Homework 4

1. (35 points) Sooners Inc., a leading company in the house decorating and maintenance industry, is expanding its product line by launching a new branch dedicated to paint production. The company has decided to begin by manufacturing two paint colors: red and blue. Each gallon of red paint will be sold for \$10, while each gallon of blue paint will be sold for \$12. The production process for these paints requires specific resources. Each gallon of red paint requires 1 kilogram of pigments and 2 gallons of water. Similarly, each gallon of blue paint requires 2 kilograms of pigments and 2 gallons of water. However, the company faces resource constraints, having only 20 gallons of water and 25 kilograms of pigments available. Additionally, due to a partnership agreement with a local store, Sooners Inc. has committed to producing at least 2 gallons of each paint color.

Formulation

Decision Variables

- x_R: Gallons of red paint produced
- x_B: Gallons of blue paint produced

$$\text{Maximize} \quad 10x_R + 12x_B$$

st:

$$x_R + 2x_B \leq 25$$
$$2x_R + 2x_B \leq 20$$
$$x_R \geq 2$$
$$x_B \geq 2$$

$$x_R \geq 0, \quad x_B \geq 0$$

(a) (5 points) Construct the dual problem of the associated LP model (use the LP formulation given before):

$$\text{Min} \quad 25y_1 + 20y_2 + 2y_3 + 2y_4$$

$y_1 \quad y_2 \quad y_3 \quad y_4$

$$1y_1 + 2y_2 + 1y_3 + 0y_4 \geq 10$$
$$2y_1 + 2y_2 + 0y_3 + 1y_4 \geq 12$$

$$y_1, y_2, y_3, y_4 \geq 0$$

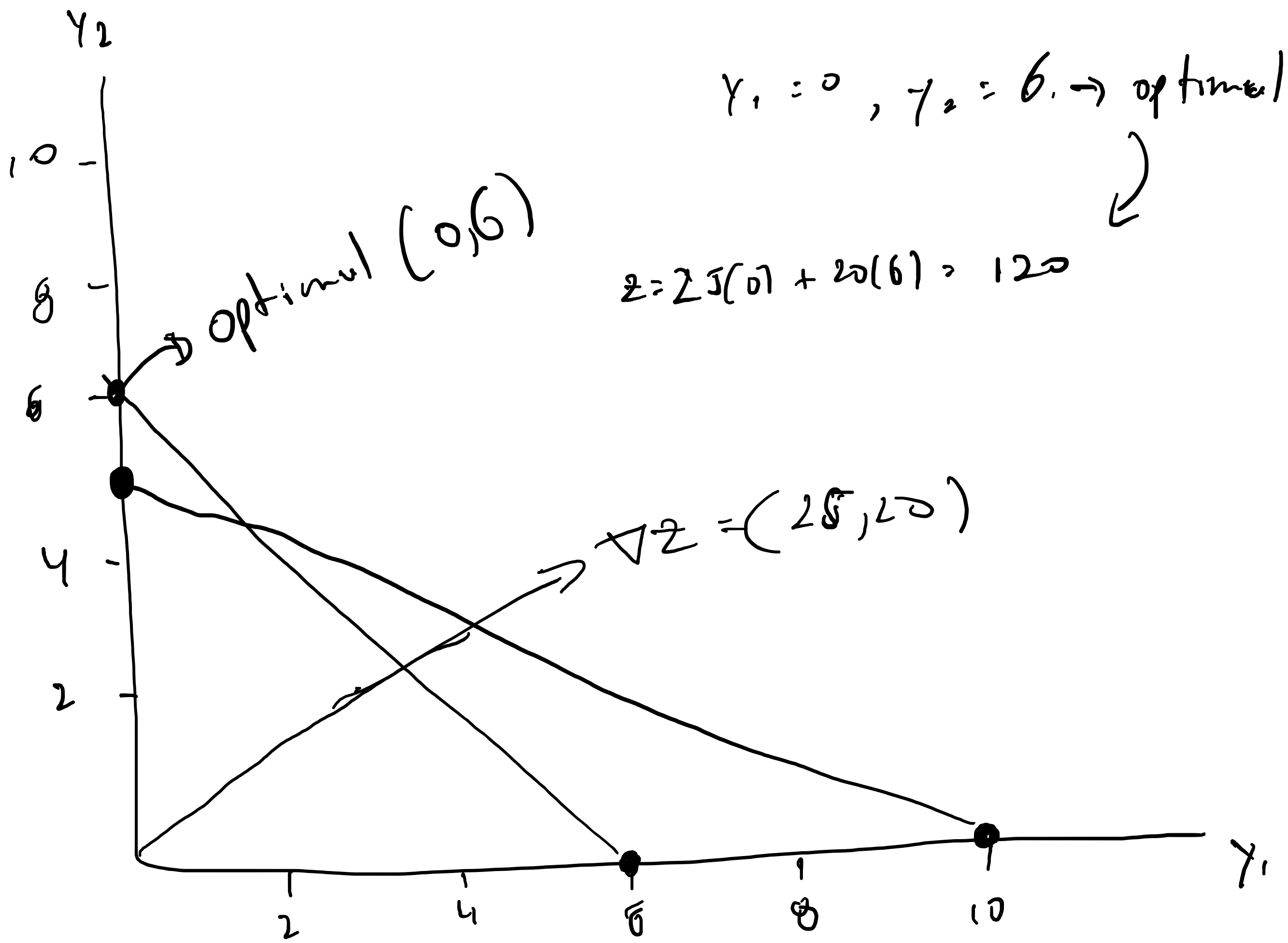
(b) (3 points) Formulate the dual problem corresponding to the primal problem. However, to plot and solve graphically, focus only on the first two dual variables, which correspond to the first two primal constraints (representing the maximum resource availability). Plot the gradient and the feasible region for this restricted dual problem, clearly indicating all relevant constraints and shading the feasible region. Solve this dual problem graphically, and indicate the values of the variables and the objective function associated with the optimal solution.

