Introduction to Astrophysics and Cosmology

Relativistic Cosmology

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Particle motion in the Schwarzshield metric

A **particle will move along a geodesic in the Schwarzschild metric**. One can use the **geodesic** equation (12.51) to study the motions of particles. Here we present a discussion starting more from the basics.

Since the Schwarzschild metric is **spherically symmetric**, a particle moving in this metric should **always lie in a plane passing through the origin**. We can choose the plane of motion **to be the equatorial plane** in which $\theta = \pi/2$ and $\sin\theta = 1$.

A standard convention in general relativity is to **choose units of length and time such that *c* and *G* turn out to be 1**. Setting $c = 1$ and $G = 1$, it follows from (13.13) that the metric lying in the equatorial plane is given by

$$ds^2 = -d\tau^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2\, d\phi^2. \qquad (13.15)$$

$$\frac{d^2 x^m}{ds^2} = -\Gamma^m_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds}, \qquad (12.51)$$

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2\, dt^2 + \frac{dr^2}{(1 - 2GM/c^2 r)} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (13.13)$$

Particle motion in the Schwarzshield metric

If a particle moves **from a spacetime point *A* to a spacetime point *B*,** then the path length between them (which turns out to be the proper time measured in a clock carried with the particle) is given by

$$\int_A^B d\tau = \int_A^B L\, d\lambda, \qquad (13.16)$$

where λ is a parameter measured along the path of the particle and L is given by

$$L = \sqrt{\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{(dr/d\lambda)^2}{\left(1 - \frac{2M}{r}\right)} - r^2\left(\frac{d\phi}{d\lambda}\right)^2}. \qquad (13.17)$$

The basic idea of general relativity is that the **particle should follow a geodesic** along which the integral given by (13.16) has to be an extremum. This requirement implies that L given by (13.17) **should satisfy the Lagrange equation**

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial(dq^i/d\lambda)}\right) - \frac{\partial L}{\partial q^i} = 0,$$

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where q^i can be t, r or ϕ. It is seen from (13.17) that **L is independent of t and ϕ**. This suggests that we shall have the following **two constants of motion**

$$\frac{\partial L}{\partial(dt/d\lambda)} = \frac{\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}}{L} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau},$$

$$\frac{\partial L}{\partial(d\phi/d\lambda)} = -\frac{r^2\frac{d\phi}{d\lambda}}{L} = -r^2\frac{d\phi}{d\tau},$$

since $L = d\tau/d\lambda$. We denote these **constants of motion by e and $-l$**, i.e.

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}, \tag{13.18}$$

$$l = r^2\frac{d\phi}{d\tau}. \tag{13.19}$$

Particle motion in the Schwarzshield metric

Dividing (13.15) by $d\tau^2$ and using these constants of motion, we get

$$\frac{e^2}{\left(1-\frac{2M}{r}\right)} - \frac{(dr/d\tau)^2}{\left(1-\frac{2M}{r}\right)} - \frac{l^2}{r^2} = 1.$$

On rearranging terms a little bit, this can be put in the form

$$\boxed{\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r), \qquad (13.20)}$$

Where

$$\boxed{V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}. \qquad (13.21)}$$

Particle motion in the Schwarzshield metric

It is to be noted that the problem is now **reduced to a one-dimensional problem of r as a function of τ**.

To proceed further, it is now instructive to make some **comparisons with the problem of particle motion in an inverse-square law force field in classical mechanics**. This problem is often referred to as the ***Kepler problem***.

The classical Kepler problem also has **two constants of motion – the angular momentum and the energy**. Our constant of motion l given by (13.19) is clearly **the general relativistic generalization of the classical angular momentum**.

To interpret e defined by (13.18), we consider the motion of a particle in the faraway regions where the gravitational field is weak. Then the relation between dt and $d\tau$ can be obtained from (12.74). On using (12.74) and (13.11), it readily follows from (13.18) that

$$e \approx 1 + \frac{\Phi}{c^2} + \frac{1}{2}\frac{v^2}{c^2}. \tag{13.22}$$

Here we have not set c equal to 1 to make the physics clearer. It is obvious that **e multiplied by mc^2 would give the sum of the rest mass, potential and kinetic energies in the non-relativistic limit**.

Particle motion in the Schwarzshield metric

It follows from (13.22) that

$$\frac{e^2-1}{2} \approx \frac{\Phi}{c^2} + \frac{1}{2}\frac{v^2}{c^2}.$$

The **right-hand side is the total energy** (sum of potential and kinetic energies) used in classical mechanics calculations. We thus identify **$(e^2 - 1)/2$ as the relativistic generalization of the classical energy**.

Now it is easy to interpret (13.20). The term **$(1/2)(dr/d\tau)^2$ is like the kinetic energy**. Then (13.20) implies that **$(e^2 - 1)/2$, has to be equal to the sum of the kinetic energy and an effective potential $V_{\mathrm{eff}}(r)$**. The classical Kepler problem also gives rise to a one-dimensional equation exactly similar to (13.20), except that the effective potential does not have the last term $-Ml^2/r^3$ appearing in (13.21). It is this **last term $-Ml^2/r^3$ which makes results of general relativity different** from the classical Kepler problem.

When **r is much larger than r_S** (which is equal to $2M$ in our units), this last term in (13.21) becomes negligible compared to the previous term $l^2/2r^2$ and the **general relativistic effects disappear.**

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\mathrm{eff}}(r), \tag{13.20}$$

Particle motion in the Schwarzshield metric

The values of r at which $V_{\text{eff}}(r)$ has extrema can be obtained from

$$\frac{dV_{\text{eff}}}{dr} = 0,$$

which gives

$$r = \frac{l^2}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{l}\right)^2}\right]. \tag{13.23}$$

If $l^2 > 12M^2$, then $V_{\text{eff}}(r)$ has extrema for two real values of r given by (13.23). On the other hand, **if $l^2 < 12M^2$,** then there is **no extremum** for any real value of r.

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Figures 13.1(a) and 13.1(b) respectively show plots of Veff(r) for one case with $l^2 > 12M^2$ and one case with $l^2 < 12M^2$. In

Figure 13.1(a), a possible value of $(e^2 - 1)/2$ is indicated by a horizontal dashed line, which cuts the curve Veff(r) at two points r_1 and r_2. Since $(dr/d\tau)^2$ in (13.20) is always positive, we easily see that (13.20) can be satisfied **only if r lies between r_1 and r_2.** We must have **$r = r_1$ and $r = r_2$ as the two turning points within which the orbit of the particle should be confined**.

