December 2019

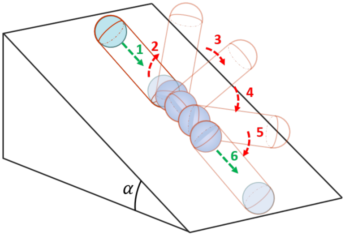
Jumping Beans

Problem 14

Universidad Industrial de Santander Team

We particularly thanks to Stephanie, Kevin, Nicolás, Jorge, Angie and Stephany for fruitful discussions.

Problem

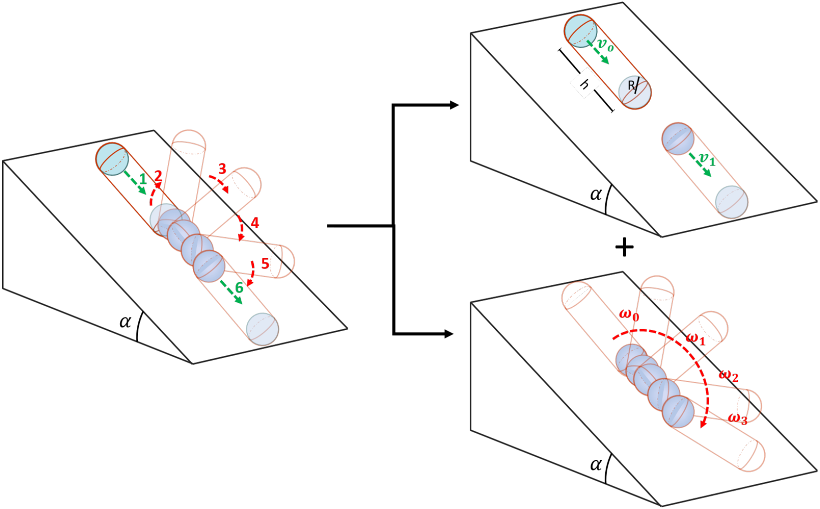


A simple toy called a "**jumping bean**" can be constructed by putting a *metal ball* inside of a *pill capsule*. Placed on an inclined surface at a certain inclination, the jumping bean will tumble down in a rather surprising way, seemingly standing up-right, flipping end to end, instead of rolling.

- Investigate its motion.
- Find the dimensions of the fastest and slowest beans for a given inclination

Theoretical approach

Motion analysis



Parameters and conditions

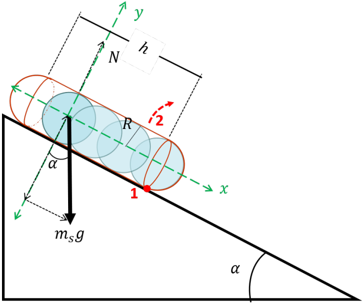
Parameter	Definition
m_s	mass of the sphere
m_p	mass of the capsule
m_T	total mass of the system
g	gravity
L	length of the capsule
h	cylindrical length of the capsule
R	sphere radius
α	inclination angle

Parameters and conditions

Conditions

- The capsule radius and the sphere radius must be the same.
- All the movement will be described by one plane.
- Rolling condition on the capsule and the sphere.

Angular momentum transfer



$$\sum F_y = N - m_s g cos(\alpha) = 0$$

$$\sum F_x = m_s g sin(\alpha) = m_s a_s$$

$$t_f = \frac{-V_i + \sqrt{V_i^2 - 2gSin(\alpha)(x_i - x_f)}}{gSin(\alpha)}$$

$$V_f = g t_f Sin(\alpha) + V_i$$

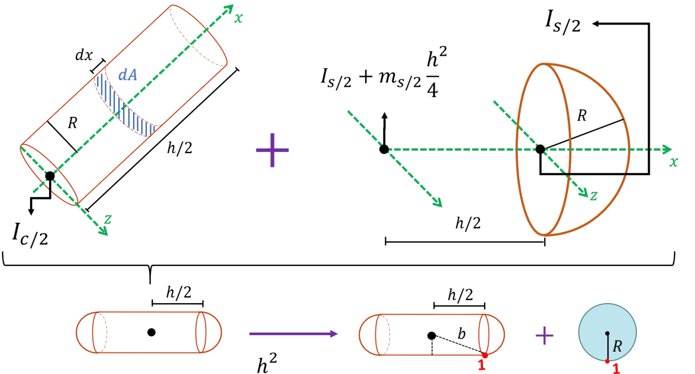
$$L_1 = m_s V_f R + \frac{2}{5} m_s R^2 = \frac{7}{5} m_s V_f R$$

$$L_2 = (I_{S1} + I_{P1})\omega_o = I_T \omega_o$$

$$L_1 = L_2$$

$$\boxed{\omega_o = \frac{7}{5 I_T} m_s V_f R}$$

Moment of inertia



$$I_{Po} = 2(I_{c/2} + (I_{s/2} + m_{s/2}\frac{h^2}{4}))$$

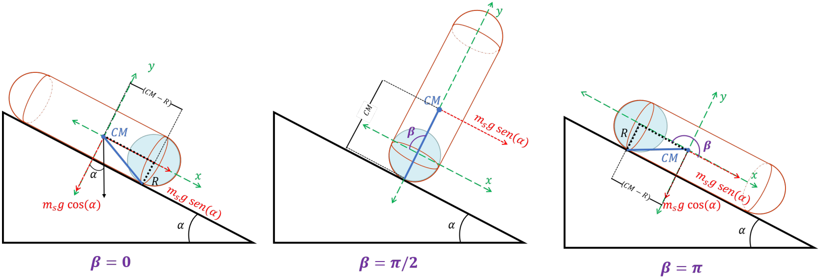
$$I_{P1} = I_{Po} + m_p b^2$$

$$I_{S1} = I_{So} + m_s R^2$$

$$\boxed{I_T = \frac{7m_sR^2}{5} + \frac{m_p(4h^3 + 24h^2R + 9hR^2 + 28R^3)}{12(h+4R)}}$$

Angular dynamics

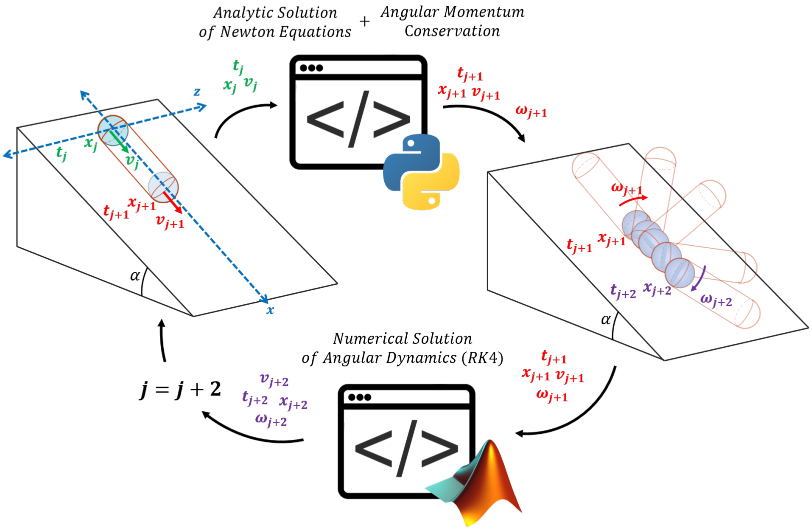
$$\omega_o = \frac{84 m_s R(4R + h) V_o}{20 h^3 m_p + 120 m_p h^2 R + 84 m_s h R^2 + 45 m_p h R^2 + 336 m_s R^3 + 140 m_p R^3}$$



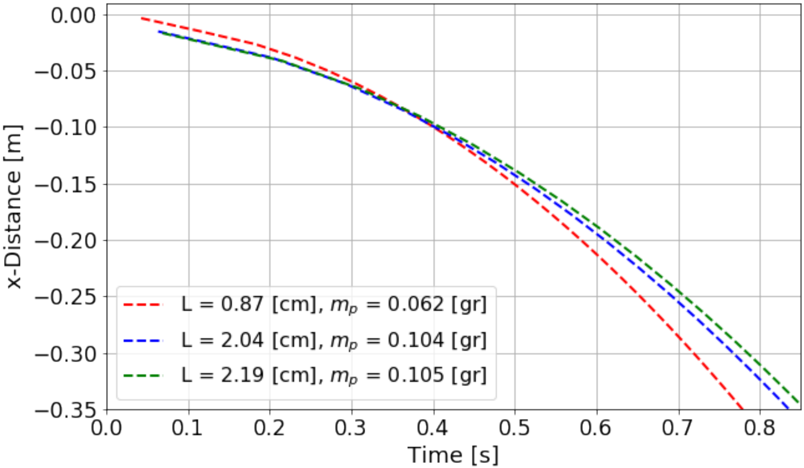
$$\sum \tau = I \frac{d^2 \beta}{dt^2}$$

$$\boxed{\frac{d^2\beta}{dt^2} = \frac{-m_T g Cos(\alpha)(CM - R)Cos(\beta) + m_T g Sin(\alpha)[R(1 - Sin(\beta) + CM Sin(\beta))]}{(I_{CM} + m_T) \cdot (2R(R - CM)|Cos(\beta)| + CM^2)}}$$

Motion analysis

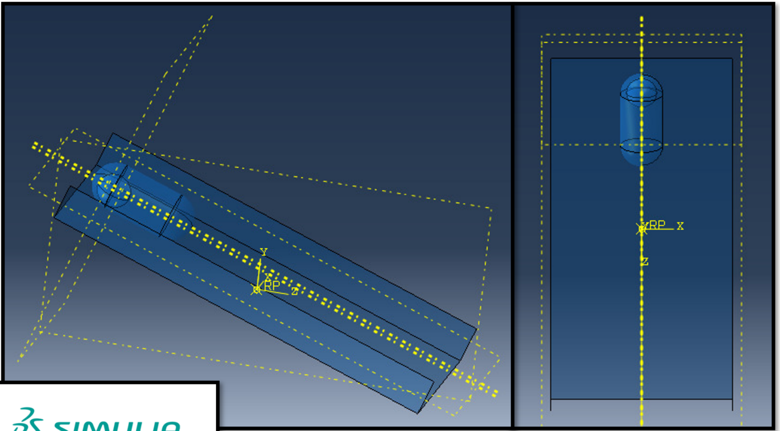


Motion analysis



Simulation approach

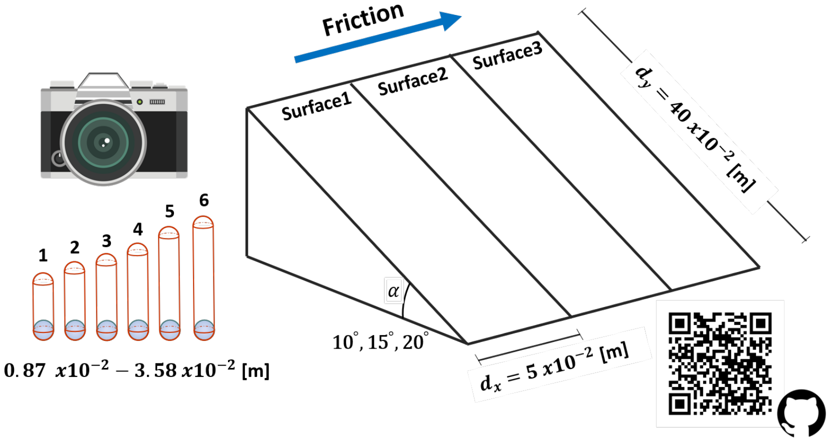
Simulation



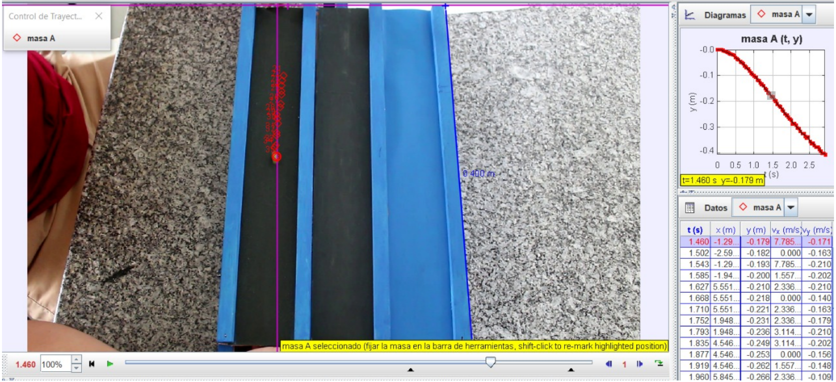
The software version is ABAQUS 6.14-1

Experimental approach

Experimental set-up



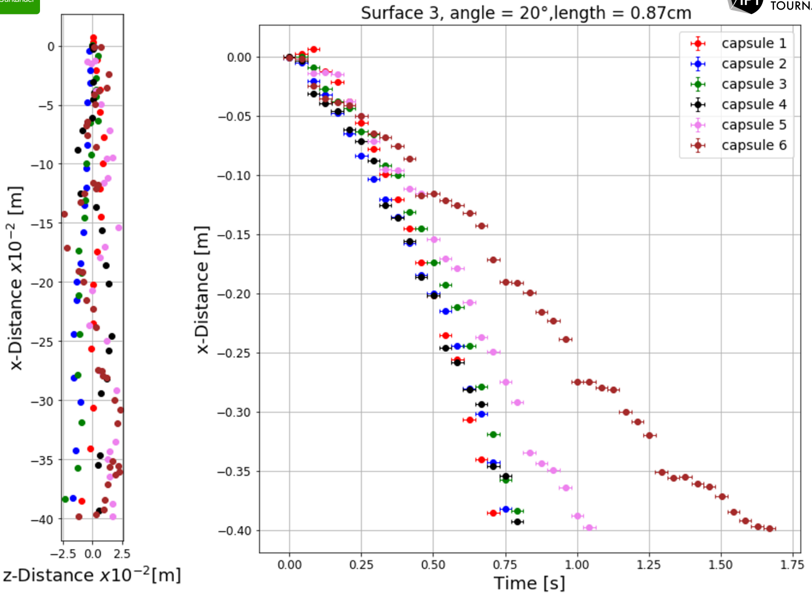
Data Collection



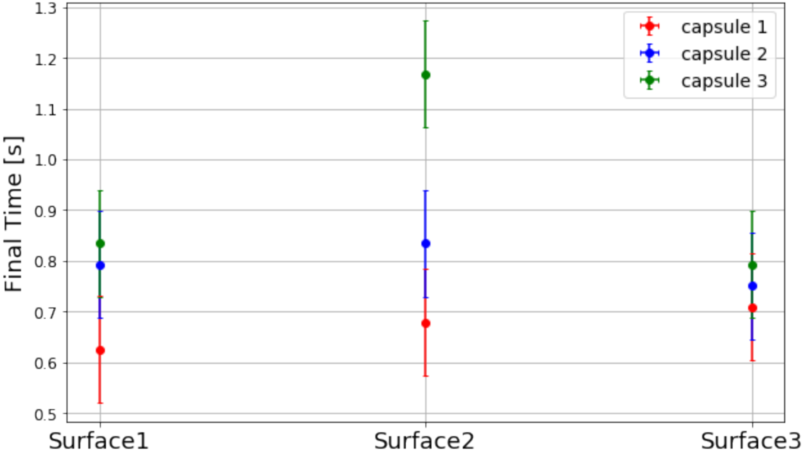
t (s)	x (m)	y (m)	v_x (m/s)	v_y (m/s)
1.460	-1.29...	-0.179	7.785...	-0.171
1.502	-2.59...	-0.182	0.000	-0.163
1.543	-1.29...	-0.193	7.785...	-0.210
1.585	-1.94...	-0.200	1.557...	-0.202
1.627	5.551...	-0.210	2.336...	-0.210
1.668	5.551...	-0.218	0.000	-0.140
1.710	5.551...	-0.221	2.336...	-0.163
1.752	1.948...	-0.231	2.336...	-0.179
1.793	1.948...	-0.236	3.114...	-0.210
1.835	4.546...	-0.249	3.114...	-0.202
1.877	4.546...	-0.253	0.000	-0.156
1.919	4.546...	-0.262	1.557...	-0.148
1.960	5.845...	-0.266	2.336...	-0.109

We used the 5.1.2 version of the software Tracker.

Inclination angle = 20^o

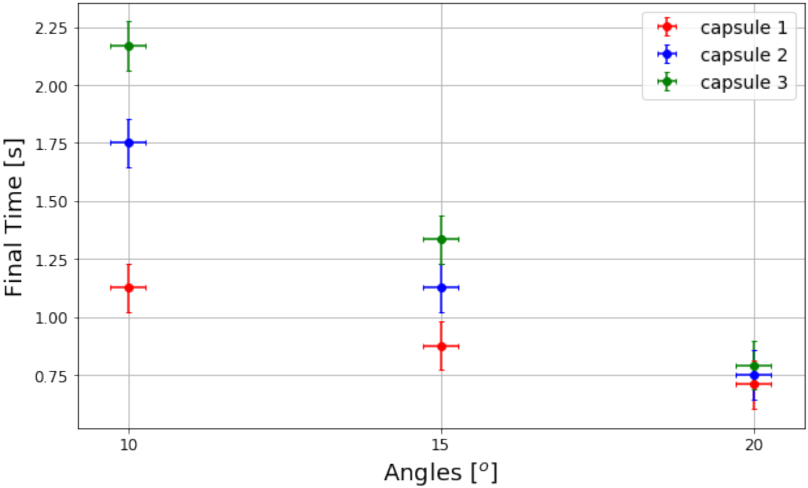


Inclination angle $= 20^o$



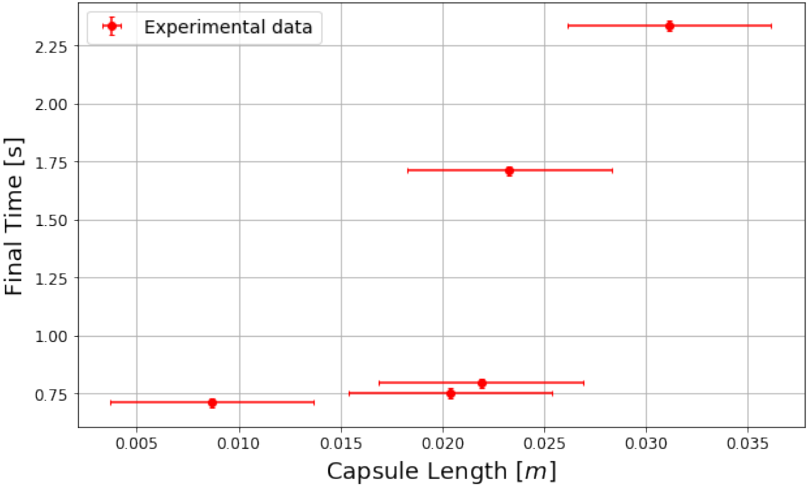
The error bars are magnified five times.

Surface 3

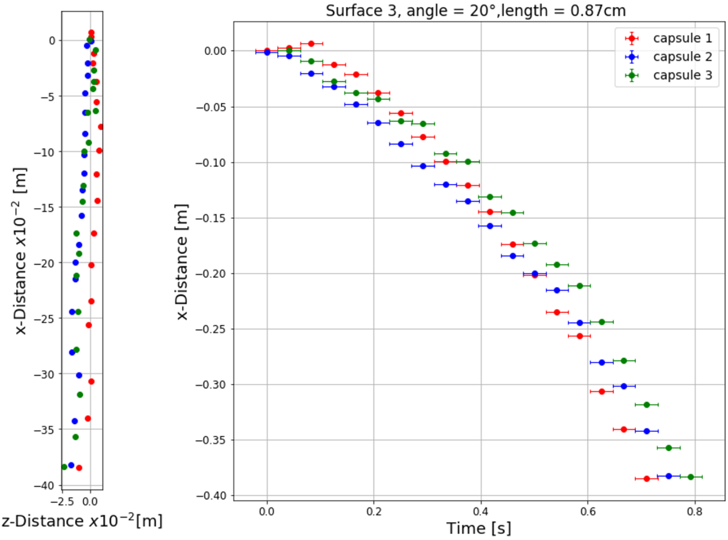


The error bars along Final Time are magnified five times.

Surface 3, angle = 20^o

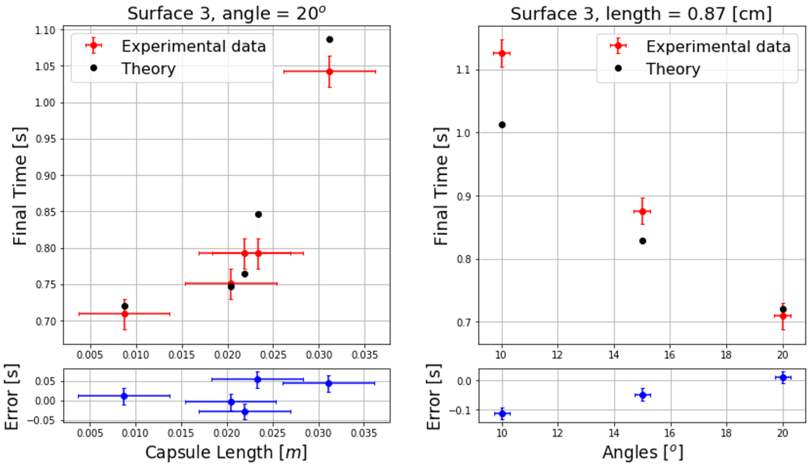


Surface 3, angle = 20^o

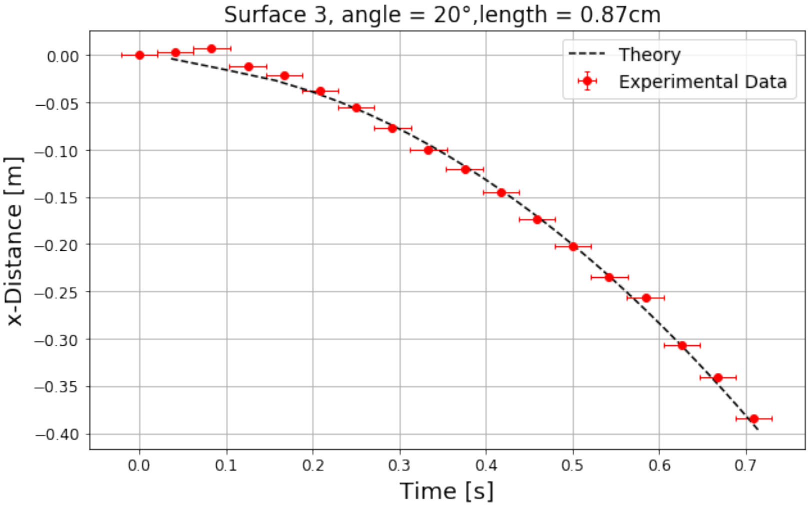


Data Vs Theory

Comparison


Surface 3, angle = 20°
Experimental data
Theory
Final Time [s]
Error [s]
Capsule Length [m]
Surface 3, length = 0.87 [cm]
Experimental data
Theory
Final Time [s]
Error [s]
Angles [°]

Comparison



Surface 3, angle = 20°,length = 0.87cm
Theory
Experimental Data
0.00
−0.05
−0.10
−0.15
−0.20
−0.25
−0.30
−0.35
−0.40
x-Distance [m]
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Time [s]

Conclusions

- The motion of the jumping beans can be explained by the angular momentum conservation and the Newtonian dynamics:

$$\boxed{\frac{d^2\beta}{dt^2} = \frac{-m_T g Cos(\alpha)(CM - R)Cos(\beta) + m_T g Sin(\alpha)[R(1 - Sin(\beta) + CMSin(\beta))]}{(I_{CM} + m_T)\cdot(2R(R - CM)|Cos(\beta)| + CM^2)}}$$

- The dimensions of the fastest and slowest beans for a given inclination will be determined by the maximum and minimums analysis of the movement equations.

- The fastest bean in our solution was:

$$\boxed{\mathrm{L} = 0.87\text{x}10^{-2}\ [\text{m}];\ m_p = 0.062\text{x}10^{-3}\ [\text{Kg}];\ R = 0.25x10^{-2}\ [\text{m}];\ m_s = 0.59x10^{-3}\ [\text{Kg}]}$$

- The slowest bean in our solution was:

$$\boxed{\mathrm{L} = 3.58\text{x}10^{-2}\ [\text{m}];\ m_p = 0.200\text{x}10^{-3}\ [\text{Kg}];\ R = 0.25x10^{-2}\ [\text{m}];\ m_s = 0.59x10^{-3}\ [\text{Kg}]}$$

THANKS!

Support material

Moment Angular Conservation

Sphere

$$I_{so} = \frac{2}{5} M_s R^2 + M_s R^2$$

Solid dial + Steiner Theorem

Pill moment

Conservation superficial density

$$M_{p/2} = M_{c/2} + M_{s/2}$$

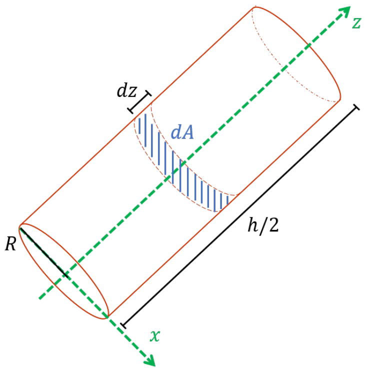
$$\frac{2M_{c/2}}{\pi R h} = \frac{M_{s/2}}{2\pi R^2}$$

$$M_{M_{c/2}} = M_{s/2} \frac{h}{4R}$$

Remplace in $M_{p/2}$

$$\boxed{M_{c/2} = \frac{hM_{p/2}}{4R + h} \qquad M_{s/2} = \frac{4RM_{p/2}}{4R + h}}$$

Integral define



$$dA = 2\pi R dz$$

$$dm = \rho dA = \frac{2M_{c/2}}{h} dz$$

Moment Angular Conservation

$$dI^z_{c/2} = dmR^2$$

Perpendicular axes Theorem

$$dI^z = dI^x + dI^y \quad and \quad dI^x = dI^y$$

$$dI^x_{c/2} = \frac{1}{2} dI^z$$

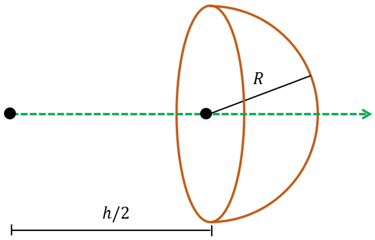
Steiner Theorem

$$dI^x_{c/2} = \frac{1}{2} dmR^2 + dmz^2$$

$$I^x_c = 2\int_0^{h/2} \left(\frac{2R^2 m_{c/2}}{2h} + \frac{2m_{c/2}}{h} z^2 \right) dz$$

$$= \frac{hM_pR^2}{2(4R+h)} + \frac{h^3 M_p}{12(4R+h)}$$

Integral define



$$I^x_{s/2} = \frac{M_s R^2}{3} + \frac{M_s h^2}{4}$$

Spherical shell + Steiner Theorem

$$I^x_{s/2} = \frac{2M_pR^3}{3(4R+h)} + \frac{M_pRh^2}{2(4R+h)}$$

$$
\begin{aligned}
I_p &= I_c^x + 2I_{s/2} \\
&= \frac{hM_pR^2}{2(4R+h)} + \frac{h^3M_p}{12(4R+h)} + \frac{4M_pR^3}{3(4R+h)} + \frac{M_pRh^2}{4R+h} \\
&= \frac{M_p(4h^3+24Rh^2+9R^2h+28R^3)}{12(4R+h)}
\end{aligned}
$$

Remplace in I_T

$$
I_T = \frac{7}{5}M_sR^2 + \frac{M_p(4h^3+24Rh^2+9R^2h+28R^3)}{12(4R+h)}
$$

By Angular moment conservation

$$
\begin{aligned}
L_1 &= L_2 \\
\frac{7}{5}m_sV_oR &= I_T\omega_o \\
\frac{7}{5}m_sV_oR &= \left(\frac{84M_sR^2(4R+h)+5M_p(4h^3+24Rh^2+9R^2h+28R^3)}{12(4R+h)}\right)\omega_o
\end{aligned}
$$

Solve ω_o

$$\omega_o = \frac{84 m_s R(4R+h)V_o}{20h^3 M_p+120M_p h^2 R+84M_s hR^2+45M_p hR^2+336M_s R^3+140M_p R^3}$$

Pill Momentum

Using:
Perpendicular axes Theorem
Steiner Theorem

$$I_c = \int_0^{h/2} \left(\frac{M_p R^2}{4R+h} + \frac{2M_p z^2}{4R+h} \right) dz$$

$$\boxed{I_c = \frac{h M_p R^2}{2(4R+h)} + \frac{h^3 M_p}{12(4R+h)}}$$

$$I_{s/2} = \frac{M_{s/2} R^2}{3} + \frac{M_{s/2} h^2}{4}$$

$$\boxed{I_s = \frac{4 M_p R^3}{3(4R+h)} + \frac{M_p R h^2}{4R+h}}$$

$$\boxed{I_{po} = \frac{M_p(4h^3 + 24Rh^2 + 9R^2 h + 28R^3)}{12(4R+h)}} \quad (1)$$

Energy

Initial Energy = Kinetic(Translational + Rotational)
$E_1 = \frac{1}{2}M_s V_o^2 + \frac{1}{2}\left(\frac{2}{5}M_s R^2\right)\omega^2$

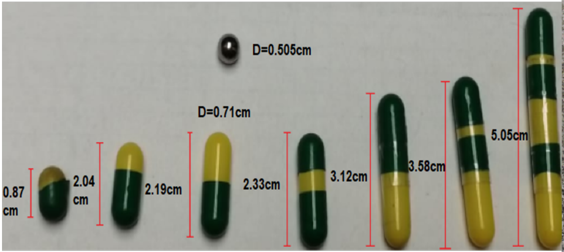
Remember $\omega = \frac{V}{R}$

$$\begin{aligned} E_1 &= \frac{1}{2}M_s V_o^2 + \frac{1}{2}\left(\frac{2}{5}M_s R^2\right)\omega^2 \\ &= \frac{1}{2}M_s V_o^2 + \frac{1}{5}M_s V_o^2 \\ &= \frac{7}{10}M_s V_o^2 \end{aligned}$$

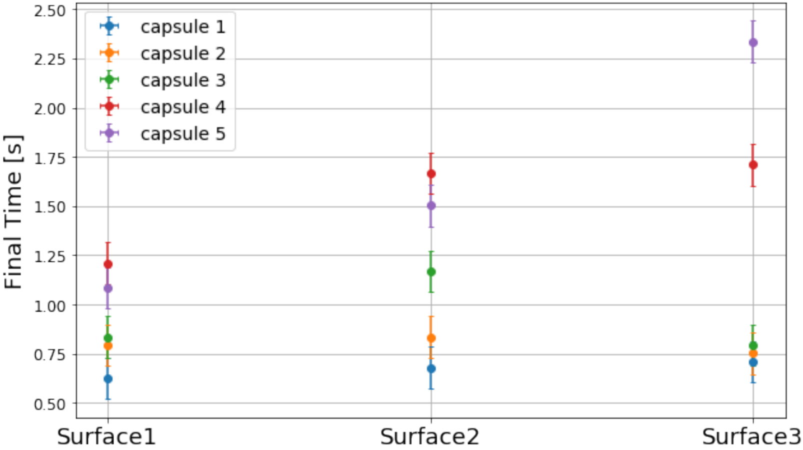
Final Energy = Kinetic Rotational

$$E_2 = \frac{1}{2}I_T\omega_o^2 = \frac{1}{2R^2}I_T V_1^2$$

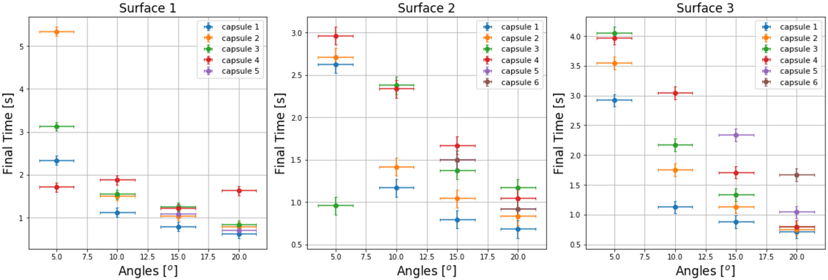
Experimental Set-up

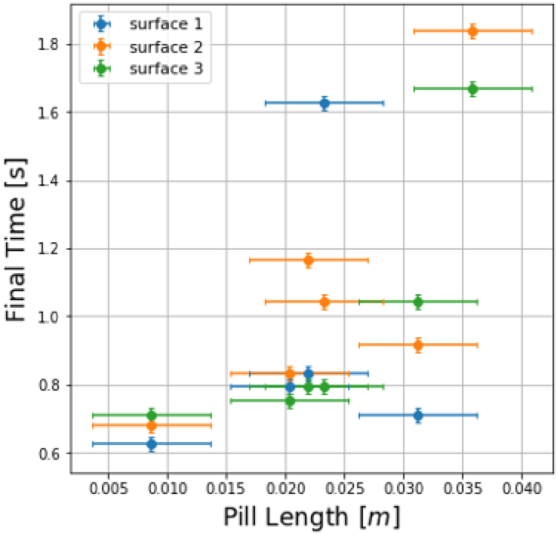


Results



Error * 5

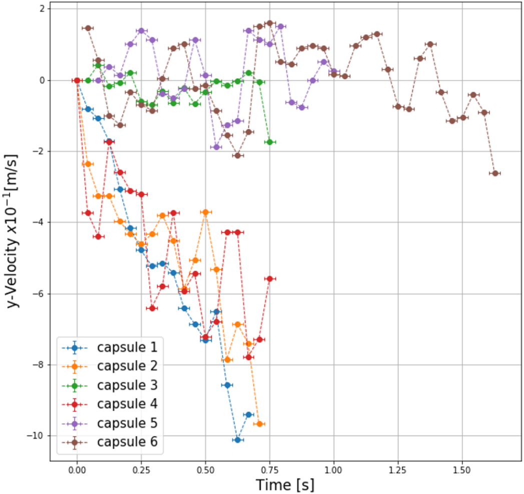

Surface 1
Surface 2
Surface 3
Final Time [s]
Angles [^{o}]
capsule 1
capsule 2
capsule 3
capsule 4
capsule 5
capsule 6



surface 1
surface 2
surface 3
1.8
1.6
1.4
1.2
1.0
0.8
0.6
Final Time [s]
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
Pill Length [m]

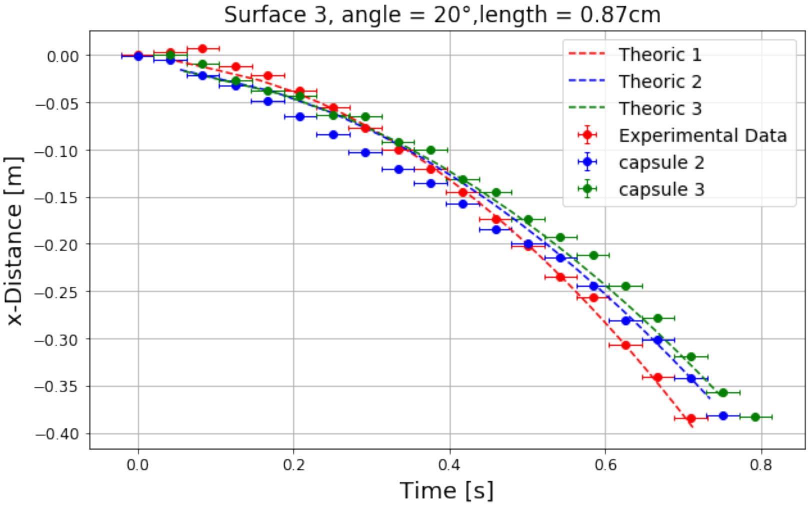
Velocity

Surface 3 – Angle 20°



Shorter pills, have faster speed and therefore arrive in less time.

Velocity



Surface 3, angle = 20°,length = 0.87cm
x-Distance [m]
Time [s]
0.00
−0.05
−0.10
−0.15
−0.20
−0.25
−0.30
−0.35
−0.40
0.0
0.2
0.4
0.6
0.8
Theoric 1
Theoric 2
Theoric 3
Experimental Data
capsule 2
capsule 3