The Balmer spectrum of the category of perfect complexes

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Contents

0 Background

Definition 0.1. In a Noetherian topological space the family of *constructible sets* is the smallest family of sets such that

1. every open set is in the family
2. it is closed by finite intersections
3. it is closed by complements

Proposition 0.2. A constructible set in a variety X is a finite union of locally closed subspaces. Where locally closed means that every point has a neighborhood

Proposition 0.3. In a qsqc scheme the complement of a quasi-compact open set is constructible.

Construction of Künneth spectral sequence
Absolute Noetherian approximation

Example 0.4.
1. boundedness is not preserved by quasi-isomorphism
2. coherence is not preserved by quasi-isomorphism

Example 0.5. There is an important example that we will be using. The derived category of $\mathbb{Z}/(4)$. It is isomorphic to the homotopy category of the full subcategory of $\mathbf{Ch}(\mathbb{Z}/(4))$ whose objects are chains of the form

$$\ldots \xrightarrow{\cdot 2} \mathbb{Z}/(4) \xrightarrow{\cdot 2} \mathbb{Z}/(4) \xrightarrow{\cdot 2} \mathbb{Z}/(4) \to 0,$$

bounded or not. Note that if the complex is unbounded, it is quasi-isomorphic

$$\ldots \to 0 \to \mathbb{Z}/(2) \to 0.$$

by the quasi-isomorphism

$$\begin{tikzcd} \ldots \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}/4 \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[d, "\mathrm{mod}2"] & 0 \\ \ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0. \end{tikzcd}$$

To remark the importance of this embedding, in this category we have the inverse quasi-isomorphism

$$\begin{tikzcd} \ldots \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r, "\cdot 2"] & \mathbb{Z}/4 \arrow[r] & 0 \\ \ldots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \mathbb{Z}/2 \arrow[r] \arrow[u, "\cdot 2"'] & 0. \end{tikzcd}$$

This does not happen, for example, for $\mathbb{Z}$. Consider the analogous chain complex quasi-isomorphism

$$\begin{tikzcd} \ldots \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{mod}2"] & 0 \\ \ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0. \end{tikzcd}$$

It is not possible that it is invertible in the category, because there is no map $\mathbb{Z}/2 \to \mathbb{Z}$. This fact is not a coincidence, it can be guessed that it depends on whether the ring R is R-injective or not (The fact that $2\mathbb{Z} \hookrightarrow \mathbb{Z}$ cannot be extended is crucial).

Example 0.6. ([2]) If k is a field, the following functor is an equivalence of categories

$$\begin{array}{rcc} \mathcal{D}(k) & \to & \{\mathbb{Z} - \text{graded } k - \text{vector spaces}\} \\ M^\bullet & \mapsto & \prod_{k=-\infty}^{k=\infty} H^k(M) \end{array}$$

1 Talk 1: Juan Martín Fajardo

1.1 Scheme-theoretical definitions

Definition 1.1. A morphism $f\colon X \to Y$ is called *quasi-separated* if the diagonal morphism is quasi-compact. A scheme X is called *quasi-separated* if the canonical morphism $X \to \mathbb{Z}$ is quasi-separated.

Proposition 1.2. TFAE:

a) The scheme X is quasi-separated

b) The intersection of two open quasi-compact sets is quasi-compact.

c) The intersection of any two affine open subsets in X is a finite union of affine subsets.

The property b) is the famous property that Hausdorff spaces verify.

Remark 1.3. Remember that a scheme $X \to \mathbb{Z}$ is called separated if it verifies one of the following equivalent conditions

a') The diagonal morphism $X \to X \times_{\mathrm{Spec}(\mathbb{Z})} X$ is a closed immersion.

b') The intersection of any two open affine subsets is affine.

Remark 1.4. We will work with quasi-compact quasi-separated schemes (qcqs schemes). Every locally Noetherian scheme is quasi-separated. Therefore the category of Noetherian schemes fits our purposes.

Remark 1.5. The bare minimum we will work with is qcqs, because we will compare the scheme X with the Balmer spectrum of a tensor triangulated category, and the later one is always qcqs (Corollary 2.15, Remark 2.7 and Proposition 2.14 in [1])

Definition 1.6. A family of line bundles $\mathcal{L}_\alpha$ on an qcqs scheme X is an *ample family of line bundles* if it satisfies the following conditions (that are equivalent according to ([SGA 6] II 2.2.3):

1. There is an affine cover of X by sets of the form X_f with $f \in \Gamma(X, \mathcal{L}_\alpha^{\otimes n})$.

Definition 1.7. A qcqs scheme X is *divisorial* if it admits an ample family of line bundles.

Remark 1.8. According to 1.6(1) and the fact that X is qc we can get that the ample family can be chosen to be finite.