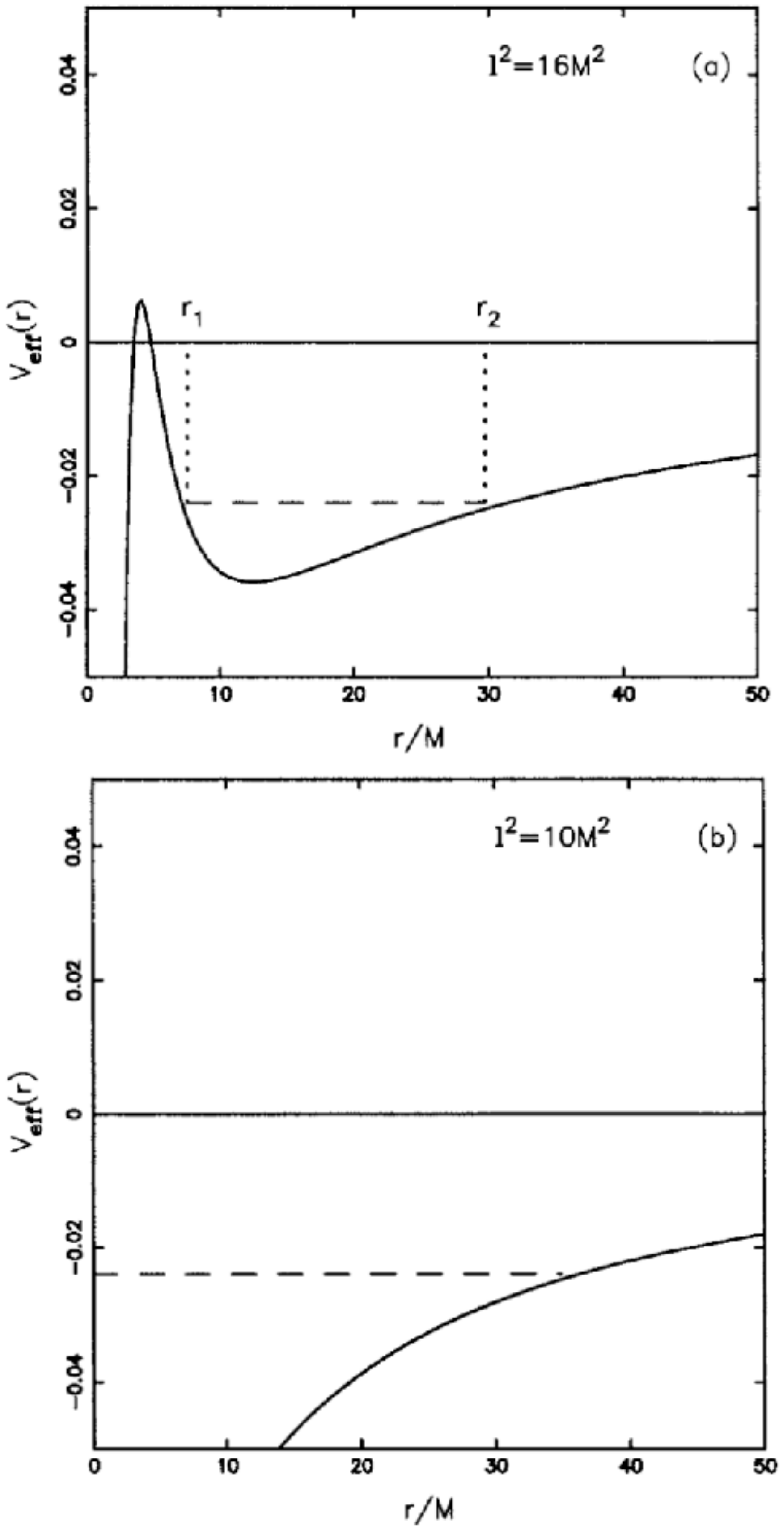


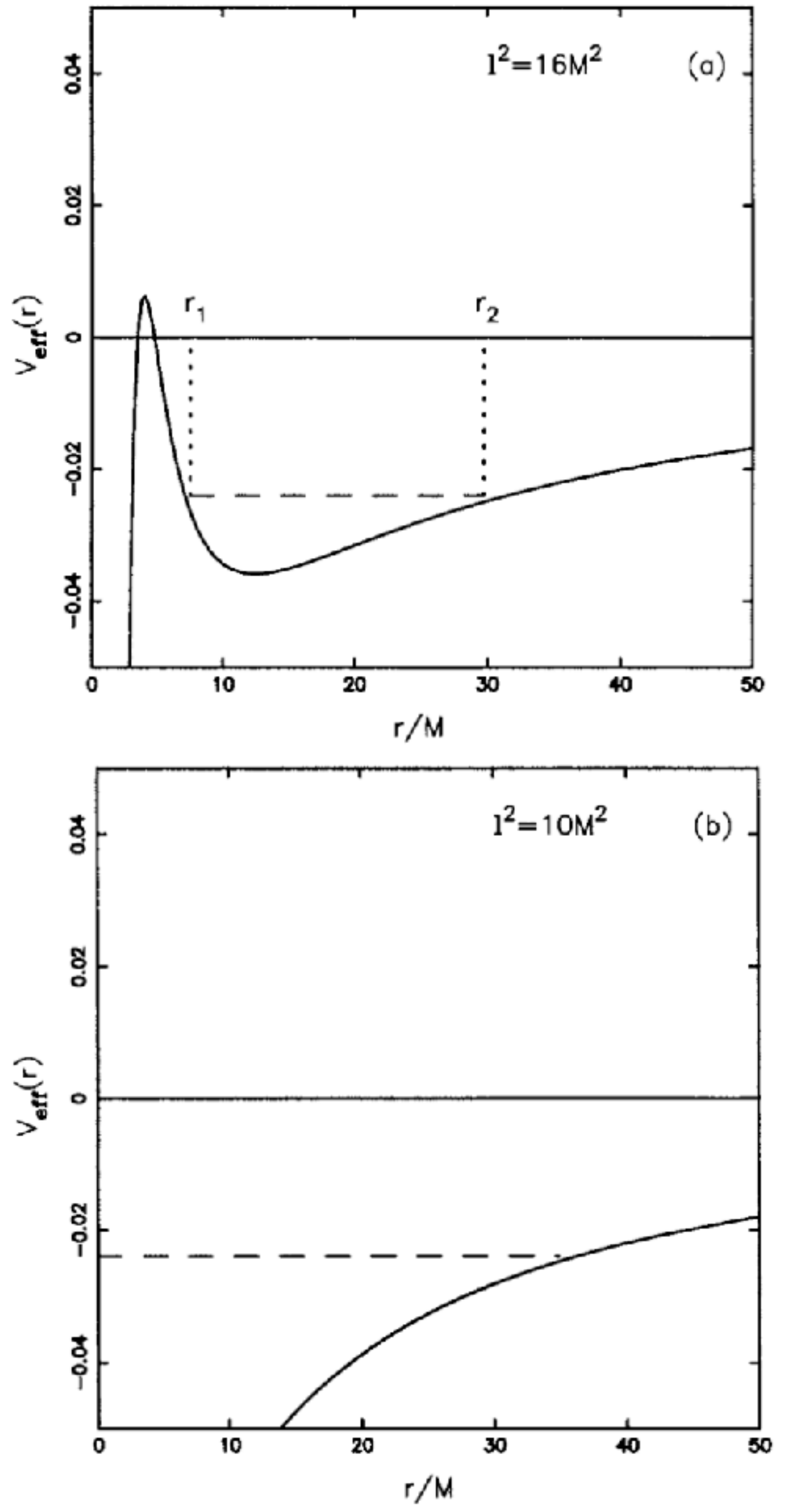
Fig. 13.1 Plots of $V_{\text{eff}}(r)$ given by (13.21) for the cases (i) $l^2 = 16M^2$ and (ii) $l^2 = 10M^2$. The dashed horizontal lines indicate possible values of $(e^2 - 1)/2$.

Particle motion in the Sch

In the case of Figure 13.1(b), **it is not possible for the orbit to be confined from the lower side**. This implies that a **particle with $l^2 < 12M^2$ should keep falling inward till it hits the central mass.**

If we throw a particle with zero angular momentum towards a gravitating mass, the particle will fall into the gravitating mass.

In general relativity that the particle has to **have an angular momentum with amplitude larger than $2\sqrt{3}M$ in order not to fall into the central mass.**



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Particle motion in the Sch

There is one other important point to be made. **When $l^2 >$ $12M^2$ and Veff(r) has two extremum points**, it is possible for a particle to have **a circular orbit if it is located at the minimum of Veff(r).** The **limiting circular orbit is obtained when $l^2 = 12M^2$.** On substituting this in (13.23), we find

$$r = 6M = 3r_S \qquad (13.24)$$

as the lowest value of r above which a circular orbit is possible. **It is not possible for a particle to go around a black hole in a circular orbit of radius less than $3r_S$.** **The circular orbit of radius $3r_S$ is called the *last stable orbit*.**

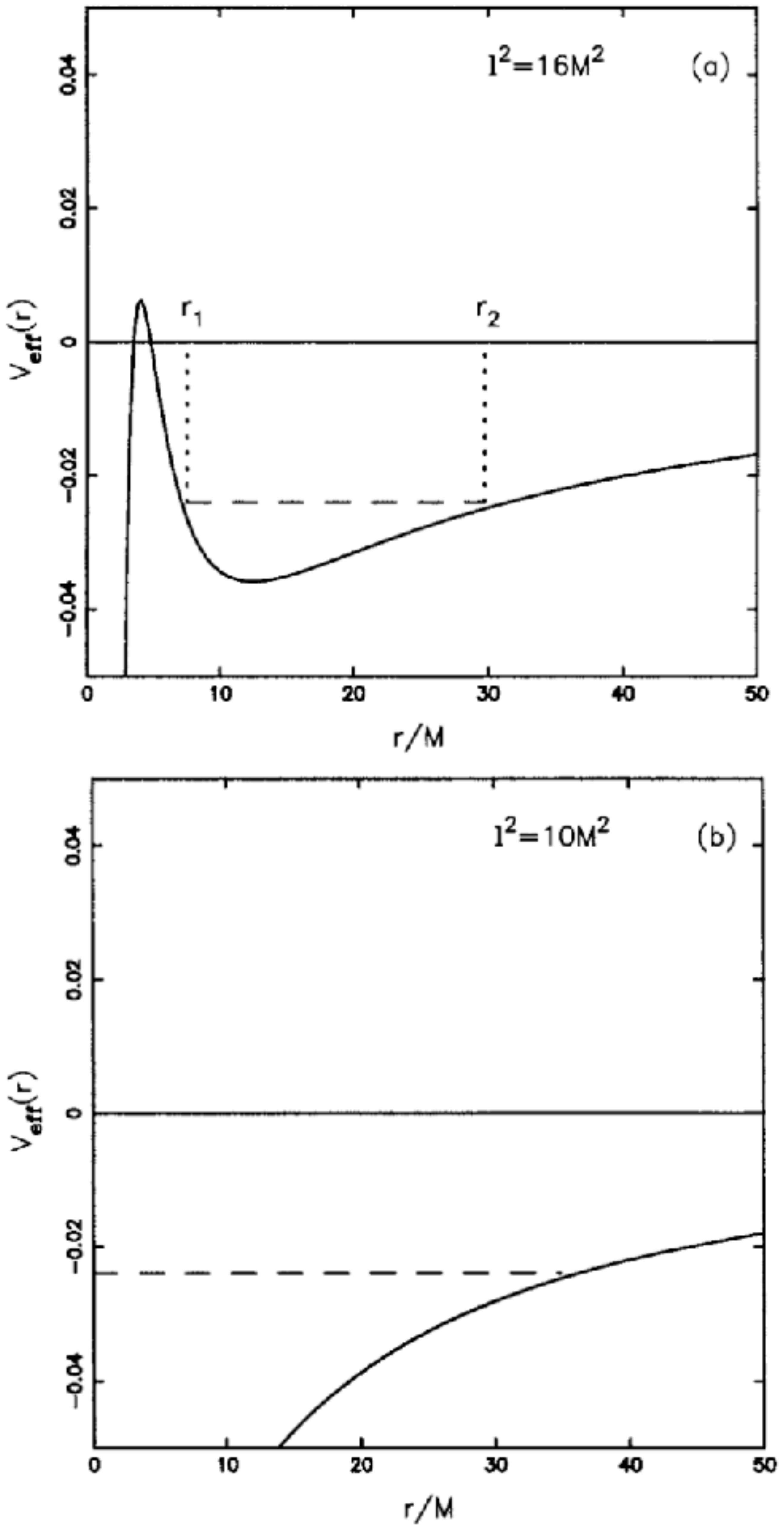


Fig. 13.1 Plots of $V_{\rm eff}(r)$ given by (13.21) for the cases (i) $l^2 = 16M^2$ and (ii) $l^2 = 10M^2$. The dashed horizontal lines indicate possible values of $(e^2 - 1)/2$.

Particle motion in the Schwarzshield metric

Finally we now want to calculate the orbit of the particle, which should be in the form of a functional relation between r and ϕ. It follows from (13.20) that

$$\frac{dr}{d\tau} = \pm\sqrt{e^2 - 1 - 2V_{\text{eff}}(r)}. \tag{13.25}$$

From (13.19) we have

$$\frac{d\phi}{d\tau} = \frac{l}{r^2}.$$

On dividing (13.25) by this and then squaring, we get

$$\left(\frac{l}{r^2}\frac{dr}{d\phi}\right)^2 = e^2 - 1 - 2V_{\text{eff}}(r). \tag{13.26}$$

Particle motion in the Schwarzshield metric

To proceed further, we make the substitution

$$r = \frac{1}{u}, \tag{13.27}$$

which is a standard procedure followed in the classical Kepler problem also. Then

$$\frac{dr}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$$

so that (13.26) becomes

$$l^2\left(\frac{du}{d\phi}\right)^2 = e^2 - 1 - 2V_{\text{eff}}(u). \tag{13.28}$$

Particle motion in the Schwarzshield metric

Differentiating both sides with respect to ϕ, we get

$$2l^2 \frac{du}{d\phi}\frac{d^2u}{d\phi^2} = -2\frac{dV_{\mathrm{eff}}}{du}\frac{du}{d\phi}. \qquad (13.29)$$

We cancel 2 $du/d\phi$ from both the sides and then write (13.21) in the form

$$V_{\mathrm{eff}}(u) = -Mu + \frac{1}{2}l^2u^2 - Ml^2u^3$$

to calculate dV_{eff}/du. This gives

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{p} + 3Mu^2, \qquad (13.30)$$

Particle motion in the Schwarzshield metric

where

$$p = \frac{l^2}{M}.$$

The orbit as a relation between u and ϕ can be obtained by solving the orbit equation (13.30).

The last term in (13.30) comes from the **last term in (13.21).** We have already identified this term as the **contribution of general relativity. If this last term were not present in the orbit equation (13.30), then it would be the equation of an ellipse,** as we find in the **classical Kepler problem.**

Let us try to solve (13.30) for a situation **where the general relativistic effect is small and the last term in (13.30) can be treated as a small perturbation compared to the other terms.** The zeroth order solution of (13.30) in the absence of this last term would be

$$u_0 = \frac{1}{p}(1 + \epsilon \cos \phi). \tag{13.32}$$

This is the equation of an **ellipse with eccentricity ε.**

Particle motion in the Schwarzshield metric

Let us now try a solution of the form

$$u = u_0 + u_1, \tag{13.33}$$

where u_0 is given by (13.32).

On substituting this in (13.30) and **approximating the small perturbation term as $3Mu^2 \approx 3Mu^2_0$**, we get

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3Mu_0^2.$$

On substituting for u_0 from (13.32), we get

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3M}{p^2}(1 + 2\epsilon \cos\phi + \epsilon^2 \cos^2\phi). \tag{13.34}$$

The term **2ε cos ϕ** in the right-hand side acts like a **resonant forcing term**, since it **varies with ϕ** the same way as u_0. As the effect of this term is going to be much more significant than that of the other two terms in the right-hand side of (13.34), we can neglect these other two terms.

Particle motion in the Schwarzshield metric

When we **keep only the 2ε cos ϕ term** in the right-hand side of (13.34), its solution can be written down as

$$u_1 = \frac{3M\epsilon}{p^2}\phi \sin\phi, \tag{13.35}$$

which can be verified by substituting in (13.34). From (13.32), (13.33) and (13.35), we have

$$u = \frac{1}{p}\left[1 + \epsilon\cos\phi + \frac{3M\epsilon}{p}\phi\sin\phi\right].$$

When $3M\phi/p$ is small compared to 1, this can be written as

$$u = \frac{1}{p}\left[1 + \epsilon\cos\left\{\phi\left(1 - \frac{3M}{p}\right)\right\}\right]. \tag{13.36}$$

If the particle followed an exact elliptical path as given by (13.32), then the value of u would be repeated when ϕ changes by 2π.