$$\text{Min} \quad 25y_1 + 20y_2$$

$$y_1 + 2y_2 \geq 10$$

$$2y_1 + 2y_2 \geq 12$$

$$y_1, y_2 \geq 0$$



(c) (4 points) Solve the dual problem using Gurobi/Python. Include a snapshot of your Gurobi/Python code and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How are the dual variables and the dual objective function connected to the primal problem?

```
import gurobipy as gp
from gurobipy import *
model = gp.Model("Sooner Inc Paints DUAL")
x1 = model.addVar(vtype= GRB.CONTINUOUS, lb= 0, ub= GRB.INFINITY, name="x1")
x2 = model.addVar(vtype= GRB.CONTINUOUS, lb= 0, ub= GRB.INFINITY, name="x2")

#Define the objective function
z = 25*x1 + 20*x2
model.setObjective(z)
model.modelSense = GRB.MINIMIZE
model.update()

#Add Constraints
model.addConstr(1*x1 + 2*x2 >= 10)
model.addConstr(x1 + 2*x2 >= 12)
model.update()

model.optimize()

#Print output

if model.status == GRB.OPTIMAL:
    print("Optimal value:", model.objVal, "USD")
    print("DUAL for red paint: ", x1.X)
    print("DUAL for blue paint: ", x2.X)
```

```
Optimize a model with 2 rows, 2 columns and 4 nonzeros
Model fingerprint: 0x4225b536
Coefficient statistics:
  Matrix range     [1e+00, 2e+00]
  Objective range  [2e+01, 2e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [1e+01, 1e+01]
Presolve removed 2 rows and 2 columns
Presolve time: 0.00s
Presolve: All rows and columns removed
Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    1.2000000e+02   0.000000e+00   0.000000e+00      0s

Solved in 0 iterations and 0.01 seconds (0.00 work units)
Optimal objective  1.200000000e+02
Optimal value: 120.0 USD
DUAL for red paint:  0.0
DUAL for blue paint:  6.0
```

Dual Objective: The dual objective function value (120 USD) provides insight into the total cost in the dual world, which relates to the pricing of constraints from the primal problem.