Remark 1.9. A single line bundle $\mathcal{L}$ determines an ample family of line bundles $\{\mathcal{L}_\alpha\}$. So any scheme admitting an ample line bundle is divisorial. In particular any quasi-projective scheme over an affine is divisorial.

Moreover, any separated regular Noetherian scheme is divisorial, according to ([SGA 6] II 2.2.7.1,).

Definition 1.10. Let X be a scheme, we define its *derived category* (denoted $\mathcal{D}(X)$) as the derived category of the abelian category of complexes of $\mathcal{O}_X-$modules. Namely

$$\mathcal{D}(X) := \mathcal{D}(\mathrm{Ch}(\mathcal{O}_X - \mathrm{mod}))$$

Remark 1.11. The derived category $\mathcal{D}(X)$ has a canonical triangulated structure. The triangles are isomorphic to the triangles of the form

$$E \xrightarrow{f} F \to \mathrm{cone}(f) \to E[1],$$

where $\mathrm{cone}(f)$ is defined as the sequence of modules $E[1] \oplus F$ and the differential is

$$d_{\mathrm{cone}(f)} = \begin{pmatrix} d_{E[1]} & 0 \\ f[1] & d_F \end{pmatrix}.$$

We can consider the analogous $\mathcal{D}^b(X), \mathcal{D}^+(X)$ and $\mathcal{D}^-(X)$. We have a tensor product in $\mathcal{D}(X)$, namely, the derived tensor product $- \otimes^L -\colon \mathcal{D}(X) \times \mathcal{D}(X) \to \mathcal{D}(X)$. With this structure, $\mathcal{D}(X)$ is a tensor triangulated category. Nevertheless, it is too big.

Lemma 1.12. Mayer-Vietoris for schemes, Lemma 3.5 [5]. Let X be a scheme. Let $i\colon U \hookrightarrow X$, $j\colon V \hookrightarrow X$ and $k\colon U \cap V \hookrightarrow X$ be open immersions. Then, for $E^\bullet, F^\bullet \in D(U \cup V)$ there is the following long exact sequence

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{D}(U\cup V)}(E,F) \arrow[r] & \mathrm{Mor}_{\mathcal{D}(U)}(i^*E, i^*F) \oplus \mathrm{Mor}_{\mathcal{D}(V)}(j^*E, j^*F) \arrow[r] & \mathrm{Mor}_{\mathcal{D}(V\cap V)}(k^*E, k^*F) \arrow[dll] \\ \mathrm{Mor}_{\mathcal{D}(U\cup V)}(\Sigma^{-1}E,F) \arrow[r] & \mathrm{Mor}_{\mathcal{D}(U)}(\Sigma^{-1}i^*E, F) \oplus \mathrm{Mor}_{\mathcal{D}(V)}(\Sigma^{-1}j^*E, F) \arrow[r] & \mathrm{Mor}_{\mathcal{D}(V\cap V)}(\Sigma^{-1}k^*E, F) \end{tikzcd}$$

Proof. If $l\colon P \hookrightarrow X$ is an open immersion, we define the functor *extension by 0* as $i_!\colon \mathcal{O}_P-\mathrm{Mod} \to \mathcal{O}_X-\mathrm{Mod}$. It sends a sheaf $\mathcal{F} \in \mathcal{O}_P - \mathrm{Mod}$ to the sheaf $i_! \in \mathcal{O}_X - \mathrm{Mod}$ defined by

$$W \subset X \mapsto \begin{cases} F(W) \text{ if } W \subset P \\ 0 \text{ otherwise} \end{cases}$$

Applying $l_!$ degree-wise, we can extend it to a functor

$$l_!\colon \mathrm{Ch}(\mathcal{O}_P - \mathrm{Mod}) \to \mathrm{Ch}(\mathcal{O}_X - \mathrm{Mod})$$

$l_!$ and l^* are adjoint on sheaves and exact, so they preserve quasi-isomorphism of complexes. So They can be even extended as adjoint in the derived category. And the adjunction

$$l_!\colon \mathcal{D}(P) \rightleftarrows \mathcal{D}(X)\colon l^*$$

leads to the isomorphism

$$\mathrm{Mor}_{\mathcal{D}(X)}(l_!l^*E, F) \cong \mathrm{Mor}_{\mathcal{D}(P)}(l^*E, l^*F) \tag{1}$$

for any $E, F \in \mathcal{D}(X)$. Note that

$$(l_!l^*E)_x = \begin{cases} E_x \text{ if } x \in P \\ 0 \text{ if } x \notin P. \end{cases}$$

