Final: CMPE 107 Spring 2017

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Question/ Problem	Points Earned	Points Available
M/M/1 and M/M/1/K Queuing System:		10
Generalized Birth-Death System		10
Discouraged Arrivals		20
Slotted ALOHA:		20
Slotted non-persistent CSMA		20
Central Limit Theorem:		10
Total		90
Extra Credit: Non-persistent CSMA		15

Contents

1 M/M/1 and M/M/1/K Queuing System

Section Grade: (/10 Points)

Question/Problem:

(a) Compute the variance of the number of users in the M/M/1 system

(b) For the M/M/1/K system show that for $\rho < 1$:

$$\pi_n = \frac{\rho^n(1-\rho)}{1-\rho^{K+1}} \qquad n = 0, 1, 2, \ldots, K-1 \qquad \rho = \frac{\lambda}{\mu}$$
$$\bar{N} = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} \tag{1}$$

Answer/Solution:

(a) The probability density function for the M/M/1 system is defined as $P_K = \rho^K(1-\rho)$ for $K = 0, 1, 2, \ldots$. The mean number of users, $\bar{N}$ in the system is $\bar{N} = E[N] = \sum_{K=0}^{\infty} K P_K$. We replace P_K by $\rho^K(1-\rho)$ and we get:

$$\begin{aligned} \bar{N} &= \sum_{K=1}^{\infty} K\rho^K(1-\rho) = \rho \sum_{K=1}^{\infty} K\rho^{K-1}(1-\rho) \\ &= \rho(1-\rho)\sum_{K=1}^{\infty} K\rho^{K-1} = \rho(1-\rho)\sum_{k=1}^{\infty} \frac{d\rho^K}{d\rho} \\ &= \rho(1-\rho)\frac{d}{d\rho}(\frac{1}{1-\rho}) = \frac{\rho}{1-\rho} \end{aligned} \tag{2}$$

Therefore, the variance of the number of users is going to be:

$$\begin{aligned} Var[N] = E[N^2] - E^2[N] &= \sum_{K=0}^{\infty}(K-\bar{N})^2 P_K \\ &= \sum_{K=0}^{\infty}(K - \frac{\rho}{1-\rho})^2 P_K \end{aligned}$$

$$
\begin{aligned}
&= \sum_{K=0}^{\infty} KP_K + \left(\frac{\rho}{1-\rho}\right)^2 - \frac{2\rho}{1-\rho}\sum_{K=1}^{\infty} KP_K \\
&= \sum_{K=0}^{\infty} K(K-1)P_K + \frac{\rho}{1-\rho} + \left(\frac{\rho}{1-\rho}\right)^2 - 2\left(\frac{\rho}{1-\rho}\right)^2 \\
&= (1-\rho)\rho^2\frac{d^2}{d\rho^2}\sum_{K=0}^{\infty}\rho^K + \frac{\rho}{1-\rho} - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{1-\rho} + \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{\rho}{(1-\rho)^2}
\end{aligned}
$$

(b) The steady state distribution of the M/M/1/K queuing model is: average rate out of n = average rate into n. For $n = 0, 1, 2$:

$$\lambda\pi_0 = \mu\pi_1 \tag{3}$$

$$\lambda\pi_1 + \mu\pi_1 = \lambda\pi_0 + \mu\pi_2 \quad \Longrightarrow \quad (\lambda+\mu)\pi_1 = \lambda\pi_0 + \mu\pi_2 \tag{4}$$

$$\lambda\pi_2 + \mu\pi_2 = \lambda\pi_1 + \mu\pi_3 \quad \Longrightarrow \quad (\lambda+\mu)\pi_2 = \lambda\pi_1 + \mu\pi_3 \tag{5}$$

For $n = K$ the system is:

$$\mu\pi_K = \lambda\pi_{K-1} \tag{6}$$

We replace $\lambda\pi_0$ by $\mu\pi_1$ and we rearrange Equation (4):

$$(\lambda+\mu)\pi_1 - \mu\pi_1 = \mu\pi_2 \quad \Longrightarrow \quad \lambda\pi_1 = \mu\pi_2 \tag{7}$$

We replace $\lambda\pi_1$ by $\mu\pi_2$ in Equation (4) and rearrange:

$$(\lambda+\mu)\pi_2 - \mu\pi_2 = \mu\pi_3 \quad \Longrightarrow \quad \lambda\pi_2 = \mu\pi_3 \tag{8}$$

