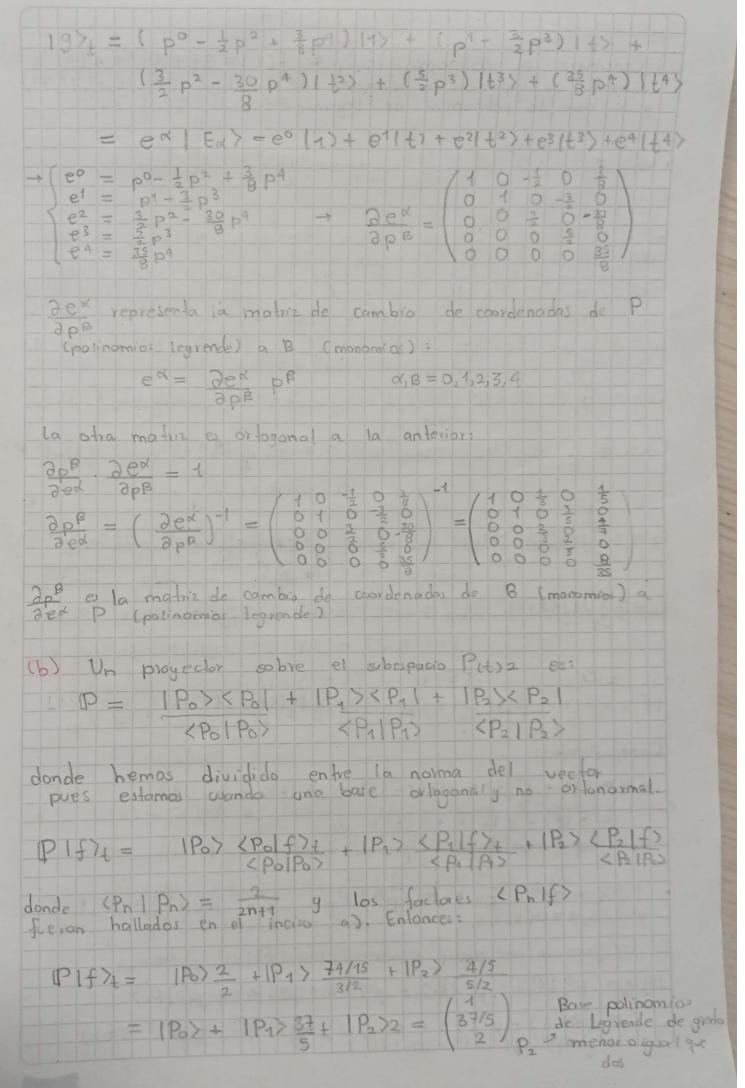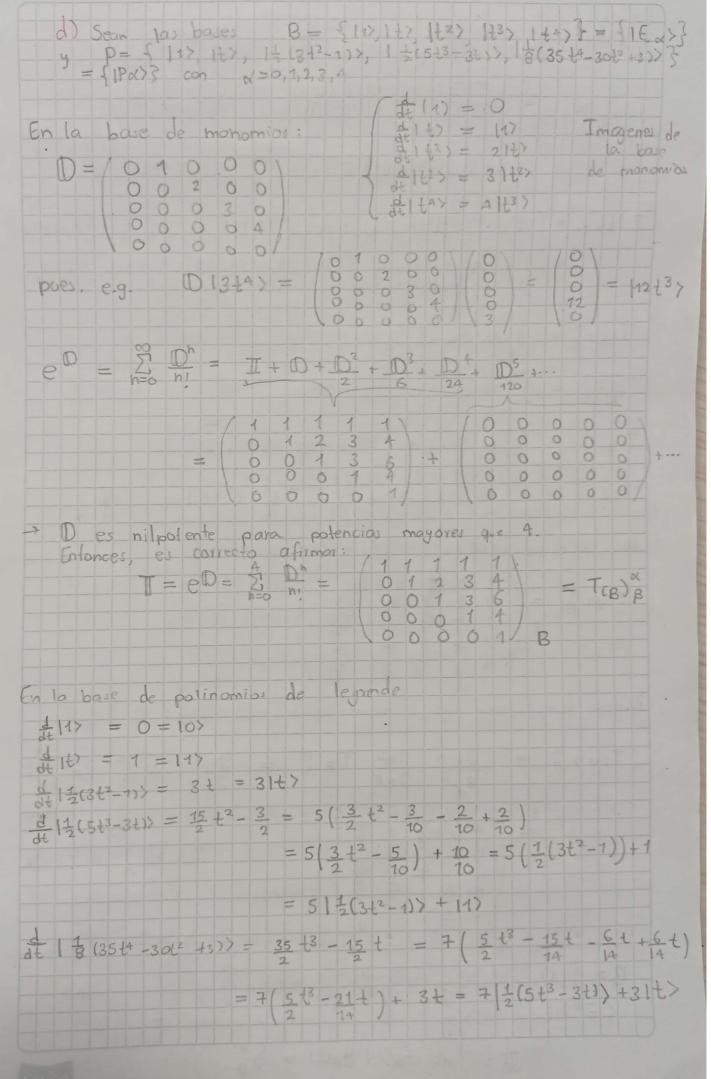4) Suponga que $AB = BA$. Demuestre que:

(a) $(A+B)^2 = A^2 + 2AB + B^2$

$(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB$

$= A^2 + AB + AB + B^2 = A^2 + 2AB + B^2$ Rta/.

(b) $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

$(A+B)^3 = (A+B)(A^2 + 2AB + B^2) = A^3 + 2A^2B + AB^2 +$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad BA^2 + 2BAB + B^3$

$= A^3 + 2A^2B + AB^2 + A^2B + 2AB^2 + B^3$

$= A^3 + 3A^2B + 3AB^2 + B^3$ Rta/.

5) Suponga que un operador L puede ser escrito como la composición de dos operadores $L = L_- L_+$ con $[L_-, L_+] = I$. Demostrar que:

• Si $L|x\rangle = \lambda|x\rangle$ y $|y\rangle = L_+|x\rangle$ entonces $L|y\rangle = (\lambda+1)|y\rangle$

Como $|y\rangle = L_+|x\rangle$:   usando $[L_-, L_+] = I$

$L|y\rangle = (L_-L_+)L_+|x\rangle = (I + L_+L_-)L_+|x\rangle = L_+|x\rangle + L_+L_-L|x\rangle$

$= |y\rangle + L_+(\lambda|x\rangle) = |y\rangle + \lambda L_+|x\rangle = |y\rangle + \lambda|y\rangle$

$= (1+\lambda)|y\rangle$ Rta/.

• Si $L|x\rangle = \lambda|x\rangle$ y $|z\rangle = L_-|x\rangle$ entonces $L|z\rangle = (\lambda-1)|z\rangle$

→ $L|z\rangle = (L_-L_+)(L_-|x\rangle) = L_-(L_+L_-)|x\rangle = L_-L|x\rangle = \lambda L_-|x\rangle = \lambda|z\rangle$

→ $L_+L_-|z\rangle = L_+L_-L_-|x\rangle = (L_-L_+ - I)L_-|x\rangle$

$= L_-(L_+L_-|x\rangle) - IL_-|x\rangle = L_-L|x\rangle - L_-|x\rangle$

$= L_-(\lambda|x\rangle) - |z\rangle = \lambda(L_-|x\rangle) - |z\rangle$

$= \lambda|z\rangle - |z\rangle = (\lambda-1)|z\rangle$

2). Considere el polinomio $|f\rangle_t \Leftrightarrow f(t) = 5t + 3t^2 + 4t^3$
encuentre su expresión en términos de las bases $B = \{1, t, t^2, t^3, t^4\}$
y la base de polinomios de Legrende $P = \{P_0, P_1, P_2, P_3, P_4\}$

$\cdot\ |f\rangle_t = 5t + 3t^2 + 4t^3 = 5|t\rangle + 3|t^2\rangle + 4|t^3\rangle = \begin{pmatrix} 0 \\ 5 \\ 3 \\ 4 \\ 0 \end{pmatrix}_B$

$\cdot$ Ahora para la base de legrende: $P = \{|1\rangle, |t\rangle, |\frac{1}{2}(3t^2-1)\rangle,$
$|\frac{1}{2}(5t^3-3t)\rangle, |\frac{1}{8}(35t^4 - 30t^2 + 3)\rangle\}$

$|f\rangle_t = \sum_{n=0}^{4} C_n |P_n\rangle$, $\qquad C_n = \int_{-1}^{1} dt\, f(t)\, P_n(t) \Big/ \left(\int dt\, P_n^2(t)\right)$
$\qquad\qquad$ con $\int_{-1}^{1} dt\, P_n^2(t) = \frac{2}{2n+1}$