Now we can prove the result. Assume without loss of generality that $X = U \cup V$. Now consider in the short exact sequence in $\mathrm{Ch}(X)$

$$0 \to k_!k^*E \to i_!i^*E \oplus j_!j^*E \to E \to 0 \tag{2}$$

defined stalk-wise

$$\begin{array}{ccccccccc} 0 & \to & (k_!k^*E)_x & \to & (i_!i^*E)_x \oplus (j_!j^*E)_x & \to & E_x & \to & 0 \\ & & f_x & \mapsto & (f_x, -f_x) & & & & \\ & & & & (f_x, g_x) & \mapsto & f_x - g_x. & & \end{array}$$

The exactness is checked stalk-wise considering the cases $x \in U \cap V$, $x \in U \setminus V$ and $x \in V \setminus U$. As 2 is a short exact sequence, it becomes a triangle in the derived category. As $\mathrm{Mor}(-, F)$ is a cohomological functor, it induces a long exact sequence. Applying the adjunctions we get the desired

□

1.2 Perfect complexes

Definition 1.13. An *algebraic vector bundle* (AVB) is an $\mathcal{O}_X$-module that is locally free of finite rank.

Proposition 1.14. The following assertions are equivalent

1. $\mathcal{F}$ is an algebraic vector bundle
2. $\mathcal{F}$ is locally projective and finitely generated.

Proof. By definition, if $\mathcal{F}$ is an algebraic vector bundle, it is locally free of finite rank and, therefore locally projective and finite.

Assume $\mathcal{F}$ is locally projective. Given $x \in X$, there exists a neighborhood U of x such that $\mathcal{F}|_U$ is projective and finite. Therefore, $F_x = (F|_U)_x$ is projective over $\mathcal{O}_x$ (projective is a local property). As $\mathcal{O}_X$ is a local ring, F_x is free. By Geometric Nakayama's Lemma, there exists a neighborhood V_x of x such that $F|_{V_x}$ is free. □

Definition 1.15. Let $E^\bullet$ be a chain complex in $\mathcal{O}_\mathbf{X} - \mathbf{Mod}$. We say that $E^\bullet$ is

1. *strictly perfect* if all the modules E^i are AVB and it is bounded, and
2. *perfect* if there exists a covering $\{U_i\}_i$ of X such that $E^\bullet|_{U_i}$ is quasi-isomorphic to a strictly perfect complex.

Proposition 1.16. If the scheme is divisorial, every perfect complex is globally quasi-isomorphic to a strictly perfect complex.

Proof. (Prop 2.3.1.(d), [6])write □

Example 1.17. For a ring R, a perfect complex turns out to be (by 1.16) a chain complex of R-modules that is quasi-isomorphic to a bounded complex of finite projective R-modules. So an R-module M is said to be perfect if $M[0] \in \mathcal{D}(R)$ is a perfect complex, i.e., if it has finite projective dimension.

Definition 1.18. We define $\mathcal{D}_{\text{perf}}(X) \subset \mathcal{D}(X)$ as the full subcategory whose complex are perfect.

Remark 1.19. We should think as perfect complexes as the derived version of vector bundles.

Corollary 1.20. $\mathcal{D}_{\text{perf}}(X) \subset \mathcal{D}^b(X)$ if X is divisorial.

Remark 1.21. In a quasi-compact scheme, every perfect complex is cohomologically bounded.

Proof. Assume $E^\bullet$ is perfect. Take $\{U_i\}_i$ such that $E|_{U_i} \simeq F_i$ strictly perfect. As X is quasi-compact. take finitely many i's. Take the maximum N such that $H^N(E|_{U_i}) \neq 0$ for some i. Then glue all the $H^{N+1}(E|_{U_i})$ to obtain $H^{N+1}(E) = 0$. Therefore, $E^\bullet$ is cohomologically bounded above. Do the analogous to see that it is cohomologically bounded bellow. □

Example 1.22.

- $\mathcal{D}_{\text{perf}}(k)$ is the subcategory of $\mathbb{Z}$-graded k-vector spaces of the form $\bigoplus_{k\in\mathbb{Z}} M^k$ with M^k of finite dimension.
- $\mathcal{D}_{\text{perf}}(\mathbb{Z}/(4))$ is the subcategory of the homotopy category of $\mathbb{Z}/(4)$ whose objects are the bounded complexes of the form

$$0 \to \mathbb{Z}/(4) \underset{\cdot 2}{\to} \mathbb{Z}/(4) \underset{\cdot 2}{\to} \ldots \to \mathbb{Z}/(4) \underset{\cdot 2}{\to} \mathbb{Z}/(4) \underset{0}{\to} 0$$

- Given a ring R, it can be proved that $\mathcal{D}_{\text{perf}}(R) = \mathcal{D}^b(R)$ if and only if $\text{gldim}(R) < \infty$.
- By Syzygie's Hilbert theorem, the global dimension of $K[x_1, \ldots, x_n]$ is n, so the theorem applies.

