Robust Hedging with Proportional Transaction Costs *

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Abstract

Duality for robust hedging with proportional transaction costs of path dependent European options is obtained in a discrete time financial market with one risky asset. Investor's portfolio consists of a dynamically traded stock and a static position in vanilla options which can be exercised at maturity. Trading of both the options and the stock are subject to proportional transaction costs. The main theorem is duality between hedging and a Monge-Kantorovich type optimization problem. In this dual transport problem the optimization is over all the probability measures which satisfy an approximate martingale condition related to consistent price systems in addition to an approximate marginal constraints.

1   Introduction

As well known super-replication in markets with transaction costs is quite costly [20, 17]. Naturally the same is even more true for the model free case in which one does not place any probabilistic assumptions on the behavior of the risky asset. However, one may reduce the hedging cost by including liquid derivatives in the super-replicating portfolio. In particular, one may use all call options (written on the underlying asset) with a price that is known to the investor initially. This leads us to the semi–static hedging introduced in the classical paper of Hobson [14] in markets without transaction costs. So following [14], we assume that all call options are traded assets and can be initially bought or sold for a known price. In addition to these static option positions, the stock is also traded dynamically. These trades, however, are subject to

transaction costs. Each option has its own cost and their general structure is outlined in the next section.

In this market, we consider the problem of robust hedging of a given path dependent European option. Robust hedging refers to super-replication of an option for all possible stock price processes. This approach has been actively researched over the past decade since the seminal paper of Hobson [14]. In particular, the optimal portfolio is explicitly constructed for special cases of European options in continuous time; barrier options in [5] and [7, 8], lookback options in [12], [13] and [14], and volatility options in [9]. The main technique that is employed in these papers is the Skorohod embedding. For more information, we refer the reader to the surveys of Hobson [15], Obłój [18] and to the reference therein.

Recently, an alternative approach is developed using the connection to optimal transport. Duality results in different types of generality or modeling have been proved in [2], [4], [10] and [12] in frictionless markets. In particular, [10] studies the continuous time models, [12] provides the connection to stochastic optimal control and a general solution methodology, [4] proves a general duality in discrete time and [2] studies the question of fundamental theorem of asset pricing in this context.

Although much has been established, the effect of frictions - in particular the impact of transaction costs - in this context is not fully studied. The classical probabilistic models with transaction costs, however, is well studied. In the classical model, a stock price model is assumed and hedging is done only through the stock and no static position in the options is used. Then, the dual is given as the supremum of "approximate" martingale measures which are equivalent to the market probability measure, see [19, 16] and the references therein. In this paper, we extend this result to the robust case. Namely, we prove that the super–replication price can be represented as a martingale optimal transport problem. The dual control problem is the supremum of the expectation of the option, over all approximate martingale measures which also satisfy an approximate marginal condition at maturity. This result is stated in Theorem 2.1 below and the definition of an approximate martingale is given in Definition 2.5. Indeed, approximate martingales are very closely related to consistent price systems which play a central role in the duality theory for markets with proportional transaction costs.

Recently Acciaio, Beiglböck and Schachermayer [2] proved the fundamental theorem of asset pricing (FTAP) in discrete time markets without transaction costs in the robust setting. Also Bouchard and Nutz [6] studies FTAP again in discrete time but in the quasi-sure setting. Our main duality result has implications towards these results as well. These corollaries are discussed in the subsection 2.5.

As in our previous paper [10] on robust hedging, our proof relies on discretization of the problem. We first show that the original robust hedging problem can be obtained as a limit of hedging problems that are defined on *finite spaces*. We exploit the finiteness of these approximate problems and directly apply an elementary Kuhn–Tucker duality theory. We then prove that any sequence of probability measures that are asymptotical maximizers of these finite problems is tight. The final step is then to directly use weak convergence and pass to the limit.

The paper is organized as follows. Main results are formulated in the next section and proved in Section 3. The final section is devoted to the proof of an auxiliary result that is used in the proof of the main results. This auxiliary result deals with super–replication under constraints and maybe of independent interest.

2   Preliminaries and main results

The financial market consists of a savings account B and a risky asset S and the trading is restricted to finitely many time points. Hence, the stock price process is S_k, $k = 0, 1, ..., N$, where

$N < \infty$ is the maturity date or the total number of allowed trades. By discounting, we normalize $B \equiv 1$. Furthermore, we normalize the initial stock price $s := S_0 > 0$ to one as well. Then, the set Ω of all possible price processes is simply all vectors $(\omega_0, \ldots, \omega_N) \in \mathbb{R}_+^{N+1}$ which satisfy $\omega_0 = 1$ and $\omega_1, \ldots, \omega_N \geq 0$. Then, any element of Ω is a possible stock price process. So we let $\mathbb{S}$ be the canonical process given by $\mathbb{S}_k(\omega) := \omega_k$ for $k = 0, \ldots, N$. Let us emphasize we make no assumptions on our financial market. In particular, we do not assume any probabilistic structure.

2.1 An assumption on the European claim

We consider general path dependent options. Hence, the pay-off is $X = G(\mathbb{S})$ with any function $G : \Omega \to \mathbb{R}$. Our approach to this problem, requires us to make the following regularity and growth assumption. Let $\|\omega\| := \max_{0 \leq k \leq n} |\omega_k|$ for $\omega \in \Omega$. We assume the following.

Assumption 2.1 *G is upper semi-continuous and bounded by a quadratic function, i.e., there exist a constant $L > 0$ such that*

$$|G(\omega)| \leq L[1 + \|\omega\|^2], \quad \forall \, \omega \in \Omega.$$

The above assumption is quite general and allows for most of the standard claims such as Asian, lookback, volatility and Barrier options. The reason for the quadratic growth choice is the volatility options. More generally, one may consider different growth conditions as well. However, in this paper, we choose not to include this extension to avoid more technicalities.

2.2 Semi static hedging with transaction costs

Let $\kappa > 0$ be a given constant. Consider a model in which every purchase or sale of the risky asset at any time is subject to a proportional transaction cost of rate κ. We assume that $\kappa < 1/4$.

Then, a *portfolio strategy* is a pair $\pi := (f, \gamma)$ where $f : \mathbb{R}_+ \to \mathbb{R}$ is a measurable function and

$$\gamma : \{0, 1, ..., N-1\} \times \Omega \to \mathbb{R}$$

is a progressively measurable map, i.e. $\gamma(i, \omega) = \gamma(i, \tilde{\omega})$ if $\omega_j = \tilde{\omega}_j$ for all $j \leq i$. The function f represents the European option with payoff $f(\mathbb{S}_N)$ that is bought at time zero for the price of $\mathscr{P}(f)$ and $\gamma(k, \mathbb{S})$ represents the number of stocks that the investor invests at time k given that the stock prices up to time k are $\mathbb{S}_0, \mathbb{S}_1, ..., \mathbb{S}_k$. Then, the portfolio value at the maturity date is given by

$$Y_N^\pi(\mathbb{S}) := f(\mathbb{S}_N) + \sum_{i=0}^{N-1} \gamma(i, \mathbb{S})(\mathbb{S}_{i+1} - \mathbb{S}_i) - \kappa \sum_{i=0}^{N-1} \mathbb{S}_i |\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})| \qquad (2.1)$$

where we set $\gamma(-1, \cdot) \equiv 0$. The initial cost of any portfolio (f, γ) is the price of the option $\mathscr{P}(f)$. Properties of this price operator $\mathscr{P}$ is given in the next subsection.

Definition 2.2 A portfolio π is called *perfect* (or *perfectly dominating*) if it super-replicates the option, i.e.,

$$Y_N^\pi(\mathbb{S}) \geq G(\mathbb{S}), \quad \forall \, \mathbb{S} \in \Omega.$$

The minimal *super-replication cost* is given by

$$V(G) = \inf \{ \mathscr{P}(f) \mid \pi := (f, \gamma) \text{ is a perfect portfolio} \}. \qquad (2.2)$$

2.3 European options and their prices

We postulate a general pricing operator $\mathscr{P}(f)$ for the initial price of the option $f(\mathbb{S}_N)$. We assume that it has the following properties.

