7 The dominated convergence theorem

Theorem 7.1 (**Dominated convergence theorem** (D.C.T.))**.** Let $X, X_1, X_2, \ldots \in \mathcal{M}$ and $Y, Y_1, Y_2, \ldots \in \mathcal{L}^1(\mu)$ such that

(i) $X_n \to X$ a.e.,

(ii) $|X_n| \le Y_n$ for all n,

(iii) $Y_n \to Y$ a.e. and

(iv) $\int Y_n d\mu \to \int Y d\mu$.

Then $X, X_1, X_2, \ldots \in \mathcal{L}^1(\mu)$ and $\int X_n d\mu \to \int X d\mu$.

Proof. It is clear that $X_1, X_2, \ldots \in \mathcal{L}^1(\mu)$; then $X \in \mathcal{L}^1(\mu)$. Indeed, by the Fatou's lemma,

$$\begin{aligned}\int |X| d\mu &\le \int \liminf |X_n| d\mu \\ &\le \liminf \int |X_n| d\mu \\ &\le \liminf \int Y_n d\mu \\ &\le \int Y d\mu.\end{aligned}$$

On one hand, $Y_n + X_n \in \mathcal{M}^+$ for every n; by the Fatou's lemma,

$$\begin{aligned}\int Y d\mu + \int X d\mu &\le \int (Y + X) d\mu \\ &\le \int \liminf (Y_n + X_n) d\mu \\ &\le \liminf \int (Y_n + X_n) d\mu \\ &\le \liminf \left(\int Y_n d\mu + \int X_n d\mu \right) \\ &\le \liminf \int Y_n d\mu \, + \, \liminf \int X_n d\mu \\ &\le \int Y d\mu + \liminf \int X_n d\mu,\end{aligned}$$

hence,

$$\int X d\mu \le \liminf \int X_n d\mu. \tag{1}$$

On the other hand, $Y_n - X_n \in \mathcal{M}^+$ for every n; and by the Fatou's lemma, analogy, we have that

$$\limsup \int X_n d\mu \leq \int X d\mu. \tag{2}$$

Finally, from (1) and (2), we conclude the desired limit. □

Remark 7.2. This version of the D.C.T. is stronger than the classical one.

Property 7.3. Let $X, Y : \Omega \to \mathbb{R}$ be integrabel functions. Then $X = Y$ a.e. iff for all $A \in \mathcal{A}$:

$$\int_A X d\mu = \int_A Y d\mu. \tag{3}$$

Proof. ($\Rightarrow$) Let $A \in \mathcal{A}$ and let $N \in \mathcal{A}$ be a null set such that $X = Y$ on N^c. Then $\mathbf{1}_A X = \mathbf{1}_A Y$ on $N^c \cup A^c$; i.e., $\mathbf{1}_A X = \mathbf{1}_A Y$ a.e.; since, $\mu(N \cap A) = 0$. Hence, we have the equation (3).

($\Leftarrow$) Since μ is σ-finite, let $A_1, A_2, \ldots \in \mathcal{A}$ be such that $A_n \uparrow \Omega$. For every n, let $X_n := \mathbf{1}_{A_n} X$ and $f_n := \mathbf{1}_{A_n}|X|$. Let $f := |X|$. Then we claim that

(i) $X_n \to X$,

(ii) $|X_n| \leq f_n$ for all n,

(iii) $f_n \to f$ and

(iv) $\int f_n d\mu \to \int f d\mu$,

where the last proposition follows from the D.C.T.; since, $f, f_1, f_2, \ldots \in \mathcal{M}^+$ and $f_n \uparrow f$. Hence, by the D.C.T.,

$$\int_{A_n} X d\mu \to \int X d\mu. \tag{4}$$

Analogously, we have that

$$\int_{A_n} Y d\mu \to \int Y d\mu. \tag{5}$$

Then from (3), (4) and (5), we conclude that

$$\int X d\mu = \int Y d\mu.$$

Therefore, as $(X - Y) \in \mathcal{L}^1$, we have that

$$\int (X - Y) d\mu = 0.$$

Finally, we conclude that $X = Y$ a.e. □