HW #2 Solutions for PHYS 410

2-12. A particle is projected vertically upward in a constant gravitational field with an initial speed v_0. Show that if there is a retarding force proportional to the square of the instantaneous speed, the speed of the particle when it returns to the initial position is

$$\frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}$$

where v_t is the terminal speed.

2-12

The equation of motion for the upward motion:

$$m\frac{d^2x}{dt^2} = -mkv^2 - mg \qquad (1)$$

Using the relation

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

we can rewrite (1)

$$\frac{v\,dv}{kv^2+g} = -dx \qquad (2)$$

Integrating (2), we find

$$\frac{1}{2k}\log(kv^2+g) = -x + C$$

where the constant C can be computed by using the initial condition that $V = V_0$ when $x=0$:

$$C = \frac{1}{2k}\log(kV_0^2+g)$$

Therefore,

$$x = \frac{1}{2k}\log\frac{kv_0^2+g}{kv^2+g} \qquad (3)$$

Now, the equation of downward motion is

$$m\frac{d^2x}{dt^2} = -mkv^2 + mg$$

This can be rewritten as

$$\frac{v\,dv}{-kv^2+g} = dx \qquad (4)$$

Integrating (4) and using the initial condition that $x=0$ at $v=0$ (we take the highest point as the origin for the downward motion), we find

$$x = \frac{1}{2k}\log\frac{g}{g-kv^2}$$

At the highest point the velocity of the particle must be zero. So we find the highest point by substituting $v=0$ in (3)

$$x_h = \frac{1}{2k}\log\frac{kv_0^2+g}{g} \qquad (5)$$

Then, substituting (5) into (4),

$$\frac{1}{2k}\log\frac{kv_0^2+g}{g} = \frac{1}{2k}\log\frac{g}{g-kv^2}$$

Solving for v,

$$v = \sqrt{\frac{\frac{g}{k}v_0^2}{v_0^2+\frac{g}{k}}}$$

We can find terminal velocity by putting $x \to \infty$

$$x = \frac{1}{2k} \log \frac{g}{g - kv^2}, \quad x \to \infty$$

This gives

$$v_t = \sqrt{\frac{g}{k}}$$

Therefore,

$$v = \frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}$$

2-17. A strong softball player smacks the ball at a height of 0.7 m above home plate. The ball leaves the player's bat at an elevation angle of 35° and travels toward a fence 2 m high and 60 m away in center field. What must the initial speed of the softball be to clear the center field fence? Ignore air resistance.

The setup for this problem is as follows:

$$x = v_0 t \cos\theta$$

$$y = y_0 + v_0 t \sin\theta - \frac{1}{2} g t^2$$

where $\theta = 35°$ and $y_0 = 0.7\,m$. The ball crosses the fence at a time $\tau = R / v_0 \cos\theta$, where $R = 60\,m$. It must be at least $h = 2\,m$ high, so we also need $h - y_0 = v_0 \tau \sin\theta - g\tau^2/2$. Solving for v_0, we obtain

$$v_0^2 = \frac{gR^2}{2\cos\theta\,[R\sin\theta - (h - y_0)\cos\theta]}$$

which gives $v_0 \cong 25.4\ m/s$.

2-25. A block of mass m = 1.62 kg slides down a frictionless incline (Figure 2-A). The block is released a height h = 3.91 m above the bottom of the loop.
(a) What is the force of the inclined track on the block at the bottom (point A)?
(b) What is the force of the track on the block at point B?
(c) At what speed does the block leave the track?
(d) How far away from point A does the block land on level ground?
(e) Sketch the potential energy $U(x)$ of the block. Indicate the total energy on the sketch.

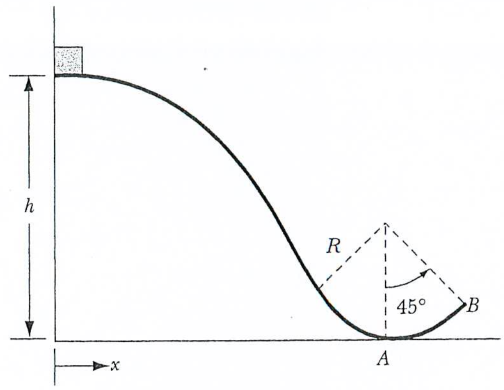


FIGURE 2-A Problem 2-25.

a) At A, the forces on the ball are

$\uparrow N$

$\downarrow mg$

The track counters the gravitational force and provides centripetal acceleration

$$N - mg = mv^2/R$$

Get v by conservation of energy:

$$E_{top} = T_{top} + U_{top} = 0 + mgh$$

$$E_A = T_A + U_A = \frac{1}{2}mv^2 + 0$$

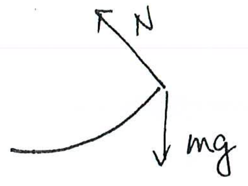
$$E_{top} = E_A \rightarrow v = \sqrt{2gh}$$

