Ejercicio 2:

2.1. Regla de Simpson de 3/8

$$\int_{x_i}^{x_{i+3}} f(x) \approx \frac{3h}{8}\left(f(x_i) + 3f(x_{i+1}) + 3f(x_{i+2}) + f(x_{i+3})\right)$$

Por Newton-Cotes tenemos:

$$\int_a^b f(x)\,dx \approx (b-a)\sum_{i=0}^{n} c_i f(x_i)$$

Así, tomando 4 puntos para cada subintervalo. Considere $[x_i, x_{i+3}]$ el intervalo con estos 4 puntos espaciados una distancia h.

El polinomio interpolador será:

$$l_i(x) = \frac{(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})}{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i+3})} = -\frac{1}{6h^3}(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})$$

$$l_{i+1}(x) = \frac{(x-x_i)(x-x_{i+2})(x-x_{i+3})}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i+3})} = \frac{1}{2h^3}(x-x_i)(x-x_{i+2})(x-x_{i+3})$$

$$l_{i+2}(x) = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+3})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})(x_{i+2}-x_{i+3})} = -\frac{1}{2h^3}(x-x_i)(x-x_{i+1})(x-x_{i+3})$$

$$l_{i+3}(x) = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+2})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} = \frac{1}{6h^3}(x-x_i)(x-x_{i+1})(x-x_{i+2})$$

$$P_4(x) = f(x_i)l_i(x) + f(x_{i+1})l_{i+1}(x) + f(x_{i+2})l_{i+2}(x) + f(x_{i+3})l_{i+3}$$
$$= \sum_{k=i}^{i+3} f(x_k)\cdot l_k(x)$$

Ahora, si tomamos $a = x_i$ y $b = x_{i+3}$, tenemos:

$$\int_{x_i}^{x_{i+3}} f(x)\,dx \approx \int_{x_i}^{x_{i+3}} P_4(x)\,dx = \int_{x_i}^{x_{i+3}} \sum_{k=i}^{i+3} f(x_k)\, L_k(x)\,dx$$

$$= \int_{x_i}^{x_{i+3}} f(x_i)\, l_i(x)\,dx + \int_{x_i}^{x_{i+3}} f(x_{i+1})\, l_{i+1}(x)\,dx + \int_{x_i}^{x_{i+3}} f(x_{i+2})\, l_{i+2}(x)\,dx + \int_{x_i}^{x_{i+3}} f(x_{i+3})\, l_{i+3}(x)\,dx$$

$$= -\frac{f(x_i)}{6h^3}\int_{x_i}^{x_{i+3}} (x-x_{i+1})(x-x_{i+2})(x-x_{i+3})\,dx + \frac{f(x_{i+1})}{2h^3}\int_{x_i}^{x_{i+3}} (x-x_i)(x-x_{i+2})(x-x_{i+3})\,dx$$
$$- \frac{f(x_{i+2})}{2h^3}\int_{x_i}^{x_{i+3}} (x-x_i)(x-x_{i+1})(x-x_{i+3})\,dx + \frac{f(x_{i+3})}{6h^3}\int_{x_i}^{x_{i+3}} (x-x_i)(x-x_{i+1})(x-x_{i+2})\,dx$$

$$= -\frac{f(x_i)}{6h^3}\int_{x_i}^{x_{i+3}}(x-x_i-h)(x-x_i-2h)(x-x_i-3h)\,dx + \frac{f(x_{i+1})}{2h^3}\int_{x_i}^{x_{i+3}}(x-x_i)(x-x_i-2h)(x-x_i-3h)\,dx$$

$$-\frac{f(x_{i+2})}{2h^3}\int_{x_i}^{x_{i+3}}(x-x_i)(x-x_i-h)(x-x_i-3h)\,dx + \frac{f(x_{i+3})}{6h^3}\int_{x_i}^{x_{i+3}}(x-x_i)(x-x_i-h)(x-x_i-2h)\,dx$$

$u = x - x_i$ Si $x = x_i \to u = 0$

$du = dx$ $\quad x = x_{i+3} = x_i + 3h \to u = 3h$

$$= -\frac{f(x_i)}{6h^3}\int_0^{3h}(u-h)(u-2h)(u-3h)\,du + \frac{f(x_{i+1})}{2h^3}\int_0^{3h}u(u-2h)(u-3h)\,du$$

$$-\frac{f(x_{i+2})}{2h^3}\int_0^{3h}u(u-h)(u-3h)\,du + \frac{f(x_{i+3})}{6h^3}\int_0^{3h}u(u-h)(u-2h)\,du$$

$$\int_0^{3h}(u-ah)(u-bh)(u-ch)\,du = \int_0^{3h}u^3-(a+b+c)hu^2+(ab+ac+bc)h^2u-abch^3\,du$$

$$= \frac{1}{4}u^4-\frac{1}{3}(a+b+c)hu^3+\frac{1}{2}(ab+ac+bc)h^2u^2-abch^3u\Big|_0^{3h}$$

$$= \frac{1}{4}(3h)^4-\frac{1}{3}(a+b+c)h(3h)^3+\frac{1}{2}(ab+ac+bc)h^2(3h)^2-abch^3(3h)$$

$$= \frac{81}{4}h^4-9(a+b+c)h^4+\frac{9}{2}(ab+ac+bc)h^4-3abch^4$$

$$= h^4\left(\frac{81}{4}-9(a+b+c)+\frac{9}{2}(ab+ac+bc)-3abc\right)$$

$$\Rightarrow \quad = -\frac{f(x_i)}{6h^3}\cdot h^4\left(\frac{81}{4}-9(6)+\frac{9}{2}(11)-3\cdot 6\right)+\frac{f(x_{i+1})}{2h^3}\cdot h^4\left(\frac{81}{4}-9(5)+\frac{9}{2}(6)\right)$$

$$-\frac{f(x_{i+2})}{2h^3}\left(\frac{81}{4}-9(4)+\frac{9}{2}(3)\right)+\frac{f(x_{i+2})}{6h^3}h^4\left(\frac{81}{4}-9(3)+\frac{9}{2}(2)\right)$$

$$= -\frac{h}{6}f(x_i)\left(-\frac{9}{4}\right)+\frac{h}{2}f(x_{i+1})\cdot\frac{9}{4}-\frac{h}{2}f(x_{i+2})\cdot\left(-\frac{9}{4}\right)+\frac{h}{6}f(x_{i+3})\left(\frac{9}{4}\right)$$

$$= \frac{9}{4}h\left(\frac{1}{6}f(x_i)+\frac{1}{2}f(x_{i+1})+\frac{1}{2}f(x_{i+2})+\frac{1}{6}f(x_{i+3})\right)$$

$$= \frac{3}{8}h\left(f(x_i)+3f(x_{i+1})+3f(x_{i+2})+f(x_{i+3})\right)$$

Así, $\displaystyle\int_{x_i}^{x_{i+3}}f(x)\,dx \approx \frac{3}{8}h\left(f(x_i)+3f(x_{i+1})+3f(x_{i+2})+f(x_{i+3})\right)$