Course: Linear Algebra(Vector Spaces)

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1 Vector Spaces

So far we have seen some useful algebraic structures. Let's focus a bit on $\mathbb{C}$, which we constructed as ordered pairs of $\mathbb{R}^2$. What we defined as sum is quite simple but powerful, but also we can expand to the multiply of an ordered pair, taking c a real number, we define $c(a,b) = (ca, cb)$. This is what we think on vectors on the real plane and even we can extend this vision to $\mathbb{R}^3, \mathbb{R}^4, \ldots$. The concept we have just created is called a vector space, and we can generalize not just in terms of numbers, but in a more general way.

1.1 What is a vector space?

Definition 1.1 (Vector Space)**.** *Let V be a nonempty set, we call it a vector space over a field K if there is an operation $+$ in V and a product operation defined from $K \times V$ such that:*

- *$(V, +)$ forms an abelian group*
- *If $v \in V$ then $kv \in V$ for any $k \in K$*
- *$k(lv) = (kl)v$ for any $v \in V$ and $k, l \in K$*
- *$k(u+v) = ku + kv$ for any $u, v \in V$ and $k \in K$*
- *$(k+l)v = kv + lv$ for any $v \in V$ and $k \in K$*

The elements in V are called vectors and the elements of K are called scalars.

So, it is easy to see that thanks to the property of real numbers, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$. Also, we can see that there are many functions that forms a vector space over $\mathbb{R}$ such as the differentiable functions or the functions that are continuos over $[0,1]$. An interesting observation is that any field k forms a vector space over itself, in where the vectors and the scalars are the same.

1.2 *Subspaces and Span*

In a similar way we did to the groups and fields, we can define substructures over vector spaces, that has the original name of subspaces(Just kidding, that is not original).

Definition 1.2 (Subspace)**.** *Let V be a vector space over K. If $W \subseteq V$ also forms a vector spaces over K with the same operations, we call it a Subspace.*

Ok, and just like we did with the groups, that we characterized certain properties a subgroup needs to be so, we can do it with a subspace. Let us introduce first an important concept.

Definition 1.3 (Linear combination)**.** *Let V be a vector space over K. Let $v_1, v_2, \dots, v_n \in V$ be vectors and $k_1, k_2, \dots, k_n \in K$ be scalars, a linear combination is just:*

$$k_1v_1 + k_2v_2 \cdots + k_nv_n$$

So, what we want is to generate with the linear combinations is the subspace itself! And with a subspace, we want to maintain linear combinations on the space. So, this is literally our theorem.

Theroem 1.1. *A subset W of a vector space V over K is a subspace of V if and only if any linear combination of elements of W is in W.*

Proof. $\Rightarrow$) Suppose that W is a subspace of V. It implies that for any $k \in K$ and $v \in W$, $kv \in W$. Also, because $+$ forms an abelian group over W, it is closed over W so for any numbers of vectors and scalars $v_1, v_2, \dots, v_n, k_1, k_2, \dots, k_n$ the linear combination $k_1v_1 + k_2v_2 + \cdots + k_nv_n$ is in W.

$\Leftarrow$) Suppose that any linear combination of elements of W is in W. Then $+$ forms an abelian group taking the linear combination $1 \cdot v_1 + 1 \cdot v_2$ it is closed on W and the other properties are inherited from V. Also, take the linear combination kv_1 seeing that property 2 indeed holds. The other three properties needed for a vector space are inherited from V and therefore are valid over W, so W also forms a vector space over K.

$\square$

Some examples of subspaces are differentiable functions that are solutions to certain differential equations, subsets of $\mathbb{R}^n$, the basic subfield with the identities $\{0, 1\}$ or polynomials. Since, all vector spaces are nothing more than sets, what happens when we intersect them?

Theroem 1.2. *If V is a vector space over K and $\{W_\alpha\}_{\alpha \in A}$ is a collection of subspaces of V, then*

$$W = \bigcap_{\alpha \in A} W_\alpha$$

is also a vector space.

Proof. Take $k_1, k_2, \ldots, k_n$ be scalars over K and let $v_1, v_2, \ldots, v_n$ be elements of W. Since, for example $v_1 \in W$, then $v_1 \in W_\alpha$ for all $\alpha \in A$, and since they are subspaces, then $k_1 v_1 \in W_\alpha$ for all $\alpha \in A$, hence $k_1 v_1 \in W$. Now, again, since $v_1, v_2 \in W$ then $v_1, v_2 \in W_\alpha$ for all $\alpha in A$, because they all form abelian groups, $v_1 + v_2 \in W_\alpha$ for all $\alpha \in A$ and therefore $v_1 + v_2 \in W$.