We write all equations in terms of π_0:

$$
\begin{aligned}
\pi_1 &= \frac{\lambda}{\mu}\pi_0 \\
\pi_2 &= \left(\frac{\lambda}{\mu}\right)^2 \pi_0 \\
\pi_3 &= \left(\frac{\lambda}{\mu}\right)^3 \pi_0 \\
\pi_K &= \left(\frac{\lambda}{\mu}\right)^K \pi_0
\end{aligned} \tag{9}
$$

Assuming steady state balance distribution, we compute π_0:

$$
\sum_{n=0}^{\infty} = 1 \qquad \Longrightarrow \qquad \pi_0 + \pi_1 + \pi_2 + \ldots + \pi_n = 1 \tag{10}
$$

We replace the expressions in terms of π_0 we obtained earlier:

$$
\begin{aligned}
&\pi_0 + \frac{\lambda}{\mu}\pi_0 + \left(\frac{\lambda}{\mu}\right)^2 \pi_0 + \left(\frac{\lambda}{\mu}\right)^3 \pi_0 + \ldots + \left(\frac{\lambda}{\mu}\right)^K \pi_0 = 1 \\
&\pi_0 \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \ldots + \left(\frac{\lambda}{\mu}\right)^K\right) = 1 \\
&\pi_0 = \left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \ldots + \left(\frac{\lambda}{\mu}\right)^K\right]^{-1} \\
&\pi_0 = \left(\sum_{n=0}^{K} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1}
\end{aligned} \tag{11}
$$

Thus we have, by considering the value of the sum above:

$$
\begin{aligned}
\pi_n &= \left(\frac{\lambda}{\mu}\right)^n \pi_0 \\
&= \left(\frac{\lambda}{\mu}\right)^n \left(\sum_{n=0}^{K} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} \\
&= \frac{\rho^n(1-\rho)}{1-\rho^{K+1}}, \quad \text{with} \quad \rho = \frac{\lambda}{\mu}
\end{aligned} \tag{12}
$$

Furthermore we compute:

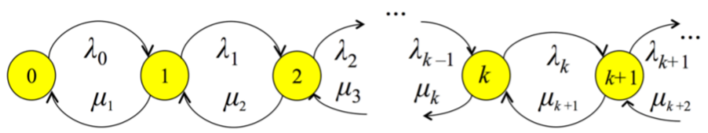
$$
\begin{aligned}
\bar{N} = E[N] &= \sum_{K=1}^{n} K\pi_K\pi_0 \\
&= \rho\pi_0 \sum_{K=1}^{n} K\rho^{K-1} \\
&= \rho\pi_0 \frac{d}{dK}\left(\rho\frac{1-\rho^K}{1-\rho}\right) \\
&= \left[(1-(K+1)\rho^K)(1-\rho)+\rho-\rho^{K+1}\right]\left(\frac{\rho\pi_0}{(1-\rho)^2}\right) \\
&= \frac{\rho\pi_0\left(1-(K+1)\rho^K+K\rho^{K+1}\right)}{(1-\rho)(1-\rho)} \\
&= \left(\frac{\rho}{1-\rho}\right)\left(\frac{1-(K+1)\rho^K+(K+1)\rho^{K+1}-\rho^{K+1}}{1-\rho^{K+1}}\right) \\
&= \left(\frac{\rho}{1-\rho}\right)\left(\frac{1+[(K+1)\rho^K+(K+1)\rho^{K+1}-\rho^{K+1}}{1-\rho^{K+1}}\right) \\
&= \left(\frac{\rho}{1-\rho}\right)\left(1+\frac{(K+1)\rho^{K+1}-(K+1)\rho^K}{1-\rho^{K+1}}\right) \\
&= \frac{\rho}{1-\rho}-\frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}
\end{aligned}
\tag{13}
$$

2 Generalized Birth-Death System

Section Grade: (/10 Points)

Question/Problem:

Consider the following birth-death system in which the arrival and departure rates depend on the number of elements in the system:



(a) Use the iteration-and-induction approach covered in class to derive an expression of the state probability for an arbitrary state n as a function of the transition probabilities and the state probability for state 0.