Assumption 2.3 *There exists $p > 2$ such that for the power function x^p, $\mathscr{P}(x^p) < \infty$. Consider the vector space*

$$\mathscr{H} := \{f : \mathbb{R}_+ \to \mathbb{R}, Borel \ mbl. \ | \ \exists\, C > 0 \ such \ that \ |f(x)| \leq C(1 + x^p), \ \forall x \in \mathbb{R}_+\}.$$

We assume that $\mathscr{P} : \mathscr{H} \to \mathbb{R}$ is a convex function and for every constant $a \in \mathbb{R}$

$$\mathscr{P}(a) = a. \tag{2.3}$$

We also assume that $\mathscr{P}$ is positively homogeneous of degree one, i.e.

$$\mathscr{P}(\lambda f) = \lambda \mathscr{P}(f), \ \ f \in \mathscr{H}, \ \ \lambda > 0. \tag{2.4}$$

Furthermore, for every sequence $\{f_n\}_{n=1}^{\infty} \subset \mathscr{H}$ converging pointwise to $f \in \mathscr{H}$

$$\mathscr{P}(f) \geq \limsup_{n \to \infty} \mathscr{P}(f_n). \tag{2.5}$$

In (2.2) we assume that the function f belongs to $\mathscr{H}$. Namely, $\mathscr{P}(f) \equiv \infty$ for $f \notin \mathscr{H}$.

We conclude this section with an elementary result.

Lemma 2.4 *The minimal super-replication cost V is sub–additive and positively homogeneous of degree one, i.e.,*

$$V(\lambda G) = \lambda V(G), \ \ \lambda > 0,$$

and

$$V(G + H) \leq V(G) + V(H).$$

Furthermore, if $G \geq 0$ and $V(G) < 0$, then $V(G) = -\infty$.

Proof. From the convexity and the positive homogeneity of $\mathscr{P}$, it follows that $\mathscr{P}$ is sub additive, i.e.

$$\mathscr{P}(f + g) \leq \mathscr{P}(f) + \mathscr{P}(g).$$

Thus, the first two properties follow immediately from (2.2). Finally, let $G \geq 0$ be a non–negative claim and assume that $V(G) < 0$. Then, there exists a perfect portfolio (f, γ) with $\mathscr{P}(f) < 0$. Clearly for any $\lambda > 1$, $(\lambda f, \lambda \gamma)$ is also a perfect portfolio. Thus from (2.4) we get

$$V(G) \leq \lim_{\lambda \to \infty} \mathscr{P}(\lambda f) = -\infty,$$

as required. ∎

2.4 The main result

To state the main result of the paper, we need to introduce the probabilistic structure as well. Recall the space Ω and the canonical process $\mathbb{S}$. Let $\mathbb{F} = (\mathscr{F}_k)_{k=1}^{N}$ be the canonical filtration generated by the process $\mathbb{S}$, i.e, $\mathscr{F}_k = \sigma(\mathbb{S}_1, \ldots, \mathbb{S}_k)$.

Definition 2.5 A probability measure $\mathbb{Q}$ on $(\Omega, \mathbb{F})$ is called a *κ-approximate martingale law* if $\mathbb{S}_0 = 1$ $\mathbb{P}$-a.s. and if the pair $(\mathbb{Q}, \tilde{\mathbb{S}})$ with

$$\tilde{\mathbb{S}}_k := \mathbb{E}_{\mathbb{Q}}[\, \mathbb{S}_N \mid \mathscr{F}_k \,],$$

is a consistent price system in the sense of [16, 19], i.e., for any $k < N$

$$(1 - \kappa)\mathbb{S}_k \leq \tilde{\mathbb{S}}_k \leq (1 + \kappa)\mathbb{S}_k \quad \mathbb{Q}\text{-a.s.} \tag{2.6}$$

We denote by $\mathscr{M}_{\kappa, \mathscr{P}}$ the set of all κ–approximate martingale laws $\mathbb{Q}$, such that

$$\mathbb{E}_{\mathbb{Q}}[\, f(\mathbb{S}_N) \,] \leq \mathscr{P}(f), \qquad \forall f \in \mathscr{H}. \tag{2.7}$$

The following theorem is the main result of the paper. We use the standard convention that the supremum over an empty set is equal to minus infinity.

Theorem 2.1 *Suppose G satisfies the Assumption 2.1 and $\mathscr{P}$ satisfies the Assumption 2.3. Then,*

$$V(G) = \sup_{\mathbb{Q} \in \mathscr{M}_{\kappa, \mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})].$$

In particular, when the set of measures $\mathscr{M}_{\kappa, \mathscr{P}}$ is empty, $V(G) = -\infty$ for every G satisfying the Assumption 2.1

Proof. In view of (2.1) and the convention $\gamma(-1, \cdot) \equiv 0$, for any portfolio $\pi = (f, \gamma)$,

$$
\begin{aligned}
Y_N^\pi(\mathbb{S}) &= f(\mathbb{S}_N) + \sum_{i=0}^{N-1} \sum_{j=0}^{i} (\gamma(j, \mathbb{S}) - \gamma(j-1, \mathbb{S}))(\mathbb{S}_{i+1} - \mathbb{S}_i) - \kappa \sum_{i=0}^{N-1} \mathbb{S}_i \, |\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})| \\
&= f(\mathbb{S}_N) + \sum_{j=0}^{N-1} (\gamma(j, \mathbb{S}) - \gamma(j-1, \mathbb{S})) \sum_{i=j}^{N-1} (\mathbb{S}_{i+1} - \mathbb{S}_i) - \kappa \sum_{i=0}^{N-1} \mathbb{S}_i \, |\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})| \\
&= f(\mathbb{S}_N) + \sum_{j=0}^{N-1} (\gamma(j, \mathbb{S}) - \gamma(j-1, \mathbb{S}))(\mathbb{S}_N - \mathbb{S}_j) - \kappa \sum_{i=0}^{N-1} \mathbb{S}_i \, |\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})|.
\end{aligned}
$$

Suppose that $\mathscr{M}_{\kappa, \mathscr{P}}$ is non-empty. Let $\mathbb{Q} \in \mathscr{M}_{\kappa, \mathscr{P}}$ and $\pi = (f, \gamma)$ be a perfect portfolio. Then, (2.6) and (2.7) yield that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] &\leq \mathbb{E}_{\mathbb{Q}}[Y_N^\pi(\mathbb{S})] \\
&\leq \mathscr{P}(f) + \sum_{i=0}^{N-1} \mathbb{E}_{\mathbb{Q}}\left[(\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})) \, (\tilde{\mathbb{S}}_i - \mathbb{S}_i) \right] \\
&\qquad - \kappa \sum_{i=0}^{N-1} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{S}_i \, |\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})| \right] \\
&\leq \mathscr{P}(f).
\end{aligned}
$$

So we have proved that

$$\sup_{\mathbb{Q} \in \mathscr{M}_{\kappa, \mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] \leq V(G). \tag{2.8}$$

Hence, to complete the proof of the theorem it suffices to show that

$$V(G) \leq \sup_{\mathbb{Q} \in \mathscr{M}_{\kappa, \mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]. \tag{2.9}$$

The proof of the second inequality is given in the next section. ∎

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Remark 2.6 Consider the following more general problem. Assume that for $0 < k \leq N$ and a set of times $0 < i_1 < i_2 < ... < i_k = N$, one can initially buy vanilla options with a payoff $f_{i_j}(\mathbb{S}_{i_j})$ with maturity date i_j for the price $\mathscr{P}_{i_j}(f_{i_j})$, where $\mathscr{P}_1, ..., \mathscr{P}_k$ satisfy similar assumptions to Assumption 2.3. Then, by using the same approach in a recursive manner we may extend Theorem 2.1 to prove that the super–replication cost in this context is equal to

$$\sup_{\mathbb{Q} \in \mathscr{M}_{\kappa, \mathscr{P}_1, ..., \mathscr{P}_k}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]$$

where $\mathscr{M}_{\kappa, \mathscr{P}_1, ..., \mathscr{P}_k}$ is the set of all κ-approximate probability laws $\mathbb{Q}$ and such that for any time $j = 1, ..., k$ and $f \in \mathscr{H}$ we have

$$\mathbb{E}_{\mathbb{Q}} f(\mathbb{S}_j) \leq \mathscr{P}_j(f).$$

Furthermore, if the set $\mathscr{M}_{\kappa, \mathscr{P}_1, ..., \mathscr{P}_k} = \emptyset$ is empty then $V(G) \equiv -\infty$. For simplicity, in this paper we deal only with the case $k = 1$.

