Parcial 2

25. Deducción teórica de la cuadratura de Laguerre para dos puntos

a. Usando la formula de Rodrigues, encuentre el polinomio de Laguerre de orden 2.

Formula de Rodrigues : $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n)$

Para el Orden 2 : $L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2} (e^{-x} x^2)$

Resolvemos la 2^{da} derivada respecto a x :

$$\frac{d}{dx}(e^{-x} x^2) = e^{-x}(2x - x^2)$$

$$\frac{d}{dx}(e^{-x}(2x - x^2)) = e^{-x}(2x - x^2)(-1) + e^{-x}(2 - 2x)$$

$$= e^{-x}(-2x + x^2 + 2 - 2x)$$

$$= e^{-x}(x^2 - 4x + 2) = \frac{d^2}{dx^2}$$

Reemplazamos:

$$L_2(x) = \frac{\cancel{e^x}}{2} (\cancel{e^{-x}}^{\,1} (x^2 - 4x + 2))$$

$$L_2(x) = \frac{1}{2}(x^2 - 4x + 2) \quad //$$

Dato: $2! = 2$ //

b. Encuentre las raíces del polinomio de orden 2, i.e. (x_0, x_1).

Raíces:

Igualar 0: $(2) \quad 0 = \frac{x^2}{2} - 2x + 1 \quad (2)$

$$0 = x^2 - 4x + 2$$

E. Cuadrática: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2(1)}$

$$x = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2}$$

$$x = \frac{4}{2} \pm \frac{\cancel{2}\sqrt{2}}{\cancel{2}} \quad , \quad x = 2 \pm \sqrt{2} \longrightarrow x_1 = 2+\sqrt{2}$$
$$\rightarrow x_2 = 2-\sqrt{2}$$

C. Encuentre los pesos de cuadratura integrando las bases cardinales con la función de peso de Laguerre $\sigma(x) = e^{-x}$

Para el caso: $x_0 = 2+\sqrt{2}$; $x_1 = 2-\sqrt{2}$

Calculos necesarios: $x_0 - x_1 = \cancel{2}+\sqrt{2} - \cancel{2} + \sqrt{2} = 2\sqrt{2}$

$x_1 - x_0 = \cancel{2} - \sqrt{2} - \cancel{2} - \sqrt{2} = -2\sqrt{2}$

$$W_0 = \int_0^\infty \sigma(x) \left(\frac{x - x_1}{x_0 - x_1} \right) dx$$

$$W_0 = \int_0^\infty e^{-x} \frac{(x - 2+\sqrt{2})}{2\sqrt{2}} dx = \frac{1}{2\sqrt{2}} \int_0^\infty e^{-x} (x - 2+\sqrt{2})\, dx$$

$$W_0 = \frac{1}{2\sqrt{2}} \left[\int_0^\infty e^{-x} x \, dx - \int_0^\infty 2e^{-x} + \int_0^\infty \sqrt{2}\, e^{-x} \, dx \right]$$

→ Calculamos las integrales:

1. $2\int_0^\infty e^{-x} = -(1)2e^{-x}\Big|_0^\infty = \frac{-2}{e^\infty} - \left(-\frac{2}{e^0} \right) = \emptyset + \frac{2}{1} = 2$

2. $\sqrt{2}\int_0^\infty e^{-x} = \sqrt{2}(-1)\, e^{-x}\Big|_0^\infty$

$$= -\sqrt{2}\left(\frac{1}{e^\infty} - \frac{1}{e^0} \right)$$
$$= -\sqrt{2}\,(\emptyset - 1) = \sqrt{2}$$

3. $\int_0^\infty \underbrace{e^{-x}}_{g'} \underbrace{x}_{f} \, dx$ $\quad f(x) = x \quad f'(x) = 1$

$\quad g'(x) = e^{-x} \quad g(x) = -e^{-x}$

$$\int_0^\infty e^{-x} x \, dx = -xe^{-x} - \int -e^{-x} \, dx$$
$$= -xe^{-x} - (-1)(-e^{-x})$$
$$= -xe^{-x} - e^{-x}$$
$$= -e^{-x}(x+1)\Big|_0^\infty$$
$$= -\frac{1}{e^x}(x+1)\Big|_0^\infty = -\frac{1}{e^\infty}(\infty+1) - \left(-\frac{1}{e^0}(0+1) \right)$$

