$x^3 y' = x^4 y^2 - 2x^2 y - 1$

①

$y_1 = x^{-2}$ $\quad y_1' = -2x^{-3}$

Sol. propuesta

$x^3(-2x^{-3}) = x^4 (x^{-2})^2 - 2x^2 x^{-2} - 1$

$-2 = x^4 x^{-4} - 2 - 1$ Comprobación de solución particular

$-2 = 1 - 2 - 1$

$-2 = -2$

②

$\left.\begin{array}{l} y = y_1 + u^{-1} \\ y = x^{-2} + u^{-1} \end{array}\right\}$ Sustitución

$y' = -2x^{-3} - u^{-2}u'$

$x^3(-2x^{-3} - u^{-2}u') = x^4(x^{-2} + u^{-1})^2 - 2x^2(x^{-2} + u^{-1}) - 1$

$-2 - x^3 u^{-2} u' = x^4(x^{-4} + 2x^{-2}u^{-1} + u^{-2}) - 2 - 2x^2 u^{-1} - 1$

$\cancel{-2} - x^3 u^{-2} u' = \cancel{1} + \cancel{2x^2 u^{-1}} + x^4 u^{-2} \cancel{-2} \cancel{-2x^2 u^{-1}} \cancel{-1}$

$-x^3 u^{-2} u' = x^4 u^{-2}$

③

$u' = \dfrac{x^4 u^{-2}}{-x^3 u^{-2}}$ $\quad u' = -x$ $\quad \dfrac{du}{dx} = -x$ $\quad du = -x\,dx$

$u = -\int x\,dx$ $\quad u = -\dfrac{x^2}{2} + c$

④

$y = x^{-2} + \left(-\dfrac{x^2}{2} + c\right)^{-1}$ $\quad y = \dfrac{1}{x^2} + \dfrac{1}{c - \frac{x^2}{2}}$ $\quad y = \dfrac{1}{x^2} + \dfrac{2}{2c - x^2}$

Aplicando la condición inicial, obtenemos el valor de c:

$y(\sqrt{2}) = 0$

$0 = \dfrac{1}{2} + \dfrac{2}{2c-2}$ $\quad -\dfrac{1}{2} = \dfrac{2}{2c-2}$ $\quad -(2c-2) = 4$

$-2c + 2 = 4$

$-2c = 2$

$\boxed{c = -1}$

Así que la solución final queda:

$$\boxed{y = \frac{1}{x^2} + \frac{2}{-2 - x^2}}$$