Chap. 13. Shor Algorithm

Ex. 13.1

2 qubit basis states $\rightarrow QFT_4$: $N = 2^n = 2^2 = 4$

$$|\psi'\rangle = QFT_4|\psi\rangle = \sum_{k=0}^{N-1} \beta_k |k\rangle$$

$$= \beta_0|0\rangle + \beta_1|1\rangle + \beta_2|2\rangle + \beta_3|3\rangle$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$|1000\rangle \quad |0100\rangle \quad |0010\rangle \quad |0001\rangle$

These states are orthonormal:

$$\langle 0|1\rangle = (1\,0\,0\,0)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 0 \checkmark \qquad \langle 0|0\rangle = (1\,0\,0\,0)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1 \checkmark$$

$$\langle 0|2\rangle = (1\,0\,0\,0)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 0 \checkmark \qquad \langle 1|1\rangle = \langle 2|2\rangle = \langle 3|3\rangle = 1 \checkmark$$

$$\langle 0|3\rangle = \langle 1|2\rangle = \langle 1|3\rangle = \langle 2|3\rangle = 0 \checkmark$$

Ex 13.2:

$$QFT_4 = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{pmatrix}$$

QFT_4 is unitary? $\rightarrow QFT_4^{-1} = QFT_4^{\dagger}$?

$$\rightarrow QFT_4^{\dagger} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & -i\end{pmatrix}$$

$\rightarrow QFT_4^{-1}\, QFT_4 = I$

$(QFT_4^{\dagger})\, QFT_4 = I$?

$$\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & -i\end{pmatrix} \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4 & 0 & 0 & 0\\ 0 & 4 & 0 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}$$

$$= I \checkmark$$

Ex. 13.3

$QFT_4|0\rangle = ?$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\sum_{k=0}^{3}|k\rangle = |\psi'\rangle$$

$\rightarrow \langle\psi'|\psi'\rangle = 4\left(\frac{1}{2}\right)^2 = 1$ ✓

$$\rightarrow QFT_4|1\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$$

$$= \frac{1}{2}\left(|0\rangle + i|1\rangle - |2\rangle - i|3\rangle\right) = |\psi'\rangle$$

$$\rightarrow \langle\psi'|\psi'\rangle = \frac{1}{4}\begin{pmatrix} 1 & -i & -1 & +i \end{pmatrix}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} = \frac{1}{4}\left(1 - i^2 + 1 - i^2\right)$$

$= 1$ ✓

$$\rightarrow QFT_4|2\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}\left(|0\rangle - |1\rangle + |2\rangle - |3\rangle\right) = |\psi'\rangle$$

$\rightarrow \langle\psi'|\psi'\rangle = \frac{1}{4}4(1)^2 = 1$ ✓

$$\rightarrow QFT_4|3\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

$$= \frac{1}{2}\left(|0\rangle - i|1\rangle - |2\rangle + i|3\rangle\right) = |\psi'\rangle$$

$$\rightarrow \langle\psi'|\psi'\rangle = \frac{1}{4}\begin{pmatrix} 1 & +i & -1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = \frac{1}{4}\left(1 - i^2 + 1 - i^2\right)$$

$= 1$ ✓

Ex. 13.4

QFT_N is unitary? $\quad (QFT_N^{-1})\, QFT_N = I$

$(QFT_N^{\dagger})\, QFT_N = I$?

$$\rightarrow QFT_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^{jk} & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & & \omega^{(N-1)(N-1)} \end{pmatrix} \qquad \begin{aligned} j &= 0, 1, \ldots, (N-1) \\ k &= 0, 1, \ldots, (N-1) \end{aligned}$$