2.5 Fundamental Theorem of Asset Pricing

Theorem 2.1 also implies results that can be seen as the Fundamental Theorem of Asset Pricing (FTAP) for this market. Indeed, when the set $\mathscr{M}_{\kappa, \mathscr{P}}$ of measures is empty, by Theorem 2.1 we conclude that the minimal super-replication cost of any G (satisfying the Assumption 2.1) is equal to minus infinity. This is a clear indication of arbitrage. However, to make a precise statement, we need to define the notion of arbitrage. Since we do not assume a probabilistic structure, there are at least two possible approaches. Indeed, in frictionless markets FTAP is proved under different assumptions and definitions in [2] and in [6]. Our result essentially implies FTAP under both definitions under the Assumption 2.3.

Definition 2.7 We say that the model admits

- *no model-independent arbitrage* (NA_{mi}), if for every $G \geq 0$ satisfying the Assumption 2.1, we have $V(G) \geq 0$.

- *no local arbitrage* (NA_{local}), if for every continuous, bounded $G \geq 0, G \not\equiv 0$, we have $V(G) > 0$.

In the above definition, NA_{mi} is similar to the notion used in [2]. Also a closely related definition is given by Cox & Obłój [7]. On the other hand NA_{local} is analogous to the one used in [6]. One may also consider other versions of NA_{local} by requiring different notions of regularity of G. In the probabilistic setting, this is related to the choice of the polar sets. There one requires that the set $\{G > 0\}$ to be non-polar (c.f. [6]). Clearly, other choices would result a similar but a different equivalent condition as proved below. We do not elaborate on different choices.

Corollary 2.8 *Suppose $\mathscr{P}$ satisfies the Assumption 2.3.*

1. *There is no model-independent arbitrage if and only if $\mathscr{M}_{\kappa, \mathscr{P}}$ is non-empty. In particular, NA_{mi} holds if and only if there is one G satisfying the Assumption 2.1 with $V(G) > -\infty$.*

2. *There is no local arbitrage if and only if for every open subset $O \subset \Omega$ there is $\mathbb{Q} \in \mathscr{M}_{\kappa, \mathscr{P}}$ with $\mathbb{Q}(O) > 0$.*

Proof. First statement follows immediately from Theorem 2.1. So we only prove the second one. First, assume that NA_{local} holds. Let $O \subset \Omega$ be an arbitrary open set. Set,

$$G_O(\omega) := \min\{1, distance(\omega, \Omega \setminus O)\}.$$

Since G_O is bounded and continuous, by NA$_{local}$, $V(G_O) > 0$. Since $0 \leq G_O \leq 1$ and $G_O = 0$ outside of O, by Theorem 2.1,

$$0 < V(G_O) = \sup_{\mathbb{Q} \in \mathcal{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G_O(\mathbb{S})] \leq \sup_{\mathbb{Q} \in \mathcal{M}_{\kappa,\mathscr{P}}} \mathbb{Q}(O).$$

Hence, there must exists a measure $\mathbb{Q} \in \mathcal{M}_{\kappa,\mathscr{P}}$ with $\mathbb{Q}(O) > 0$.

To prove the opposite implication, consider a continuous, bounded option $G \geq 0, G \not\equiv 0$. Set

$$O_G := \{\omega \in \Omega \, : \, G(\omega) > 0\}.$$

By the continuity of G, O_G is a non-empty, open set. By hypothesis, there exists $\mathbb{Q}_G \in \mathcal{M}_{\kappa,\mathscr{P}}$ with $\mathbb{Q}_G(O_G) > 0$. We estimate using Theorem 2.1 to arrive at

$$V(G) = \sup_{\mathbb{Q} \in \mathcal{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] \geq \mathbb{E}_{\mathbb{Q}_G}[G(\mathbb{S})] > 0.$$

$\blacksquare$

3   Proof of the main result

In this section, we prove (2.9).

3.1   Reduction to bounded uniformly continuous claims

We first use the elegant path-wise approach of [1] to martingale inequalities to show that the super-replication cost of certain options are asymptotically small. Indeed, for $M > 0$ consider the option,

$$\alpha_M(\mathbb{S}) := \|\mathbb{S}\|^2 \, \chi_{\{\|\mathbb{S}\| \geq M\}}.$$

Let $\mathbb{S}^*$ be the running maximum, i.e.,

$$\mathbb{S}_k^* := \max_{0 \leq i \leq k} \mathbb{S}_i.$$

Since $\mathbb{S}_k > 0$ for each k, $\|\mathbb{S}\| = \mathbb{S}_N^*$.

Lemma 3.1

$$\lim_{M \to \infty} \sup_{\mathbb{Q} \in \mathcal{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[\alpha_M(\mathbb{S})] \leq \lim_{M \to \infty} V(\alpha_M) \leq 0.$$

Proof. Let $p > 2$ be the exponent in Assumption 2.3. Since $\kappa < 1/4$, there exists $r \in (2, p)$ such that $\lambda := \kappa r c_r < 1$, with

$$c_r := \frac{r}{r-1}.$$

We now use Proposition 2.1 in [1] with the portfolio $\hat{\pi} = (\hat{f}, \hat{\gamma})$ given by

$$\hat{f}(\mathbb{S}_N) := (c_r \, \mathbb{S}_N)^r - c_r, \qquad \hat{\gamma}(i, \mathbb{S}) = -r c_r \, (\mathbb{S}_k^*)^{r-1}, \quad k < N.$$

We use (2.1) and Proposition 2.1 in [1] to arrive at

$$\begin{aligned}
Y_N^{\hat{\pi}}(\mathbb{S}) &\geq \|\mathbb{S}\|^r - \kappa \sum_{i=0}^{N-1} \mathbb{S}_i |\hat{\gamma}(i, \mathbb{S}) - \hat{\gamma}(i-1, \mathbb{S})| \\
&\geq \|\mathbb{S}\|^r - \kappa \|\mathbb{S}\| \sum_{i=0}^{N-1} (\hat{\gamma}(i-1, \mathbb{S}) - \hat{\gamma}(i, \mathbb{S})) = \|\mathbb{S}\|^r (1 - \lambda).
\end{aligned} \tag{3.1}$$

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Hence,
$$V\left((1-\lambda)\,\|\mathbb{S}\|^r\right) \le \mathscr{P}(\hat{f}).$$
Clearly, $\alpha_M(\mathbb{S}) \le \|\mathbb{S}\|^r / M^{r-2}$. Hence, by Lemma 2.4,
$$\begin{aligned} V(\alpha_M) &\le V\left(\frac{\|\mathbb{S}\|^r}{M^{r-2}}\right) = \frac{1}{(1-\lambda)M^{r-2}} V\left((1-\lambda)\,\|\mathbb{S}\|^r\right) \\ &\le \frac{1}{(1-\lambda)M^{r-2}}\,\mathscr{P}(\hat{f}). \end{aligned}$$
Since $\hat{f} \in \mathscr{H}$, $\mathscr{P}(\hat{f})$ is finite. Therefore,
$$\lim_{M\to\infty} V(\alpha_M) \le 0.$$
To complete the proof, we recall the proof of (2.8) to restate that for every M,
$$\sup_{\mathbb{Q}\in\mathscr{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}\left[\alpha_M(\mathbb{S})\right] \le V(\alpha_M).$$

∎

This result allows us to consider bounded claims. We also use a compactness argument, to obtain the following equivalence.

Theorem 3.1 *It suffices to prove* (2.9) *for non-negative, bounded, uniformly continuous claims.*

Since the proof of this result is almost orthogonal to the rest of the paper, we relegate it to the Appendix.

