6 Simple properties of the integral

Theorem 6.1. Let $X \in \mathcal{M}$. Then

(i) $X = 0$ c.t.p. $\Leftrightarrow$ $\int X d\mu = 0$.

(ii) $\int X d\mu < +\infty$ $\Rightarrow$ $X < +\infty$ c.t.p.

Proof. Exercise. □

Theorem 6.2 (Properties of the integral)**.** Let $X, Y \in \mathcal{L}^1(\mu)$ and $\alpha, \beta \in \mathbb{R}$.

(i) [Monotonicity] If $X \leq Y$ a.e., then $\int X d\mu \leq \int Y d\mu$. In particular, if $X = Y$ a.e., then $\int X d\mu = \int Y d\mu$.

(ii) [Triangular inequality] $|\int f d\mu| \leq \int |X| d\mu$.

(iii) [Linearity] $\alpha X + \beta Y \in \mathcal{L}^1(\mu)$ and

$$\int (\alpha X + \beta Y) d\mu = \alpha \int X d\mu + \beta \int Y d\mu.$$

This equation also holds if at most one of the integrals infinite.

Proof. Exercise. □

Property 6.3 (Image measure)**.** Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces, let μ be a measure on $(\Omega, \mathcal{A})$ and let $X : \Omega \to \Omega'$ be measurable. Let $\mu' = \mu \circ X^{-1}$ be the image measure of μ under the map X. Assume that $f : \Omega' \to \overline{\mathbb{R}}$ is μ'-integrable. Then $f \circ X \in \mathcal{L}^1(\mu)$ and

$$\int (f \circ X) d\mu = \int f d(\mu \circ X^{-1}).$$

In particular, if X is a random variable on $(\Omega, \mathcal{A}, \mathbf{P})$, then

$$\int (f \circ X)(\omega) \mathbf{P}[d\omega] = \int f(x) \mathbf{P}_X[dx].$$

Proof. Exercise. □

Example 6.4 (Discrete measure space)**.** Let $(\Omega, \mathcal{A})$ be a discrete measurable space and let $\mu = \sum_{\omega \in \Omega} \alpha_\omega \delta_\omega$ for certain numbers $\alpha_\omega \geq 0$, $\omega \in \Omega$. A map $f : \Omega \to \mathbb{R}$ is integrable iff $\sum_{\omega \in \Omega} |f(\omega)| \alpha_\omega < +\infty$. In this case

$$\int f d\mu = \sum_{\omega \in \Omega} f(\omega) \alpha_\omega.$$

Property 6.5. Let μ be a measure on $(\Omega, \mathcal{A})$ and let $f : \Omega \to [0, +\infty[$ be a measurable map. Then the map $\nu : \mathcal{A} \to [0, +\infty]$,

$$\nu(A) := \int_A f d\mu,$$

is a measure on $(\Omega, \mathcal{A})$.

Proof. *Exercise.* □

Definition 6.6. Let μ and ν be a measures on $(\Omega, \mathcal{A})$. A measurable function $f : \Omega \to [0, +\infty[$ is called a **density** of ν with respect to μ if

$$\nu(A) = \int_A f d\mu,$$

for every $A \in \mathcal{A}$, and we write

$$\nu = f\mu \quad \text{and} \quad f = \frac{d\nu}{d\mu}.$$

Example 6.7. The normal distribution $\nu = \mathcal{N}_{0,1}$ has the density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ with respect to the Lebesgue measure $\mu = \lambda$ on $\mathbb{R}$.

Property 6.8 (Uniqueness of the density)**.** Let ν de σ-finite. If f_1 and f_2 are densities of ν with respect to μ, then $f_1 = f_2$ μ-almost everywhere. In particular, the density $\frac{d\nu}{d\mu}$ is unique up to equality μ-almost everywhere.

Property 6.9. Let μ be a measure on $(\Omega, \mathcal{A})$, and let $f : \Omega \to [0, +\infty[$ and $g : \Omega \to \overline{\mathbb{R}}$ be measurable maps. Then $g \in \mathcal{L}^1(f\mu)$ iff $gf \in \mathcal{L}^1(\mu)$. In this case,

$$\int g \, d(f\mu) = \int gf \, d\mu.$$

Proof. *Exercise.* □

Definition 6.10. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space (Ω is a nonempty set) and $\overline{\mathbb{R}} := [-\infty, +\infty]$. For measurable $f : \Omega \to \overline{\mathbb{R}}$, define

$$|f|_p := \begin{cases} \left(\int |f|^p d\mu\right)^{1/p} & \text{if } p \in [1, +\infty[\\ \inf\{K \geq 0 \; ; \; \mu(|f| > K) = 0\} & \text{if } p = +\infty. \end{cases}$$

Further, for any $p \in [1, +\infty]$, define the vector space

$$\mathcal{L}^p(\mu) := \{f : \Omega \to \overline{\mathbb{R}} \; ; \; f \text{ is measurable and } |f|_p < +\infty\}.$$

Property 6.11. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f \in \mathcal{L}^1(\mu)$. Then it holds:

(i) $(\mathcal{L}^1(\mu), |\ |_1)$ is a seminormed vector space.

(ii) $|f|_1 = 0 \quad \Rightarrow \quad f = 0$ a.e.

Proof. *Exercise.* □

Property 6.12. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, let $\mu(\Omega) < +\infty$ and let $1 \leq p' \leq p \leq +\infty$. Then it holds:

(i) $\mathcal{L}^p(\mu) \subset \mathcal{L}^{p'}(\mu)$.

(ii) The canonical inclusion $i : \mathcal{L}^p(\mu) \hookrightarrow \mathcal{L}^{p'}(\mu)$ is continuous.

Proof. *Exercise.* $\square$

Exercise 6.1. Let μ and ν be a measures on $(\Omega, \mathcal{A})$; let $f : \Omega \to [0, +\infty[$ be a measurable function. If $\nu = f\mu$, then for all $A \in \mathcal{A}$

$$\mu(A) \quad \Rightarrow \quad \nu(A).$$