Theorem 1.23. (Serre, Main Theorem 4.4.16, [7]) A local ring $A_\mathfrak{p}$ is regular if and only if $\text{gl.dim}(A_\mathfrak{p}) < \infty$.

Remark 1.24. The category $\mathcal{D}_{\text{perf}}(X)$ informs us about the singularities. That motivates the definition of the category of singularities as the Verdier quotient $\mathcal{D}(X)/\mathcal{D}_{\text{perf}}(X)$.

Singularity is D_{coh}/D_{perf}

Remark 1.25. Recall that the global dimension of a ring R is the maximum k such that $\text{Ext}^k_R(-,-) \neq 0$.

1.3 The triangulated structure

Proposition 1.26. The full subcategory $\mathcal{D}_{\text{perf}}(X) \subset \mathcal{D}(X)$ is a triangulated subcategory.

Proof. We have to check TS1 and TS2.

1. TS1. The suspension of a perfect complex is perfect: this is immediate because the suspension of a perfect complex consists just on shifting it.

2. TS2. Given an exact triangle $X \to Y \to Z \to \Sigma X$, if two out of three are perfect, so is the third. We reduce this part to prove that if $X^\bullet \to Y^\bullet$ is a morphism of perfect complexes, then so is $cone(f)$. The cone of a morphism is defined as $X[1] \oplus Y$. As before, $X[1]$ is perfect. And the sum of perfect complexes is perfect. Take U open set such that $(X[1])|_U$ and $Y|_U$ are quasi-isomorphic to strictly perfect complexes. Then, $(X[1] \oplus Y)|_U$ is quasi isomorphism to a sum of strictly perfect complexes and, therefore, it is strictly perfect (because the sum of projective modules is projective).

□

Proposition 1.27. The tensor product is well defined in $\mathcal{D}_{\text{perf}}(X)$.

Proof. We define the tensor product in $\mathcal{D}_{\text{perf}}(X)$ as the tensor product inherited from $\mathcal{D}(X)$.

Now we show that the tensor product of two perfect complexes is a perfect complex. Again, we can take an open set U where both E and F are strictly perfect. And as they are bounded, we can shrink U so that the modules are free. So we need to prove the local case: Take $E^\bullet$ and $F^\bullet$ strictly perfect complexes. The tensor product $E^\bullet \otimes F^\bullet$ is defined by the modules $\bigoplus_{i+j=k}(E^i \otimes F^j)$. So it is obviously finite. As being free is preserved by tensor products and sums, and the modules of $E^\bullet$ and $F^\bullet$ are free. □

Corollary 1.28. $\mathcal{D}_{\text{perf}}(X)$ is a tensor triangulated category and the unit is given by the complex

$$\ldots \to 0 \to \mathcal{O}_X \to 0 \to \ldots.$$

Proposition 1.29. Let $\mathcal{K}$ be a tt-subcategory of $\mathcal{D}(X)$. Assume

- $\mathcal{K}$ is stable under direct summands
- $\mathcal{O}_X \in \mathcal{K}$

Then every strictly perfect complex is in $\mathcal{K}$.

Example 1.30. $E, F, G \in \mathcal{D}(X)$. Let E or G be perfect. Then the morphism

$$RHom(E, F) \otimes G \to RHom(E, F \otimes G)$$

is an isomorphism.

Definition 1.31. Given an essentially small triangulated category $\mathcal{K}$, define the *Grothendieck group* of $\mathcal{K}$ (denoted $K_0(\mathcal{K})$ as the free abelian group generated by the set of isomorphism classes of objects of $\mathcal{K}$ quotient out by the relations

$$[B] = [A] + [C]$$

if $A \to B \to C \to \Sigma A$ is an exact triangle.

Now we state certain properties of the Grothendieck group.

a) Universal property

b) $K_0(-)$ is a covariant functor (for triangulated functors)

c) $[A \oplus B] = [A] + [B]$ (because of $A \to A \oplus B \to B \to \Sigma A$)

d) $[A] + [0] = [A]$

e) $-[A] = [\Sigma A]$ (because of $A \to 0 \to \Sigma A \to \Sigma A$

Unlike when one considers the Grothendieck group of an abelian category, all the elements in $K_0(\mathcal{K})$ are represented by the class of an element and not by a sum.

1.4 Representing perfect complexes

1.4.1 As compact objects

Definition 1.32. A compact object in a triangulated category $\mathcal{T}$ is an object $\mathcal{C}$ such that for every family of objects $\mathcal{E}$ in $\mathcal{T}$ the canonical morphism

$$\bigoplus_{E\in\mathcal{E}} \mathrm{Hom}(C,E) \to \mathrm{Hom}(C, \oplus_{E\in\mathcal{E}} E)$$

is an isomorphism.

