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Escuela de Física, Facultad de Ciencias
Universidad Industrial de Santander
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On the relativistic function of velocity distribution of Maxwell-Jüttner

Juan Diego Figueroa Hernández - 2200815
Escuela de Física
Universidad Industrial de Santander
Bucaramanga, Colombia

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Índice

1. Abstract

In this work, the expression for the relativistic velocity distribution of Maxwell-Jüttner was derived using an undergraduate-level formalism. The main expected values were obtained and compared with those of the Maxwell-Boltzmann distribution. Additionally, the distribution was analyzed in a non-relativistic regime, showing that both distributions converged to the same result. Applications of both models were made to compare their predictions graphically in the cases of an ideal gas at 10 times the temperature of the Sun's interior, an ideal gas at the temperature of red giants' interiors, and an ideal gas at the temperature found in quasars. It was found that the Maxwell-Jüttner velocity distribution fully corrects the physically meaningless non-zero probabilities predicted by the classical ideal gas. It was also found that the value of the ratio between the particle's rest mass and the system's temperature determines whether or not a relativistic treatment is necessary for the velocity distribution function of the system.

2. Introduction

The Maxwell-Boltzmann distribution describes the probability density of velocities in an ideal gas according to Maxwell-Boltzmann statistics. It provides information on how the velocities of molecules or particles are distributed in an ideal gas at a given temperature. The distribution is given by the function:

$$D_{MB}(v) = 4\pi(\frac{m}{2\pi k_B T})^{3/2} v^2 e^{\frac{-mv^2}{2k_B T}} \tag{1}$$

where v is the velocity, k_B is the Boltzmann constant, m is the mass of the constituent particle of the gas, and T is the temperature.[1].

A fundamental issue is the fact that the distribution of velocities in Maxwell-Boltzmann distribution allows for a non-zero probability of finding particles with velocities greater than the speed of light in a vacuum c. However, according to the theory of relativity, only velocities lower than c have physical meaning for an inertial observer. Therefore, in this article, we will analyze a velocity distribution function capable of addressing this problem: the distribution function proposed by Ferencz Jüttner. For this purpose, in Section 3, we will derive this distribution function. In Section 4, we will derive the expected values and compare them with the respective values for the Maxwell-Boltzmann distribution. In Section 5, we will analyze the limit in which both distributions converge to the same result. In Section 6, we will present specific graphical results for both distributions. Finally, in Section 7, we will present the conclusions and final remarks.

3. Deduction of the Maxwell-Jüttner distribution function

We begin by performing an initial procedure analogous to the derivation of the Maxwell-Boltzmann distribution function in statistical physics. We take a differential of probability for the Maxwell-Jüttner distribution function and equate it to the differential given by the probability

$$P(\vec{p}) = \frac{e^{-\frac{E}{k_B T}}}{Z},$$

where Z is the normalization factor. Thus, we have

$$D_{MJ}(p)dp = P(\vec{p})d\vec{p} = P(\vec{p})4\pi p^2 dp = 4\pi \frac{1}{Z} p^2 e^{-\frac{E}{k_B T}} dp. \tag{2}$$

Notice the similarity between Equation 2 and Equation 1. Next, we need to find the normalization factor Z by carrying out the following procedure:

$$\int_0^\infty D_{MJ}(p)dp = 1 = \int_0^\infty 4\pi \frac{1}{Z} p^2 e^{-\frac{E}{k_B T}} dp \to Z = 4\pi \int_0^\infty p^2 e^{-\frac{E}{k_B T}} dp,$$

However, in this case, $E \neq \frac{p^2}{2m}$ since, according to the theory of relativity, $E = \sqrt{m_0 c^2 + p^2 c^2} = m_0 c^2 \sqrt{1 + \frac{p^2}{m_0^2 c^2}}$ where $p = \gamma m_0 v$ [2]. Therefore, the function we are looking for is

$$D_{MJ}(p)dp = \frac{p^2 e^{\frac{-m_0 c^2}{k_B T}\sqrt{1+p^2/m_0^2 c^2}}}{\int_0^\infty p^2 e^{\frac{-m_0 c^2}{k_B T}\sqrt{1+p^2/m_0^2 c^2}} dp} dp.$$

