8 Product spaces

Definition 8.1 (Product space)**.** Let $(\Omega_i, i \in I)$ be an arbitrary family of sets. We denote by $\Omega = \times_{i \in I} \Omega_i$ the sets of maps $\omega : I \to \bigcup_{i \in I} \Omega_i$ such that $\omega(i) \in \Omega_i$ for all $i \in I$. Ω is called the **product** of the spaces $(\Omega_i, i \in I)$, or briefly the **product space**. If $\Omega_i = \Omega_0$ for all $i \in I$, then we write $\Omega = {\Omega_0}^I$.

Example 8.2.

(i) If $\Omega_1 = \{1, 2\}$ and $\Omega_2 = \{1, 2, 3\}$, then $\Omega_1 \times \Omega_2$ is isomorphic to

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}.$$

(ii) If $\Omega_0 = \mathbb{R}$ and $I = \{1, 2\} \times \{1, 2, 3\}$, then $\mathbb{R}^I$ is isomorphic to $\mathcal{M}(2, 3)$.

(iii) If $\Omega_0 = \mathbb{R}$ and $I = \{1, 2\}$, then $\mathbb{R}^{\{1,2\}}$ is isomorphic to the customary $\mathbb{R}^2$.

(iv) If $\Omega_0 = \mathbb{R}$ and $I = \mathbb{N}$, then $\mathbb{R}^{\mathbb{N}}$ is the space of sequences $(\omega(n), n \in \mathbb{N})$ in $\mathbb{R}$.

(v) If $\Omega_0 = \mathbb{R}$ and $I = \mathbb{R}$, then $\mathbb{R}^{\mathbb{R}}$ is the set of maps $\mathbb{R} \to \mathbb{R}$.

Definition 8.3 (Coordinate maps)**.** Let $(\Omega_i, i \in I)$ be an arbitrary family of sets and let Ω be the product of the spaces $(\Omega_i, i \in I)$. For every $i \in I$, $X_i : \Omega \to \Omega_i$, $\omega \mapsto \omega(i)$, denotes the ith **coordinate map**.

Definition 8.4 (Product σ-algebra)**.** Let $(\Omega_i, \mathcal{A}_i)$, $i \in I$, be measurable spaces and let Ω be the product of the spaces $(\Omega_i, i \in I)$. The **product σ-algebra**

$$\mathcal{A} = \bigotimes_{i \in I} \mathcal{A}_i$$

is the smallest σ-algebra on Ω such that for every $i \in I$, the coordinate map X_i is measurable with respect to $\mathcal{A} - \mathcal{A}_i$, i.e.,

$$\mathcal{A} = \sigma(X_i, i \in I).$$

If $(\Omega_i, \mathcal{A}_i) = (\Omega_0, \mathcal{A}_0)$ for all $i \in I$, then we write $\mathcal{A} = {\mathcal{A}_0}^{\otimes I}$.