$C_0 = \frac{1}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)(1) = \frac{1}{2}\left[\frac{5}{2}t^2 + t^3 + t^4\right]_{-1}^{1} = 1$

$C_1 = \frac{3}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)(t) = \frac{3}{2}\left[\frac{5}{3}t^3 + \frac{3}{4}t^4 + \frac{4}{5}t^5\right]_{-1}^{1} = \frac{37}{5}$

$C_2 = \frac{5}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)(\frac{1}{2}(3t^2-1)) = \frac{5}{4}\left[15t^4\frac{9}{5}t^5 + \frac{12}{6}t^6 - \frac{5}{2}t^2 - t^3 - t^4\right]_{-1}^{1}$
$\qquad = 2$

$C_3 = \frac{7}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)(\frac{1}{2}(5t^3-3t)) = \frac{7}{4}\left[5t^5 + \frac{15}{6}t^6 + \frac{20}{7}t^7 - 5t^3 - \frac{9}{4}t^4 - \frac{12}{5}t^5\right]_{-1}^{1}$
$\qquad = \frac{8}{5}$

$C_4 = \frac{9}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)\frac{1}{8}(35t^4 - 30t^2 + 3) = \frac{9}{16}\left[\frac{35}{2}x^8 + 15x^7 + \frac{55}{6}x^6 - 18x^5\right.$
$\qquad\qquad\qquad\left. - \frac{69}{2}x^4 + 3x^3 + \frac{15}{2}x^2\right]_{-1}^{1}$
$\qquad = 0$

$|f\rangle_t = |1\rangle + \frac{37}{5}|t\rangle + 2|\frac{1}{2}(3t^2-1)\rangle + \frac{8}{5}|\frac{1}{2}(5t^3-3t)\rangle = \begin{pmatrix} 1 \\ 37/5 \\ 2 \\ 8/5 \\ 0 \end{pmatrix}_P$

Comprobación:

$|f\rangle_t = |1\rangle + \frac{37}{5}|t\rangle + 3|t^2\rangle - |1\rangle + 4|t^3\rangle - \frac{24}{10}|t\rangle$

$\qquad = 5|t\rangle + 3|t^2\rangle + 4|t^3\rangle$ ✓

$\cdot\ |g\rangle_t = e^\alpha |E_\alpha\rangle = p^\beta |P_\beta\rangle$ $\quad$ con $\{|E_\alpha\rangle\} = \{1, t, t^2, t^3, t^4\}$
$\quad$ y $\quad \alpha, \beta = 0, 1, 2, 3, 4$

Si $|g\rangle_t$ es un polinomio genérico de $P_{(t)4}$, entonces:

$|g\rangle_t = p^0|P_0\rangle + p^1|P_1\rangle + p^2|P_2\rangle + p^3|P_3\rangle + p^4|P_4\rangle$
$\qquad = p^0(1) + p^1(t) + p^2(\frac{1}{2}(3t^2-1)) + p^3(\frac{1}{2}(5t^3-3t))$
$\qquad\quad + p^4(\frac{1}{8}(35t^4 - 30t^2 + 3))$

$$|g\rangle_t = \left(p^0 - \tfrac{1}{2}p^2 + \tfrac{3}{8}p^4\right)|1\rangle + \left(p^1 - \tfrac{3}{2}p^3\right)|t\rangle +$$

$$\left(\tfrac{3}{2}p^2 - \tfrac{30}{8}p^4\right)|t^2\rangle + \left(\tfrac{5}{2}p^3\right)|t^3\rangle + \left(\tfrac{35}{8}p^4\right)|t^4\rangle$$

$$= e^\alpha |E_\alpha\rangle = e^0|1\rangle + e^1|t\rangle + e^2|t^2\rangle + e^3|t^3\rangle + e^4|t^4\rangle$$

$$\rightarrow \begin{cases} e^0 = p^0 - \tfrac{1}{2}p^2 + \tfrac{3}{8}p^4 \\ e^1 = p^1 - \tfrac{3}{2}p^3 \\ e^2 = \tfrac{3}{2}p^2 - \tfrac{30}{8}p^4 \\ e^3 = \tfrac{5}{2}p^3 \\ e^4 = \tfrac{35}{8}p^4 \end{cases} \rightarrow \frac{\partial e^\alpha}{\partial p^\beta} = \begin{pmatrix} 1 & 0 & -\tfrac{1}{2} & 0 & \tfrac{3}{8} \\ 0 & 1 & 0 & -\tfrac{3}{2} & 0 \\ 0 & 0 & \tfrac{3}{2} & 0 & -\tfrac{30}{8} \\ 0 & 0 & 0 & \tfrac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & \tfrac{35}{8} \end{pmatrix}$$