Note that if we had employed only the kinetic energy given by $E_k = m_0 c^2(\gamma - 1)$, the last expression would have been the same. Continuing, since $p = \gamma m_0 v$, we have

$$dp = m_0(1 - v^2/c^2)^{-3/2} \left(v^2/c^2 + 1 - v^2/c^2\right) dv = m_0 \gamma^3 dv,$$

substituting this expression, we obtain

$$D_{MJ}(v)dv = \frac{\gamma^2 m_0^2 v^2 e^{\frac{-m_0c^2}{k_BT}\sqrt{1+\gamma^2 m_0^2 v^2/m_0^2c^2}} m_0\gamma^3}{\int_0^\infty \gamma^2 m_0^2 v^2 e^{\frac{-m_0c^2}{k_BT}\sqrt{1+\gamma^2 m_0^2 v^2/m_0^2c^2}} m_0\gamma^3 dv} dv.$$

Also, observe that

$$1 + \frac{\gamma^2 m_0^2 v^2}{m_0^2 c^2} = 1 + \frac{v^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2} = \gamma^2,$$

which, upon substitution and simplification, yields the following expression:

$$D_{MJ}(v)dv = \frac{v^2\gamma^5 e^{\frac{-m_0c^2\gamma}{k_BT}} dv}{\int_0^c v^2\gamma^5 e^{\frac{-m_0c^2\gamma}{k_BT}} dv}, \tag{3}$$

In the integral, it is evident that the upper limit changes as $p \to \infty$, which implies $v \to c$. Focusing on the integral, it is nontrivial. To solve it, we make the following substitutions:

$$\beta = v/c \to tanh(\theta) = \beta \to d\beta = \frac{d\theta}{cosh^2(\theta)},$$

and we take $\epsilon = \frac{m_0c^2}{k_BT}$. It is important to note that from now on, for simplicity of calculations, we will analyze β instead of v. Additionally, this approach allows us to visualize the results more easily. With these substitutions, the integral takes the following form:

$$Z_{MJ} = \int_0^\infty e^{-\epsilon cosh(\theta)} sinh^2(\theta) cosh(\theta) d\theta,$$

because $\gamma^5 = (\frac{1}{\sqrt{1-\tanh^2(\theta)}})^5 = \cosh^5(\theta)$. Using the following trigonometric identity:

$$sinh^2(\theta)cosh(\theta) = \frac{1}{4}(cosh(3\theta) - cosh(\theta)),$$

the integral becomes:

$$Z_{MJ} = \frac{1}{4}\left(\int_0^\infty e^{-\epsilon cosh(\theta)} cosh(3\theta) d\theta - \int_0^\infty e^{-\epsilon cosh(\theta)} cosh(\theta) d\theta\right),$$

Integrals of this form are well-known, as they correspond to modified Bessel functions of the second kind with parameters ν and ϵ, given by:

$$K_\nu(\epsilon) = \int_0^\infty e^{-\epsilon cosh(t)} cosh(\nu t) dt, \tag{4}$$

Hence, we have $Z_{MJ} = \frac{1}{4}(K_3(\epsilon) - K_1(\epsilon))$. Using the following recurrence relation:

$$K_{\nu+1}(\epsilon) - K_{\nu-1}(\epsilon) = \frac{2\nu}{\epsilon} K_\nu(\epsilon),$$

taking $\nu = 2$, we obtain $Z_{MJ} = \frac{K_2(\epsilon)}{\epsilon}$. Finally, we obtain the definitive expressions for the Maxwell-Jüttner distribution function in terms of β and v [3]:

$$D_{MJ}(\beta)d\beta = \frac{\epsilon\beta^2\gamma^5 e^{-\epsilon\gamma}}{K_2(\epsilon)} d\beta, \tag{5}$$

$$D_{MJ}(v)dv = \frac{\epsilon v^2\gamma^5 e^{-\epsilon\gamma}}{c^3 K_2(\epsilon)} dv. \tag{6}$$

