Structured Ring Spectra

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1 Introduction

In higher Homotopy Theory, we can regard the spectra as the analogous of abelian groups. The mapping spaces in Spectra are abelian groups, so Sp is an abelian category. To go on with our analogy, we would like to develope a theory of Ring Spectra. A priori, one could think that homotopy ring spectra are a candidate for this.

Definition 1.1. A *homotopy ring spectrum* is a spectrum E together with two maps $\mu\colon E \otimes E \to E$ and $\eta\colon \mathbf{S} \to E$, such that E becomes a monoid object in $h\mathrm{Sp}$. This is, the following diagrams commute up to homotopy

$$\begin{tikzcd}
E \otimes E \otimes E \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"] & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[dr, "\mathrm{id}"'] & E \otimes E \arrow[d, "\mu"] & E \arrow[l, "\eta\otimes\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\
& E &
\end{tikzcd}$$

Nevertheless, we do not want associativity and unitality up to homotopy, but up to coherent homotopy. For that we need to use strong machinery.

Remark 1.2.
1. Recall that for $0 \leq k \leq \infty$, tE_k denotes the topological k-little disks operad, and that $E_k^{\otimes}$ is the k-little disks ∞-operad.
2. Moreover, $E_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$ and $E_\infty^{\otimes} \simeq \mathrm{Comm}^{\otimes}$.
3. There is a sequence $E_0^{\otimes} \to E_1^{\otimes} \to E_2^{\otimes} \to \ldots \to E_\infty^{\otimes}$.
4. The underlying ∞-categoryof $E_k^{\otimes}$ is contractible.

2 Definitions

Definition 2.1. Let $0 \leq k \leq \infty$. An E_k-ring spectra is an E_k-algebra object in Sp. We denote by Alg^k the ∞-categoty $\mathrm{Alg}_{E_k}(\mathrm{Sp})$ of E_k-rings.

Remark 2.2. An algebra object in Sp is a functor $\alpha\colon E_k \to \mathrm{Sp}$ such that

$$\begin{tikzcd} E_k \arrow[rr, "\alpha"] \arrow[dr] & & \mathrm{Sp} \arrow[dl] \\ & N(\mathrm{Fin}_*) & \end{tikzcd}$$

commutes and it takes inert morphisms in E_k to inert morphisms in Sp.

Remark 2.3. There are cannonical maps $E_0 \to E_1 \to E_2 \to \ldots \to E_\infty$. So if R is an E_k-ring, then it can be cannonically regarded as an $E_{k'}$-ring for all $k' \leq k$.

Remark 2.4. There is a forgetful functor $U\colon \mathrm{Alg}^k \to \mathrm{Sp}$, defined by evaluating in the color.

We unfold the definition for $k = 1$ and $k = \infty$.

2.1 Case $k = 1$

We know that $E_1^{\otimes} = \mathrm{Assoc}^{\otimes}$.

Classical Assoc: Recall that **Assoc** (as a colored operad) is defined as follows:

1. **Assoc** has a single object $\mathfrak{a}$.
2. For every finite set I, $\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathfrak{a}_{i\ \in I}, \mathfrak{a})$ is the set of linear orderings on I.

Then, if $\mathcal{C}$ is a symmetric monoidal category and $F\colon \mathrm{Assoc} \to \mathcal{C}$ is a map of operads, $F(\mathfrak{a})$ is an object A in $\mathcal{C}$. Given a linear ordering of a finite set I, the corresponding element of $\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathfrak{a}_{i\ \in I}, \mathfrak{a})$ determines a map

$$A^{\otimes I} \to A$$

in $\mathcal{C}$. If $I = \phi$, we have a map $\mathbf{1} \to A$ and if $I = \{1 < 2\}$ we obtain $m\colon A \otimes A \to A$, that defines an associative multiplication.

Higher Assoc:

Definition 2.5. $\mathrm{Assoc}^{\otimes} = N(\mathbf{Assoc}^{\otimes})$

Remark 2.6.

