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Homework 10 CS375

Juan Alejandro Ormaza November 10 2021

```
clear all; clc; close all;  
format long g;
```

Problem 1

a,b,c,d see attachments

Problem 2

```
% a: see attachments.
```

Problem 2.b

```
f=@(x) x*sin(x);  
gauss_int(f,0,pi)
```

ans =

3.14377435398307

Problem 2.c

```
comp_gauss_int(f,0,pi,10)
```

ans =

3.141592655093

Problem 2.d

```
exact_integral=pi;  
n=[4,8,16,32];
```

```

p=zeros(1,length(n));
error = zeros(1,length(n));
h = zeros(1,length(n));
integral_results = zeros(1,length(n));

for i=1:length(n)
    integral_results(i) = comp_gauss_int(f,0,pi,n(i));
    error(i) = abs(exact_integral-comp_gauss_int(f,0,pi,n(i)));
    h(i)=pi/n(i);
    if(i>=2)
        p(i)=log(error(i)/error(i-1))/log(h(i)/h(i-1));
    end
end

fprintf(" n approx integral\t error\t\t\t convergence p\n");
fprintf("%2d %2.10f\t\t %2.16f\t %2.10f\n",[n;integral_results ;error; p])

```

n	approx integral	error	convergence p
4	3.1415930272	0.0000003735699172	0.0000000000
8	3.1415926593	0.0000000057451479	6.0228904358
16	3.1415926537	0.0000000000894151	6.0056810619
32	3.1415926536	0.0000000000013953	6.0018426554

CS375 HW10

Juan Alejandro Ormaza

November 10 2021

Problem 1

a) Explain (show) why Simpson's rule is exact for $f(x)=1$, $f(x)=x$, $f(x)=x^2$

$$\int_a^b f(x) dx \approx \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$$

$f(x)=1$

$$\int_a^b 1 dx = \underline{b-a} = \frac{b-a}{6} (1 + 4(1) + 1) = \underline{b-a}$$

they are the same

$f(x)=x$

$$\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2} = \frac{b-a}{6} \left(a + 4\left(\frac{a+b}{2}\right) + b \right)$$

$$\frac{b-a}{6} a + 4 \frac{b^2-a^2}{12} + \frac{b-a}{6} b$$

$$\frac{ba}{6} - \frac{a^2}{6} + \frac{b^2}{3} - \frac{a^2}{3} + \frac{b^2}{6} - \frac{ab}{6}$$

$$\frac{b^2}{3} + \frac{b^2}{6} - \frac{a^2}{6} - \frac{a^2}{3}$$

$$\underline{\frac{b^2}{2}} - \underline{\frac{a^2}{2}}$$

they are the same

$$f(x) = x^2$$

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3} = \frac{b-a}{6} (a^2 + 4 \frac{(a+b)^2}{4} + b^2)$$

$$\frac{b-a}{6} (a^2 + a^2 + 2ab + b^2 + b^2)$$

$$\frac{b-a}{6} (2a^2 + 2ab + 2b^2)$$

$$\frac{b-a}{3} (a^2 + ab + b^2)$$

$$\frac{b-a}{3} a^2 + \frac{b-a}{3} ab + \frac{b-a}{3} b^2$$

$$\cancel{\frac{ba^2}{3}} - \frac{a^3}{3} + \cancel{\frac{b^2a}{3}} - \cancel{\frac{ba^2}{3}} + \frac{b^3}{3} - \cancel{\frac{ab^2}{3}}$$

$$\frac{b^3}{3} - \frac{a^3}{3}$$

They are the
Same
 \therefore It is exact.

b. $f(x) = x^3$

$$\int_a^b x^3 dx = \frac{b^4}{4} - \frac{a^4}{4} = \frac{b-a}{6} (a^3 + 4 \frac{(a+b)^3}{8} + b^3)$$

$$\frac{b-a}{6} (a^3 + \frac{1}{2} (a^3 + 3a^2b + 3ab^2 + b^3) + b^3)$$

$$\frac{b-a}{6} (\frac{3}{2}a^3 + \frac{3}{2}a^2b + \frac{3}{2}ab^2 + \frac{3}{2}b^3)$$

$$\frac{b-a}{4} a^3 + \frac{b-a}{4} a^2 b + \frac{b-a}{4} a b^2 + \frac{b-a}{4} b^3$$

$$\frac{\cancel{b}a^3}{4} - \frac{a^4}{4} + \frac{\cancel{b}a^2}{4} - \frac{\cancel{a}^3b}{4} + \frac{\cancel{b}^3a}{4} - \frac{\cancel{a}^2b^2}{4} + \frac{b^4}{4} - \frac{\cancel{b}^3a}{4}$$

$$\frac{b^4}{4} - \frac{a^4}{4} = \int_a^b x^3 dx$$

∴ it is exact

C. $\int_a^b f(x) dx \approx S(f(x)) = \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$

Show that $S(\alpha f(x) + \beta g(x)) = \alpha S(f(x)) + \beta S(g(x))$
for functions $f(x)$ and $g(x)$ and scalars α and β .

let:

$$S(f(x)) = \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$$

$$S(g(x)) = \frac{b-a}{6} \left(g(a) + 4g\left(\frac{a+b}{2}\right) + g(b) \right)$$

multiply both by α and β respectively

$$\alpha S(f(x)) = \alpha \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$$

$$\beta S(g(x)) = \beta \frac{b-a}{6} \left(g(a) + 4g\left(\frac{a+b}{2}\right) + g(b) \right)$$

Sum both sides

$$\alpha S(f(x)) + \beta S(g(x)) = \frac{b-a}{6} \left(\alpha f(a) + \beta g(a) + 4\alpha f\left(\frac{a+b}{2}\right) + 4\beta g\left(\frac{a+b}{2}\right) + \alpha f(b) + \beta g(b) \right)$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$S(\alpha f(x) + \beta g(x))$$

d. α, β is scalar

$p(x)$ up to degree 3

any $p(x)$ can be written as a sum of other polynomials and sums of monomials.

for example:

$$f(x) = \alpha g(x) + \beta C(x)$$

we know

$$\int_a^b \alpha f(x) + \beta g(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx \quad (1)$$

in the previous examples we proved

1. $S(f(x))$ for $f(x) = x^3$ is exact.