4. Expected values of the Maxwell-Jüttner distribution function

To obtain the most probable value of β, we set

$$\frac{\partial D_{MJ}(\beta)}{\partial \beta} = 0 \to \frac{\epsilon}{K_2(\epsilon)} \frac{\partial}{\partial \beta} \left(\frac{\beta^2 e^{-\epsilon/\sqrt{1-\beta^2}}}{(1-\beta^2)^{5/2}} \right) = 0,$$

from now on, we will simplify the calculations using tools such as MAPLE or Wolfram Mathematica, given the extent of the expressions to be developed. We have

$$e^{-\epsilon/\sqrt{1-\beta^2}} \beta \left(\beta^2 (1 - \epsilon\sqrt{1-\beta^2}) - 3\beta^4 + 2 \right) = 0,$$

The root of interest between 0 and 1 is given by

$$\beta_p = \sqrt{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}, \tag{7}$$

with

$$\begin{gathered} \alpha_1 = -\tfrac{\epsilon^2}{27} + \tfrac{\epsilon^4}{27\sqrt[3]{\xi}} \\ \alpha_2 = \tfrac{2\epsilon^2}{9\sqrt[3]{\xi}} \\ \alpha_3 = \tfrac{\sqrt[3]{\xi}}{27} \\ \alpha_4 = -\tfrac{1}{9} + \tfrac{25}{3\sqrt[3]{\xi}} \\ \xi = -\epsilon^6 - 9\epsilon^4 - 351\epsilon^2 + 54\sqrt{3}\sqrt{-\epsilon^6 - 13\epsilon^4 - 375\epsilon^2} + 3375. \end{gathered}$$

Now we will find the average value of β:

$$< \beta >= \int_0^1 \beta D_{MJ}(\beta) d\beta = \int_0^1 \frac{\epsilon \beta^3 \gamma^5 e^{-\epsilon\gamma}}{K_2(\epsilon)} d\beta = \frac{\epsilon}{K_2(\epsilon)} \left(\frac{\Gamma(3,\epsilon)}{\epsilon^3} - \frac{e^{-\epsilon}}{\epsilon} \right), \tag{8}$$

where $\Gamma(a,b) = \int_b^\infty t^{a-1} e^{-t} dt$, is the incomplete gamma function. We continue with the root mean square:

$$\beta_{rms} = \sqrt{\langle \beta^2 \rangle} = \sqrt{\int_0^1 \beta^2 D_{MJ}(\beta) d\beta} = \sqrt{\int_0^1 \frac{\epsilon \beta^4 \gamma^5 e^{-\epsilon\gamma}}{K_2(\epsilon)} d\beta} = \sqrt{1 - \frac{\epsilon K_1(\epsilon)}{K_2(\epsilon} + \frac{\epsilon\pi}{2K_2(\epsilon)} (1 - \epsilon K_0(\epsilon) L_{-1}(\epsilon) - \epsilon K_1(\epsilon) L_0(\epsilon))}, \tag{9}$$

where $L_\nu(\epsilon)$ are the modified Struve functions. In order to compare these expected values with the corresponding ones for the Maxwell-Boltzmann distribution function, we also express this function in terms of β and ϵ:

$$D_{MB}(\beta) d\beta = 4\pi (\frac{m_0}{2\pi k_B T})^{3/2} \beta^2 c^2 e^{-\frac{m_0 \beta^2 c^2}{2k_B T}} c d\beta = \sqrt{2/\pi} \epsilon^{3/2} \beta^2 e^{-\epsilon\beta^2/2} d\beta, \tag{10}$$

and as we found the expected values for the Maxwell-Jüttner distribution function, we do the same for the Maxwell-Boltzmann function, obtaining:

$$\begin{gathered} \langle \beta \rangle_{MB} = \sqrt{\tfrac{8}{\pi\epsilon}}, \\ \beta_{p(MB)} = \sqrt{2/\epsilon}, \\ \beta_{rms(MB)} = \sqrt{3/\epsilon}, \end{gathered}$$

So, in summary, we have the following comparative table:

Comparison between Maxwell-Boltzmann and Maxwell-Jüttner distributions.