In view of the above Theorem, in the sequel we assume that the claim G is non-negative, bounded and is uniformly continuous. So we assume that there exists a constant $K > 0$ and a modulus of continuity, i,e., a continuous function $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $m(0) = 0$, satisfying,

$$0 \le G(\omega) \le K, \qquad |G(\omega) - G(\tilde{\omega})| \le m\left(\|\omega - \tilde{\omega}\|\right), \qquad \forall\, \omega, \tilde{\omega} \in \Omega. \tag{3.2}$$

If $V(G) = -\infty$ then (2.9) is clear. Thus in view of Lemma 2.4 it follows that without loss of generality, we can assume that

$$V(G) \ge 0. \tag{3.3}$$

3.2 Discretization of the space

Next, we introduce a modification of the original super–replication problem. Fix $n \in \mathbb{N}$ and set $h = 1/n$ and $U_n = \{kh,\ k = 0, 1, ...\}$. Denote

$$\mathscr{H}_n := \{f : U_n \to \mathbb{R} | \exists C > 0 \text{ such that } |f(x)| \le C(1 + x^p)\ \forall x\}.$$

For any $g : U_n \to \mathbb{R}$, define a function $\mathscr{L}^{(n)}(g) : \mathbb{R}_+ \to \mathbb{R}$ by,

$$\mathscr{L}^{(n)}(g)(x) := (1-\alpha)g\left(\lfloor nx\rfloor h\right) + \alpha g\left((\lfloor nx\rfloor + 1)h\right), \quad \alpha = nx - \lfloor nx\rfloor, \qquad x \in \mathbb{R}_+, \tag{3.4}$$

where for a real number y, $\lfloor y\rfloor$ is the largest integer less than or equal to y. Observe that

$$\mathscr{L}^{(n)} : \mathscr{H}_n \to \mathscr{H},$$

is a bounded, linear map.

Set $\Omega_n = (U_n)^N$. Clearly $\Omega_n \subset \Omega$ and we consider a financial market where the set of possible stock price processes is the set Ω_n. Then, this restriction lowers the minimal super-replication cost. However, we restrict the admissible portfolios as well. Indeed, for a constant $M > 0$, we define the set of *admissible* portfolio strategies below.

Definition 3.2 For any $M > 1$, we say that $\pi := (g, \gamma)$ is an (*M*-)*admissible portfolio,* if $g \in \mathcal{H}_n$ and $\gamma : \{0, 1, ..., N-1\} \times \Omega_n \to \mathbb{R}$ is a progressively measurable map sastisfying

$$|\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})| \le M, \qquad \forall i \ge 0, \ \mathbb{S} \in \Omega_n.$$

We denote by $\mathscr{A}_M^n$ the set of all admissible portfolios. A portfolio $\pi \in \mathscr{A}_M^n$ is called *perfect* (or perfectly super-replicating) if

$$Y_N^\pi(\mathbb{S}) \ge G(\mathbb{S}), \ \ \forall \mathbb{S} \in \Omega_n,$$

where Y_N^π is given by (2.1).

The minimal super–replication cost is given by

$$V^{n,M}(G) := \inf \left\{ \mathscr{P}^{(n)}(g) \mid \pi := (g, \gamma) \in \mathscr{A}_M^n \text{ is a perfect portfolio} \right\}, \tag{3.5}$$

where we choose the price function as

$$\mathscr{P}^{(n)}(g) := \mathscr{P}\left(\mathscr{L}^{(n)}(g)\right). \tag{3.6}$$

The following provides the crucial connection between the original and the discretized problems. Recall that $h = 1/n$.

Proposition 3.3 *Assume G satisfies* (3.2) *with a modulus function m. Then, for any $M > 0$ and $n \in \mathbb{N}$,*
$$V(G) \le V^{n,M}(G) + (N + 2\kappa)MNh + m(h).$$

Proof. Assume that we have a perfect hedge $\pi = (g, \gamma) \in \mathscr{A}_M^n$ in the sense of Definition 3.2. We continue by lifting this portfolio to a portfolio $\tilde{\pi} = (f, \tilde{\gamma})$ that is defined on Ω.

Let $f = \mathscr{L}^{(n)}(g)$ be as in (3.4) and define $\tilde{\gamma}$ by

$$\tilde{\gamma}(k, \omega) := \gamma(k, \omega_0, \lfloor n\omega_1 \rfloor h, \ldots, \lfloor n\omega_N \rfloor h), \ \ \forall k < N, \ \omega = (\omega_0, \ldots, \omega_N) \in \Omega,$$

where as before $h = 1/n$ and $\lfloor y \rfloor$ is the integer part of y. Clearly $\tilde{\gamma} : \{0, 1, ..., N-1\} \times \Omega \to \mathbb{R}$ is progressively measurable, and $|\tilde{\gamma}(i, \mathbb{S}) - \tilde{\gamma}(i-1, \mathbb{S})| \le M$ for any $i \ge 0$ and $\mathbb{S} \in \Omega$.

For $\mathbb{S} \in \Omega$ define $\mathbb{S}^{(1)}, \mathbb{S}^{(2)}$ by

$$\mathbb{S}_k^{(1)} := \lfloor n\mathbb{S}_k \rfloor h, \ \mathbb{S}_k^{(2)} := \mathbb{S}_k^{(1)} + h\delta_k^N, \ \text{ for all } k \le N,$$

where δ_k^N is equal to one when $k = N$ and zero otherwise. Then, there exists $\lambda \in [0, 1]$ such that $\mathbb{S}_N = \lambda \mathbb{S}_N^{(1)} + (1 - \lambda)\mathbb{S}_N^{(2)}$. Also both $\|\mathbb{S}^{(1)} - \mathbb{S}\|$ and $\|\mathbb{S}^{(2)} - \mathbb{S}\|$ are less than $h = 1/n$. Moreover, $\gamma(k, \mathbb{S}^{(1)}) = \gamma(k, \mathbb{S}^{(2)}) = \tilde{\gamma}(k, \mathbb{S})$ for every $k < N$. We use these together with (2.1), (3.4) and the fact that $\gamma \in [-MN, MN]$. The result is

$$
\begin{aligned}
Y_N^{\tilde{\pi}}(\mathbb{S}) &\ge \lambda Y_N^\pi(\mathbb{S}^{(1)}) + (1 - \lambda)Y_N^\pi(\mathbb{S}^{(2)}) - (N + 2\kappa)MNh \qquad (3.7) \\
&\ge \lambda G(\mathbb{S}^{(1)}) + (1 - \lambda)G(\mathbb{S}^{(2)}) - (N + 2\kappa)MNh \\
&\ge G(\mathbb{S}) - m(h) - (N + 2\kappa)MNh,
\end{aligned}
$$

where the last inequality follows from (3.2). Thus $(f + m(h) + (N + 2\kappa)MNh, \tilde{\gamma})$ is a perfect hedge in the sense of Definition 2.2. This together with the equality $\mathscr{P}_n(g) = \mathscr{P}(f)$ completes the proof. ∎

3.3 Analysis of $V^{n,\sqrt{n}}(G)$

Fix $n > 0$. From Proposition 3.3 and (3.3), it follows that for sufficiently large n

$$V^{n,\sqrt{n}}(G) \geq -1. \tag{3.8}$$

Fix $n \in \mathbb{N}$ sufficiently large such that (3.8) holds true. We introduce three auxiliary sets. Let $\mathcal{W}_n$ be the set of all functions $g \in \mathcal{H}_n$ which satisfy the growth condition

$$\|g\|_* := \sup_{\{x \in U_n\}} \left\{ \frac{|g(x)|}{(1+x)^p} \right\} \leq n.$$

Let $\mathcal{Q}_n$ be the set of all probability measures $\mathbb{Q}$ on Ω_n which satisfy

$$\mathbb{E}_{\mathbb{Q}}\left[\|\mathbb{S}\|^p\right] < \infty.$$

Finally, let $\hat{\mathcal{Q}}_n$ be set of all probability measures $\mathbb{Q} \in \mathcal{Q}_n$ which satisfy

$$\mathbb{E}_{\mathbb{Q}}\left[\, g(\mathbb{S}_N)\,\right] \leq \mathscr{P}^{(n)}(g) + \frac{K+1}{n}\,\|g\|_*, \qquad \forall g \in \mathcal{W}_n, \tag{3.9}$$

where

$$K := \sup_{\mathbb{S} \in \Omega} G(\mathbb{S}).$$

We shall show in the below proof that in view of (3.8), the set of measures $\hat{\mathcal{Q}}_n$ is non-empty for all sufficiently large n.

The following provides an upper bound for the super-replication cost $V^{n,\sqrt{n}}$ defined by (3.5).