$\rightarrow \omega = e^{2\pi i/N}$

$\omega^* = e^{-2\pi i/N} = \omega^{-1}$

$$\rightarrow QFT_N^{\dagger} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^* & \omega^{*2} & \cdots & \omega^{*N-1} \\ 1 & \omega^{*2} & \omega^{*jk} & \cdots & \omega^{*2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{*N-1} & \omega^{*2(N-1)} & & \omega^{*(N-1)(N-1)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-jk} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & & \omega^{-(N-1)(N-1)} \end{pmatrix} \frac{1}{\sqrt{N}}$$

$$\rightarrow QFT_N^{\dagger}\, QFT_N = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-jk} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & & \omega^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^{jk} & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & & \omega^{(N-1)(N-1)} \end{pmatrix}$$

$$= \frac{1}{N} \begin{pmatrix} \omega^{-0\cdot 0} & \omega^{-0\cdot 1} & \omega^{-0\cdot 2} & \cdots & \omega^{-0(N-1)} \\ \omega^{-1\cdot 0} & \omega^{-1\cdot 1} & \omega^{-1\cdot 2} & \cdots & \omega^{-1(N-1)} \\ \omega^{-2\cdot 0} & \omega^{-2\cdot 1} & \omega^{-jk} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \\ \omega^{-(N-1)\cdot 0} & \omega^{-(N-1)\cdot 1} & \omega^{-(N-1)\cdot 2} & & \omega^{-(N-1)(N-1)} \end{pmatrix}$$

$$* \begin{pmatrix} \omega^{0\cdot 0} & \omega^{0\cdot 1} & \omega^{0\cdot 2} & \cdots & \omega^{0(N-1)} \\ \omega^{1\cdot 0} & \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1(N-1)} \\ \omega^{2\cdot 0} & \omega^{2\cdot 1} & \omega^{jk} & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \\ \omega^{(N-1)\cdot 0} & \omega^{(N-1)\cdot 1} & \omega^{(N-1)\cdot 2} & & \omega^{(N-1)(N-1)} \end{pmatrix}$$

$$= \frac{1}{N}\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0(N-1)} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1(N-1)} \\ a_{20} & a_{21} & a_{mn} & \cdots & a_{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(N-1)0} & a_{(N-1)1} & a_{(N-1)2} & \cdots & a_{(N-1)(N-1)} \end{pmatrix}$$

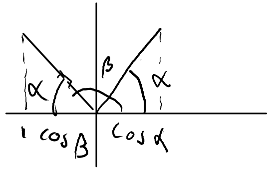
If $m=n$:

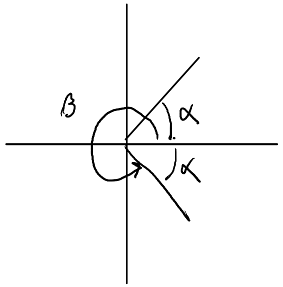
$$a_{mn} = \sum_{p=0}^{N-1} e^{p(n-m)2\pi i/N} = N$$

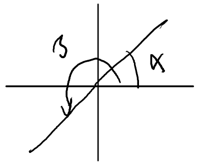
$$\to a_{mn} = \sum_{p=0}^{N-1} \bar{\omega}^{mp}\omega^{pn} = \sum_{p=0}^{N-1} \omega^{p(n-m)} = \sum_{p=0}^{N-1} e^{p(n-m)2\pi i/N}$$

$$= \sum_{p=0}^{N-1}\left(\cos\left[\frac{p(n-m)2\pi}{N}\right] + i\sin\left[\frac{p(n-m)2\pi}{N}\right]\right)$$

$$= \sum_{p=0}^{N-1}\cos\left[\frac{p(n-m)2\pi}{N}\right] + i\sum_{p=0}^{N-1}\sin\left[\frac{p(n-m)2\pi}{N}\right]$$







$\cos\alpha + \cos\beta = 0$

$\rightarrow \alpha + \beta = (\text{odd})\pi$

$\rightarrow N = 2^n \leftarrow \text{even}$

$\sin\alpha + \sin\beta = 0$

$\alpha + \beta = (\text{even})\pi$

$\rightarrow \beta - \alpha = (\text{odd})\pi$

$\rightarrow \sin\alpha + \sin\beta = 0$

$\rightarrow \cos\alpha + \cos\beta = 0$

$$p = \overbrace{0, 1, \dots, \frac{N}{2} - 1}^{\text{1st half}}, \overbrace{\frac{N}{2}, \dots, N-1}^{\text{2nd half}}$$