	Maxwell-Boltzmann	Maxwell-Jüttner
Distribution	$\sqrt{2/\pi}\epsilon^{3/2}\beta^2 e^{-\epsilon\beta^2/2}$	$\frac{\epsilon\beta^2\gamma^5 e^{-\epsilon\gamma}}{K_2(\epsilon)}$
β_p	$\sqrt{\frac{2}{\epsilon}}$	$\sqrt{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$
$\langle\beta\rangle$	$\sqrt{\frac{8}{\pi\epsilon}}$	$\frac{\epsilon}{K_2(\epsilon)}\left(\frac{\Gamma(3,\epsilon)}{\epsilon^3} - \frac{e^{-\epsilon}}{\epsilon}\right)$
β_{rms}	$\sqrt{\frac{3}{\epsilon}}$	$\sqrt{1 - \frac{\epsilon K_1(\epsilon)}{K_2(\epsilon} + \frac{\epsilon\pi}{2K_2(\epsilon)}(1 - \epsilon K_0(\epsilon)L_{-1}(\epsilon) - \epsilon K_1(\epsilon)L_0(\epsilon))}$

5. Limiting Case when $v/c \ll 1$

In this limiting case, the following approximations hold:

$$(1-\beta^2)^{-1/2} \approx 1 + \beta^2/2,$$
$$K_2(\epsilon) \approx \sqrt{\frac{\pi}{2\epsilon}} e^{-\epsilon},$$
$$D_{MJ}(\beta) \approx \epsilon\beta^2 e^{-\epsilon}\sqrt{\frac{2\epsilon}{\pi}} e^{\epsilon}\gamma^5 e^{-\epsilon\gamma} \approx \sqrt{\frac{2}{\pi}}\epsilon^{3/2}\beta^2(1+\frac{5\beta^2}{2})e^{-\epsilon(\gamma-1)} \approx \sqrt{\frac{2}{\pi}}\epsilon^{3/2}\beta^2 e^{-\epsilon\beta^2/2},$$

However, when comparing this last result with the one obtained in equation 10, it can be seen that they are the same. Therefore, as expected, in the classical limit, both distributions converge to the same result, i.e.,

$$D_{MJ}(\beta) \approx D_{MB}(\beta).$$

6. Specific Results

6.1. Ideal Gas at 10 Times the Temperature of the Solar Interior

In order to analyze a case close to our experience, let's consider a plasma of deuterons and electrons. According to the Lawson criterion [4], the minimum temperature for nuclear fusion to occur is estimated to be $1{,}5 \times 10^8[K]$, which is approximately 10 times the temperature of the solar interior [5]. We know that electrons, depending on the plasma temperature, can reach considerable velocities relative to c, so it is a good case to analyze with both models.