(b) Use the fact that the system must be in some state to derive an expression for the state probability for state 0 as a function of transition probabilities and provide an expression of the state probability for state n as a function of transition probabilities.

Answer/Solution:

(a) We know that the average rate going out of state n = average rate going into state n. We thus have:

$$\lambda_0\pi_0 = \mu_1\pi_1 \tag{14}$$

$$\lambda_1\pi_1 + \mu_1\pi_1 = \lambda_0\pi_0 + \mu_2\pi_2 \tag{15}$$

$$\lambda_2\pi_2 + \mu_2\pi_2 = \lambda_1\pi_1 + \mu_3\pi_3 \tag{16}$$

Rearranging Equation (15) and replacing π_1 by using Equation (14) we get:

$$\pi_2 = \frac{(\lambda_1 + \mu_1)\pi_1 - \lambda_0\pi_0}{\mu_2} = \frac{(\lambda_1 + \mu_1)\pi_1 - \mu_1\pi_1}{\mu_2} = \frac{\lambda_1}{\mu_2}\pi_1 \tag{17}$$

This suggests that for $n \geq 1$ we might have:

$$\pi_n = \frac{\lambda_{n-1}}{\mu_n}\pi_{n-1} \tag{18}$$

Assuming this is true, we have:

$$\pi_{n+1} = \frac{(\lambda_n + \mu_n)\pi_n - \lambda_{n-1}\pi_{n-1}}{\mu_{n+1}} = \frac{(\lambda_n + \mu_n)\pi_n - \mu_n\pi_n}{\mu_{n+1}} = \frac{\lambda_n}{\mu_{n+1}\pi_n} \tag{19}$$

Now we have to rearrange Equations (14), (15) and (16) in order to write them in terms of π_0 so that we can compute it later.

$$\lambda_1 = \frac{\lambda_0}{\mu_1}\pi_0 \tag{20}$$

$$\begin{aligned} (\lambda_1 + \mu_1)\pi_1 = \lambda_0\pi_0 + \mu_2\pi_2 \quad &\implies \quad \frac{(\lambda_1 + \mu_1)\pi_1 - \mu_1\pi_1}{\mu_2} = \pi_2 \\ \implies \frac{\lambda_1\pi_1}{\mu_2} = \pi_2 \quad &\implies \quad \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0 = \pi_2 \end{aligned} \tag{21}$$

$$\begin{aligned} (\lambda_2 + \mu_2)\pi_2 = \lambda_1\pi_1 + \mu_2\pi_2 \quad &\implies \quad \frac{(\lambda_2 + \mu_2)\pi_2 - \mu_2\pi_2}{\mu_3} = \pi_3 \\ \implies \frac{\lambda_2\pi_2}{\mu_3} = \pi_3 \quad &\implies \quad \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}\pi_0 = \pi_3 \end{aligned} \tag{22}$$

With this result we can assume:

$$\pi_n = \frac{\prod_{i=0}^{n-1}\lambda_i}{\prod_{i=1}^{n}\mu_i}\pi_0 \tag{23}$$

(b) Assuming that the birth and death rates are such that we already have well-defined steady state probabilities, $\sum_{n=0}^{\infty}\pi_n = 1$, so:

$$\sum_{n=0}^{\infty}\left(\frac{\prod_{i=0}^{n-1}\lambda_i}{\prod_{i=1}^{n}\mu_i}\pi_0\right) = 1 \quad \implies \quad \pi_0 = \left(\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{n-1}\lambda_i}{\prod_{i=1}^{n}\mu_i}\right)^{-1} \tag{24}$$

Therefore, in a more general sense, we can replace π_0 in Equation (23) and obtain:

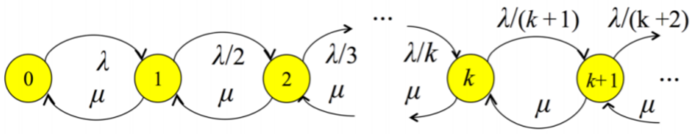
$$\pi_n = \left(\prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} \right) \left(\sum_{n=0}^{\infty} \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right)^{-1} \tag{25}$$

3 Discouraged Arrivals

Section Grade: (/20 Points)

Question/Problem:

Consider the queuing system given by the following state-transition diagram. New arrivals are discouraged as the number of customers in the system increases, but the departure rate remains constant.



