# 2 Ordered sets

When the partial (total) order is understood, we will usually write $E$ instead of $(E, \leq)$. Henceforward we consider the sets $\mathbb{R}$ and $\overline{\mathbb{R}}$ with the total order relations defined in the examples above. A subset $A$ of a partially (totally) ordered set $E$ is a partially (totally) ordered set with the partial (total) order inherited from $E$.

**Example 2.1.** $\mathbb{N}$ is a totally ordered sets with the relation inherited from $\mathbb{R}$.

**Example 2.2.** The total order of $\mathbb{R}$ coincides with the one inherited from $\overline{\mathbb{R}}$.

**Definition 2.3.** Let $(E, \leq)$ be a partially ordered set. Let $A$ be subset of $E$. An element $c$ of $A$ is called the **least element** (**greatest element**) of $A$ if $c \leq x$ $(c \geq x)$ for every $x$ in $E$.

When a least or greatest element of a subset $A$ of $E$ exist it is unique.

**Definition 2.4.** Let $(E, \leq)$ be a partially ordered set. Let $A$ be subset of $E$. An element $c$ of $E$ is called a **lower bound** (an **upper bound**) of $A$ if $c \leq x$ $(c \geq x)$ for every $x$ in $E$. We say that $A$ is **bounded bellow** (**bounded above**) if it has a lower bound (an upper bound).

**Definition 2.5.** Let $E$ be a partially ordered set. Let $A$ be subset of $E$. A lower bound (an upper bound) $c$ of $E$ is called the **infimum** (**supremum**) of $A$, and we denote it as $\inf A$ $(\sup A)$, if $c$ is the greatest (least) element of the set of all lower (upper) bounds of $A$.

When a infimum or supremum of a subset $A$ of $E$ exist it is unique.

**Property 2.6.** Let $(E, \leq)$ be a totally ordered set. Let $A$ be a subset of $E$. An element $c$ of $E$ is the infimum (supremum) of $A$ if and only if for every $x$ in $E$:

$$c < x(x < c) \;\Rightarrow\; x \text{ is not a lower bound (an upper bound).}$$

For every bounded below (above) subset $A$ of $\mathbb{R}$ the infimum (supremum) of $A$ exits.

**Property 2.7.** For every subset $A$ of $\overline{\mathbb{R}}$ the infimum and supremum of $A$ exit.

Let $x$ and $y$ be two elements of a totally ordered set $(E, \leq)$. We say that an element $z$ of $E$ is **beteween** $x$ and $y$ if $x \leq z \leq y$ or $x \geq z \geq y$.

**Definition 2.8.** Let $E$ be a totally ordered set. We say that a subset $I$ of $E$ is an **interval** if it contains every element between two elements of it.

**Property 2.9.** Intervals in $\mathbb{R}$ could be characterized as one of the following sets:

$$\begin{aligned}
]-\infty,\infty[ &:= \mathbb{R}, \\
]-\infty,b] &:= \{x \in \mathbb{R} \; ; \; x \leq b\}, \\
]-\infty,b[ &:= \{x \in \mathbb{R} \; ; \; x < b\}, \\
[a,\infty[ &:= \{x \in \mathbb{R} \; ; \; x \geq a\}, \\
]a,\infty[ &:= \{x \in \mathbb{R} \; ; \; x > a\}, \\
[a,b] &:= \{x \in \mathbb{R} \; ; \; a \leq x \leq b\}, \\
[a,b[ &:= \{x \in \mathbb{R} \; ; \; a \leq x < b\}, \\
]a,b] &:= \{x \in \mathbb{R} \; ; \; a < x \leq b\}, \\
]a,b[ &:= \{x \in \mathbb{R} \; ; \; a < x < b\}
\end{aligned}$$

with $a$ and $b$ in $\mathbb{R}$.

**Property 2.10.** Intervals in $\overline{\mathbb{R}}$ could be characterized as one of the following sets:

$$\begin{aligned}
[-\infty,\infty] &:= \overline{\mathbb{R}}, \\
[-\infty,\infty[ &:= \{x \in \overline{\mathbb{R}} \; ; \; x < \infty\}, \\
]-\infty,\infty] &:= \{x \in \overline{\mathbb{R}} \; ; \; x > -\infty\}, \\
]-\infty,\infty[ &:= \{x \in \overline{\mathbb{R}} \; ; \; -\infty < x < \infty\}, \\
[-\infty,b[ &:= \{x \in \overline{\mathbb{R}} \; ; \; x < b\}, \\
[-\infty,b] &:= \{x \in \overline{\mathbb{R}} \; ; \; x \leq b\}, \\
]-\infty,b[ &:= \{x \in \overline{\mathbb{R}} \; ; \; -\infty < x < b\}, \\
]-\infty,b] &:= \{x \in \overline{\mathbb{R}} \; ; \; -\infty < x \leq b\}, \\
[a,\infty] &:= \{x \in \overline{\mathbb{R}} \; ; \; x \geq a\}, \\
]a,\infty] &:= \{x \in \overline{\mathbb{R}} \; ; \; x > a\}, \\
[a,\infty[ &:= \{x \in \overline{\mathbb{R}} \; ; \; a \leq x < \infty\}, \\
]a,\infty[ &:= \{x \in \overline{\mathbb{R}} \; ; \; a < x < \infty\}, \\
[a,b] &:= \{x \in \overline{\mathbb{R}} \; ; \; a \leq x \leq b\}, \\
[a,b[ &:= \{x \in \overline{\mathbb{R}} \; ; \; a \leq x < b\}, \\
]a,b] &:= \{x \in \overline{\mathbb{R}} \; ; \; a < x \leq b\}, \\
]a,b[ &:= \{x \in \overline{\mathbb{R}} \; ; \; a < x < b\}
\end{aligned}$$

with $a$ and $b$ in $\mathbb{R}$.

For every $a$ and $r$ in $\mathbb{R}$ with $r > 0$ we called the set $]a-r, a+r[\subset \overline{\mathbb{R}}$ as the open interval centered at $a$ with radius $r$.