving against gravity, we know that a positive total energy E would imply that it will move forever and escape to infinity, whereas a negative energy implies that it will eventually fall back due to the attraction of gravity.

We expect similar considerations to hold here also. Noting from (10.26) that k has a sign opposite of E, we can at once draw a very important conclusion.

If $k = -1$, then the Universe will expand forever.

On the other hand, **if $k = +1$, then the expansion of the Universe will eventually be halted, making the Universe fall back and collapse** (provided the cosmological constant is zero). Such a Universe will last for a finite time before it ends in a big crunch.

We have already seen in §10.3 that a Universe with positive curvature (i.e. **$k = +1$) has a finite volume**, whereas a Universe with negative curvature (i.e. **$k = -1$) is infinite.**

This leads to an interesting statement. **A Universe finite in space (with $k = +1$) should last for a finite time. On the other hand, a Universe infinite in space (with $k = -1$) will last for infinite time.**

Friedman equation for the scale factor

Whether the Universe will expand forever or not must **depend on the density of the Universe,** which determines the strength of the gravitational attraction.

The value of density for which the Universe will lie exactly on the borderline between these two possibilities **(with $k = 0$) is called the *critical density*** and is denoted by ρ_c. On putting $k = 0$ in (10.27) and using (10.23), the critical density is given by

$$\boxed{\rho_c = \frac{3H^2}{8\pi G}. \qquad (10.28)}$$

On substituting the present-day value of the Hubble constant as given by (9.17), **the present-day value of the critical density** turns out to be

$$\rho_{c,0} = 1.88 \times 10^{-26} h^2 \text{ kg m}^{-3}. \qquad (10.29)$$

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}, \qquad (9.17)$$

Friedman equation for the scale factor

If the average density of our present Universe is less than this, then it should expand forever. On the other hand, a higher average density is expected to make the Universe eventually fall back.

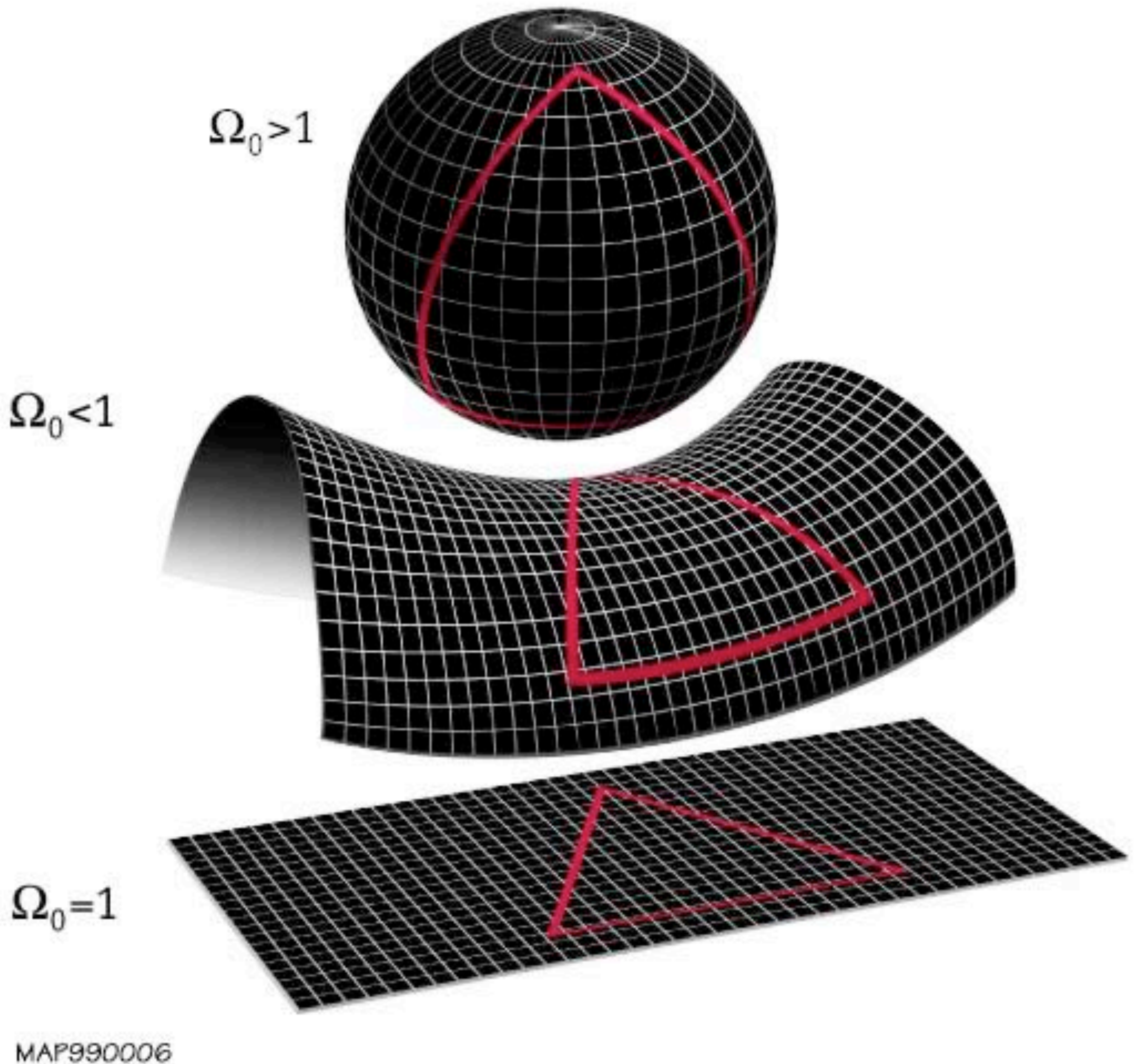
We shall come to the question of the value of average density of the Universe in the next section where we discuss the contents of the Universe.