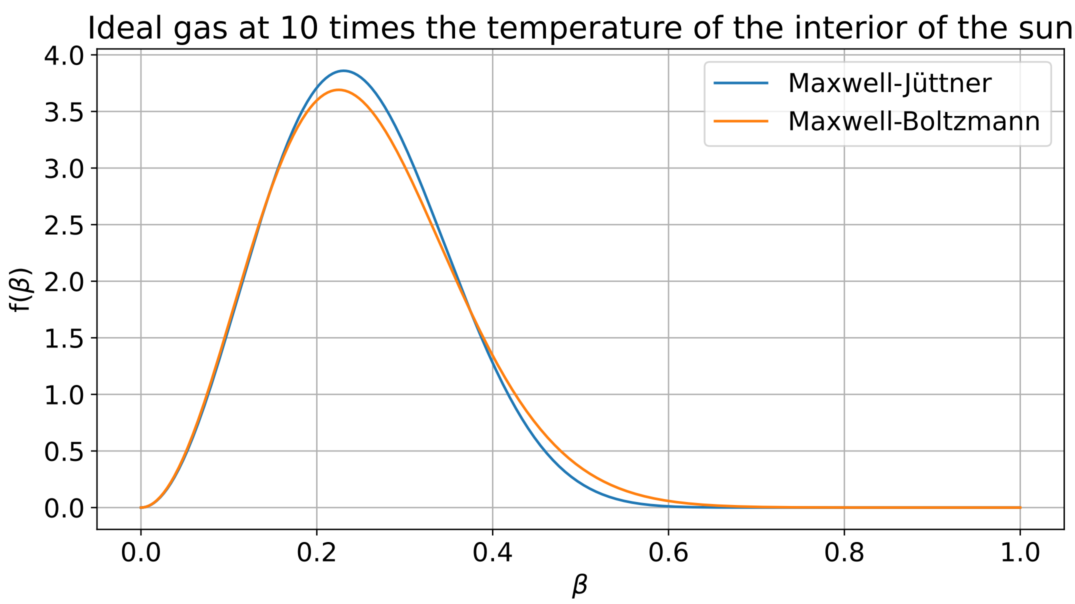

Figura 1: This figure shows the Maxwell-Jüttner and Maxwell-Boltzmann distributions for an ideal gas of electrons at the temperature of the Lawson criterion, $1{,}5 \times 10^8[K]$. Note that the axes are dimensionless due to the normalization performed in the theoretical development.

As shown in Figure 1, there is not a significant difference between both distributions. Now let's analyze the same case with only 5 times the minimum temperature required for such fusion according to the Lawson criterion, which is approximately the temperature of the interior of red giants [6].

In Figure 2, it can be observed that now there is a clear difference between the two distributions. A significant portion of the Maxwell-Boltzmann curve appears for values of $\beta > 1$, which would imply electron

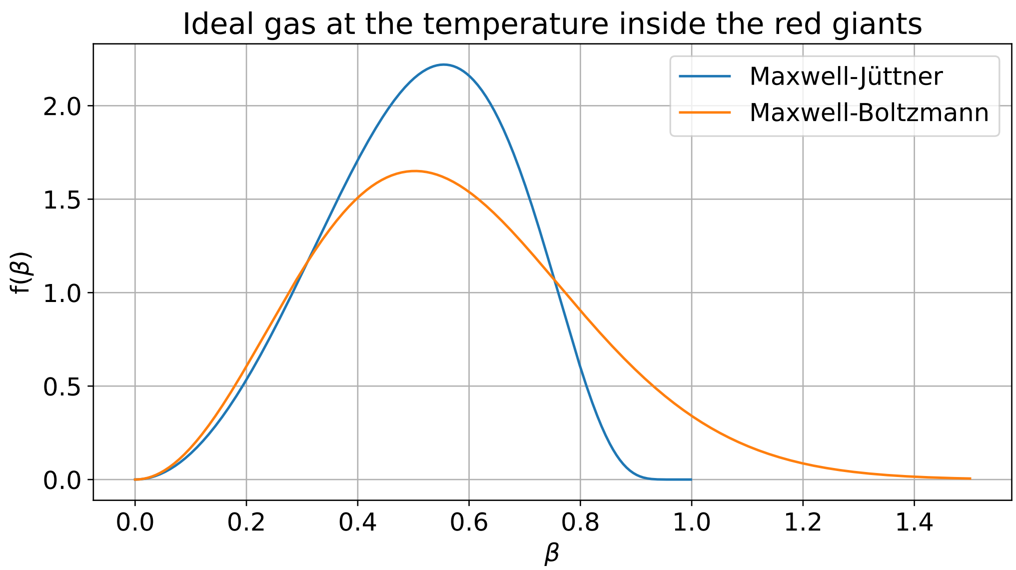

Figura 2: This figure shows the Maxwell-Jüttner and Maxwell-Boltzmann distributions for an ideal gas of electrons at a temperature of approximately $6 \times 10^8[K]$.

velocities greater than the speed of light, according to the definition of $\beta = v/c$. This does not occur with the Maxwell-Jüttner distribution.