Particle motion in the Schwarzshield metric

It follows from (13.36) that u **would repeat when ϕ changes by $2\pi + \delta\phi$,** where

$$\delta\phi = 2\pi \frac{3M}{p} = 6\pi \frac{M^2}{l^2}$$

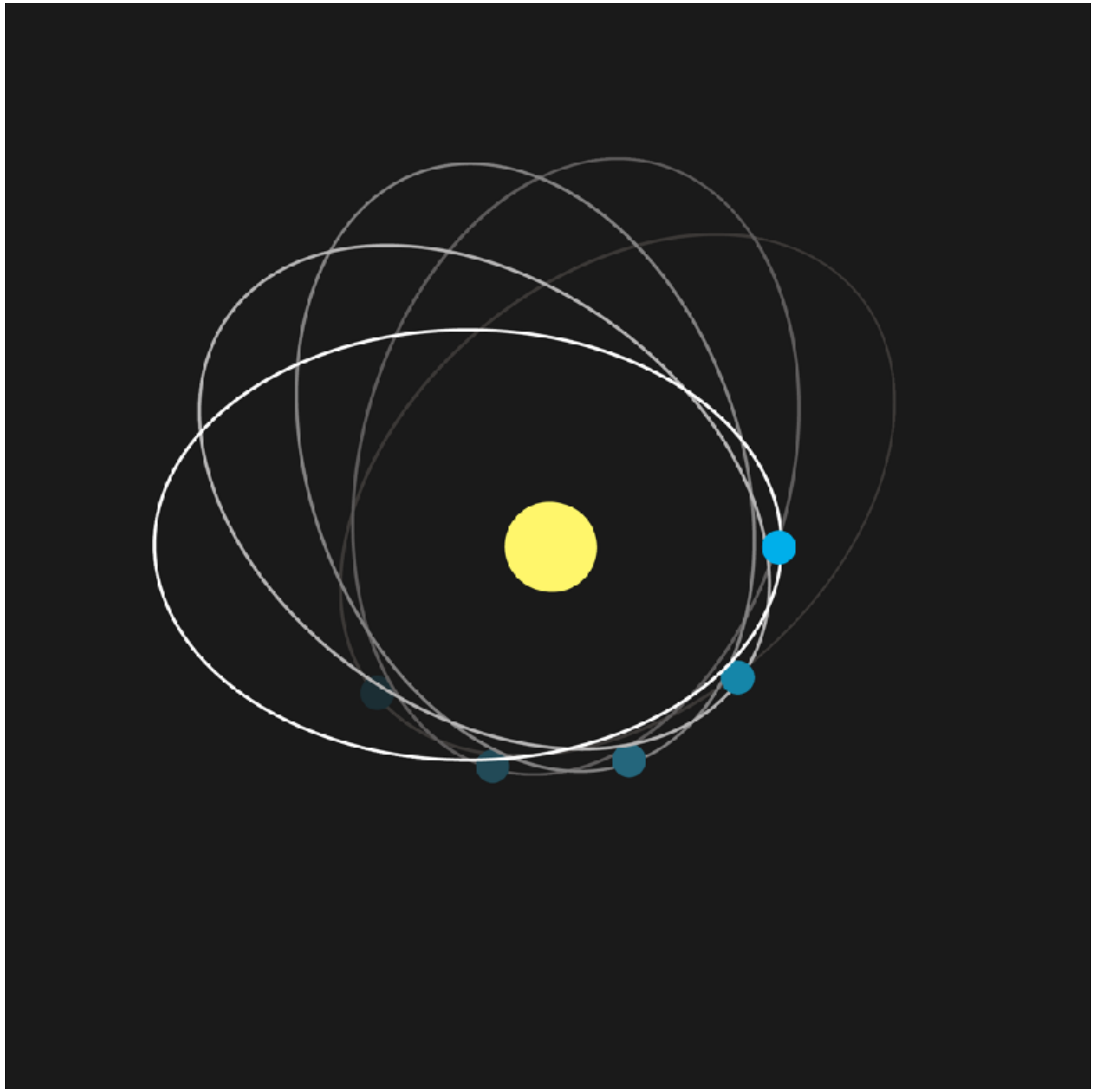
on using (13.31). Clearly **$\delta\phi$ is the angle by which the perihelion of the particle precesses during one revolution.** If one puts back G and c which were set to 1 in our analysis, then the expression for the perihelion precession is given by

$$\boxed{\delta\phi = 6\pi \left(\frac{GM}{c\,l}\right)^2 . \qquad (13.37)}$$

For the planet **Mercury, the perihelion precession rate turns out to be 43$''$ per century**. This provided one of the famous tests of general relativity.

Particle motion in the Schwarzshield metric

Under Newtonian physics, an object in an (isolated) two-body system, consisting of the object orbiting a spherical mass, would trace out an ellipse with the center of mass of the system at a focus of the ellipse. The point of closest approach, called the periapsis (or when the central body is the Sun, perihelion), is fixed. Hence the major axis of the ellipse remains fixed in space. Both objects orbit around the center of mass of this system, so they each have their own ellipse. However, a number of effects **in the Solar System cause the perihelia of planets to precess (rotate) around the Sun in the plane of their orbits,** or equivalently, cause the major axis to rotate about the center of mass, hence changing its orientation in space. The principal cause is **the presence of other planets which perturb one another's orbit.**



The motion of massless particles

A **photon or a massless particle moving with speed c follows a special geodesic** for which

$$\boxed{ds^2 = 0.}$$

A geodesic with this property is called a ***null geodesic***. For such a massless particle moving in the equatorial plane, (13.15) becomes

$$\left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2 = 0. \tag{13.38}$$

Even for a massless particle, we expect the **energy and the angular momentum to be conserved** because of the symmetry with respect to t and ϕ. However, e and l as defined in (13.18) and (13.19) tend to be infinite, since $d\tau \to 0$ along the trajectory of the particle as its mass goes to zero. But the ratio

$$\frac{e}{l} = \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \frac{dt}{d\phi} \tag{13.39}$$

would remain finite and constant even when the mass tends to zero.

The motion of massless particles

In the case of **a particle with mass, we had used the proper time τ** as a label to mark the trajectory of the particle. For the massless particle, $d\tau = 0$ and τ can no longer be used to label the trajectory. So we introduced an ***affine parameter* λ** which **increases along the trajectory of the massless particle** in such a way that

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad (13.40)$$

remains **constant**. Then

$$l = r^2 \frac{d\phi}{d\lambda} \qquad (13.41)$$

also **has to be a constant** to make the ratio given in (13.39) a constant.

Thus, **for a massless particle, we define e and l with the help of the affine parameter λ** rather than the proper time τ as done in (13.18) and (13.19).

The motion of massless particles

Dividing (13.38) by $d\lambda^2$, we get

$$\frac{e^2 - (dr/d\lambda)^2}{1 - 2M/r} = \frac{l^2}{r^2}$$

on using (13.40) and (13.41). From this

$$\frac{e^2}{l^2} - \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right),$$

which can be written as

$$\boxed{\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + \mathcal{Q}_{\mathrm{eff}}(r), \qquad (13.42)}$$

where ***b* = *l*/*e* is clearly a constant of motion** and

$$\boxed{\mathcal{Q}_{\mathrm{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right), \qquad (13.43)}$$

The motion of massless particles

$Q_{eff}(r)$ is plotted in Figure 13.2, has a **maximum at $r = 3M$ with a value $(27\,M^2)^{-1}$**.

Figure 13.2 also shows a dashed horizontal line indicating a possible value of $1/b^2$ less than $(27\,M^2)^{-1}$.

From (13.42) follows that **r has to be restricted to a lower limit r_1 if the trajectory is on the right side of the $Q\mathrm{eff}(r)$ curve**.

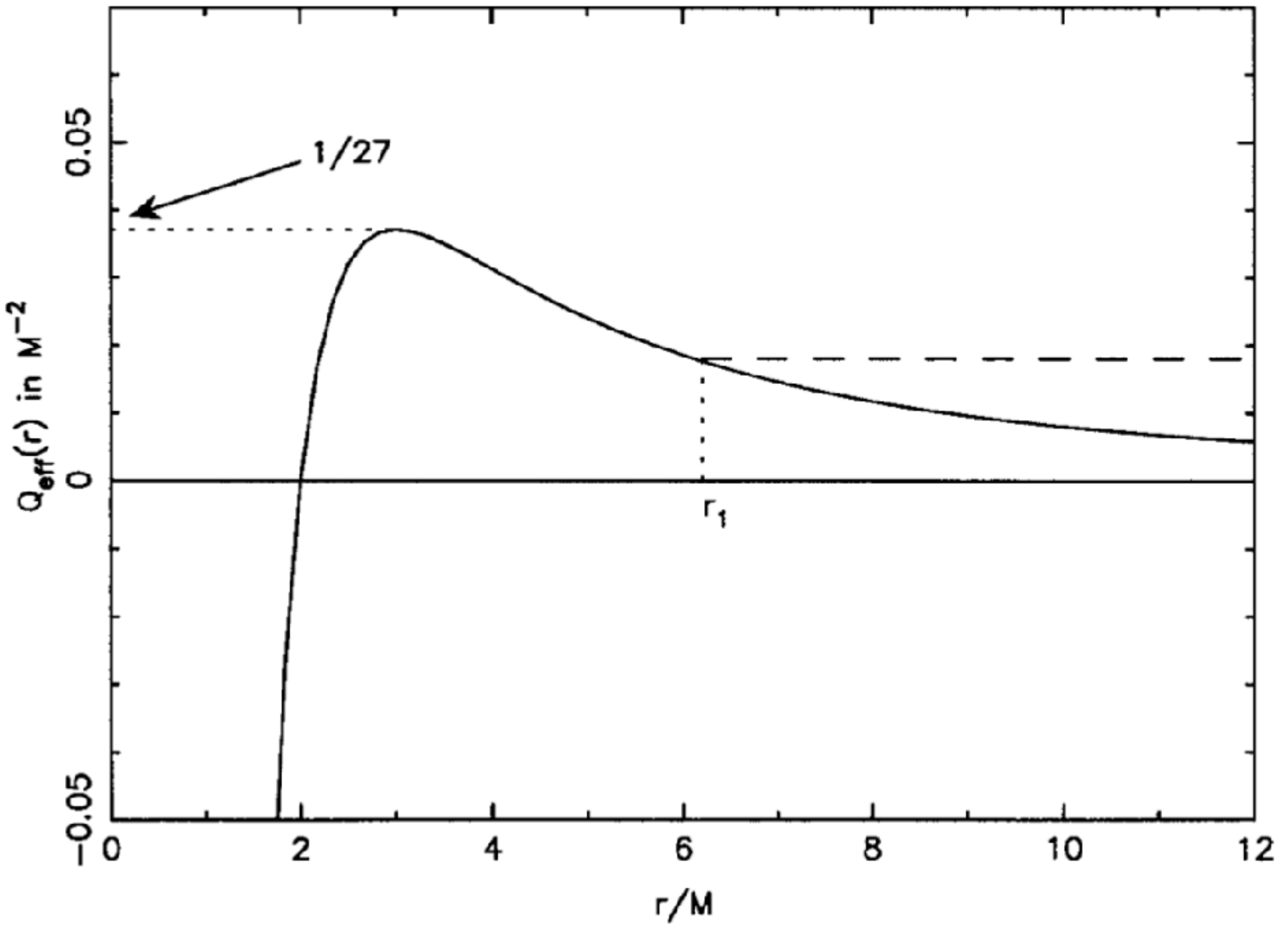


Fig. 13.2 A plot of the effective potential $Q_{\mathrm{eff}}(r)$ for a massless particle given by (13.43). The dashed horizontal line indicates a possible value of $1/b^2$.

The motion of massless particles

In other words, a massless particle coming from infinity would not approach any closer than r_1.

