Chemical Engineering Science, 1967, Vol. 22, pp. 653–660. Pergamon Press Ltd., Oxford. Printed in Great Britain.

Slow viscous motion of a sphere parallel to a plane wall—II Couette flow

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(*Received* 7 *September* 1966; *accepted* 6 *October* 1966)

Abstract—Using bipolar co-ordinates, an exact solution of Stokes equations is obtained for the translational and rotational velocities of a neutrally buoyant sphere moving in proximity to a single plane wall under the influence of a simple shearing flow. The solution, valid for small shear Reynolds numbers, applies for all ratios of sphere radius to distance of its center from the wall. This formal solution is supplemented by two asymptotic solutions: (i) a lubrication-theory-like approximation applicable to the case where the sphere is very near to the wall; (ii) a "method of reflections" approximation, valid for the opposite case. Agreement with limited experimental data currently available in the literature is shown to be good, though the question of the true, limiting behavior of a sphere "touching" a wall remains unresolved.

1. INTRODUCTION

THE IDEALIZED model of a neutrally buoyant spherical particle in motion near a plane wall bounding a semi-infinite viscous fluid subjected to a uniform shear flow is useful in a variety of disciplines. Among the various applications we cite the following: the lubrication characteristics of spherical bearings; the rheology of dilute suspensions in a Couette flow; the motion of particles in a laminar boundary layer in flow visualization experiments; the movement of silt in river beds. This paper presents an exact solution of the title problem on the basis of Stokes equations, fluid inertia being neglected. By exact is meant that the results are valid for all ratios of sphere radius, a, to distance h of its center from the wall—including the limiting case $a/h = 1$ where the sphere touches the wall.

In view of the linearity of Stokes equations and of the boundary conditions, the motion may be resolved into two distinct contributions: (i) a translational and rotational motion of the sphere near a plane wall in a fluid at rest at infinity; (ii) a shearing flow past an immobilized sphere in proximity to a plane wall. Exact, bipolar co-ordinate system solutions are already available for the first class of problems.* Accordingly, attention is here directed only to the second problem—that of computing the shearing force and couple on the immobilized sphere. Rather remarkably, this calculation can be brought to fruition merely by an appropriate quadrature (BRENNER [1, 2]) of the former results for translation and rotation of a sphere near a plane wall in a *quiescent* fluid. Thus, it is not required

* DEAN and O'NEILL [3] consider rotation of the sphere about an axis passing through its center and lying parallel to the wall. Numerical corrections to their work are given be GOLDMAN, COX and BRENNER [4] (hereafter referred to as Part I), who also present an asymptotic, "lubrication-theory" formula for the case where the sphere is very near to the wall. O'NEILL'S [5] analysis pertains to the situation where the sphere translates parallel to the wall. His solution is supplemented by a lubrication-theory formula in Part I. JEFFERY [6] analyzes the symmetrical rotation of a sphere about an axis perpendicular to the wall. MAUDE [7] and BRENNER [8] independently solve the axisymmetric case of quasistatic translation of the sphere perpendicular to the wall. Lubrication-theory results for the Jeffery and Maude–Brenner configurations are given by COX and BRENNER [9], who also take account of both inertial and unsteady-state effects for the Maude–Brenner case. Because of the linearity of Stokes equations, the various solutions for motion parallel and perpendicular to the wall may be combined to furnish the general solution for motion of the sphere at an arbitrary angle of attack, though such generality is not, in fact, required for the problem at hand.

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that we effect a detailed solution of Stokes equations for the shear flow case.

Section 2 outlines the strategy to be employed in solving the title problem. The force and couple on a stationary sphere in proximity to a plane wall in a uniform shear flow are obtained in Section 3 using the quadrature scheme alluded to above. Section 4 is devoted to a calculation of the translational and angular velocities of a neutrally buoyant sphere near a wall in a simple shear flow as a function of a, h, and S—the undisturbed shear rate. These results are shown to be in good agreement with existing experimental data in Section 5.