Dual Variables: These represent the marginal values of increasing the resources in the primal. If a dual variable is zero, it means the associated constraint is not binding in the primal problem (additional resources would not improve the objective value).

(d) (5 points) Construct the dual problem of the standard form of the LP model
Formulation Decision Variables

$$\text{Maximize} \quad 10x_R + 12x_B$$

$$\begin{aligned} x_R + 2x_B + s_1 &= 25 \\ 2x_R + 2x_B + s_2 &= 20 \\ x_R - s_3 &= 2 \\ x_B - s_4 &= 2 \\ x_R, x_B, s_1, s_2, s_3, s_4 &\geq 0 \end{aligned}$$

$$\text{Min} \quad 25y_1 + 20y_2 + 2y_3 + 2y_4$$

$$y_1 + 2y_2 + y_3 \geq 10$$

$$2y_1 + 2y_2 + y_4 \geq 12$$

$$y_1, y_2, y_3, y_4 \geq 0$$

(e) (4 points) Solve this dual problem using Gurobi/Python. Include a snapshot of your Gurobi/Python code and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How do these compare to the results obtained in part b? Explain in detail.

[36…

```python
import gurobipy as gp
from gurobipy import *

# Create a new model
model = gp.Model("Sooner Inc Paints")

# Define decision variables
x1 = model.addVar(vtype=GRB.CONTINUOUS, lb=-GRB.INFINITY, ub=GRB.INFINITY, name="x1")
x2 = model.addVar(vtype=GRB.CONTINUOUS, lb=-GRB.INFINITY, ub=GRB.INFINITY, name="x2")
x3 = model.addVar(vtype=GRB.CONTINUOUS, lb=-GRB.INFINITY, ub=GRB.INFINITY, name="x3")
x4 = model.addVar(vtype=GRB.CONTINUOUS, lb=-GRB.INFINITY, ub=GRB.INFINITY, name="x4")

# Define the objective function
z = 25*x1 + 20*x2 + 2*x3 + 2*x4
model.setObjective(z)
model.modelSense = GRB.MINIMIZE
model.update()

# Add Constraints
model.addConstr(1*x1 + 2*x2 + 1*x3 >= 10)
model.addConstr(2*x1 + 2*x2 + 1*x4 >= 12)
model.addConstr(x1 >= 0)
model.addConstr(x2 >= 0)
model.addConstr(x3 >= 0)
model.addConstr(x4 <= 0)
model.update()

# Optimize the model
model.optimize()

# Print output
if model.status == GRB.OPTIMAL:
    print("Optimal value:", model.objVal, "USD")
    print("pigments: ", x1.X)
    print("water: ", x2.X)
    print("y3: ", x3.X)
    print("y4: ", x4.X)
```

```
Gurobi Optimizer version 11.0.3 build v11.0.3rc0 (mac64[arm] - Darwin 23.6.0 23G93)

CPU model: Apple M2
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 6 rows, 4 columns and 10 nonzeros
Model fingerprint: 0x7ec0a01c
Coefficient statistics:
  Matrix range     [1e+00, 2e+00]
  Objective range  [2e+00, 2e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [1e+01, 1e+01]
Presolve removed 6 rows and 4 columns
Presolve time: 0.00s
Presolve: All rows and columns removed
Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    1.2000000e+02   0.000000e+00   0.000000e+00      0s

Solved in 0 iterations and 0.00 seconds (0.00 work units)
Optimal objective  1.200000000e+02
Optimal value: 120.0 USD
pigments:  0.0
water:  6.0
y3:  0.0
y4:  0.0
```

They are the same results found in part b.