However, **if we have $b < 3\sqrt{3}M$, making $1/b^2$ larger than $(27\,M^2)^{-1}$**, then the **horizontal line** corresponding to $1/b^2$ would be **above the maximum of $Q_{\rm eff}(r)$ and a massless particle coming from infinity would fall into the gravitating mass M.**

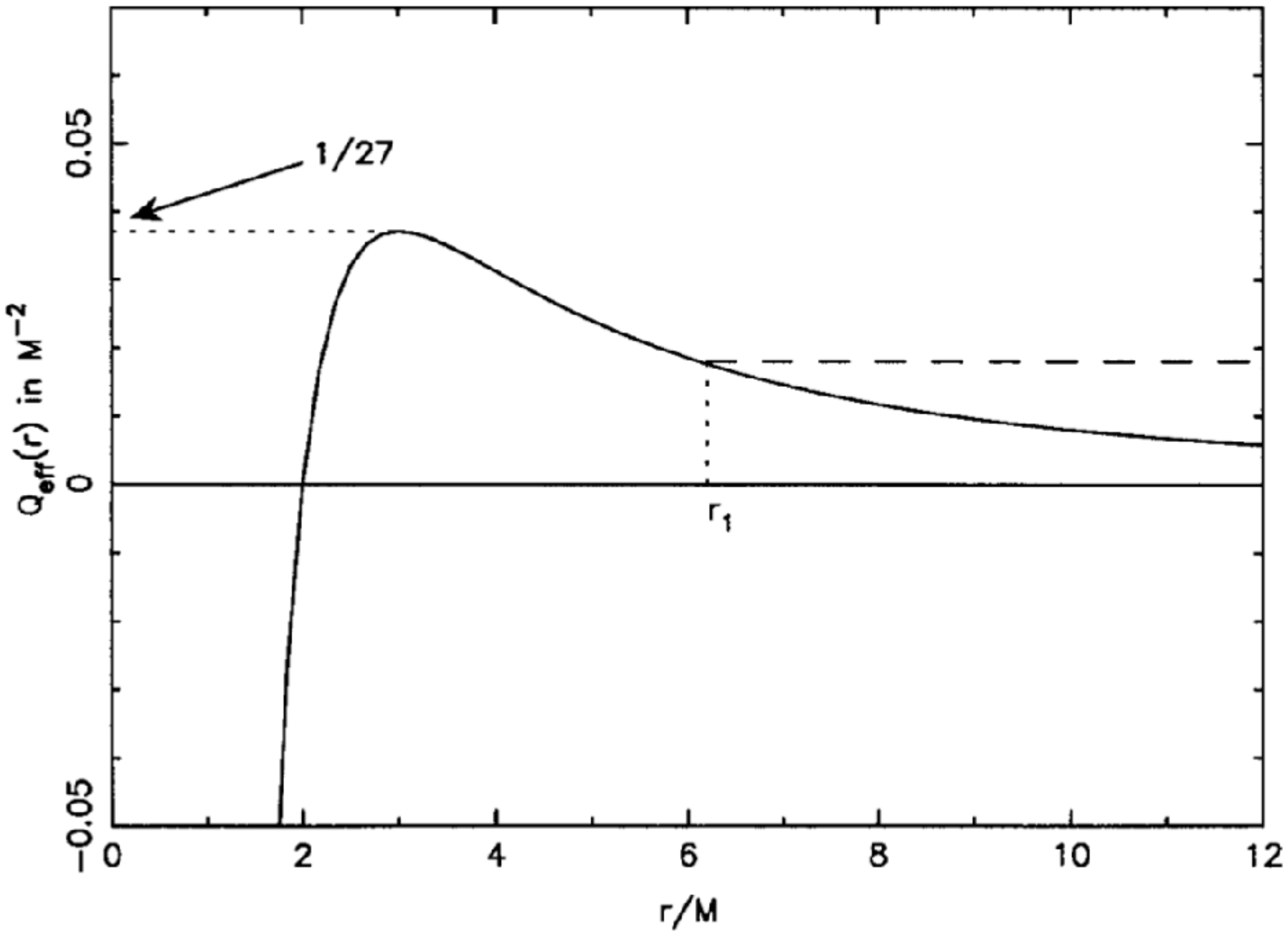


Fig. 13.2 A plot of the effective potential $Q_{\rm eff}(r)$ for a massless particle given by (13.43). The dashed horizontal line indicates a possible value of $1/b^2$.

The motion of massless particles

To understand the significance of the important result that a massless particle coming from infinity with $b < 3\sqrt{3}M$ would be **captured by the mass M**,

let us try to figure out the **physical meaning of b. At r much larger than the Schwarzschild radius $2M$**, it follows from (13.40) and (13.41) that

$$\boxed{b \approx r^2 \frac{d\phi}{dt}. \qquad (13.44)}$$

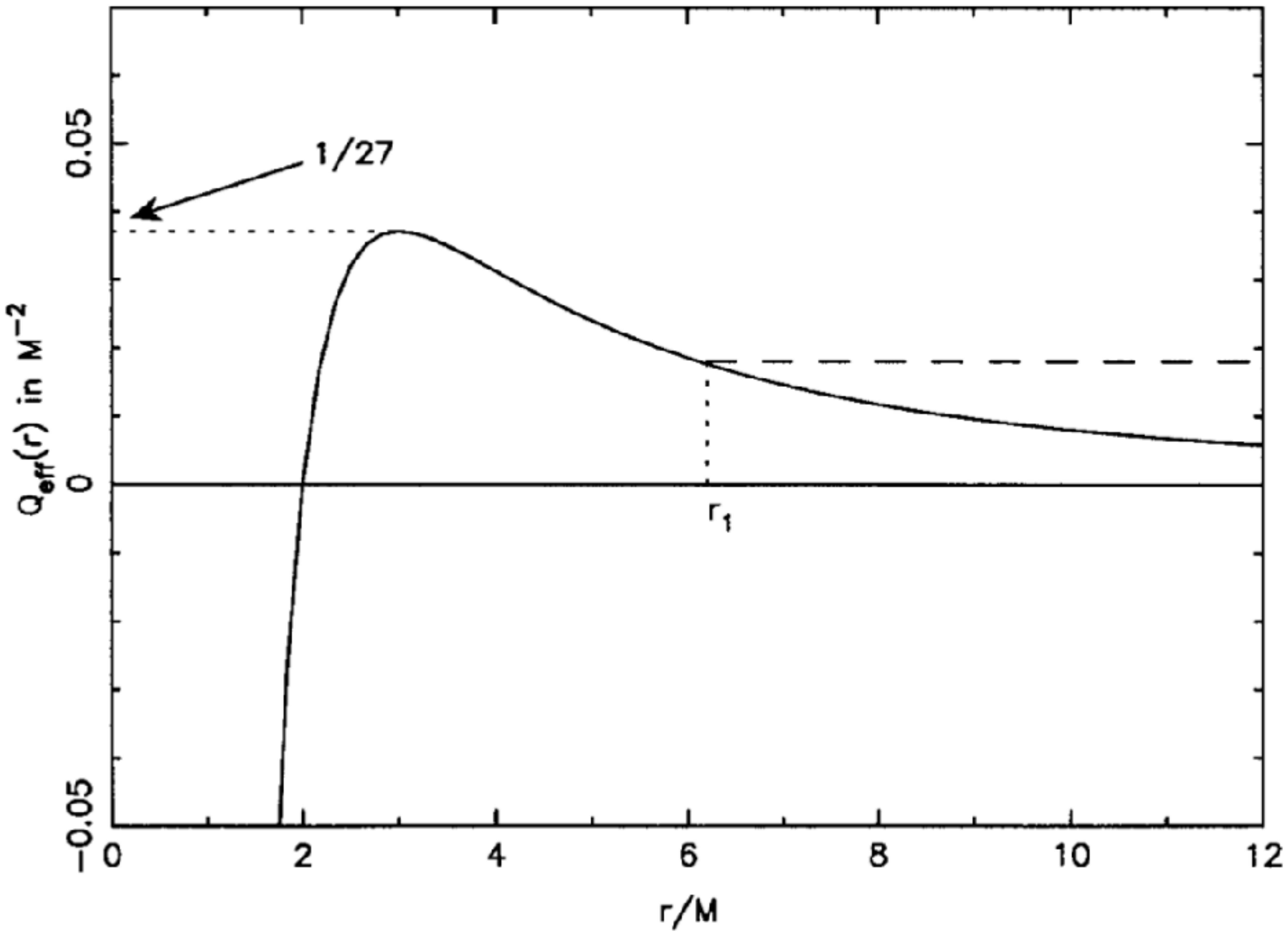


Fig. 13.2 A plot of the effective potential $Q_{\mathrm{eff}}(r)$ for a massless particle given by (13.43). The dashed horizontal line indicates a possible value of $1/b^2$.

The motion of massless particles

We now consider a **massless particle approaching the central mass *M*** from a large distance with the **impact parameter *h*** as shown in Figure 13.3.

The x axis is chosen in such a way that the **particle moves in the negative *x* direction**. If the polar angle ϕ is measured with respect to the x axis, then we have

$$\tan\phi = \frac{h}{x},$$

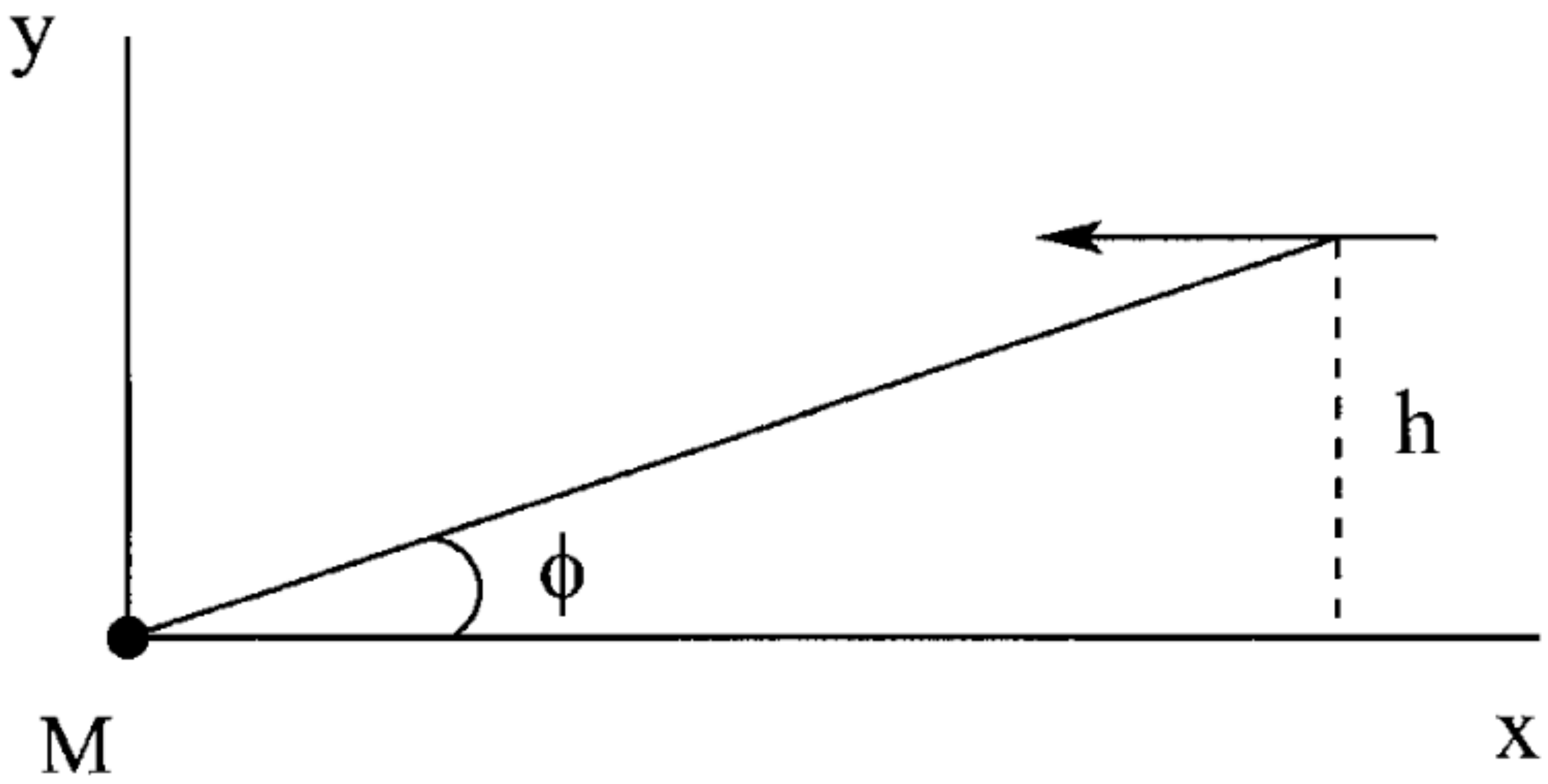
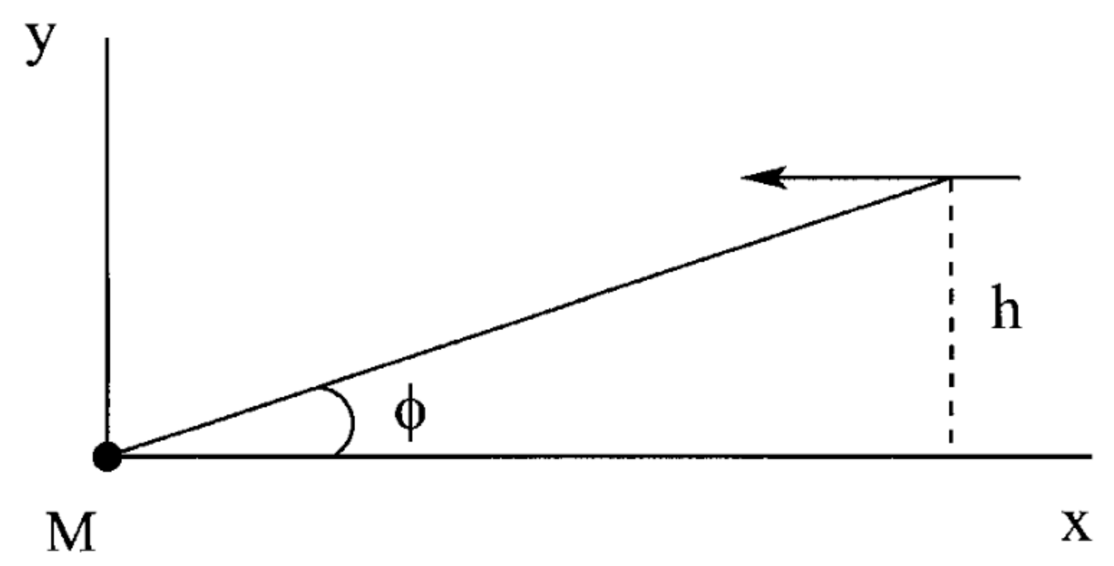