1. Objects of $\mathbf{Assoc}^{\otimes}$ = Objects of $N(\mathrm{Fin}_*)$.
2. If $\langle n\rangle, \langle m\rangle$ in Fin_*, a morphism $\langle m\rangle \to \langle n\rangle$ in $\mathbf{Assoc}^{\otimes}$ is a pair $(\alpha, \{\alpha_i\}_{1\leq i\leq n})$ where $\alpha\colon \langle m\rangle \to \langle n\rangle$ is a map of pointed finite sets, and α_i is a linear ordering of $\alpha^{-1}(i)$.
3. The underlying ∞-category of $\mathrm{Assoc}^{\otimes}$ is equivalent to a point.

Remark 2.7. Let $\mathcal{C}$ be a monoidal ∞-category. Evalutation on the object $\mathfrak{a}$ determines a forgetful functor

$$\theta\colon \mathrm{Alg}_{\mathrm{Assoc}^{\otimes}}(\mathcal{C}) \to \mathcal{C}.$$

So we denote an algebra A by $\theta(A) \in \mathcal{C}$. Again, given an ordering of $\{1, \ldots, n\}$, the active morphism $\{a\}_{1\leq i\leq n} \to \mathfrak{a}$ in $\mathrm{Assoc}^{\otimes}$ induces

$$A^{\otimes n} \to A$$

in $\mathcal{C}$. So taking $\{1, 2\}$ we obtain a morphism $m\colon A \otimes A \to A$.

2.2 Case $k = \infty$

We know that $E_\infty^\otimes = N(\mathrm{Fin}_*)$. The discussion is the same as before. But this time there is no distinctions between the orderings of finite sets. Namely, all the pairs $(\alpha, \{\alpha_i\}_{1\leq i\leq n})$ are equivalent if they share the first coordinate.

2.3 Case $k = 0$

A few lectures ago, it was stated that $E_0^\otimes = N(\mathrm{Fin}_*^{\mathrm{inj}}) \hookrightarrow N(\mathrm{Fin}_*)$.

Before going to the examples, we are going to state a result that argues roughly that, in some cases, all the cases $k = 2, 3, \ldots, \infty$ are equivalent at the level of algebras.

Proposition 2.8. Let $k \geq 0$. For every pair of integers $m, n \geq 0$, the map of topological spaces

$$\mathrm{Map}_{tE_k^\otimes}(\langle m\rangle, \langle n\rangle) \to \mathrm{Hom}_{\mathrm{Fin}_*}(\langle m\rangle, \langle n\rangle)$$

is $(k-1)$-connective.

Corollary 2.9. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Let $n \geq 1$, and asume that the underlying ∞-category $\mathcal{C}$ is equivalent to an n-category. Then the map $E_k^\otimes \to N(\mathrm{Fin}_*)$ induces an equivalence of ∞-categories

$$\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}_{E_k}(\mathcal{C})$$

for $k > n$.

Proof. C, D objets in $C^\otimes$ corresponding to sequences $(X_1, \ldots, X_m), (Y_1, \ldots, Y_{m'})$ of objects in $\mathcal{C}$.

$$\mathrm{Map}_{\mathcal{C}^\otimes}(C, D) \simeq \coprod_{\alpha\langle m\rangle \to \langle n\rangle} \prod_{1\leq j\leq m'} \mathrm{Map}_{\mathcal{C}}(\bigotimes_{\alpha(i)=j} X_i, Y_j)$$

Then $\mathrm{Map}_{\mathcal{C}^\otimes}(C, D)$ is $(n-1)$-truncated. So $C^\otimes$ is equivalent to an n-category.

By the above proposition, if $k > n$, $E_k^\otimes \to N(\mathrm{Fin}_*)$ induces a homotopy equivalence on the n-truncations. Therefore we have that

$$\theta\colon \mathrm{Fun}_{N(\mathrm{Fin}_*)}(N(\mathrm{Fin}_*), \mathcal{C}^\otimes) \to \mathrm{Fun}_{N(\mathrm{Fin}_*)}(E_k^\otimes, \mathcal{C}^\otimes)$$

is an equivalence after n-truncation, but they are already n-categories, because so is $C^\otimes$. □

3 Examples

3.1 Sphere Spectrum

We would like to define the sphere ring spectrum as the sphere spectrum $\mathbf{S}$ equipped with the maps

$$\mathbf{S} \otimes \mathbf{S} \xrightarrow{\mu_2} \mathbf{S} \qquad \mathbf{S} \xrightarrow{\mathrm{Id}} \mathbf{S},$$

where μ_2 is the equivalence given by the symmetric monoidal structure. But we cannot do it in this way.