(f) (7 points) Sooners Inc. is exploring various options to enhance the profitability of their new paint branch. They have identified potential new suppliers that could lower their production costs. One supplier offers an additional supply of water at a cost of \$3 per gallon, while another supplier offers additional pigments at a cost of \$4 per kilogram. However, they can only choose one supplier. Your task is to analyze the situation and provide a recommendation. How much is Sooners Inc. willing to pay per unit of water and per unit of pigments? Which supplier should they choose to maximize their profit?

Water: The shadow price for water is \$6, which is greater than the supplier's price of \$3 per gallon. This means that purchasing additional water would increase profitability by \$3 per gallon (\$6 value - \$3 cost).

Pigments: The shadow price for pigments is \$0, which is less than the supplier's price of \$4 per kilogram. This means that purchasing additional pigments would not increase profitability.

Recommendation:

Sooners Inc. should purchase additional water from the supplier, as it will increase profitability. They should not purchase additional pigments, as there is no benefit in doing so (the shadow price is zero, indicating that the pigment constraint is not binding).

(g) (7 points) Sooners Inc. has decided to expand their product line by adding a new paint color: Black. The price for the black paint is \$9 per gallon, and it requires $\frac{1}{5}$ kilogram of pigments and $\frac{1}{5}$ gallon of water per gallon produced. After contracting new suppliers, the resource limits have increased to 45 gallons of water and 40 kilograms of pigments. Formulate this new linear optimization problem, incorporating the new black paint. Using the algebraic sensitivity analysis (Simplex tableau) discussed in class, determine whether the previous optimal basis —before the inclusion of the new paint color— will remain optimal after adding the new paint. Provide a comprehensive explanation of your conclusion, detailing the factors that influence the optimality status and the potential effects of incorporating the new color.

X_K = production of black paint

Primal:

$$\text{Max} \quad 10X_R + 12X_B + 9X_K$$

s.t

$$X_R + 2X_B + \frac{1}{5}X_K \leq 40$$

$$2X_R + 2X_B + \frac{1}{5}X_B \leq 45$$

$$X_R, X_B, X_K \geq 0$$

Basic	z	xr	xb	s1	s2	s3	s4	sol
z	1	0	0	0	6	2	0	116
s1	0	0	0	1	-1	-1	0	7
s4	0	0	0	0	0.5	1	1	6
xr	0	1	0	0	0	-1	0	2
xb	0	0	1	0	0.5	1	0	8

Basic	z	xr	xb	xk	s1	s2	s3	s4	sol
z	1	0	0	-9	0	6	2	0	116
s1	0	0	0	0.2	1	-1	-1	0	7
s4	0	0	0	0.2	0	0.5	1	1	6
xr	0	1	0	0	0	0	-1	0	2
xb	0	0	1	0	0	0.5	1	0	8

