DETERMINISTIC SYSTEMS MODELS/SYSTEMS OPTIMIZATION
ISE 4623/5023
EXAM 2
Fall 2022

Last name: ____

First name: ____

Student ID: ____

Section (mark with X): _______ ISE 4623
_______ ISE 5023

Pledge: *"On my honor, I have neither given nor received inappropriate assistance in the completion of this Exam."*

Student signature: ____

Problem 1 (45 points)

Seeds Inc. is a company that processes and sells bags of corn seeds. For this purpose, the company has a production plant in the state of Oklahoma, in which the company processes and sells corn of two varieties: Hard and Serrated. For each ton of seeds they process and sell, the net profit is *four* thousand dollars for hard seeds, and *three* thousand dollars for serrated seeds.

To process the seeds the plant uses large amounts of water: *one* thousand liters per ton of seeds processed (independently of the type of seed). Due to state regulations, the plant cannot use more than *five* thousand liters of water per month. The following table provides a summary of these data:

Resource	Hard seeds (units of resource per ton of seeds)	Serrated seeds (units of resource per ton of seeds)	Maximum availability (units of resource)
Water (thousands of liters)	1	1	5

Profit per seed ton of seeds (thousands of dollars)	4	3

Additionally, demand for serrated seeds is always very high, so they always sell all the serrated seeds processed. However, hard seeds have a monthly demand of *two* tons, so you want to guarantee that you do not process more than that amount.

To determine the processing plan that maximizes Seeds Inc.'s profit, you constructed and solved the LP model below, where x_1 and x_2 indicate the number of tons of hard and serrated seeds, respectively, to process each month.

Maximize $z = 4x_1 + 3x_2$ (obj. function: maximize profit, in thousands of dollars)

s.t.

$x_1 + x_2 \leq 5$ (constraint 1: Water utilization/availability)

$x_1 \leq 2$ (constraint 2: Maximum demand of hard seeds)

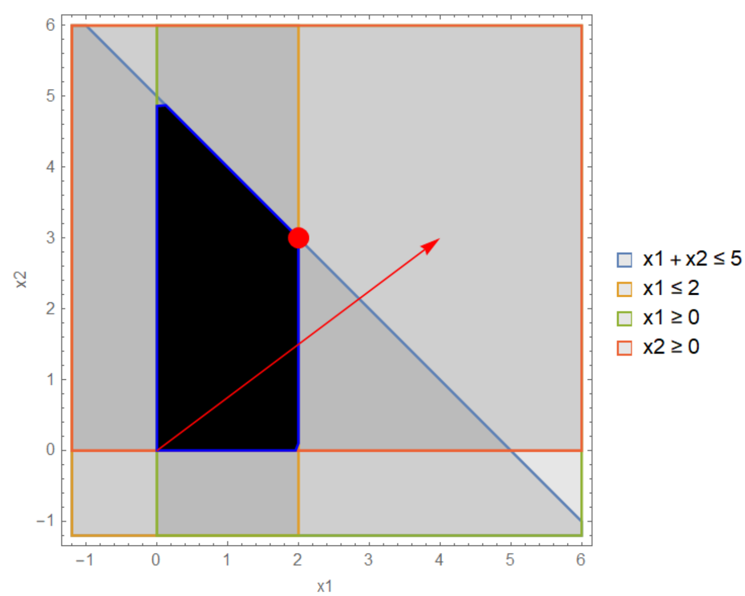
$x_1, x_2 \geq 0$

Optimal solution	
Basic variables	$\{x_1, x_2\}$
Non-basic variables	$\{s_1, s_2\}$
B	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
c_B^T	$(4 \quad 3)$
B^{-1}	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$
$B^{-1}b$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$c_B^T B^{-1}$	$(3 \quad 1)$
$c_B^T B^{-1} b$	17