6.2. Ideal Gas at Quasar Temperature

Quasars are highly luminous astronomical objects located in the centers of some galaxies. They consist of a black hole and a massive gas cloud orbiting it. The gravitational attraction of the black hole causes the gas to be accreted, and in this process, gas particles reach very high temperatures and velocities close to c. Some studies estimate that quasars can reach temperatures on the order of $10^{13}[K]$ [7]. Such high temperatures can lead to very high velocities of the atoms and ions that make up the gas surrounding the black hole. Therefore, it is another good case to analyze with both models, assuming an ideal gas at that temperature composed of protons.

As observed in Figure 3, the Maxwell-Boltzmann distribution model predicts values of β_p, $\langle\beta\rangle$, and β_{rms} above 1, indicating $v > c$. Therefore, it is a case where the Maxwell-Boltzmann model fails, at least considering only the results of relativity. However, it should be noted that the Jüttner model does not predict such values, but provides a function that is only defined for $0 < \beta < 1$ as presented.

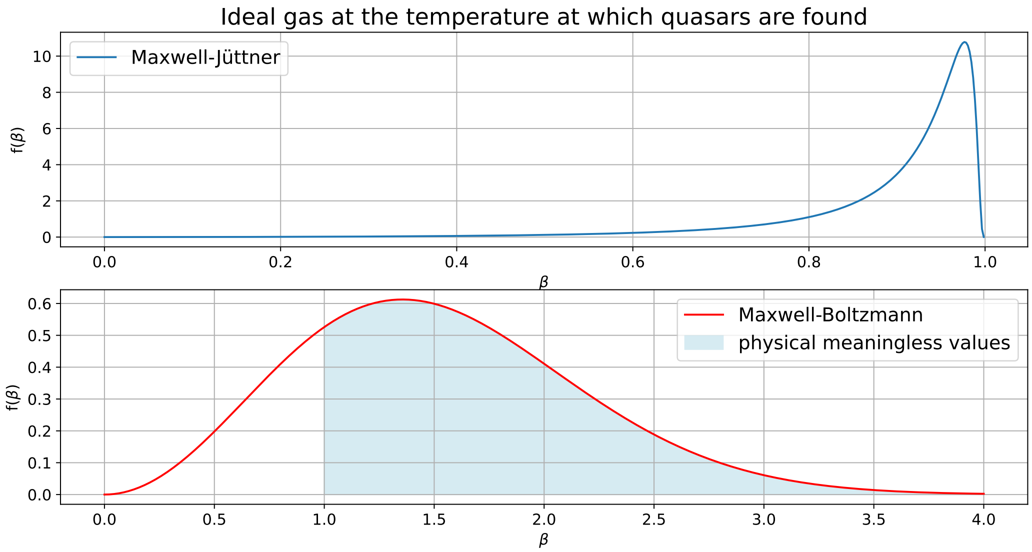

Figura 3: This figure shows the Maxwell-Jüttner and Maxwell-Boltzmann distributions for an ideal gas of protons at a temperature of $10^{13}[K]$.

7. Conclusions

Based on the presented theoretical development and the respective analysis of results, the following conclusions are drawn:

- The Maxwell-Jüttner velocity distribution effectively corrects the unphysical non-zero probabilities predicted by the classical ideal gas model by incorporating relativistic results in its formulation.
- The Maxwell-Jüttner velocity distribution shares some limitations of the distributions for the classical ideal gas, such as neglecting interactions and quantum effects, and overlooking the existence of antiparticles.
- It was found that the value of the ratio between the rest mass of the considered particle and the temperature of the system determines the need to consider a relativistic treatment for the velocity distribution function of the system.

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