SBM model with outputs

Juan Díaz Amorin

February 6, 2019

1 Theory

The following program implements the slack-based measure of efficiency model (SBM)using only outputs. Let the set of Decision making units (DMUs) be $\mathbf{J} = \{1, 2, \cdots, n\}$, each DMU has s outputs denoted by $\boldsymbol{y}_j = (y_{1j}, y_{2j}, \cdots, y_{sj})^T$. We define the output matrix by:

$$\boldsymbol{Y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n) \quad \in \quad R^{sxn}$$

We also assume that all data are positive, i.e, $\boldsymbol{Y} > 0$

The production possibility set is defined using the nonnegative combination of DMUs in the set J as:

$$\boldsymbol{P} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq \sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j, \quad \mathbf{0} \leq \boldsymbol{y} \leq \sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j, \quad \boldsymbol{\lambda} \geq 0 \right\}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ is called the intensity vector.

These inequalities can be transformed into equalities by introducing slacks variables as follows:

$$\begin{aligned} \boldsymbol{x} &= \sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j + \boldsymbol{s}^-, \\ \boldsymbol{y} &= \sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j + \boldsymbol{s}^+, \end{aligned}$$

$$\boldsymbol{s}^- \geq 0, \quad \boldsymbol{s}^+ \geq 0$$

where $\boldsymbol{s}^- = (s_1^-, s_2^-, \cdots, s_m^-)^T \in \mathrm{I\!R}^{\mathrm{m}}$ and $\boldsymbol{s}^+ = (s_1^+, s_2^+, \cdots, s_s^+)^T \in \mathrm{I\!R}^{\mathrm{s}}$ are, respectively called input and output slacks.

1.1 Output-Oriented SBM only with outputs

Here we consider only the output vector $\boldsymbol{y}$. Each DMU$_o$ has a vector of s outputs $\boldsymbol{y}_o = (y_0^1, y_0^2, \cdots, y_s^s)$. y_o^i represents the i-th output of DMU$_o$. The output-oriented SBM efficiency $\boldsymbol{\rho_o^*}$ of DMU$_o = (\boldsymbol{x}_o, \boldsymbol{y}_o)$ is defined as

$$\begin{aligned}\frac{1}{\rho_o^*} &= \max_{\lambda, \boldsymbol{s}^+} \quad 1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}} \\ \text{subject to} & \\ y_{ro} &= \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ \quad (r = 1, \cdots, s), \\ \sum_{j=1}^{n}\lambda_j &= 1\end{aligned}$$

$$\lambda_j \geq 0 (\forall j), \quad s_r^+ \geq 0 (\forall r)$$

Let an optimal solution of the model be $\boldsymbol{\lambda}^*, \boldsymbol{s}^{+*}$.
This system will produce a vector $(s_o^1, \cdots, s_o^s, \lambda_{1o}, \cdots, \lambda_{No})$ for DMU$_o$

- **SBM-Output-Efficient.** A DMU$_o$ = $(\boldsymbol{x}_0, \boldsymbol{y}_o)$ is called SBM-output-efficient if $\rho_o^* = 1$ holds. This means $\boldsymbol{s}^{+*} = \mathbf{0}$, i.e, all outputs slacks are equal to zero.

2 Matlab implementation

2.1 Matlab linprog.m

This procedure solves the problem

$$\min_x f^T \quad \text{such that} \quad \begin{cases} A.x \leq b, \\ Aeq.x = beq \\ lb \leq x \end{cases}$$

where x is the (n+s) x 1 vector of parameters:

$$x = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ s_1^+ \\ \vdots \\ s_s^* \end{bmatrix}$$

2.2 Code structure

- Feed matrix **Y** of dimension n x s (each DMU with s outputs) $\boldsymbol{Y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_s)$, where $\boldsymbol{y}_r = (y_{r1}, y_{r2}, \cdots, y_{rn})'$ for $r = 1, \cdots, s$
- For DMU$_o$ define:

$$f^T = [\cdots 0(-1/y_{1o}) \cdots (-1/y_{so})]_{1x(n+s)}$$

$$Aeq = \begin{bmatrix} \boldsymbol{Y}' & -\boldsymbol{I}_s \\ \mathbf{1} & \mathbf{0} \end{bmatrix}_{(s+1)\ x\ (n+s)}$$

where $\mathbf{1}$ is (1 x n), $\mathbf{0}$ is (1 x s), and diag(y_o) is a zero matrix with the elements of y_o in the diagonal.

$$\text{beq} = \begin{bmatrix} y_0^{'} \\ 1 \end{bmatrix}_{(s+1)\ x\ 1}$$

$$\text{lb} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{(n+s)\ x\ 1}$$