Contents

Homework 3

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CS375
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CS375: HW3

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Q1.

a) Use taylor series expansion to show that

$$\frac{f(a+h)-f(a-h)}{2h} = f'(a) + O(h^2)$$

let's use taylor series for $f(a+h)$ & $f(a-h)$

we find:

$$f(a+h) = \cancel{f(a)} + hf'(a) + \frac{h^2}{2}f''(a) + \frac{h^3}{6}f'''(\xi)$$

$$f(a-h) = \cancel{f(a)} - hf'(a) + \frac{h^2}{2}f''(a) - \frac{h^3}{6}f'''(\xi)$$

$$f(a+h) - f(a-h) = 2hf'(a) + 2\frac{h^3}{6}f'''(\xi)$$

$$\frac{f(a+h)-f(a-h)}{2h} = f'(a) + \underbrace{\frac{h^{\cancel{3}\,2} f'''(\xi)}{3 \cdot 2 \cancel{h}}}$$

$$\boxed{\frac{f(a+h)-f(a-h)}{2h} = f'(a) + O(h^2)}$$

b) $f(x) = e^{1.5x}$ and $a=0$

$f'(x) = 1.5e^{1.5x}$ $\quad f'(x) = 1.5$

C.i Show that the truncation error is bounded C_1h^2 for some constant C_1

notice we found in a) that our central difference is given by

$$f'(a) + \frac{h^2}{6}f'''(\xi) = \frac{f(a+h) - f(a-h)}{2h}$$

where $\frac{h^2}{6}f'''(\xi)$ is our truncation error which can be rewritten as C_1h^2

C_1

C.ii $\frac{fl(f(a+h) - f(a-h))}{2h} = \frac{f(a+h) - f(a-h)(1+\epsilon)}{2h}$ → The error is $\propto$ to $C_1\frac{\epsilon}{h}$ with C being a constant

C.iii

$$\boxed{\text{Total error} = C_1h^2 + \frac{C_2\epsilon}{h}}$$

d) $\frac{dT}{dh} = 0 = 2C_1h - \frac{C_2\epsilon}{h^2}$

$2C_1h^3 = C_2\epsilon$ → $h_{opt} = \underbrace{\left(\frac{C_2}{2C_1}\right)^{1/3}}_{C}\epsilon^{1/3}$

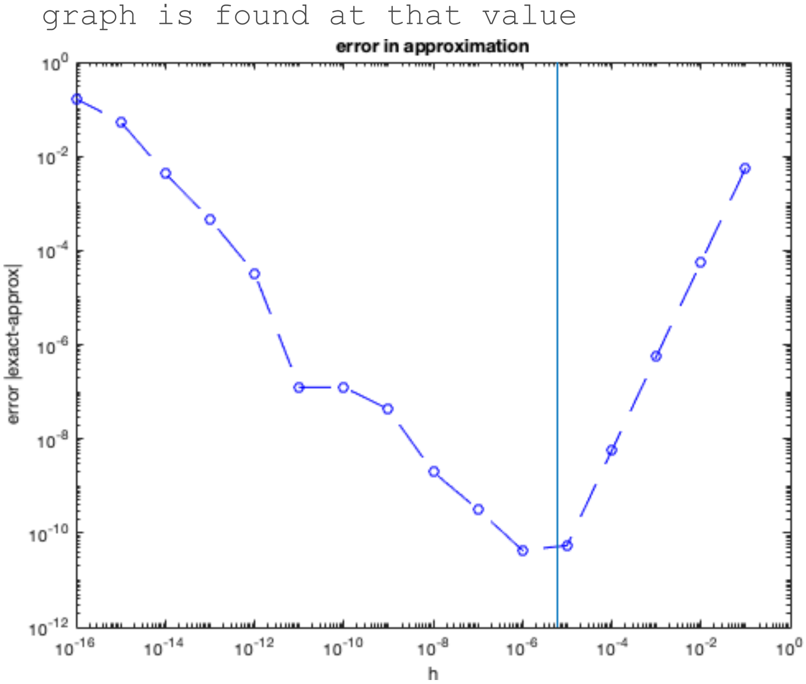
$$\boxed{h_{opt} = C\epsilon^{1/3}}$$

```
clear all; clc; format long e; close all;
```

Problem 1

```
central_difference=@(h) (exp(1.5*h)-exp(1.5*-h))./(2*h);

exact_solution = 1.5*exp(1.5*0);

error=@(exact,approx) abs(exact-approx);

h=logspace(-1,-16,16);

approx_array=central_difference(h);

error_array=error(exact_solution,approx_array);

figure(1)
loglog(h,error_array,'b--o');
hold on
line([power(eps,1/3) power(eps,1/3)],[get(gca,'ylim')]);
title('error in approximation')
xlabel('h');
ylabel('error |exact-approx|')

fprintf('Looking at the graph, we can say that result from problem 1d agrees with the graph\n')
fprintf('this is because the calculation yields an error of C*machine_epsilon^1/3 and the\n')
fprintf('lowest point of the graph is found at that value\n')
```

```
Looking at the graph, we can say that result from problem 1d
agrees with the graphthis is because the calculation yields an
error of C*machine_epsilon^1/3 and the lowest point of the
graph is found at that value
```



Q2.

a) Let a be an $n \times m$ matrix and B be an $m \times p$ matrix

what are the dimensions of $(AB)^T$?

