4.1.4

4) Suponga que $AB = BA$. Demuestre que:

(a) $(A+B)^2 = A^2 + 2AB + B^2$

$$(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB$$
$$= A^2 + AB + AB + B^2 = A^2 + 2AB + B^2 \quad \text{Rta/.}$$

(b) $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

$$(A+B)^3 = (A+B)(A^2 + 2AB + B^2) = A^3 + 2A^2B + AB^2 + BA^2 + 2BAB + B^3$$
$$= A^3 + 2A^2B + AB^2 + A^2B + 2AB^2 + B^3$$
$$= A^3 + 3A^2B + 3AB^2 + B^3 \quad \text{Rta/.}$$

5) Suponga que un operador L puede ser escrito como la composición de dos operadores $L = L_- L_+$ con $[L_-, L_+] = \mathbb{I}$
Demostrar que:

- Si $L|x\rangle = \lambda|x\rangle$ y $|y\rangle = L_+|x\rangle$ entonces $L|y\rangle = (\lambda+1)|y\rangle$

Como $|y\rangle = L_+|x\rangle$: usando $[L_-, L_+] = I$

$$L|y\rangle = (L_-L_+)L_+|x\rangle = (I + L_+L_-)L_+|x\rangle = L_+|x\rangle + L_+L|x\rangle$$
$$= |y\rangle + L_+(\lambda|x\rangle) = |y\rangle + \lambda L_+|x\rangle = |y\rangle + \lambda|y\rangle$$
$$= (1+\lambda)|y\rangle \quad \text{Rta/.}$$

- Si $L|x\rangle = \lambda|x\rangle$ y $|z\rangle = L_-|x\rangle$ entonces $L|z\rangle = (\lambda-1)|z\rangle$

$$\rightarrow L|z\rangle = (L_-L_+)(L_-|x\rangle) = L_-(L_+L_-)|x\rangle = L_-L|x\rangle = \lambda L_-|x\rangle = \lambda|z\rangle$$

$$\rightarrow L_+L_-|z\rangle = L_+L_-L_-|x\rangle = (L_-L_+ - I)L_-|x\rangle$$
$$= L_-(L_+L_-|x\rangle) - IL_-|x\rangle = L_-L|x\rangle - L_-|x\rangle$$
$$= L_-(\lambda|x\rangle) - |z\rangle = \lambda(L_-|x\rangle) - |z\rangle$$
$$= \lambda|z\rangle - |z\rangle = (\lambda-1)|z\rangle$$

(1) Considere los siguientes operadores:
$A = A^{+}$ hermítico, $K = -K^{+}$ antihermítico;
$U^{-1} = U^{+}$ unitario, P y Q los operadores genéricos. Pruebe lo siguiente:

(a) $I (P^{+})^{-1} = (P^{-1})^{+}$

recordando que

$<x|y> = <x|I|y>$
$= <x|PP^{-1}|y>$
$= <y|(P^{-1})^{+}P^{+}|x>^{*}$
$= <y|x>^{*}$

Por lo tanto:
$(P^{-1})^{+} . P^{+} = I \rightarrow (P^{-1})^{+} P^{+} (P^{+})^{-1} = I (P^{+})^{-1}$

$(P^{-1})^{+} = (P^{+})^{-1}$

Observación:
Se utilizo la propiedad $(MB)^{+} = B^{+} M^{+}$ demostrada en clase

II $(PQ)^{-1} = Q^{-1} P^{-1}$

$\Rightarrow (PQ)^{-1} PQ = I \rightarrow (PQ)^{-1} P = I . Q^{-1}$

$\boxed{(PQ)^{-1} = Q^{-1} P^{-1}}$

III Si $[P, Q] = 0$ entonces $PQ^{-1} = Q^{-1} P$

$[P, Q] = PQ - QP = 0 \rightarrow PQ = QP$
$PQQ^{-1} = QPQ^{-1} \rightarrow \boxed{Q^{-1} P = PQ^{-1}}$ ✓

IV $(e^{P})^{+} = e^{P^{+}}$

$\downarrow$

$\left[I + P + \frac{P^2}{2!} + \ldots \frac{P^n}{n!} \ldots\right]^{+} = \left[I^{+} + P^{+} + \frac{(P^{+})^2}{2!} \ldots + \frac{(P^{+})^n}{n!} \ldots\right]$

