2) Dados los tensores

$$R^i_j = \begin{pmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{pmatrix}, \quad T^i = \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix}, \quad g^{ij} = g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Hallar parte simétrica $S^i_j$ y antisimétrica $A^i_j$ de $R^i_j$

$$S^i_j = \frac{1}{2}(R^i_j + R^j_i) = \frac{1}{2}\left[\begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ 1 & \frac{5}{2} & 4 \\ \frac{3}{2} & 3 & \frac{9}{2} \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ \frac{5}{2} & \frac{7}{2} & \frac{9}{2} \end{pmatrix} \quad \text{RTA/}$$

donde $R^j_i$ representa a la transpuesta de la matriz asociada al tensor $R^i_j$

$$A^i_j = \frac{1}{2}(R^i_j - R^i_j) = \frac{1}{2}\left[\begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ 1 & \frac{5}{2} & 4 \\ \frac{3}{2} & 3 & \frac{9}{2} \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \quad \text{RTA/}$$

b) (I) $R_{kj} = g_{ik} R^i_j$ (II) $R^{ki} = g^{jk} R^i_j$ (III) $T_j = g_{ij} T^i$

(I) $R_{kj} = g_{ik} R^i_j$

↳ Trasponemos $g_{ik}$ para conseguir notación de producto matricial:

$$(g_{ik})^T R^i_j = g_{ki} R^i_j,$$

que denotan matrices: $A_{ki} B_{ij}, = C_{kj},$

Iguales

Como:

$$(g_{ik})^T R^i_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ -2 & -\frac{5}{2} & -3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix}$$

Entonces:

$$R_{kj} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ -2 & -\frac{5}{2} & -3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} \quad \text{RTA/}$$

(II) $R^{ki} = g^{jk} R^i_j = R^i_j g^{jk}$

↳ Verificamos que halla producto matricial en $R^i_j g^{jk}$: $A_{ij} B_{jk} = C_{ik}$, donde el resultado es una matriz $C_{ik}$.

Iguales

Así, debemos trasponer $C_{ik}$ para hallar $R^{ki}$:

$R^{ki} = (C_{ik})^T,$

$$C_{ik} = R^i_{\ j}\, g^{jk} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} & 3 \\ \frac{7}{2} & -4 & \frac{9}{2} \end{pmatrix}$$

Entonces:

$$R^{ki} = \begin{pmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ -1 & -\frac{5}{2} & -4 \\ \frac{3}{2} & 3 & \frac{9}{2} \end{pmatrix}$$ RTA/.

(III) $T_j = g_{ij} T^i = T^i g_{ij}$

↳ Si trasponemos a $T^i$, tendríamos matrices $A_{1i}\, B_{ij} = C_{1j}$

Iguales

donde $C_{1j}$ indica a una matriz de 1 fila y columnas con índices $j = 1, 2, 3$.

Así:

$$T_j = (T^i)^T g_{ij} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix}$$ RTA/.

(c) (i) $R^i_{\ j} T_i$, (ii) $R^i_{\ j} T^j$ (iii) $R^i_{\ j} T_i T^j$

(i) $R^i_{\ j} T_i = M_j$ forma / covector

Así:

$$M_j = T_i R^i_{\ j} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ 1 & \frac{5}{2} & 4 \\ \frac{3}{2} & 3 & \frac{9}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{3} & \frac{8}{3} & 3 \end{pmatrix}$$ RTA/.

(ii) $R^i_{\ j} T^j = N^i$ vector

Así:

$$N^i = R^i_{\ j} T^j = \begin{pmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ 1 & \frac{5}{2} & 4 \\ \frac{3}{2} & 3 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 16/3 \\ 25/3 \end{pmatrix}$$ RTA/.

(iii) $R^i_{\ j} T_i T^j = \alpha$ escalar

Así:

$$\alpha = T_i R^i_{\ j} T^j = M_j T^j = \begin{pmatrix} \frac{7}{3} & \frac{8}{3} & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} = \frac{7}{9} + \frac{16}{9} + 3 = \frac{50}{9}$$ RTA/.