(a) Use the iteration-and-induction approach covered in class to derive an expression of the state probability for an arbitrary state n as a function of the transition probabilities and the state probability for state 0.

(b) Use the fact that the system must be in some state to derive an expression for the state probability for state 0 as a function of transition probabilities and show that it equals $\pi_0 = e^{-\lambda/\mu}$

(c) Show that the state probability for state n equals $\pi_k = \frac{(\lambda/\mu)^k}{k!}e^{-\lambda/\mu}$ for $k = 0, 1, 2, \ldots$

(d) Obtain the average number of customers in the system and the average time a customer spends in the system.

Answer/Solution:

(a) We have the following:

$$\lambda\pi_0 = \mu\pi_1 \tag{26}$$

$$\frac{\lambda}{2}\pi_1 + \mu\pi_1 = \lambda\pi_0 + \mu\pi_2 \quad \Longrightarrow \quad \frac{\lambda}{2}\pi_1 = \mu\pi_2 \tag{27}$$

$$\frac{\lambda}{3}\pi_2 + \mu\pi_2 = \frac{\lambda}{2}\pi_1 + \mu\pi_3 \quad \Longrightarrow \quad \frac{\lambda}{3}\pi_2 = \mu\pi_3 \tag{28}$$

$$\frac{\lambda}{4}\pi_3 + \mu\pi_3 = \frac{\lambda}{3}\pi_2 + \mu\pi_4 \quad \implies \quad \frac{\lambda}{4}\pi_3 = \mu\pi_4 \tag{29}$$

We rearrange the expressions in terms of π_0:

$$\pi_1 = \frac{\lambda}{\mu}\pi_0 \tag{30}$$

$$\pi_2 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 \pi_0 \tag{31}$$

$$\pi_3 = \frac{1}{6}\left(\frac{\lambda}{\mu}\right)^3 \pi_0 \tag{32}$$

$$\pi_2 = \frac{1}{24}\left(\frac{\lambda}{\mu}\right)^4 \pi_0 \tag{33}$$

This in turn suggests the form:

$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 \tag{34}$$

(b) Assuming that we have well defined steady state probabilities, $\sum_{n=0}^{\infty}$, we have:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 = 1 \\ &\implies \pi_0 = \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} \\ &\implies \pi_0 = e^{-(\lambda/\mu)} \end{aligned} \tag{35}$$

(c) Gathering our previous results we have:

$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} \tag{36}$$

Considering the expression of the Taylor series of the exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we can rewrite the equation for π_n as:

$$\begin{aligned} \pi_n &= \frac{(\lambda/\mu)^n}{n!} \left(e^(\lambda/\mu)\right)^{-1} \\ \implies \pi_n &= \frac{(\lambda/\mu)^n}{n!} e^{-\lambda/\mu} \end{aligned} \tag{37}$$

(d) The probability density function for this queuing system is defined as $P_K = \pi_n$ for $K = 0, 1, 2, \ldots$. The number of users $\bar{N}$ in the system is then:

$$\begin{aligned} \bar{N} = E[N] &= \sum_{K=0}^{\infty} K P_K = \sum_{K=0}^{\infty} K \pi_K \\ &= \sum_{K=0}^{\infty} K \frac{(\lambda/\mu)^K}{k!} e^{-(\lambda/\mu)} \\ &= e^{-(\lambda/\mu)} \sum_{K=0}^{\infty} K \frac{(\lambda/\mu)^K}{K!} \\ &= \frac{\lambda}{\mu} e^{-(\lambda/\mu)} \sum_{K=0}^{\infty} \frac{(\lambda/\mu)^{K-1}}{(K-1)!} \\ &= e^{-(\lambda/\mu)} \frac{\lambda}{\mu} e^{(\lambda/\mu)} \\ &= \frac{\lambda}{\mu} \end{aligned} \tag{38}$$

To calculate the average time a customer spends in the system, we need $E[T]$:

$$E[T] = \bar{T} = \frac{\bar{N}}{\lambda} = \frac{\lambda/\mu}{\lambda} = \frac{1}{\mu} \tag{39}$$

The average time a customer spends in the system is thus $1/\mu$

4 Slotted ALOHA

Section Grade: (/20 Points)

Question/Problem:

Consider the slotted ALOHA protocol. The channel is time-slotted with a time slot lasting a packet time T. Use the same assumptions we made for the discussion of ALOHA in class, namely: Poisson arrivals with parameter λ, fixed packet lengths of duration T seconds, a fully connected network with all nodes having the same propagation delay from each other. Show that the throughput of slotted ALOHA is Ge^{-G} using a three-state Markov chain as we discussed in class for pure ALOHA. See set 13 of lecture notes for details on the transmission periods in slotted ALOHA.