Remark 1.33. FACT: The compact objects of a category form a set.

Definition 1.34. Given a complex of $\mathcal{O}_X-$modules, we define its *homology sheaves* as the sheafification of the presheaf

$$U \mapsto H^k((E(U)^\bullet) = \frac{\ker(d^k\colon E^k(U) \to E^{k+1}(U))}{\mathrm{Im}(d^{k-1}\colon E^{k-1}(U) \to E^k(U))}.$$

Remark 1.35. Note that $H^k(E^\bullet_x) = H^k(E^\bullet)_x$ because localization commutes with quotients. And

Definition 1.36. Define $\mathcal{D}(X)_{\mathrm{Qcoh}}$ as the full subcategory of $\mathcal{D}(X)$ consisting on the objects $E^\bullet$ whose cohomology $H^k(E^\bullet)$ are quasi-coherent $\mathcal{O}_X-$modules.

Remark 1.37.
1. If X is qcqs, $\mathcal{D}_{\mathrm{perf}}(X) \hookrightarrow \mathcal{D}_{\mathrm{Qcoh}}(X)$.
2. If X is Noetherian, $\mathcal{D}_{\mathrm{perf}}(X) \hookrightarrow \mathcal{D}_{\mathrm{Coh}}(X)$.

Remark 1.38. If the scheme X is Noetherian, $\mathcal{D}_{\mathrm{perf}}(X) \hookrightarrow \mathcal{D}_{\mathrm{Coh}}(X)$. Because the terms E^k of a perfect complex are locally free of finite type, and the category $\mathrm{Coh}(X)$ is abelian.

Remark 1.39. Due to Theorem 22.35 in [3], in qc semi separated (exists an open covering of affine closed under finite intersections. For example, a ring) schemes, the natural functor $\mathcal{D}(\mathrm{QCoh}) \hookrightarrow \mathcal{D}_{\mathrm{Qcoh}}(X)$ is an equivalence of categories.

Proposition 1.40. *E is perfect in $\mathcal{D}(X)$ if and only if E is a compact object in $\mathcal{D}_{\mathrm{Qcoh}}(X)$*

[4], Lemma 3.5. . □

Important to get essentially small

1.4.2 As dualizable objects

Remark 1.41. Remark that an object E is dualizable if and only exists an F and maps $\eta : E \to E \otimes F$ and $\epsilon\colon F \otimes F \to F$. Remember that the category is rigid iff every object is dualizable. Equivalently, if exists a functor

$$D\colon \mathcal{K}^{op} \to \mathcal{K}$$

and natural isomorphisms

$$\mathrm{Hom}_{\mathcal{K}}(a \otimes b, c) \cong \mathrm{Hom}_{\mathcal{K}}(b, Da \otimes c)$$

for every objects a, b, c.

Definition 1.42. If $E \in \mathcal{D}(X)$, we define its dual as $E^\vee := RHom_{\mathcal{O}_X}(E, \mathcal{O}_X)$

There does exists a functorial map $E \to (E^\vee)^\vee$.

Proposition 1.43. *$E \in \mathcal{D}_{\mathrm{perf}}(X)$. Then*

a) *$E^\vee$ perfect and $E \to (E^\vee)^\vee$ is an iso.*

b) *$E^\vee \otimes G \to RHom(E, G)$ is an iso*

c) For all $F, G \in \mathcal{D}(X)$, we have an isomorphism

$$RHom_{\mathcal{O}_X}(E^\vee \otimes F, G) \to RHom_{\mathcal{O}_X}(F, E \otimes G)$$

Proof. We can work locally, (a) can be easily checked on perfect complexes. (b)
(c).

$$Rhom(E^\vee \otimes F, G) \cong^{\text{adjunction}} RHom(F, RHom(E^\vee, G)) =$$
$$=^{b)} RHom(F, (E^\vee)^\vee \otimes G) =^{a)} RHom(F, E \otimes G)$$

□

Corollary 1.44. The category is rigid

Proposition 1.45. E is perfect in $\mathcal{D}(X)$ if and only if E is a dualizable object.

[3], Remark 21.149. □

1.5 Main theorem

The main goal of $\mathcal{D}_{\text{perf}}(X)$ in this talk is to prove the theorem

Theorem 1.46. ([1]) Given a topologically noetherian scheme X, the the Balmer spectrum of $\mathcal{D}_{\text{perf}}(X)$ is homeomorphic to X.