(d) (1) $R^i_j \, S^j_i$ (2) $R^i_j \, A^j_i$ (3) $A^j_i \, T^i \, T_j$

(1) $R^i_j \, S^j_i = R^1_j S^j_1 + R^2_j S^j_2 + R^3_j S^j_3$

$$= \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{5}{2} \end{pmatrix} + \begin{pmatrix} 2 & \frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \end{pmatrix} + \begin{pmatrix} \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \left( \frac{1}{4} + \frac{3}{2} + \frac{15}{4} \right) + \left( 3 + \frac{25}{4} + \frac{21}{2} \right) + \left( \frac{35}{4} + \frac{28}{2} + \frac{81}{4} \right)$$

$$= \frac{277}{4} = 69.25$$ RTA/.

↳ escalar

(2) $R^i_j \, A^j_i = R^1_j A^j_1 + R^2_j A^j_2 + R^3_j A^j_3$

$$= \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & \frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

$$= \left( \frac{1}{2} + \frac{3}{2} \right) + \left( -1 + \frac{3}{2} \right) + \left( -\frac{7}{2} - 2 \right) = -3$$ RTA/.

↳ escalar

(3) $A^j_i \, T^i \, T_j = T_j \, A^j_i \, T^i$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$$

$$= \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$ RTA/.

↳ escalar

e) (I) $R^i_j - 2\,\delta^i_j \, R^l_l$ (II) $(R^i_j - 2\,\delta^i_j \, R^l_l)\, T_i$

(III) $(R^i_j - 2\,\delta^i_j \, R^l_l)\, T_i \, T^j$

(I) $R^i_j - 2\,\delta^i_j \, R^l_l$

Observemos que $R^l_l$ corresponde a un tensor de rango 0 o escalar y representa la traza del tensor $R^i_j$:

$$R^l_l = R^1_1 + R^2_2 + R^3_3 = \frac{1}{2} + \frac{5}{2} + \frac{9}{2} = \frac{15}{2}.$$

Así:

$$-2\,\delta^i_j \, R^l_l = -2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{15}{2} = \begin{pmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{pmatrix},$$

Ahora:

$$R^i_j - 2\,\delta^i_j \, R^l_l = \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} + \begin{pmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{pmatrix} = \begin{pmatrix} -\frac{29}{2} & 1 & \frac{3}{2} \\ 2 & -\frac{25}{2} & 3 \\ \frac{7}{2} & 4 & -\frac{21}{2} \end{pmatrix}$$ RTA/.

$$(\mathrm{II})\ \underbrace{(R^i_j - 2\,\delta^i_j\, R^l_l)}_{M^i_j} T_i = M^i_j\, T_i = T_i\, M^i_j$$

$$\rightarrow T_i M^i_j = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} -\frac{29}{2} & 1 & \frac{3}{2} \\ 2 & -25/2 & 3 \\ \frac{7}{2} & 4 & -\frac{21}{2} \end{pmatrix} = \begin{pmatrix} -\frac{8}{3} & \frac{38}{3} & -12 \end{pmatrix} \quad \text{RTA/.}$$

$$(\mathrm{III})\ (R^i_j - 2\delta^i_j R^l_l) T_i T_j = (T_i M^i_j) T_j = N_j T_j$$

$$= \begin{pmatrix} -\frac{8}{3} & \frac{38}{3} & -12 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} = -\frac{8}{9} + \frac{76}{9} - 12 = -\frac{40}{9} \quad \text{RTA/.}$$

$\hookrightarrow$ escalar

D M A

(a) $q^1 = x+y \qquad q^2 = x-y \qquad q^3 = 2z$

$$\begin{pmatrix} x+y \\ x-y \\ 2z \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\boxed{|u_1\rangle}} + y\underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\boxed{|u_2\rangle}} + z\underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{\boxed{|u_3\rangle}}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \qquad |u_1\rangle \perp |u_2\rangle \perp |u_3\rangle$$

b)

$$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\end{pmatrix}$$

c) $g_{ij} = \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 4\end{pmatrix}$

Sea $|a\rangle = a^1 \hat{i} + a^2 \hat{j} + a^3 \hat{k} = a^{1'}(\hat{i}+\hat{j}) + a^{2'}(\hat{i}-\hat{j}) + a^{3'}(2\hat{k})$

$$dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j$$

$a^1 = a^{1'} + a^{2'}$ $\quad a^3 = 2a^{3'}$

$a^2 = a^{1'} - a^{2'}$

$$\Rightarrow \frac{\partial x^i}{\partial \tilde{x}^j} = \begin{pmatrix}1 & 1 & 0\\1 & -1 & 0\\0 & 0 & 2\end{pmatrix}$$