The ratio of the density to the critical density is called the ***density parameter*** and is **denoted by Ω**

$$\Omega = \frac{\rho}{\rho_c}. \qquad (10.30)$$

On making use of (10.23), (10.28) and (10.30), we can write (10.27) in the form

$$\frac{kc^2}{a^2H^2} = \Omega - 1. \qquad (10.31)$$



Friedman equation for the scale factor

It should be noted that ***a*, *H* and Ω all evolve with time.** Their values at any particular epoch t have to be related by (10.31).

Their values at the **present epoch are denoted by a_0, H_0 and Ω_0.**

It should be clear from (10.30) and (10.31) that **$k = 0$, $k = +1$ and $k = -1$ respectively correspond to the cases $\rho = \rho_c$, $\rho > \rho_c$ and $\rho < \rho_c$.**

This is consistent with the conclusion we have already drawn that a Universe with $k = +1$ should eventually fall back, whereas a Universe with $k = -1$ should expand forever.

Only when the density is more than the critical density ρ_c, will the gravitational force be strong enough to pull back the Universe eventually ($k = +1$ case). On the other hand, a density less than the critical density ρ_c corresponds to a weak gravitational pull that cannot halt the expansion of the Universe ($k = -1$ case).

Friedman equation for the scale factor

In the Newtonian expression (10.25) of the spherical shell, it is possible for E to have any real value. However, it follows from (10.26) that **general relativistic considerations constrain E to have only three values corresponding to the three values of k.** This may seem surprising at first sight.

It is to be noted that the variables like a, $\dot{a}$ and ρ appearing in (10.27) can have continuous possible values. The three possible values of k basically force a, $\dot{a}$ and ρ to satisfy three possible relationships amongst themselves which follow from (10.27) on substituting the values of k. **Since it is the mass-energy which creates the curvature of spacetime in general relativity, we do expect such relationships.**

On the other hand, when we apply purely Newtonian considerations to the spherical shell of Figure 10.3, its radius a does not have to be related to the density ρ. We could consider spherical shells of different radii and write down equations of the form (10.25) for them. The value of E can be different for them. **In relativistic cosmology, on the other hand, a is like the radius of curvature of the Universe and general relativity certainly imposes some extra constraints which would not be present in the Newtonian formulation of the expanding shell.**

While Newtonian considerations lead us to the equation (10.25) having the same form as what we get from general relativity, these subtle differences should be kept in mind.

Friedman equation for the scale factor

We shall discuss in the next section **how ρ varies with $\boldsymbol{a}$**.

Here we merely point out how we proceed to solve (10.27) when we have ρ as a function of a. It is particularly easy to handle (10.27) when $\boldsymbol{k} = \mathbf{0}$. Since $\dot{a}$ has to be positive for an expanding Universe, we get

$$\dot{a} = \sqrt{\frac{8\pi G\rho}{3}}a. \tag{10.32}$$

This is very easy to integrate when we have ρ as a function of a.

When $\boldsymbol{k} = \pm\mathbf{1}$, it is useful to change from t to another time-like variable η defined through

$$c\,dt = a\,d\eta. \tag{10.33}$$

Friedman equation for the scale factor

On using this variable η, the Robertson–Walker metric (10.19) would have the form

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]. \qquad (10.34)$$

Keeping in mind that a dot denotes a differentiation with respect to t, a differentiation with respect to η has to be indicated explicitly. We find from (10.33) that

$$\dot{a} = \frac{c}{a} \frac{da}{d\eta}.$$

Substituting in (10.27), we get

$$\frac{da}{d\eta} = \pm \sqrt{\frac{8\pi G}{3c^2} \rho a^4 - ka^2}.$$

Friedman equation for the scale factor

from which
$$\frac{da}{d\eta} = \pm\sqrt{\frac{8\pi G}{3c^2}\rho a^4 - ka^2}.$$

This can be put in the form of a quadrature

$$\eta = \pm\int \frac{da}{\sqrt{\frac{8\pi G}{3c^2}\rho a^4 - ka^2}}. \tag{10.35}$$

Once ρ is given as a function of a, one can work out this quadrature to find **how a varies with the time-like variable η.**

If one is interested in determining the variation of a with t, then it is further necessary to relate η to t by solving (10.33) after obtaining a as a function of η. We shall carry out some calculations of this type explicitly in §10.6.