Answer/Solution:

We know that there are going to be three states: π_0, corresponding to idle time (no parcels, failure), π_1, corresponding to a busy time (one parcel, success), and π_2 corresponding to a collision (failure). We have the following properties:

$$\begin{aligned} &\pi_0 + \pi_1 + \pi_2 = 1 \\ &\text{Poisson:} \quad P(K\text{in T seconds}) = \frac{(\lambda T)^K}{K!}e^{-\lambda T} \\ &P_{i0} + P_{i1} + P_{i2} = 1 \quad \text{for} \quad i = 0, 1, 2 \end{aligned} \tag{40}$$

Furthermore, since we have T_1, TP1 is always successful because we are considering the slotted case. This means that in the expression for the throughput (time spent in success state divided by total time) we have $P_s = 1$, and thus:

$$S = \frac{\pi_1 P_s T_1}{\sum_{i=1}^{2} \pi_i T_i} = \frac{\pi_1 T_1}{\pi_0 T_1 + \pi_1 T_1 + \pi_2 T_1} = \frac{\pi_1 T_1}{(\pi_0 + \pi_1 + \pi_2) T_1} = \pi_1 \tag{41}$$

We write the balance equations for each state:

$$\begin{aligned} \pi_0(P_{01} + P_{02}) &= \pi_1 P_{10} + \pi_2 P_{20} \\ \pi_1(P_{10} + P_{12}) &= \pi_0 P_{01} + \pi_2 P_{21} \\ \pi_2(P_{20} + P_{21}) &= \pi_0 P_{02} + \pi_1 P_{12} \end{aligned} \tag{42}$$

We derive the transition probabilities from the Poisson equation, which gives us, for $i = 0, 1, 2$ and with $G = \lambda T$:

$$
\begin{aligned}
&P\{0 \quad \text{arrivals in} \quad T\} = P_{i0} = e^{-\lambda T} = e^{-G} \\
&P\{1 \quad \text{arrival in} \quad T\} = P_{i1} = \lambda T e^{-\lambda T} = G e^{-G} \\
&P\{> 1 \quad \text{arrivals in} \quad T\} = P_{i2} = 1 - (e^{-\lambda T} + \lambda T e^{-\lambda T}) = 1 - (e^{-G} + G e^{-G})
\end{aligned} \tag{43}
$$

We replace these quantities in the balanced equations and we get:

$$
\begin{aligned}
&\pi_0 \left(G e^{-G} + 1 - \left(e^{-G} + G e^{-G} \right) \right) = \pi_1 e^{-G} + \pi_2 e^{-G} \\
&\pi_0 \left(e - e^{-G} \right) = (\pi_1 + \pi_2) G e^{-G} \\
&\pi_1 \left(e^{-G} + 1 - \left(e^{-G} + G e^{-G} \right) \right) = \pi_2 G e^{-G} + \pi_0 G e^{-G} \\
&\implies \pi_1 \left(1 - G e^{-G} \right) = (\pi_0 + \pi_2) G e^{-G}
\end{aligned} \tag{44}
$$

Finally, we recall the equation $\pi_0 + \pi_1 + \pi_2 = 1$, which leads to:

$$
\begin{aligned}
&\pi_1 \left(1 - G e^{-G} \right) = (1 - \pi_1) G e^{-G} \\
&\implies \pi_1 = G e^{-G}
\end{aligned} \tag{45}
$$

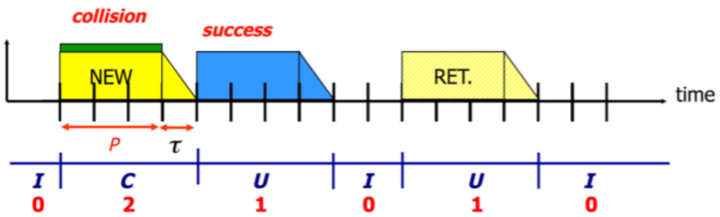
And this proves that the throughput for the slotted ALOHA protocol is $S = G e^{-G}$

