# 4 Classes of sets

We say that property ... is **closed under** certain property if ...

**Definition 4.1.** A subset $\mathcal{C}$ of $2^E$ is called a **$\pi$-system** if it is closed under intersection.

**Definition 4.2.** A subset $\mathcal{C}$ of $2^E$ is called a **semi-ring** if it is a $\pi$-system such that $A \setminus B$ could be expressed as a finite union of disjoint elements of $\mathcal{C}$ for every $A$ and $B$ in $\mathcal{C}$.

**Definition 4.3.** A subset $\mathcal{C}$ of $2^E$ is called a **semi-algebra** over $E$ if it is a $\pi$-system such that $E$ is in $\mathcal{C}$ and $E \setminus A$ could be expressed as a finite union of disjoint elements of $\mathcal{C}$ for every $A$ in $\mathcal{C}$.

Semi-algebras could be characterized as those semi-rings that have the universe set. Notice that we talk about a semi-algebra over some universal set, in contrast with $\pi$-systems and semi-rings. Every semi-algebra contains the empty set and the **trivial semi-algebra** over $E$: $\{\emptyset, E\}$ is the smallest semi-algebra over $E$.

**Definition 4.4.** A subset $\mathcal{C}$ of $2^E$ is called an **algebra** over $E$ if it is a $\pi$-system such that $E$ is in $\mathcal{C}$ and $E \setminus A$ is in $\mathcal{C}$ for every $A$ in $\mathcal{C}$.

Every algebra contains the empty set. For one thing, the **trivial algebra** over $E$: $\{\emptyset, E\}$ is the least element between the algebras over $E$. For the other, the **discrete algebra** over $E$: $2^E$ is the greatest element between the algebras over $E$. Algebras over $E$ could be characterized as those subsets of $2^E$ that contains $E$ and are closed under complements and unions. Moreover, every algebra is closed under all the set operations over the universal set.

**Property 4.5.** Let $\mathcal{C}$ be a semi-algebra over $E$. Let $\mathcal{C}_1$ be the subset of $2^E$ whose elements are the finite unios of elements of $\mathcal{E}$. Let $\mathcal{C}_2$ be the subset of $2^E$ whose elements are the finite and disjoint unions of elements of $\mathcal{C}$. Then $\mathcal{C}_1$ is the least element between the the algebras over $E$ that contains $\mathcal{C}$ and $\mathcal{C}_1 = \mathcal{C}_2$.

**Definition 4.6.** A subset $\mathcal{C}$ of $2^E$ is called a **$\lambda$-system** over $E$ if $E$ is in $\mathcal{C}$ and is closed under proper differences and unions of nondecreasing sequences.

$\lambda$-systems over $E$ are closed under disjoint unions. $\lambda$-systems over $E$ could be characterized as those subsets of $2^E$ that contains $E$ and are closed under proper differences and unions of disjoint sequences.

**Definition 4.7.** A subset $\mathcal{C}$ of $2^E$ is called a **$\sigma$-algebra** over $E$ if $E$ is in $\mathcal{C}$ and is closed under complements and unions of sequences.

$\sigma$-algebras could be characterized as those $\lambda$-systems that are $\pi$-systems. Moreover, $\sigma$-algebras over $E$ could be characterized as those subsets of $2^E$ that contains $E$ and are closed under complements and intersections of sequences or those $\pi$-systems that contain $E$ and are closed under complements and unions of increasing sequences.

Since, the intersection of an arbitrary family of $\lambda$-systems and $\sigma$-systems over $E$ is still a $\lambda$-system and $\sigma$-system over $E$, respectively, giving a subset $\mathcal{C}$ of $2^E$ we can talk about the smallest $\lambda$-system and the smallest $\sigma$-system over $E$ that contains $\mathcal{C}$.

**Definition 4.8.** Let $\mathcal{C}$ be a subset of $2^E$. The smallest $\lambda$-system ($\sigma$-system) over $E$ that contains $\mathcal{C}$ is called the **$\lambda$-system generated** (**$\sigma$-algebra generated**) by $\mathcal{C}$ and are denoted by $\sigma\,\mathcal{C}$ and $\lambda\,\mathcal{C}$, respectively.

The following theorem links... The proof of this follows from the two lemmas bellow.

**Theorem 4.9** (**Dynkin's lemma**). Let $\mathcal{C} \subset 2^E$ be a $\pi$-system and $\mathcal{E}$ be a $\lambda$-system over $E$ that contains $\mathcal{C}$. Then $\sigma\,\mathcal{C}$ is contained in $\mathcal{E}$.

**Lemma 4.10.** If $\mathcal{C} \subset 2^E$ is a $\pi$-system, then $\lambda\,\mathcal{C}$ is still a $\pi$-system.

*Proof.* Fisrt, let $B$ be in $\mathcal{C}$. Then $\mathcal{C}_1 := \{A \in \lambda\,\mathcal{C}\,;\ A \cap B \in \lambda\,\mathcal{C}\}$ is a $\lambda$-system that contains $\mathcal{C}$. Second, let $B$ be in $\lambda\,\mathcal{C}$. Then $\mathcal{C}_2 := \{A \in \lambda\,\mathcal{C}\,;\ A \cap B \in \lambda\,\mathcal{C}\}$ is a $\lambda$-system that contains $\mathcal{C}$. $\square$

**Lemma 4.11.** If $\mathcal{C} \subset 2^E$ is a $\pi$-system, then $\lambda\,\mathcal{C} = \sigma\,\mathcal{C}$.