③ Demostrar la ecuación:

$$u_{i,j}^{l+1} = \left(\frac{\alpha\Delta t}{\Delta\rho}\right)^2\left[u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l} + \frac{\Delta\rho}{\rho[i]}(u_{i,j}^{l} - u_{i-1,j}^{l}) + \left(\frac{\Delta\rho}{\Delta\phi\,\rho[i]}\right)^2(u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l})\right]\ldots$$

$$\ldots + 2u_{i,j}^{l} - u_{i,j}^{l-1}$$

La ecuación de laplace en 2D en coordenadas cilíndricas es la siguiente

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = \frac{\partial^2 u}{\partial t^2}$$

Lo cual se puede reescribir como:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2\left[\frac{\partial^2 u}{\partial\rho^2} + \frac{\partial^2 u}{\partial\varphi^2} + \frac{1}{\rho}\frac{\partial^2 u}{\partial\rho^2}\right]$$ Con las condiciones del problema

Ahora, usando el operador derivada central

$$\frac{u_{i,j}^{l+1} - 2u_{i,j}^{l} + u_{i,j}^{l-1}}{(\Delta t)^2} = \alpha^2\frac{u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l}}{(\Delta\rho)^2} + \left(\frac{\alpha^2}{\rho[i]^2}\right)\frac{u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l}}{(\Delta\varphi)^2} + \left(\frac{\alpha^2}{\rho[i]}\right)\frac{u_{i,j}^{l} - u_{i-1,j}^{l}}{\Delta\rho}$$

$u_{i,j}^{l+1}$ despejado queda:

$$u_{i,j}^{l+1} = 2u_{i,j}^{l} - u_{i,j}^{l-1} + \left[\frac{(\Delta t\alpha)^2}{(\Delta\rho)^2}\right](u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l}) + \left(\frac{(\Delta t\alpha)^2}{(\Delta\varphi\,\rho[i])^2}\right)(u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l}) + \left[\frac{(\Delta t\alpha)^2}{\rho[i]\Delta\rho}\right](u_{i,j}^{l} - u_{i-1,j}^{l})$$

$$u_{i,j}^{l+1} = \left[\frac{\alpha\Delta t}{\Delta\rho}\right]^2\left[(u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l}) + \left[\frac{\Delta\rho}{\Delta\varphi}\right]^2\left[\frac{1}{\Delta\varphi\,\rho[i]}\right]^2(u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l}) + \left(\frac{\Delta\rho}{\rho[i]}\right)(u_{i,j}^{l} - u_{i-1,j}^{l})\right] + 2u_{i,j}^{l} - u_{i,j}^{l-1}$$

$$\lambda = \frac{\Delta\rho}{\Delta\varphi} \qquad v = \frac{\alpha\Delta t}{\Delta\rho}$$

$$u_{i,j}^{l+1} = v^2\left[u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l} + \frac{\Delta\rho}{\rho[i]}(u_{i,j}^{l} - u_{i-1,j}^{l}) + \left(\frac{\lambda}{\rho[i]}\right)^2(u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l})\right] + 2u_{i,j}^{l} - u_{i,j}^{l-1}$$