Fig. 13.3 A massless particle approaching the central mass M from a large distance with an impact parameter h.

The motion of massless particles



which on differentiation with respect to t gives

$$\sec^2 \phi \frac{d\phi}{dt} = -\frac{h}{x^2}\frac{dx}{dt}.$$

Here **$-dx/dt$ is the speed of the particle**, which is $c = 1$ in our units. Hence

$$\frac{d\phi}{dt} = \frac{h}{(x \sec \phi)^2} = \frac{h}{r^2}.$$

Comparing it with (13.44), we at once see that

$$b = h,$$

which means that the parameter ***b* is the impact parameter with which the massless particle approaches the mass *M***. When this impact parameter **is less than $3\sqrt{3}M$ the massless particle or the photon gets captured by the central mass *M*.**

The orbit of light

If the massless particle has an **impact parameter much larger than** $3\sqrt{3}M$**, then its trajectory will be slightly bent.**

To calculate this bending, we first have to derive the orbit equation. From (13.42), we get

$$\frac{dr}{d\lambda} = \pm l \sqrt{\frac{1}{b^2} - Q_{\text{eff}}(r)}.$$

Substituting for l from (13.41), we have

$$\frac{dr}{d\phi} = \pm r^2 \sqrt{\frac{1}{b^2} - Q_{\text{eff}}(r)},$$

so that

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - Q_{\text{eff}}(r), \qquad (13.45)$$

The orbit of light

which can be compared with (13.26). We proceed to solve it in exactly the same way we solved (13.26). By introducing the **variable $u = 1/r$** and using (13.43) to write

$$Q_{\text{eff}}(u) = u^2 - 2Mu^3,$$

we put (13.45) in the form

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} - u^2 + 2Mu^3.$$

Differentiating with respect to ϕ and cancelling $2du/d\phi$ from both sides, we finally get the orbit equation

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2, \tag{13.46}$$

which has to be solved to find u as a function of ϕ giving the orbit.

The orbit of light

It is easy to check that the **term $3Mu^2$** on the right-hand side of (13.46) **is the general relativistic effect.** When this term is small, we can solve (13.46) by following the same perturbative approach which we followed to solve the orbit equation (13.30) for particles with non-zero mass. **When the $3Mu^2$ term is neglected, we can write the zeroth order solution as**

$$u_0 = \frac{\cos\phi}{R}, \tag{13.47}$$

where ***R* is the distance of the closest approach** and ϕ is defined in such a way that we have **ϕ = 0 at the point of closest approach.**

Again writing u in the form (13.33), we find that u_1 should satisfy the equation

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3Mu_0^2 = \frac{3M}{R^2}\cos^2\phi,$$

of which a solution is

$$u_1 = \frac{M}{R^2}(1 + \sin^2\phi). \tag{13.48}$$

The orbit of light

Then the **full solution** can be written down by adding (13.47) and (13.48), i.e.

$$\boxed{u = \frac{\cos\phi}{R} + \frac{M}{R^2}(1 + \sin^2\phi). \qquad (13.49)}$$

Since we have chosen $\phi = 0$ at the point of closest approach, the **incoming and the outgoing directions** of the massless particle would have been **$-\pi/2$ and $\pi/2$** respectively **if the particle travelled in a straight line**. This will be clear from Figure 13.4.

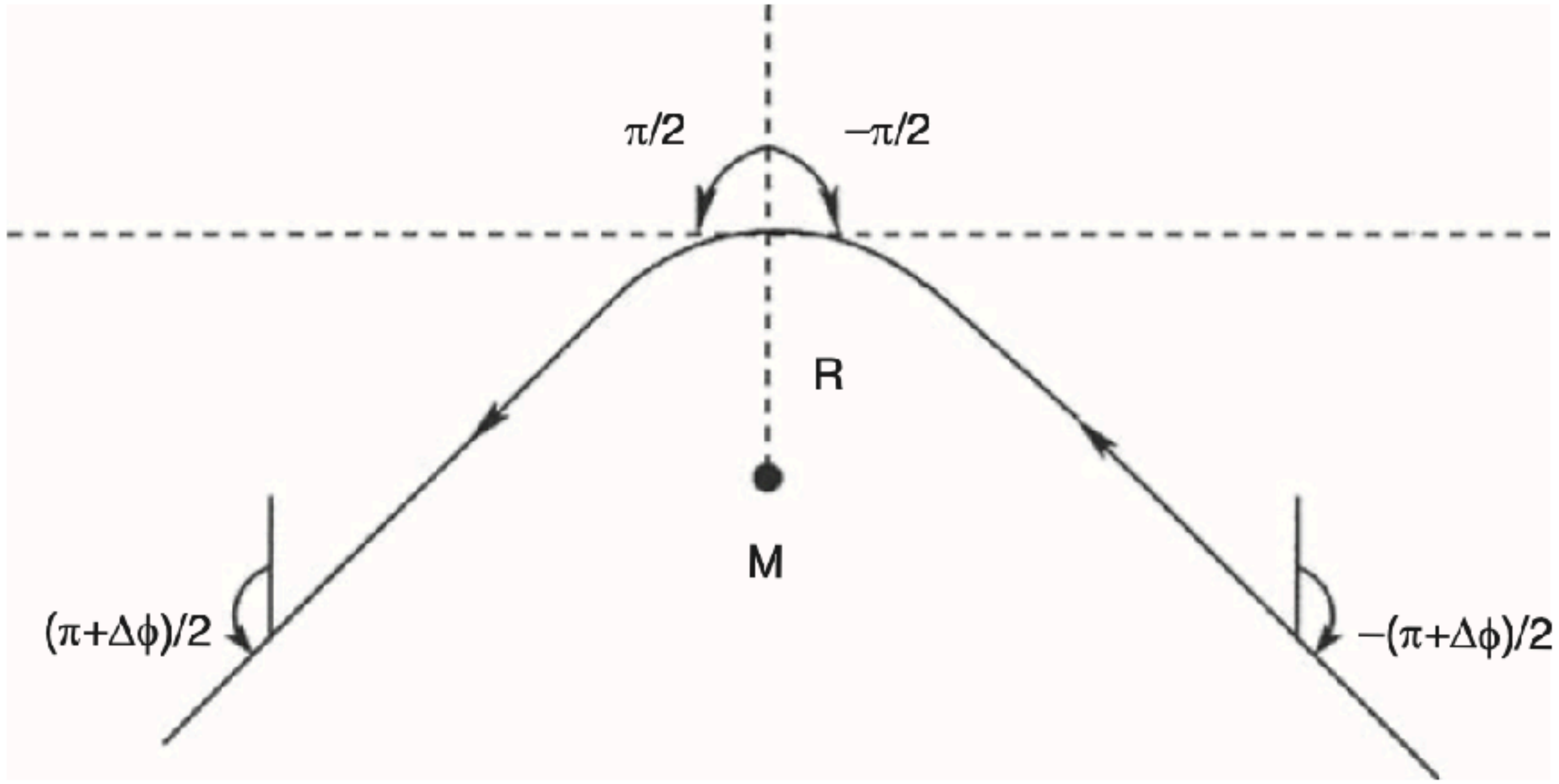


Fig. 13.4 The trajectory of light bent by an angle $\Delta\phi$ while passing by the side of the mass M. The angle ϕ is measured with respect to the vertical direction such that $\phi = 0$ at the point of closest approach.

The orbit of light

If the particle undergoes a **deflection** $\Delta\phi$, then the **incoming and outgoing directions** are

$$\phi = \mp\left(\frac{\pi}{2} + \frac{\Delta\phi}{2}\right).$$

Then we have

$$\cos\phi = -\sin\frac{\Delta\phi}{2} \approx -\frac{\Delta\phi}{2}$$

on **assuming** $\Delta\phi$ **to be small**.

When the massless particle initially starts from infinity or finally reaches infinity, we have $u \approx 0$ and $\sin\phi \approx 1$. Hence (13.49) gives