Definition 1.47. Let X be a scheme, $E^\bullet$ a complex of $\mathcal{O}_X$-modules. The *cohomological support* of $E^\bullet$ is the subspace

$$\text{Supph}(E^\bullet) = \{x \in X : E^\bullet_x \text{ is not acyclic}\}$$

Definition 1.48. (Support data, Definition 3.1 [1]) A *support data* on a tensor triangulated category is a pair (X, σ) where X is a netherian topological space and σ is an assignment which associates to every object in the category a a closed subset $\sigma(a) \subset X$ satisfying the following conditions

1. $\sigma(0) = \varnothing$ and $\sigma(1) = X$,
2. $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$,
3. $\sigma(\Sigma a) = \sigma(a)$
4. $\sigma(a) \subset \sigma(b) \cup \sigma(c)$ for any exact triangle $a \to b \to c \to \Sigma a$.
5. $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$.

Definition 1.49. (Classifying support data, Definition 5.1 [1]) A support data (X, σ), is a *classifying support data* if

a) The topological space X is Noetherian and any not irreducible subset $Z \subset X$ has a unique generic point.

b) There is a bijection

$$\begin{array}{lccc} \theta\colon & \{Y \subset X : y \in Y \Rightarrow \bar{\{y\}} \subset Y\} & \to & \{\mathcal{J} \subset \mathcal{K} : \mathcal{J} \text{ radical thick } \otimes\text{-ideal}\} \\ & Y & \mapsto & \{a \in \mathcal{K} : \sigma(a) \subset Y\} \\ & \bigcup_{a \in \mathcal{J}} \sigma(a) := \sigma(\mathcal{J}) & <-| & \mathcal{J} \end{array}$$

Theorem 1.50. (Theorem 5.2, [1]) Suppose that (X, σ) is a classifying support data on a tt-category $\mathcal{K}$. Then the canonical map $f\colon X \to \text{Spc}(\mathcal{K})$ is a homeomorphism.

Proof. Proof done by a colleague in a former talk. □

Remark 1.51. The map $f\colon X \to \text{Spc}(X)$ is defined as $f(x) = \{a \in \mathcal{K} : x \notin \sigma(a)\}$ for $x \in X$.

Theorem 1.52. (Theorem 5.5,[1]) The pair (X, Supph) (X top noetherian) is a classifying support data on $\mathcal{D}_{\text{perf}}(X)$.

Proof. Theorem 3.15 [5]. This is precisely the core of the next talk. □

Corollary 1.53. (Corrolary 5.6, [1]). There is a homeomorphism $f\colon X \to \mathrm{Spc}(\mathcal{D}_{\mathrm{perf}}(X))$ with

$$f(x) = \{a \in \mathcal{D}_{\mathrm{perf}}(X) : a_x \text{ is acyclic } \} = \{a \in \mathcal{D}_{\mathrm{perf}}(X) : a_x \simeq 0 \text{ in } \mathcal{D}_{\mathrm{perf}}(X)\}.$$

So now we focus in the classification of thick $\otimes-$subcategories of $\mathcal{D}_{\mathrm{perf}}(X)$.

1.6 More properties

Theorem 1.54. (Lemma 3.3, [5]) Let X be a qcqs scheme and $E^\bullet$ a perfect complex on X.

a) For any $x \in X$, $E^\bullet_x$ is an acyclic complex of $\mathcal{O}_{X,x}$-modules if and only if $E^\bullet \otimes^L_{\mathcal{O}_X} k(x)$ is an acyclic complex of $k(x)-$modules.

b) If Y is a qcqs scheme and $f\colon Y \to X$ is a morphism of schemes, then

$$\mathrm{Supph}(Lf^*E) = f^{-1}\mathrm{Supph}(E)$$

c) $\mathrm{Supph}(E^\bullet)$ is closed in X and $X \setminus \mathrm{Supph}(E^\bullet)$ is quasi-compact.

Proof. a) Consider the strong converging Künneth spectral sequence.

$$E^2_{p,q} = \mathrm{Tor}^p_{\mathcal{O}_{X,x}}(H_q(E^\bullet), k(x)) \underset{p}{\Rightarrow} H_{p+q}(E^\bullet \otimes^L_{\mathcal{O}_{X,x}} k(x))$$

Assume first that E_x is acyclic, i.e. $H_\bullet(E^\bullet_x) = 0$, then $E^2_{p,q} = 0 = E^\infty_{p,q}$. This means the following: as $H_\bullet(E^\bullet \otimes^L_{\mathcal{O}_{X,x}} k(x))$ is a graded module filtered by $\Phi^p H_\bullet$ with a finite filtration, the quotients $\Phi^p H_n/\Phi^{p-1}H_n = 0$ for every p, n $\Phi^p H_n = \Phi^{p-1}H_n$. And due to the finiteness of the filtration, $H_n = 0$ for every n.