2. DESCRIPTION OF PROBLEM

The configuration of the system is illustrated in Fig. 1. The center, O, of a neutrally buoyant sphere, S, of radius a is situated at a distance h above a stationary plane wall ($z=0$), P, bounding a semi-infinite viscous fluid ($z>0$). x, y, and z, in that order, constitute a right-handed system of rectangular Cartesian co-ordinates, the x axis lying parallel to the streamlines of the undisturbed shear flow at infinity,

$$\mathbf{v}_\infty = \mathbf{i}_x S z \tag{2.1}$$

$\mathbf{i}_k$ denotes a unit vector along the kth coordinate axis. In the neutrally buoyant case the translational velocity $\mathbf{u}_O$ of the sphere center, and angular velocity $\boldsymbol{\omega}$ of the sphere are not independent parameters; rather, they are determined by the direction and magnitude of the shear, among other things. It is readily shown on the basis of symmetry arguments, in conjunction with the linearity of Stokes equations, that the translational and angular sphere velocities are necessarily parallel to the x and y axes, respectively (GOLDMAN [10]). Accordingly, we may set

$$\mathbf{u}_O = \mathbf{i}_x U, \qquad \boldsymbol{\omega} = \mathbf{i}_y \Omega \tag{2.2a, b}$$

where U and Ω are positive scalars. At higher Reynolds numbers, where inertial effects are sensible, a translational velocity of the sphere center perpendicular to the wall would also be anticipated.

Stokes equations are

$$\frac{1}{\mu}\nabla p = \nabla^2 \mathbf{v}, \qquad \nabla \cdot \mathbf{v} = 0 \tag{2.3a, b}$$

The boundary conditions appropriate to the present problem are:

$$\mathbf{v}_S = \mathbf{u}_O + \boldsymbol{\omega} \times \mathbf{r}_{OS}, \qquad \mathbf{v}_P = \mathbf{0} \tag{2.4a, b}$$

$$\mathbf{v} \to \mathbf{v}_\infty \ \text{as} \ |\mathbf{r}| \to \infty \tag{2.4c}$$

where $\mathbf{r}_{OS}$ is the position vector of a point on the sphere surface S relative to an origin at the sphere center. The quantities of immediate interest are

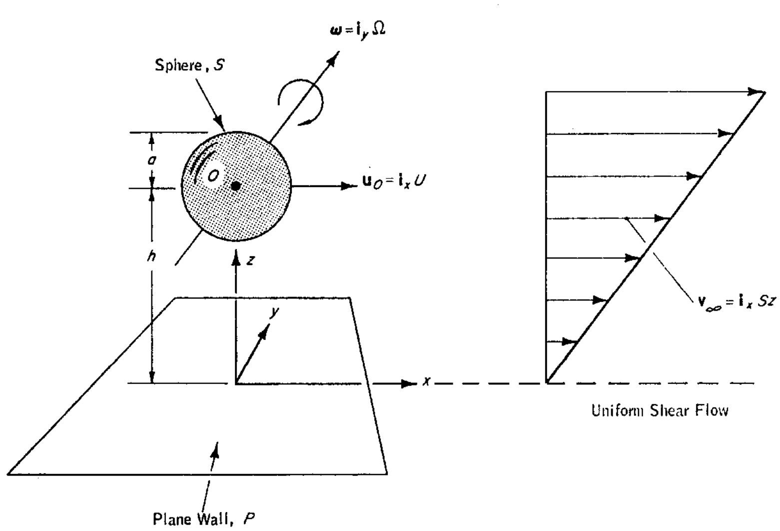

FIG. 1. Stationary sphere in a shearing flow near a plane wall.

the hydrodynamic force and torque (about the sphere center) exerted by the fluid on the sphere,

$$\mathbf{F}=\int_S\int d\mathbf{S}\cdot\mathbf{P},\quad \mathbf{T}_O=\int_S\int \mathbf{r}_{OS}\times(d\mathbf{S}\cdot\mathbf{P}) \tag{2.5a, b}$$

in which $\mathbf{P}$ is the pressure tensor for an incompressible fluid,

$$\mathbf{P}=-\mathbf{I}p+\mu[\nabla\mathbf{v}+(\nabla\mathbf{v})^{\dagger}] \tag{2.6}$$

and $d\mathbf{S}$ is a directed element of surface area pointing into the fluid.

