CS375 HW10

Juan Alejandro Ormaza

November 10 2021

Problem 1

a) Explain (show) why Simpson's rule is exact for $f(x)=1$, $f(x)=x$, $f(x)=x^2$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$f(x)=1$

$$\int_a^b 1\,dx = \underline{b-a} = \frac{b-a}{\cancel{6}}\big(\underbrace{1+4(1)+1}_{\cancel{6}}\big) = \underline{b-a}$$

they are the same

$f(x)=x$

$$\int_a^b x\,dx = \frac{b^2}{2}-\frac{a^2}{2} = \frac{b-a}{6}\left(a+4\left(\frac{a+b}{2}\right)+b\right)$$

$$\frac{b-a}{6}a + 4\,\frac{b^2-a^2}{12} + \frac{b-a}{6}b$$

$$\cancel{\frac{ba}{6}} - \frac{a^2}{6} + \frac{b^2}{3} - \frac{a^2}{3} + \frac{b^2}{6} - \cancel{\frac{ab}{6}}$$

$$\underbrace{\frac{b^2}{3}+\frac{b^2}{6}} - \underbrace{\frac{a^2}{6}-\frac{a^2}{3}}$$

$$\frac{b^2}{2} - \frac{a^2}{2}$$ they are the same

$f(x) = x^2$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3} = \frac{b-a}{6}\left(a^2 + 4\frac{(a+b)^2}{4} + b^2\right)$$

$$\frac{b-a}{6}\left(a^2 + a^2 + 2ab + b^2 + b^2\right)$$

$$\frac{b-a}{6}\left(2a^2 + 2ab + 2b^2\right)$$

$$\frac{b-a}{3}\left(a^2 + ab + b^2\right)$$

$$\frac{b-a}{3}a^2 + \frac{b-a}{3}ab + \frac{b-a}{3}b^2$$

$$\cancel{\frac{ba^2}{3}} - \frac{a^3}{3} + \cancel{\frac{b^2a}{3}} - \cancel{\frac{ba^2}{3}} + \frac{b^3}{3} - \cancel{\frac{ab^2}{3}}$$

$$\frac{b^3}{3} - \frac{a^3}{3}$$

They are the Same

$\therefore$ It is exact.

b. $f(x) = x^3$

$$\int_a^b x^3\,dx = \frac{b^4}{4} - \frac{a^4}{4} = \frac{b-a}{6}\left(a^3 + 4\frac{(a+b)^3}{8} + b^3\right)$$

$$\frac{b-a}{6}\left(a^3 + \frac{1}{2}(a^3 + 3a^2b + 3ab^2 + b^3) + b^3\right)$$

$$\frac{b-a}{6}\left(\frac{3}{2}a^3 + \frac{3}{2}a^2b + \frac{3}{2}ab^2 + \frac{3}{2}b^3\right)$$

$$\frac{b-a}{4}a^3 + \frac{b-a}{4}a^2b + \frac{b-a}{4}ab^2 + \frac{b-a}{4}b^3$$

$$\cancel{\frac{ba^3}{4}} - \frac{a^4}{4} + \cancel{\frac{b^2a^2}{4}} - \cancel{\frac{a^3b}{4}} + \cancel{\frac{b^3a}{4}} - \cancel{\frac{a^2b^2}{4}} + \frac{b^4}{4} - \cancel{\frac{b^3a}{4}}$$

$$\frac{b^4}{4} - \frac{a^4}{4} = \int_a^b x^3\,dx$$

$\therefore$ it is exact

C. $\displaystyle\int_a^b f(x)dx \approx S(f(x)) = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$

Show that $S(\alpha f(x) + \beta g(x)) = \alpha S(f(x)) + \beta S(g(x))$
for functions $f(x)$ and $g(x)$ and scalars α and β

let;

$$S(f(x)) = \frac{b-a}{6}\left(f(a) + f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$S(g(x)) = \frac{b-a}{6}\left(g(a) + g\left(\frac{a+b}{2}\right) + g(b)\right.$$

multiply both by α and β respectively

$$\alpha S(f(x)) = \alpha\,\frac{b-a}{6}\left(f(a) + f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$\beta S(g(x)) = \beta\,\frac{b-a}{6}\left(g(a) + g\left(\frac{a+b}{2}\right) + g(b)\right)$$

