11 Lebesgue's decomposition theorem

Throughout this section $(\Omega, \mathcal{A})$ will denote a measurable space, and μ and ν will denote measures on this space.

Property 11.1. Let $f : \Omega \to [0, +\infty[$ be a density of ν with respect to μ. Then

$$\forall A \in \mathcal{A} : \quad \mu(A) \quad \Rightarrow \quad \nu(A) \tag{11.1}$$

Definition 11.2. We say that

(i) ν is called **absolutely continuous** with respect to μ (symbolically $\nu \ll \mu$) if 11.1 holds,

(ii) μ and ν are **equivalent** (symbolically $\mu \approx \nu$) if $\nu \ll \mu$ and $\mu \ll \nu$,

(iii) μ is **singular** to ν (symbolically $\mu \perp \nu$) if there exists an $A \in \mathcal{A}$ such that

$$\mu(A) = 0 \quad \text{and} \quad \nu(\Omega \setminus A).$$

Remark 11.3.

(i) $\approx$ is an equivalence relation.

(ii) $\mu \perp \nu \Leftrightarrow \nu \perp \mu$.

Let $f : \Omega \to [0, +\infty[$ be a measurable function. If $\nu = f\mu$, then ν is absolutely continuous with respect to μ. The situation is quite the opposite for, e.g., the Poisson distribution $\mu = \mathrm{Poi}_\lambda$ with parameter $\lambda > 0$ and $\nu = \mathcal{N}_{0,1}$. Here μ is singular to ν. The main goal of this chapter is to show that if μ and ν are σ-finite measures on $(\Omega, \mathcal{A})$, then ν can be decomposed into a part that is singular to μ and a part that is absolutely continuous with respect to μ.

Theorem 11.4 (Lebesgue's decomposition theorem)**.** If μ and ν are σ-finites, then ν can be uniquely decomposed into an absolutely continuous part ν_a and a singular part ν_s (with respect to μ):

$$\nu = \nu_a + \nu_s,$$

where ν_a has a density with respect to μ, and $\frac{d\nu_a}{d\mu}$ is $\mathcal{A}$-measurable and finite μ-a.e.

Corollary 11.5 (Radon–Nikodym theorem)**.** If μ and ν are σ-finites, then

$$\nu \text{ has density w.r.t. } \mu \quad \Leftrightarrow \quad \nu \ll \mu.$$

In this case, $\frac{d\nu}{d\mu}$ is $\mathcal{A}$-measurable and finite μ-a.e. And $\frac{d\nu}{d\mu}$ is called the **Radon–Nikodym derivative** of ν w.r.t. μ.