EX: $N = 2$

$p = 0, 1$

$\downarrow$

$\frac{N}{2} - 1$

$N = 4$

$p = 0, 1, 2, 3$

$\downarrow$

$\frac{N}{2} - 1$

$N = 8$

$p = 0, 1, 2, 3, 4, 5, 6, 7$

$\downarrow$

$\frac{N}{2} - 1$

$$\rightarrow \sum_{p=0}^{N-1} \cos\left[\frac{p(n-m)2\pi}{N}\right] =$$

$$\cos\left[0\,\frac{(n-m)}{N}2\pi\right] + \cos\left[\frac{(n-m)}{N}2\pi\right] + \dots + \cos\left[\left(\frac{N}{2}-2\right)\frac{(n-m)}{N}2\pi\right]$$

$$+ \cos\left[\left(\frac{N}{2}-1\right)\frac{(n-m)}{N}2\pi\right]$$

$$+ \cos\left[\left(\frac{N}{2}\right)\frac{(n-m)}{N}2\pi\right] + \cos\left[\left(\frac{N}{2}+1\right)\frac{(n-m)}{N}2\pi\right] + \dots + \cos\left[(N-2)\frac{(n-m)}{N}2\pi\right]$$

$$+ \cos\left[(N-1)\frac{(n-m)}{N}2\pi\right]$$

If $n - m = \text{odd} \neq 0$

$$\rightarrow \left(\frac{\cancel{N}}{\cancel{2}}\right)\frac{(n-m)}{\cancel{N}}\cancel{2}\pi + 0 = (n-m)\pi$$

$$\rightarrow \frac{(n-m)}{N}2\pi + \left(\frac{N}{2}-1\right)\frac{(n-m)}{N}2\pi = (n-m)\pi$$

$$\rightarrow 2\,\frac{(n-m)}{N}2\pi + \left(\frac{N}{2}-2\right)\frac{(n-m)}{N}2\pi = (n-m)\pi$$

$$\rightarrow \left(\frac{N}{2}+1\right)\frac{(n-m)}{N}2\pi + (N-1)\frac{(n-m)}{N}2\pi = \underbrace{3(n-m)}_{\text{odd}}\pi$$

$$\rightarrow \left(\frac{N}{2}+2\right)\frac{(n-m)}{N}2\pi - (N-2)\frac{(n-m)}{N}2\pi = \underbrace{3(n-m)}_{\text{odd}}\pi$$

$$\rightarrow \sum_{p=0}^{N-1} \cos\left[\frac{p(n-m)2\pi}{N}\right] =$$

$$\cancelto{1}{\cos\left[0\frac{(n-m)}{N}2\pi\right]} + \cancel{\cos\left[\frac{(n-m)}{N}2\pi\right]} + \cdots + \cancel{\cos\left[\left(\frac{N}{2}-2\right)\frac{(n-m)}{N}2\pi\right]}$$

$$+ \cancel{\cos\left[\left(\frac{N}{2}-1\right)\frac{(n-m)}{N}2\pi\right]}$$

$$+ \cancelto{-1}{\cos\left[\left(\frac{N}{2}\right)\frac{(n-m)}{N}2\pi\right]} + \cancel{\cos\left[\left(\frac{N}{2}+1\right)\frac{(n-m)}{N}2\pi\right]} + \cdots + \cancel{\cos\left[(N-2)\frac{(n-m)}{N}2\pi\right]}$$

$$+ \cancel{\cos\left[(N-1)\frac{(n-m)}{N}2\pi\right]}$$

$$\rightarrow \sum_{p=0}^{N-1} \cos\left[\frac{p(n-m)2\pi}{N}\right] = \emptyset$$

$$\rightarrow \sum_{p=0}^{N-1} \sin\left[\frac{p(n-m)2\pi}{N}\right] =$$

$$\sin\left[0\frac{(n-m)}{N}2\pi\right] + \sin\left[\frac{(n-m)}{N}2\pi\right] + \cdots + \sin\left[\left(\frac{N}{2}-2\right)\frac{(n-m)}{N}2\pi\right]$$