Proposition 3.1. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Then $\mathrm{CAlg}(\mathcal{C})$ has an initial object. Moreover a commutative algebra object A of $\mathcal{C}$ is initial in $\mathrm{CAlg}(\mathcal{C})$ if and only if the map $1_{\mathcal{C}} \to A$ is an equivalence in $\mathcal{C}$.

Proof. ([**HA**],Corollary 3.2.1.9) □

We argue in this case that there is an initial object on Calg(Sp) and we define it to be the sphere ring spectrum, that we denote by $A_{\mathbf{S}}$.

Because of Proposition 3.1.3.13 in [**HA**], the forgetful functor $U\colon \mathrm{Calg(Sp)} \to \mathrm{Sp}$ defined by evaluating an algebra $A\colon N(\mathrm{Fin}_*) \to \mathrm{Sp}$ in $\langle 1\rangle$ has a left adjoint $\mathrm{F} = \mathrm{Free}\colon \mathrm{Sp} \to \mathrm{CAlg(Sp)}$. This is the *free algebra functor*. Take the *zero spectrum* (0), that can be represented by one point in every coordinate. (0) is initial in Sp. As $F\colon \mathrm{Sp} \to \mathrm{CAlg}(\mathrm{Sp})$ is left adjoint, it preserves colimits. So it sends initial objects to initial objects. Therefore, we call $A_{\mathbf{S}} := \mathrm{Free}(0)$ the initial object, the sphere spectrum.

By Example 3.1.3.14 in [**HA**],

$$U \circ \mathrm{Free}(X) \simeq \coprod_{n \geq 0} \mathrm{Sym}^n(X) \simeq \bigvee_{n \geq 0} (X^{\otimes n})_{h\Sigma_n}$$

Taking $X = (0)$, for $n \geq 1$ $X^{\otimes n} = 0$ and for $n = 0$, $X^{\otimes}0 = \mathbf{S}$ and Σ_0 is trivial, so $U \circ \mathrm{Free}((0)) \simeq \mathbf{S}$.

How does this map look like?

As $A_{\mathbf{S}}$ sends $\langle 1 \rangle$ to $\mathbf{S}$, the Segal condition gives us that $\langle n \rangle$ goes to $\mathbf{S}^{\otimes n}$, and the active maps $\langle n \rangle \to \langle 1 \rangle$ go to

$$\mu_n \colon \mathbf{S}^{\otimes n} \to \mathbf{S}.$$

For $n = 1$ we have the identity and for $n = 2$ we have the isomorphism given by the symmetric monoidal structure.

3.2 Spherical monoidal rings

Recall that $\Sigma^\infty \colon \mathrm{An}^\times \to \mathrm{Sp}^\otimes$ is strongly monoidal and $\mathrm{CAlg}(\mathrm{An}^\times) \simeq \mathrm{CMod}(\mathrm{An})$, then Σ^∞ upgrades to a functor

$$\Sigma^\infty \colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CAlg}$$

Start with a monoid M in the category An. This is E_1-algebra in the category of anima, i.e. map $M\colon \mathrm{Assoc}^\otimes \to \mathrm{An}$ satisfying the Segal condition. We consider on An and An_* the cartesian monoidal structure with $\times$. The functor $()_+ \colon \mathrm{An} \to \mathrm{An}_*$ is symmetric monoidal. And it was proven that there is an essentially unique symmetric monoidal structure on Sp such that the suspension functor

$$\Sigma^\infty \colon \mathrm{An}_* \to \mathrm{Sp}$$

is symmetric monoidal.

If we compose, we obtain

$$\mathrm{Ass}^\otimes \xrightarrow{M} \mathrm{An} \xrightarrow{(-)_+} \mathrm{An}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}.$$

And this is an E_1-ring spectrum.

When we evaluate this ring spectrum in the point of the underlying ∞-category of $\mathrm{Assoc}^\otimes$, we obtain the suspension spectrum Σ^∞_+, and when we evaluate on the order $\{1 < 2\}$ we obtain a map

$$\Sigma^\infty_+ M \otimes \Sigma^\infty_+ M \xrightarrow{\simeq} \Sigma^\infty_+ M.$$

4 Left-modules and algebras over ring spectra

We construct an operad $\mathcal{LM}$. We define two colors: $\{\mathfrak{a}, \mathfrak{m}\}$. And given a finite sequence of colors $\{X_i\}_{i \in I}$, we define

$$\mathrm{Mul}(\{X_i\}_{1 \leq i \leq n}, \mathfrak{a}) = \begin{cases} \text{linear orderings of } I & \text{if } X_i = \mathfrak{a} \text{ for all } i, \\ \phi & \text{otherwise} \end{cases}$$

$$\mathrm{Mul}(\{X_i\}_{1 \leq i \leq n}, \mathfrak{m}) = \text{linear orderings } \{i_1 < \ldots < i_n\} \text{ on } I \text{ such that } X_{i_n} = \mathfrak{m} \text{ and } X_{i_j} = \mathfrak{a} \text{ for all } j < n.$$

Remark 4.1. The subcolored operad with the color $\mathfrak{a}$ is equivalent to Assoc.

Definition 4.2. We define the ∞-operad $\mathcal{LM} := N(\mathcal{LM}^\otimes)$. Observe that that its underlying ∞-category is equivalent to $\Delta^0 \sqcup \Delta^0 = \{\mathfrak{a}, \mathfrak{m}\}$.

Definition 4.3. Let R be an algebra object R in $\mathrm{Alg}(\mathcal{C})$, a *left R-module in $\mathcal{C}$* is a map of ∞-operads $M\colon \mathcal{LM}^\otimes \to \mathcal{C}^\otimes$ such that $M|_{\mathrm{Assoc}^\otimes} = R$. We usually refer to the object $M(\mathfrak{m})$ as the module.

We denote the ∞-category of R-module objects in Sp by LMod_R.

Proposition 4.4. If R is an E_{k+1} ring (with $k \geq 1$), then LMod_R can be regarded as a E_k-monoidal category.

Proof. (Idea) There is an equivalence $\mathrm{Alg}_{E_n}(\mathcal{C}) \simeq \mathrm{Alg}_{E_1}(\mathrm{Alg}_{E_{n-1}}(\mathcal{C}))$ $\square$

Theorem 4.5. If R is a E_1-ring, then LMod_R is a stable ∞-category.

Proposition 4.6. Let R be an E_1-connective ring. Then

1. $(\mathrm{LMod}_R^{\geq 0}, \mathrm{LMod}^{\leq 0})_R$ determine a t-structure on LMod_R,
2. this structure is compatible with the E_k-monoidal structure, and
3. the functor π_0 determines an equivalence $\mathrm{LMod}_R^{\heartsuit} \to N(\text{discrete } \pi_0 R\text{-modules})$.

Proof. ([**HA**], Lemma 7.1.3.10) $\square$

Definition 4.7. Let $k \geq 0$, and let R in Alg^k be an E_{k+1}-ring. We denote Alg_R^k the ∞-category $\mathrm{Alg}_{E_k}(\mathrm{LMod}_R)$ of *E_k-algebra objects over R.*

Remark 4.8. The forgetful functor $\mathrm{Alg}_{\mathbf{S}}^k \to \mathrm{Alg}^k$ is an equivalence.

Definition 4.9. Let $0 \leq k \leq \infty$. Let R be a connective E_{k+1}-ring. An algebra A in Alg_R^k is *discrete* if it is connective and 0-truncated. We denote the ∞-category of discrete R-algebras as $\mathrm{Alg}_R^{k,\mathrm{disc}}$.

Proposition 4.10. (Proposition 7.1.3.18, HA) Let $1 \leq k \leq \infty$ and R be a connective E_{k+1}-ring.

- If $k = 1$, the construction $A \to \pi_0 A$ induces an equivalence

$$\mathrm{Alg}_R^{k,\mathrm{disc}} \longrightarrow N(\text{Discrete associative algebras over } \pi_0 R)$$

- If $k \geq 2$, $A \to \pi_0 A$ induces an equivalence

$$\mathrm{Alg}_R^{k,\mathrm{disc}} \longrightarrow N(\text{discrete commutative algebras over } \pi_0 R)$$

Proof. First of all, we prove that there is an equivalence

$$\mathrm{Alg}_R^{k,\mathrm{disc}} \simeq \{E_k\text{-algebra objects of } \mathrm{LMod}_R^{\heartsuit}\}$$

From the fact that the t-structure on LMod_R is compatible with the E_k-monoidal structure, it is deduced that the localization functor

$$\tau_{\leq n} \colon \mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LMod}_R^{\mathrm{cn}}$$

is also compatible with the E_k-structure. Therefore (Prop 2.2.1.9 [**HA**]) we have an E_k-monoidal structure on $\mathrm{LMod}_R^{\mathrm{cn}} \cap (\mathrm{LMod}_R)_{\leq n}$ and an identification

$$\mathrm{Alg}_{E_k}(\mathrm{LMod}_R^{\mathrm{cn}} \cap (\mathrm{LMod}_R)_{\leq n}) \simeq \tau_{\leq n}^k \mathrm{Alg}_R^{(k),\mathrm{cn}}.$$

Taking $n = 0$, we gen the result.

- For $k = 1$, the proposition 4.6 implies that $\mathrm{Alg}_R^{1,\mathrm{disc}}$ is equivalent to the ∞-category of associative objects of $N(\text{discrete } \pi_0\text{-modules})$, and therefore to $N(\text{Discrete associative algebras over } \pi_0 R)$.
- For $k \geq 2$, the the proposition 4.6 implies that $\mathrm{Alg}_R^{k,\mathrm{disc}}$ is equivalent to the ∞-category of E_k-algebra objects of $N(\text{discrete } \pi_0\text{-modules})$, but according to corollary 2.9 this is precisely $N(\text{disc. comm. algebras over } \pi_0 R)$.

$\square$

Definition 4.11. Taking in the above proposition $R = \mathbf{S}$, $\pi_0 \mathbf{S} = \mathbb{Z}$, so we have a correspondence

$$\mathrm{Alg}^{1,\mathrm{disc}} \simeq \mathrm{Alg}_{\mathbf{S}}^{1,\mathrm{disc}} \simeq N(\text{discrete associative algebras over } \mathbb{Z}) \simeq N(\text{discrete rings})$$

So, taking a discrete ring R, we get a correspontent E_1-algebra HR, called *Eilenberg-Mc Lane Ring Spectrum.*

5 Ring structure on $\pi_* R$

Let R be an E_1-ring spectra. Recall that for any $n \in \mathbb{Z}$

$$\pi_n R := \pi_0 \mathrm{Map}_{Spc}(\mathbf{S}[n], R)$$

Then we have maps

$$\mathrm{Map}_{Spc}(\mathbf{S}[n], R) \times \mathrm{Map}_{Spc}(\mathbf{S}[m], R) \to \mathrm{Map}_{Spc}(\mathbf{S}[n] \otimes \mathbf{S}[m], R \otimes R) \to \mathrm{Map}_{Spc}(\mathbf{S}[m+n], R)$$

$$\pi_n R \times \pi_m R \to \pi_{m+n} R$$

This a structure of graded associative ring on $\pi_* R = \bigoplus_n \pi_n R$.

We have that $\pi_0 R$ is an associative ring and $\pi_n R$ is a $\pi_0 R$-bimodule.