So

$$N = mg + m2gh/R$$

$$N = mg\left(1 + \frac{2h}{R}\right)$$

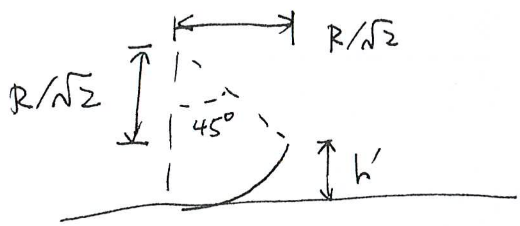
b) At B the forces are:



$$N = mv^2/R + mg\cos 45^\circ$$
$$= mv^2/R + mg/\sqrt{2} \qquad (1)$$

Get v by conservation of energy. From a), $E_{total} = mgh$.

At B, $E = \frac{1}{2}mv^2 + mgh'$, $R = \frac{R}{\sqrt{2}} + h' = R(1 - \frac{1}{\sqrt{2}})$



So $E_{total} = T_B + U_B$ becomes:

$$mgh = mgR(1 - \frac{1}{\sqrt{2}}) + \frac{1}{2}mv^2$$

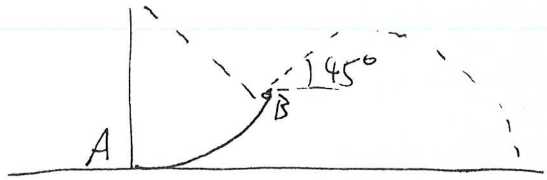
$$v^2 = 2[gh - gR(1 - \frac{1}{\sqrt{2}})]$$

Substituting into (1):

$$N = mg[\frac{2h}{R} + (\frac{3}{\sqrt{2}} - 2)]$$

c) From b) $v_B^2 = 2g[h - R + R/\sqrt{2}]$

d)



Put the origin at A

The equations:

$$x = x_0 + v_{x0} t$$
$$y = y_0 + v_{y0} t - \frac{1}{2} g t^2$$

become

$$x = \frac{R}{\sqrt{2}} + \frac{v_B}{\sqrt{2}} t \qquad (2)$$
$$y = h' + \frac{v_B}{\sqrt{2}} t - \frac{1}{2} g t^2 \qquad (3)$$

Solve (3) for t when $y=0$ (ball lands)

$$g t^2 - \sqrt{2} v_B t - 2h' = 0$$
$$t = \frac{\sqrt{2} v_B \pm \sqrt{2 v_B^2 + 8 g h'}}{2g}$$

We dicard the negative root since it gives a negative time. Substituting into (2):

$$x = \frac{R}{\sqrt{2}} + \frac{v_B}{\sqrt{2}} \left[\frac{\sqrt{2} v_B \pm \sqrt{2 v_B^2 + 8 g h'}}{2g} \right]$$

Using previous expressions for v_B and h' yields

$$x = (\sqrt{2} - 1) R + h + \left[h^2 - \frac{3}{2} R^2 + \sqrt{2} R^2 \right]^{1/2}$$

e) $U(x) = mg\, y(x)$, with $y(0) = h$, so $U(x)$ has the shape of the track.

2-41. A train moves along the tracks at a constant speed u. A woman on the train throws a ball of mass m straight ahead with a speed v with respect to herself. **(a)** What is the kinetic energy gain of the ball as measured by a person on the train? **(b)** by a person standing by the railroad track? **(c)** How much work is done by the woman throwing he ball and **(d)** by the train?

a) As measured on the train:

$$T_i = 0 \; ; \quad T_f = \frac{1}{2} m v^2$$

$$\Delta T = \frac{1}{2} m v^2$$

b) As measured on the ground:

$$T_i = \frac{1}{2} m u^2 \; ; \quad T_f = \frac{1}{2} m (v+u)^2$$

$$\Delta T = \frac{1}{2} m v^2 + m v u$$

c) The woman does an amount of work equal to the kinetic energy gain of the ball as measured in her frame.

$$W = \frac{1}{2} m v^2$$

d) The train does work in order to keep moving at a constant speed u. (If the train did no work, its speed after the woman threw the ball would be slightly less than u, and the speed of the ball relative to the ground would not be $u+v$.) The term mvu is the work that must be supplied by the train.

$$W = m v u$$

2-48. Two gravitationally bound stars with equal masses m, separated by a distance d, revolve about their center of mass in circular orbits. Show that the period τ is proportional to $d^{3/2}$ (Kepler's Third Law) and find the proportionality constant.

2-48 In equilibrium, the gravitationa force and the eccentric force acting on each star must be equal

$$\frac{Gm^2}{d^2} = \frac{mv^2}{d/2} \Rightarrow v = \sqrt{\frac{mG}{2d}} \Rightarrow \tau = \frac{\pi d}{v} = \frac{\sqrt{2}\,\pi\, d^{3/2}}{\sqrt{mG}}$$

Proportionality constant is therefore : $\frac{\sqrt{2}\,\pi}{\sqrt{mG}}$