$$0 \approx -\frac{\Delta\phi}{2R} + \frac{M}{R^2}(1+1),$$

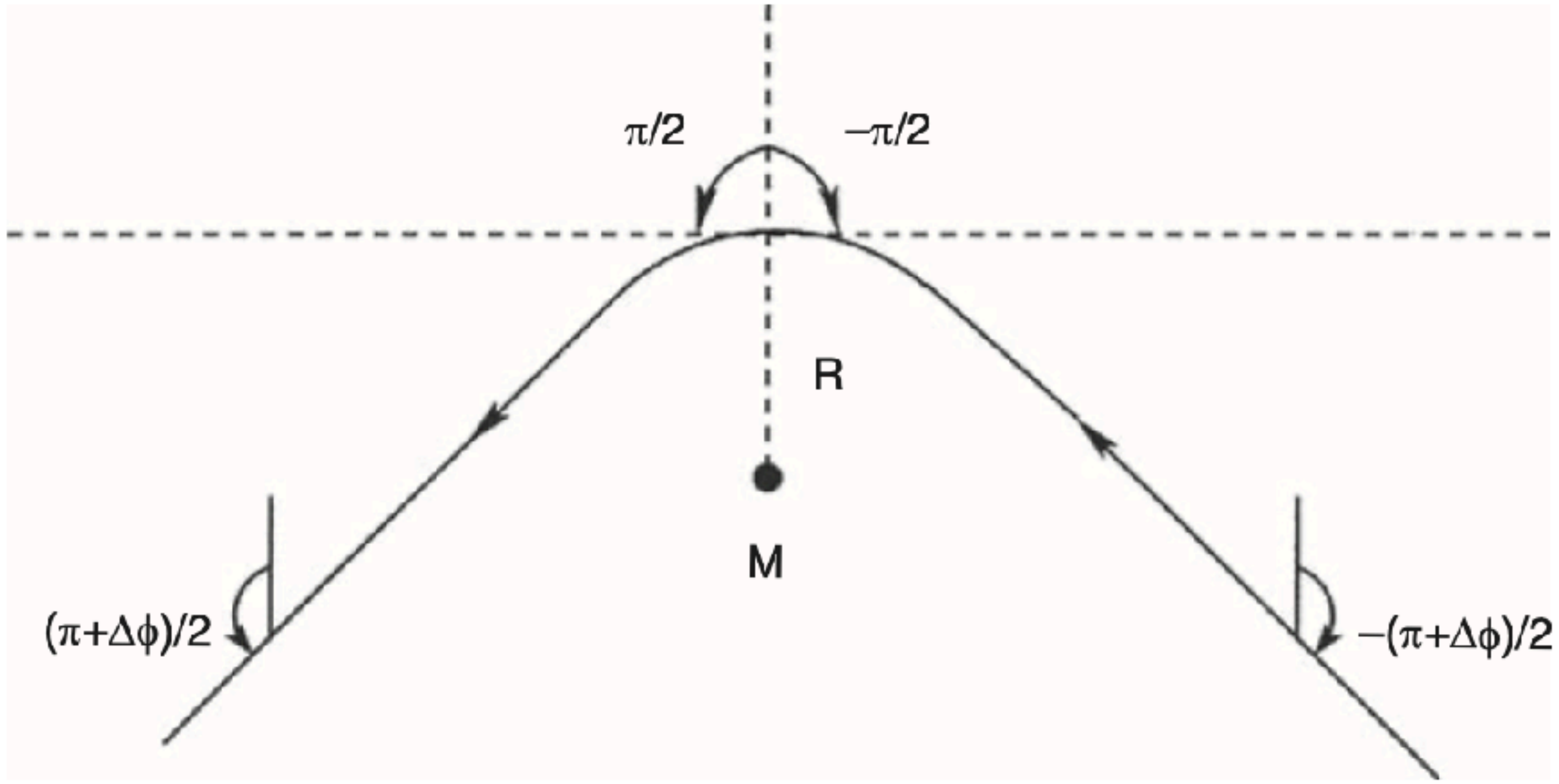


Fig. 13.4 The trajectory of light bent by an angle $\Delta\phi$ while passing by the side of the mass M. The angle ϕ is measured with respect to the vertical direction such that $\phi = 0$ at the point of closest approach.

The orbit of light

from which we finally have

$$\Delta\phi = \frac{4M}{R}.$$

On putting back G and c, this becomes

$$\boxed{\Delta\phi = \frac{4GM}{c^2 R}. \qquad (13.50)}$$

If a light ray passes by the side of a mass *M* such that the closest distance of approach is *R*, then the light ray is bent by the amount $\Delta\phi$ given by the relation (13.50).

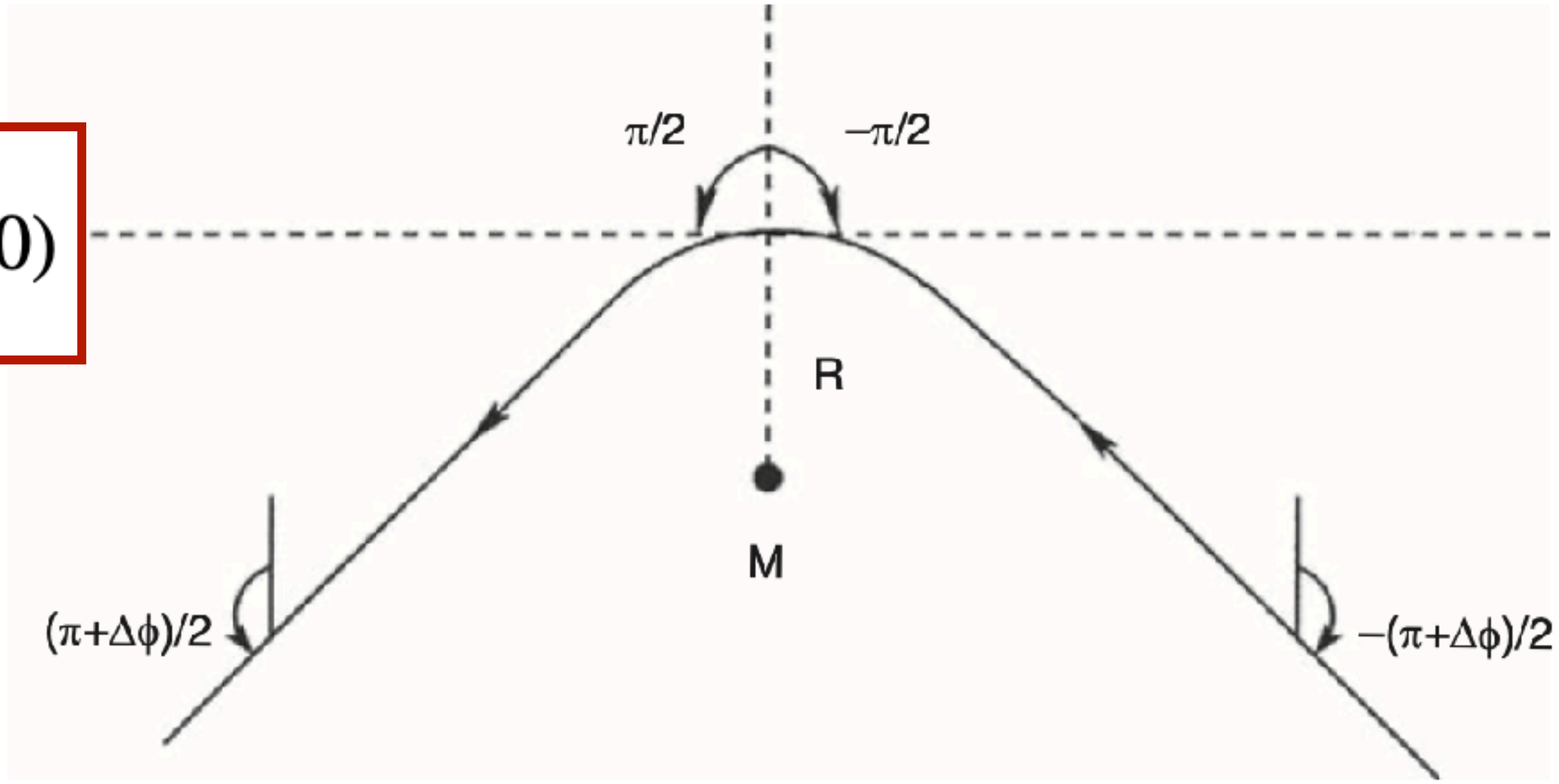
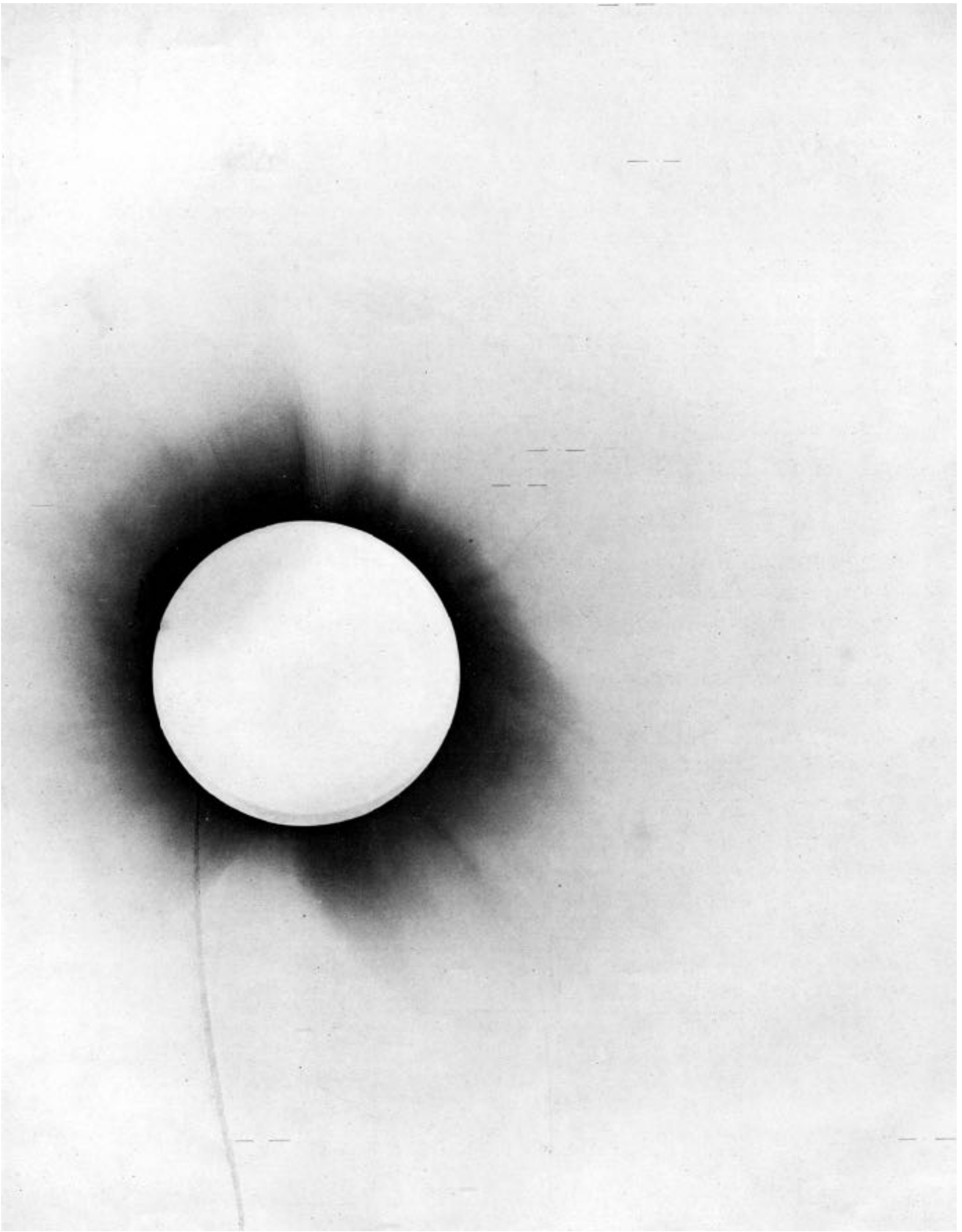


Fig. 13.4 The trajectory of light bent by an angle $\Delta\phi$ while passing by the side of the mass M. The angle ϕ is measured with respect to the vertical direction such that $\phi = 0$ at the point of closest approach.

The orbit of light

Image of the Solar eclipse in 1919 - shows the position of the stars



If we **substitute** the mass and radius of **the Sun for *M* and *R*** in (13.50), then $\Delta\phi$ **turns out to be 1.75$''$**. This means that **light from a star at the edge of the solar disk will be bent in such a way that the star will appear to be shifted outward from the centre of the solar disk by 1.75$''$**.

We of course cannot see stars at the edge of the solar disk under normal circumstances. Such stars, however, may become visible at the time of the **total solar eclipse**. Comparing their positions around the eclipsed Sun with their usual positions, one can determine whether a shift has taken place.

Eddington and his colleagues carried out this exercise **during an eclipse in 1919** and announced the **discovery of the bending of light in agreement with general relativity.**

Gravitational lensing

Important application: ***gravitational lensing.***

Suppose there is a **massive object M** exactly **between a source S and an observer O** as shown in Figure 13.5. Light rays from the source S passing by M **on different sides will be deflected by the angle $\Delta\phi$**. The observer will then see the **source S in the form of a ring called *Einstein ring***. A few examples of nearly perfect Einstein rings are known.