2. $\alpha S(f(x)) + \beta S(g(x)) = S(\alpha f(x) + \beta g(x))$

use 2. on (1)

$$S(\alpha f(x) + \beta g(x)) = \alpha S(f(x)) + \beta S(g(x))$$

↑

use 1. to prove that
as long as $f(x)$ is of
degree 3 or less

the combination of

$$\alpha S(f(x)) + \beta S(g(x))$$

is exact, and if
 $g(x)$ is another polynomial
of degree 3 or less
then the linear combination
of functions and scalars
can create all polynomials

e.g

next page

e.g.

$$\alpha S(f(x) = x^3) + \beta S(g(x) = x^2) + \gamma S(k(x) = x) + \varphi S(c(x) = 1)$$

for all $\alpha, \beta, \gamma, \varphi$ scalars can create any polynomial to a degree 3 and we have proven that $S(f(x))$ is precise up to $f(x) = x^3$

2. a)

$$\int_{-1}^1 f(x) dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

weights due to Legendre polynomials

$\nearrow \frac{5}{9} \quad \nearrow x_0 = -\sqrt{\frac{3}{5}} \quad \nearrow \frac{5}{9}$
 $\downarrow \frac{8}{9} \quad \downarrow x_1 = 0 \quad \downarrow x_2 = \sqrt{\frac{3}{5}}$

$\int_{-\pi}^{\pi} x \cdot \sin(x) dx$
 $\xrightarrow{b} -1$
 $\xrightarrow{a} 0$

let

$$x(t) = \frac{b-a}{2}t + \frac{b+a}{2} = \frac{\pi}{2}t + \frac{\pi}{2}$$

$$dx = \frac{b-a}{2} dt = \frac{\pi}{2}$$

$$\begin{aligned} \int_0^{\pi} x \cdot \sin(x) dx &= \frac{\pi}{2} \left[\frac{5}{9} \left(\frac{\pi}{2} \left(-\sqrt{\frac{3}{5}} \right) + \frac{\pi}{2} \right) \sin \left(\frac{\pi}{2} \left(-\sqrt{\frac{3}{5}} \right) + \frac{\pi}{2} \right) \right. \\ &\quad \left. + \frac{8}{9} \left(\frac{\pi}{2} \right) \sin \left(\frac{\pi}{2} \right) \right. \\ &\quad \left. + \frac{5}{9} \left(\frac{\pi}{2} \left(\sqrt{\frac{3}{5}} \right) + \frac{\pi}{2} \right) \sin \left(\frac{\pi}{2} \left(\sqrt{\frac{3}{5}} \right) + \frac{\pi}{2} \right) \right] \\ &= 3.14 \end{aligned}$$

2.6

$$\int_0^{\pi} x \cdot \sin(x) dx =$$

exact integral is $3.1415... = \pi$

$$u = x \quad dv = \sin(x)$$

$$du = 1 dx \quad v = -\cos(x)$$

◦◦ Solution in Matlab gives the same answer as hand calculations with certain loss of precision.

$$= -x \cos x \Big|_0^{\pi} - \int_0^{\pi} \cos(x) dx$$

$$= -x \cos x \Big|_0^{\pi} + \sin(x) \Big|_0^{\pi}$$

$$= -\pi(-1) + 0 + 0(1) - 0$$

$$= \pi$$

2.d

Since Gaussian quadrature gives exact integration for polynomials of degree $2n+1$ and less, we find that for 3 points ($n=2$) we get $2(2)+1=5$

and in the results thrown by Matlab we get $p=6$ which slightly differs from the expected convergence.

This could be due to the composite nature of the integration. However, compared to composite trapezoid, this will converge much faster as $p=6 > p=2$.

Ormaza Juan Comp_Gauss_int

```
function integral = comp_gauss_int(f,a,b,n)

x0=a;
h=(b-a)/n;
sum = 0;
xi=0;
x_interval = [x0,xi];

for i=1:n
    xi=x0+i*h;
    x_interval(2)=xi;
    sum=sum+gauss_int(f,x_interval(1),x_interval(2));
    x_interval(1)=xi;
end

integral = sum;

end
```

Not enough input arguments.

Error in comp_gauss_int (line 5)
x0=a;

Ormaza Juan Gauss_int

```
function integral = gauss_int(f,a,b)

w0=5/9; w1=8/9; w2=5/9;
t0=-sqrt(3/5); t1=0; t2=-t0;

x_t=@(t) ((b-a)/2)*t + (b+a)/2;
dx = (b-a)/2;

integral = dx*(w0*f(x_t(t0))+w1*f(x_t(t1))+w2*f(x_t(t2)));

end
```

Not enough input arguments.

Error in gauss_int (line 9)
dx = (b-a)/2;