Lemma 3.4 *Suppose that G satisfies (3.2)–(3.3). Then, for all sufficiently large n,*

$$V^{n,\sqrt{n}}(G) \leq \sup_{\mathbb{Q} \in \hat{\mathcal{Q}}_n} \mathbb{E}_{\mathbb{Q}}\left(G(\mathbb{S}) - \sqrt{n} \sum_{k=0}^{N-1} \left(\left| \mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N | \mathscr{F}_k) - \mathbb{S}_k \right| - \kappa \mathbb{S}_k \right)^+ \right).$$

Proof. Define $H : \mathcal{W}_n \times \mathcal{Q}_n \to \mathbb{R}$ by

$$H(g, \mathbb{Q}) := \mathbb{E}_{\mathbb{Q}}\left(G(\mathbb{S}) - g(\mathbb{S}_N) - \sqrt{n} \sum_{k=0}^{N-1} \left(\left| \mathbb{E}_{\mathbb{Q}}[\mathbb{S}_N | \mathscr{F}_k] - \mathbb{S}_k \right| - \kappa \mathbb{S}_k \right)^+ \right) + \mathscr{P}^{(n)}(g).$$

Since $\mathscr{P}^{(n)}$ is finite on $\mathcal{W}_n$, in view of the definitions of $\mathcal{W}_n$ and $\mathcal{Q}_n$, H is well defined. We now use Theorem 4.1 that will be proved in the next section with $F(\mathbb{S}) := G(\mathbb{S}) - g(\mathbb{S}_N)$ with an arbitrary $g \in \mathcal{W}_n$. This yields that

$$V^{n,\sqrt{n}}(G) \leq \sup_{\mathbb{Q} \in \mathcal{Q}_n} H(g, \mathbb{Q}), \qquad \forall \, g \in \mathcal{W}_n.$$

Hence,

$$V^{n,\sqrt{n}}(G) \leq \inf_{g \in W_n} \sup_{\mathbb{Q} \in \mathcal{Q}_n} H(g, \mathbb{Q}). \tag{3.10}$$

Since the functions in $\mathcal{W}_n$ are restricted to satisfy the growth condition, the above is possibly an inequality and not an equality.

Next, we continue by interchanging the order of the above infimum and supremum. For that purpose, consider the vector space $\mathbb{R}^{U_n}$ of all functions $g : U_n \to \mathbb{R}$ induced with the topology of point-wise convergence. This space is locally convex and since U_n is countable, $\mathcal{W}_n \subset \mathbb{R}^{U_n}$

is compact. Also, the set $\mathscr{Q}_n$ can be naturally considered as a convex subspace of the vector space $\mathbb{R}^{\Omega_n}$. In order to apply a min-max theorem, we need to show continuity and concavity. In view of Assumption 2.3, in the first variable H is convex and is therefore continuous due to the dominated convergence theorem. We also claim that H is concave in the second variable. For this, it is sufficient to show that for any $k < N$ the functional $\mathbb{E}_{\mathbb{Q}}\left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N|\mathscr{F}_k) - \mathbb{S}_k| - \kappa\mathbb{S}_k\right)^+$ is convex in $\mathbb{Q}$. Indeed, this convexity follows from the following representation

$$\mathbb{E}_{\mathbb{Q}}\left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N|\mathscr{F}_k) - \mathbb{S}_k| - \kappa\mathbb{S}_k\right)^+$$
$$= \sum_{(z_1,\ldots,z_k)\in U_n^k}\left(\left|\sum_{z_N\in U_n} z_N\mathbb{Q}(A(z_1,\ldots,z_k,z_N)) - z_k\mathbb{Q}(B(z_1,\ldots,z_k))\right| - \kappa z_k\mathbb{Q}(B(z_1,\ldots,z_k))\right)^+$$

where

$$A(z_1,\ldots,z_k,z_N) = \{\mathbb{S}_1 = z_1,\ldots,\mathbb{S}_k = z_k,\mathbb{S}_N = z_N\}, \quad B(z_1,\ldots,z_k) = \{\mathbb{S}_1 = z_1,\ldots,\mathbb{S}_k = z_k\}.$$

We now apply Theorem 2 in [4] to the function H. The result is

$$\inf_{g\in\mathscr{W}_n}\sup_{\mathbb{Q}\in\mathscr{Q}_n} H(g,\mathbb{Q}) = \sup_{\mathbb{Q}\in\mathscr{Q}_n}\inf_{g\in\mathscr{W}_n} H(g,\mathbb{Q}).$$

We combine the above inequality with the previous one to obtain,

$$V^{n,\sqrt{n}}(G) \leq \sup_{\mathbb{Q}\in\mathscr{Q}_n}\inf_{g\in\mathscr{W}_n} H(g,\mathbb{Q}). \tag{3.11}$$

Now suppose that $\mathbb{Q}\in\mathscr{Q}_n$ but not in $\hat{\mathscr{Q}}_n$. Then, there is $g^*\in\mathscr{W}_n$ so that

$$\mathbb{E}_{\mathbb{Q}}[g^*(\mathbb{S}_N)] > \mathscr{P}^{(n)}(g^*) + \frac{K+1}{n}\|g^*\|_*.$$

By the positive homogeneity of $\mathscr{P}$, we may assume that $\|g^*\|_* = n$. Then,

$$\mathbb{E}_{\mathbb{Q}}[g^*(\mathbb{S}_N)] > \mathscr{P}^{(n)}(g^*) + K + 1,$$

and recall that $K = \sup_{\Omega} G$. The definition of H yields that

$$H(g^*,\mathbb{Q}) \leq \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] - \mathbb{E}_{\mathbb{Q}}[g^*(\mathbb{S}_N)] + \mathscr{P}^{(n)}(g^*) < \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] - K - 1 \leq -1.$$

In view of (3.8), we conclude that there must exists measures in $\hat{\mathscr{Q}}_n$. Additionally, we may restrict the maximization in (3.11) over the probability measures $\mathbb{Q}\in\hat{\mathscr{Q}}_n$. We use this restricted version of (3.11) to arrive at

$$V^{n,\sqrt{n}}(G) \leq \sup_{\mathbb{Q}\in\hat{\mathscr{Q}}_n}\inf_{g\in\mathscr{W}_n} H(g,\mathbb{Q}) \leq \sup_{\mathbb{Q}\in\hat{\mathscr{Q}}_n} H(0,\mathbb{Q}).$$

Since $\mathscr{P}(0) = 0$, the above is exactly the statement of the lemma. ∎

3.4   Last step of the proof

We combine Proposition 3.3 and Lemma 3.4 to conclude that

$$V(G) \leq \liminf_{n\to\infty}\beta_n, \tag{3.12}$$

where

$$\beta_n := \sup_{\mathbb{Q}\in\hat{\mathscr{Q}}_n}\mathbb{E}_{\mathbb{Q}}\left(G(\mathbb{S}) - \sqrt{n}\sum_{k=0}^{N-1}\left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N|\mathscr{F}_k) - \mathbb{S}_k| - \kappa\mathbb{S}_k\right)^+\right).$$

Thus in order to complete the proof of inequality (2.9) it is sufficient to establish the following.

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Lemma 3.5 *Suppose that G satisfies (3.2)–(3.3). Then,*

$$\liminf_{n\to\infty} \beta_n \leq \sup_{\mathbb{Q}\in\mathcal{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]. \tag{3.13}$$

Proof. From (3.3) and (3.12) it follows that for sufficiently large n, $\beta_n \geq -1$. Therefore, for all sufficiently large $n \in \mathbb{N}$, there exists $\mathbb{Q}_n \in \hat{\mathcal{Q}}_n$ so that

$$\mathbb{E}^{(n)} \sum_{k=0}^{N-1} \left(\left(|\mathbb{E}^{(n)}(\mathbb{S}_N|\mathscr{F}_k) - \mathbb{S}_k| - \kappa\mathbb{S}_k \right)^+ \right) \leq \frac{K+1}{\sqrt{n}} \tag{3.14}$$

and

$$\beta_n \leq \frac{1}{n} + \mathbb{E}^{(n)} \left(G(\mathbb{S}) - \sqrt{n} \sum_{k=0}^{N-1} \left(|\mathbb{E}^{(n)}(\mathbb{S}_N|\mathscr{F}_k) - \mathbb{S}_k| - \kappa\mathbb{S}_k \right)^+ \right), \tag{3.15}$$

where $\mathbb{E}^{(n)}$ denotes the expectation with respect to $\mathbb{Q}_n$. From (3.6) and (3.9) we get

$$\mathbb{E}^{(n)} \left[\mathbb{S}_N^p \right] \leq \mathscr{P}^{(n)}(x^p) + \frac{K+1}{n} \|x^p\|_* \leq \mathscr{P}(2 + 2x^p) + K + 1 < \infty.$$

Hence,

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{(n)} \left[\mathbb{S}_N^p \right] < \infty. \tag{3.16}$$

We claim that the probability measures $\mathbb{Q}_n$, $n \in \mathbb{N}$ are tight. Indeed, in view of the uniform second moment estimate (3.16), tightness would follow from the uniform integrability which states that for any $A > 0$,

$$\lim_{A\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}^{(n)}(\mathbb{S}_k \, \chi_{\{\mathbb{S}_k>A\}}) = 0, \qquad \forall\, k = 1,...,N-1.$$

Since $\mathbb{S}_k$ is $\mathbb{Q}_n$ integrable, the above would follow from

$$\lim_{M\to\infty} \lim_{A\to\infty} \sup_{n\geq M} \mathbb{E}^{(n)}(\mathbb{S}_k \, \chi_{\{\mathbb{S}_k>A\}}) = 0, \qquad \forall\, k = 1,...,N-1. \tag{3.17}$$

We continue by proving (3.17). Fix positive integers $k < N$ and n. Set $X := (1-\kappa)\mathbb{S}_k$, $Y := \mathbb{E}^{(n)}(\mathbb{S}_N|\mathscr{F}_k)$. In view of (3.14), $\mathbb{E}^{(n)}((X-Y)^+) \leq (K+1)/\sqrt{n}$. Therefore, by Cauchy-Schwarz and the Markov inequality, we obtain that for any $A > 0$,

$$
\begin{aligned}
\mathbb{E}^{(n)}[X\,\chi_{\{X>A\}}] &\leq \mathbb{E}^{(n)}\left[((X-Y)^+ + Y)\chi_{\{X>A\}}\right] \leq \mathbb{E}^{(n)}\left[(X-Y)^+\right] + \mathbb{E}^{(n)}\left[Y\,\chi_{\{X>A\}}\right] \\
&\leq \frac{K+1}{\sqrt{n}} + \mathbb{E}^{(n)}\left[Y\,\chi_{\{X>A\}}\right] \leq \frac{K+1}{\sqrt{n}} + \sqrt{\mathbb{E}^{(n)}[Y^2]}\,\sqrt{\mathbb{Q}_n(X>A)} \\
&\leq \frac{K+1}{\sqrt{n}} + \sqrt{\mathbb{E}^{(n)}[\mathbb{S}_N^2]}\,\sqrt{\mathbb{E}^{(n)}[X]}\,\frac{1}{\sqrt{A}} \\
&\leq \frac{K+1}{\sqrt{n}} + \frac{1}{\sqrt{A}}\sqrt{\mathbb{E}^{(n)}[\mathbb{S}_N^2]}\,\sqrt{\frac{K+1}{\sqrt{n}} + \mathbb{E}^{(n)}[Y]} \\
&\leq \frac{K+1}{\sqrt{n}} + \frac{1}{\sqrt{A}}\sqrt{\mathbb{E}^{(n)}[\mathbb{S}_N^2]}\,\sqrt{\frac{K+1}{\sqrt{n}} + \mathbb{E}^{(n)}[\mathbb{S}_N]}.
\end{aligned}
$$

This together with (3.16) yields (3.17) and hence, the uniform integrability of the sequence $\mathbb{Q}_n$.

In view of the Prohorov's Theorem (see [3]), there exists a subsequence $\mathbb{Q}_{n_l}$, $l \in \mathbb{N}$ which converge weakly to a probability measure $\tilde{\mathbb{Q}}$. Then, (3.15) implies that

$$
\begin{aligned}
\tilde{\mathbb{E}} G(\mathbb{S}) &= \lim_{l \to \infty} \mathbb{E}^{(n_l)} G(\mathbb{S}) \\
&\geq \liminf_{n \to \infty} \sup_{\mathbb{Q} \in \mathscr{D}_n} \mathbb{E}_{\mathbb{Q}} \left(G(\mathbb{S}) - \sqrt{n} \sum_{k=0}^{N-1} \left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N | \mathscr{F}_k) - \mathbb{S}_k| - \kappa \mathbb{S}_k \right)^+ \right),
\end{aligned}
$$

where $\tilde{\mathbb{E}}$ denotes the expectation with respect to $\tilde{\mathbb{Q}}$. Then, Proposition 3.3 and Lemma 3.4 imply (2.9) provided that $\tilde{\mathbb{Q}} \in \mathscr{M}_{\kappa,\mathscr{P}}$.

Thus, in order to complete the proof of this lemma, it suffices to show that for the limiting probability measure $\tilde{\mathbb{Q}}$ is in $\mathscr{M}_{\kappa,\mathscr{P}}$.

Fix k and let $h : \mathbb{R}^k \to \mathbb{R}_+$ be a continuous bounded function. Denote by $\|\cdot\|$ the sup norm on $\mathbb{R}^k$. By (3.14), it follows that

$$
\begin{aligned}
\mathbb{E}^{(n)} (\mathbb{S}_N h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) &= \mathbb{E}^{(n)} (\mathbb{E}_n(\mathbb{S}_N | \mathscr{F}_k) h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) \\
&\leq \mathbb{E}^{(n)} ((1+\kappa)\mathbb{S}_k h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) + \frac{(K+1)\|h\|}{\sqrt{n}}.
\end{aligned}
$$

Similarly, we conclude that

$$
\begin{aligned}
\mathbb{E}^{(n)} (\mathbb{S}_N h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) &= \mathbb{E}^{(n)} (\mathbb{E}_n(\mathbb{S}_N | \mathscr{F}_k) h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) \\
&\geq \mathbb{E}^{(n)} ((1-\kappa)\mathbb{S}_k h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) - \frac{(K+1)\|h\|}{\sqrt{n}}.
\end{aligned}
$$

We next take the limit $n_l \to \infty$, and use (3.16), (3.17). The result is

$$
(1-\kappa)\tilde{\mathbb{E}} (\mathbb{S}_k h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) \leq \tilde{\mathbb{E}} (\mathbb{S}_N h(\mathbb{S}_1, \ldots, \mathbb{S}_k)) \leq (1+\kappa)\tilde{\mathbb{E}} (\mathbb{S}_k h(\mathbb{S}_1, \ldots, \mathbb{S}_k)).
$$

The above holds for any non-negative, continuous and bounded function h. Then, by a standard density argument we arrive at

$$
(1-\kappa)\mathbb{S}_k \leq \tilde{\mathbb{E}}(\mathbb{S}_N | \mathscr{F}_k) \leq (1+\kappa)\mathbb{S}_k, \quad k = 0, \ldots, N-1.
$$

Hence, $\tilde{\mathbb{Q}}$ is an κ-approximate martingale law.

We continue by showing that $\tilde{\mathbb{Q}}$ satisfies (2.7). From (3.16) it follows that

$$
\tilde{\mathbb{E}}\left[\mathbb{S}_N^p\right] < \infty. \tag{3.18}
$$

Let $g \in \mathscr{H}$ and let $C > 0$ be such that $g(x) \leq C(1+x)^p$ for all $x \geq 0$. There exists a sequence of continuous functions $\{g_k\}_{k=1}^\infty \subset \mathscr{H}$ which convergence pointwise to g and satisfy $|g_k(x)| \leq C(1+x^p)$, for all $x \geq 0$ and $n \in \mathbb{N}$. Moreover, by (3.18) and the dominated convergence theorem,

$$
\tilde{\mathbb{E}}[g(\mathbb{S}_N)] = \lim_{k \to \infty} \tilde{\mathbb{E}}[g_k(\mathbb{S}_N)].
$$

Thus, to prove $\tilde{\mathbb{E}}[g(\mathbb{S}_N)] \leq \mathscr{P}(g)$, it is sufficient to show that $\tilde{\mathbb{E}}[g_n(\mathbb{S}_N)] \leq \mathscr{P}(g_n)$ for any n. Therefore, without loss of generality we may assume that g is a continuous function. Set $f_n := g_{|U_n}$ and $h_n = \mathscr{L}_n(f_n)$, $n \in \mathbb{N}$. Observe that for sufficiently large n, $f_n \in \mathscr{W}_n$. Since g is continuous, $g(x) = \lim_{n \to \infty} f_n(x_n)$ for any $x \geq 0$ and a sequence $x_n \geq 0$, $n \in \mathbb{N}$ which converge to x. Furthermore the sequence h_n, $n \in \mathbb{N}$ convergence pointwise to g. We use the Skorohod representation theorem, (2.5) and (3.6), to conclude that

$$
\tilde{\mathbb{E}}[g(\mathbb{S}_N)] = \lim_{n \to \infty} \mathbb{E}^{(n)}[f_n(\mathbb{S}_N)] \leq \lim_{n \to \infty} \mathscr{P}_n(f_n) = \lim_{n \to \infty} \mathscr{P}(h_n) \leq \mathscr{P}(g)
$$

as desired.

∎

4 Hedging with Constraints and Transaction costs

This section is devoted to the proof of an auxiliary result that is used in Lemma 3.4.

Fix $n \in \mathbb{N}$ and recall $\Omega_n = \{ kh \mid k = 0, 1, \dots \}$ with $h = 1/n$ as defined in the subsection 3.2. In this section, we do not allow to buy vanilla options, but only to trade the stock with proportional transaction costs. Furthermore, the number of the stocks that the investor is allowed to buy should lie in the interval $[-M, M]$. Therefore, in this section a portfolio strategy is a pair $\tilde{\pi} = (x, \gamma)$ where $x \in \mathbb{R}$ is the initial capital and $\gamma : \{0, 1, ..., N-1\} \times \Omega_n \to \mathbb{R}$ is a progressively measurable map which satisfy $|\gamma(i, S) - \gamma(i-1, S)| \le M$ for all i, S. The portfolio value for any $\mathbb{S} \in \Omega_n$ is given by

$$\tilde{Y}_N^{\tilde{\pi}}(\mathbb{S}) = x + \sum_{i=0}^{N-1} \gamma(i, \mathbb{S})(\mathbb{S}_{i+1} - \mathbb{S}_i) - \kappa \sum_{i=0}^{N-1} \mathbb{S}_i |\gamma(i, \mathbb{S}) - \gamma(i-1, \mathbb{S})|,$$

where as before we set $\gamma(-1, S) \equiv 0$.

Consider a European option with the payoff $\hat{X} = F(\mathbb{S})$ where $F : \Omega_n \to \mathbb{R}$. We do not make any assumptions on the function F. The super–replication price is defined by

$$\tilde{V}(F) = \inf\{x \mid \exists \tilde{\pi} = (x, \gamma) \text{ such that } \tilde{Y}_N^{\tilde{\pi}}(\mathbb{S}) \ge F(\mathbb{S}), \ \forall \, \mathbb{S} \in \Omega_n\}.$$

Theorem 4.1 *For any $F : \Omega_n \to \mathbb{R}$,*

$$\tilde{V}(F) = \sup_{\mathbb{Q} \in \tilde{\mathscr{Q}}_n} \mathbb{E}_{\mathbb{Q}} \left(F(\mathbb{S}) - M \sum_{k=0}^{N-1} \left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N | \mathscr{F}_k) - \mathbb{S}_k| - \kappa \mathbb{S}_k \right)^+ \right),$$

where $\tilde{\mathscr{Q}}_n$ is the set of all probability measures on Ω_n, which are supported on a finite set.

Proof. We start with establishing the inequality

$$\tilde{V}(F) \le \sup_{\mathbb{Q} \in \tilde{\mathscr{Q}}_n} \mathbb{E}_{\mathbb{Q}} \left(F(\mathbb{S}) - M \sum_{k=0}^{N-1} \left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N | \mathscr{F}_k) - \mathbb{S}_k| - \kappa \mathbb{S}_k \right)^+ \right). \tag{4.1}$$

In fact in Lemma 3.4 we used only the above inequality. Without loss of generality we assume that the right hand side of (4.1) is finite.

For a positive integer $J \in \mathbb{N}$, consider the finite set $\Omega_n^J := \{0, h, 2h, \dots, Jh\}^N$ with as before $h = 1/n$. Define the minimal super–replication cost

$$\tilde{V}^J(F) = \inf\{x \mid \exists \tilde{\pi} = (x, \gamma) \text{ such that } \tilde{Y}_N^{\tilde{\pi}}(\mathbb{S}) \ge F(\mathbb{S}), \ \forall \mathbb{S} \in \Omega_n^J\}.$$

The cost $\tilde{V}^J(G)$ is in fact equal to the minimal super-replication cost in the multinomial model which is supported on the set Ω_n^J. Thus, we are in a position to apply Theorem 3.1 in [11] with the penalty function

$$g(\tilde{s}, v) = \begin{cases} \kappa \tilde{s} |v|, & \text{if } |v| \le M, \\ +\infty, & \text{else.} \end{cases} \tag{4.2}$$

The function g is convex in the second variable. Moreover, the convex dual of g is given by

$$\hat{G}(\tilde{s}, y) = \sup_{v \in \mathbb{R}} vy - g(\tilde{s}, v) = M(|y| - \kappa s)^+.$$

Therefore, Theorem 3.1 in [11] implies that

$$\begin{aligned} \tilde{V}^J(F) &= \sup_{\mathbb{Q} \in \mathscr{Q}_n^J} \mathbb{E}_{\mathbb{Q}} \left(F(\mathbb{S}) - M \sum_{k=0}^{N-1} \left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N | \mathscr{F}_k) - \mathbb{S}_k| - \kappa \mathbb{S}_k \right)^+ \right) \\ &\le \sup_{\mathbb{Q} \in \tilde{\mathscr{Q}}_n} \mathbb{E}_{\mathbb{Q}} \left(F(\mathbb{S}) - M \sum_{k=0}^{N-1} \left(|\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_N | \mathscr{F}_k) - \mathbb{S}_k| - \kappa \mathbb{S}_k \right)^+ \right), \end{aligned} \tag{4.3}$$

where $\mathscr{Q}_n^J$ is the set of all probability measures on Ω_n^J.

Now, for every $J \in \mathbb{N}$ there exists a super-replicating portfolio $\tilde{\pi}_J = \left(\tilde{V}^J(F) + 1/J, \gamma_J\right)$ for the multinomial model supported on Ω_n^J. Namely, $\gamma_J : \{0, 1, ..., N-1\} \times \Omega_n \to \mathbb{R}$ is a progressively measurable map such that $|\gamma_J(i, \mathbb{S}) - \gamma_J(i-1, \mathbb{S})| \leq M$ for any $i, \mathbb{S}$ and $\tilde{Y}_N^{\tilde{\pi}_J}(\mathbb{S}) \geq F(\mathbb{S})$, for every $\mathbb{S} \in \Omega_n^J$. By using standard a diagonal procedure, we construct a subsequence $\{\gamma_{J_i}\}_{i=1}^\infty$ such that for any $j = 0, 1, ..., N-1$ and $\mathbb{S} \in \Omega_n$, $\lim_{i \to \infty} \gamma_{J_i}(j, \mathbb{S})$ exists. We denote this limit by $\gamma(j, \mathbb{S})$. Let $x = \liminf_{i \to \infty} \tilde{V}^{J_i}(F)$. Then, clearly $\gamma\{0, 1, ..., N-1\} \times \Omega_n \to \mathbb{R}$ is a progressively measurable map and the portfolio which is given by $\tilde{\pi} = (x, \gamma)$ satisfy $|\gamma_J(i, \mathbb{S}) - \gamma_J(i-1, \mathbb{S})| \leq M$ for any i, $\mathbb{S}$. Moreover, $\tilde{Y}_N^{\tilde{\pi}}(\mathbb{S}) \geq F(\mathbb{S})$, for every $\mathbb{S} \in \Omega_n$. This together with (4.3) yields that

$$\tilde{V}(F) \leq x \leq \sup_{\mathbb{Q} \in \tilde{\mathscr{Q}}_n} \mathbb{E}_\mathbb{Q}\left(F(\mathbb{S}) - M \sum_{k=0}^{N-1}\left(|\mathbb{E}_\mathbb{Q}(\mathbb{S}_N|\mathscr{F}_k) - \mathbb{S}_k| - \kappa\mathbb{S}_k\right)^+\right),$$

and (4.1) follows.

Finally, by using similar arguments to the arguments on page 9 in [11], we prove the inequality

$$\tilde{V}(F) \geq \sup_{\mathbb{Q} \in \tilde{\mathscr{Q}}_n} \mathbb{E}_\mathbb{Q}\left(F(\mathbb{S}) - M \sum_{k=0}^{N-1}\left(|\mathbb{E}_\mathbb{Q}(\mathbb{S}_N|\mathscr{F}_k) - \mathbb{S}_k| - \kappa\mathbb{S}_k\right)^+\right),$$

and complete the proof. ∎

5 Appendix

In this appendix, we prove Theorem 3.1. We proceed in several lemmas. We first use Lemma 3.1 to reduce the problem to bounded claims. Then, using a compactness argument as in [4], we further reduce it to bounded and continuous claims.