These equations ultimately yield general expressions for $\mathbf{F}$ and $\mathbf{T}_O$ for arbitrary values of U and Ω. Since $\mathbf{F}$ and $\mathbf{T}_O$ vanish for a neutrally buoyant sphere, these expressions thereby furnish a pair of simultaneous equations to be solved for the two unknown velocities, U and Ω, for given values of a, h and S. Dimensional arguments show that the functional form of the final results is

$$\frac{U}{aS}\ \text{or}\ \frac{U}{hS}=\text{function}\left(\frac{a}{h}\right),\quad \frac{\Omega}{S}=\text{function}\left(\frac{a}{h}\right) \tag{2.7a, b}$$

At Reynolds numbers above the Stokes regime, fluid inertia will contribute the shear Reynolds number, $\mathrm{Re}=a^2S/\nu$, as an additional, independent parameter to these functional relationships. The use of Stokes equations is thus tantamount to assuming the shear Reynolds number to be small compared with unity.

Linearity permits us to express the solutions of Eqs. (2.3)–(2.5) each as the sum of a translational (t), rotational (r), and shear (s) contribution:

$$\mathbf{v}=\mathbf{v}^t+\mathbf{v}^r+\mathbf{v}^s,\qquad p=p^t+p^r+p^s \tag{2.8a, b}$$

$$\mathbf{F}=\mathbf{F}^t+\mathbf{F}^r+\mathbf{F}^s,\qquad \mathbf{T}_O=\mathbf{T}_O^t+\mathbf{T}_O^r+\mathbf{T}_O^s \tag{2.9a, b}$$

Here, $(\mathbf{v}^t, p^t)$, $(\mathbf{v}^r, p^r)$, and $(\mathbf{v}^s, p^s)$ each separately satisfy Stokes equations and the following boundary conditions:

Translation:

$$\mathbf{v}_S^t=\mathbf{i}_xU,\qquad \mathbf{v}_P^t=\mathbf{0},\quad \mathbf{v}_\infty^t=\mathbf{0} \tag{2.10a, b, c}$$

Rotation:

$$\mathbf{v}_S^r=(\mathbf{i}_y\times\mathbf{r}_{OS})\Omega,\quad \mathbf{v}_P^r=\mathbf{0},\quad \mathbf{v}_\infty^r=\mathbf{0} \tag{2.11a, b, c}$$

Shear:

$$\mathbf{v}_S^s=\mathbf{0},\qquad \mathbf{v}_P^s=\mathbf{0},\quad \mathbf{v}_\infty^s=\mathbf{i}_xSz \tag{2.12a, b, c}$$

It is clear from symmetry that

$$(\mathbf{F}^t,\ \mathbf{F}^r,\ \mathbf{F}^s)=\mathbf{i}_x(F_x^t,\ F_x^r,\ F_x^s) \tag{2.13}$$

and

$$(\mathbf{T}_O^t,\ \mathbf{T}_O^r,\ \mathbf{T}_O^s)=\mathbf{i}_y(T_y^t,\ T_y^r,\ T_y^s) \tag{2.14}$$

in which the translational and rotational force and torque components are negative scalars, whereas the shear force and torque components are positive scalars.

The solutions of the translational and rotational problems corresponding to the boundary conditions (2.10) and (2.11), respectively, are discussed in Part I. Results are tabulated there for the nondimensional forces

$$F_x^{t*}=F_x^t/6\pi\mu aU,\qquad F_x^{r*}=F_x^r/6\pi\mu a^2\Omega \tag{2.15a, b}$$

and torques

$$T_y^{t*}=T_y^t/8\pi\mu a^2U,\qquad T_y^{r*}=T_y^r/8\pi\mu a^3\Omega \tag{2.16a, b}$$

vs. h/a in the range $1\le h/a<\infty$. In the next section we compute the comparable shear-induced forces and torques stemming from the boundary conditions (2.12).

