ISE 4623/5023: Deterministic Systems Models / Systems Optimization University of Oklahoma School of Industrial and Systems Engineering Fall 2024

Individual Assignment 4: Duality and Sensitivity Analysis

1. (35 points) Sooners Inc., a leading company in the house decorating and maintenance industry, is expanding its product line by launching a new branch dedicated to paint production. The company has decided to begin by manufacturing two paint colors: red and blue. Each gallon of red paint will be sold for \$10, while each gallon of blue paint will be sold for \$12. The production process for these paints requires specific resources. Each gallon of red paint requires 1 kilogram of pigments and 2 gallons of water. Similarly, each gallon of blue paint requires 2 kilograms of pigments and 2 gallons of water. However, the company faces resource constraints, having only 20 gallons of water and 25 kilograms of pigments available. Additionally, due to a partnership agreement with a local store, Sooners Inc. has committed to producing at least 2 gallons of each paint color.

   Formulation

   Decision Variables

   - x_R: Gallons of red paint produced
   - x_B: Gallons of blue paint produced

   $$\text{Maximize} \quad 10x_R + 12x_B$$

   st:

   $$x_R + 2x_B \leq 25$$
   $$2x_R + 2x_B \leq 20$$
   $$x_R \geq 2$$
   $$x_B \geq 2$$

   $$x_R \geq 0, \quad x_B \geq 0$$

   (a) (5 points) Construct the dual problem of the associated LP model (use the LP formulation given before):

   (b) (3 points) Formulate the dual problem corresponding to the primal problem. However, to plot and solve graphically, focus only on the first two dual variables, which correspond to the first two primal constraints (representing the maximum resource availability). Plot the gradient and the feasible region for this restricted dual problem, clearly indicating all relevant constraints and shading the feasible region. Solve this dual problem graphically, and indicate the values of the variables and the objective function associated with the optimal solution.

   (c) (4 points) Solve the dual problem using Gurobi/Python. Include a snapshot of your Gurobi/Python code and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How are the dual variables and the dual objective function connected to the primal problem?

   (d) (5 points) Construct the dual problem of the standard form of the LP model

   Formulation Decision Variables

   - x_R: Gallons of red paint produced
   - x_B: Gallons of blue paint produced
   - s_1: Slack constraint 1
   - s_2: Slack constraint 2
   - s_2: Slack constraint 3

- s_2: Slack constraint 4

$$\text{Maximize} \quad 10x_R + 12x_B$$

st:

$$x_R + 2x_B + s_1 = 25$$
$$2x_R + 2x_B + s_2 = 20$$
$$x_R - s_3 = 2$$
$$x_B - s_4 = 2$$
$$x_R, x_B, s_1, s_2, s_3, s_4 \geq 0$$

(e) (4 points) Solve this dual problem using Gurobi/Python. Include a snapshot of your Gurobi/Python code and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How do these compare to the results obtained in part b? Explain in detail.

(f) (7 points) Sooners Inc. is exploring various options to enhance the profitability of their new paint branch. They have identified potential new suppliers that could lower their production costs. One supplier offers an additional supply of water at a cost of \$3 per gallon, while another supplier offers additional pigments at a cost of \$4 per kilogram. However, they can only choose one supplier. Your task is to analyze the situation and provide a recommendation. How much is Sooners Inc. willing to pay per unit of water and per unit of pigments? Which supplier should they choose to maximize their profit?

(g) (7 points) Sooners Inc. has decided to expand their product line by adding a new paint color: Black. The price for the black paint is \$9 per gallon, and it requires $\frac{1}{5}$ kilogram of pigments and $\frac{1}{5}$ gallon of water per gallon produced. After contracting new suppliers, the resource limits have increased to 45 gallons of water and 40 kilograms of pigments. Formulate this new linear optimization problem, incorporating the new black paint. Using the algebraic sensitivity analysis (Simplex tableau) discussed in class, determine whether the previous optimal basis —before the inclusion of the new paint color— will remain optimal after adding the new paint. Provide a comprehensive explanation of your conclusion, detailing the factors that influence the optimality status and the potential effects of incorporating the new color.

