CS375 HW 12

Juan Alejandro Ormaza

Nov 30th 2021

1.a) $|\det(BC)| = |\det(B)||\det(C)|$ (i)

The absolute value of determinant of a unitary (orthogonal in real arithmetic) matrix is 1. (ii)

The determinant of a diagonal matrix is the product of the diagonal entries (iii)

use these three facts to show that if $A \in \mathbb{R}^{n \times n}$ (square matrix) then

$$|\det(A)| = \prod_{i=1}^{n} \sigma_i$$

Where σ_i is the ith Singular value. In other words, the absolute value of the determinant of a square matrix is the product of its singular values.

$$A = USV^T$$

$$|\det(A)| = |\det(USV^T)|$$

Product of diagonal entries

$$|\det(A)| = |\cancel{\det(U)}|^{1}\,|\det(S)|\,|\cancel{\det(V^T)}|^{1}$$

Since V & U are orthogonal & $\underbrace{V^TV = I \quad U^TU = I}_{\substack{U^{-1} = U^T \\ V^{-1} = V^T}}$

$$\boxed{|\det(A)| = \prod_{i=1}^{n} \sigma_i \qquad S = \sigma}$$

b)

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & -5 \\ -5 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda = \lambda(\lambda - 10)$$

$$\lambda_1 = 10$$
$$\lambda_2 = 0$$

$$S^2 = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \quad S = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$$

$\lambda_1 = 10$

$$\begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix} v_1 = 0 \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} v_1 = 0 \quad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\lambda_2 = 0$

$$\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} v_2 = 0 \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} v_2 = 0 \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$$

$\lambda_1 = 10$

$$\begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix} u_1 = 0 \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} u_1 = 0$$

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} u_1 = 0$$

$$u_1 = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$\lambda_2 = 0$

$$\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} u_2 = 0 \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

2\. a) Assume you are given the QR factorization of an nxn invertible matrix, i.e., $A = QR$, where Q is unitary (orthogonal in real arithmetic) and R is upper triangle. Explain each step for using Q and R to solve $Ax = b$ for x

to solve $Ax = b$ with $A = QR$

we want to first solve $Rx = C$ using backward substitution. where $C = Q^T b$

this can be seen when we plug $A = QR$ into $Ax = b$

$$QRx = b$$

$$\underbrace{Q^T Q}_{I} \underbrace{Rx}_{C} = Q^T b$$

$$IC = Q^T b \quad C = Q^T b$$

4. a) Suppose that an ordinary differential equation is solved numerically on an interval $[a,b]$ and that the local truncation error is ch^p. Show that if all truncation errors have the same sign (the worst possible case), then the total truncation error is $(b-a)ch^{p-1}$, where $h=\frac{b-a}{n}$

$$\text{Total error} = \sum_{i=1}^{n} ch^p = ch^p + ch^p + ch^p + \dots + ch^p = n\,ch^p$$

$$n = \frac{b-a}{h}$$

$$\boxed{\frac{b-a}{h}ch^p = (b-a)ch^{p-1}}$$

b) By hand) consider the following ODE:

$$\begin{cases} u' = -u^2 - 2\sin(2t) + \cos^2(2t) & t \in [0,1] \quad (1) \\ u(0) = 1 \end{cases}$$

verify that $u(t) = \cos(2t)$ satisfies both the ODE and the initial conditions

ODE:

Plug in (1)

$u(t) = \cos(2t)$

$u'(t) = -2\sin(2t)$

$$-2\cdot\sin(2t) = \cancel{-\cos^2(2t)} - 2\sin(2t) \cancel{+\cos^2(2t)}$$

$$-2\sin(2t) = -2\sin(2t) \checkmark$$

$\cos(2t)$ is a solution

initial conditions:

$U(0) = 1$

$u(t=0) = \cos(2\cdot 0) = 1$

$u'(t=0) = -2\sin(2\cdot 0) = 0$

$u' = -u^2 - 2\sin(2t) + \cos^2(2t)$

$0 = \cancel{-1} - 0 + \cancel{1}$

$0 = 0$

initial conditions are satisfied $\therefore$ $u(t) = \cos(2t)$ Satisfies both the ODE and the initial condition.