3. Force and Torque on a Stationary Sphere near a Wall in a Shear Field

The components of the hydrodynamic force and torque exerted by the fluid on the sphere, for *arbitrary* boundary conditions $\mathbf{v}_S$ on the sphere surface, and an *arbitrary* condition $\mathbf{v}_\infty$ at infinity, are given by the expressions (Brenner [1])

$$F_x=\int_S\int d\mathbf{S}\cdot\mathbf{P}^{tx}\cdot(\mathbf{v}_S-\mathbf{v}_\infty) \tag{3.1}$$

and

$$T_y=\int_S\int d\mathbf{S}\cdot\mathbf{P}^{ry}\cdot(\mathbf{v}_S-\mathbf{v}_\infty) \tag{3.2}$$

with similar expressions for the remaining components. Here, $\mathbf{P}^{tx}$ is the pressure dyadic arising from translational motion of the sphere with *unit* velocity in the x direction in a fluid at rest at infinity. Specifically, if $(\mathbf{v}^{tx}, p^{tx})$ is the Stokes flow satisfying the boundary conditions

$$\mathbf{v}_S^{tx}=\mathbf{i}_x,\qquad \mathbf{v}_P^{tx}=\mathbf{0},\qquad \mathbf{v}_\infty^{tx}=\mathbf{0} \tag{3.3a, b, c}$$

then $\mathbf{P}^{tx}$ is given by an equation of the form (2.6) in which p^{tx} and $\mathbf{v}^{tx}$ appear. Similarly, $\mathbf{P}^{ry}$ is the pressure dyadic arising from rotation of the sphere with unit angular velocity about its center, parallel to the y axis, the fluid at infinity being at rest. Specifically, $\mathbf{P}^{ry}$ derives from Eq. (2.6) for the Stokes flow ($\mathbf{v}^{ry}$, p^{ry}) satisfying the boundary conditions

$$\mathbf{v}_S^{ry}=\mathbf{i}_y\times\mathbf{r}_{OS}, \quad \mathbf{v}_P^{ry}=\mathbf{0}, \quad \mathbf{v}_\infty^{ry}=\mathbf{0} \qquad (3.4\text{a, b, c})$$

For the particular problem at hand, namely that of computing the shear-induced force and torque, F_x^s and T_y^s, the appropriate boundary conditions are tabulated in Eq. (2.12). Thus, Eqs. (3.1) and (3.2) become

$$F_x^s=-S\int\!\!\int_S d\mathbf{S}\cdot\mathbf{P}^{tx}\cdot\mathbf{i}_x z \qquad (3.5)$$

$$T_y^s=-S\int\!\!\int_S d\mathbf{S}\cdot\mathbf{P}^{ry}\cdot\mathbf{i}_x z \qquad (3.6)$$

Since the translational and rotational velocity and pressure fields required in the computation of the pressure dyadics appearing in the preceding equations are already known (see Part I) from the prior work of O'NEILL [5] and DEAN and O'NEILL [3], our sole problem is that of effecting the quadrature of Eqs. (3.5) and (3.6). Details of the integration scheme, and the resulting closed-form expressions obtained therefrom can be found in GOLDMAN'S [10] thesis. Numerical values of the normalized, nondimensional, shear-induced forces and torques

$$F_x^{s*}=F_x^s/6\pi\mu ahS, \quad T_y^{s*}=T_y^s/4\pi\mu a^3 S \quad (3.7\text{a, b})$$

calculated from these expressions are tabulated in Table 1 as a function of h/a.