[48]:

```
import gurobipy as gp
from gurobipy import *

# Create a new model for the primal problem
model = gp.Model("Sooner Inc Paints")

# Define decision variables
x1 = model.addVar(vtype=GRB.CONTINUOUS, lb=0, ub=GRB.INFINITY, name="x1")  # Red Paint
x2 = model.addVar(vtype=GRB.CONTINUOUS, lb=0, ub=GRB.INFINITY, name="x2")  # Blue Paint
x3 = model.addVar(vtype=GRB.CONTINUOUS, lb=0, ub=GRB.INFINITY, name="x3")  # Black Paint

# Define the objective function (maximize profit)
z = 10*x1 + 12*x2 + 9*x3
model.setObjective(z, GRB.MAXIMIZE)
model.update()

# Add Constraints
model.addConstr(2*x1 + 2*x2 + (1/5)*x3 <= 45)  # Water constraint
model.addConstr(1*x1 + 2*x2 + (1/5)*x3 <= 40)  # Pigment constraint
model.addConstr(x1 >= 2)  # Minimum red paint production
model.addConstr(x2 >= 2)  # Minimum blue paint production

# Optimize the model
model.optimize()

# Print output
if model.status == GRB.OPTIMAL:
    print("Optimal value (Total Profit):", model.objVal, "USD")
    print("Production quantity for red paint: ", x1.X)
    print("Production quantity for blue paint: ", x2.X)
    print("Production quantity for black paint: ", x3.X)
```

```
Gurobi Optimizer version 11.0.3 build v11.0.3rc0 (mac64[arm] - Darwin 23.6.0 23G93)

CPU model: Apple M2
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 4 rows, 3 columns and 8 nonzeros
Model fingerprint: 0xae29957a
Coefficient statistics:
  Matrix range     [2e-01, 2e+00]
  Objective range  [9e+00, 1e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [2e+00, 4e+01]
Presolve removed 2 rows and 1 columns
Presolve time: 0.01s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    1.7090000e+03   1.676813e+00   0.000000e+00      0s
       3    1.5740000e+03   0.000000e+00   0.000000e+00      0s

Solved in 3 iterations and 0.01 seconds (0.00 work units)
Optimal objective  1.574000000e+03
Optimal value (Total Profit): 1574.0 USD
Production quantity for red paint:  2.0
Production quantity for blue paint:  2.0
Production quantity for black paint:  170.0
```

The addition of black paint introduces a new variable that uses fewer resources (water and pigments) per gallon compared to red and blue paints. Since the reduced cost of the black paint variable is negative (-9), black paint will enter the basis, improving the objective function and leading to a new optimal solution, which can be seen in the gurobi result is xr,xb,xk.

2. (35 points) As a financial advisor, your role is to help clients achieve their savings goals while minimizing the cost of their total investment. Today, a couple has approached you to create an investment portfolio consisting of stocks from Company A and Company B. Their goal is to make at least \$20 in profit by the end of the year, in addition to recovering their initial investment. You know that for each dollar invested in Stock A, they will receive \$1.50 at the end of the year, while each dollar invested in Stock B will yield \$1.20. Due to regulatory constraints, they are allowed to invest at most 1.5 times more in Stock A than in Stock B. Additionally, the couple prefers Stock B for its lower risk, despite its lower return, and has requested that at least \$5 be invested in Stock B. The total budget available for their investment is \$100.

Formulation

Decision Variables

- x_A: Dollars invested in stock A.
- x_B: Dollars invested in stock B.

$$\text{Minimize} \quad x_A + x_B$$

st:

$$0.5x_A + 0.2x_B \geq 20$$
$$x_A \leq 1.5x_B$$
$$x_B \geq 5$$
$$x_A + x_B \leq 100$$

$$x_A \geq 0, \quad x_B \geq 0$$

(a) (6 points) Construct the dual problem of the associated LP model (use the formulation given above).

$$\text{Max } 20y_1 + 0y_2 + 5y_3 + 100y_4$$

s.t

$$0.5y_1 + y_2 + 0y_3 + y_4 \leq 1$$

$$0.2y_1 + 1.5y_2 + y_3 + y_4 \leq 1$$

$$y_1, y_3 \geq 0$$

$$y_2, y_4 \leq 0$$

(b) (4 points) Solve the dual problem using Excel/Solver. Include a snapshot of your Excel/Solver and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How are the dual variables and the dual objective function connected to the primal problem?