$$+ \sin\left[\left(\frac{N}{2}-1\right)\frac{(n-m)}{N}2\pi\right]$$

$$+ \sin\left[\left(\frac{N}{2}\right)\frac{(n-m)}{N}2\pi\right] + \sin\left[\left(\frac{N}{2}+1\right)\frac{(n-m)}{N}2\pi\right] + \cdots + \sin\left[(N-2)\frac{(n-m)}{N}2\pi\right]$$

$$+ \sin\left[(N-1)\frac{(n-m)}{N}2\pi\right]$$

$$\rightarrow \frac{(n-m)}{N}2\pi + (N-1)\frac{(n-m)}{N}2\pi = \underbrace{(n-m)2\pi}_{\text{even}}$$

$$\rightarrow \frac{2(n-m)}{N}2\pi + (N-2)\frac{(n-m)}{N}2\pi = (n-m)2\pi$$

$$\rightarrow \left(\frac{N}{2}-2\right)\frac{(n-m)}{N}2\pi + \left(\frac{N}{2}+2\right)\frac{(n-m)}{N}2\pi = (n-m)2\pi$$

$$\rightarrow \left(\frac{N}{2}-1\right)\frac{(n-m)}{N}2\pi + \left(\frac{N}{2}+1\right)\frac{(n-m)}{N}2\pi = (n-m)2\pi$$

$$\rightarrow \sum_{p=0}^{N-1} \sin\left[\frac{p(n-m)2\pi}{N}\right] =$$

$$\cancelto{0}{\sin\left[0\,\frac{(n-m)}{N}2\pi\right]} + \cancel{\sin\left[\frac{(n-m)}{N}2\pi\right]} + \ldots + \cancel{\sin\left[\left(\frac{N}{2}-2\right)\frac{(n-m)}{N}2\pi\right]}$$

$$+ \cancel{\sin\left[\left(\frac{N}{2}-1\right)\frac{(n-m)}{N}2\pi\right]}$$

$$+ \cancelto{0}{\sin\left[\left(\frac{N}{2}\right)\frac{(n-m)}{N}2\pi\right]} + \cancel{\sin\left[\left(\frac{N}{2}+1\right)\frac{(n-m)}{N}2\pi\right]} + \ldots + \cancel{\sin\left[(N-2)\frac{(n-m)}{N}2\pi\right]}$$

$$+ \cancel{\sin\left[(N-1)\frac{(n-m)}{N}2\pi\right]}$$

$$\rightarrow \sum_{p=0}^{N-1} \sin\left[\frac{p(n-m)2\pi}{N}\right] = \emptyset$$

$$\rightarrow a_{mn} = \sum_{p=0}^{N-1} e^{p(n-m)2\pi i/N} = \emptyset$$

If $n-m = \text{even} \neq 0$

$$\rightarrow \sum_{p=0}^{N-1} \cos\left[\frac{p(n-m)2\pi}{N}\right] =$$

$$\cancelto{1}{\cos\left[0\,\frac{(n-m)}{N}2\pi\right]} + \cos\left[\frac{(n-m)}{N}2\pi\right] + \ldots + \cos\left[\left(\frac{N}{2}-2\right)\frac{(n-m)}{N}2\pi\right]$$

$$+ \cos\left[\left(\frac{N}{2}-1\right)\frac{(n-m)}{N}2\pi\right]$$

$$+ \cancelto{1}{\cos\left[\left(\frac{N}{2}\right)\frac{(n-m)}{N}2\pi\right]} + \cos\left[\left(\frac{N}{2}+1\right)\frac{(n-m)}{N}2\pi\right] + \ldots + \cos\left[(N-2)\frac{(n-m)}{N}2\pi\right]$$

$$+ \cos\left[(N-1)\frac{(n-m)}{N}2\pi\right]$$

$$\vdots$$

$$\rightarrow \sum_{p=0}^{N-1} \cos\left[\frac{p(n-m)2\pi}{N}\right] = \emptyset \qquad \rightarrow \sum_{p=0}^{N-1} \sin\left[\frac{p(n-m)2\pi}{N}\right] = \emptyset$$