6 Localizations of ring spectra

Definition 6.1. Let R be an associative ring, and $S \subset R$. We say that S satisfies the *Ore conditions* if

1. $1_R \in S$ and S is closed under multiplication.
2. For every $x \in R$ and $s \in S$, exists $y \in R$ and $t \in S$, such that $tx = ys$.
3. For for some $x \in X$ and $s \in S$ we have $xs = 0$, then exists $t \in S$ such that $tx = 0$.

Remark 6.2. The Ore conditions are the classical conditions that we need to define a localization in a ring in the non-commutative context.

Definition 6.3. Let R be an E_1-ring and $S \subset \pi_* R$ set of homogeneus elements satisfying the Ore condition. And let M be a left R-module.

1. M is *S-nilpotent* if for all $x \in \pi_* M$ exists some $s \in S$ such taht $sx = 0$.
2. M is *S-local* if for all $s \in S$ the map
$$\pi_* M \xrightarrow{\cdot s} \pi_* M$$
is an isomorphism of graded abelian groups.
3. We denote $\mathrm{LMod}^{S\text{-nil}}$ the full subcategory of LMod_R spanned by the S-nilpotent left R-modules
4. We denote $\mathrm{LMod}^{\mathrm{loc}(S)}$ the full subcategory of LMod_R spanned by the S-local left R-modules

Proposition 6.4. Let R be an E_1-ring and $S \subset \pi_* R$ a homogeneous subset veryfing the Ore conditions.

1. Then $\mathrm{LMod}_R^{S\text{-nil}}$ and $\mathrm{LMod}_R^{\mathrm{loc}(S)}$ are stable subcategories of LMod_R.
2. $(\mathrm{LMod}_R^{S\text{-nil}}, \mathrm{LMod}_R^{\mathrm{loc}(S)})$ form an accesible t-structure on LMod_R with trivial heart.

Corollary 6.5. Every object $M \in \mathrm{LMod}_R$ determines a fibre sequence
$$M' \to M \xrightarrow{\phi} M''$$
up to contractible ambiguity, where M' is S-nilpotent and M'' is S-local. We denote $S^{-1}M := M''$.

Remark 6.6. It can be proved that $S^{-1}M$ admits the structure of a E_1-ring.

Now we are going to prove a proposition that is the analogous to the universal property of localization in this higher context.

Proposition 6.7. (Proposition 7.2.3.27 HA) Let R be an E_1-ring. Let $S \subset \pi_* R$ be a set of homogeneous elements satisfying the Ore condition. Then there exists a map $\phi\colon R \to R[S^{-1}]$ such that, given an E_1-ring A, composition with φ induces a fully faithful map of Kan complexes
$$\theta\colon \mathrm{Map}_{\mathrm{Alg}^1}(R[S^{-1}], A) \to \mathrm{Map}_{\mathrm{Alg}^1}(R, A),$$
whose essential image is the collection of E_1-rings $\psi\colon R \to A$ such that $\psi(s)$ is invertible in $\pi_* A$ for each $s \in S$.

Proof. (Sketch)

Definition 6.8. Given a E_1-ring R, denote the E_1-ring classifying endomorphisms of $S^{-1}R$ in $\mathrm{LMod}_R^{\mathrm{loc}(S)}$. And there is a canonical map $R \to R[S^{-1}]$

Remark 6.9. It can be shown that $S^{-1}R$ is a compact generator for the stable ∞-category $\mathrm{LMod}_R^{\mathrm{loc}(S)}$. So by Schwede-Shipley Theorem, there is an equivalence of categories $\mathrm{LMod}_R^{\mathrm{loc}} \to \mathrm{LMod}_{R[S^{-1}]}$ that sends $S^{-1}R$ to $R[S^{-1}]$.

1. θ is fully faithful: Let $F\colon \mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{loc}(S)}$ the localization functor (defined to be the left adjoint to the inclusion). Then we have a fully faithful embedding

$$\mathrm{LFun}(\mathrm{LMod}_R^{\mathrm{loc}(S)}, \mathrm{LMod}_A) \to \mathrm{LFun}(\mathrm{LMod}_R, \mathrm{LMod}_A)$$

   (the L means that we are taking the full subcategory of functors that preserve small colimits). Using the equivalence $\mathrm{LMod}_R^{\mathrm{loc}} \to \mathrm{LMod}_{R[S^{-1}]}$ and some formal properties, we have the claim.