5 Slotted non-persistent CSMA

Section Grade: (/20 Points)

Question/Problem:

Consider the slotted CSMA protocol with no persistence after carrier sensing. The channel is time-slotted with a time slot lasting a propagation delay τ. Use the same assumptions we made for the discussion of ALOHA with all nodes having the same propagation delay τ from each other. Obtain the throughput of slotted non-persistent CSMA using a three-state Markov chain as we discussed in class for pure ALOHA. The type of transmission period that happens next is defined only by the number of arrivals in the last time slot of the previous transmission period. An example of the operation of slotted CSMA over time is shown in the following figure.



Answer/Solution:

We note here that the average length of the transmission perior is not the same for all states. We have:

$$\begin{aligned} T_0 &= \tau \\ T_1 &= T_2 = P + \tau, \quad \text{where} \quad P = \text{packet length} \end{aligned} \tag{46}$$

Once again, as for the slotted ALOHA case, we have three possible states: π_0, corresponding to idle time (no parcels, failure), π_1, corresponding to a busy time (one parcel, success), and π_2 corresponding to a collision (failure). Since the propagation delay is the same as the slot size, we have the throughput:

$$S = \frac{\pi_1 P}{\pi_0 T_0 + \pi_1 T_1 + \pi_2 T_2} \tag{47}$$

Furthermore, we use the same balanced equations as in the slotted ALOHA case:

$$
\begin{aligned}
&\pi_0 + \pi_1 + \pi_2 = 1 \\
&\pi_0(P_{01} + P_{02}) = \pi_1 P_{10} + \pi_2 P_{20} \\
&\pi_1(P_{10} + P_{12}) = \pi_0 P_{01} + \pi_2 P_{21} \\
&\pi_2(P_{20} + P_{21}) = \pi_0 P_{02} + \pi_1 P_{12}
\end{aligned} \tag{48}
$$

Also, the transition probabilities are going to be the same:

$$
\begin{aligned}
&P\{0 \quad \text{arrivals in} \quad T\} = P_{i0} = e^{-\lambda\tau} \\
&P\{1 \quad \text{arrival in} \quad T\} = P_{i1} = \lambda\tau e^{-\lambda\tau} \\
&P\{> 1 \quad \text{arrivals in} \quad T\} = P_{i2} = 1 - (e^{-\lambda\tau} + \lambda\tau e^{-\lambda\tau})
\end{aligned} \tag{49}
$$

We replace these quantities into the balanced equations and we get:

$$
\begin{aligned}
&\pi_0(1 - e^{-\lambda\tau}) = (\pi_1 + \pi_2)e^{-\lambda\tau} \\
&\pi_0(1 - e^{-\lambda\tau}) = (1 - \pi_0)e^{-\lambda\tau} \\
&\pi_0 = e^{-\lambda\tau}
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&\pi_1 \left(1 - \lambda\tau e^{-\lambda\tau}\right) = (\pi_0 + \pi_2)\lambda\tau e^{-\lambda\tau} \\
&\pi_1 \left(1 - \lambda\tau e^{-\lambda\tau}\right) = (1 - \pi_1)\lambda\tau e^{-\lambda\tau} \\
&\pi_1 = \lambda\tau e^{-\lambda\tau}
\end{aligned} \tag{51}
$$

$$
\pi_2 = 1 - (1 + \lambda\tau)e^{-\lambda\tau} \tag{52}
$$

We substitute these quantities into the expression for the throughput and we have:

$$
\begin{aligned}
S &= \frac{P\lambda\tau e^{-\lambda\tau}}{\pi_0 T_0 + \pi_1 T_1 + \pi_2 + T_2} \\
&= \frac{P\lambda\tau e^{-\lambda\tau}}{\tau e^{-\lambda\tau} + (P + \tau)\lambda\tau e^{-\lambda\tau} + (P + \tau)\left(1 - e^{-\lambda\tau} - \lambda\tau e^{-\lambda\tau}\right)} \\
&\implies S = \frac{P\lambda\tau e^{-\lambda\tau}}{P + \tau - Pe^{-\lambda\tau}}
\end{aligned} \tag{53}
$$

6 Central Limit Theorem

Section Grade: (/10 Points)

Question/Problem:

Come up with an engineering problem in which you can showhow the central limit theorem is used. Your problem cannot be too similar to any of the problems in the textbook.