$\dfrac{\partial e^\alpha}{\partial p^\beta}$ representa la matriz de cambio de coordenadas de P (polinomios legrende) a B (monomios):

$$e^\alpha = \frac{\partial e^\alpha}{\partial p^\beta} p^\beta \qquad \alpha, \beta = 0,1,2,3,4$$

La otra matriz es ortogonal a la anterior:

$$\frac{\partial p^\beta}{\partial e^\alpha} \cdot \frac{\partial e^\alpha}{\partial p^\beta} = 1$$

$$\frac{\partial p^\beta}{\partial e^\alpha} = \left(\frac{\partial e^\alpha}{\partial p^\beta}\right)^{-1} = \begin{pmatrix} 1 & 0 & -\tfrac{1}{2} & 0 & \tfrac{3}{8} \\ 0 & 1 & 0 & -\tfrac{3}{2} & 0 \\ 0 & 0 & \tfrac{3}{2} & 0 & -\tfrac{30}{8} \\ 0 & 0 & 0 & \tfrac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & \tfrac{35}{8} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & \tfrac{1}{3} & 0 & \tfrac{1}{5} \\ 0 & 1 & 0 & \tfrac{3}{5} & 0 \\ 0 & 0 & \tfrac{2}{3} & 0 & \tfrac{4}{7} \\ 0 & 0 & 0 & \tfrac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & \tfrac{8}{35} \end{pmatrix}$$

$\dfrac{\partial p^\beta}{\partial e^\alpha}$ es la matriz de cambio de coordenadas de B (monomios) a P (polinomios legrende)

(b) Un proyector sobre el subespacio $P_{(t)2}$ es:

$$\mathbb{P} = \frac{|P_0\rangle\langle P_0|}{\langle P_0|P_0\rangle} + \frac{|P_1\rangle\langle P_1|}{\langle P_1|P_1\rangle} + \frac{|P_2\rangle\langle P_2|}{\langle P_2|P_2\rangle}$$

donde hemos dividido entre la norma del vector pues estamos usando una base ortogonal y no ortonormal.

$$\mathbb{P}|f\rangle_t = \frac{|P_0\rangle\langle P_0|f\rangle_t}{\langle P_0|P_0\rangle} + \frac{|P_1\rangle\langle P_1|f\rangle_t}{\langle P_1|P_1\rangle} + \frac{|P_2\rangle\langle P_2|f\rangle}{\langle P_2|P_2\rangle}$$

donde $\langle P_n|P_n\rangle = \dfrac{2}{2n+1}$ y los factores $\langle P_n|f\rangle$ fueron hallados en el inciso a). Entonces:

$$\mathbb{P}|f\rangle_t = \frac{|P_0\rangle\, 2}{2} + \frac{|P_1\rangle\, 74/15}{3/2} + \frac{|P_2\rangle\, 4/5}{5/2}$$

$$= |P_0\rangle + |P_1\rangle\tfrac{37}{5} + |P_2\rangle 2 = \begin{pmatrix} 1 \\ 37/5 \\ 2 \end{pmatrix}_{P_2}$$

Base polinomios de Legrende de grado menor o igual que dos

d) Sean las bases $\quad B = \{|1\rangle, |t\rangle, |t^2\rangle, |t^3\rangle, |t^4\rangle\} = \{|E_\alpha\rangle\}$

y $\quad P = \{|1\rangle, |t\rangle, |\frac{1}{2}(3t^2-1)\rangle, |\frac{1}{2}(5t^3-3t)\rangle, |\frac{1}{8}(35t^4-30t^2+3)\rangle\}$

$= \{|P_\alpha\rangle\}$ con $\quad \alpha = 0,1,2,3,4$

En la base de monomios:

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} \frac{d}{dt}|1\rangle = 0 \\ \frac{d}{dt}|t\rangle = |1\rangle \\ \frac{d}{dt}|t^2\rangle = 2|t\rangle \\ \frac{d}{dt}|t^3\rangle = 3|t^2\rangle \\ \frac{d}{dt}|t^4\rangle = 4|t^3\rangle \end{cases}$$

Imágenes de la base de monomios

pues, e.g. $\quad D|3t^4\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 12 \\ 0 \end{pmatrix} = |12t^3\rangle$

$$e^D = \sum_{h=0}^{\infty} \frac{D^h}{h!} = \underbrace{\mathbb{I} + D + \frac{D^2}{2} + \frac{D^3}{6} + \frac{D^4}{24}}_{} + \underbrace{\frac{D^5}{120}}_{} + \cdots$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \cdots$$

$\rightarrow$ D es nilpotente para potencias mayores que 4.