The gradient (of the given objective function) is: {4, 3}

The optimal value of the objective function is: 17

The optimum solution is: {x1 → 2, x2 → 3}



Simplex Tableau for the optimal solution (Primal problem)

basic	z	x1	x2	s1	s2	sol
z	1	0	0	3	1	17
x1	0	1	0	0	1	2
x2	0	0	1	1	-1	3

Simplex Tableau for the optimal solution (Dual problem)

basic	z	y1	y2	sd1	sd2	sol
z	1	0	0	-2	-3	17
y1	0	1	0	0	-1	3
y2	0	0	1	-1	1	1

In other words, based on your LP model, you determined that the optimal monthly processing plan is *two* tons of hard seeds and *three* tons of serrated seeds, for a monthly profit of *17* thousand dollars.

a. (5 points) What are the dual variables associated with the optimal solution? What is their meaning in the context of this problem? Explain in detail.

Dual variable	**Value**
y_1	
y_2	

b. (8 points) For what range of monthly demand of hard seeds would processing both types of seeds remain optimal? Show the steps taken to determine this range. <u>Hint</u>: since the problem has only two variables, you can easily determine this range (the maximum and minimum values) from the plot by seeing how much the associated constraint could move before the optimal basis changes.

Range limits (to maintain optimality of current basis)	**Value**
Max (of monthly demand of serrated seeds)	
Min (of monthly demand of serrated seeds)	

c. (8 points) For what range of water availability would processing both types of seeds remain optimal? Show the steps taken to determine this range. Hint: same as in part b.

Range limits (to maintain optimality of current basis)	**Value**
Max (of monthly water availability)	
Min (of monthly water availability)	

d. (8 points) For what range of profit per ton of hard seeds would processing both types of seeds remain optimal? Show the steps taken to determine this range. Hint: since the problem has only two variables, you can easily determine this range (the maximum and minimum values) from the plot by seeing how much the gradient could change until the optimal solution would change.

Range limits (to maintain optimality of current solution)	**Value**
Max (of profit per ton of hard seeds)	
Min (of profit per ton of hard seeds)	

e. (8 points) For what range of profit per ton of serrated seeds would processing both types of seeds remain optimal? Show the steps taken to determine this range. Hint: same as in part d.

Range limits (to maintain optimality of current solution)	**Value**
Max (of profit per ton of serrated seeds)	
Min (of profit per ton of serrated seeds)	

f. (8 points) Seeds Inc. is considering processing a third type of corn: Soft. For each ton of Soft corn processed, the company expects a profit of $2.5 thousands. Processing a ton of soft corn would also require one thousand liters of water, and its demand is expected to be very large (so you can assume that all soft corn processed will be sold). Determine if the previous optimal solution ($x_1 = 2; x_2 = 3$) would remain optimal. Explain your answer in detail. Hint: calculate the reduced cost of the new variable.

Problem 2 (35 points + 5 extra credit points)