2. Essential image: A map $\psi\colon R \to A$ is in the essential image if and only if for every map of R-modules $M \to M$ inducing an equivalence $S^{-1}M \to S^{-1}M$, the induced map $M \otimes A \to M \otimes A$ is an equivalence. This is equivalent (by passing to the fibers) to: for every S-nilpotent R-module M, we have $A \otimes_R M = 0$.

   Fact: The collection of S-nilpotent R-module is generated under colimits by R-modules of the form $(R/Rs)[n]$, with s homogeneous. Where R/Rs is the cofiber of the map $R[d] \to R$ given by s ($d = \deg(s)$).

   But $A \otimes_R M$ is the cofiber of the map $A[d] \xrightarrow{\cdot s} A$. So $A \otimes_R M \simeq 0$ if and only if $\psi(s) \in \pi_d A$ is invertible in $\pi_* A$

□

7 Completions of ring spectra

Definition 7.1. Take an E_2-ring R. Take an ideal $I \subset \pi_0 R$. Take a $M \in \mathrm{LMod}_R$, we say that

1. M is I-nilpotent if for every $x \in I$, $C[x^{-1}]$ vanishes.
2. M is I-local if for every I-nilpotent N, $\mathrm{Map}(M, N)$ is I-contractible.
3. M is I-complete if for every I-local N, $\mathrm{Map}(M, N)$ is I-local.

Proposition 7.2. Let $R \in \mathrm{Alg}^2$ and $I \subset \pi_0 R$ be a finitely generated ideal. Then $(\mathrm{LMod}_R^{\mathrm{loc}(I)}, \mathrm{LMod}_R^{\mathrm{Cp}(I)})$ determine a t-structure on LMod_R. In particular, for every object $C \in \mathrm{LMod}_R$, there is a unique fiber sequence

$$C' \to C \to C'',$$

were C is I-local and C'' is I-complete.

Definition 7.3. Define the I-completion of C as C''. Then the inclusion $\mathrm{LMod}_R^{\mathrm{Cp}(I)} \hookrightarrow \mathrm{LMod}_R$ admits a left adjoint, and we call it the I-completion functor

$$C \mapsto C_{\hat{I}}.$$

Theorem 7.4. Let R be an E_2-ring and $I \subset \pi_0 R$ a finitely generated ideal. Then the following conditions are equivalent:

1. The left R-module M is I-complete
2. For every integer k, the homotopy group $\pi_k M$ is I-complete, when regarded as a discrete module over the commutative ring $\pi_0 R$.

Proposition 7.5. Let R be an E_2-ring, and let $I \subset \pi_0 R$ a finitely generated ideal, and let $\alpha\colon M \to M'$ an equivalence of R-modules inducing an equivalence $M_{\hat{I}} \simeq M'_{\hat{I}}$. Then for any left R-module N, the induced maps

$$(M \otimes N)_{\hat{I}} \to (M' \otimes N)_{\hat{I}} \quad (N \otimes M)_{\hat{I}} \to (N \otimes M')_{\hat{I}}$$

are equivalences.

Proof. A map α induces an equivalence of I-completion if and only if its fiber is local. Then we just need to prove that if either P or Q are local, then $P \otimes_R Q$ is local.

Fact: The full subcategory $\mathcal{C}^{\mathrm{Loc}} \subset \mathcal{C}$ is closed under small colimits.

Assume P is I-local. Then the collection of those left R-modules K such that $P \otimes_R K$ is I-local contains R and it is closed under small colimits and desuspensions, and therefore it contains all left R-modules. □

Corollary 7.6. (Corollary 7.3.5.2, SAG) Let R be an E_2 ring and let $I \subset \pi_0 R$ be a finitely generated ideal. Then there is an essentially unique monoidal structure on the ∞-category $\mathrm{LMod}_R^{\mathrm{Cp(I)}}$, for which the I-completion functor $M \mapsto M_{\hat{I}}$ is monoidal.