Entonces, es correcto afirmar:

$$\mathbb{I} = e^D = \sum_{h=0}^{4} \frac{D^h}{h!} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_B = T_{(B)}{}^\alpha{}_\beta$$

En la base de polinomios de legende

$\frac{d}{dt}|1\rangle = 0 = |0\rangle$

$\frac{d}{dt}|t\rangle = 1 = |1\rangle$

$\frac{d}{dt}|\frac{1}{2}(3t^2-1)\rangle = 3t = 3|t\rangle$

$\frac{d}{dt}|\frac{1}{2}(5t^3-3t)\rangle = \frac{15}{2}t^2 - \frac{3}{2} = 5\left(\frac{3}{2}t^2 - \frac{3}{10} - \frac{2}{10} + \frac{2}{10}\right)$

$\qquad = 5\left(\frac{3}{2}t^2 - \frac{5}{10}\right) + \frac{10}{10} = 5\left(\frac{1}{2}(3t^2-1)\right) + 1$

$\qquad = 5|\frac{1}{2}(3t^2-1)\rangle + |1\rangle$

$\frac{d}{dt}|\frac{1}{8}(35t^4-30t^2+3)\rangle = \frac{35}{2}t^3 - \frac{15}{2}t = 7\left(\frac{5}{2}t^3 - \frac{15}{14}t - \frac{6}{14}t + \frac{6}{14}t\right)$

$\qquad = 7\left(\frac{5}{2}t^3 - \frac{21}{14}t\right) + 3t = 7|\frac{1}{2}(5t^3-3t)\rangle + 3|t\rangle$

$$\mathbb{D} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{T} = e^{\mathbb{D}} = \sum_{n=0}^{4} \frac{\mathbb{D}^n}{n!} = \begin{pmatrix} 1 & 1 & 3/2 & 7/2 & 75/8 \\ 0 & 1 & 3 & 15/2 & 41/2 \\ 0 & 0 & 1 & 5 & 35/2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_P = T_{(P)}{}^{\alpha}{}_{\beta}$$

¿Cómo transforman las representaciones matriciales de $\mathbb{T}$?

$$T_{(P)}{}^{\alpha}{}_{\beta} = \frac{\partial p^{\alpha}}{\partial e^{\mu}} \frac{\partial e^{\nu}}{\partial p^{\beta}} T_{(B)}{}^{\mu}{}_{\nu} = \frac{\partial p^{\alpha}}{\partial e^{\mu}} T_{(B)}{}^{\mu}{}_{\nu} \frac{\partial e^{\nu}}{\partial p^{\beta}}$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & 0 & 1/5 \\ 0 & 1 & 0 & 3/5 & 0 \\ 0 & 0 & 2/3 & 0 & 4/7 \\ 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 8/35 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1/2 & 0 & 3/8 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 3/2 & 0 & -15/4 \\ 0 & 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 0 & 35/8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 3/2 & 7/2 & 75/8 \\ 0 & 1 & 3 & 15/2 & 41/2 \\ 0 & 0 & 1 & 5 & 35/2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = T_{(B)}{}^{\alpha}{}_{\beta} \checkmark$$

$$\Rightarrow T_{(P)}{}^{\alpha}{}_{\beta} = \frac{\partial p^{\alpha}}{\partial e^{\mu}} \frac{\partial e^{\nu}}{\partial p^{\beta}} T_{(B)}{}^{\mu}{}_{\nu}$$

Igualmente y análogamente: $T_{(B)}{}^{\mu}{}_{\nu} = \frac{\partial e^{\mu}}{\partial p^{\alpha}} \frac{\partial p^{\beta}}{\partial e^{\nu}} T_{(P)}{}^{\alpha}{}_{\beta}$