Assume now that $E^\bullet_x$ is not acyclic. As $E^\bullet$ is a perfect complex, it is locally bounded, so $E^\bullet_x$ is bounded, therefore cohomologically bounded. Take the least N s.t. $H_N(E^\bullet_x) \neq 0$. For $q < N$ we have $H_q(E^\bullet_x) = 0$. Therefore, $E^2_{p,q} = 0$ for $q < N$ and $p < 0$, and we get a corner in the second page of the spectral sequence.

$$H_N(E^\bullet_x) \otimes k(x) = E^2_{0,N} = E^\infty_{0,N} = \Phi^0 H_N/\Phi^{-1}H_N = \Phi^0 H_N = H_N(E^\bullet \otimes^L_{\mathcal{O}_{X,x}} k(x))$$

As $E^\bullet_x$ is perfect and $H_q(E^\bullet_x) = 0$ for $q < N$ Proposition 2.6 and Lemma 2.2.3 [6] we get that $H_N(E^\bullet_x)$ is finitely generated $\mathcal{O}_{X,x}-$module. So we can apply Nakayama's Lemma (generators of the localization can be obtained from generators of the module). And obtain that, as $H_N(E^\bullet_x) \neq 0$, $H_N(E^\bullet_x) \otimes_{\mathcal{O}_{X,x}} k(x) \neq 0$. So $E^\bullet_x \otimes^L_{\mathcal{O}_X} k(x)$ is not acyclic.

b) By (a), $y \in \mathrm{Supph}(Lf^*E) \Leftrightarrow (Lf^*E) \otimes^L_{\mathcal{O}_Y} k(y)$ is not acyclic. Take $x = f(y)$.

$$(Lf^*E) \otimes^L_{\mathcal{O}_Y} k(y) \cong (E^\bullet_x \otimes^L_{\mathcal{O}_X} k(x)) \otimes^L_{k(x)} k(y)$$

$k(y)$ is an extension field of $k(x)$ so $k(y)$ is faithfully flat over $k(x)$ [a sequence of $k(x)-$modules is exact if and only if tensoring by $- \otimes_{k(x)} k(y)$ is exact.] So

$$(Lf^*E) \otimes_{\mathcal{O}_Y} k(y) \text{ not acyclic } \Leftrightarrow$$

$$(E^\bullet \otimes^L_{\mathcal{O}_X} k(x)) \otimes^L_{k(x)} k(y) \text{ not acyclic } \Leftrightarrow_{k(y) \text{ faithfully flat}}$$

$$E^\bullet \otimes^L_{\mathcal{O}_X} k(x) \text{ not acyclic} \Leftrightarrow_{(a)}$$

$$E^\bullet_x \text{ not acyclic} \Leftrightarrow$$

$$x \in \mathrm{Supph}(E^\bullet).$$

c) First assume X is Noetherian. As $E^\bullet$ is perfect, it is cohomologically bounded (Remark 1.21.) As X is Noetherian, the $\mathcal{O}_X-$modules $H^k(E^\bullet)$ are coherent $\mathcal{O}_X-$modules(1.38). As the support of quasi-coherent modules of finite type (in particular, if it coherent) is closed, $\mathrm{Supp}(H^n(E^\bullet)) = \{x \in X : H^n(E^\bullet)_x \neq 0\}$ is closed. As $E^\bullet$ is cohomologically bounded, $\mathrm{Supph}(E^\bullet) = \bigcup_n \mathrm{Supp}(H^n(E^\bullet))$ is a finite union and hence closed. As X is Noetherian, any subspace is quasi-compact, and therefore $X \backslash \mathrm{Supph}(E^\bullet)$ is quasi-compact.

Consider now a general qcqs scheme X. By absolute Noetherian approximation, there exist a map $g\colon X \to X'$ where X' is Noetherian, and a perfect complex F' on X' such that $E = Lg^*F$. By (b) $\mathrm{Supph}(E^\bullet) = g^{-1}\mathrm{Supph}(F^\bullet)$ and therefore is closed in X as $\mathrm{Supph}(F^\bullet)$ is closed in X', that is Noetherian. As g is affine, it is quasi-compact, so $X \setminus \mathrm{Supph}(E^\bullet) = g^{-1}(X' \setminus \mathrm{Supph}(F^\bullet))$ is quasi-compact.

□

Remark 1.55. Comment on Noetherian I: The final statement we want to prove does not need that X is Noetherian, but topologically Noetherian.