Basic	z	y1	y2^	y3	y4^	s1	s2	sol	Min ratio test
Z	1	-20	0	-5	100	0	0	0	
s1	0	0.5	-1	0	-1	1	0	1	2
s2	0	0.2	1.5	1	0	0	1	1	5

Basic	z	y1	y2^	y3	y4^	s1	s2	sol	
Z	1	0	-40	-5	60	40	0	40	
y1	0	1	-2	0	-2	2	0	2	-1
s2	0	0	1.9	1	0.4	-0.4	1	0.6	0.315789

Basic	z	y1	y2^	y3	y4^	s1	s2	sol
Z	1	0	0	16.05263	68.42105	31.57895	21.05263	52.63158
y1	0	1	0	1.052632	-1.57895	1.578947	1.052632	2.631579
y2^	0	0	1	0.526316	0.210526	-0.21053	0.526316	0.315789

(c) (6 points) Construct the dual problem of the standard form of the LP model (i.e., use the LP formulation given in the solution of Problem 2(a) of Individual Assignment 3 as the primal).

Primal: **Formulation** Decision Variables

- x_A: Dollars invested in stock A.
- x_B: Dollars invested in stock B.
- s_1: Slack constraint 1
- s_2: Slack constraint 2
- s_2: Slack constraint 3
- s_2: Slack constraint 4

$$\text{Minimize} \quad x_A + x_B$$

st:

$$0.5x_A + 0.2x_B - s_1 = 20$$
$$x_A - 1.5x_B + s_2 = 0$$
$$x_B - s_3 = 5$$
$$x_A + x_B + s_4 = 100$$
$$x_A, x_B, s_1, s_2, s_3, s_4 \geq 0$$

Max $\quad 20y_1 + 5y_3 + 100y_4$

$0.5y_1 + y_2 + y_4 \leq 1$

$0.2y_1 + (-1.5y_2) + y_3 + y_4 \leq 1$

y_1, y_2, y_3, y_4 free

$y_1, y_3 \geq 0 \qquad y_2, y_4 \leq 0$

(d) (3 points) Solve this dual problem using Excel/Solver. Include a snapshot of your Excel/Solver and obtained results, and discuss the results (both of the dual variables and the dual objective function) and their meaning. How do these compare to the results obtained in part b? Explain in detail.

(e) (16 points) After determining the optimal solution, determine the price range(coefficient in the objective function), separately for stock A and B, that would maintain the same optimal solution. Add the total investment range as well.

```python
import gurobipy as gp
from gurobipy import GRB

# Create a new model
model = gp.Model("investment_portfolio")

# Decision variables: xA (investment in stock A), xB (investment in stock B)
xA = model.addVar(vtype=GRB.CONTINUOUS, name="xA", lb=0)
xB = model.addVar(vtype=GRB.CONTINUOUS, name="xB", lb=0)

# Set objective function: Minimize total investment xA + xB
model.setObjective(xA + xB, GRB.MINIMIZE)

# Add constraints
model.addConstr(0.5 * xA + 0.2 * xB >= 20, "ProfitConstraint")
model.addConstr(xA <= 1.5 * xB, "RatioConstraint")
model.addConstr(xB >= 5, "MinStockBConstraint")
model.addConstr(xA + xB <= 100, "BudgetConstraint")

# Optimize the model
model.optimize()

# Display the results
if model.status == GRB.OPTIMAL:
    print(f"Optimal Investment in Stock A: {xA.X}")
    print(f"Optimal Investment in Stock B: {xB.X}")
    print(f"Total Investment: {xA.X + xB.X}")

    # Sensitivity analysis for objective function coefficients
    print("\nSensitivity Analysis for Stock A and Stock B Objective Coefficients:")

    # Price range for stock A
    print(f"Stock A objective coefficient range: [{xA.SAObjLow}, {xA.SAObjUp}]")

    # Price range for stock B
    print(f"Stock B objective coefficient range: [{xB.SAObjLow}, {xB.SAObjUp}]")

    # Sensitivity analysis for RHS of total investment constraint
    budget_constraint = model.getConstrByName("BudgetConstraint")
    print(f"\nTotal Investment range: [{budget_constraint.SARHSLow}, {budget_constraint.SARHSUp}]")
else:
    print("No optimal solution found.")
```

```
Gurobi Optimizer version 11.0.3 build v11.0.3rc0 (mac64[arm] - Darwin 23.6.0 23G93)

CPU model: Apple M2
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 4 rows, 2 columns and 7 nonzeros
Model fingerprint: 0x9d0cbb5c
Coefficient statistics:
  Matrix range     [2e-01, 2e+00]
  Objective range  [1e+00, 1e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+02]
Presolve removed 1 rows and 0 columns
Presolve time: 0.01s
Presolved: 3 rows, 2 columns, 6 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    4.3000000e+01   3.812500e+00   0.000000e+00      0s
       1    5.2631579e+01   0.000000e+00   0.000000e+00      0s

Solved in 1 iterations and 0.01 seconds (0.00 work units)
Optimal objective  5.263157895e+01
Optimal Investment in Stock A: 31.578947368421055
Optimal Investment in Stock B: 21.05263157894737
Total Investment: 52.631578947368425

Sensitivity Analysis for Stock A and Stock B Objective Coefficients:
Stock A objective coefficient range: [-0.6666666666666665, 2.5]
Stock B objective coefficient range: [0.4, inf]

Total Investment range: [52.631578947368425, inf]
```

