Based on the notions of measure spaces and measurable maps, we introduce the integral of a measurable map with respect to a general measure. This generalizes the Lebesgue integral that can be found in textbooks on calculus. Furthermore, the integral is a cornerstone in a systematic theory of probability that allows for the definition and investigation of expected values and higher moments of random variables. In this chapter, we define the integral by an approximation scheme with simple functions. Then we deduce basic statements such as Fatou's lemma.

5 Construction of the integral

In what follows, $(\Omega, \mathcal{A}, \mu)$ would be a measure space σ-finite; using the following convention: $0 \cdot (+\infty) = (+\infty) \cdot 0 = 0$; we define

$$\begin{aligned} \mathcal{S} &:= \{\, f\colon \Omega \to \mathbb{R}\ ;\ f \text{ is measurable and } f(\Omega) \text{ is finite}\,\}, \\ \mathcal{S}^+ &:= \{\, f \in \mathcal{S}\ ;\ f \geq 0\,\}, \\ \mathcal{M} &:= \{\, X\colon \Omega \to \overline{\mathbb{R}}\ ;\ X \text{ is measurable}\,\} \text{ and} \\ \mathcal{M}^+ &:= \{\, X \in \mathcal{M}\ ;\ X \geq 0\,\}; \end{aligned}$$

and we say that a certain property holds **almost everywhere** (a.e.) if there exists a null set $N \in \mathcal{A}$ such that this property holds on N^c.

Let $f \in \mathcal{S}^+$. If

$$f = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{A_i} \tag{1}$$

for some $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \left]0, +\infty\right[$, and mutually disjoint sets $A_1, \ldots, A_m \in \mathcal{A}$, then (1) is called **a normal representation** of f.

Lemma 5.1. Let $f \in \mathcal{S}^+$. If $f = \sum\limits_{i=1}^{m} \alpha_i \mathbf{1}_{A_i}$ and $f = \sum\limits_{i=1}^{n} \beta_i \mathbf{1}_{B_i}$ are two normal representations of f, then

$$\sum_{i=1}^{m} \alpha_i \mu(A_i) = \sum_{i=1}^{n} \beta_i \mu(B_i).$$

Proof.

Exercise 5.1. □

This lemma allows us to do the following definition.

Definition 5.2. Let $f \in \mathcal{S}^+$. Then, we define **the integral** of f **with respect to** μ as

$$\int f d\mu := \sum_{i=1}^{m} \alpha_i \mu(A_i)$$

if f has normal representation $f = \sum\limits_{i=1}^{m} \alpha_i \mathbf{1}_{A_i}$.

Lemma 5.3. Let $f, g \in \mathcal{S}^+$ and $\alpha \in [0, +\infty[$. Then the following propositions holds:

(i) $\int (\alpha f) d\mu = \alpha \int f d\mu$.

(ii) $\int (f+g) d\mu = \int f d\mu + \int g d\mu$.

(iii) $f \leq g \ \Rightarrow \ \int f d\mu \leq \int g d\mu$

Proof.

Exercise 5.2. □

Lemma 5.4.

$$\mathcal{M}^+ = \{\, X \colon \Omega \to [0, +\infty] \ ; \ X \text{ is measurable} \,\}$$

Proof.

Exercise 5.3. □

Definition 5.5. Let $X \in \mathcal{M}^+$. Then, we define **the integral of X with respect to μ** as

$$\int X d\mu := \sup \left\{ \int f d\mu \ ; \ f \in \mathcal{S}^+, f \leq X \right\}.$$

It could be verified that this last definition extends the previous one (

Exercise 5.4.)

Lemma 5.6. Let $X, Y, X_1, X_2, \ldots \in \mathcal{M}^+$ y $\alpha, \beta \in [0, +\infty]$. Then the following propositions holds:

(i) $X \leq Y \ \Rightarrow \ \int X d\mu \leq \int Y d\mu$

(ii) [Monotone convergence theorem (M.C.T.)] $X_n \uparrow X \ \Rightarrow \ \int X_n d\mu \uparrow \int X d\mu$.

(iii) $\int (\alpha X + \beta Y) d\mu = \alpha \int X d\mu + \beta \int Y d\mu$.

Proof.

Exercise 5.5. □

Lemma 5.7 (Fatou's lemma)**.** Let $X_1, X_2, \ldots \in \mathcal{M}^+$. Then

$$\int \liminf X_n d\mu \leq \liminf \int X_n d\mu.$$

Proof. For every $n \in \mathbb{N}$ sea $Y_n = \inf\{\, X_k \ ; \ k \geq n \,\}$. On the one hand, we have $Y_n \in \mathcal{M}^+$ for all n and $Y_n \uparrow \liminf X_n$; Then, by the M.C.T.,

$$\int Y_n d\mu \uparrow \int \liminf X_n d\mu. \tag{2}$$

On the other hand, $Y_n \leq X_n$ for all n; then,

$$\liminf \int Y_n d\mu \leq \liminf \int X_n d\mu. \tag{3}$$

Finally, from (2) and (3) we have the desired inequality. $\square$

Remark 5.8. It could be verified that the Fatou's lemma and the M.C.T. are equivalent (

Exercise 5.6.)

Finally, we give our last definition of integral for measurable functions.

Definition 5.9. We say that $X \in \mathcal{M}$ is μ-**integrable** if $\int |X| d\mu < +\infty$. Then we define

$$\mathcal{L}^1(\mu) := \mathcal{L}^1(\Omega, \mathcal{A}, \mu) := \{ X \in \mathcal{M} \, ; \int |X| d\mu < +\infty \}.$$

Let $X \in \mathcal{L}^1(\mu)$. Now, we define **the integral of X with respect to μ** as

$$\int X d\mu := \int X^+ d\mu - \int X^- d\mu. \tag{4}$$

If we only have either $\int X^+ d\mu < +\infty$ or $\int X^- d\mu < +\infty$, then we define $\int X d\mu$ as (4). Finally, for every $A \in \mathcal{A}$ we define

$$\int_A X d\mu := \int (\mathbf{1}_A X) d\mu.$$