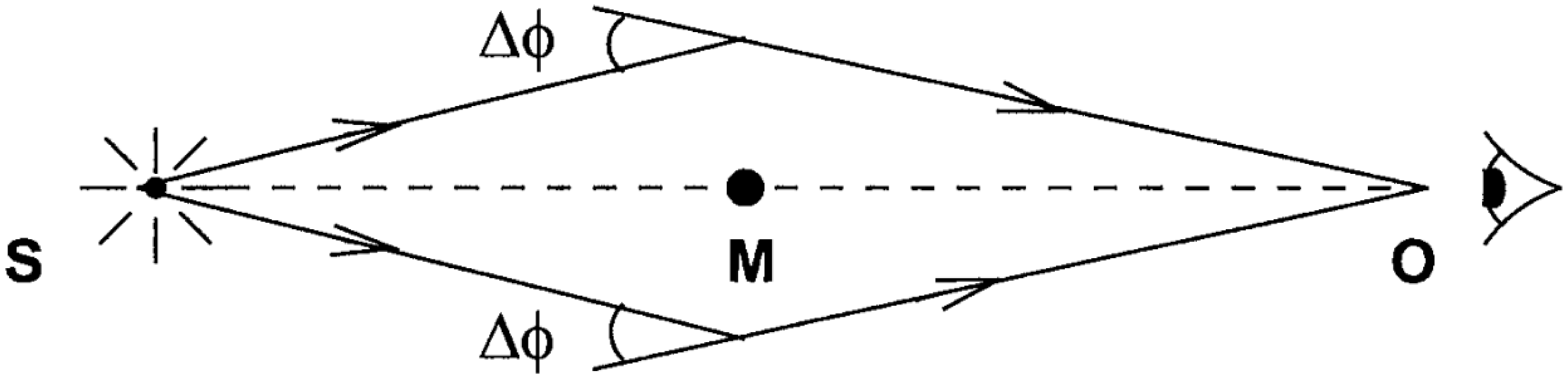
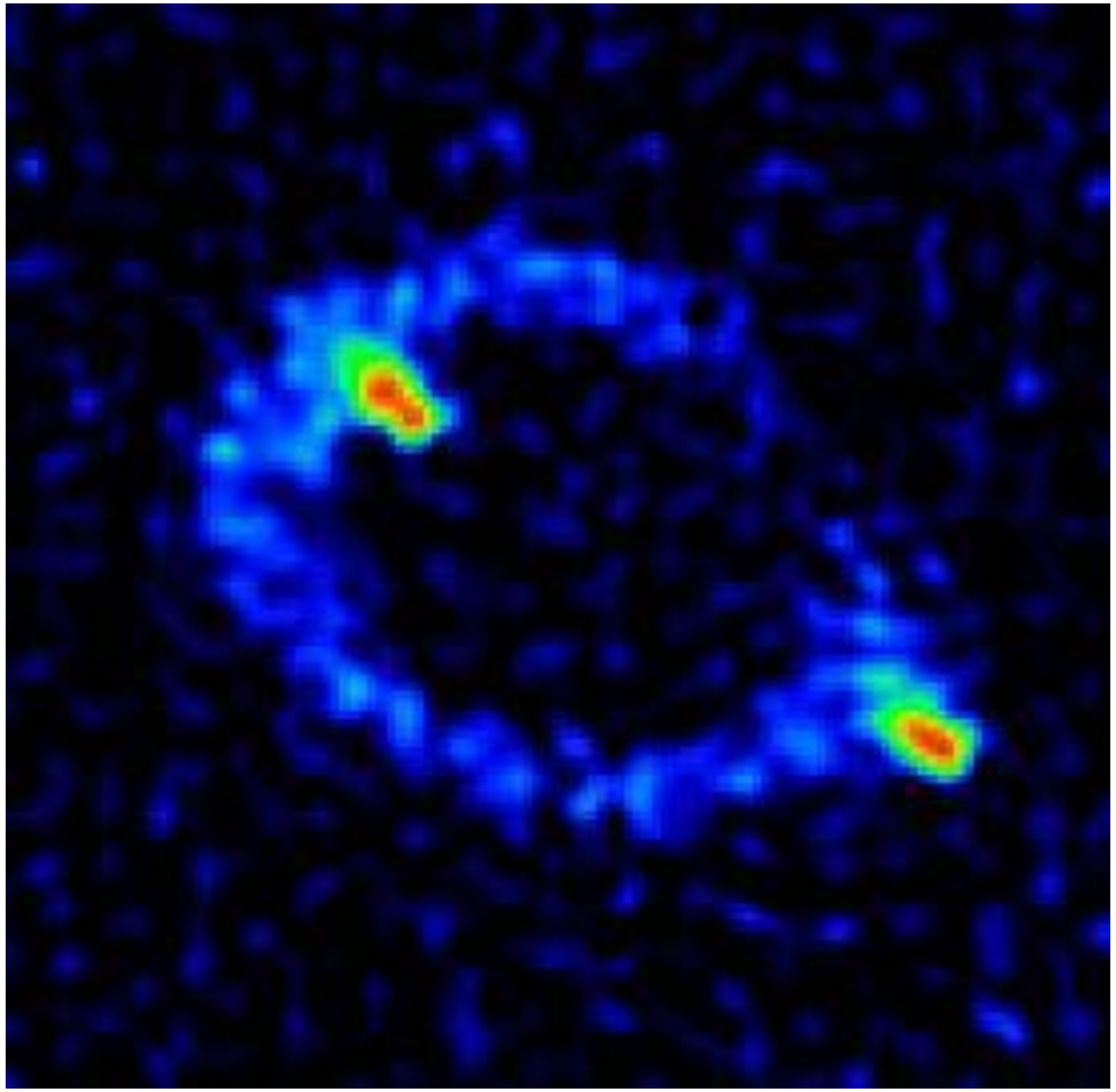


Fig. 13.5 An illustration of gravitational lensing by the mass M located symmetrically between the source S and the observer O.

Gravitational lensing

The Figure shows an almost complete Einstein ring.

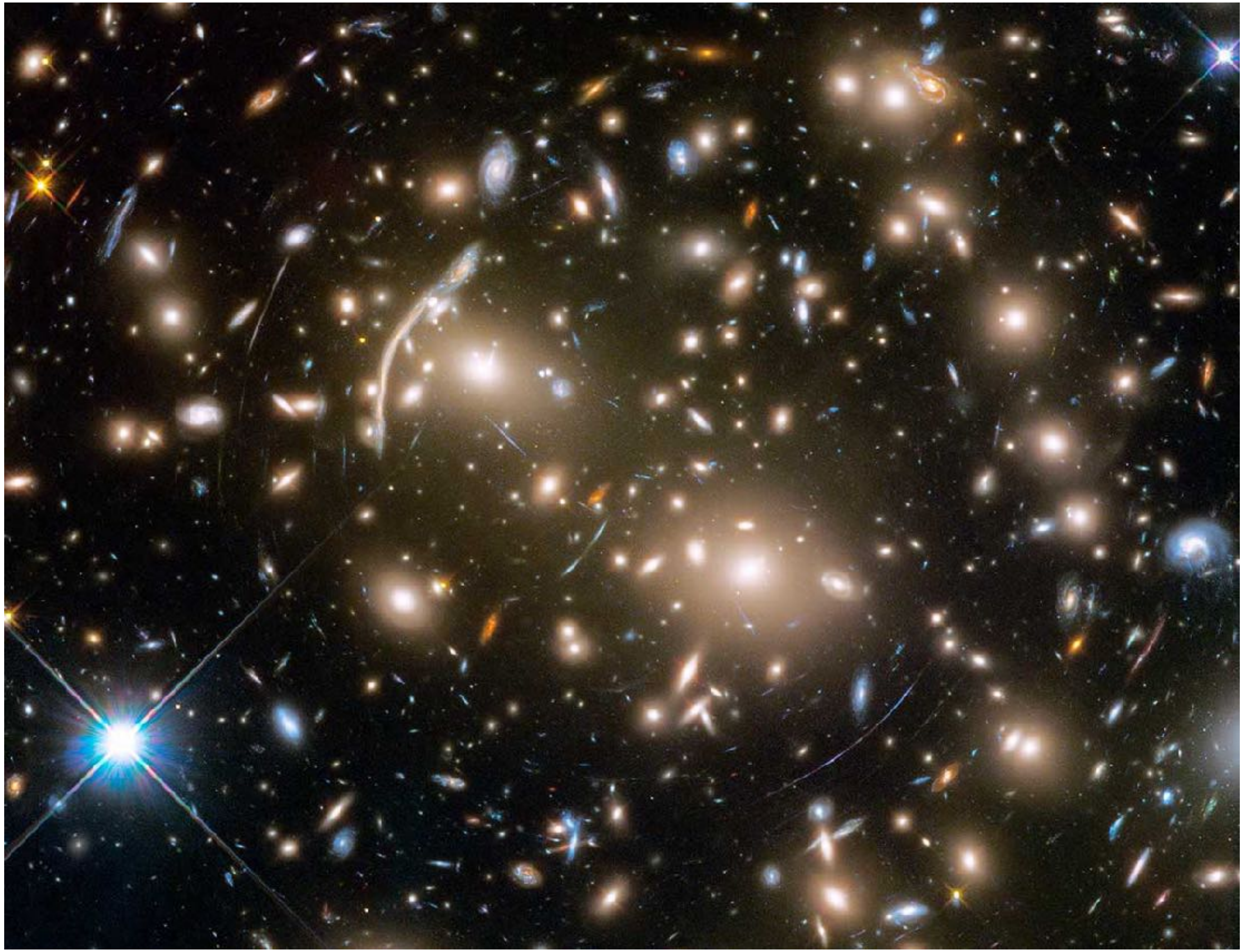
We expect to see **a perfect ring only if the lensing-producing mass M is located symmetrically on the line of sight between S and O.**



A radio image of MG 1131+0456, the first known Einstein ring observed in 1987 using the Very Large Array.