Answer/Solution:

In order to illustrate a possible application of the Central Limit Theorem in an engineering context, we could consider a problem related to signal analysis. For example, let us consider m independent measurements of a given signal, denoted X_i, with $i = 1, 2, \ldots, m$. We suppose that each measurement is polluted by a given noise, which we denote by Y_i, with $i = 1, 2, \ldots, m$ (we assume these are also independent random variables presenting the same distribution). We can also assume, without loss of generality, that these noise variables are normally distributed: $N(0, \sigma^2)$ (with mean $\mu = 0$ and variance σ^2). Now, instead of considering each measurement separately, we calculate the distribution of the average:

$$\bar{X} = \frac{X_1 + \ldots X_m}{m} = \frac{(S + Y_1) + \ldots + (S + Y_m)}{m} = S + \frac{Y_1 + \ldots Y_m}{m} \tag{54}$$

By the Central Limit Theorem, if the number of measurements is large enough, the distribution of the mean is going to be approximately a normal distribution, with the same mean value as each of the X_i variables. Furthermore, looking at the right hand side of the expression above, we see that the second term is also going to have be normally distributed. The mean of the distribution is going to be the same as for a single Y_i - 0, but the variance is going to be σ^2/m. This implies in turn that if we take a sufficient number of measurements of the given signal, the variance of the resulting noise for the average of our measurements is going to decrease by a factor m.

7 Non-persistent CSMA

Extra Credit: (/15 Points)

Question/Problem:

Compute the throughput of non-persistent CSMA without time slotting using a two-state or a three-state Markov chain. The analysis is similar to what we discussed in class for pure ALOHA for the case of a two-state Markov chain.

Answer/Solution:

Using a two-state Markov chain similarly to pure ALOHA, we have two states: busy and idle. In addition, we have three possibilities within: I (idle), U (useful) and B (busy). We also have a packet size P=T, a parameter λ and time τ in seconds. The average durations of the possible periods are: T_I, T_U and T_B. The throughput is going to be given by:

$$S = \frac{\pi_1 T_U}{\pi_0 T_I + \pi_1 T_B} \tag{55}$$

From the Poisson distribution we have:

$$\begin{aligned} \pi_0 &= e^{-\lambda T} \\ \pi_1 &= 1 - e^{-\lambda T} \end{aligned} \tag{56}$$

We can now compute the durations of the periods:

$$T_I = \int_0^\infty T(\text{pdf}_I)dT = \int_0^\infty Te^{-\lambda T}dT = \frac{1}{\lambda} \tag{57}$$

In the above we have used: $\text{pdf}_I = \frac{d(1-e^{-\lambda T})}{d\lambda T} = e^{-\lambda T}$. The useful period is:

$$T_U = T \times P[\text{success}] = Te^{-\lambda T} \tag{58}$$

For the busy period, we must take into account the case where we have only one or multiple arrivals. In the first case, the busy period is going to be the length of the packet plus the delay τ. In the second case, we must add an additional term, L, corresponding to the lag of the previous transmission, which is going to follow a Poisson distribution.

$$T_B = T + \tau + e^{-\lambda T} \tag{59}$$

Putting everything together, we get the expression for the throughput:

$$\begin{aligned} S &= \frac{\left(1 - e^{-\lambda T}\right) T e^{-\lambda T}}{e^{-\lambda T}/\lambda + (1 - e^{-\lambda T})(T + \tau + e^{-\lambda T})} \\ &= \frac{(1 - e^{-G} G e^{-G})}{e^{-G} + \lambda(1 - e^{-G}(T + \tau + e^{-G}))} \end{aligned} \tag{60}$$

References

[1] *"CS485 Sylabus". Mathcs.emory.edu. N.p., 2017. Web. 14 June 2017*

[2] Sztrik, János *Basic Queueing Theory. Debrecen: University of Debrecen, Faculty of Informatics, 2017. Print*

[3] *Collaboration on Piazza Slack Group*

[4] *Collaboration with Kunal Dhillon - Student ID: 1426632*