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CS375: HW7
October 19 2021

4. What is the rate of convergence that you observe for each method? Explain your reasoning for your observed convergence rate. Remember the numerical data may be a bit noisy.

- Bisection method

let $\delta_0 = b - a$

$\delta_1 = \frac{1}{2}\delta_0$

$\delta_2 = \frac{1}{2}\delta_1 = \frac{1}{4}\delta_0$

$\delta_n = \left(\frac{1}{2}\right)^n \delta_0$

$$\frac{\delta_n}{\delta_0} = \left(\frac{1}{2}\right)^n$$

$$|x_* - c_n| \le (b_n - a_n)/2 = \left(\frac{1}{2}\right)^{n+1} \delta_0$$

$$\boxed{\frac{|e_{n+1}|}{|e_n|} = \frac{\cancel{\left(\frac{1}{2}\right)^{n+1}}\cancel{\delta_0}}{\cancel{\left(\frac{1}{2}\right)^{n}}\cancel{\delta_0}} = \left(\frac{1}{2}\right)}$$

Convergence rate agrees with results in matlab code.

- Newton's method

$$f(x_*) = f(x_k) + (x_* - x_k)f'(x_k) + \frac{1}{2}(x_* - x_k)^2 f''(\xi) = 0$$

$$\underbrace{\left(\frac{f(x_k)}{f'(x_k)} - x_k\right)}_{-x_{k+1}} + x_* + (x_* - x_k)^2\left(\frac{1}{2}\right)\frac{f''(\xi)}{f'(x_k)} = 0$$

$$\frac{|x_* - x_{k+1}|}{|x_* - x_k|^2} = \left(\frac{1}{2}\right)\left|\frac{f''(\xi)}{f'(x_k)}\right|$$

when $f'(x_k) \to f'(r)$ convergence rate diverges because denominator $\to 0$

- Bisection Method

↳ Converges with $r=1.62$
will also diverge as $x_k \to x_*$
because denominator becomes 0.

5.

Prove what the general convergence behavior of Newton's method is when used to find cube roots with

$$f(x) = x^3 - a$$

for any number $a \in \mathbb{R}$, $a \neq 0$. You may use any existing convergence theorems in the textbook or from the slides, and make assumptions like having a "suitable starting guess"

$$f'(x_k) = 3x_k^2 \qquad \left(\frac{1}{2}\right)\left|\frac{6\xi}{3x_k^2}\right|$$

$$f''(\xi) = 6\xi$$

as the interval $x_* \leq \xi \leq x_k$
gets smaller $x \approx x_k \approx \xi$
Where $x = \sqrt[3]{a}$ and $a \neq 0$

We thus find that $\lim \dfrac{|e_{k+1}|}{|e_k|^r} \leq C$

is always bounded.

Thus, in this case we find that there is always a constant C such that

$$\left(\frac{1}{2}\right)\frac{6 \cdot x}{3x^2}$$ is always bounded as long as $x = \sqrt[3]{a}$ and $a \neq 0$

for example with $a=8$ $x=2$

and $\boxed{\left(\frac{1}{2}\right) \cdot \frac{6(2)}{3(4)} = \frac{1}{2}}$ which (without the nase) was close to the magnitude of $\frac{e_{k+1}}{e_k^r}$ found in MatLab.

b) What is the convergence rate of Newton's method if $a=0$? Explain.

The convergence rate at $a=0$ is undefined and will cause the method to diverge because there will be a 0 at the denominator.

$\therefore$ $\left(\frac{1}{2}\right)\frac{6}{3(0)}$ ∴ diverges