TABLE 1. FORCE AND TORQUE ON A STATIONARY SPHERE IN A SHEAR FIELD

α†	h/a	F_x^{s*}	T_y^{s*}
∞	∞	1·0000	1·00000
3·0	10·0677	1·0587	0·99981
2·0	3·7622	1·1671	0·99711
1·5	2·3524	1·2780	0·99010
1·0	1·5431	1·4391	0·97419
0·5	1·1276	1·6160	0·95374
0·3	1·0453	1·6682	0·94769
0·1	1·005004	1·6969	0·94442
0·08	1·003202	1·6982	0·94427
0	1·0000	1·7005	0·94399

† The bipolar co-ordinate parameter α is defined as $\alpha=\cosh^{-1} h/a$ [17].

It can be shown that the shear-induced force and torque both approach finite limits as the sphere contacts the wall. Thus, the results tabulated in Table 1 were extrapolated to obtain the values $F_x^{s*}=1{\cdot}7005$ and $T_y^{s*}=0{\cdot}9440$ for the limiting case $h/a=1$. These results are, respectively, only 70 per cent greater and 5·6 per cent less than would arise in the absence of wall effects.

Using the "method of reflections," valid for small a/h, it is shown in the Appendix that

$$F_x^{s*}\sim 1+\frac{9}{16}\frac{a}{h}, \qquad T_y^{s*}\sim 1-\frac{3}{16}\left(\frac{a}{h}\right)^3 \qquad (3.8\text{a, b})$$

These asymptotic formulas agree well with the exact values in Table 1 at the larger values of h/a.

4. FREE MOTION OF A NEUTRALLY BUOYANT SPHERE NEAR A PLANE WALL IN A SHEAR FIELD

When the densities of the sphere and fluid are matched the hydrodynamic force and torque on the sphere must each be zero. In accordance with Eqs. (2.9), (2.13) and (2.14), this requires that

$$F_x^t+F_x^r+F_x^s=0, \qquad T_y^t+T_y^r+T_y^s=0 \quad (4.1\text{a, b})$$

Alternatively, in terms of the various nondimensional force and torque components, the preceding are equivalent to the relations

$$F_x^{t*}\left(\frac{U}{aS}\right)+F_x^{r*}\left(\frac{\Omega}{S}\right)=-F_x^{s*}\left(\frac{h}{a}\right) \qquad (4.2)$$

and

$$T_y^{t*}\left(\frac{U}{aS}\right)+T_y^{r*}\left(\frac{\Omega}{S}\right)=-\frac{1}{2}T_y^{s*} \qquad (4.3)$$

Simultaneous solution of these equations for the normalized translational and angular velocities of the sphere yields

$$\frac{\Omega}{\frac{1}{2}S}=\frac{2(h/a)F_x^{s*}T_y^{t*}-F_x^{t*}T_y^{s*}}{F_x^{t*}T_y^{r*}-F_x^{r*}T_y^{t*}} \qquad (4.4)$$

$$\frac{U}{hS}=\frac{\frac{1}{2}(a/h)F_x^{r*}T_y^{s*}-F_x^{s*}T_y^{r*}}{F_x^{t*}T_y^{r*}-F_x^{r*}T_y^{t*}} \qquad (4.5)$$

Values of the dimensionless translational and rotational forces and torques as a function of h/a are tabulated in Part I; numerical values of the shearing force and torque are tabulated in the present paper. These were employed to prepare Table 2, which gives U/hS and $\Omega/\frac{1}{2}S$, as well as $a\Omega/U$, as a function of h/a.

TABLE 2. FREE MOTION OF A SPHERE NEAR A PLANE WALL IN A SHEAR FIELD

α	h/a	U/hS	$\Omega/\frac{1}{2}S$	$a\Omega/U$
∞	∞	1·00000	1·00000	0
3·0	10·0677	0·99962	0·99952	0·049659
2·0	3·7622	0·99436	0·99430	0·13289
1·5	2·3524	0·97768	0·97780	0·21257
1·0	1·5431	0·92185	0·92368	0·32468
0·5	1·1276	0·76692	0·77916	0·45050
0·3	1·0453	0·65375	0·67462	0·49360
0·1	1·005004	0·47861	0·50818	0·52825
0·08	1·003202	0·45291	0·48300	0·53152