Sum both sides

$$\underline{\alpha S(f(x)) + \beta S(g(x))} = \frac{b-a}{6}\left(\underbrace{\alpha f(a) + \beta g(a)} + \underbrace{\alpha f\left(\frac{a+b}{2}\right) + \beta g\left(\frac{a+b}{2}\right)} + \underbrace{\alpha f(b) + \beta g(b)}\right)$$

$$S(\alpha f(x) + \beta g(x))$$

✓

d. α, β is scalar

$p(x)$ up to degree 3

any $p(x)$ can be written as a sum of other polynomials and sums of monomials.

for example:

$f(x) = \alpha g(x) + \beta C(x)$

we know

$$\int_a^b \alpha f(x) + \beta g(x)\, dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \quad (1)$$

in the previous examples we proved

1. $S(f(x))$ for $f(x) = x^3$ is exact.
2. $\alpha S(f(x)) + \beta S(g(x)) = S(\alpha f(x) + \beta g(x))$

use 2. on (1)

$$S(\alpha f(x)) + \beta g(x)) = \alpha S(f(x)) + \beta S(g(x))$$

use 1. to prove that as long as $f(x)$ is of degree 3 or less the combination of

$\alpha S(f(x)) + \beta S(g(x))$

is exact, and if $g(x)$ is another polynomial of degree 3 or less then the linear combination of functions and scalars can create all polynomials

e.g

next page

e.g.

$$\alpha S(f(x)=x^3) + \beta S((g(x)=x^2) + \xi S(k(x)=x) + \varphi S(c(x)=1)$$

for all $\alpha, \beta, \xi, \varphi$ scalars can create any polynomial to a degree 3 and we have proven that $S(f(x))$ is precise up to $f(x)=x^3$.

2. a)

$\frac{5}{9}$ $\quad x_0 = -\sqrt{\frac{3}{5}}$ $\quad \frac{5}{9}$ weights due to Lagendre polynomials

$$\int_{-1}^{1} f(x)\,dx \approx \omega_0 f(x_0) + \omega_1 f(x_1) + \omega_2 f(x_2)$$

$\frac{8}{9}$ $\quad x_1 = 0$ $\quad x_2 = \sqrt{\frac{3}{5}}$

$$\int_0^{\pi} x \cdot \sin(x)\,dx \qquad \pi \to b,\ 0 \to a$$

let

$$x(t) = \frac{b-a}{2}t + \frac{b+a}{2} = \frac{\pi}{2}t + \frac{\pi}{2}$$

$$dx = \frac{b-a}{2}dt = \frac{\pi}{2}$$

$$\int_0^{\pi} x \cdot \sin(x)\,dx = \frac{\pi}{2}\Bigg[\frac{5}{9}\left(\frac{\pi}{2}\left(-\sqrt{\frac{3}{5}}\right) + \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\left(-\sqrt{\frac{3}{5}}\right) + \frac{\pi}{2}\right)$$

$$+ \frac{8}{9}\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$

$$+ \frac{5}{9}\left(\frac{\pi}{2}\left(\sqrt{\frac{3}{5}}\right) + \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\left(\sqrt{\frac{3}{5}}\right) + \frac{\pi}{2}\right)\Bigg]$$

$$\boxed{= 3.14}$$

2.b

$$\int_0^{\pi} x \cdot \sin(x)\,dx =$$

$\boxed{\text{exact integral is } 3.1415... = \text{Pi}}$

$u = x \qquad dv = \sin(x)$

$du = 1dx \qquad v = -\cos(x)$

$$= -x\cos x \Big|_0^{\pi} - -\int_0^{\pi} \cos(x)\,dx$$

$$= -x\cos x \Big|_0^{\pi} + \sin(x)\Big|_0^{\pi}$$

$$= -\pi(-1) + 0 + 0(1) - 0$$

$$\boxed{= \pi}$$

$\therefore$ Solution in Matlab gives the same answer as hand Calculations with certain loss of precision.

2.d

Since Gaussian quadrature gives exact integration for polynomials of degree $2n+1$ and less, we find that for 3 points $(n=2)$ we get $2(2)+1 = 5$ and in the results thrown by Matlab we get $p=6$ which slightly differs from the expected convergence. This could be due to the composite nature of the integration. However, compared to composite trapezoid, this will converge much faster as $p=6 > p=2$.