$= e^{P^{+}}$

V ① $P e^{Q} P^{-1} = e^{PQP^{-1}}$ ②

$\downarrow$

① $P\left[I + Q + \frac{Q^2}{2!} + \ldots\right]P^{-1} = \left[I + PQP^{-1} + \boxed{\frac{PQ^2P^{-1}}{2!}} + \ldots\right]$

② $e^{PQP^{-1}} = \left[I + PQP^{-1} + \frac{PQP^{-1}PQP^{-1}}{2!} + \ldots\right]$

y así sucesivamente $\boxed{PQ^2P^{-1}}$

$\Rightarrow$ Entonces ya que lo es ① = ②

$$\boxed{Pe^{Q}P^{-1} = e^{PQP^{-1}}}$$

b) Si $A = A^{+}$ entonces $\hat{A} = U^{-1}AU$ también será hermítico?

$(\hat{A})^{+} = (U^{-1}AU)^{+} = U^{+}A^{+}(U^{-1})^{+} = U^{-1}A(U^{-1})^{-1}$
$= U^{-1}AU = \hat{A}$

Por lo tanto $(\hat{A})^{+} = \hat{A}$

(c) Si $A = A^{+}$ entonces $e^{iA} \Rightarrow (e^{iA})^{+} = (e^{iA})^{-1}$

observese que:

$(iI)^{+} = i^{*} I = -iI$, tambien $[iA, -iA] = A^2 - (-i) A i A = 0$

$$(e^{iA})^{+} . e^{iA} = e^{A^{+}(iI)^{+}} . e^{iA} = e^{-Ai} . e^{iA} = e^{-Ai} e^{Ai}$$

$$= e^{-Ai + Ai} e^{\frac{1}{2}[-Ai, Ai]} = I$$

por lo tanto $(e^{iA})^{-1} = (e^{iA})^{+}$

(d) Si $K = -K^{+}$ entonces $\tilde{K} = U^{-1} K U$ lo será

en particular $\tilde{K} = iA$

$$-\tilde{K}^{+} = -[U^{-1} K U]^{+} = -U^{+} K^{+} (U^{-1})^{+} = -U^{-1} . -K . U$$

$$= U^{-1} K U = \tilde{K}$$

por lo tanto $\tilde{K} = -\tilde{K}^{+}$

(e) Dados A y B tal que $A = A^{+}$, $B = B^{+}$

AB será hermitica si A y B conmutan

$(AB)^{+} = B^{+} A^{+} = B.A$ si queremos que

$(AB)^{+} = AB$ entonces BA debe ser AB

por lo tanto deben conmutar

(f) Si $S = -S^{+}$ y I operador Identidad

$I (I - S)(I + S) = (I + S)(I - S)$?

① $(I-S)(I+S) = I + S - S - S^2 = I - S^2$

② $(I+S)(I-S) = I - S + S - S^2 = I - S^2$

① = ② Luego $(I-S)(I+S) = (I+S)(I-S)$

$[(I-S)(I+S)]^t = (I+S)^t(I-S)^t$

$= (I+S^t)(I-S^t) = (I-S)(I+S)$

ya que $(I-S)(I+S) = [(I-S)(I+S)]^t$

Entonces es simétrico

$[(I-S)(I+S)^{-1}]^t [(I-S)(I+S)^{-1}]$

$= (I+S^t)^{-1}(I-S^t)(I-S)(I+S)^{-1}$

$= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1}$

$= (I-S)^{-1}(I-S)(I+S)(I+S)^{-1}$

$= I \cdot I = I$ ✓

⑨ Sea $R = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}$

$R = (I-S)(I+S)^{-1}$

$R(I+S) = (I-S)$

$$\begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1+S_{11} & S_{12} \\ S_{21} & 1+S_{22} \end{pmatrix} = \begin{pmatrix} 1-S_{11} & -S_{12} \\ -S_{21} & 1-S_{22} \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta + \cos\theta S_{11} + \operatorname{sen}\theta S_{21} & \cos\theta S_{12} + \operatorname{sen}\theta + \operatorname{sen}\theta S_{22} \\ -\operatorname{sen}\theta - \operatorname{sen}\theta S_{11} + \cos\theta S_{21} & -\operatorname{sen}\theta S_{12} + \cos\theta + \cos\theta S_{22} \end{pmatrix}$$