D M A

$$dx^{i} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & z \end{pmatrix} \cdot \begin{pmatrix} dx^{j} \\ dy^{j} \\ dz^{j} \end{pmatrix} = \begin{pmatrix} dx^{j} + dy^{j} \\ dx^{j} - dy^{j} \\ z\, dz^{j} \end{pmatrix}$$

$$dx\, dy\, dz = (dx^{j} + dy^{j})(dx^{j} - dy^{j})\, z\, dz^{j}$$
$$= \left[(dx^{j})^{2} - (dy^{j})^{2}\right] z\, dz^{j}$$

d) Encuentre las expresiones en el sistema $(q^{1}, q^{2}, q^{3})$ para los vectores

* $A = 2\hat{j}$

$$\begin{array}{cccc} a^{1)} & a^{2)} & a^{3)} & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & z & 0 \end{array}\right) \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

d) Encuentre las expresiones en el sistema $(a^1, a^2, a^3)$ para los vectores

* $A = 2\hat{j}$

$$\begin{array}{cccc} a^{1)} & a^{2)} & a^{3)} & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Luego $A = 2\hat{j} = (1)|u_1\rangle - 1|u_2\rangle$

* $B = \hat{i} + 2\hat{j}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Luego $B = \hat{i} + 2\hat{j} = \frac{3}{2}|u_1\rangle - \frac{1}{2}|u_2\rangle$

* $C = \hat{i} + 7\hat{j} + 3\hat{k}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 7 \\ 0 & 0 & 2 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

Luego
$C = \hat{i} + 7\hat{j} + 3\hat{k}$
$= 4|u_1\rangle - 3|u_2\rangle + \frac{3}{2}|u_3\rangle$

e) Encuentre en el sistema $(q^1, q^2, q^3)$ las expresiones para $A \times B$, $A \cdot C$, $(A \times B) \cdot C$

$$* \; A \times B: \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(-2)$$

$$A \times B = -2\hat{k} = -1 u_3 >$$

$$\begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 1 & -1 & 0 & | & 0 \\ 0 & 0 & 2 & | & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}$$

$* \; A \cdot C = (|u_1\rangle - |u_2\rangle) \cdot (4|u_1\rangle - 3|u_2\rangle + \frac{3}{2}|u_3\rangle)$

$\downarrow$

$14 = (2\hat{j}) \cdot (\hat{i} + 7\hat{j} + 3\hat{k})$

$14 = 14$

$* \; (A \times B) \cdot C = (-|u_3\rangle) \cdot (4|u_1\rangle - 3|u_2\rangle + \frac{3}{2}|u_3\rangle)$

$\downarrow$

$(-2\hat{k}) \cdot (\hat{i} + 7\hat{j} + 3\hat{k}) = -\frac{3}{2} \, \| |u_3\rangle \|^2$

$-6 = -\frac{3}{2}(4)$

$-6 = -6$

$\Rightarrow$ Aunque las expresiones sean distintas al operar se llega al mismo resultado, como 2 observadores en distintas condiciones midiendo lo mismo

F) Considere los siguientes tensores y vectores en coordenadas cartesianas:

$$R^i_{\ j} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} \qquad t^i = \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix}$$

$$g^{ij} = g_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

⇒ encuentre sus expresiones para el nuevo sistema de coordenadas $(q^{1'}, q^{2'}, q^{3'})$

recordando que: $|u_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $|u_2\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$|u_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$$R^{k'}_{m'} = \frac{\partial x^{k'}}{\partial x^i} \frac{\partial x^j}{\partial x^{m'}} R^i_j$$

Sea $|a\rangle = a^j |\hat{e}_j\rangle = a^{m'} |u_m\rangle$

$= a^1 \hat{\imath} + a^2 \hat{\jmath} + a^3 \hat{k} = a^{1'}(\hat{\imath} + \hat{\jmath}) + a^{2'}(\hat{\imath} - \hat{\jmath})$