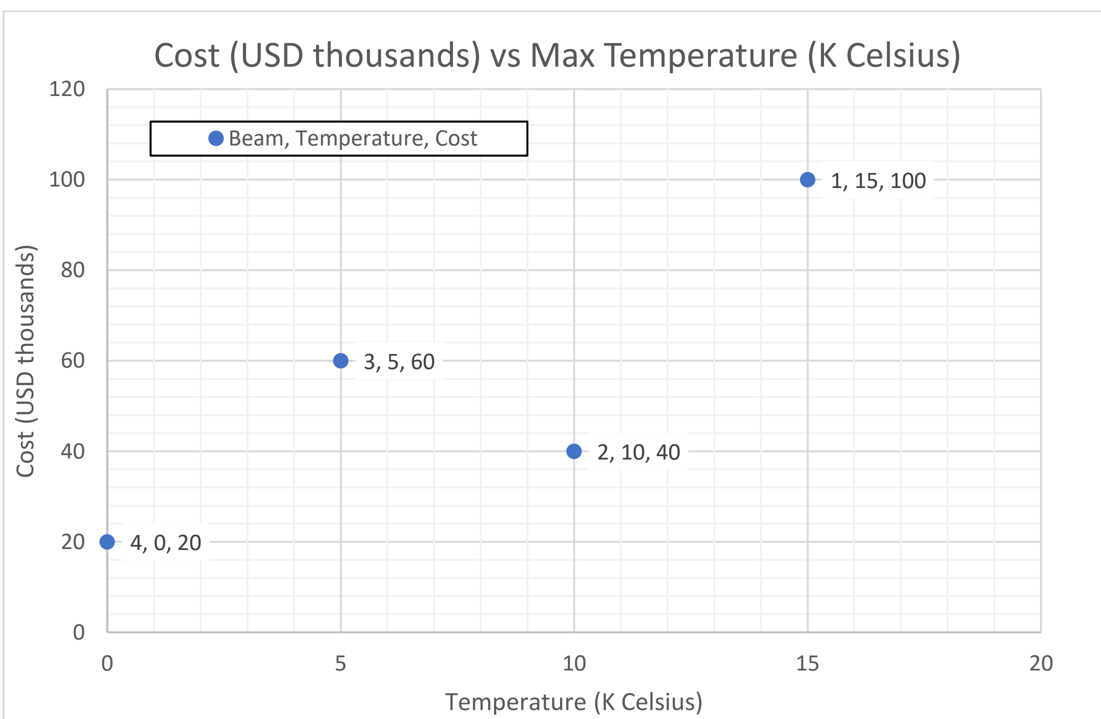
Suppose that you are designing a construction beam that will be subjected to high temperatures, but to produce them you incur in large costs. The information below is related to the production costs (in USD thousands) and the maximum temperatures (in Kilo Celsius) that four different commercial types of beams can withstand (of which you are currently producing type 3).

Beam type	Temperature (K Celsius)	Cost (USD thousands)
1	15	100
2	10	40
3	5	60
4	0	20

You would like to design a beam with the minimum cost possible, while making sure that it can withstand temperatures of at least 5 thousand Celsius degrees. To estimate how much you could improve the capacity of your beam, you decide to use Data Envelopment Analysis (DEA). The associated DEA efficiency model is detailed below, with the input being temperature and the output being the beam capacity:

$$
\begin{aligned}
&\text{Minimize } \phi \\
&\text{s.t.} \\
&15\lambda_1 + 10\lambda_2 + 5\lambda_3 + 0\,\lambda_4 \geq 5 \\
&100\lambda_1 + 40\lambda_2 + 60\lambda_3 + 20\lambda_4 - 60\phi \leq 0 \\
&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
&\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \\
&\phi \text{ is free}
\end{aligned}
$$

a. (10 points) Plot the efficiency frontier, indicating the beam types that would be in it.



b. (10 points) Indicate the reference set for your beam (type 3). What would be the optimal values of $\lambda_1, \lambda_{2,}\lambda_3$, and λ_4? What do these indicate?

c. (5 points) What would be the optimal value of ϕ? What does it indicate? Is beam type 3 efficient?

d. (10 points) Construct the dual problem associated with the described DEA efficiency model.

e. (5 points – extra credit) Write the problem in standard form (so that it contains **only non-negative constants in the right-hand side, non-negative variables, and equality constraints).** Indicate the variables, objective function, the associated constraints, and the nature of the variables (if they are non-negative, non-positive, etc.).

Problem 3 (20 points)
Suppose you have the following LP model, associated with a minimum cost flow problem

- Sets:

N: Set of nodes $\{1,2,3,\dots,n\}$
A: Set of arcs

- Parameters:

c_{ij}: unit cost of sending a commodity through arc $(i,j) \in A$
b_i: demand/supply of commodities in node $i \in N$
u_{ij}: maximum flow through arc $(i,j) \in A$
l_{ij}: minimum flow through arc $(i,j) \in A$

- Variables:

x_{ij}: flow through arc $(i,j) \in A$

- Objective function:

$$\min z = \sum_{(i,j)\in A} c_{ij} x_{ij} \qquad \text{(o1)}$$

- Constraints:

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i, \qquad \forall i \in N \qquad \text{(c1)}$$

$$x_{ij} \le u_{ij}, \qquad \forall (i,j) \in A \qquad \text{(c2)}$$

$$x_{ij} \ge l_{ij}, \qquad \forall (i,j) \in A \qquad \text{(c3)}$$

You were given a particular instance of this problem to be solved in Gurobi/Python. The code to initialize the sets and parameters has been given to you (below).

```
#Set N, parameter b
N, b= multidict({
    ('node1'): 5,
    ('node2'): 3,
    ('node3'):5,
    ('node4'):0,
    ('node5'):-5,
    ('node6'):-8
})

#Set A, Parameters l, u, c

A, l, u, c = multidict({
    ('node1','node2'): [2,5,1],
    ('node1','node3'): [2,8,4],
    ('node3','node2'): [2,8,2],
    ('node2','node4'): [0,7,3],
    ('node3','node5'): [1,8,1],
    ('node4','node5'): [1,9,5],
    ('node4','node6'): [1,6,2],
    ('node5','node6'): [0,6,3]
})
```

Now you want to write and solve the associated LP model. In particular, you must:

a. (2 points) Write the code in Python/Gurobi to import “Gurobipy”

b. (2 points) Write the code in Python/Gurobi to define/create a Gurobi model named “Exam2_Problem3”

c. (2 points) Write the code in Python/Gurobi to add the decision variables to the Gurobi model. Do not forget to indicate their respective coefficients (for the objective function) when defining the variables.

d. (2 points) Write the code in Python/Gurobi to set the “sense” of the Gurobi model to “Minimize”

e. (10 points) Write the code in Python/Gurobi to add the constraints (c1, c2, and c3) to the Gurobi model

f. (2 points) Write the code in Python/Gurobi to solve (optimize) the Gurobi model

Problem 4 (30 points – Extra credit)
Suppose you have the following LP model

$$\begin{aligned}
&\text{Maximize } z = -2x_1 - x_2 \\
&\text{s.t.} \\
&\qquad 2x_1 + x_2 \leq 4 \\
&\qquad x_1 + 4x_2 \geq 4 \\
&\qquad x_1, x_2 \geq 0
\end{aligned}$$

You would like to solve it using Simplex Tableau. Since you need to standardize the LP model before using Simplex to solve it, you start by constructing its associated standard form, from now on called **P4_SF** (below):

$$\begin{aligned}
&\text{Maximize } z = -2x_1 - x_2 \\
&\text{s.t.} \\
&\qquad 2x_1 + x_2 + s_1 = 4 \\
&\qquad x_1 + 4x_2 - s_2 = 4 \\
&\qquad x_1, x_2, s_1, s_2 \geq 0
\end{aligned}$$

a. (5 points extra credit) If you want to use the big M method to initialize and solve the problem P4_SF, what would be the associated big M optimization problem to be solved?

b. (5 points extra credit) If you want to initialize and solve the problem P4_SF with the two-phase method, what would be the associated phase-1 optimization problem to be solved?

c. (10 points extra credit) Using Simplex Tableau solve the associated phase-1 optimization problem and complete the table below with the optimal basic solution for the phase-1 problem. Don't forget to write all the steps/iterations made, including the row operations used.

Basic	z	x_1	x_2				Solution
z							

Optimal value of objective function of phase-1 problem

Variable	Value	Basic or non-basic?
x_1		
x_2		

d. (10 points extra credit) Initialize the phase-2 problem (using the solution from part c) and solve it to optimality using Simplex Tableau. Complete the table below with the associated basic solution. Don't forget to write all the steps/iterations made, including the row operations used. Is this solution feasible for P4_SF? Is this solution optimal for P4_SF? Explain your answers in detail.

Basic	z	x_1	x_2			Solution
z						

Optimal value of objective function of phase-2 problem

Variable	Value	Basic or non-basic?
x_1		
x_2		

Is this basic solution feasible (with respect to the problem in part a)? Yes / No
Explanation:

Is this basic solution optimal (with respect to the problem in part a)? Yes / No
Explanation: