1 Binary relations

Definition 1.1. Let n be a positive integer greater than 1. The **cartesian product** of nonempty sets $E_1, E_2, \ldots \text{and} E_n$, denoted as $E_1 \times E_2 \times \ldots \times E_n$, is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_i \in E_i$ for every $i = 1, 2, \ldots, n$. The ith element of an n-tuple is called the i**th coordinate** of the n-tuple. Finally, we say that two n-tuples are equal if their respective coordeinates are.

Definition 1.2. A **binary relation** over E and F is a subset of $E \times F$.

When $E = F$, we say a binary relation over E instead of a binary relation over E and F. Henceforward, when we talk about a binary relation R over E, we will abuse the notation by writing $xR\,y$ instead of $(x, y) \in R$.

Definition 1.3. We say that a binary relation R over E is **reflexive**, **symmetric**, **antisymmetric** and **transitive** if for every x, y and z in E:

$$\begin{aligned}
&xR\,x;\\
&xR\,y \;\Rightarrow\; yR\,x;\\
&xR\,y\,,\, yR\,x \;\Rightarrow\; x = y \text{ and}\\
&xR\,y\,,\, yR\,z \;\Rightarrow\; xR\,z,
\end{aligned}$$

respectively.

Example 1.4. We say that a binary relation R over E is an **equivalence realtion** over E if it is reflexive, symmetric and transitive.

Example 1.5. We say that a binary relation $\leq$ over E is a **partial order** over E if it is reflexive, antisymmetric and transitive.

Example 1.6. We say that a binary relation $\leq$ over E is a **total order** over E if it is antisymmetric, transitive and $x \leq y$ or $y \leq x$ for every x and y in E.

Henceforth, when we talk about a partial or total order relation $\leq$ and write $x < y$, we mean that $x \leq y$ and $x \neq y$. Moreover, $y \geq x$ and $y > x$ do mean the same thing as $x \leq y$ and $x < y$, respectively.

Definition 1.7. A set E together with a partial order (total order) $\leq$, denoted as $(E, \leq)$, is called a **partially ordered set** (**totally ordered set**.)

Every totally ordered set is a partially ordered one. Total orders over E could be characterized as those partial orders $\leq$ over E such that $x \leq y$ or $x \geq y$ for every x and y in E.

Example 1.8. $(2^E, \subset)$ with the ususal set inclusion relation is a partially ordered set.

Example 1.9. $(\mathbb{R}, \leq)$ with the usual inequality relation $\leq$ is a totally ordered set.

Example 1.10. To the set $\mathbb{R}$ we add two points: $-\infty$ and ∞. Then we extend the total order relation defined above in the following fashion: $-\infty \leq x \leq \infty$ for every x in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. Thus, $(\overline{\mathbb{R}}, \leq)$ is totally ordered set.