When h/a is large we have by the method of reflections (Part I), to terms of lowest order in a/h, that the translational and rotational forces and torques are

$$F_x^{t*} \sim -\left[1-\frac{9}{16}\left(\frac{a}{h}\right)\right]^{-1},$$

$$T_y^{t*} \sim \frac{3}{32}\left(\frac{a}{h}\right)^4 \tag{4.6a, b}$$

and

$$F_x^{r*} \sim \frac{1}{8}\left(\frac{a}{h}\right)^4,$$

$$T_y^{r*} \sim -\left[1+\frac{5}{16}\left(\frac{a}{h}\right)^3\right] \tag{4.7a, b}$$

In conjunction with the comparable values of F_x^{s*} and T_y^{s*} noted in Eq. (3.8), these yield, upon substitution into Eqs. (4.4) and (4.5),

$$\frac{U}{hS} \sim 1-\frac{5}{16}\left(\frac{a}{h}\right)^3, \quad \frac{\Omega}{\frac{1}{2}S} \sim 1-\frac{5}{16}\left(\frac{a}{h}\right)^3 \tag{4.8a, b}$$

For large h/a these approximate formulas are in good agreement with the exact values cited in Table 2.

When the gap width, $\delta = h-a$, is very small compared with the sphere radius, one must employ the asymptotic, lubrication-theory formulas developed in Part I for the translational and rotational forces and torques,

$$F_x^{t*} \sim \frac{8}{15}\ln\left(\frac{\delta}{a}\right)-0\cdot9588 \tag{4.9a}$$

$$T_y^{t*} \sim -\frac{1}{10}\ln\left(\frac{\delta}{a}\right)-0\cdot1895 \tag{4.9b}$$

and

$$F_x^{r*} \sim -\frac{2}{15}\ln\left(\frac{\delta}{a}\right)-0\cdot2526 \tag{4.10a}$$

$$T_y^{r*} \sim \frac{2}{5}\ln\left(\frac{\delta}{a}\right)-0\cdot3817 \tag{4.10b}$$

as well as the limiting values of the shearing force and torque, $F_x^{s*} = 1\cdot7005$ and $T_y^{s*} = 0\cdot9440$, valid in the limit as $\delta/a \to 0$. In conjunction with Eqs. (4.4) and (4.5) these yield

$$\frac{U}{hS} \sim \frac{0\cdot7431}{0\cdot6376-0\cdot200\ln(\delta/a)} \tag{4.11}$$

and

$$\frac{\Omega}{\frac{1}{2}S} \sim \frac{0\cdot8436}{0\cdot6376-0\cdot200\ln(\delta/a)} \tag{4.12}$$

Numerical values based upon these formulas are tabulated in Table 3, as is $a\Omega/U$ too.

TABLE 3. FREE MOTION OF A SPHERE ALMOST TOUCHING A PLANE WALL IN A SHEAR FIELD

δ/a	U/hS	$\Omega/\frac{1}{2}S$	$a\Omega/U$
0·003202	0·4529	0·4830	0·5315
10^{-3}	0·3966	0·4268	0·5375
10^{-4}	0·3185	0·3468	0·5445
10^{-5}	0·2659	0·2918	0·5488
10^{-6}	0·2283	0·2520	0·5518
10^{-7}	0·2000	0·2216	0·5539
10^{-8}	0·1779	0·1976	0·5556
0	0	0	0·5676

Tables 2 and 3 furnish a complete description of the shear-induced motion of the sphere. In all cases the translational and angular velocities of the sphere lag those of the undisturbed flow; that is, U/hS and $\Omega/\frac{1}{2}S$ are consistently less than unity for all h/a. However, for $\delta/a > 1$ the lag is less than about 3 per cent. In the limit as $\delta/a \to 0$ the translational and angular velocities both approach zero, though the ratio $a\Omega/U$, of circumferential to centerline speeds, approaches a definite limit, namely 0·5676. Were the sphere to roll along the wall without slipping this ratio would be unity. Hence, the sphere appears to slip as it rolls, at least according to the theory. Whether or not this and related conclusions apply to actual experiments remains to be demonstrated.

