HW 5 Systems optimization

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1 Question 1

1.1 a.

- Revenue vs Workers: Sooners Inc. has about 4 workers and generates a revenue of around 1900 USD. When compared to other companies, there are companies with fewer workers (around 2-3) that generate higher revenue (closer to 3500–5000 USD). On the other hand, there are companies with more workers (5–10) that either perform similarly or better in terms of revenue. Conclusion: Sooners Inc. seems to be inefficient in terms of worker allocation because other companies can generate higher revenue with fewer workers.

- Revenue vs Publicity: Sooners Inc. has a publicity value of around 42 and generates a revenue of about 1900 USD. There are companies with similar levels of publicity (around 40–45) that are generating significantly higher revenue (above 3500 USD), and even some companies with lower publicity (around 30) outperform Sooners Inc. Conclusion: Sooners Inc. is likely inefficient in the way it is utilizing publicity, as other companies with similar or lower publicity investments are generating more revenue.

- Revenue vs Inventory Management: Sooners Inc. has an inventory management score of around 15 and generates around 1900 USD. Other companies with similar inventory management scores (around 15) tend to generate higher revenues, and some companies with lower scores (around 12) outperform Sooners Inc. Conclusion: Sooners Inc. is not efficiently managing its inventory compared to peers, as other companies with similar or even lower inventory scores are achieving better revenue outcomes.

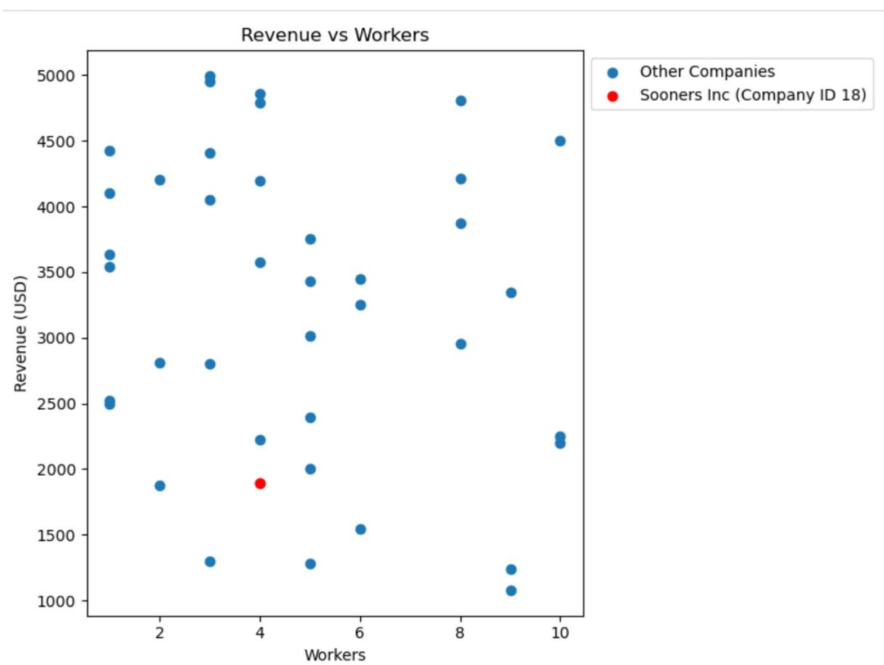

Figure 1

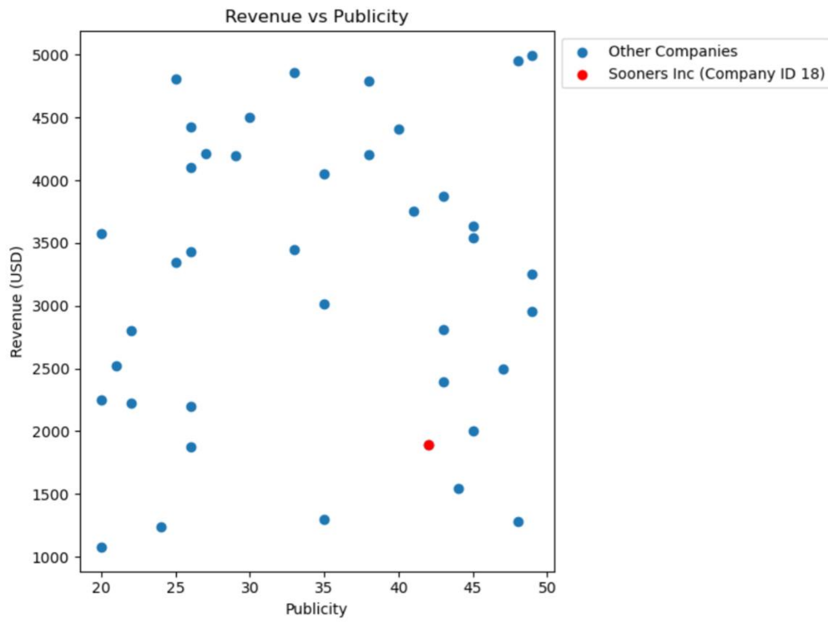

Figure 2

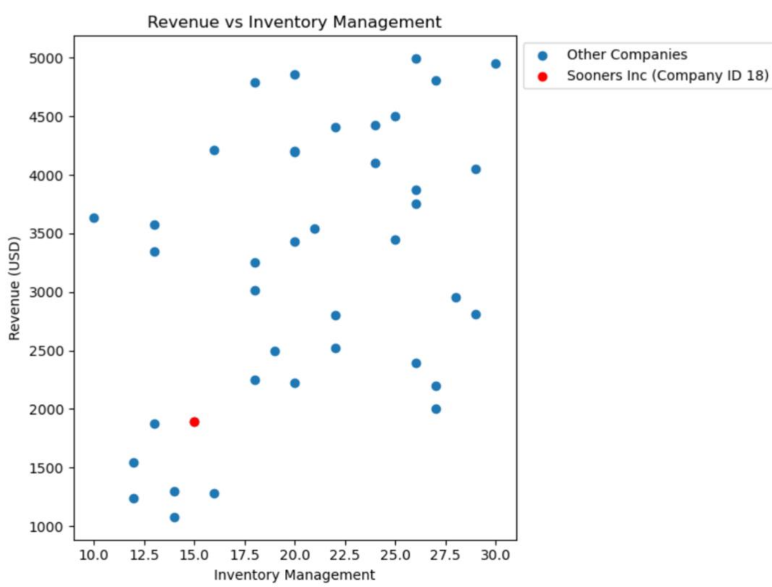

Figure 3

1.2 b.

Sets

- J: Set of decision-making units (DMUs), indexed by j, where $j \in \{1, 2, \ldots, 40\}$
- I: Set of inputs, indexed by i, where $i \in \{\text{workers, publicity, inventory management}\}$
- O: Set of outputs, indexed by o, where $o \in \{\text{revenue}\}$

Parameters

- x_{ij}: Amount of input i used by DMU j

- y_{oj}: Amount of output o produced by DMU j

Decision Variables

- λ_j: Weight assigned to DMU j
- ϕ: Output expansion factor for DMU 18 (target DMU)

Objective Function

The objective is to maximize the output expansion factor ϕ for DMU 18:

$$\max_{\phi,\lambda} \ \phi \tag{1}$$

Constraints

- **Input constraints**: The weighted sum of the inputs across all DMUs should be less than or equal to the inputs of DMU 18.
$$\sum_{j \in J} \lambda_j x_{ij} \leq x_{i18}, \quad \forall i \in I \tag{2}$$
- **Output constraints**: The weighted sum of the outputs across all DMUs should be greater than or equal to the expanded output of DMU 18.
$$\sum_{j \in J} \lambda_j y_{oj} \geq \phi y_{o18}, \quad \forall o \in O \tag{3}$$
- **Convexity constraint**: The sum of the weights should equal 1 to ensure convexity.
$$\sum_{j \in J} \lambda_j = 1 \tag{4}$$
- **Non-negativity constraint**: The weights and the output expansion factor should be non-negative.
$$\lambda_j \geq 0, \quad \phi \geq 1, \quad \forall j \in J \tag{5}$$

```
Optimal objective  2.300158395e+00
Optimal phi (Efficiency Score) for Sooners Inc: 2.300158394931362
Lambda_1: 0.0
Lambda_2: 0.0
Lambda_3: 0.0
Lambda_4: 0.0
Lambda_5: 0.0
Lambda_6: 0.0
Lambda_7: 0.0
Lambda_8: 0.0
Lambda_9: 0.0
Lambda_10: 0.0
Lambda_11: 0.625
Lambda_12: 0.0
Lambda_13: 0.0
Lambda_14: 0.0
Lambda_15: 0.0
Lambda_16: 0.375
Lambda_17: 0.0
Lambda_18: 0.0
Lambda_19: 0.0
Lambda_20: 0.0
Lambda_21: 0.0
Lambda_22: 0.0
Lambda_23: 0.0
Lambda_24: 0.0
Lambda_25: 0.0
Lambda_26: 0.0
Lambda_27: 0.0
Lambda_28: 0.0
Lambda_29: 0.0
Lambda_30: 0.0
Lambda_31: 0.0
Lambda_32: 0.0
Lambda_33: 0.0
Lambda_34: 0.0
Lambda_35: 0.0
Lambda_36: 0.0
Lambda_37: 0.0
Lambda_38: 0.0
Lambda_39: 0.0
Lambda_40: 0.0
Sooners Inc. is inefficient.
```

Figure 4

We can conclude that sooners inc is inefficient and has a growing factor of approximately 2.3

1.3 c.i

```
Optimal objective  2.300158395e+00
Optimal phi (Efficiency Score) for Sooners Inc: 2.300158394931362
Lambda_1: 0.0
Lambda_2: 0.0
Lambda_3: 0.0
Lambda_4: 0.0
Lambda_5: 0.0
Lambda_6: 0.0
Lambda_7: 0.0
Lambda_8: 0.0
Lambda_9: 0.0
Lambda_10: 0.0
Lambda_11: 0.625
Lambda_12: 0.0
Lambda_13: 0.0
Lambda_14: 0.0
Lambda_15: 0.0
Lambda_16: 0.375
Lambda_17: 0.0
Lambda_18: 0.0
Lambda_19: 0.0
Lambda_20: 0.0
Lambda_21: 0.0
Lambda_22: 0.0
Lambda_23: 0.0
Lambda_24: 0.0
Lambda_25: 0.0
Lambda_26: 0.0
Lambda_27: 0.0
Lambda_28: 0.0
Lambda_29: 0.0
Lambda_30: 0.0
Lambda_31: 0.0
Lambda_32: 0.0
Lambda_33: 0.0
Lambda_34: 0.0
Lambda_35: 0.0
Lambda_36: 0.0
Lambda_37: 0.0
Lambda_38: 0.0
Lambda_39: 0.0
Lambda_40: 0.0
Slack in workers: 1.125
Slack in publicity: 1.375
Slack in inventory management: 0.0
Slack in output (revenue shortfall): 0.0
Sooners Inc. is inefficient.
```

Figure 5

slacks mean inefficiencies.

- Reduce Workforce: The company could reduce its workforce by 1.125 workers while still maintaining the same level of revenue. This suggests that there is some inefficiency in its use of human resources, and it could improve by downsizing slightly.
- Reduce Publicity Spending: Sooners Inc. is spending more on publicity than necessary and could cut its publicity spending by 1.375 units and still generate the same revenue.
- Efficient in Inventory Management: There is no excess in the inventory management resources, meaning Sooners Inc. is utilizing this resource efficiently.

1.4 c.ii

Objective Function:

$$\max \left(s^{+}_{\text{output}} + s^{-}_{\text{workers}} + s^{-}_{\text{publicity}} + s^{-}_{\text{inventory}} \right)$$

Subject to:

$$\sum_{j \in S} \lambda_j \cdot y_{o,j} - s^+_{\text{output}} = \phi \cdot y_{o,18}, \quad \forall o \in O$$

$$\sum_{j \in S} \lambda_j \cdot x_{\text{workers},j} + s^-_{\text{workers}} = x_{\text{workers},18}, \quad \forall i \in I$$

$$\sum_{j \in S} \lambda_j \cdot x_{\text{publicity},j} + s^-_{\text{publicity}} = x_{\text{publicity},18}, \quad \forall i \in I$$

$$\sum_{j \in S} \lambda_j \cdot x_{\text{inventory},j} + s^-_{\text{inventory}} = x_{\text{inventory},18}, \quad \forall i \in I$$

$$\sum_{j \in S} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad \forall j \in S$$

$$s^+_{\text{output}} \geq 0, \quad s^-_{\text{workers}} \geq 0, \quad s^-_{\text{publicity}} \geq 0, \quad s^-_{\text{inventory}} \geq 0$$

```
Slack in output (revenue shortfall): 4356.5
Slack in workers: 1.125
Slack in publicity: 1.375
Slack in inventory management: 0.0
```

Figure 6

when we maximize the slacks, we get the same results, meaning that when we found the growth factor, we got to the point of optimality in which the maximum values of slacks are the ones found alread.

1.5 d.

Objective Function:

$$\min \ \phi_{18}$$

Subject to:

$$\sum_{j \in S} \lambda_j \cdot c_j \leq \phi_{18} \cdot c_{18}$$

$$\sum_{j \in S} \lambda_j \cdot x_{i,j} \leq x_{i,18}, \quad \forall i \in I$$

$$\sum_{j \in S} \lambda_j \cdot y_j \geq 2.3 \cdot y_{18}$$

$$\sum_{j \in S} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad \forall j \in S$$

Parameters:

- $x_{i,j}$: Input i (workers, publicity, or inventory management) for company j
- y_j: Revenue output for company j
- c_j: Cash for company j

Decision Variables:

- λ_j: Weight assigned to company j

- ϕ_{18}: Cash proportion scaling factor for the target company (Company 18)

```
Optimal objective  3.731865581e-01
Optimal cash proportion: 0.37318655808343554
Lambda_1: 0.0
Lambda_2: 0.0
Lambda_3: 0.0
Lambda_4: 0.0
Lambda_5: 0.0
Lambda_6: 0.0
Lambda_7: 0.0
Lambda_8: 0.0
Lambda_9: 0.0
Lambda_10: 0.0
Lambda_11: 0.624772497472194
Lambda_12: 0.0
Lambda_13: 0.0
Lambda_14: 0.0
Lambda_15: 0.0006066734074826745
Lambda_16: 0.37462082912032335
Lambda_17: 0.0
Lambda_18: 0.0
Lambda_19: 0.0
Lambda_20: 0.0
Lambda_21: 0.0
Lambda_22: 0.0
Lambda_23: 0.0
Lambda_24: 0.0
Lambda_25: 0.0
Lambda_26: 0.0
Lambda_27: 0.0
Lambda_28: 0.0
Lambda_29: 0.0
Lambda_30: 0.0
Lambda_31: 0.0
Lambda_32: 0.0
Lambda_33: 0.0
Lambda_34: 0.0
Lambda_35: 0.0
Lambda_36: 0.0
Lambda_37: 0.0
Lambda_38: 0.0
Lambda_39: 0.0
Lambda_40: 0.0
```

Figure 7

we can see in image 7 the results for this reduced cash proportion, the result is cash can be reduced to 0.37 the initial value.

Incorporating the proportion of revenue from cash transactions as a new output variable changes the DEA model's structure by adding a new objective aimed at minimizing this proportion while maintaining the efficient revenue found in the previous analysis. This introduces an additional trade-off between efficiency and the reliance on cash-based revenue, making the model more complex. The new output variable shifts the focus from solely maximizing efficiency to also reducing dependency on cash transactions, which may affect how resources are allocated to achieve the same revenue target while minimizing cash transactions.

1.6 e.

Let μ_i be the dual variable associated with the input constraint:

$$\sum_{j \in S} x_{i,j} \leq x_{i,18}$$

Let w_o be the dual variable associated with the output constraint:

$$\sum_{j \in S} y_{o,j} \geq y_{o,18} \cdot \phi_{18}$$

Let π be the dual variable associated with the convexity constraint:

$$\sum_{j \in S} \lambda_j = 1$$

Dual Problem

Objective Function:

$$\text{Minimize} \quad \sum_{i \in I} \mu_i x_{i,18} - \sum_{o \in O} w_o y_{o,18} + \pi$$

Subject to:

$$\sum_{i \in I} \mu_i x_{i,j} + \sum_{o \in O} w_o y_{o,j} + \pi \geq 0, \quad \forall j \in S$$

$$\sum_{o \in O} w_o y_{o,18} = 1$$

$$\mu_i \geq 0, \quad \forall i \in I$$

$$w_o \leq 0, \quad \forall o \in O$$

$$\pi \text{ is free}$$

2 Question 2

2.1 a.

Sets:

$$F = \{A, B, C, D\} \quad \text{(factories)}$$
$$C = \{1, 2, \ldots, 15\} \quad \text{(customers)}$$

Parameters:

$$d_j \quad \text{demand for customer } j \in C$$
$$c_{ij} \quad \text{cost of shipping from factory } i \text{ to customer } j$$

Decision Variables:

$$x_{ij} \quad \text{number of units shipped from factory } i \text{ to customer } j$$

Objective: Minimize Total Transportation Cost

$$\min \sum_{i \in F} \sum_{j \in C} c_{ij} \cdot x_{ij}$$

Subject to:

Demand Constraints:

$$\sum_{i \in F} x_{ij} = d_j \quad \forall j \in C$$

Non-Negativity Constraints:

$$x_{ij} \geq 0 \quad \forall i \in F, \forall j \in C$$

```
Optimal objective  2.889000000e+03
Optimal objective value (minimum transportation cost): 2889.0
Ship 64.0 cans from A to Customer 5
Ship 19.0 cans from A to Customer 7
Ship 98.0 cans from A to Customer 8
Ship 72.0 cans from A to Customer 12
Ship 53.0 cans from A to Customer 15
Ship 97.0 cans from B to Customer 1
Ship 17.0 cans from B to Customer 4
Ship 73.0 cans from B to Customer 6
Ship 79.0 cans from B to Customer 10
Ship 17.0 cans from B to Customer 11
Ship 39.0 cans from B to Customer 13
Ship 24.0 cans from C to Customer 2
Ship 19.0 cans from D to Customer 3
Ship 87.0 cans from D to Customer 9
Ship 96.0 cans from D to Customer 14
```

Figure 8

2.2 b.

Sets:

$$
\begin{aligned}
F &= \{A, B, C, D\} \quad \text{(factories)} \\
C &= \{1, 2, \ldots, 15\} \quad \text{(customers)}
\end{aligned}
$$

Parameters:

$$
\begin{aligned}
s_i &\quad \text{supply available from factory } i \in F \\
d_j &\quad \text{demand for customer } j \in C \\
c_{ij} &\quad \text{cost of shipping from factory } i \text{ to customer } j \\
u_j &\quad \text{cost of undelivered units for customer } j
\end{aligned}
$$

Decision Variables:

$$
\begin{aligned}
x_{ij} &\quad \text{number of units shipped from factory } i \text{ to customer } j \\
z_j &\quad \text{undelivered units for customer } j
\end{aligned}
$$

Objective: Minimize Total Transportation Cost and Undelivered Penalty Cost

$$\min \sum_{i \in F} \sum_{j \in C} c_{ij} \cdot x_{ij} + \sum_{j \in C} u_j \cdot z_j$$

Subject to:

Supply Constraints (factories cannot supply more than available supply):

$$\sum_{j \in C} x_{ij} \leq s_i \quad \forall i \in F$$

Demand Constraints (demand must be met or undelivered units penalized):

$$\sum_{i \in F} x_{ij} + z_j = d_j \quad \forall j \in C$$

Non-Negativity Constraints:

$$x_{ij} \geq 0 \quad \forall i \in F, \forall j \in C$$

$$z_j \geq 0 \quad \forall j \in C$$

```
Optimal objective  2.449300000e+04
Optimal objective value (minimum cost): 24493.0
Ship 52.0 cans from A to Customer 12
Ship 48.0 cans from A to Customer 15
Ship 43.0 cans from B to Customer 1
Ship 73.0 cans from B to Customer 6
Ship 79.0 cans from B to Customer 10
Ship 5.0 cans from B to Customer 15
Ship 28.0 cans from C to Customer 1
Ship 64.0 cans from C to Customer 5
Ship 19.0 cans from C to Customer 7
Ship 39.0 cans from C to Customer 13
Ship 26.0 cans from D to Customer 1
Ship 19.0 cans from D to Customer 3
Ship 17.0 cans from D to Customer 4
Ship 17.0 cans from D to Customer 11
Ship 96.0 cans from D to Customer 14
Customer 2 has 24.0 undelivered units, with penalty cost.
Customer 8 has 98.0 undelivered units, with penalty cost.
Customer 9 has 87.0 undelivered units, with penalty cost.
Customer 12 has 20.0 undelivered units, with penalty cost.
```

Figure 9

In part 2.b, the mathematical formulation was updated to include factory capacity constraints and penalties for undelivered units, which changed the delivery strategy compared to part 2.a. While in part 2.a, all customer demands were fully met because supply was unlimited, in part 2.b, the limited supply from each factory meant that some customer demands could not be fully satisfied. The model had to balance between minimizing transportation costs and avoiding high penalties for undelivered units. As a result, some customers received fewer units than requested, and penalties for undelivered units were included in the total cost. This led to a shift in the strategy, with a focus on optimizing the available supply and incurring penalties where it was more cost-effective, resulting in a significantly higher overall cost.

2.3 c.

Let y_j be the dual variable associated with the demand constraint for customer j.

Let w_i be the dual variable associated with the supply constraint for factory i.

The objective function aims to maximize the total dual contribution from both demand and supply constraints.

Objective Function:

$$\text{Maximize} \quad \sum_{j \in N} d_j y_j + \sum_{i \in F} b_i w_i$$

Subject to:

$$\sum_{j \in N} C_{ij} y_j - w_i \leq 0, \quad \forall i \in F$$

$$y_j \text{ is free for each customer } j \in N$$

$$w_i \leq 0 \quad \text{for each factory } i \in F$$