$\cos\theta + \cos\theta S_{11} + S_{11} - 1 + \mathrm{Sen}\theta S_{21} = 0$

① $S_{11}[\cos\theta + 1] + S_{21}[\mathrm{Sen}\theta] = 1 - \cos\theta$

② $S_{12}[1 + \cos\theta] + S_{22}\mathrm{Sen}\theta = -\mathrm{Sen}\theta$

③ $-\mathrm{Sen}\theta S_{11} + S_{21}[1 + \cos\theta] = \mathrm{Sen}\theta$

④ $-\mathrm{Sen}\theta S_{12} + S_{22}[1 + \cos\theta] = 1 - \cos\theta$

$$\begin{pmatrix} \cos\theta+1 & 0 & \mathrm{Sen}\theta & 0 \\ 0 & 1+\cos\theta & 0 & \mathrm{Sen}\theta \\ -\mathrm{Sen}\theta & 0 & 1+\cos\theta & 0 \\ 0 & -\mathrm{Sen}\theta & 0 & 1+\cos\theta \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{12} \\ S_{21} \\ S_{22} \end{pmatrix} = \begin{pmatrix} 1-\cos\theta \\ -\mathrm{Sen}\theta \\ \mathrm{Sen}\theta \\ 1-\cos\theta \end{pmatrix} \checkmark$$

entonces $\rightarrow S_{11} = 0$ ✓, $\rightarrow S_{22} = 0$

Con $S_{12} = -S_{21}$ donde:

$$S_{12} = \frac{-2\sin\theta}{\sin^2\theta + \cos^2\theta + 2\cos\theta + 1} = \frac{-2\sin\theta}{2(1+\cos\theta)}$$

$$\rightarrow S_{12} = \frac{-\sin\theta}{(1+\cos\theta)} \checkmark$$

$$S_{21} = \frac{2\sin\theta}{\sin^2\theta + \cos^2\theta + 2\cos\theta + 1} = \frac{2\sin\theta}{2[1+\cos\theta]}$$

$$\rightarrow S_{21} = \frac{\sin\theta}{(1+\cos\theta)} \checkmark$$

Sección 4.3.8

2). Considere el polinomio $|f\rangle_t \Leftrightarrow f(t) = 5t + 3t^2 + 4t^3$ encuentre su expresión en términos de las bases $B=\{1, t, t^2, t^3, t^4\}$ y la base de polinomios de Legrende $P=\{P_0, P_1, P_2, P_3, P_4\}$

• $|f\rangle_t = 5t + 3t^2 + 4t^3 = 5|t\rangle + 3|t^2\rangle + 4|t^3\rangle = \begin{pmatrix} 0 \\ 5 \\ 3 \\ 4 \\ 0 \end{pmatrix}_B$ //

• Ahora para la base de legrende: $P=\{|1\rangle, |t\rangle, |\frac{1}{2}(3t^2-1)\rangle, |\frac{1}{2}(5t^3-3t)\rangle, |\frac{1}{8}(35t^4-30t^2+3\rangle\}$

$$|f\rangle_t = \sum_{n=0}^{4} C_n |P_n\rangle, \quad C_n = \int_{-1}^{1} dt\, f(t)\, P_n(t) / \left(\int_{-1}^{1} dt\, P_n^2(t)\right)$$

con $\int_{-1}^{1} dt\, P_n^2(t) = \frac{2}{2n+1}$

$$C_0 = \frac{1}{2}\int_{-1}^{1} dt\, (5t+3t^2+4t^3)(1) = \frac{1}{2}\left[\frac{5}{2}t^2 + t^3 + t^4\right]_{-1}^{1} = 1$$

$$C_1 = \frac{3}{2}\int_{-1}^{1} dt\, (5t+3t^2+4t^3)(t) = \frac{3}{2}\left[\frac{5}{3}t^3 + \frac{3}{4}t^4 + \frac{4}{5}t^5\right]_{-1}^{1} = \frac{37}{5}$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)\left(\tfrac{1}{2}(3t^2-1)\right) = \frac{5}{4}\left[\frac{15}{4}t^4 + \frac{9}{5}t^5 + \frac{12}{6}t^6 - \frac{5}{2}t^2 - t^3 - t^4\right]_{-1}^{1}$$
$$= 2$$