5. COMPARISON WITH EXPERIMENTAL DATA AND DISCUSSION

Experiments performed by DARABANER [11] show good agreement with those limited aspects of the present theory to which they pertain.† Measurements were made of the angular velocities of neutrally buoyant spheres of various sizes in a Couette viscometer as a function of the distance of the sphere from the nearer wall of the viscometer. In the limiting case where the sphere touched the wall, special precautions were taken to attain intimate contact. Preliminary results† are shown in Fig. 2. Theoretical values are represented by the solid curve. Considering that the data have not been corrected for wall curvature nor for the presence of a second wall, and in view of the difficulties in maintaining and measuring the sphere's distance from the wall, the agreement is quite good.

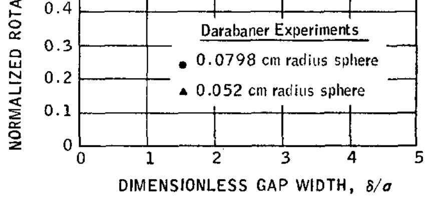

FIG. 2. Rotation of a neutrally buoyant sphere near a plane wall in a Couette flow.

As discussed in Part I, the theoretical calculations of the translational and rotational forces and torques are at odds with existing experimental data for very small gap widths. Accordingly, the results of Table 3 cannot be taken too seriously, despite the apparent agreement of the theory with Darabaner's experiments at the larger gap widths. From an experimental viewpoint, one can hardly be expected to distinguish between the various δ/a values listed in Table 3. Yet, with the sole exception of $a\Omega/U$, the present theory indicates a critical dependence of the various dependent variables upon the δ/a ratio.

RAASCH [13] presents an exact solution of Stokes equations for the comparable *two-dimensional* problem of the shear-induced motion of a neutrally buoyant circular cylinder near a plane wall bounding a semi-infinite fluid.‡ He shows that the cylinder will not move when in contact with the wall, that the linear and angular velocities vary as the square root of the gap width, and that the $a\Omega/U$ ratio is exactly $\frac{1}{2}$ for *all* gap widths. Hence, the same general trends exist for both the cylinder and sphere problems. However, the square-root dependence of the linear and angular velocities upon gap width in the two-dimensional case is in marked contrast to the corresponding logarithmic dependence in the three-dimensional case.

The present results may be applied to the motion of an eccentrically positioned sphere near the wall of a circular tube (radius $= R_o$) within which a Poiseuille flow is occurring. Provided that $a/R_o \ll 1$ and $a/(R_o - b) = O(1)$ (b = distance from cylinder axis to sphere center), it appears intuitively clear that the hydrodynamic force and torque exerted on the sphere will be the same as for the plane wall case if we set $h = R_o - b$ and $S = 4V_m/R_o$ (V_m = mean velocity of flow through the tube), the latter being the shear rate at tube wall. This supplements the solution of BRENNER and HAPPEL [16], valid for the case where $a/R_o \ll 1$ and $a/(R_o - b) \ll 1$.

† More complete data, experimental details, and further interpretation will be found elsewhere (DARABANER and MASON [12]).

‡ This is an extension of FRAZER's analysis of the translation and rotation of a circular cylinder parallel to a plane wall bounding a fluid at rest at infinity [14].

Acknowledgments—During the course of this work A. J. G. was supported by the United States Atomic Energy Commission through a fellowship in Nuclear Science and Engineering administered by the Oak Ridge Institute of Nuclear Studies. R. G. C. and H. B. were supported by the National Science Foundation (Grant No. NSF GK-56). The authors are also grateful to the United Nuclear Corporation for assistance in the preparation of this manuscript.

