CS375 HW5

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1. Prove or provide a counter-example for each of the following statements. You can use Matlab to find counter-examples. Assume A is a matrix and c is a scalar.

a)

$$\|cA\| = |c|\|A\|$$

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

because of vector norms we know

$\|\lambda x\| = |\lambda|\|x\|$ if $\lambda \in \mathbb{R}, x \in V$

$$\|cA\| = \max_{x\neq 0} \frac{\|(cA)x\|}{\|x\|} = \max_{x\neq 0} \frac{\|Acx\|}{\|x\|}$$

$\therefore$

$$= \max_{x\neq 0} \frac{|c|\|Ax\|}{\|x\|}$$

$$= |c| \underbrace{\max_{x\neq 0} \frac{\|Ax\|}{\|x\|}}_{\|A\|}$$

$$\boxed{\therefore \|cA\| = |c|\|A\|}$$

b) $k(cA) = k(A)$ for any nonzero constant c

let $k(A) = \|A\|\|A^{-1}\|$

$$k(cA) = \|cA\| \|(cA)^{-1}\|$$

$$k(cA) = |c| \|A\| |\tfrac{1}{c}| \|A^{-1}\|$$

$$k(cA) = |\cancel{c}| |\tfrac{1}{\cancel{c}}| \cdot \underbrace{\|A\|\|A^{-1}\|}_{k(A)}$$

c) $k(A)$ is the same for every matrix norm

$$\text{cond}(A) = \|A\|_1 \|A^{-1}\|_1 \neq \|A\|_2 \|A^{-1}\|_2$$

false

see matlab attached for counter Example.

2.

$$A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix} \qquad \text{where} \quad k \neq 1$$

a) Find $\|A\|_1$ in terms of k.

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| = \begin{array}{cc} j=1 & j=2 \\ |1|+|1| & |1|+|k| \end{array}$$

$$\|A\|_1 = \begin{cases} 2 & \text{if } k \in (-1, 1) \\ |1| + |k| & \text{for } k > 1 \text{ or } k < -1 \end{cases}$$

b) find $\|A^{-1}\|_1$ in terms of k. remember that

$$\|A^{-1}\| = \left\| \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \right\|$$

$$\|A^{-1}\|_1 = \left| \frac{1}{\det(A)} \right| \underbrace{\left\| \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \right\|_1}$$

$$\left\| \begin{pmatrix} 1 & -k \\ -1 & 1 \end{pmatrix} \right\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| = \begin{array}{c} \overbrace{|1| + |1|}^{j=1} \\ \underbrace{|k| + |1|}_{j=2} \end{array}$$

$$\|A^{-1}\| = \left| \frac{1}{1-k} \right| \cdot \begin{cases} 2 & \text{if } k \in (-1, 1) \\ |1| + |k| & \text{for } k > 1 \text{ or } k < -1 \end{cases}$$

c)

$$\kappa(A) = \|A\|_1 \|A^{-1}\|_1$$

recall

$$\|A\|_1 = \begin{cases} 2 & \text{for } k \in (-1,1) \\ |1| + |k| & \text{for } k>1 \text{ or } k<-1 \end{cases}$$

and

$$\|A^{-1}\|_1 = \begin{cases} \left|\frac{1}{1-k}\right| 2 & \text{for } k \in (-1,1) \\ \left|\frac{1}{1-k}\right|(|1|+|k|) & \text{for } k>1 \quad k<-1 \end{cases}$$

$\therefore$

$$\kappa(A) = \left|\frac{1}{1-k}\right| 4 \quad \text{for } k \in (-1,1)$$

&

$$\kappa(A) = \left|\frac{1}{1-k}\right| (|1|+|k|)^2 \quad \text{for } k>1 \text{ or } k<-1$$

d) Explain geometrically why the condition number grows as $k \to 1$

as k reaches one $\kappa(A)$ for both conditions will approach infinity because of the $\left|\frac{1}{1-k}\right|$ term.

e) If $\kappa(A) = 10^k$ then you can expect to lose at least k digits of precision in solving the system $Ax=b$

$\therefore$ we have $10^k = 10^6$ because we have 16 digits of accuracy and we want 10 digits accuracy it follows that

$$\kappa(A) = 10^6 = \left|\frac{1}{1-k}\right| 4$$

$$\frac{10^6}{4} = \left|\frac{1}{1-k}\right| = \frac{1}{\sqrt{(1-k)^2}}$$

$$\sqrt{(1-k)^2} = \frac{4}{10^6}$$

$$(1-k) = \sqrt{\frac{4^2}{10^{12}}}$$

$$-k = \sqrt{\frac{16}{10^{12}}} - 1$$

$$\boxed{k = 1 - \sqrt{\frac{16}{10^{12}}}}$$

3. Let

$$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix}$$

a) Find the determinant of A

$$\det(A) = 1 - (1+\epsilon)(1-\epsilon) = 1 - (1-\epsilon^2)$$

b) Using double precision floating point arithmetic, for what values of ϵ will $\det(A)$ equal to 0?