We just apply this recursively to determine that the linear combination

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$$

is in W. And by the last theorem, W is a subspace. □

Great! So far we have seen some useful properties about subspaces. But the magic comes with the concept of $Span$.

Definition 1.4 (Span and generation)**.** *Let V be a vector space over K. If $A \subseteq V$, we call $Span(A)$ to the intersection of all subspaces of V that contain A. We say that A spans or generates V if and only $Span(A) = V$.*

If we would want to describe in formal terms $Span(A)$, then suppose that X is the set defined as:

$$X := \{W \subseteq V : A \subseteq W \wedge \text{W forms a vector space over } K\}$$

And then $\bigcap X$ is $Span(A)$ and by the second theorem we proved, it is also a vector space.For example, over $\mathbb{R}^2$ we see that

$$A = \{(x, 0) : x \in \mathbb{R}\}$$

can't generate $\mathbb{R}^2$ since for any element $(a, b) \in \mathbb{R}^2$ if $b \neq 0$, no matter what we do, the linear combination of vectors over A is just $(k_1 x_1 + k_2 x_2 + \ldots k_n x_n, 0)$. In other example, the set

$$A = \{(0, 1), (1, 0)\}$$

indeed can generate $\mathbb{R}^2$. The argument is that any linear combination of elements of A is also in $\mathbb{R}^2$ and for any element in $\mathbb{R}^2$ we can find a linear combination to create it.

1.3 *Exercises*

The next exercises are taken from Abstract Linear Algebra by Morton L. Curtis.

Problem 1. *Show that $V = \{f : \mathbb{R} \to \mathbb{R} | f \text{ is continuous}\}$ is a vector space over $\mathbb{R}$ if you define:*

$$(f+g)(t) = f(t) + g(t)$$
$$(kf)(t) = k(f(t))$$

Then prove that:

1. *Let $t_0 \in R$ and let $W = \{f \in V : f(t_0) = 0\}$, W is a subspace of V*
2. *Let $U = \{f \in V : \forall t \in \mathbb{R}, f(t^2) = (f(t))^2\}$, show that U is not a subspace of V*
3. *Let $X = \{f \in V : f \text{ is differentiable}\}$ then X is a subspace of V*

Proof. First, note that if f and g are continuous, so is $f+g$ by limit properties. So, we have a closed operation over V, and thanks to the properties of real numbers, we can assure associativity, commutativity and we define the neutral element as $e : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in \mathbb{R}$, so the inverse element is just $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = -f(x)$. Also, by limit properties, we have that $kf(t)$ is also a continuous function and they obey the other laws since the images are just real numbers, so the same properties are valid. So, V is a vector space.

1. If $k \in \mathbb{R}$ then $(kf)(t) = kf(t)$ and in special, $kf(t_0) = k \cdot 0 = 0$, so $kf(t) \in W$ if $f \in W$. If f and g are in W then $f+g$ is also a function, and in special, $(f+g)(t_0) = f(t_0) + g(t_0) = 0+0 = 0$. So, any linear combination of elements in W is also in W, and hence W is a subspace of V.
2. Suppose that f and g are functions in U. Then $f(t^2) = (f(t))^2$ and $g(t^2) = (g(t))^2$ for all t^2. So, $(f+g)(t^2) = f(t^2) + g(t^2) = (f(t))^2 + (g(t))^2$. But $(f(t)+g(t))^2 = (f(t))^2 + 2(f(t))(g(t)) + (g(t))^2$ so $f+g \notin U$ and it cannot form an abelian group so U is not a subspace of V.
3. If f and g are differentiable, then $f+g$ is also differentiable and it is just $\frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. If f is differentiable, then $kf(x)$ is differentiable and its derivate is $\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$. So, any linear

combination of elements in X is also an element of X and therefore X is a subspace of V.