1. $\underbrace{n \times m \quad m \times p}_{\text{inner dimensions agree.}}$

2. $\therefore\ n \times m \cdot m \times p = n \times p$

3. $\boxed{\therefore\ (AB)^T = [p \times n]}$

b) Suppose

$$A = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 & 3 \\ -2 & 6 & 11 \end{pmatrix}$$

Show that $(AB)^T = B^T A^T$

$$A \cdot B = \underset{2\times 2}{\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} -1 & 1 & 3 \\ -2 & 6 & 11 \end{pmatrix}} = \begin{pmatrix} -4+2 & 4-6 & 12-11 \\ 3-4 & -3+12 & -9+22 \end{pmatrix} = \begin{pmatrix} -2 & -2 & 1 \\ -1 & 9 & 13 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} -2 & -1 \\ -2 & 9 \\ 1 & 13 \end{pmatrix}$$

$$B^T = \begin{pmatrix} -1 & -2 \\ 1 & 6 \\ 3 & 11 \end{pmatrix} \quad A^T = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} \quad B^T A^T = \begin{pmatrix} -1 & -2 \\ 1 & 6 \\ 3 & 11 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -4+2 & 3-4 \\ 4-6 & -3+12 \\ 12-11 & -9+22 \end{pmatrix}$$

$$\boxed{B^T A^T = \begin{pmatrix} -2 & -1 \\ -2 & 9 \\ 1 & 13 \end{pmatrix} = (AB)^T \ \checkmark}$$

c) Suppose $A = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}$

Show that $AB \neq BA$

AB

$$\begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 4-15 & -2+35 \\ 2-6 & -1-14 \end{pmatrix} = \begin{pmatrix} -11 & 33 \\ -4 & -15 \end{pmatrix}$$

BA

$$\begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 4-1 & 10+2 \\ 6-7 & 15-14 \end{pmatrix} = \begin{pmatrix} 3 & 12 \\ -1 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} -11 & 33 \\ -4 & -15 \end{pmatrix} \neq \begin{pmatrix} 3 & 12 \\ -1 & 1 \end{pmatrix} \checkmark}$$

d) Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$\det(A) = a_{11}a_{22} - a_{12}a_{21}$

and

$$B = \frac{1}{\det(A)}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

• Verify that $B = A^{-1}$

to verify $B = A^{-1}$ we will use the property $AA^{-1} = I$

1. Calculate B & Det(A)

$\text{Det}\,A = a_{11}a_{22} - a_{12}a_{21}$

$$B = \begin{pmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & -\frac{a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \dfrac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} & -\dfrac{a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \\[2ex] \dfrac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \dfrac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \end{pmatrix}$$

$$AB = \begin{pmatrix} \cancelto{1}{\dfrac{a_{22}a_{11} - a_{21}a_{12}}{a_{11}a_{22} - a_{12}a_{21}}} & \cancelto{0}{\dfrac{-a_{12}a_{11} + a_{12}a_{11}}{a_{11}a_{22} - a_{12}a_{21}}} \\[3ex] \cancelto{0}{\dfrac{a_{21}a_{22} - a_{21}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}} & \cancelto{1}{\dfrac{-a_{12}a_{21} + a_{11}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}} \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \therefore B = A^{-1}$$

Problem 3

```
alpha = 8;

A=[1 5 -3;4 -1 9;7 -2 alpha];
B=[1;2;3];
```

Problem 3.A

```
%i
x=A\B

%ii
determinant=det(A)

%iii
rank_of_matrix=rank(A)
```

```
x =

     4.047619047619047e-01
     1.547619047619048e-01
     5.952380952380956e-02


determinant =

   168

rank_of_matrix =

     3
```

Problem 3.B

```
alpha = 16;

A=[1 5 -3;4 -1 9;7 -2 alpha];

%i
x=A\B

%ii
determinant=det(A)

%iii
rank_of_matrix=rank(A)
```

```
Warning: Matrix is close to singular or badly scaled. Results may be
inaccurate. RCOND =  3.916763e-18.

x =

    -1.232205101418091e+15
     6.161025507090458e+14
     6.161025507090458e+14


determinant =

     1.623106411179671e-14


rank_of_matrix =

     2
```

problem 3.b.4

problem iv

Notice that for problem 3.B.iii answer, we find that the rank is 2. This means that the number of independent columns or rows in the matrix is 2. This means, one column is most likely a linear combination of the 2 independent columns. We find that indeed, that is the case. Column three is a linear combination of 1 and 2. To prove this we calculate alpha and beta (see written solution) to calculate the value of these coefficients.

$$\alpha * (x1, x2, x3) + \beta * (y1, y2, y3) = (z1, z2, z3)$$

note

Notice, MatLab also has trouble find the determinant of the matrix because a singular matrix has no determinant.

```
%iv


alpha=2;
beta=-1;

column1 = [1;4;7];
column2 = [5;-1;-2];
fprintf('the result of adding the first and second column with coefficients 2 an -1 is:')
column3= alpha*column1 + beta*column2
fprintf('the third row of the matrix is:')
A(:,3)
fprintf('therefore, the third column is dependent on the 1st and the 2nd')
```

```
the result of adding the first and second column with coefficients 2 an -1 is:
column3 =

    -3
     9
    16

the third row of the matrix is:
ans =

    -3
     9
    16

therefore, the third column is dependent on the 1st and the 2nd
```

Problem 3.6.4

$$\alpha\begin{pmatrix}1\\4\\7\end{pmatrix}+\beta\begin{pmatrix}5\\-1\\-2\end{pmatrix}=\begin{pmatrix}-3\\9\\16\end{pmatrix}$$

Notice that when $\alpha=2$ and $\beta=-1$ the expression above is true.

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