$\rightarrow a_{mn} = \sum_{p=0}^{N-1} e^{p(n-m)2\pi i/N} = 0$

$$\rightarrow QFT_N^{\dagger}\, QFT_N = \frac{1}{N}\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0(N-1)} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1(N-1)} \\ a_{20} & a_{21} & a_{mn} & \cdots & a_{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(N-1)0} & a_{(N-1)1} & a_{(N-1)2} & \cdots & a_{(N-1)(N-1)} \end{pmatrix}$$

So!

$$\rightarrow QFT_N^{\dagger}\, QFT_N = \frac{1}{N}\begin{pmatrix} N & 0 & 0 & \cdots & 0 \\ 0 & N & 0 & \cdots & 0 \\ 0 & 0 & N & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \end{pmatrix}$$

$\rightarrow QFT_N^{\dagger}\, QFT_N = I$ ✓ → Unitary

Ex. 13.5:

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|01000\rangle + |10000\rangle + |11000\rangle\right) = \frac{1}{\sqrt{3}}\left(|8\rangle + \underbrace{|16\rangle}_{r=8} + \underbrace{|24\rangle}_{r=8}\right)$$

$\rightarrow n = 5 \rightarrow N = 2^5 = 32$

$j = 0, \ldots, 31$

$\rightarrow |\psi'\rangle = QFT_{32}|\psi\rangle = \sum_{k=0}^{31} \beta_k |k\rangle$ $\quad \beta_0, \ldots, \beta_{31} \leftarrow 31$ calculations

$$\beta_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \alpha_j e^{2\pi i jk/N}$$

$\rightarrow \beta_0 = \frac{1}{\sqrt{32}} \sum_{j=0}^{N-1} \alpha_j \cancel{e^{2\pi i j(0)/32}} \rightarrow 1 = \frac{1}{\sqrt{32}}\left(1\right.$

$\rightarrow \beta_0 = \frac{1}{\sqrt{32}}$

By (13.18):

$$|\psi\rangle = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} |x_0 + kr\rangle = \frac{1}{\sqrt{3}}\left(|8+0\cdot 8\rangle + |8+1\cdot 8\rangle + |8+2\cdot 8\rangle\right)$$

$\rightarrow M = 3 \rightarrow x_0 = 8 \rightarrow r = 8$

$$|\psi\rangle = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} |8 + k8\rangle$$

$\to |\varphi'\rangle = QFT_{32}|\varphi\rangle$

Using (13.20):

$$|\varphi'\rangle = \frac{1}{\sqrt{32\cdot 3}} \sum_{y=0}^{31} \left(e^{2\pi 8 y i/32} \sum_{k=0}^{2} e^{2\pi 8 y k i/32} \right) |y\rangle$$

By (13.21)

$$P_y = \left| \frac{1}{\sqrt{32\cdot 3}} \sum_{k=0}^{2} e^{2\pi 8 k y i/32} \right|^2 = \left| \frac{1}{\sqrt{96}} \sum_{k=0}^{2} e^{\pi k y i/2} \right|^2$$

$$P_y = \left| \frac{1}{\sqrt{96}} \left(e^{\pi 0 y i/2} + e^{\pi 1 y i/2} + e^{\pi 2 y i/2} \right) \right|^2$$

$$P_y = \frac{1}{96} \left| \left(1 + e^{y\pi i/2} + e^{y\pi i} \right) \right|^2$$

$P_0 = \frac{1}{96}|3|^2$ $\quad P_1 = \frac{1}{96}|1+i-1|^2$ $\quad P_2 = \frac{1}{96}|1-1+1|^2$

$P_3 = \frac{1}{96}|1-i-1|^2$ $\quad P_4 = \frac{1}{96}|1+1+1|^2$ $\quad P_5 = \frac{1}{96}|1+i-1|^2$

$\vdots$

$P_0 = \frac{9}{96}$ $\quad P_1 = \frac{1}{96}$ $\quad P_2 = \frac{1}{96}$

$P_3 = \frac{1}{96}$ $\quad P_4 = \frac{9}{96}$ $\quad P_5 = \frac{1}{96}$

$\vdots$

max values

$4 - 0 = \frac{N}{r}$

$r = \frac{32}{4} = 8$ ✓