$$C_3 = \frac{7}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)\left(\tfrac{1}{2}(5t^3-3t)\right) = \frac{7}{4}\left[5t^5 + \frac{15}{6}t^6 + \frac{20}{7}t^7 - 5t^3 - \frac{9}{4}t^4 - \frac{12}{5}t^5\right]_{-1}^{1}$$
$$= \frac{8}{5}$$

$$C_4 = \frac{9}{2}\int_{-1}^{1} dt\,(5t+3t^2+4t^3)\left(\tfrac{1}{8}(35t^4-30t^2+3)\right) = \frac{9}{16}\left[\frac{35}{2}x^8 + 15x^7 + \frac{55}{6}x^6 - 18x^5 - \frac{69}{2}x^4 + 3x^3 + \frac{15}{2}x^2\right]_{-1}^{1}$$
$$= 0$$

$$|f\rangle_t = |1\rangle + \frac{37}{5}|t\rangle + 2\left|\tfrac{1}{2}(3t^2-1)\right\rangle + \frac{8}{5}\left|\tfrac{1}{2}(5t^3-3t)\right\rangle = \begin{pmatrix} 1 \\ 37/5 \\ 2 \\ 8/5 \\ 0 \end{pmatrix}_P$$

Comprobación:

$$|f\rangle_t = \cancel{|1\rangle} + \frac{37}{5}|t\rangle + 3|t^2\rangle - \cancel{|1\rangle} + 4|t^3\rangle - \frac{24}{10}|t\rangle$$
$$= 5|t\rangle + 3|t^2\rangle + 4|t^3\rangle \quad W$$

• $|g\rangle_t = e^\alpha |E_\alpha\rangle = p^\beta |P_\beta\rangle$ con $\{|E_\alpha\rangle\} = \{1, t, t^2, t^3, t^4\}$
y $\alpha, \beta = 0, 1, 2, 3, 4$

Si $|g\rangle_t$ es un polinomio genérico de $P_{(t)4}$, entonces:

$$|g\rangle_t = p^0|P_0\rangle + p^1|P_1\rangle + p^2|P_2\rangle + p^3|P_3\rangle + p^4|P_4\rangle$$
$$= p^0(1) + p^1(t) + p^2\left(\tfrac{1}{2}(3t^2-1)\right) + p^3\left(\tfrac{1}{2}(5t^3-3t)\right) + p^4\left(\tfrac{1}{8}(35t^4-30t^2+3)\right)$$

$$|g\rangle_t = \left(p^0 - \tfrac{1}{2}p^2 + \tfrac{3}{8}p^4\right)|1\rangle + \left(p^1 - \tfrac{3}{2}p^3\right)|t\rangle +$$
$$\left(\tfrac{3}{2}p^2 - \tfrac{30}{8}p^4\right)|t^2\rangle + \left(\tfrac{5}{2}p^3\right)|t^3\rangle + \left(\tfrac{35}{8}p^4\right)|t^4\rangle$$

$$= e^\alpha |E_\alpha\rangle = e^0|1\rangle + e^1|t\rangle + e^2|t^2\rangle + e^3|t^3\rangle + e^4|t^4\rangle$$

$$\rightarrow \begin{cases} e^0 = p^0 - \frac{1}{2}p^2 + \frac{3}{8}p^4 \\ e^1 = p^1 - \frac{3}{2}p^3 \\ e^2 = \frac{3}{2}p^2 - \frac{30}{8}p^4 \\ e^3 = \frac{5}{2}p^3 \\ e^4 = \frac{35}{8}p^4 \end{cases} \rightarrow \frac{\partial e^\alpha}{\partial p^\beta} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} & 0 & -\frac{30}{8} \\ 0 & 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{35}{8} \end{pmatrix}$$

$\frac{\partial e^\alpha}{\partial p^\beta}$ representa la matriz de cambio de coordenadas de P (polinomios legrende) a B (monomios):

$$e^\alpha = \frac{\partial e^\alpha}{\partial p^\beta} p^\beta \qquad \alpha, \beta = 0,1,2,3,4$$

La otra matriz es ortogonal a la anterior:

$$\frac{\partial p^\beta}{\partial e^\alpha} \cdot \frac{\partial e^\alpha}{\partial p^\beta} = 1$$

$$\frac{\partial p^\beta}{\partial e^\alpha} = \left(\frac{\partial e^\alpha}{\partial p^\beta}\right)^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} & 0 & -\frac{30}{8} \\ 0 & 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{35}{8} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{4}{7} \\ 0 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{35} \end{pmatrix}$$

$\frac{\partial p^\beta}{\partial e^\alpha}$ es la matriz de cambio de coordenadas de B (monomios) a P (polinomios legrende)

(b) Un proyector sobre el subespacio $P_{(t)2}$ es:

$$\mathbb{P} = \frac{|P_0\rangle\langle P_0|}{\langle P_0|P_0\rangle} + \frac{|P_1\rangle\langle P_1|}{\langle P_1|P_1\rangle} + \frac{|P_2\rangle\langle P_2|}{\langle P_2|P_2\rangle}$$

donde hemos dividido entre la norma del vector pues estamos usando una base ortogonal y no ortonormal.

$$\mathbb{P}|f\rangle_t = |P_0\rangle \frac{\langle P_0|f\rangle_t}{\langle P_0|P_0\rangle} + |P_1\rangle \frac{\langle P_1|f\rangle_t}{\langle P_1|P_1\rangle} + |P_2\rangle \frac{\langle P_2|f\rangle}{\langle P_2|P_2\rangle}$$

donde $\langle P_n|P_n\rangle = \frac{2}{2n+1}$ y los factores $\langle P_n|f\rangle$ fueron hallados en el inciso a). Entonces:

$$\mathbb{P}|f\rangle_t = |P_0\rangle \frac{2}{2} + |P_1\rangle \frac{74/15}{3/2} + |P_2\rangle \frac{4/5}{5/2}$$

$$= |P_0\rangle + |P_1\rangle \frac{37}{5} + |P_2\rangle 2 = \begin{pmatrix} 1 \\ 37/5 \\ 2 \end{pmatrix}_{P_2}$$

Base polinomios de Legrende de grado menor o igual que dos

d) Sean las bases $B = \{|1\rangle, |t\rangle, |t^2\rangle, |t^3\rangle, |t^4\rangle\} = \{|E_\alpha\rangle\}$
y $P = \{|1\rangle, |t\rangle, |\frac{1}{2}(3t^2-1)\rangle, |\frac{1}{2}(5t^3-3t)\rangle, |\frac{1}{8}(35t^4-30t^2+3)\rangle\}$
$= \{|P\alpha\rangle\}$ con $\alpha = 0,1,2,3,4$

En la base de monomios:

$$\mathbb{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} \frac{d}{dt}|1\rangle = 0 \\ \frac{d}{dt}|t\rangle = |1\rangle \\ \frac{d}{dt}|t^2\rangle = 2|t\rangle \\ \frac{d}{dt}|t^3\rangle = 3|t^2\rangle \\ \frac{d}{dt}|t^4\rangle = 4|t^3\rangle \end{cases}$$

Imagenes de la base de monomios

pues, e.g. $\mathbb{D}|3t^4\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 12 \\ 0 \end{pmatrix} = |12t^3\rangle$

$$e^{\mathbb{D}} = \sum_{n=0}^{\infty} \frac{\mathbb{D}^n}{n!} = \underbrace{\mathbb{I} + \mathbb{D} + \frac{\mathbb{D}^2}{2} + \frac{\mathbb{D}^3}{6} + \frac{\mathbb{D}^4}{24}} + \underbrace{\frac{\mathbb{D}^5}{120} + \cdots}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \cdots$$

→ $\mathbb{D}$ es nilpotente para potencias mayores que 4.
Entonces, es correcto afirmar:

$$\mathbb{T} = e^{\mathbb{D}} = \sum_{n=0}^{4} \frac{\mathbb{D}^n}{n!} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_B = T_{(B)}{}^{\alpha}_{\beta}$$

En la base de polinomios de legendre

$\frac{d}{dt}|1\rangle = 0 = |0\rangle$

$\frac{d}{dt}|t\rangle = 1 = |1\rangle$

$\frac{d}{dt}|\frac{1}{2}(3t^2-1)\rangle = 3t = 3|t\rangle$

$\frac{d}{dt}|\frac{1}{2}(5t^3-3t)\rangle = \frac{15}{2}t^2 - \frac{3}{2} = 5\left(\frac{3}{2}t^2 - \frac{3}{10} - \frac{2}{10} + \frac{2}{10}\right)$

$= 5\left(\frac{3}{2}t^2 - \frac{5}{10}\right) + \frac{10}{10} = 5\left(\frac{1}{2}(3t^2-1)\right) + 1$

$= 5|\frac{1}{2}(3t^2-1)\rangle + |1\rangle$

$\frac{d}{dt}|\frac{1}{8}(35t^4-30t^2+3)\rangle = \frac{35}{2}t^3 - \frac{15}{2}t = 7\left(\frac{5}{2}t^3 - \frac{15}{14}t - \frac{6}{14}t + \frac{6}{14}t\right)$

$= 7\left(\frac{5}{2}t^3 - \frac{21}{14}t\right) + 3t = 7|\frac{1}{2}(5t^3-3t)\rangle + 3|t\rangle$

Escaneado con CamScanner

$$\mathbb{D} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{T} = e^{\mathbb{D}} = \sum_{n=0}^{4} \frac{\mathbb{D}^n}{n!} = \begin{pmatrix} 1 & 1 & 3/2 & 7/2 & 75/8 \\ 0 & 1 & 3 & 15/2 & 41/2 \\ 0 & 0 & 1 & 5 & 35/2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_P = T_{(P)}{}^{\alpha}{}_{\beta}$$

¿Como transforman las representaciones matriciales de $\mathbb{T}$?

$$T_{(P)}{}^{\alpha}{}_{\beta} = \frac{\partial p^{\alpha}}{\partial e^{\mu}} \frac{\partial e^{\nu}}{\partial p^{\beta}} T_{(B)}{}^{\mu}{}_{\nu} = \frac{\partial p^{\alpha}}{\partial e^{\mu}} T_{(B)}{}^{\mu}{}_{\nu} \frac{\partial e^{\nu}}{\partial p^{\beta}}$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & 0 & 1/5 \\ 0 & 1 & 0 & 3/5 & 0 \\ 0 & 0 & 2/3 & 0 & 4/7 \\ 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 8/35 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1/2 & 0 & 3/8 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 3/2 & 0 & -15/4 \\ 0 & 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 0 & 35/8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 3/2 & 7/2 & 75/8 \\ 0 & 1 & 3 & 15/2 & 41/2 \\ 0 & 0 & 1 & 5 & 35/2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = T_{(B)}{}^{\alpha}{}_{\beta} \checkmark$$

$$\Rightarrow T_{(P)}{}^{\alpha}{}_{\beta} = \frac{\partial p^{\alpha}}{\partial e^{\mu}} \frac{\partial e^{\nu}}{\partial p^{\beta}} T_{(B)}{}^{\mu}{}_{\nu}$$

Igualmente y análogamente: $T_{(B)}{}^{\mu}{}_{\nu} = \dfrac{\partial e^{\mu}}{\partial p^{\alpha}} \dfrac{\partial p^{\beta}}{\partial e^{\nu}} T_{(P)}{}^{\alpha}{}_{\beta}$

- $Tr(T_{(P)}{}^{\alpha}{}_{\beta}) = T_{(P)}{}^{\alpha}{}_{\alpha} = T_{(P)}{}^{0}{}_{0} + T_{(P)}{}^{1}{}_{1} + T_{(P)}{}^{2}{}_{2} + T_{(P)}{}^{3}{}_{3} + T_{(P)}{}^{4}{}_{4}$
  $= 1+1+1+1+1 = 5$
- $Tr(T_{(B)}{}^{\alpha}{}_{\beta}) = T_{(B)}{}^{\alpha}{}_{\alpha} = T_{(B)}{}^{0}{}_{0} + T_{(B)}{}^{1}{}_{1} + T_{(B)}{}^{2}{}_{2} + T_{(B)}{}^{3}{}_{3} + T_{(B)}{}^{4}{}_{4}$
  $= 1+1+1+1+1 = 5$

↳ Ambas trazas son iguales.