Remark 1.56. Comment on Noetherian II: Nevertheless, this generality is worth it, because a more general result (cite) allows us to prove the main theorem (cite) for non-necessarily topologically Noetherian spaces. write

2 As finite tor amplitude and pseudo coherent complexes

Definition 2.1.

a) A map of chain complexes $f\colon E^\bullet \to F^\bullet$ is an *m-quasi-isomorphism* if $H_k(f)$ is an isomorphism for $k > m$ and an epimorphism for $k = m$.

b) A complex of $\mathcal{O}_X-$modules $E^\bullet$ is *m-pseudo coherent* if there is a covering $\{U_i\}_i$ of X and for each i and $m-$isomorphism $\mathcal{F}_i \to E|_{U_i}$ with $\mathcal{F}_i$ strictly perfect on U_i.

c) A complex of $\mathcal{O}_X-$modules E is *pseudo-coherent* if it is $m-$pseudo-coherent for all m.

d) We define the derived category of pseudo-coherent modules as the full subcategory of $\mathcal{D}_{\text{pseudo-coh}}(X) \hookrightarrow \mathcal{D}(X)$ whose objects are the pseudo-coherent modules.

Remark 2.2. Note that pseudo-coherence is just an approximation to perfect complexes. As we make $m \to -\infty$, the open sets defining local quasi-isomorphisms with the perfect complexes tend to shrink.

Remark 2.3. If the scheme X is quasi-compact, $\mathcal{D}_{\text{pseudo-coh}}(X) \hookrightarrow \mathcal{D}^-(X)$.

Proposition 2.4. (Example 2.2.8, [5]) X noetherian. Let $E^\bullet$ be a complex of $\mathcal{O}_X-$modules. Then E is pseudo-coherent if and only if E is cohomologically bounded above and all the $H^k(E^\bullet)$ are coherent $\mathcal{O}_X-$modules.

Proof. We just need the only if statement. As $\mathcal{D}_{\text{pseudo-coh}}(X) \hookrightarrow \mathcal{D}^-(X)$, a pseudo coherent complex is obviously cohomologically bounded above. Let's see now that it has coherent cohomology revisit ☐

Definition 2.5. A complex $E \in \mathcal{D}(X)$ is said to have *finite Tor-amplitude* in the interval $[a, b]$ if for all $k \notin [a, b]$ and $\mathcal{F} \in \mathcal{D}(X)$ the groups $H^k(E \otimes^L \mathcal{F}) = 0$

Theorem 2.6. [3](Theorem 21.174) A complex $E \in \mathcal{D}(X)$ is perfect if and only if it is pseudo-coherent and locally on X it has finite tor-amplitude

Example 2.7. ([3]Example 22.49) Let X be a scheme and $E \in \mathcal{D}_{\text{perf}}(X)$. Then E is isomorphic to a vector bundle if and only if E has tor amplitude in $[0, 0]$.

2.1 Extra

Proposition 2.8. Extension lemma (Thomason 97)

Lemma 2.9. (Lemma 3.4) X qcqs, and $Y \subset X$ closed subset such that $X \setminus Y$ is q.c., then there exists a perfect complex $E^\bullet$ on X such that $\mathrm{Supph}(E^\bullet) \subset Y$.

Theorem 2.10. X qcqs, $E^\bullet$ perfect complex on X and $F^\bullet$ a complex of $\mathcal{O}_X$-modules with quasi-coherent cohomology. Let $f\colon E^\bullet \to F^\bullet$ morphism in $\mathcal{D}(X)$. Hypothesis: ...

Proposition 2.11. (Thomason and Thorough, K-Theory,C.9) If R is a Noetherian ring and X is a qcqs scheme over $\mathrm{Spec}(R)$, then X can be writen as the inverse limit of finitely presented schemes (over $\mathrm{Spec}(R)$).

Remark 2.12. Absolute Noetherian approximation (lemma 3.4)

Example 2.13. $D^{\mathrm{perf}}(\mathbb{Z}/(4))$ has only the trivial subcategory.

3 Talk 2

Goal 1: Prove the following theorem

Theorem 3.1. (Theorem 3.15, Thomason 97) Let X be a quasicompact and quasi-separated scheme. Denote by $\mathfrak{C}$ the set of thick triangulated $\otimes$-subcategories (3.9) of the derived category $D(X)_{\mathrm{parf}}$ of perfect complexes (3.1) on X. Denote by $\mathfrak{S}$ the set of those subspaces $Y \subset X$ such that $Y = \bigcup_\alpha Y_\alpha$ is a union of closed subspaces Y of X such that $X \setminus Y$ is quasi-compact. Then there is a bijective correspondence between $\mathfrak{C}$ and S. The bijection $\varphi\colon \mathfrak{S} \to \mathfrak{C}$ sends a subspace $Y \subset X$ to the thick subcategory whose objects are those perfect E such that $Supph(E) \subset Y$, i.e. which are acyclic off Y . The inverse bijection '$\psi\colon \mathfrak{C} \to \mathfrak{S}$ sends a triangulated subcategory A D(X)parf to the subspace Y = SE 2A Supph(E).

Goal 2: Rewrite the theorem in terms of Balmer spectrum

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