2. (35 points) As a financial advisor, your role is to help clients achieve their savings goals while minimizing the cost of their total investment. Today, a couple has approached you to create an investment portfolio consisting of stocks from Company A and Company B. Their goal is to make at least \$20 in profit by the end of the year, in addition to recovering their initial investment.You know that for each dollar invested in Stock A, they will receive \$1.50 at the end of the year, while each dollar invested in Stock B will yield \$1.20. Due to regulatory constraints, they are allowed to invest at most 1.5 times more in Stock A than in Stock B. Additionally, the couple prefers Stock B for its lower risk, despite its lower return, and has requested that at least \$5 be invested in Stock B.The total budget available for their investment is \$100.

Formulation

Decision Variables

- x_A: Dollars invested in stock A.
- x_B: Dollars invested in stock B.

$$\text{Minimize} \quad x_A + x_B$$

st:

$$0.5x_A + 0.2x_B \geq 20$$
$$x_A \leq 1.5x_B$$
$$x_B \geq 5$$
$$x_A + x_B \leq 100$$

$$x_A \geq 0, \quad x_B \geq 0$$

(a) (6 points) Construct the dual problem of the associated LP model (use the formulation given above).

(b) (4 points) Solve the dual problem using Excel/Solver. Include a snapshot of your Excel/Solver and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How are the dual variables and the dual objective function connected to the primal problem?

(c) (6 points) Construct the dual problem of the standard form of the LP model (i.e., use the LP formulation given in the solution of Problem 2(a) of Individual Assignment 3 as the primal).

Primal: **Formulation** Decision Variables

- x_A: Dollars invested in stock A.
- x_B: Dollars invested in stock B.
- s_1: Slack constraint 1
- s_2: Slack constraint 2
- s_2: Slack constraint 3
- s_2: Slack constraint 4

$$\text{Minimize} \quad x_A + x_B$$

st:

$$0.5x_A + 0.2x_B - s_1 = 20$$
$$x_A - 1.5x_B + s_2 = 0$$
$$x_B - s_3 = 5$$
$$x_A + x_B + s_4 = 100$$
$$x_A, x_B, s_1, s_2, s_3, s_4 \geq 0$$

(d) (3 points) Solve this dual problem using Excel/Solver. Include a snapshot of your Excel/Solver and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How do these compare to the results obtained in part b? Explain in detail.

(e) (16 points) After determining the optimal solution, determine the price range(coefficient in the objective function), separately for stock A and B, that would maintain the same optimal solution. Add the total investment range as well.

3. (30 points) For the following problem, provide the mathematical programming formulation that would find the optimal solution to it. Clearly, indicate all **set(s)** (6 points), **parameter(s)** (6 points), **variable(s)** (6 points), **objective function** (6 points), and **constraint(s)** (6 points).

You are a door-to-door vendor of cookies, and you have a list of clients denoted by the set C. Each client i has a probability α_{it} of purchasing your product, which depends on both the time t and the specific client. The quantity that each client i will buy is denoted by β_i, which is client-dependent but not time-dependent. Your objective is to maximize the expected value of sales during your shift.

However, there are several constraints to consider. Your workday lasts 8 hours, and you can only visit two clients during each hour. Additionally, you can only visit each client once. The city is divided into two zones, Zone A and Zone B. If you choose to visit a client in Zone A, you cannot visit a client in Zone B during the same hour.Furthermore, there is a subset of clients $M \subseteq C$ who can only be visited during the afternoon, after $t > 4$). You must ensure that these clients are visited at the appropriate times. Lastly, to cover transportation costs, the expected sales value from the clients visited during each hour must exceed a minimum threshold γ_t.