$+ a^{3'}(2\hat{k})$

$a^1 = a^{1'} + a^{2'}$, $a^2 = a^{1'} - a^{2'}$, $a^3 = 2a^{3'}$

$$\frac{\partial x^j}{\partial x^{m'}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

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Sea $|a\rangle = a^j |e_j\rangle = a^{m'} |u_m\rangle$

$= a^1 \hat{\imath} + a^2 \hat{\jmath} + a^3 \hat{k} = a^{1'}(\hat{\imath} + \hat{\jmath}) + a^{2'}(\hat{\imath} - \hat{\jmath}) + a^{3'}(2\hat{k})$

$a^1 = a^{1'} + a^{2'}$, $a^2 = a^{1'} - a^{2'}$, $a^3 = 2a^{3'}$

$$\frac{\partial x^j}{\partial x^{m'}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\to a^{1'} = a^1 - a^{2'} = a^1 + (a^2 - a^{1'})$

$2a^{1'} = a^1 + a^2 \to ⊛\, a^{1'} = \frac{a^1}{2} + \frac{a^2}{2}$

$⊛\, a^{2'} = a^1 - a^{1'} = a^1 - \frac{a^1}{2} - \frac{a^2}{2}$

$= \frac{a^1}{2} - \frac{a^2}{2}$

$⊛\, a^{3'} = \frac{a^3}{2}$

Luego $\to \frac{\partial x^{k'}}{\partial x^i} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

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$$R^{k'}{}_{m'} = \frac{\partial x^{k'}}{\partial x^{i}} \frac{\partial x^{j}}{\partial x^{m'}} R^{i}{}_{j} = \frac{\partial x^{k'}}{\partial x^{i}} R^{i}{}_{j} \frac{\partial x^{j}}{\partial x^{m'}}$$

$$R^{k'}{}_{m'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$R^{k'}{}_{m'} = \begin{pmatrix} 3 & -\frac{1}{2} & \frac{9}{2} \\ -\frac{3}{2} & 0 & -\frac{3}{2} \\ \frac{15}{4} & -\frac{1}{4} & \frac{9}{2} \end{pmatrix}$$

$$g_{k'm'} = \frac{\partial x^i}{\partial x^{k'}} \frac{\partial x^j}{\partial x^{m'}} g_{ij} = \frac{\partial x^i}{\partial x^{k'}} g_{ij} \frac{\partial x^j}{\partial x^{m'}}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

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$$T^{k'} = \frac{\partial x^{k'}}{\partial x^{i}} T^{i} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$$

$$T^{k'} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{2} \end{pmatrix}$$

6) Si consideramos al tensor de maxwell como:

$$F_{\mu\alpha} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -cB^z & cB^y \\ E^y & cB^z & 0 & -cB^x \\ E^z & -cB^y & cB^x & 0 \end{pmatrix}, \quad \text{con} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

donde $\vec{E} = (E^x, E^y, E^z)$, $\vec{B} = (B^x, B^y, B^z)$ son los campos eléctricos y magnéticos para un observador $O$.

(a) $\vec{E} = E^x \hat{\imath}$, $\vec{\beta} = v\hat{\imath} \rightarrow$ otro observador

Entonces:

$$F_{\mu\alpha} = \begin{pmatrix} 0 & -E^x & 0 & 0 \\ E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$ Así el tensor $F_{\mu\alpha}$ transformará como:

$$F_{\mu'\alpha'} = \Lambda^{\nu}_{\mu'} \Lambda^{\beta}_{\alpha'} F_{\nu\beta}$$

Ahora, dado que $F_{\mu\alpha} = -F_{\alpha\mu}$, porque $F_{\mu\alpha}$ es antisimétrica, entonces:

$$F_{\mu'\alpha'} = \Lambda^{0}_{\mu'} \Lambda^{1}_{\alpha'} F_{01} + \Lambda^{1}_{\mu'} \Lambda^{0}_{\alpha'} F_{10}$$

$\hookrightarrow$ dado que $F_{10}, F_{01}$ son las únicas componentes no nulas de $F_{\mu\alpha}$

$$= (\Lambda^{0}_{\mu'} \Lambda^{1}_{\alpha'} - \Lambda^{1}_{\mu'} \Lambda^{0}_{\alpha'})\, F_{01}$$

$\hookrightarrow$ aplicando $F_{10} = -F_{01}$ y factorizando $F_{01}$

Ahora, sabemos que: $\Lambda^{0}_{0'} = \gamma$, $\Lambda^{i}_{0'} = \gamma v^{i}$, $\Lambda^{0}_{i'} = \gamma v_{i'}$,

$\Lambda^{i}_{j'} = \delta^{i}_{j'} + v^{i} v_{j'} \dfrac{\gamma - 1}{|v|^2}$ para $i, j = 1,2,3$

Como $v^{i} = (v, 0, 0)$:

Obs: $\Lambda^{1}_{1'} = \delta^{1}_{1'} + v^{1} v_{1'} \dfrac{\gamma-1}{v^2} = 1 + v^2 \dfrac{\gamma - 1}{v^2} = \gamma$

$$\Lambda^{\alpha}_{\beta'} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ $\rightarrow$ Sus componentes distintas de cero son: $\Lambda^{0}_{0'}$, $\Lambda^{1}_{0'}$, $\Lambda^{0}_{1'}$, $\Lambda^{1}_{1'}$, $\Lambda^{2}_{2'}$, $\Lambda^{3}_{3'}$

Luego:

$$F_{0'0'} = (\Lambda^0_{0'}\Lambda^1_{0'} - \Lambda^1_{0'}\Lambda^0_{0'})F_{01} = (-\gamma^2 v + \gamma^2 v)F_{01} = \emptyset$$

$$F_{1'1'} = (\Lambda^0_{1'}\Lambda^1_{0'} - \Lambda^1_{0'}\Lambda^0_{1'})F_{01} = (-\gamma^2 v^2 + \gamma^2 v^2)F_{01} = \emptyset$$

$$F_{1'0'} = (\Lambda^0_{1'}\Lambda^1_{0'} - \Lambda^1_{1'}\Lambda^0_{0'})(-E^x) = (\gamma^2 v^2 - \gamma^2)(-E^x)$$

analogamente, $F_{0'1'} = (\gamma^2 v^2 - \gamma^2)E^x$

Como las demás componentes de $F_{\mu'\alpha'}$ tendrán necesariamente un elemento $\Lambda^\alpha_{\beta'}$ igual a cero, entonces:

$$F_{\mu'\alpha'} = \begin{pmatrix} 0 & -E^{x'} & -E^{y'} & -E^{z'} \\ E^{x'} & 0 & cB^{z'} & -cB^{y'} \\ E^{y'} & -cB^{z'} & 0 & cB^{x'} \\ E^{z'} & cB^{y'} & -cB^{x'} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (v^2-1)\gamma^2 E^x & 0 & 0 \\ -(v^2-1)\gamma^2 E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Así: $\vec{E}' = (E^{x'}, E^{y'}, E^{z'}) = (-(v^2-1)\gamma^2 E^x, 0, 0)$

$\vec{B}' = (B^{x'}, B^{y'}, B^{z'}) = (0,0,0)$

donde $E^{x'} = -(v^2-1)\gamma^2 E^x = \underbrace{\frac{(1-v^2)}{c^2-v^2}}_{\text{negativo}} c^2$

negativo: $\frac{1-v^2}{c^2-v^2} < 0$ para $1 < v < c$

RTA: Los campos que un observador $O'$ percibe es solo el eléctrico $\vec{E} = ((1-v^2)\gamma^2 E^x, 0, 0)$, pues $\vec{B} = \vec{0}$

b) Expresando las ecuaciones de Maxwell para la ley de gauss de electroestática y la ley de Ampere-Maxwell se tiene:

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}, \qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (*)$$

Queremos mostrar que las ecuaciones $(*)$ pueden expresarse como:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = F^{\mu\nu}{}_{,\nu} = c\mu_0 J^\mu$$

donde $J^\mu = (c\rho, \vec{J})$, $\vec{J} = (J^1, J^2, J^3)$, $x^\nu = (ct, x, y, z)$,
$\vec{E} = (E^x, E^y, E^z)$ $\vec{B} = (B^x, B^y, B^z)$.

