# 9 Finite products

Consider now the situation of finitely many $\sigma$-finite measure spaces $(\Omega_i, \mathcal{A}_i, \mu_i)$, $i = 1, \dots, n$.

**Theorem 9.1** (Finite product measures)**.** There exists a unique $\sigma$-finite measure $\mu$ on $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ such that

$$\mu(A_1 \times \dots \times A_n) = \mu_1(A_1) \cdots \mu_n(A_n)$$

for $A_i \in \mathcal{A}_i$, $i = 1, \dots, n$. $\bigotimes_{i=1}^n \mu_i := \mu_1 \otimes \dots \otimes \mu_n := \mu$ is called the **product measure** of the $\mu_i$. If all spaces involved equal $(\Omega_0, \mathcal{A}_0, \mu_0)$, then we write $\mu_0{}^{\otimes n} := \bigotimes_{i=1}^n \mu_0$.

*Proof. Exercise.* □

**Example 9.2.** Let $(\Omega_i, \mathcal{A}_i, \mathbf{P}_i)$ be probability spaces , $i = 1, \dots, n$. On the space $(\Omega, \mathcal{A}, \mathbf{P}) = (\times_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{A}_i, \bigotimes_{i=1}^n \mathbf{P}_i)$ , the coordinate maps $X_i : \Omega \to \Omega_i$ are independent with distribution $\mathbf{P}_{X_i} = \mathbf{P}_i$.

**Theorem 9.3** (Fubini)**.** Let $(\Omega_i, \mathcal{A}_i, \mu_i)$ be $\sigma$-finite measure spaces, $i = 1, 2$. On the space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$ let $f : (\Omega_1 \times \Omega_2) \to \overline{\mathbb{R}}$ be measurable. If one of the following conditions holds:

(i) $f \geq 0$

(ii) $f \in \mathcal{L}^1(\mu_1 \otimes \mu_2)$

then

$$\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2) \mu_2(d\omega_2) \text{ is } \mathcal{A}_1\text{-measurable,}$$
$$\omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2) \mu_1(d\omega_1) \text{ is } \mathcal{A}_2\text{-measurable,}$$

and

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \otimes \mu_2) &= \int_{\Omega_1} \left( \int_{\Omega_2} f(\omega_1, \omega_2) \mu_2(d\omega_2) \right) \mu_1(d\omega_1 = \\ &= \int_{\Omega_2} \left( \int_{\Omega_1} f(\omega_1, \omega_2) \mu_1(d\omega_1) \right) \mu_2(d\omega_2) \end{aligned}$$

*Proof. Exercise.* □