matriz triángular

- $\det(\mathbb{T}_{(P)}) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (1)(1)(1)(1)(1) = 1$

$$\det(\mathbb{T}_{(B)}) = \begin{pmatrix} 1 & 1 & 3/2 & 7/2 & 75/8 \\ 0 & 1 & 3 & 15/2 & 41/2 \\ 0 & 0 & 1 & 5 & 35/2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (1)(1)(1)(1)(1) = 1$$

matriz triangular

ambos determinantes coinciden

Escaneado con CamScanner

$$\mathbb{P}|f\rangle_t = |1\rangle + \frac{37}{5}|t\rangle + 2\left(\tfrac{1}{2}(3|t^2\rangle - |1\rangle)\right)$$

$$= |1\rangle + \frac{37}{6}|t\rangle + 3|t^2\rangle - |1\rangle = \frac{37}{5}|t\rangle + 3|t^2\rangle$$

$$= \begin{pmatrix} 0 \\ 37/5 \\ 3 \end{pmatrix}_{B_2}$$, Base de monomios de grado menor o igual que dos

→ Como vemos los coeficientes de la proyección de $|f\rangle_t$ en $P(t)_2$ son diferentes para las dos bases.
Además, realizamos el operador proyección solamente con la base de polinomios de Legendre pues solo podemos proyectar en una base ortogonal.

(c) $\mathbb{T} = e^{\mathbb{D}} = \exp(\mathbb{D})$ con $\mathbb{D} = \frac{d}{dt}$

Ahora: $e^{\mathbb{D}} = \sum_{n=0}^{4} \frac{\mathbb{D}^n}{n!}$

$\mathbb{T}^{-1} = ?$

Tomemos: $e^{\mathbb{D}} e^{-\mathbb{D}} = e^{\mathbb{D} + (-\mathbb{D})} e^{[\mathbb{D}, -\mathbb{D}]}$

$[\mathbb{D}, -\mathbb{D}] = \mathbb{D}(-\mathbb{D}) - (-\mathbb{D})\mathbb{D} = -\mathbb{D}^2 + \mathbb{D}^2 = \mathbb{O}$ → operador nulo

↪ conmutan

$$e^{\mathbb{D}} e^{-\mathbb{D}} = e^{\mathbb{D} - \mathbb{D}} e^{\mathbb{O}}$$
$$= e^{\mathbb{O}} \mathbb{I}$$
$$= \mathbb{I}$$

$\Rightarrow e^{-\mathbb{D}}$ es la inversa de $e^{\mathbb{D}}$, con $e^{-\mathbb{D}} = \sum_{n=0}^{4} \frac{(-\mathbb{D})^n}{n!}$

$\Rightarrow \mathbb{T}^{-1} = e^{-\mathbb{D}} = 1\mathbb{I} - 1\mathbb{D}^1 + \frac{1}{2}\mathbb{D}^2 - \frac{1}{6}\mathbb{D}^3 + \frac{1}{24}\mathbb{D}^4$ RTA/.

$\mathbb{T}^{\dagger} = ?$

$\mathbb{T}^{\dagger} = (e^{\mathbb{D}})^{\dagger} = \left(\sum_{n=0}^{4} \frac{\mathbb{D}^n}{n!}\right)^{\dagger} = \sum_{n=0}^{4} \left(\frac{(\mathbb{D}^n)^{\dagger}}{n!}\right)$

$$(\mathbb{D}^n)^{\dagger} = (\underbrace{\mathbb{D}\mathbb{D}\cdots\mathbb{D}}_{n\text{-veces}})^{\dagger} = \underbrace{\mathbb{D}^{\dagger}\mathbb{D}^{\dagger}\cdots\mathbb{D}^{\dagger}}_{n\text{-veces}} = (\mathbb{D}^{\dagger})^n$$

Como $\mathbb{T}$ opera sobre $P_{(t)4}$, si $|f\rangle_t \in P_{(t)4}$ → $|f\rangle_t$ es una función real por ser un polinomio de $P_{(t)4}$. Entonces:

$$\mathbb{D}^{\dagger}|f\rangle_t = \frac{d}{dt}^{\dagger}|f\rangle_t = \frac{d}{dt}|f\rangle_t = \mathbb{D}|f\rangle_t$$ → $\mathbb{D}$ es hermítico.

$\mathbb{T}^{\dagger} = \sum_{n=0}^{4} \frac{(\mathbb{D}^n)^{\dagger}}{n!} = \sum_{n=0}^{4} \frac{\mathbb{D}^n}{n!} = e^{\mathbb{D}} = \mathbb{T} \Rightarrow \mathbb{T}^{\dagger} = \mathbb{T}$ → $\mathbb{T}$ es hermítico