recall

$\det(A) = 1-(1-\epsilon^2)$

If $\det(A)=0 \Rightarrow$ $\cancel{-}(1-\epsilon) = \cancel{-}1$

$fl(1-\epsilon^2) = fl(1)$

$fl(1-\epsilon^2) = 1$

$(1-\epsilon^2)(1+\epsilon) = 1$

$1+\epsilon-\epsilon^2-\epsilon^3 = 1$

$\epsilon-\epsilon^2-\epsilon^3 \geq \epsilon_m$

So that

$1+\underbrace{(\epsilon-\epsilon^2-\epsilon^3)}_{<\epsilon_m} = 1$

c) Find the LU factorization of A without pivoting

$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix} \Rightarrow \quad R_2-(1-\epsilon)R_1 \begin{bmatrix} 1 & 1+\epsilon \\ 0 & 1-(1-\epsilon^2) \end{bmatrix}$

$$\boxed{A = \underset{L}{\begin{bmatrix} 1 & 0 \\ (1-\epsilon) & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1+\epsilon \\ 0 & 1-(1-\epsilon^2) \end{bmatrix}}}$$

4. Let ϵ be some small positive number less than half machine epsilon and

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

a) compute $1 - \frac{1}{\epsilon}$ using floating-point arithmetic.

we know

$1 + \epsilon_m > 1$

$1 + \epsilon = 1$ with $|\epsilon| < \epsilon_m$

let's use $\epsilon = \frac{\epsilon_m}{4}$ based on the description of the problem, then ....

$1 + \frac{\epsilon_m}{4} = 1 \rightarrow \frac{1}{\epsilon_m} + \frac{1}{4} = \frac{1}{\epsilon_m}$

notice that $\frac{1}{\epsilon_m} >>> \frac{1}{4}$ or any constant $k > 2$

↳ recall $\epsilon < \frac{\epsilon_m}{2}$

$\therefore$

$1 - \frac{1}{\epsilon} \rightarrow 1 - \frac{k}{\epsilon_m}$

has this term which is larger than 1 and not negligible making

$1 - \frac{1}{\epsilon}$ with $|\epsilon| < \frac{\epsilon_m}{2}$

a negative number of the order of 10^{16} or larger.

and essentially we could approximate

$1 - \frac{1}{\epsilon} \approx -\frac{1}{\epsilon_m}$

b)

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1/\epsilon & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \epsilon & 1 \\ 0 & 1-\frac{1}{\epsilon} \end{bmatrix}}_{U}$$

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow[R_2 - \frac{1}{\epsilon}R_1]{} \begin{bmatrix} \epsilon & 1 \\ 0 & 1-\frac{1}{\epsilon} \end{bmatrix}$$

c) Compute LU for the L and U matrices found in the previous part.

$$\overset{L}{\begin{bmatrix} 1 & 0 \\ 1/\epsilon & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} \epsilon & 1 \\ 0 & 1-\frac{1}{\epsilon} \end{bmatrix}} = \begin{bmatrix} \epsilon + 0 & 1 + \overset{0}{\cancel{0(1-\frac{1}{\epsilon})}} \\ 1+0 & \cancel{\frac{1}{\epsilon}} + 1 - \cancel{\frac{1}{\epsilon}} \end{bmatrix} = \overset{A}{\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}}$$

d) Permute the rows of A, compute the LU factorization of the resulting matrix in floating-point arithmetic, and confirm the product LU is correct.

$$LU = PA$$

$$\longrightarrow P = I$$

5. Consider a symmetric positive definite matrix

$$E = \begin{bmatrix} A & \alpha \\ \alpha^T & c \end{bmatrix}$$

where c is a scalar, α is a $n \times 1$ vector, and A is an $n \times n$ matrix.

a) prove that c is positive and A is positive definite

If A is positive it follows that

$x^T A x > 0$ for all $x \in \mathbb{R}^n$

$$[x_1, x_2, \dots x_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$[x_1, x_2, \dots x_n] \begin{bmatrix} x_1 a_{11} + x_2 a_{22} + \dots x_n a_{1n} \\ \vdots \\ x_1 a_{n1} + \dots\dots x_n a_{nn} \end{bmatrix}$$

$$x_1(x_1 a_{11} + x_2 a_{22} + \dots x_n a_{1n}) + x_2(x_1 a_{21} + \dots x_n a_{2n}) + \dots x_n(x_1 a_{n1} + \dots x_n a_{nn})$$

If A is non-zero and nonsingular then the condition that follows is

$A = LDL^T$

where $x = L^{-T} e_i$ and e_i is the i-th element of I

$\therefore x^T A x = d_i > 0$

b) by the above, the cholesky factorization $A = LL^T$ exists. Assuming $(L^{-1}\alpha)^t(L^{-1}\alpha) < c$, find the cholesky factorization of E in terms of L, α, c

$$E = LDL^T = \begin{bmatrix} A & \alpha \\ \alpha^T & c \end{bmatrix}$$

$$L = \begin{bmatrix} I & 0 \\ \alpha^T A^{-1} & I \end{bmatrix} \rightarrow L^T = \begin{bmatrix} I & A^{-1}\alpha \\ 0 & I \end{bmatrix}$$

$$D = \begin{bmatrix} A & 0 \\ 0 & c - \alpha^T A^{-1}\alpha \end{bmatrix}$$

$$E = \begin{bmatrix} I & 0 \\ \alpha^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & c - \alpha^T A^{-1}\alpha \end{bmatrix} \begin{bmatrix} I & A^{-1}\alpha \\ 0 & I \end{bmatrix}$$