3 The extended real line

At this point we will endowed with a topology to the set $\overline{\mathbb{R}}$ in order to talk about convergence in this setting. Moreover, we will show that this topological space is second countable, metrizable, compact and connected.

Property 3.1. The collection of all open intervals centered at rational numbers with rational radius forms a countable base for the usual topology over $\mathbb{R}$.

Taking property 3.1 as inspiration, we define a topology over $\overline{\mathbb{R}}$: the topology that has the collection

- of all open intervals centered at rational numbers with rational radius and
- all intervals of the form: $[-\infty, q[$ or $]q, \infty]$ with q in $\mathbb{Q}$

as a base. Henceforward we will consider this topology for $\overline{\mathbb{R}}$.

Property 3.2. The topological space $\overline{\mathbb{R}}$ is second countable.

Now, in order to show that this topology is metrizable, we consider the following biyective funcion $\varphi : \overline{\mathbb{R}} \to [-1, 1]$,

$$\varphi(x) = \begin{cases} -1 & \text{if } x = -\infty \\ x/(1 + |x|) & \text{if } x \in \mathbb{R} \\ 1 & \text{if } x = \infty, \end{cases}$$

and define the metric d over $\overline{\mathbb{R}}$ as $d(x, y) = |\varphi x - \varphi y|$ for every x and y in $\overline{\mathbb{R}}$.

Property 3.3. The topological space $\overline{\mathbb{R}}$ is metrizable.

Proof. The topology induced by the metric d defined above coincides with the usual topology over $\overline{\mathbb{R}}$. $\square$

Property 3.4. The spaces $\overline{\mathbb{R}}$ and $[-1, 1]$ are homeomorphic.

Proof. The function φ defined above is an homeomorphism between $\overline{\mathbb{R}}$ and $[-1, 1]$. $\square$

Property 3.5. The topological space $\overline{\mathbb{R}}$ is compact, Hausdorff and connected.

Proof. Since $[-1, 1]$ is compact, Hausdorff and connected, $\overline{\mathbb{R}}$ is too. $\square$

Property 3.6. The connected subsets of $\overline{\mathbb{R}}$ are precisely the intervals.

Since $\overline{\mathbb{R}}$ is a metrizable topological space, $[0, \infty]$ and $[0, \infty] \times [0, \infty]$ are too.

Property 3.7. The functions ξ and η, from $[0,\infty] \times [0,\infty]$ into $[0,\infty]$,

$$\xi(x,y) = \begin{cases} x+y & \text{if } x<\infty \text{ and } y<\infty \\ \infty & \text{if } x=\infty \text{ or } y=\infty, \end{cases}$$

$$\eta(x,y) = \begin{cases} xy & \text{if } x<\infty \text{ and } y<\infty \\ \infty & \text{if } x=\infty \text{ or } y=\infty, \end{cases}$$

are continuous.

Hereafter we will write $x+y$ instead of $\xi(x,y)$ and xy instead of $\eta(x,y)$.