Lemma 5.1 *Suppose* (2.9) *holds for all bounded, upper semi-continous continuous functions. Then, it also holds for all G satisfying Assumption 2.1.*

Proof. Suppose that G satisfies Assumption 2.1. Let φ be any smooth function satisfying

$$0 \leq \varphi \leq 1, \quad \varphi(\mathbb{S}) = 1, \ \forall \ \|\mathbb{S}\| \leq 1, \quad \varphi(\mathbb{S}) = 0, \ \forall \ \|\mathbb{S}\| \geq 2.$$

For a constant $M > 1$, set

$$\varphi_M(\mathbb{S}) := \varphi(\mathbb{S}/M), \qquad G_M := G\varphi_M.$$

G_M is bounded and upper semi-continuous. Then, by the hypothesis, the inequality (2.9) and the duality formula holds for G_M. In view of Assumption (2.1),

$$|G(\mathbb{S}) - G_M(\mathbb{S})| \leq L(1 + \|\mathbb{S}\|^2)\chi_{\{\|\mathbb{S}\| \geq M\}}.$$

Let α_M be as in Lemma 3.1. Then, for all sufficiently large M,

$$|G(\mathbb{S}) - G_M(\mathbb{S})| \leq 2L\alpha_M(\mathbb{S}).$$

Since G_M satisfies (2.9),

$$V(G_M) \leq \sup_{\mathbb{Q} \in \mathscr{M}_{\kappa,\mathscr{P}}} \mathbb{E}_\mathbb{Q}[G_M(\mathbb{S})] \leq \sup_{\mathbb{Q} \in \mathscr{M}_{\kappa,\mathscr{P}}} \mathbb{E}_\mathbb{Q}[G(\mathbb{S})] + 2L \sup_{\mathbb{Q} \in \mathscr{M}_{\kappa,\mathscr{P}}} \mathbb{E}_\mathbb{Q}[\alpha_M(\mathbb{S})].$$

By the subadditivity of the minimal super-replication cost V,

$$V(G) \leq V(G_M) + 2L\,V(\alpha_M).$$

Combining the above inequalities and Lemma 3.1, we arrive at

$$
\begin{aligned}
V(G) \quad &\leq \quad \liminf_{M\to\infty}\ [V(G_M) + 2L\,V(\alpha_M)]\\[2mm]
&\leq \quad \sup_{\mathbb{Q}\in\mathcal{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] + 2L \liminf_{M\to\infty}\left[V(\alpha_M) + \sup_{\mathbb{Q}\in\mathcal{M}_{\kappa,\mathscr{P}}}\mathbb{E}_{\mathbb{Q}}[\alpha_M(\mathbb{S})]\right]\\[2mm]
&\leq \quad \sup_{\mathbb{Q}\in\mathcal{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})].
\end{aligned}
$$

∎

The above proof also yields the following equivalence.

Lemma 5.2 *Suppose* (2.9) *holds for all, non-negative, bounded, uniformly continuous functions. Then, it also holds for all G that are bounded and continuous.*

Proof. Let G be a bounded continuous function. By adding G an appropriate constant, we may assume that it is nonnegative as well. Given an integer N, define G_N as before. Since G_N is compactly supported and continuous, it is also uniformly continuous. We then proceed exactly as in the previous lemma to conclude the proof. ∎

We need the following elementary result.

Lemma 5.3 *Let G be bounded and upper semicontinuous. Then, there exists a sequence of uniformly bounded, continuous functions $G_n : \mathbb{R}_+^d \to \mathbb{R}$, so that $G_n \geq G$ and*

$$\limsup_{n\to\infty} G_n(x_n) \leq G(x), \tag{5.1}$$

for every $x \in \mathbb{R}_+^d$ and every sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}_+^d$ with $\lim_{n\to\infty} x_n = x$.

Proof. For a positive integer n, consider the grid $O_n = \left\{\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right), k_1, ..., k_d \in \mathbb{Z}_+\right\}$. Define the function $G_n : O_n \to \mathbb{R}_+$ by

$$G_n(x) = \sup_{\{u\in\mathbb{R}_+^d\ |\ \|u-x\|\leq\frac{2}{n}\}} G(u), \qquad x \in O_n.$$

Next, we extend G_n to the domain $\mathbb{R}_+^d$.

For any $k_1, ..., k_d \in \mathbb{Z}_+$ and a permutation $\sigma : \{1,...,d\} \to \{1,...,d\}$ consider the d–simplex

$$U_{k_1,...,k_d}^\sigma = \left\{(x_1,...,x_d) : \frac{k_i}{n} \leq x_i \leq \frac{k_i+1}{n},\ i=1,...,d\right\} \bigcap \left\{(x_1,...,x_d) : x_{\sigma(i)} \leq x_{\sigma(j)}, \forall i<j\right\}.$$

Fix a simplex $U_{k_1,...,k_d}^\sigma$. Any $u \in U_{k_1,...,k_d}^\sigma$ can be represented uniquely as a convex combination of the simplex vertices $u_1,...,u_{d+1}$ (which belong to O_n). Thus define a continuous function $G_{k_1,...,k_d}^{n,\sigma} : U_{k_1,...,k_d}^\sigma \to \mathbb{R}$ by $G_{k_1,...,k_d}^{n,\sigma}(u) = \sum_{i=1}^{d+1} \lambda_i G_n(u_i)$ where $\lambda_1,...,\lambda_{d+1} \in [0,1]$ with $\sum_{i=1}^{d+1} \lambda_i = 1$ and $\sum_{i=1}^{d+1} \lambda_i u_i = u$, are uniquely determined.

Any element $u \in \mathbb{R}_+^d$ belongs to at least one simplex of the above form. Observe that if u belongs to two simplexes, say $U_{k_1,...,k_d}^\sigma$ and $U_{k_1',...,k_d'}^{\sigma'}$ then $G_{k_1,...,k_d}^{n,\sigma}(u) = G_{k_1',...,k_d'}^{n,\sigma'}(u)$. Thus we can

extend the function $G_n : O_n \to \mathbb{R}$ to a function $G_n : \mathbb{R}^d_+ \to \mathbb{R}$ by setting $G_n(u) = G^{n,\sigma}_{k_1,...,k_d}(u)$ for $u \in U^\sigma_{k_1,...,k_d}$, where $k_1, ..., k_d \in \mathbb{Z}_+$ and $\sigma : \{1, ..., d\} \to \{1, ..., d\}$ is a permutation.

This sequence has the desired properties. ■

The following result completes the proof of theorem 3.1

Lemma 5.4 *Suppose* (2.9) *holds for all bounded, continuous functions. Then, it also holds for all bounded, upper semi-continuous G.*

Proof. Let G be bounded and upper semi-continuous. Let G_n be the sequence of bounded, continuous functions constructed in the previous lemma. Hence (2.9) and Theorem 2.1 holds for G_n.

Using Theorem 2.1, we choose a sequence of probability measures $\mathbb{Q}_n \in \mathscr{M}_{\kappa,\mathscr{P}}$ satisfying,

$$\mathbb{E}^{(n)}[G_n(\mathbb{S})] > V(G_n) - \frac{1}{n}. \tag{5.2}$$

Using similar compactness arguments as in Lemma 3.5, we construct a subsequence $\mathbb{Q}_{n_l}$, $l \in \mathbb{N}$ which converge weakly to a probability measure $\tilde{\mathbb{Q}} \in \mathscr{M}_{\kappa,\mathscr{P}}$. Recall that G_n's are uniformly bounded. Thus, by (5.1) and the Skorohod representation theorem,

$$\limsup_{l \to \infty} \mathbb{E}^{(n_l)}[G_{n_l}(\mathbb{S})] \leq \tilde{\mathbb{E}}[G(\mathbb{S})].$$

This together with (5.2) yields that

$$V(G) \leq \liminf_{n \to \infty} V(G_n) \leq \liminf_{n \to \infty} \mathbb{E}^{(n)}[G_n(\mathbb{S})] \leq \tilde{\mathbb{E}}[G(\mathbb{S})] \leq \sup_{\mathbb{Q} \in \mathscr{M}_{\kappa,\mathscr{P}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})].$$

This completes the proof. ■

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