□

Problem 2. *If U, W are subspaces of the vector space V, show that the sum of U and W*

$$U + W := \{u + w : u \in U \wedge w \in W\}$$

is also a subspace of V

Proof. Suppose that $u + w$ is in $U + W$ and $k \in K$, then $k(u + w) = ku + kw$ and since U, W are subspaces $ku \in U$ and $kw \in W$, $ku + kw \in U + W$. Also, if $u + w, a + b \in U + W$ then the sum $(u + w) + (a + b) = (u + a) + (w + b)$ and since they are subspaces, $u + a \in U$ and $w + b \in W$, so $(u + w) + (a + b) \in U + W$. So any linear combination of elements in $U + W$ is also in $U + W$ so $U + W$ is also a subspace. □

Problem 3. *If U and W are subspaces of V, show that $U \cup W$ need not be a subspace. However, if $U \cup W$ is a subspace, show that either $U \subseteq W$ or $W \subseteq U$.*

Proof. Since U and W are subspaces, it would be easy to say that $ku \in U$ and $lw \in W$, but the operation $+$ is just closed on U and W alone, so $ku + lw$ might not be in some of them, even when it is indeed on V. So, if it happens, $U \cup W$ is not a subspace.

Suppose that $U \not\subseteq W$ and $W \not\subseteq U$, then there are elements u, w such that $u \notin W$ and $w \notin U$. Now, suppose that $U \cup W$ is a vector space, then since $u, w \in U \cup W$, $u + w \in U \cup W$. But it implies that $u + w \in U$ or $u + w \in W$, in both cases we would conclude that $u \in W$ or $w \in U$ which is a contradiction, so $U \cup W$ cannot be a vector space. □

Problem 4. *Suppose A and B are subsets of the vector space V; Show that if $A \subseteq B$ then $Span(A) \subseteq Span(B)$*

Proof. Suppose that $w \in Span(A)$, so for any vector space W, if that space contains A, $w \in W$. For all vector space U such that $B \subseteq W$,

$A \subseteq W$ and so $w \in U$. Since it applies for all vector spaces that contains B, then $w \in Span(B)$ and therefore $Span(A) \subseteq Span(B)$. $\square$

Problem 5. *Consider 2×2 squares arrays of real numbers. We denote the set of them as:*

$$M_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

We make $M_2(\mathbb{R})$ a vector space defining:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & f \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+f \end{pmatrix} \quad r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

1. *A matriz $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is diagonal if $b = c = 0$. Show that the set of all diagonal matrices D is a subspace of $M_2(\mathbb{R})$. Do the same for the set T of upper triangular matrices($c = 0$).*
2. *A matriz $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is singular if $ad - bc = 0$ and otherwise is not singular. Prove that the set of all singular matrices in $M_2(\mathbb{R})$ does not form a subspace, and do the same for the set of all not singular matrices.*

Proof. First, if you want to show that the set $M_2(\mathbb{R})$ indeed forms a vector space, it is easy to see that it forms an abelian group under $+$ just because $\mathbb{R}$ do it. By definition, for a matrix A and a scalar r, rA is a matrix, and the prove for the other properties is obvious.

1. First, if A is a diagonal matrix, then $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ so if you multiply it by an scalar r it will be:

$$\begin{aligned} rA &= \begin{pmatrix} ra & r0 \\ r0 & rb \end{pmatrix} \\ &= \begin{pmatrix} ra & 0 \\ 0 & rb \end{pmatrix} \end{aligned}$$

so rA is also a diagonal matrix. If A and B are diagonal matrices, such that $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ then its sum is:

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} &= \begin{pmatrix} a+c & 0+0 \\ 0+0 & b+d \end{pmatrix} \\ &= \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} \end{aligned}$$

so $A + B$ is also a diagonal matrix. Now, any linear combination of diagonal matrices is also a diagonal matrix, so D is a subspace of $M_2(\mathbb{R})$. The proof for T is exactly the same, so T is also a subspace of $M_2(\mathbb{R})$.

2. First, suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc = 0$, with $a \neq 0$ and $a \neq b$. Now, the matrix $A' = \begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix}$ is also a singular matrix. So, if you add the two matrices:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix} &= \begin{pmatrix} a+0 & b+0 \\ c+a & d+a \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c+a & d+a \end{pmatrix} \end{aligned}$$

And if you verify for the singularity of $A + A'$ you would have:

$$\begin{aligned} a(d+a) - b(c+a) &= ad + a^2 - bc - ba \\ &= ad - bc + a^2 - ba \\ &= a^2 - ba \\ &= a(a-b) \end{aligned}$$

And since neither $a = 0$ or $a - b = 0$, then the matrix is not singular, and the sum of matrices is not closed on this set. Therefore it cannot be a subspace. For the no singular matrices, since the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is singular, it cannot be in the set, but it is the identity element for the sum of matrices, so it cannot be a vector space since it cannot form an abelian group.

$\square$