• $\text{Tr}\left(T_{(P)}{}^{\alpha}{}_{\beta}\right) = T_{(P)}{}^{\alpha}{}_{\alpha} = T_{(P)}{}^{0}{}_{0} + T_{(P)}{}^{1}{}_{1} + T_{(P)}{}^{2}{}_{2} + T_{(P)}{}^{3}{}_{3}$
$$+ T_{(P)}{}^{4}{}_{4}$$

$$= 1+1+1+1+1 = 5$$

• $\text{Tr}\left(T_{(B)}{}^{\alpha}{}_{\beta}\right) = T_{(B)}{}^{\alpha}{}_{\alpha} = T_{(B)}{}^{0}{}_{0} + T_{(B)}{}^{1}{}_{1} + T_{(B)}{}^{2}{}_{2} + T_{(B)}{}^{3}{}_{3} + T_{(B)}{}^{4}{}_{4}$

$$= 1+1+1+1+1 = 5$$

↳ Ambas trazas son iguales.

matriz triángulor

• $\det\left(T_{(P)}\right) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (1)(1)(1)(1)(1) = 1$

$$\det(\mathbb{T}_{(B)}) = \begin{pmatrix} 1 & 1 & 3/2 & 7/2 & 75/8 \\ 0 & 1 & 3 & 15/2 & 41/2 \\ 0 & 0 & 1 & 5 & 35/2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

↱ matriz triángular

$= (1)(1)(1)(1)(1) = 1$

↳ ambos determinantes coinciden

$$\mathbb{P}|f\rangle_t = |1\rangle + \frac{37}{5}|t\rangle + 2\left(\frac{1}{2}(3|t^2\rangle) - |1\rangle\right)$$

$$= |1\rangle + \frac{37}{5}|t\rangle + 3|t^2\rangle - |1\rangle = \frac{37}{5}|t\rangle + 3|t^2\rangle$$

$$= \begin{pmatrix} 0 \\ 37/5 \\ 3 \end{pmatrix}_{B_2} , \quad \text{Base de monomios de grado menor o igual que dos}$$

→ Como vemos, los coeficientes de la proyección de $|f\rangle_t$ en $P(t)_2$ son diferentes para las dos bases. Además, realizamos el operador proyección solamente con la base de polinomios de Legende pues solo podemos proyectar en una base ortogonal.

(c) $\quad T = e^{\mathbb{D}} \equiv \exp(\mathbb{D}) \quad$ con $\mathbb{D} = \frac{d}{dt}$

Ahora: $\quad e^{\mathbb{D}} = \sum\limits_{n=0}^{4} \frac{\mathbb{D}^n}{n!}$

$T^{-1} = ?$

Tomemos: $\quad e^{\mathbb{D}} e^{-\mathbb{D}} = e^{\mathbb{D}+(-\mathbb{D})} e^{[\mathbb{D},-\mathbb{D}]}$

$[\mathbb{D}, -\mathbb{D}] = \mathbb{D}(-\mathbb{D}) - (-\mathbb{D})\mathbb{D} = -\mathbb{D}^2 + \mathbb{D}^2 = \mathbb{O}$ → operador nulo

↳ conmutan

$$e^{\mathbb{D}} e^{-\mathbb{D}} = e^{\mathbb{D}-\mathbb{D}} e^{\mathbb{O}}$$
$$= e^{\mathbb{O}} \mathbb{I}$$
$$= \mathbb{I}$$

$\Rightarrow e^{-\mathbb{D}}$ es la inversa de $e^{\mathbb{D}}$, con $e^{-\mathbb{D}} = \sum\limits_{n=0}^{4} \frac{(-\mathbb{D})^n}{n!}$

$\Rightarrow T^{-1} = e^{-\mathbb{D}} = 1\mathbb{I} - 1\mathbb{D}^1 + \frac{1}{2}\mathbb{D}^2 - \frac{1}{6}\mathbb{D}^3 + \frac{1}{24}\mathbb{D}^4$ RTA/.

$T^\dagger = ?$

$T^\dagger = (e^{\mathbb{D}})^\dagger = \left(\sum\limits_{n=0}^{4} \frac{\mathbb{D}^n}{n!}\right)^\dagger = \sum\limits_{n=0}^{4} \frac{(\mathbb{D}^n)^\dagger}{n!}$

$(\mathbb{D}^n)^\dagger = (\underbrace{\mathbb{D}\mathbb{D}\cdots\mathbb{D}}_{n-veces})^\dagger = \underbrace{\mathbb{D}^\dagger \mathbb{D}^\dagger \cdots \mathbb{D}^\dagger}_{n-veces} = (\mathbb{D}^\dagger)^n$

Como T opera sobre $P(t)_4$, Si $|f\rangle_t \in P(t)_4$ → $|f\rangle_t$ es una función real por ser un polinomio de R_{t14}. Entonces:

$\mathbb{D}^\dagger |f\rangle_t = \frac{d}{dt}^\dagger |f\rangle_t = \frac{d}{dt}|f\rangle_t = \mathbb{D}|f\rangle_t \quad$ → $\mathbb{D}$ es hermítico.

$T^\dagger = \sum\limits_{n=0}^{4} \frac{\mathbb{D}^{n\dagger}}{n!} = \sum\limits_{n=0}^{4} \frac{\mathbb{D}^n}{n!} = e^{\mathbb{D}} = T \Rightarrow T^\dagger = T$ → T es hermítico.

① Considere los siguientes operadores:

$A = A^+$ hermítico, $K = -K^+$ antihermítico;
$U^{-1} = U^+$ unitario, P y Q los operadores
genéricos. Pruebe lo siguiente:

ⓐ $\underline{I}$ $(P^+)^{-1} = (P^{-1})^+$

recordando que

$$\langle x|y\rangle = \langle x|\underline{I}|y\rangle$$
$$= \langle x|PP^{-1}|y\rangle$$
$$= \langle y|(P^{-1})^+ P^+|x\rangle^*$$
$$= \langle y|x\rangle^*$$

Por lo tanto:

$$(P^{-1})^+ \cdot P^+ = \underline{I} \rightarrow (P^{-1})^+ P^+ (P^+)^{-1} = \underline{I}(P^+)^{-1}$$

$$(P^{-1})^+ = (P^+)^{-1}$$

Observación:

Se utilizo la propiedad $(MB)^+ = B^+ M^+$
demostrada en clase

$\underline{II}$ $(PQ)^{-1} = Q^{-1}P^{-1}$

$\Rightarrow (PQ)^{-1}PQ = \underline{I} \Rightarrow (PQ)^{-1}P = \underline{I}\cdot Q^{-1}$

$\boxed{(PQ)^{-1} = Q^{-1}P^{-1}}$

$\underline{III}$ Si $[P,Q] = 0$ entonces $PQ^{-1} = Q^{-1}P$

$[P,Q] = PQ - QP = 0 \Rightarrow PQ = QP$
$PQQ^{-1} = QPQ^{-1} \Rightarrow \boxed{Q^{-1}P = PQ^{-1}}$ ✓

IV $(e^P)^+ = e^{P^+}$

$\downarrow$

$$\left[I + P + \frac{P^2}{2!} + \cdots \frac{P^n}{n!} \cdots \right]^+ = \left[I^+ + P^+ + \frac{(P^+)^2}{2!} \cdots \right.$$
$$\left. + \frac{(P^+)^n}{n!} \cdots \right]$$

$= e^{P^+}$

V $P e^Q P^{-1} = e^{PQP^{-1}}$ ②

$\downarrow$

① $P\left[I + Q + \frac{Q^2}{2!} + \cdots \right] P^{-1} = \left[I + PQP^{-1} + \frac{PQ^2P^{-1}}{2!} + \cdots \right]$

② $e^{PQP^{-1}} = \left[I + PQP^{-1} + \frac{PQP^{-1} \cdot PQP^{-1}}{2!} + \cdots \right]$

y así sucesivamente $\boxed{PQ^2P^{-1}}$

$\Rightarrow$ Entonces ya que lo es ① = ②

$\boxed{P e^Q P^{-1} = e^{PQP^{-1}}}$

(b) Si $A = A^+$ entonces $\hat{A} = U^{-1} A U$ también será hermítico?

$(\hat{A})^+ = (U^{-1} A U)^+ = U^+ A^+ (U^{-1})^+ = U^{-1} A (U^{-1})^{-1}$
$= U^{-1} A U = \hat{A}$

Por lo tanto $(\hat{A})^+ = \hat{A}$

© Si $A=A^+$ entonces $e^{iA} \Rightarrow (e^{iA})^+ = (e^{iA})^{-1}$

Obsérvese que:

$(iI)^+ = i^*I = -iI$, también $[iA,-iA] = A^2 - (-i)AiA = 0$

$(e^{iA})^+ \cdot e^{iA} = e^{A^+(iI)^+} \cdot e^{iA} = e^{-Ai} \cdot e^{iA} = e^{-Ai}e^{Ai}$

$= e^{-Ai+Ai} e^{\frac{1}{2}[-A,A]} = I$

por lo tanto $(e^{iA})^{-1} = (e^{iA})^+$

Ⓓ Si $K = -K^+$ entonces $\tilde{K} = U^{-1}KU$ también
en particular $K = iA$

$-\tilde{K}^+ = -[U^{-1}KU]^+ = -U^+K^+(U^{-1})^+ = -U^{-1}\cdot -K\cdot U$

$= U^{-1}KU = \tilde{K}$

por lo tanto $\tilde{K} = -\tilde{K}^+$

Ⓔ Dados A,B tal que $A=A^+$ y $B=B^+$
AB será hermítico si A y B conmutan

$(AB)^+ = B^+A^+ = B\cdot A$ si queremos que
$(AB)^+ = AB$ entonces BA debe ser AB
por lo tanto deben conmutar

Ⓕ Si $S = -S^+$ y $\#$ operador Identidad
$\frac{1}{I}(I-S)(I+S) = (I+S)(I-S)$?

$(I-S)(I+S) = I + S - S - S^2 = I - S^2$

$(I+S)(I-S) = I - S + S - S^2 = I - S^2$

$0 = 0$ Luego $(I-S)(I+S) = (I+S)(I-S)$

\# $[(I-S)(I+S)]^+ = (I+S)^+(I-S)^+$

$= (I + S^+)(I - S^+) = (I-S)(I+S)$

ya que $(I-S)(I+S) = [(I-S)(I+S)]^+$

Entonces es simétrico

* $[(I-S)(I+S)^{-1}]^+ [(I-S)(I+S)^{-1}]$

$= (I+S)^{-1}(I-S)(I-S)(I+S)^{-1}$

$= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1}$

$= (I-S)^{-1}(I-S)(I+S)(I+S)^{-1}$

$= I \cdot I = I$ ✓

⑨ Sea $\mathbb{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\mathbb{R} = (I-S)(I+S)^{-1}$

$R(I+S) = (I-S)$

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1+S_{11} & S_{12} \\ S_{21} & 1+S_{22} \end{pmatrix} = \begin{pmatrix} 1-S_{11} & -S_{12} \\ -S_{21} & 1-S_{22} \end{pmatrix}$

$\begin{pmatrix} \cos\theta + \cos\theta S_{11} + \sin\theta S_{21} & \cos\theta S_{12} + \sin\theta + \sin\theta S_{22} \\ -\sin\theta - \sin\theta S_{11} + \cos\theta S_{21} & -\sin\theta S_{12} + \cos\theta + \cos\theta S_{22} \end{pmatrix}$

$$\cos\theta + \cos\theta S_{11} + S_{11} - 1 + \sin\theta S_{21} = 0$$

① $\quad S_{11}[\cos\theta + 1] + S_{21}[\sin\theta] = 1 - \cos\theta$

② $\quad S_{12}[1 + \cos\theta] + S_{22}\sin\theta = -\sin\theta$

③ $\quad -\sin\theta S_{11} + S_{21}[1 + \cos\theta] = \sin\theta$

④ $\quad -\sin\theta S_{12} + S_{22}[1 + \cos\theta] = 1 - \cos\theta$

$$\begin{pmatrix} \cos\theta+1 & 0 & \sin\theta & 0 \\ 0 & 1+\cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & 1+\cos\theta & 0 \\ 0 & -\sin\theta & 0 & 1+\cos\theta \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{12} \\ S_{21} \\ S_{22} \end{pmatrix} = \begin{pmatrix} 1-\cos\theta \\ -\sin\theta \\ \sin\theta \\ 1-\cos\theta \end{pmatrix}$$

adecos $\rightarrow S_{11} = 0 \checkmark$, $\rightarrow S_{22} = 0$

Con $\quad S_{12} = -S_{21}$ $\qquad$ donde:

$$S_{12} = \frac{-2\sin\theta}{\sin^2\theta + \cos^2\theta + 2\cos\theta + 1} = \frac{-2\sin\theta}{2(1+\cos\theta)}$$

$$\rightarrow S_{12} = \frac{-\sin\theta}{(1+\cos\theta)} \checkmark$$

$$S_{21} = \frac{2\sin\theta}{\sin^2\theta + \cos^2\theta + 2\cos\theta + 1} = \frac{2\sin\theta}{2[1+\cos\theta]}$$

$$\rightarrow S_{21} = \frac{\sin\theta}{(1+\cos\theta)} \checkmark$$