

# SBM model with outputs

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## 1 Theory

The following program implements the slack-based measure of efficiency model (SBM) using only outputs. Let the set of Decision making units (DMUs) be  $\mathbf{J} = \{1, 2, \dots, n\}$ , each DMU has  $s$  outputs denoted by  $\mathbf{y}_j = (y_{1j}, y_{2j}, \dots, y_{sj})^T$ . We define the output matrix by:

$$\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n) \in R^{s \times n}$$

We also assume that all data are positive, i.e,  $\mathbf{Y} > 0$

The production possibility set is defined using the nonnegative combination of DMUs in the set  $J$  as:

$$\mathbf{P} = \left\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \geq \sum_{j=1}^n \lambda_j \mathbf{x}_j, \mathbf{0} \leq \mathbf{y} \leq \sum_{j=1}^n \lambda_j \mathbf{y}_j, \boldsymbol{\lambda} \geq 0 \right\}$$

where  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$  is called the intensity vector.

These inequalities can be transformed into equalities by introducing slacks variables as follows:

$$\begin{aligned} \mathbf{x} &= \sum_{j=1}^n \lambda_j \mathbf{x}_j + \mathbf{s}^-, \\ \mathbf{y} &= \sum_{j=1}^n \lambda_j \mathbf{y}_j + \mathbf{s}^+, \end{aligned}$$

$$\mathbf{s}^- \geq 0, \mathbf{s}^+ \geq 0$$

where  $\mathbf{s}^- = (s_1^-, s_2^-, \dots, s_m^-)^T \in \mathbb{R}^m$  and  $\mathbf{s}^+ = (s_1^+, s_2^+, \dots, s_s^+)^T \in \mathbb{R}^s$  are, respectively called input and output slacks.

### 1.1 Output-Oriented SBM only with outputs

Here we consider only the output vector  $\mathbf{y}$ . Each DMU<sub>*o*</sub> has a vector of  $s$  outputs  $\mathbf{y}_o = (y_o^1, y_o^2, \dots, y_o^s)$ .  $y_o^i$  represents the *i*-th output of DMU<sub>*o*</sub>. The output-oriented SBM efficiency  $\rho_o^*$  of DMU<sub>*o*</sub> =  $(\mathbf{x}_o, \mathbf{y}_o)$  is defined as

$$\begin{aligned}
\frac{1}{\rho_o^*} &= \max_{\lambda, \mathbf{s}^+} 1 + \frac{1}{s} \sum_{r=1}^s \frac{s_r^+}{y_{ro}} \\
\text{subject to} \\
y_{ro} &= \sum_{j=1}^n y_{rj} \lambda_j - s_r^+ \quad (r = 1, \dots, s), \\
\sum_{j=1}^n \lambda_j &= 1 \\
\lambda_j &\geq 0 (\forall j), \quad s_r^+ \geq 0 (\forall r)
\end{aligned}$$

Let an optimal solution of the model be  $\lambda^*, \mathbf{s}^{+*}$ .

This system will produce a vector  $(s_o^1, \dots, s_o^s, \lambda_{1o}, \dots, \lambda_{No})$  for  $\text{DMU}_o$

- **SBM-Output-Efficient.** A  $\text{DMU}_o = (\mathbf{x}_o, \mathbf{y}_o)$  is called SBM-output-efficient if  $\rho_o^* = 1$  holds. This means  $\mathbf{s}^{+*} = \mathbf{0}$ , i.e, all outputs slacks are equal to zero.

## 1.2 Nonoriented SBM only with outputs

Nonoriented or both oriented SBM efficiency  $\rho_{IO}^*$  is defined by

$$\begin{aligned}
\rho_{IO}^* &= \min_{\lambda, \mathbf{s}^+} \frac{1}{1 + (1/s) \sum_{r=1}^s (s_r^+ / y_{ro})} \\
\text{subject to} \\
y_{ro} &= \sum_{j=1}^n y_{rj} \lambda_j - s_r^+ \quad (r = 1, \dots, s), \\
\sum_{j=1}^n \lambda_j &= 1 \\
\lambda_j &\geq 0 (\forall j), \quad s_r^+ \geq 0 (\forall r)
\end{aligned}$$

- **SBM-Output-Efficient.** A  $\text{DMU}_o = (\mathbf{x}_o, \mathbf{y}_o)$  is called SBM-efficient if  $\rho_{IO}^* = 1$  holds. This means  $\mathbf{s}^{+*} = \mathbf{0}$ , i.e, all outputs slacks are equal to zero.

## 2 Matlab implementation