3. (30 points) For the following problem, provide the mathematical programming formulation that would find the optimal solution to it. Clearly, indicate all **set(s)** (6 points), **parameter(s)** (6 points), **variable(s)** (6 points), **objective function** (6 points), and **constraint(s)** (6 points).

   You are a door-to-door vendor of cookies, and you have a list of clients denoted by the set C. Each client i has a probability α_{it} of purchasing your product, which depends on both the time t and the specific client. The quantity that each client i will buy is denoted by β_i, which is client-dependent but not time-dependent. Your objective is to maximize the expected value of sales during your shift.

   However, there are several constraints to consider. Your workday lasts 8 hours, and you can only visit two clients during each hour. Additionally, you can only visit each client once. The city is divided into two zones, Zone A and Zone B. If you choose to visit a client in Zone A, you cannot visit a client in Zone B during the same hour.Furthermore, there is a subset of clients $M \subseteq C$ who can only be visited during the afternoon, after $t > 4$). You must ensure that these clients are visited at the appropriate times. Lastly, to cover transportation costs, the expected sales value from the clients visited during each hour must exceed a minimum threshold γ_t.

1. Sets, Parameters, and Variables

Sets:

C: The set of clients i that can be visited.

$T = \{1, 2, \ldots, 8\}$: The set of hours t during the workday.

$Z = \{A, B\}$: The set of zones, Zone A and Zone B.

$M \subseteq C$: The subset of clients that can only be visited after hour 4.

Parameters:

α_{it}: The probability that client i will purchase if visited at time t.

β_i: The quantity that client i will buy if they decide to purchase (client-dependent, not time-dependent).

γ_t: The minimum expected sales value that must be achieved during hour t.

$q_i \in Z$: Indicates the zone Z (A or B) where client i is located.

$N = 2$: Maximum number of clients that can be visited per hour.

Variables:

$x_{it} \in \{0, 1\}$: Binary decision variable where $x_{it} = 1$ if client i is visited during hour t, and $x_{it} = 0$ otherwise.

Objective Function and Constraints

$$\text{Maximize } \sum_{i \in C} \sum_{t \in T} \alpha_{it} \beta_i x_{it}$$

$$\sum_{i \in C} x_{it} \leq 2 \quad \forall t \in T$$

$$\sum_{t \in T} x_{it} \leq 1 \quad \forall i \in C$$

$$\sum_{i \in C : q_i = A} x_{it} + \sum_{i \in C : q_i = B} x_{it} \leq 1 \quad \forall t \in T$$

$$x_{it} = 0 \quad \forall i \in M, t \leq 4$$

$$\sum_{i \in C} \alpha_{it} \beta_i x_{it} \geq \gamma_t \quad \forall t \in T$$

$$x_{it} \in \{0, 1\} \quad \forall i \in C, t \in T$$