Ahora,

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} = \eta^{\mu\alpha}F_{\alpha\beta}\eta^{\beta\nu}$$

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & cB^z & -cB^y \\ E^y & -cB^z & 0 & cB^x \\ E^z & cB^y & -cB^x & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ -E^x & 0 & cB^z & -cB^y \\ -E^y & -cB^z & 0 & cB^x \\ -E^z & cB^y & -cB^x & 0 \end{pmatrix} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & cB^z & -cB^y \\ -E^y & -cB^z & 0 & cB^x \\ -E^z & cB^y & -cB^x & 0 \end{pmatrix}$$

Así:

(1) Para la ley de Gauss para electroestática

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} \equiv F^{\mu\nu}{}_{,\nu} = F^{\mu 0}{}_{,0} + F^{\mu 1}{}_{,1} + F^{\mu 2}{}_{,2} + F^{\mu 3}{}_{,3}$$

$$\rightarrow F^{0\nu}{}_{,\nu} = F^{00}{}_{,0} + F^{01}{}_{,1} + F^{02}{}_{,2} + F^{03}{}_{,3} = 0 + \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z}$$

$$= \vec{\nabla}\cdot\vec{E} \underset{\text{Ley Gauss}}{=} \frac{\rho}{\varepsilon_0} = \frac{c\rho}{c\varepsilon_0} = \frac{J^0}{c\frac{1}{\mu_0 c^2}} = \frac{J^0}{\frac{1}{\mu_0 c}} = c\mu_0 J^0 \quad \checkmark \quad (1)$$

(2) Para la ley Ampere-Maxwell:

$$F^{1\nu}{}_{,\nu} = F^{10}{}_{,0} + F^{11}{}_{,1} + F^{12}{}_{,2} + F^{13}{}_{,3} = -\frac{1}{c}\frac{\partial E^x}{\partial t} + c\frac{\partial B^z}{\partial y} - c\frac{\partial B^y}{\partial z}$$

$$= c\left(-\frac{1}{c^2}\frac{\partial E^x}{\partial t} + \frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z}\right) = c\left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x \underset{\text{ley Ampere-Maxwell}}{=} c\mu_0 J^1 \quad \checkmark \quad (2)$$

donde observe que, de acuerdo a la ley de Ampere-Maxwell:

$$\mu_0 J^1 = \varepsilon^{1jk}\partial_j B_k - \frac{1}{c^2}\frac{\partial E^1}{\partial t} = (\partial_2 B_3 - \partial_3 B_2) - \frac{1}{c^2}\frac{\partial E^1}{\partial t}$$

$$= \frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z} - \frac{1}{c^2}\frac{\partial E^x}{\partial t} = \underbrace{\left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x}_{\text{componente } x}$$

$$F^{2\nu}{}_{,\nu} = F^{20}{}_{,0} + F^{21}{}_{,1} + F^{22}{}_{,2} + F^{23}{}_{,3} = -\frac{1}{c}\frac{\partial E^y}{\partial t} - c\frac{\partial B^z}{\partial x} + c\frac{\partial B^x}{\partial z}$$

$$= c\left(-\frac{1}{c^2}\frac{\partial E^y}{\partial t} - \frac{\partial B^z}{\partial x} + \frac{\partial B^x}{\partial z}\right) = c\left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_y = c\mu_0 J^2 \quad \checkmark \quad (3)$$

$$= (\partial_3 B_1 - \partial_1 B_3) - \frac{1}{c^2}\frac{\partial E^2}{\partial t}$$

donde $\mu_0 J^2 = \varepsilon^{2jk}\partial_j B_k - \frac{1}{c^2}\frac{\partial E^2}{\partial t}$

$$= \frac{\partial B^x}{\partial z} - \frac{\partial B^z}{\partial x} - \frac{1}{c^2}\frac{\partial E^y}{\partial t}$$

$$F^{3\nu}{}_{,\nu} = F^{30}{}_{,0} + F^{31}{}_{,1} + F^{32}{}_{,2} + F^{33}{}_{,3} = -\frac{1}{c}\frac{\partial E^z}{\partial t} + c\frac{\partial B^y}{\partial x} - c\frac{\partial B^x}{\partial y}$$

$$= c\left(-\frac{1}{c^2}\frac{\partial E^z}{\partial t} + \frac{\partial B^y}{\partial x} - \frac{\partial B^x}{\partial y}\right) = c\mu_0 J^3 \quad \checkmark \quad (4)$$

donde $\mu_0 J^3 = \varepsilon^{3jk}\partial_j B_k - \frac{1}{c^2}\frac{\partial E^3}{\partial t} = (\partial_1 B_2 - \partial_2 B_1) - \frac{1}{c^2}\frac{\partial E^3}{\partial t}$

$$= \frac{\partial B^y}{\partial x} - \frac{\partial B^x}{\partial y} - \frac{1}{c^2}\frac{\partial E^z}{\partial t}$$

RTA/.
Por lo tanto, por (1), (2), (3), (4) se concluye que: $F^{\mu\nu}{}_{,\nu} = c\mu_0 J^\mu$ conteniendo a las leyes de Gauss y Ampere-Maxwell

(c) Considerando la identidad de Bianchi de la forma:

$$F_{\mu\nu,\gamma} + F_{\nu\gamma,\mu} + F_{\gamma\mu,\nu} = 0 \qquad (*)$$

Queremos demostrar que las ecuaciones:

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0,$$

están contenidas en la ecuación $F^{\mu\nu}{}_{,\nu} = c\mu_0 J^\mu$

Ahora:

Por (*): $F_{\mu\nu,\gamma} + F_{\nu\gamma,\mu} + F_{\gamma\mu,\nu} = 0$

$$F_{13,2} + F_{32,1} + F_{21,3} = 0$$

$$-c\frac{\partial B^y}{\partial y} - c\frac{\partial B^x}{\partial x} - c\frac{\partial B^z}{\partial z} = 0$$

$$\frac{\partial B^x}{\partial x} + \frac{\partial B^y}{\partial y} + \frac{\partial B^z}{\partial z} = 0 \rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

Luego:

$$F_{\mu\nu,0} + F_{\nu 0,\mu} + F_{0\mu,\nu} = 0$$

Tomando $\mu=2,\ \nu=3$: $F_{23,0} + F_{30,2} + F_{02,3} = 0$

$$\frac{\cancel{c}}{\cancel{c}}\frac{\partial B^x}{\partial t} + \frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z} = 0 \rightarrow (\dot{\vec{B}} + \vec{\nabla}\times\vec{E})_x = 0 \quad \checkmark \ (1)$$

donde: $(\vec{\nabla}\times\vec{E})_x = \varepsilon^{1jk}\partial_j E_k = \partial_2 E_3 - \partial_3 E_2 = \frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z}$. $\checkmark$

Tomando $\mu=1,\ \nu=3$: $F_{13,0} + F_{30,1} + F_{01,3} = 0$

$$-\frac{\cancel{c}}{\cancel{c}}\frac{\partial B^y}{\partial t} + \frac{\partial E^z}{\partial x} - \frac{\partial E^x}{\partial z} = 0 \rightarrow (-\dot{\vec{B}} - \vec{\nabla}\times\vec{E})_y = 0 \quad \checkmark$$

donde: $(-\vec{\nabla}\times\vec{E})_y = -\varepsilon^{2jk}\partial_j E_k = -(\partial_3 E_1 - \partial_1 E_3) = -\frac{\partial E^x}{\partial z} + \frac{\partial E^z}{\partial x}$ $\checkmark$ (2)

Tomando $\mu=2$ $\nu=1$: $F_{21,0} + F_{10,2} + F_{02,1} = 0$

$$-\frac{c}{c}\frac{\partial B^z}{\partial t} + \frac{\partial E^x}{\partial y} - \frac{\partial E^y}{\partial x} = 0 \rightarrow (-\vec{B} - \vec{\nabla}\times\vec{E})_z = 0 \quad (3)$$

donde: $(-\vec{\nabla}\times\vec{E})_z = -\varepsilon^{3jk}\partial_j E_k = -(\partial_1 E_2 - \partial_2 E_1) = -\frac{\partial E^y}{\partial x} + \frac{\partial E^x}{\partial y}$

Por ende, debido a (1), (2), (3) podemos concluir que: la identidad de Bianchi contiene a las ecuaciones

$$\vec{\nabla}\cdot\vec{B} = 0 \quad , \quad \vec{B} + \vec{\nabla}\times\vec{E} = 0$$

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