$= -0(\infty) + 1(1) = 1$

$W_0 = \frac{1-x_1}{-2\sqrt{2}} = \frac{1+\sqrt{2}}{2\sqrt{2}}$

Con:

$W_1 = {}_0\int^{\infty} \sigma(x) \left(\frac{x - x_0}{x_1 - x_0}\right) dx = {}_0\int^{\infty} e^{-x} \left(\frac{x - 2 - \sqrt{2}}{x - \sqrt{2} - x - \sqrt{2}}\right) dx$

$W_1 = {}_0\int^{\infty} e^{-x} \left(\frac{x - 2 - \sqrt{2}}{-2\sqrt{2}}\right) dx = -\frac{1}{2\sqrt{2}} {}_0\int^{\infty} e^{-x} (x - 2 - \sqrt{2}) \, dx$

$W_1 = -\frac{1}{2\sqrt{2}} {}_0\int^{\infty} e^{-x} x - {}_0\int^{\infty} 2e^{-x} - {}_0\int^{\infty} \sqrt{2}\, e^{-x} \, dx$

Calculamos las integrales:

1. ${}_0\int^{\infty} e^{-x} x \, dx$ $\quad f(x) = x \quad f'(x) = 1$
$\quad g'(x) = e^{-x} \quad g(x) = -e^{-x}$

${}_0\int^{\infty} e^{-x} x = -xe^{-x} - \int -e^{-x} dx$

$= -xe^{-x} - (-1)(-e^{-x})$

$= -xe^{-x} - e^{-x} = -e^{-x}(x+1) \Big|_0^{\infty}$

$= -\frac{1}{e^x}(x+1) = -\frac{1}{\infty}(\infty) - \left(-\frac{1}{e^0}(1)\right)$

$= -0(\infty) + \frac{1}{1}(1) = 1$

2. ${}_0\int^{\infty} 2e^{-x} dx = 2\,{}_0\int^{\infty} e^{-x} dx = -2e^{-x} \Big|_0^{\infty} = \cancel{-\frac{2}{e^{\infty}}}^{\,\emptyset} - \left(-\frac{2}{e^0}\right)$

$= \frac{2}{1} = 2$

3. ${}_0\int^{\infty} \sqrt{2}\, e^{-x} dx = \sqrt{2}\,{}_0\int^{\infty} e^{-x} dx = -\sqrt{2}\, e^{-x} \Big|_0^{\infty} = \cancel{-\frac{\sqrt{2}}{e^{\infty}}}^{\,\emptyset} - \left(-\frac{\sqrt{2}}{e^0}\right)$

$= \frac{\sqrt{2}}{1} = \sqrt{2}$

$$W_1 = -\frac{1}{2\sqrt{2}}\left[1 - 2 - \sqrt{2}\right] = -\frac{1}{2\sqrt{2}} + \frac{\cancel{2}}{2\sqrt{2}} + \frac{\cancel{\sqrt{2}}}{2\sqrt{2}} = -\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2}$$

$$W_1 = \frac{1 - x_0}{2\sqrt{2}} = \frac{-1+\sqrt{2}}{2\sqrt{2}} \;//$$

d. Muestre que la regla es exacta para un polinomio de grado tres

$$\int_0^{\infty} e^{-x} x^3 \, dx = \sum_{i=0}^{1} w_i f(x_i) = 6$$

$$x_0^3 = (2-\sqrt{2})^3 = 20 - 14\sqrt{2}$$

$$x_1^3 = (2+\sqrt{2})^3 = 20 + 14\sqrt{2}$$

Suma:

$$W_0 x_0^3 + W_1 x_1^3 = \underbrace{\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)(20-14\sqrt{2})}_{1.} + \overbrace{\left(\frac{-1+\sqrt{2}}{2\sqrt{2}}\right)(20+14\sqrt{2})}^{2.}$$

1. $\dfrac{(1+\sqrt{2})(20-14\sqrt{2})}{2\sqrt{2}} = \dfrac{-8+6\sqrt{2}}{2\sqrt{2}}$

2. $\dfrac{(-1+\sqrt{2})(20+14\sqrt{2})}{2\sqrt{2}} = \dfrac{8+6\sqrt{2}}{2\sqrt{2}}$

Quedaría: $-\dfrac{8+6\sqrt{2}}{2\sqrt{2}} + \dfrac{8+6\sqrt{2}}{2\sqrt{2}} = \dfrac{(-8+6\sqrt{2}) + (8+6\sqrt{2})}{2\sqrt{2}} = \dfrac{12\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}} = 6$