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Appendix

In this Appendix we employ the method of reflections to derive approximate formulas for the shear force and torque on a stationary sphere, valid for large h/a. These may be derived by the following elementary arguments. As a consequence of the shear flow, the stationary sphere experiences a force and torque which in the absence of wall effects (i.e. to terms of lowest order in a/h) are given by the expressions

$$F_x^s = 6\pi\mu a U', \qquad T_y^s = 8\pi\mu a^3 \tfrac{1}{2} S \quad \text{(A.1a, b)}$$

in which $U' = Sh$. At large distances from the sphere, the dominant term in the velocity field arising from the "reflection" of the shear field from the sphere surface stems entirely from the force term, Eq. (A.1a). The subsequent reflection of this reflected velocity field from the wall is required to vanish on the plane wall and at infinity. Now, a sphere *translating* (without rotation) with velocity U' in the *negative* x direction, parallel to a plane wall bounding a quiescent fluid, experiences the same force (in the positive x direction) as noted in Eq. (A.1a), at least to terms of lowest order in a/h. But this translational motion is required to satisfy the same boundary conditions of zero velocity on the plane and at infinity as does the previous reflected velocity field. It immediately follows that the wall effects arising from the translational motion U' (Faxén [15]),

$$F_x = 6\pi\mu a U' \frac{9}{16}\left(\frac{a}{h}\right) \quad \text{(A.2a)}$$

$$T_y = -8\pi\mu a^2 U' \frac{3}{32}\left(\frac{a}{h}\right)^4 \quad \text{(A.2b)}$$

are precisely those arising in the shear flow case too, at least to terms of the lowest orders in a/h. Hence, setting $U' = Sh$ in Eqs. (A.2), and adding these values to Eqs. (A.1) yields

$$F_x^s = 6\pi\mu a Sh\left[1 + \frac{9}{16}\left(\frac{a}{h}\right)\right] \quad \text{(A.3a)}$$

$$T_y^s = 4\pi\mu a^3 S\left[1 - \frac{3}{16}\left(\frac{a}{h}\right)^3\right] \quad \text{(A.3b)}$$

In normalized form these results are equivalent to Eqs. (3.8).

Résumé—Par l'utilisation de co-ordonnées bipolaires, on obtient une solution exacte des équations de Stokes pour les vélocités de translation et de rotation d'une sphère flottant en neutralité se déplaçant à proximité d'une paroi simple plane sous l'influence d'un simple flot transversal. La solution, valable pour des nombres Reynolds de faible force transversale, s'applique à tous les rapports des rayons de la sphère à la distance de son centre à partir de la paroi. A cette solution formelle on a ajouté deux autre solutions asymptotiques: (a) une approximation du genre théorie de lubrification applicable au cas où la sphère est très près de la paroi; (b) une approximation du type "méthode de reflexions", valable pour le cas contraire. Il est démontré que ceci concorde bien avec les données expérimentales limitées, couramment disponibles dans la littérature, bien que la question du comportement vrai, limité de la sphère "Touchant" la paroi demeure sans solution.

Zusammenfassung—Unter Anwendung bipolarer Koordinationen wird eine präzise Lösung der Stokes Gleichungen für die Übertragungs- und Rotationsgeschwindigkeiten einer neutral schwebenden Kugel gewonnen, die sich in der Nähe einer einzigen ebenen Wand unter dem Einfluss einer einfachen Querströmung bewegt. Die Lösung, die für Reynolds-Zahlen bei geringer Querwirkung gültig ist, lässt sich auf alle Verhältnisse des Kugelhalbmessers zu dem Abstand der Kugel von der Wand beziehen. Diese formale Lösung wird durch zwei asymptotische Lösungen ergänzt:

(i) Eine an die Schmierungs-Theorie anschliessende Näherung, die für den Fall anwendbar ist, in dem sich die Kugel sehr nahe der Wand befindet.

(ii) eine auf der "Reflektionsmethode" beruhende Annäherung, die für den entgegengesetzten Fall zutrifft.

Es wurde erwiesen, dass die Übereinstimmung mit den beschränkten Versuchsdaten, die zur Zeit im Schrifttum verfügbar sind, gut ist, obwohl die Frage des wahren Grenzverhaltens einer Kugel, die eine Wand "berührt", nach wie vor ungelöst ist.