Gravitational lensing

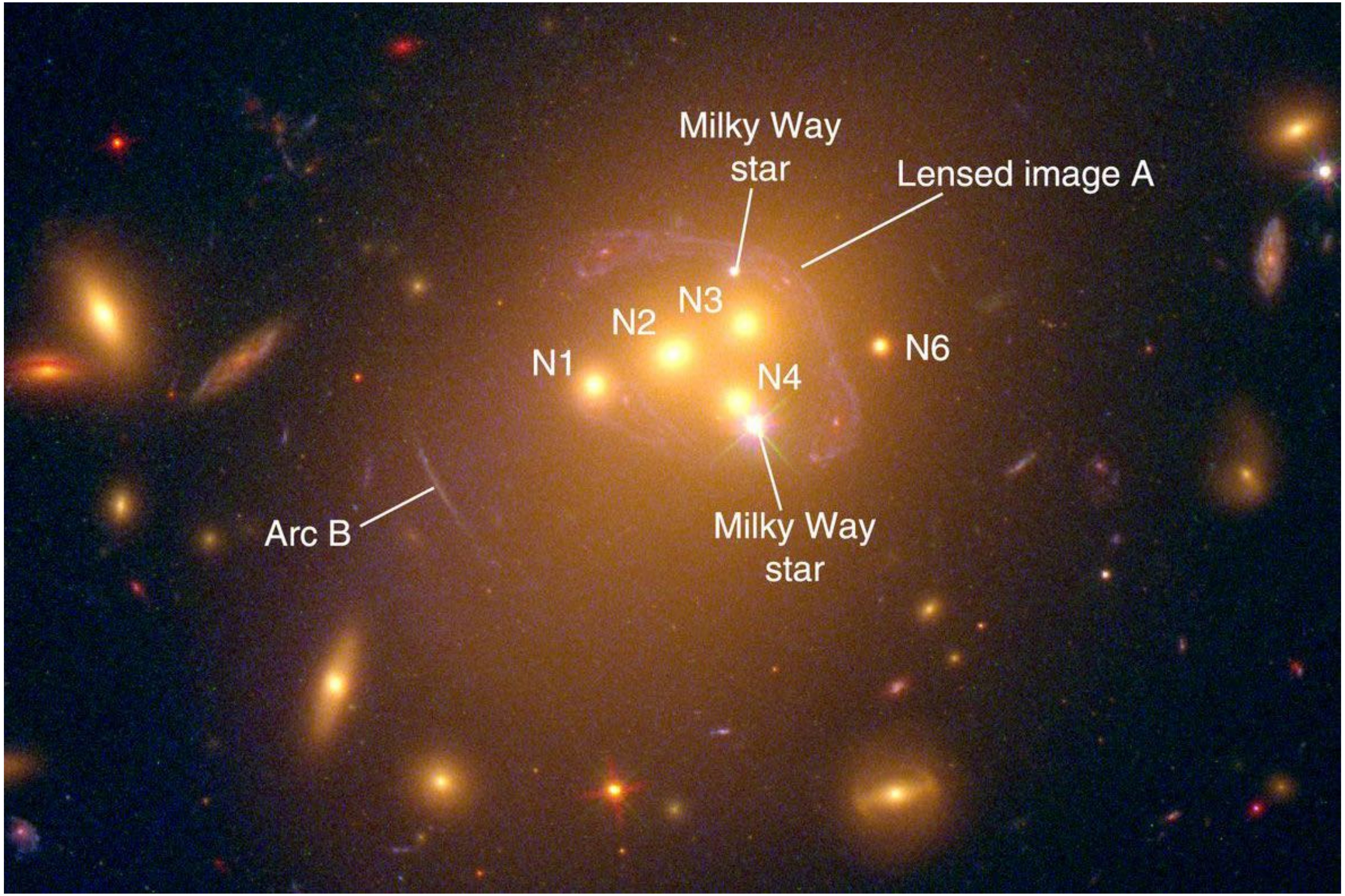
Galaxy cluster Abell 370



In a **less symmetric situation, we would see arcs of the ring rather than the whole ring.** Many images of extragalactic sources in the forms of extended arcs are known, suggesting that gravitational lensing is a **quite common phenomenon** in the extragalactic world.

Gravitational lensing

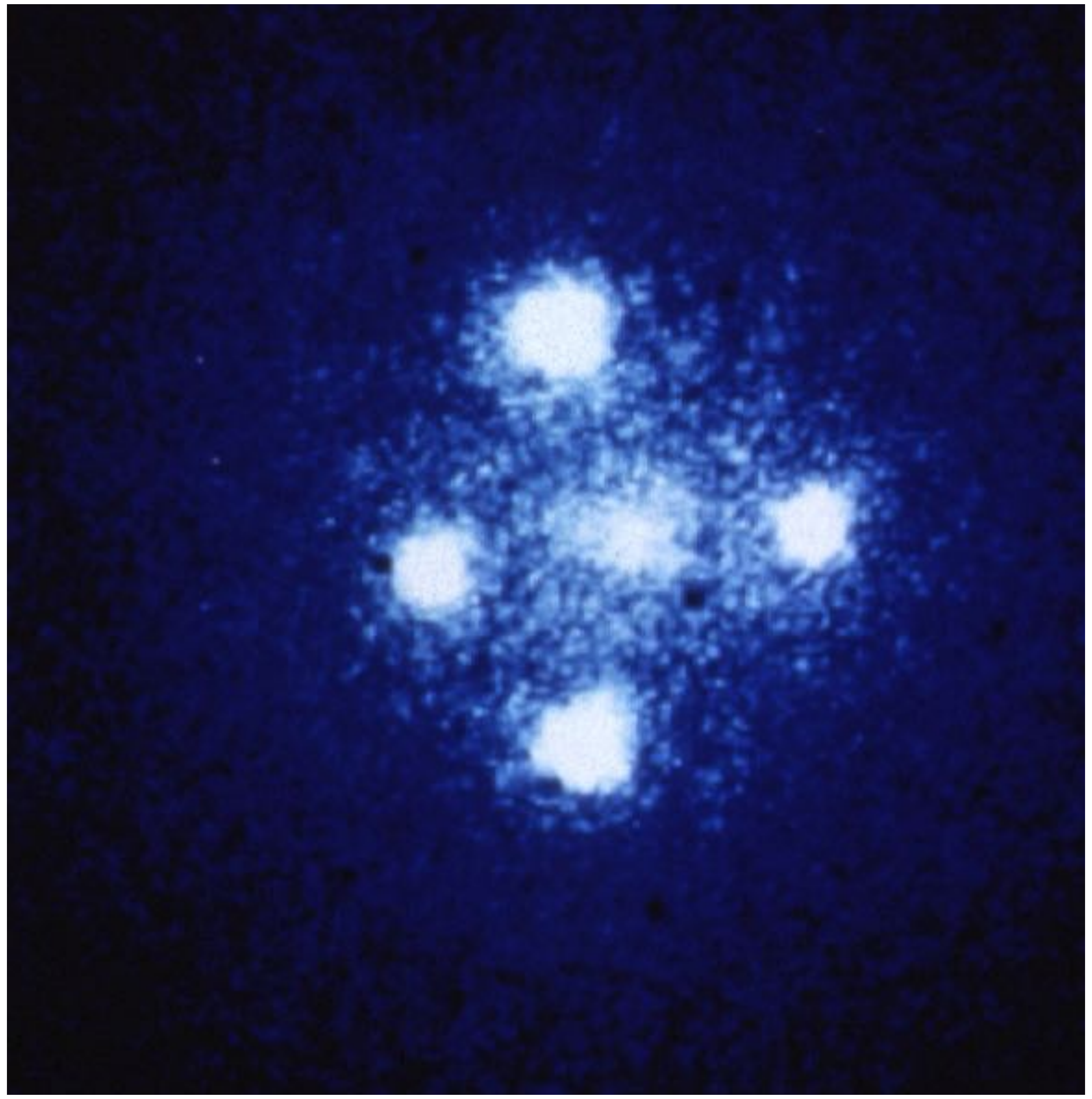
This Hubble Space Telescope image of the galaxy cluster Abell 3827 shows the ongoing collision of four bright galaxies and one faint central galaxy, as well as foreground stars in our Milky Way galaxy and galaxies behind the cluster (Arc B and Lensed image A) that are distorted because of normal and dark matter within the cluster.



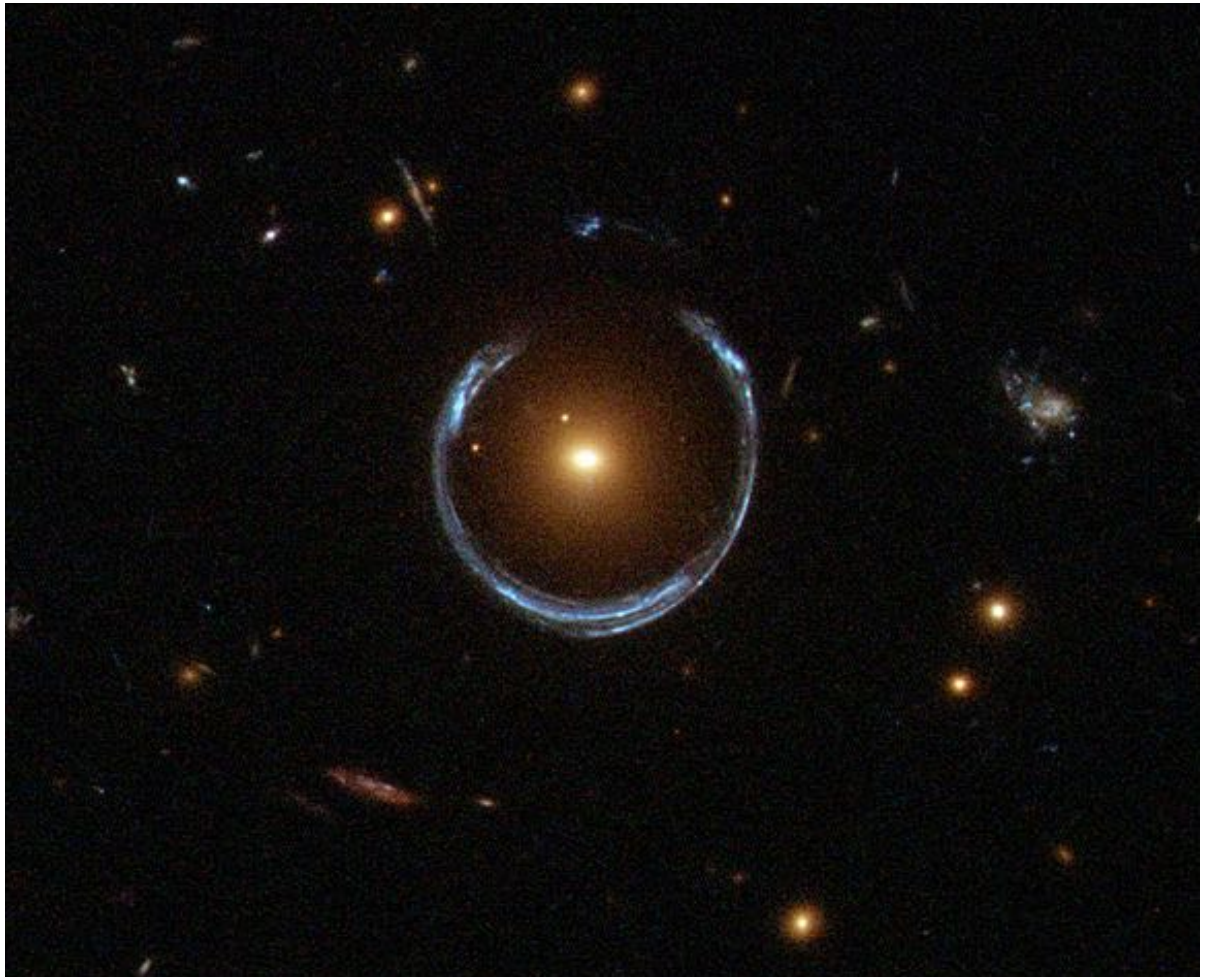
The Curvature of spacetime

More modern observations of the bending of light:
gravitational lensing

Multiple images like this are called Einstein crosses.



Rings like this are now known as Einstein Rings



The Curvature of spacetime

More modern observations of the bending of light:
gravitational lensing

https://www.eso.org/public/videos/eso1522b/

Gravitational lensing

Another kind of gravitational lensing: **The rotation curves of spiral galaxies suggest the presence of dark matter** associated with these galaxies. **One possibility** is that the dark matter exists in the form of **massive compact objects** called MACHOs (from something like a large planet to something having a few solar masses) **in the halo of the galaxy**.

Suppose one **such object** in the halo of our Galaxy **comes between us and a star in a nearby galaxy.** As **gravitational lensing would amplify the light from the star**, the star would appear brighter as long as the compact object remains between us and the star. This is called **gravitational microlensing.**

Since such an event would be very **rare** and one cannot predict when a particular star is likely to be lensed, the best way of detecting such lensing events is to **monitor a rich field of extragalactic stars for a long time** to see if the brightness of any star is temporarily enhanced. Events of this kind in which stars in the Large Magellanic Cloud temporarily appeared brighter **typically by 1 magnitude for a few days** were first reported simultaneously by two groups. From a study of such events, it seems that a part of the dark matter associated with our Galaxy, but probably not the whole of it, is in the form of **compact objects in the galactic halo.**

Gravitational lensing

A **MAssive Compact Halo Object (MACHO)** is a kind of astronomical body that might explain the apparent presence of dark matter in galaxy halos. A MACHO is a body that **emits little or no radiation and drifts through interstellar space unassociated with any planetary system** (and may or may not be composed of normal baryonic matter). Since MACHOs are **not luminous**, they are hard to detect.

MACHO candidates include:

- **black holes**
- **neutron stars**
- **brown dwarfs**
- **unassociated planets**
- **White dwarfs**
- **very faint red dwarfs**

Several groups have searched for MACHOs by searching for the microlensing amplification of light. These groups have ruled out dark matter being explained by MACHOs with mass in the range 1×10^{-8} solar masses to 100 solar masses. These **searches have ruled out the possibility that these objects make up a significant fraction of dark matter in our galaxy.** Therefore, the missing mass problem is not solved by MACHOs.